Expectation and Variance of the Generation Interval Distribution

The intrinsic generation interval distribution is derived following CHAMPREDON et al. 2018, *Equivalence of the Erlang-Distributed SEIR Epidemic Model and the Renewal Equation.*

Derive the Intrinsic Generation Interval from an SEIR model

The intrinsic generation-interval distribution for the Erlang SEIR model for k infectious compartments is given in equation 2.6 from Champredon et al. (2018) as $g(\tau)=\frac{\beta \sum_{k=1}^{n} F_k(\tau)}{\mathcal{R}_0}$.

Transmission matrix, β, from our two-Population SEIR model

In[1]:=
```
β = {{βi, βij}, {βij, βj}}
```
Out[1]= $\{\{\beta i, \beta ij\}, \{\beta ij, \beta j\}\}$

In[2]:=
```
Evaluesβ = Eigenvalues[β]
```
Out[2]= $\left\{\frac{1}{2}\left(\beta i + \beta j - \sqrt{\beta i^2 + 4\,\beta ij^2 - 2\,\beta i\,\beta j + \beta j^2}\right), \frac{1}{2}\left(\beta i + \beta j + \sqrt{\beta i^2 + 4\,\beta ij^2 - 2\,\beta i\,\beta j + \beta j^2}\right)\right\}$

In[3]:=
```
βcomp = Evaluesβ[[2]]
```
Out[3]= $\frac{1}{2}\left(\beta i + \beta j + \sqrt{\beta i^2 + 4\,\beta ij^2 - 2\,\beta i\,\beta j + \beta j^2}\right)$

The instantaneous reproductive number can be calculated from an SEIR model as $\mathcal{R}_t = \beta\, S(t)\, D$, where D is the duration of infection and equal to γ^{-1} for each subpopulation:

In[4]:=
```
S = {Si, Sj}
```
Out[4]= $\{Si, Sj\}$

In[5]:=
```
Rtsubpop = β.S γ^-1
```
Out[5]= $\left\{\frac{Si\,\beta i + Sj\,\beta ij}{\gamma}, \frac{Si\,\beta ij + Sj\,\beta j}{\gamma}\right\}$

Find $F(\tau)$

$F_k(\tau)$ is the probability one individual is alive and in the infectious state τ time units after being infected. Champredon et al. (2018) solve for the probability $F_k(\tau)$ for and SEIR model in equation A.10 for a single population. In our model we have k=1 infected state within each population, but two sub-populations with infected states that occur simultaneously that make up proportions π_i and π_j of the total population. We assume the latency and recover periods are equivalent in populations i and j.

In[6]:=
```
F_SEIR = FullSimplify[
  πi (σi/(σi - γi) (Exp[-γi t] - Exp[-σi t])) + πj (σj/(σj - γj) (Exp[-γj t] - Exp[-σj t])) /.
   {σi → σ, σj → σ, γi → γ, γj → γ}, Assumptions → {πi + πj == 1}]
```

Out[6]=
$$-\frac{\left(e^{-t\gamma}-e^{-t\sigma}\right)\sigma}{\gamma-\sigma}$$

In[7]:=
```
Integrate[F_SEIR, {t, 0, Infinity}]
```

Out[7]=
$$\frac{1}{\gamma} \text{ if } \mathrm{Re}[\gamma] > 0 \,\&\&\, \mathrm{Re}[\sigma] > 0$$

Find $\mathcal{R}_0$

The definition of $\mathcal{R}_0$ given in Champredon et al. (2018) is $\mathcal{R}_0=\beta\int_0^\infty \sum_{k=1}^n F_k(\tau)\,d\tau$.

In[8]:=
```
R0matrix = β γ^-1
```

Out[8]=
$$\left\{\left\{\frac{\beta i}{\gamma},\frac{\beta ij}{\gamma}\right\},\left\{\frac{\beta ij}{\gamma},\frac{\beta j}{\gamma}\right\}\right\}$$

In[9]:=
```
EvaluesR0 = Eigenvalues[R0matrix]
```

Out[9]=
$$\left\{\frac{\beta i+\beta j-\sqrt{\beta i^2+4\,\beta ij^2-2\,\beta i\,\beta j+\beta j^2}}{2\,\gamma},\frac{\beta i+\beta j+\sqrt{\beta i^2+4\,\beta ij^2-2\,\beta i\,\beta j+\beta j^2}}{2\,\gamma}\right\}$$

$\mathcal{R}_0$ from the NEst Generation Method when S_i and S_j and treated as proportions $\frac{S_i}{N_i}\to 1$ and $\frac{S_j}{N_j}\to 1$ is

In[10]:=
```
R0NextGen =
  (Si0 βi + Sj0 βj + Sqrt[Si0^2 βi^2 + 4 Si0 Sj0 βij^2 - 2 Si0 Sj0 βi βj + Sj0^2 βj^2]) / (2 γ) /. {Si0 → 1, Sj0 → 1}
```

Out[10]=
$$\frac{\beta i+\beta j+\sqrt{\beta i^2+4\,\beta ij^2-2\,\beta i\,\beta j+\beta j^2}}{2\,\gamma}$$

Find $g(\tau)$

In[11]:=
```
gSEIR = FullSimplify[βcomp FSEIR / R0NextGen]
```
Out[11]=

$$-\frac{\left(e^{-t\gamma}-e^{-t\sigma}\right)\gamma\,\sigma}{\gamma-\sigma}$$

In[12]:=
```
Integrate[ gSEIR, {t, 0, Infinity}]
```
Out[12]=

1 if $\mathrm{Re}[\gamma] > 0$ && $\mathrm{Re}[\sigma] > 0$

Find the expectation and variance of $g(\tau)$

In[13]:=
```
EgSEIR = Integrate[t gSEIR, {t, 0, Infinity}]
```
Out[13]=

$\frac{1}{\gamma}+\frac{1}{\sigma}$ if $\mathrm{Re}[\gamma] > 0$ && $\mathrm{Re}[\sigma] > 0$

In[14]:=
```
EgSEIR2 = Integrate[t^2 gSEIR, {t, 0, Infinity}]
```
Out[14]=

$\frac{2\left(\gamma^2+\gamma\,\sigma+\sigma^2\right)}{\gamma^2\,\sigma^2}$ if $\mathrm{Re}[\gamma] > 0$ && $\mathrm{Re}[\sigma] > 0$

In[15]:=
```
VgSEIR = FullSimplify[EgSEIR2 - EgSEIR^2]
```
Out[15]=

$\frac{1}{\gamma^2}+\frac{1}{\sigma^2}$ if $\mathrm{Re}[\gamma] > 0$ && $\mathrm{Re}[\sigma] > 0$