Two-Pop Model $\mathcal{R}_0$ using Next Generation Matrix (NGM) Method

$\mathcal{R}_0$ is the dominant eigen value of $K = -\mathcal{F}\mathcal{V}^{-1}$

Find the disease free equilibria (DFE)

In[1]:=
```
DFE = {Si → Si0, Sj → Sj0, Ei → 0, Ej → 0, Ii → 0, Ij → 0};
```

Build vectors $\mathcal{F}$ and $\mathcal{V}$ to linearize the *infection subsystem*

$\mathcal{F}$ represents the rate of appearance of new infections in compartment i, $\mathcal{V}_i^+(x)$ represents the rate of transfer of individuals into compartment i by all other means, and $\mathcal{V}_i^-(x)$ represents the rate of transfer of individuals out of compartment i, where $\mathcal{V}_i(x)=\mathcal{V}_i^-(x)-\mathcal{V}_i^+(x)$. The infected compartments x={E,I} and the non-infected compartments are y={S,R}. Thus, this can be written as $\frac{dx}{dt}=\mathcal{F}(x)-\mathcal{V}(x)$.
https://www.ncbi.nlm.nih.gov/pmc/articles/PMC2871801/

In[2]:=
```
𝓕 = {βi Si Ii + βij Si Ij, βj Sj Ij + βij Sj Ii, 0, 0}
```
Out[2]=
```
{Ii Si βi + Ij Si βij, Ii Sj βij + Ij Sj βj, 0, 0}
```

In[3]:=
```
MatrixForm[𝓕]
```
Out[3]//MatrixForm=

$$\begin{pmatrix} Ii\,Si\,\beta i + Ij\,Si\,\beta ij \\ Ii\,Sj\,\beta ij + Ij\,Sj\,\beta j \\ 0 \\ 0 \end{pmatrix}$$

In[4]:=
```
𝓥 = {-Ei σ, -Ej σ, Ei σ - Ii γ, Ej σ - Ij γ}
```
Out[4]=
```
{-Ei σ, -Ej σ, -Ii γ + Ei σ, -Ij γ + Ej σ}
```

In[5]:=
```
MatrixForm[𝓥]
```
Out[5]//MatrixForm=

$$\begin{pmatrix} -Ei\,\sigma \\ -Ej\,\sigma \\ -Ii\,\gamma + Ei\,\sigma \\ -Ij\,\gamma + Ej\,\sigma \end{pmatrix}$$

Find the Jacobian of $\mathcal{F}$ and $\mathcal{V}$ around the DFE

In[6]:= `F = Grad[ℱ, {Ei, Ej, Ii , Ij}]`

Out[6]= $\{\{0, 0, Si\,\beta i, Si\,\beta ij\}, \{0, 0, Sj\,\beta ij, Sj\,\beta j\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[7]:= `Fdfe = F /. DFE`

Out[7]= $\{\{0, 0, Si0\,\beta i, Si0\,\beta ij\}, \{0, 0, Sj0\,\beta ij, Sj0\,\beta j\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[8]:= `MatrixForm[Fdfe]`

Out[8]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & Si0\,\beta i & Si0\,\beta ij \\ 0 & 0 & Sj0\,\beta ij & Sj0\,\beta j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[9]:= `V = Grad[𝒱, {Ei, Ej, Ii , Ij}]`

Out[9]= $\{\{-\sigma, 0, 0, 0\}, \{0, -\sigma, 0, 0\}, \{\sigma, 0, -\gamma, 0\}, \{0, \sigma, 0, -\gamma\}\}$

In[10]:= `MatrixForm[V]`

Out[10]//MatrixForm=

$$\begin{pmatrix} -\sigma & 0 & 0 & 0 \\ 0 & -\sigma & 0 & 0 \\ \sigma & 0 & -\gamma & 0 \\ 0 & \sigma & 0 & -\gamma \end{pmatrix}$$

In[11]:= `Vdfe = V /. DFE`

Out[11]=

$\{\{-\sigma, 0, 0, 0\}, \{0, -\sigma, 0, 0\}, \{\sigma, 0, -\gamma, 0\}, \{0, \sigma, 0, -\gamma\}\}$

In[12]:= `MatrixForm[Vdfe]`

Out[12]//MatrixForm=

$$\begin{pmatrix} -\sigma & 0 & 0 & 0 \\ 0 & -\sigma & 0 & 0 \\ \sigma & 0 & -\gamma & 0 \\ 0 & \sigma & 0 & -\gamma \end{pmatrix}$$

Take the inverse of V

In[13]:= `Vinverse = Inverse[Vdfe]`

Out[13]=

$$\left\{\left\{-\frac{1}{\sigma}, 0, 0, 0\right\}, \left\{0, -\frac{1}{\sigma}, 0, 0\right\}, \left\{-\frac{1}{\gamma}, 0, -\frac{1}{\gamma}, 0\right\}, \left\{0, -\frac{1}{\gamma}, 0, -\frac{1}{\gamma}\right\}\right\}$$

In[14]:= **MatrixForm[Vinverse]**

Out[14]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{\sigma} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sigma} & 0 & 0 \\ -\frac{1}{\gamma} & 0 & -\frac{1}{\gamma} & 0 \\ 0 & -\frac{1}{\gamma} & 0 & -\frac{1}{\gamma} \end{pmatrix}$$

In[15]:= **K = -Fdfe.Vinverse**

Out[15]=

$$\left\{\left\{\frac{\text{Si0}\,\beta\text{i}}{\gamma}, \frac{\text{Si0}\,\beta\text{ij}}{\gamma}, \frac{\text{Si0}\,\beta\text{i}}{\gamma}, \frac{\text{Si0}\,\beta\text{ij}}{\gamma}\right\}, \left\{\frac{\text{Sj0}\,\beta\text{ij}}{\gamma}, \frac{\text{Sj0}\,\beta\text{j}}{\gamma}, \frac{\text{Sj0}\,\beta\text{ij}}{\gamma}, \frac{\text{Sj0}\,\beta\text{j}}{\gamma}\right\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\right\}$$

In[16]:= **MatrixForm[K]**

Out[16]//MatrixForm=

$$\begin{pmatrix} \frac{\text{Si0}\,\beta\text{i}}{\gamma} & \frac{\text{Si0}\,\beta\text{ij}}{\gamma} & \frac{\text{Si0}\,\beta\text{i}}{\gamma} & \frac{\text{Si0}\,\beta\text{ij}}{\gamma} \\ \frac{\text{Sj0}\,\beta\text{ij}}{\gamma} & \frac{\text{Sj0}\,\beta\text{j}}{\gamma} & \frac{\text{Sj0}\,\beta\text{ij}}{\gamma} & \frac{\text{Sj0}\,\beta\text{j}}{\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[17]:= **Det[K]**

Out[17]=

0

In[18]:= **Evalues = Eigenvalues[K]**

Out[18]=

$$\left\{0, 0, \frac{\text{Si0}\,\beta\text{i} + \text{Sj0}\,\beta\text{j} - \sqrt{\text{Si0}^2\,\beta\text{i}^2 + 4\,\text{Si0}\,\text{Sj0}\,\beta\text{ij}^2 - 2\,\text{Si0}\,\text{Sj0}\,\beta\text{i}\,\beta\text{j} + \text{Sj0}^2\,\beta\text{j}^2}}{2\,\gamma}, \frac{\text{Si0}\,\beta\text{i} + \text{Sj0}\,\beta\text{j} + \sqrt{\text{Si0}^2\,\beta\text{i}^2 + 4\,\text{Si0}\,\text{Sj0}\,\beta\text{ij}^2 - 2\,\text{Si0}\,\text{Sj0}\,\beta\text{i}\,\beta\text{j} + \text{Sj0}^2\,\beta\text{j}^2}}{2\,\gamma}\right\}$$

In[19]:= **R0 = Evalues[[4]]**

Out[19]=

$$\frac{\text{Si0}\,\beta\text{i} + \text{Sj0}\,\beta\text{j} + \sqrt{\text{Si0}^2\,\beta\text{i}^2 + 4\,\text{Si0}\,\text{Sj0}\,\beta\text{ij}^2 - 2\,\text{Si0}\,\text{Sj0}\,\beta\text{i}\,\beta\text{j} + \text{Sj0}^2\,\beta\text{j}^2}}{2\,\gamma}$$