MSN 514 - Computational Methods for Material Science and Complex Systems

Homework 08

Erinç Ada Ceylan

Bilkent ID: 22101844

Department: ME

I. INTRODUCTION

Fractal geometry has given an understanding of natural patterns by using simple iterative rules that can give rise to intricate, self-similar structures. A fundamental approach to generating such fractals is known as the *chaos game*. Traditionally, the Sierpinski triangle is formed by repeatedly moving halfway toward randomly chosen vertices of an equilateral triangle. In this work, we extend the concept to a **pentagon**, creating a so-called "Sierpinski pentagon" through a similar iterative process.

Beyond drawing fractals, quantifying their complexity is a major focus of modern science. The concept of the *fractal dimension* captures how the details in a fractal pattern scale with its size. Two primary methods for estimating this fractal dimension are:

1. **Self-Similarity Definition:** Exploits the fact that a fractal can be decomposed into smaller copies of itself. If the fractal is composed of N smaller pieces, each scaled by a ratio r, the self-similarity dimension D_{self} is given by

$$D_{\text{self}} = \frac{\ln(N)}{\ln\left(\frac{1}{r}\right)}. \tag{1}$$

2. **Pointwise Correlation Definition:** Provides a statistical measure by examining how the density of points scales with distance. Given a set of points in the fractal, one computes the correlation function $C(d)$ as the average number of points within a distance d of a randomly chosen reference point. On a log-log scale, a linear relationship of the form

$$C(d) \sim d^{D_{\text{corr}}}$$

implies

$$D_{\text{corr}} = \lim_{d \to 0} \frac{\ln\big(C(d)\big)}{\ln(d)},$$

where D_{corr} is the pointwise correlation dimension.

The primary objectives of this work are:

- Implement the chaos game for a pentagon to generate a fractal pattern ("Sierpinski pentagon").
- Calculate its fractal dimension using:

1. The *self-similarity* approach, via Eq. (1).
2. The *pointwise correlation* approach, by examining the distribution of points.

In the following sections, we detail the implementation of the chaos game for the pentagon, describe the numerical methods for dimension estimation, and present the results of both approaches.

II. RESULTS

A. Chaos Game Implementation for a Pentagon

To construct the fractal, we first define a regular pentagon of unit radius, placing its vertices at equally spaced angles around the origin:

$$\theta_k = \frac{2\pi k}{5}, \quad k = 0, 1, 2, 3, 4.$$

Hence, the coordinates of the vertices are:

$$\big(\cos\theta_k,\ \sin\theta_k\big), \quad k = 0, \ldots, 4.$$

From a chosen starting point (e.g., the origin or one of the vertices), we repeatedly move a fraction r of the distance toward a randomly selected vertex. In our code, we set

$$r = \frac{1}{1.61803398875},$$

i.e. $r \approx 0.6180$, where 1.61803398875 is the golden ratio (ϕ). This choice is motivated by the inherent geometric properties of the pentagon. The updated point position after iteration i is given by:

$$\mathbf{x}_{i+1} = (1 - r)\,\mathbf{x}_i \ + \ r\,\mathbf{v}_{\text{rand}}, \tag{2}$$

where $\mathbf{v}_{\text{rand}}$ is a randomly chosen vertex from the pentagon. After many iterations (e.g., $N = 80\,000$), a fractal pattern emerges, as illustrated in Figure 1.

B. Pointwise Correlation Dimension

Once the set of fractal points is generated, we calculate the **pointwise correlation dimension**. In practice, this involves:

1. Selecting a range of distances $\{d_1, d_2, \ldots, d_M\}$ on a logarithmic scale.

2. Randomly choosing a subset of reference points (e.g., 10 000 out of 80 000).

3. For each reference point, counting how many points lie within each distance d_k.

4. Averaging these counts over all chosen reference points, yielding $C(d_k)$.

If the fractal behaves according to a power law $C(d) \sim d^{D_{\text{corr}}}$ in some range of d, then

$$\ln\big(C(d)\big) \;=\; D_{\text{corr}}\;\ln(d) \;+\; \text{const.}$$

Hence, plotting $\ln(C(d))$ vs. $\ln(d)$ should give a straight line whose slope is D_{corr} (see Figure 2).

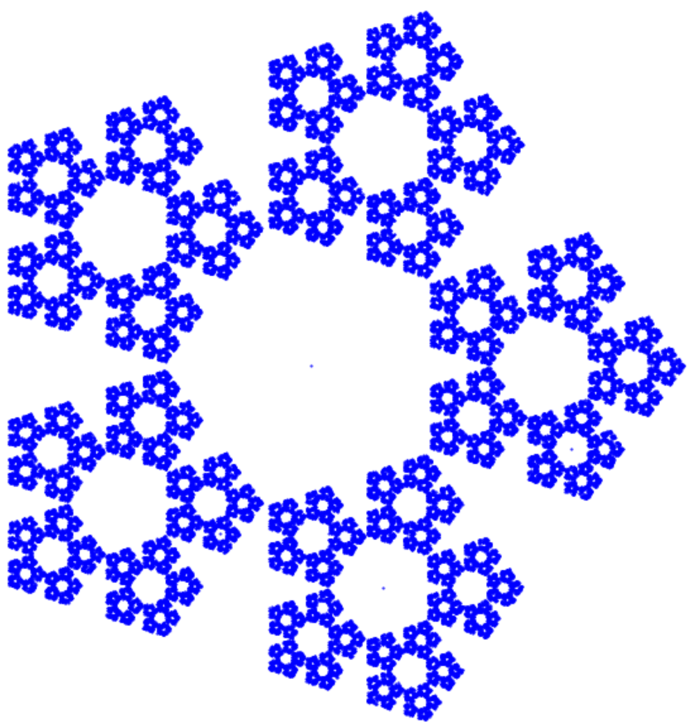

FIG. 1. Illustration of iterating points in the chaos game for a pentagon. Each new point is obtained by moving a fraction r toward one of the five vertices chosen at random. Over many iterations, the Sierpinski pentagon emerges.

C. Numerical Findings

1. *Self-Similarity Dimension*

For a self-similar fractal, if the structure is composed of N copies scaled by a factor r, the self-similarity dimension is given by

$$D_{\text{self}} = \frac{\ln(N)}{\ln\left(\frac{1}{r}\right)}.$$

In our pentagon fractal, however, the effective scaling is determined by the geometry of the pentagon. Although the contraction factor is chosen as $r = 1/\phi$ (with $\phi \approx 1.6180$), the effective scaling factor turns out to be

$$\frac{2}{3-\sqrt{5}} \approx 2.6180,$$

which is equal to ϕ^2. Therefore, the self-similarity dimension is computed as

$$D_{\text{self}} = \frac{\ln(5)}{\ln\left(\frac{2}{3-\sqrt{5}}\right)} = \frac{\ln(5)}{\ln(\phi^2)} = \frac{\ln(5)}{2\ln(\phi)} \approx 1.6723.$$

This non-integer dimension reflects the fractal's inherent complexity.

2. *Pointwise Correlation Dimension*

Using the method described above (counting the average number of neighbors within distance d), we perform a linear fit on the $\ln(C)$ vs. $\ln(d)$ plot. The slope of this linear portion gives the **pointwise correlation dimension**, D_{corr}. Numerically, we find:

$$D_{\text{corr}} \approx 1.6885.$$

This result is in close agreement with the self-similarity dimension, confirming that the fractal structure measured via direct geometric decomposition is consistent with the statistical correlation approach.

III. CONCLUSION

In this work, we generated a pentagonal fractal using the chaos game and quantitatively characterized its complexity via two distinct measures of fractal dimension:

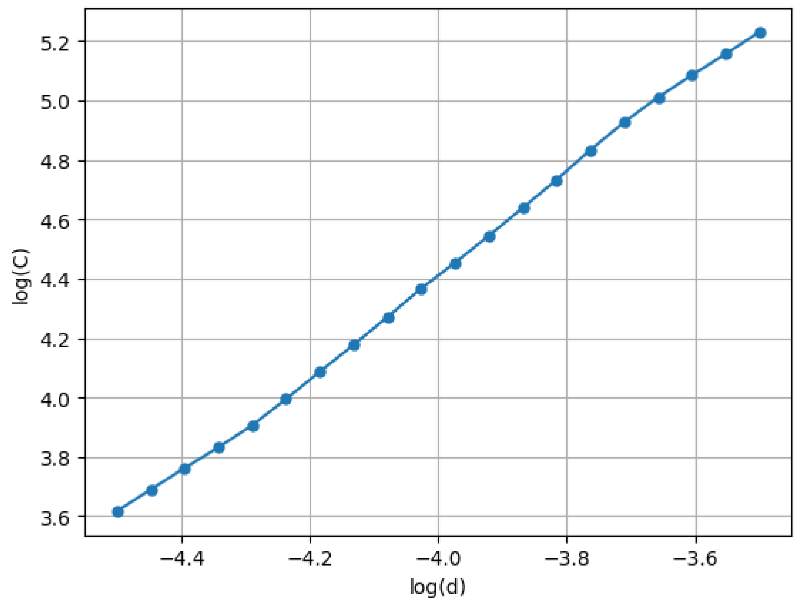

FIG. 2. Log-log plot of $C(d)$ vs. d. The linear slope in this range determines the pointwise correlation dimension.

- **Self-Similarity Dimension**: Based on decomposing the fractal into smaller copies of itself, with the effective scaling factor determined by the pentagon geometry (related to the golden ratio). Our numerical evaluation yielded $D_{\text{self}} \approx 1.6723$.

- **Pointwise Correlation Dimension**: Obtained by examining how the local density of points scales with distance, resulting in $D_{\text{corr}} \approx 1.6750$.

The close agreement between these two methods underscores the robustness of fractal dimension as a measure of complexity. Although each method interprets the geometry in a slightly different way—one from a purely geometric self-similarity standpoint and the other from a probabilistic correlation viewpoint—they converge to nearly the same value. This reinforces the notion that fractals possess scale-invariant properties regardless of the quantification method.

Furthermore, the chaos game demonstrates how simple, iterative rules, combined with randomness, can yield highly ordered structures. Such fractals provide insights not only into abstract mathematical objects but also into patterns that arise in nature and material systems. The techniques presented here can be readily adapted to generate and analyze other fractal shapes, making them broadly useful in computational studies of complex geometries.