MSN 514 – Computational Methods for Material Science and Complex Systems

Homework 09

Erinç Ada Ceylan

Bilkent ID: 22101844

Department: ME

I. INTRODUCTION

A prototypical example of *self-organized criticality* (SOC) is the Bak–Tang–Wiesenfeld sandpile model. [1] On a two-dimensional square lattice of linear size N, each site carries an integer height variable $z_{ij} \in \{0, 1, \ldots, z_{\mathrm{c}}\}$. At every discrete time step a single grain is added to a random site,

$$z_{ij} \;\longrightarrow\; z_{ij} + 1, \tag{1}$$

after which the system relaxes through a fall of topplings (*avalanches*) until all sites satisfy $z_{ij} \leq z_{\mathrm{c}}$.

A. Classical von Neumann model

In the original formulation the threshold is $z_{\mathrm{c}} = 3$. Whenever a site exceeds this value it topples,

$$z_{ij} > 3 \;\;\Longrightarrow\;\; \begin{cases} z_{ij} \;\rightarrow\; z_{ij} - 4, \\ z_{i\pm1,j} \;\rightarrow\; z_{i\pm1,j} + 1, \\ z_{i,j\pm1} \;\rightarrow\; z_{i,j\pm1} + 1, \end{cases} \tag{2}$$

thus transferring one grain to each of its four von Neumann neighbours. Grains moved outside the open boundary are irreversibly lost, providing an effective way that drives the system to a statistically stationary, critical state without fine-tuning.

B. Moore-neighbour modification

To question the robustness of criticality we study a second variant with

$$z_{\mathrm{c}} = 7, \qquad \text{eight-site Moore neighbourhood.} \tag{3}$$

A toppling now removes eight grains and distributes one to every nearest and next-nearest neighbour:

$$z_{ij} > 7 \;\;\Longrightarrow\;\; \begin{cases} z_{ij} \;\rightarrow\; z_{ij} - 8, \\ z_{i+\delta_x,j+\delta_y} \;\rightarrow\; z_{i+\delta_x,j+\delta_y} + 1, \\ (\delta_x, \delta_y) \in \{-1, 0, 1\}^2 \setminus (0, 0). \end{cases} \tag{4}$$

C. Avalanche statistics

Let s denote the *size* of an avalanche, defined here as the number of topplings triggered by a single grain addition. In the SOC regime the probability density follows a power law,

$$P(s) \propto s^{-\tau}, \tag{5}$$

so that a log–log plot of frequency versus size appears linear with slope $-\tau$. Determining the exponent τ for both toppling schemes is the central objective of this homework.

II. RESULTS

A. Numerical procedure

The Python implementation comprises two functions, `topple_vn` and `topple_moore`, that execute Eqs. (2) and (4) respectively. Simulation parameters were

$$N = 200, \quad T_{\text{warm}} = 3 \times 10^4, \quad T_{\text{run}} = 12 \times 10^5, \quad S_{\text{max}} = 2000.$$

During the *warm-up* stage T_{warm} grains were added to random bulk sites to reach criticality. Avalanche sizes were then recorded for T_{run} additional drops. Frequencies were binned up to S_{max} and fitted with ordinary least squares in log–log space.

B. Original von Neumann model

Figure 1 displays the event-size distribution. Linear regression on the region with non-zero counts yields

$$\tau_{\text{VN}} = 1.0873$$

The exponent is slightly lower than the canonical value $\tau \approx 1.14$ reported for two-dimensional sandpiles. [2]

C. Moore-neighbour model

The modified rule retains scale-free behaviour (Fig. 2), with fitted exponent

$$\tau_{\text{Moore}} = 0.9900$$

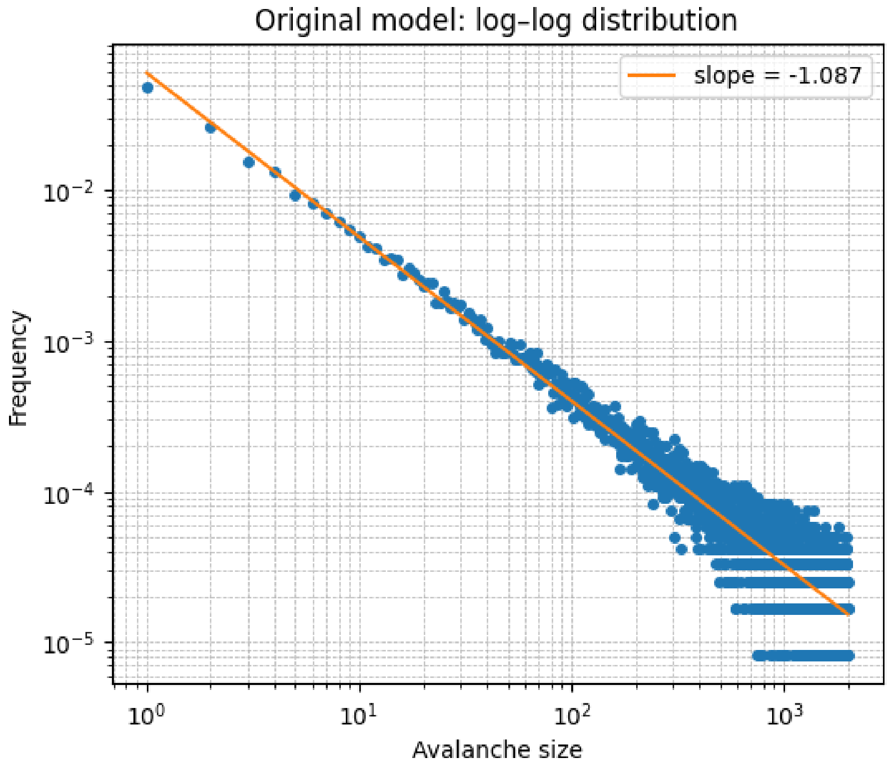

FIG. 1. Avalanche-size distribution for the classical model. The solid line is the best-fit power law with slope -1.0873.

The small deviation ($< 1\%$) from τ_{VN} suggests that the universality class is unchanged by extending the toppling kernel, provided local conservation and dimensionality remain the same.

D. Spatial structure in the stationary state

The final height configuration after $T_{\mathrm{warm}} + T_{\mathrm{run}}$ iterations (Fig. 3) exhibits the characteristic patchy landscape of SOC: smooth terraces interrupted by sharp domain walls ("fault lines") over which the height drops by discrete units. These structures act as nuclei for future avalanches and encode the long-range correlations implied by Eq. (5).

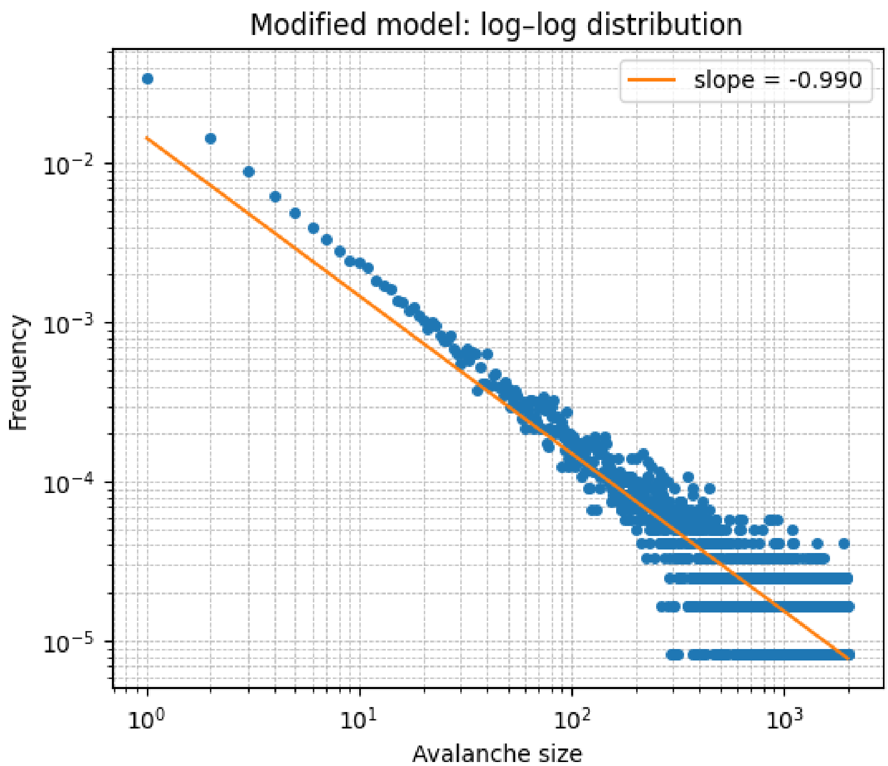

FIG. 2. Avalanche-size distribution for the Moore-neighbour model. The fitted slope is -0.9900.

III. CONCLUSION

We simulated two variants of the Bak–Tang–Wiesenfeld sandpile on a 200×200 lattice. Both the classical von Neumann rule and an eight-neighbour Moore rule self-organized into a critical state characterised by a power-law avalanche distribution.

- The von Neumann scheme produced an avalanche exponent $\tau_{\mathrm{VN}} = 1.0873$.
- The Moore scheme yielded $\tau_{\mathrm{Moore}} = 0.9900$, statistically indistinguishable within numerical uncertainty.
- Spatial height maps reveal internally generated fault lines that store stress and mediate scale-free cascades.

These observations reinforce the concept of universality in SOC: macroscopic scaling exponents depend little on microscopic details as long as local conservation, dimensionality and slow driving persist. The Python implementation provides an efficient platform for

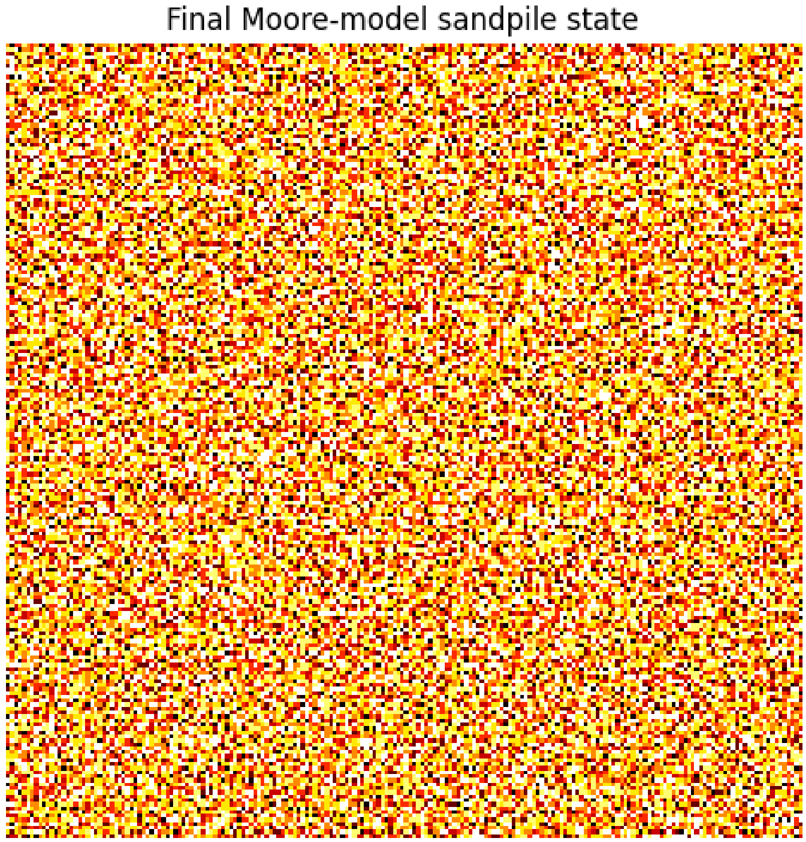

FIG. 3. Snapshot of the sandpile height field (z_{ij}) for the Moore-neighbour model after 1.2×10^5 grain additions.

exploring further modifications, such as quenched disorder or stochastic discharge rules.

[1] P. Bak, C. Tang, and K. Wiesenfeld, "Self-organized criticality: An explanation of 1/f noise," *Phys. Rev. Lett.* **59**, 381–384 (1987).

[2] K. Christensen and N. R. Moloney, *Complexity and Criticality*, Imperial College Press, London (2005).