Clancey et al. MechanisticModel (STOPPP)

SIR Model with Waning Antibodies and Waning Immunity

N is the total population size, *S* is the number of susceptible individuals, *I* is the number of infected individuals, R_A is the number of seropositive and recovered individuals, and R_T is the number of seronegative and recovered individuals.

```
In[1]:= Ṅ = Ṡ + İ + RȦ + RṪ
Out[1]= RȦ + RṪ + Ṡ + İ

In[2]:= Ṡ = b N - β S I - S (μ + k N) + ω_T RT
Out[2]= -S β I + b N - S (μ + k N) + RT ω_T

In[3]:= İ = β S I - γ I - I (μ + k N)
Out[3]= S β I - γ I - I (μ + k N)

In[4]:= RȦ = γ I - RA (μ + k N) - ω_A RA
Out[4]= γ I - RA (μ + k N) - RA ω_A

In[5]:= RṪ = ω_A RA - RT (μ + k N) - ω_T RT
Out[5]= -RT (μ + k N) + RA ω_A - RT ω_T
```

Equilibria and $\mathcal{R}_0$

Disease Free Equilibria (DFE)

```
In[6]:= Ṅ = FullSimplify[Ṡ + İ + RȦ + RṪ] /. {RA + RT + S + I → N}
Out[6]= b N - N (μ + k N)

In[7]:= Solve[Ṅ == 0, N]
```

Out[7]= $\left\{\{N \to 0\}, \left\{N \to \frac{b-\mu}{k}\right\}\right\}$

In[8]:=
```
Solve[(Ṡ /. {E → 0, I → 0, RA → 0, RT → 0}) == 0, S] /. {N → (b - μ)/k}
```
Out[8]=
$$\left\{\left\{S \to \frac{b-\mu}{k}\right\}\right\}$$

In[9]:=
```
DFE = {S → (b - μ)/k, E → 0, I → 0, RA → 0, RT → 0};
```

Pathogen Present Equilibria

In[10]:=
```
FullSimplify[Solve[{0 == Ṡ, 0 == İ, 0 == RȦ, 0 == RṪ}, {S, I, RA, RT}] /. {N → (b - μ)/k}]
```
Out[10]=
$$\left\{\left\{S \to \frac{b-\mu}{k}, I \to 0, RA \to 0, RT \to 0\right\}, \left\{S \to \frac{b+\gamma}{\beta}, I \to -\frac{(b(k-\beta)+k\gamma+\beta\mu)(b+\omega_A)(b+\omega_T)}{k\beta((b+\gamma)(b+\omega_T)+\omega_A(b+\gamma+\omega_T))},\right.\right.$$
$$\left.\left. RA \to -\frac{\gamma(b(k-\beta)+k\gamma+\beta\mu)(b+\omega_T)}{k\beta((b+\gamma)(b+\omega_T)+\omega_A(b+\gamma+\omega_T))}, RT \to -\frac{\gamma(b(k-\beta)+k\gamma+\beta\mu)\omega_A}{k\beta((b+\gamma)(b+\omega_T)+\omega_A(b+\gamma+\omega_T))}\right\}\right\}$$

Calculate $\mathcal{R}_0$ using the Next Generation Method (NGM)

Build vectors $\mathcal{F}$ and $\mathcal{V}$ to linearize the *infection subsystem*

$\mathcal{F}$ represents the rate of appearance of new infections in compartment i, $\mathcal{V}_i^+(x)$ represents the rate of transfer of individuals into compartment i by all other means, and $\mathcal{V}_i^-(x)$ represents the rate of transfer of individuals out of compartment i, where $\mathcal{V}_i$(x)=$\mathcal{V}_i^-(x)$ -$\mathcal{V}_i^+(x)$. The infected compartments x={I} and the non-infected compartments are y={S,RA,RT}. Thus, this can be written as $\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{V}(x)$.

In[11]:=
```
𝓕 = {β S I}
```
Out[11]=
$$\{S\,\beta\,I\}$$

In[12]:=
```
𝓥 = -{-γ I - I (μ + k N)}
```
Out[12]=
$$\{\gamma I + I(\mu + k N)\}$$

Find the Jacobian of $\mathcal{F}$ and $\mathcal{V}$ around the DFE

In[13]:=
```
F = Grad[𝓕, {I}]
```
Out[13]=
$$\{\{S\,\beta\}\}$$

In[14]:=
```
Fdfe = F /. DFE
```
Out[14]=

$$\left\{\left\{\frac{\beta\ (b-\mu)}{k}\right\}\right\}$$

In[15]:=
```
MatrixForm[Fdfe]
```
Out[15]//MatrixForm=

$$\left(\frac{\beta\ (b-\mu)}{k}\right)$$

In[16]:=
```
V = Grad[𝒱, {I}]
```
Out[16]=

$$\{\{\gamma+\mu+k\,N\}\}$$

In[17]:=
```
Vdfe = V /. DFE
```
Out[17]=

$$\{\{\gamma+\mu+k\,N\}\}$$

In[18]:=
```
MatrixForm[Vdfe]
```
Out[18]//MatrixForm=

$$(\gamma+\mu+k\,N)$$

Take the inverse of $\mathcal{V}$

In[19]:=
```
Vinverse = Inverse[Vdfe]
```
Out[19]=

$$\left\{\left\{\frac{1}{\gamma+\mu+k\,N}\right\}\right\}$$

In[20]:=
```
MatrixForm[Vinverse]
```
Out[20]//MatrixForm=

$$\left(\frac{1}{\gamma+\mu+k\,N}\right)$$

In[21]:=
```
FVinverse = Fdfe.Vinverse
```
Out[21]=

$$\left\{\left\{\frac{\beta\ (b-\mu)}{k\ (\gamma+\mu+k\,N)}\right\}\right\}$$

In[22]:=
```
MatrixForm[FVinverse]
```
Out[22]//MatrixForm=

$$\left(\frac{\beta\ (b-\mu)}{k\ (\gamma+\mu+k\,N)}\right)$$

Find the dominant eigen value of $\mathcal{F}\,\mathcal{V}^{-1}$ to find $\mathcal{R}_0$

In[23]:=
```
FullSimplify[Eigenvalues[FVinverse]]
```
Out[23]=

$$\left\{\frac{\beta\ (b-\mu)}{k\ (\gamma+\mu+k\,N)}\right\}$$

In[24]:=
```
R0 = β (b - μ) / (k (b + γ))
```
Out[24]=

$$\frac{\beta\ (b-\mu)}{k\ (b+\gamma)}$$

Solution for *I* as a Function of R_A

Solution for *I* for the Mechanistic Model in Counts

$$\hat{i}(t) = \frac{R_A\ (\mu + k\ N + \omega_A) + \dot{R}_A}{\gamma}$$

In[25]:=
```
bfunc = g Exp[-s Cos[π f t - ψ]^2]
```
Out[25]=

$$e^{-s\ \mathrm{Cos}[f\ \pi\ t - \psi]^2}\ g$$

In[26]:=
```
bPars = {g → 0.004441452, s → 4, f → 2 / 365, ψ → 1}
```
Out[26]=

$$\left\{g \to 0.00444145,\ s \to 4,\ f \to \frac{2}{365},\ \psi \to 1\right\}$$

In[27]:=
```
bave = N[Integrate[bfunc /. bPars, {t, 0, 365}]] / 365
```
Out[27]=

```
0.00137022
```

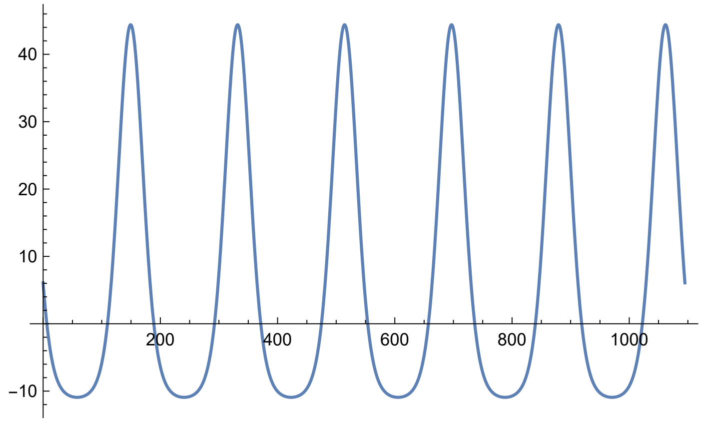
In[28]:=
```
plottime = 365 * 3
```
Out[28]=

```
1095
```

In[29]:=
```
Pars = {b → bfunc /. bPars, β → .0006, γ → 1 / 10,
   μ → 1 / 1095, k → (1 / 730 - 1 / 1095) / 1000, ωA → 1 / 90, ωT → 1 / 365};
```

In[30]:=
```
R0Plot = Plot[{R0 /. Pars}, {t, 0, plottime}]
```

Out[30]=

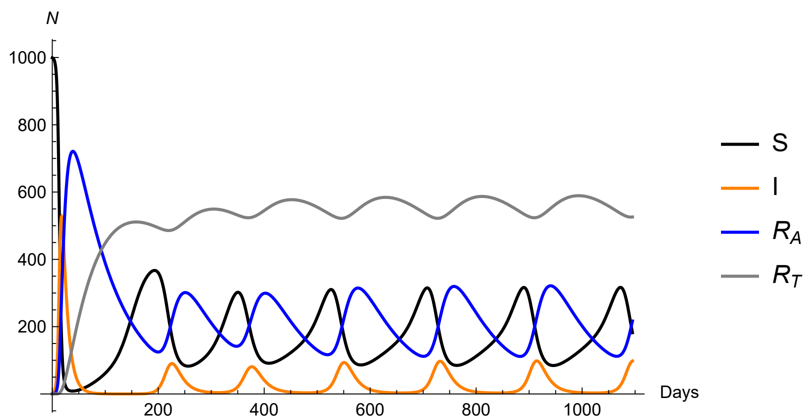


In[31]:=
```
Sol1 = NDSolve[{S'[t] == b (S [t] + I[t] + RA[t] + RT[t]) -
      β S [t] × I[t] - S[t] (μ + k (S [t] + I[t] +  RA[t] + RT[t])) + ω_T RT[t],
    I'[t] == β S [t] × I[t] - γ I[t] - I[t] (μ + k (S [t] + I[t] + RA[t] + RT[t])),
    RA'[t] == γ I[t] - RA[t] (μ + k (S [t] + I[t] +  RA[t] + RT[t])) - ω_A RA[t],
    RT'[t] == ω_A RA[t] - RT[t] (μ + k (S [t] + I[t] +  RA[t] + RT[t])) - ω_T RT[t],
    S[0] == 999, I[0] == 1, RA[0] == 0,  RT[0] == 0} /. Pars,
  {S[t], I[t], RA[t], RT[t]}, {t, 0, plottime}]
```

Out[31]=
```
{{S[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  I[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  RA[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  RT[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t]}}
```

In[32]:=
```
Plot[{S[t] /. Sol1, I[t] /. Sol1, RA[t] /. Sol1, RT[t] /. Sol1},
 {t, 0, plottime}, PlotLegends → {"S", "I", "R_A", "R_T"},
 PlotStyle → {Black, Orange, Blue, Gray}, AxesLabel → {Days, N}]
```

Out[32]=

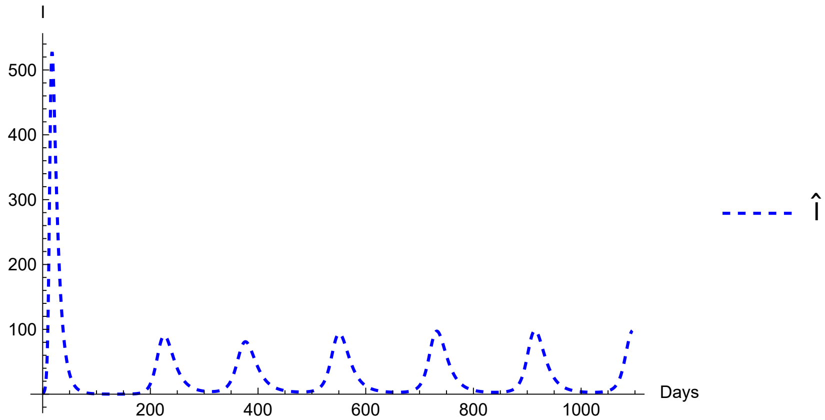


In[33]:=
```
IPred = 1/γ (D[RA[t] /. Sol1, t] +
     (ω_A + μ + k ((S[t] /. Sol1) + (I[t] /. Sol1) + (RA[t] /. Sol1) + (RT[t] /. Sol1)))
      RA[t] /. Sol1) /. Pars
```

Out[33]=
```
{{10 (InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t]
    (79/6570 + 1/2190000 (InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t] +
         InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t] +
         InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t] +
         InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t])) +
    InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t])}}
```

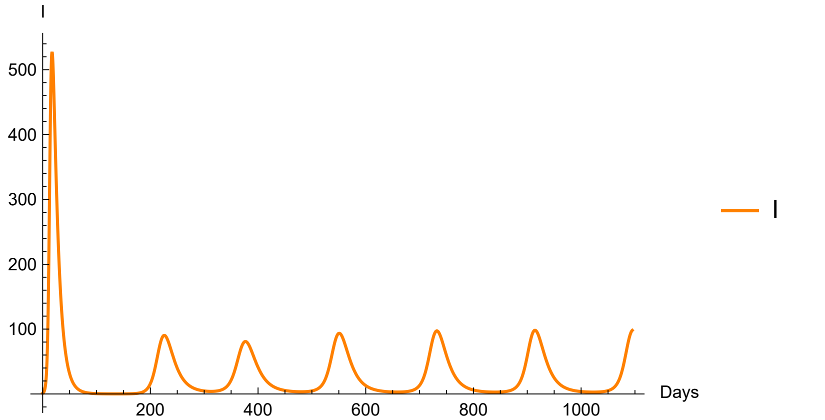
In[34]:=
```
Pred = Plot[{IPred /. Sol1}, {t, 0, plottime}, PlotStyle → {Blue, Dashed},
  PlotRange → All, AxesLabel → {Days, I}, PlotLegends → {"Î"}]
```

Out[34]=



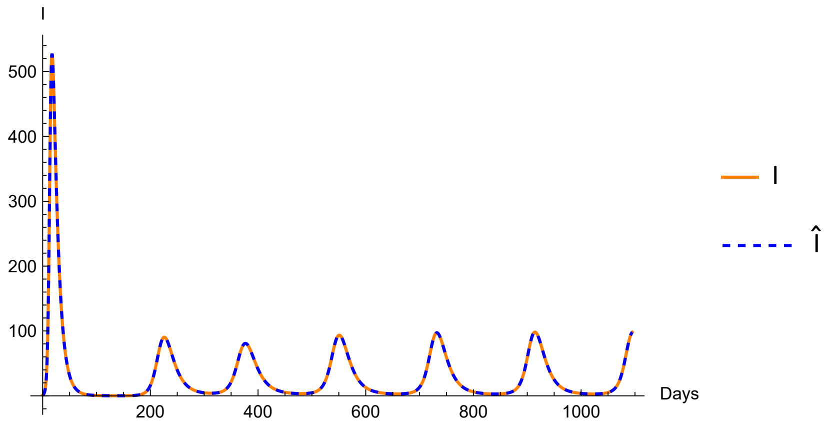
In[35]:=
```
TRUE = Plot[{I[t] /. Sol1}, {t, 0, plottime}, PlotRange → All,
  PlotStyle → {Orange}, AxesLabel → {Days, I}, PlotLegends → {"I"}]
```

Out[35]=



In[36]:=
```
Show[TRUE, Pred]
```

Out[36]=



Solution for *I* for the Mechanistic Model in Proportions

N is the total population size, *s* is the number of susceptible individuals, *ι* is the number of infected individuals, r_A is the number of seropositive and recovered individuals, and r_T is the number of seronegative and recovered individuals.

Change of Variables (COV) from Counts to Proportions using the quotient rule

In[37]:=

$$\dot{s} = \text{FullSimplify}\left[\text{FullSimplify}\left[\frac{\dot{S}\,N - \dot{N}\,S}{N^2}\right] /. \{S \to s\,N,\ I \to \iota\,N,\ RA \to rA\,N,\ RT \to rT\,N\}\right]$$

Out[37]=

$$b - b\,s - s\,\beta\,\iota\,N + rT\,\omega_T$$

In[38]:=

$$\dot{\iota} = \text{FullSimplify}\left[\text{FullSimplify}\left[\frac{\dot{I}\,N - \dot{N}\,I}{N^2}\right] /. \{S \to s\,N,\ I \to \iota\,N,\ RA \to rA\,N,\ RT \to rT\,N\}\right]$$

Out[38]=

$$-\iota\,(b + \gamma - s\,\beta\,N)$$

In[39]:=

$$\dot{rA} = \text{FullSimplify}\left[\text{FullSimplify}\left[\frac{\dot{RA}\,N - \dot{N}\,RA}{N^2}\right] /. \{S \to s\,N,\ I \to \iota\,N,\ RA \to rA\,N,\ RT \to rT\,N\}\right]$$

Out[39]=

$$-b\,rA + \gamma\,\iota - rA\,\omega_A$$

In[40]:=

$$\dot{rT} = \text{FullSimplify}\left[\text{FullSimplify}\left[\frac{\dot{RT}\,N - \dot{N}\,RT}{N^2}\right] /. \{S \to s\,N,\ I \to \iota\,N,\ RA \to rA\,N,\ RT \to rT\,N\}\right]$$

Out[40]=

$$rA\,\omega_A - rT\,(b + \omega_T)$$

In[41]:=

$$\text{FullSimplify}\left[\dot{s} + \dot{\iota} + \dot{rA} + \dot{rT}\right] /. \{rA + rT + s + \iota \to 1\}$$

Out[41]=

0

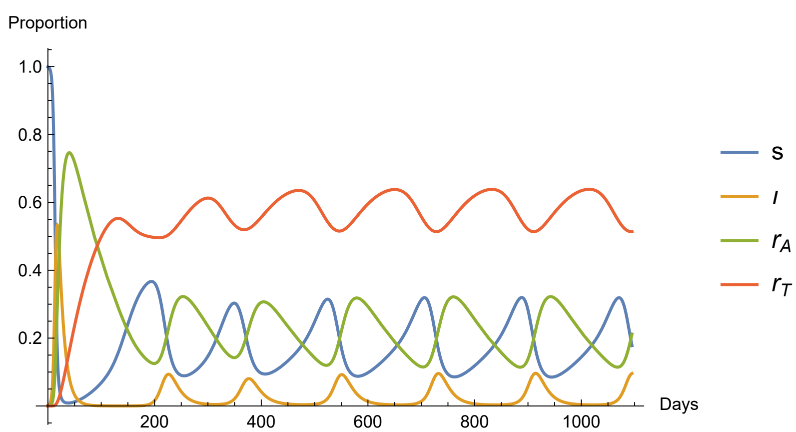
Check Proportion (COV) Solution Equals Count Solution

In[42]:=
```
Sol2 =
 NDSolve[{s'[t] == b - b s[t] - s[t] β ι[t] × N[t] + rT[t] ωT, ι'[t] == - ι[t] (b + γ - s[t] β N[t]),
    rA'[t] == γ ι[t] - rA[t] (b + ωA), rT'[t] == rA[t] ωA - rT[t] (b + ωT),
    N'[t] == b N[t] - N[t] (μ + k N[t]), N[0] == 1000, s[0] == 999 / N[0],
    ι[0] == 1 / N[0], rA[0] == 0 / N[0], rT[0] == 0 / N[0]} /. Pars,
  {s[t], ι[t], rA[t], rT[t], N[t]}, {t, 0, plottime}]
```

Out[42]=
```
{{s[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  ι[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  rA[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  rT[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  N[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t]}}
```

In[43]:=
```
COVPlot = Plot[{s[t] /. Sol2, ι[t] /. Sol2, rA[t] /. Sol2, rT[t] /. Sol2}, {t, 0, plottime},
  PlotLegends → {"s", "ι", "rA", "rT"}, AxesLabel → {Days, Proportion}]
```

Out[43]=

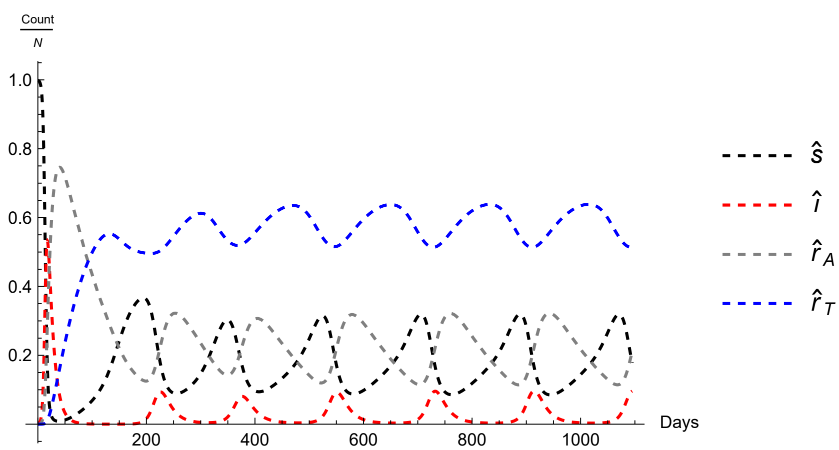


In[44]:=
```
CountSol2 = NDSolve[{S'[t] == b N[t] - β S[t] × I[t] - S [t] (μ + k N[t]) + ωT RT[t], I'[t] ==
      β S[t] × I[t] - γ I[t] - I[t] (μ + k N[t] ), RA'[t] == γ I[t] - RA[t] (μ + k N[t]) - ωA RA[t],
     RT'[t] == ωA RA[t] - RT[t] (μ + k N[t] ) - ωT RT[t], N'[t] == b N[t] - N[t] (μ + k N[t]),
     N[0] == 1000, S[0] == 999, I[0] == 1, RA[0] == 0, RT[0] == 0} /. Pars,
   {S[t], I[t], RA[t], RT[t], N[t]}, {t, 0, plottime}]
```

Out[44]=
```
{{S[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  I[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  RA[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  RT[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t],
  N[t] → InterpolatingFunction[ Domain: {{0., 1100.}} Output: scalar ][t]}}
```

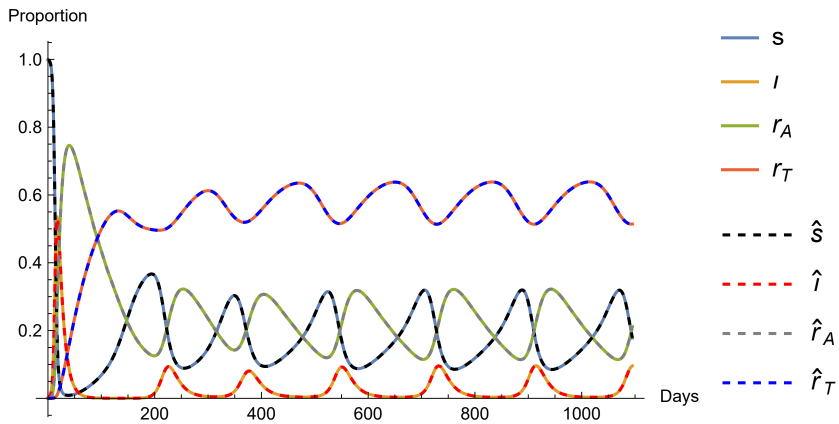
In[45]:=
```
CountPlot =
 Plot[{S[t] / (S[t] + I[t] + RA[t] + RT[t]) /. CountSol2, I[t] / (S[t] + I[t] + RA[t] + RT[t]) /. CountSol2,
   RA[t] / (S[t] + I[t] + RA[t] + RT[t]) /. CountSol2, RT[t] / (S[t] + I[t] + RA[t] + RT[t]) /. CountSol2},
  {t, 0, plottime}, PlotLegends → {"ŝ", "î", "r̂A", "r̂T"}, AxesLabel → {Days, Count / N},
  PlotStyle → {{Black, Dashed}, {Red, Dashed}, {Gray, Dashed}, {Blue, Dashed}}]
```

Out[45]=



In[46]:=
```
Show[{COVPlot, CountPlot}]
```

Out[46]=



$$\hat{\imath}(t) = \frac{r_A\,(b+\omega_A)+\dot{r}_A}{\gamma}$$

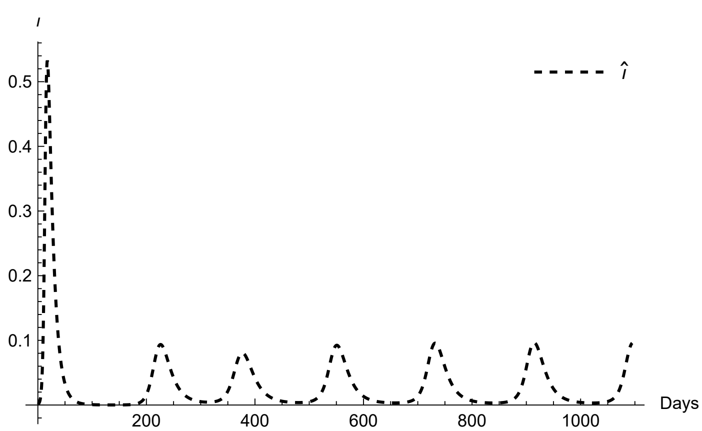
In[47]:=
```
IPredProp = (D[rA[t] /. Sol2, t] + (rA[t] /. Sol2) (b + ω_A)) / γ /. Pars
```

Out[47]=

$$\left\{10\left(\left(\frac{1}{90}+0.00444145\,e^{-4\,\mathrm{Cos}\left[1-\frac{2\pi t}{365}\right]^2}\right)\mathrm{InterpolatingFunction}[\ \text{Domain: }\{\{0.,1100.\}\}\ \text{Output: scalar}\ ][t]+\mathrm{InterpolatingFunction}[\ \text{Domain: }\{\{0.,1100.\}\}\ \text{Output: scalar}\ ][t]\right)\right\}$$

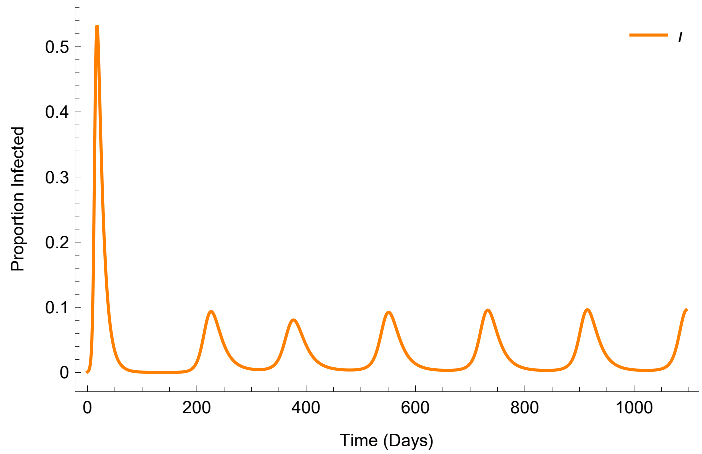
In[48]:=
```
PredProp = Plot[{IPredProp}, {t, 0, plottime}, PlotStyle → {Black, Dashed},
  PlotRange → All, AxesLabel → {Days, ι}, PlotLegends → Placed[{"ι̂"}, {Right, Top}]]
```

Out[48]=



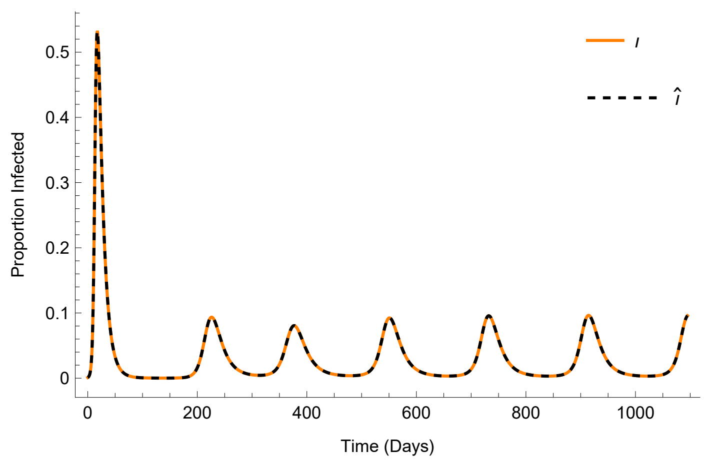
In[49]:=
```
TRUEProp = Plot[{ι[t] /. Sol2}, {t, 0, plottime}, PlotRange → All,
  PlotStyle → {Orange}, Axes → False, Frame → {{True, False}, {True, False}},
  FrameLabel → {{"Proportion Infected", None}, {"Time (Days)", None}},
  FrameTicks → All, PlotLegends → Placed[{"ι"}, {Right, Top}]]
```

Out[49]=



In[50]:=
```
TrueBirth = Show[TRUEProp, PredProp]
```

Out[50]=



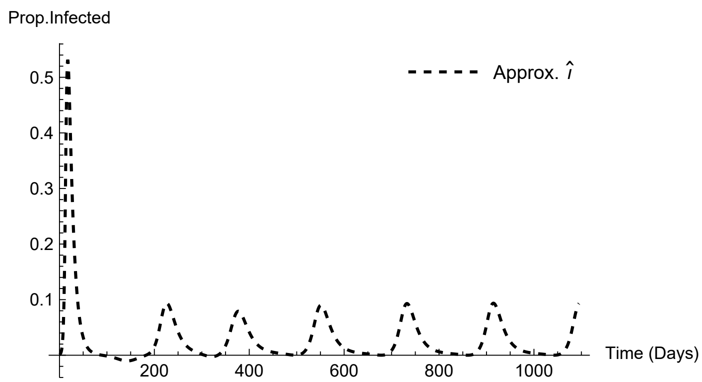
The Solution for ι̂(t) can be approximated without the need for b(t) as long as the birth rate at any time point (t) is << ω_a.

In[51]:=
$$\text{IPredPropapprox} = \frac{\text{D[rA[t] /. Sol2, t] + (rA[t] /. Sol2)}\ \omega_A}{\gamma} \text{ /. Pars}$$

Out[51]=
$$\left\{10\left(\frac{1}{90}\ \text{InterpolatingFunction}\left[\text{Domain: \{\{0., 1100.\}\} Output: scalar}\right][t] + \text{InterpolatingFunction}\left[\text{Domain: \{\{0., 1100.\}\} Output: scalar}\right][t]\right)\right\}$$

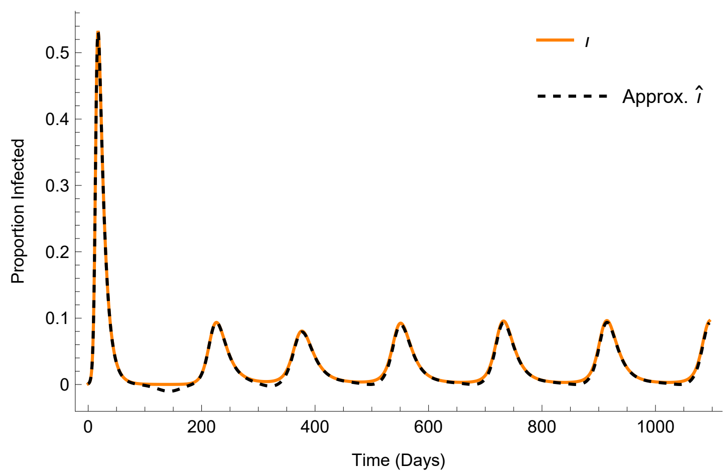
In[52]:=
```
PredPropapprox = Plot[{IPredPropapprox}, {t, 0, plottime}, PlotStyle → {Black, Dashed},
  PlotRange → All, AxesLabel → {"Time (Days)", Prop. Infected},
  PlotLegends → Placed[{"Approx. î"} , {Right, Top}]]
```

Out[52]=



In[53]:=
```
NoBirth = Show[TRUEProp, PredPropapprox]
```

Out[53]=



Sensitivity Analysis for $\hat{\imath}(t)$ and $\hat{\imath}_{peak}$

The sensitivity index of $\hat{\imath}(t)$ with respect to parameter ξ is given by $\hat{\imath}(t)_\xi = \frac{\partial \hat{\imath}(t)}{\partial \xi}$. To find the sensitivity of the predicted peak timing with to parameter ξ, we find where $\hat{\imath}(t)_\xi$ is equal to zero.

In[54]:=
```
rAdata = C2 - C1 Exp[-s Cos[π f t - ψ]^2]
```

Out[54]=

$$C2 - C1\, e^{-s \operatorname{Cos}[f \pi t - \psi]^2}$$

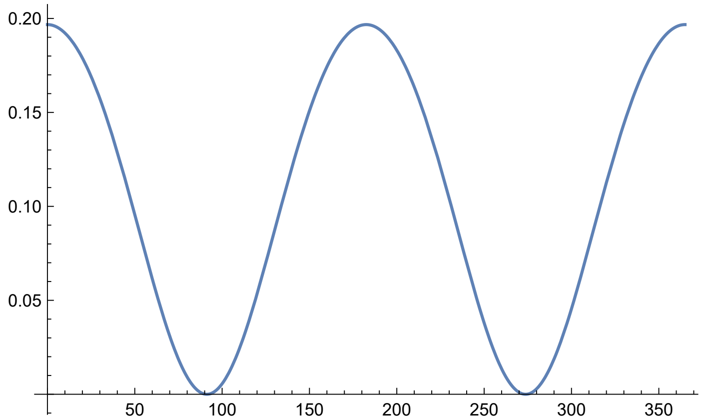
In[55]:=
```
Parsnew = {C1 → 1/2, C2 → 1/2, γ → 1/10, f → 2/365, ψ → 0, ωA → 1/90, s → 0.5}
```

Out[55]=

$$\left\{C1 \to \frac{1}{2},\ C2 \to \frac{1}{2},\ \gamma \to \frac{1}{10},\ f \to \frac{2}{365},\ \psi \to 0,\ \omega_A \to \frac{1}{90},\ s \to 0.5\right\}$$

In[56]:=
```
Plot[rAdata /. Parsnew, {t, 0, 365}]
```

Out[56]=



In[57]:=

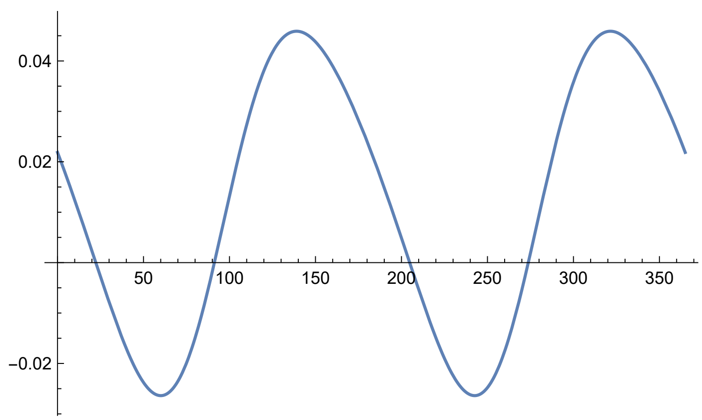
$$\iota\text{hat} = \frac{\text{rAdata}\,(b + \omega_A) + D[\text{rAdata}, t]}{\gamma}$$

Out[57]=

$$\frac{-2\,C1\,e^{-s\,\text{Cos}[f\,\pi\,t-\psi]^2}\,f\,\pi\,s\,\text{Cos}[f\,\pi\,t-\psi]\,\text{Sin}[f\,\pi\,t-\psi] + \left(C2 - C1\,e^{-s\,\text{Cos}[f\,\pi\,t-\psi]^2}\right)\,(b + \omega_A)}{\gamma}$$

In[58]:=
```
Plot[{ιhat /. Parsnew /. b → 0}, {t, 0, 365}]
```

Out[58]=



In[59]:=

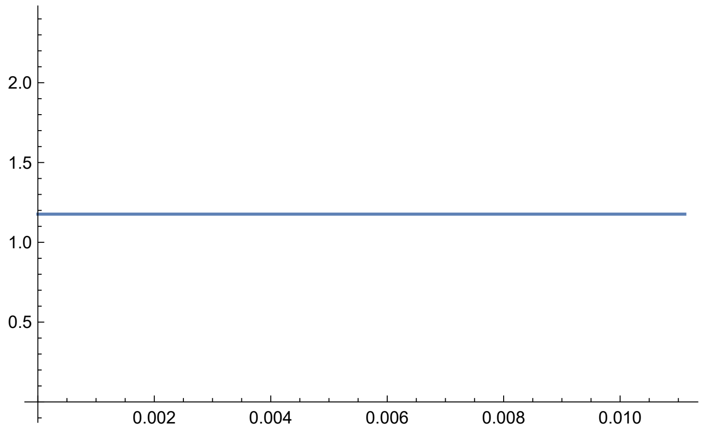
$$\iota\text{hatD} = D[\iota\text{hat}, \omega_A]$$

Out[59]=

$$\frac{C2 - C1\,e^{-s\,\text{Cos}[f\,\pi\,t-\psi]^2}}{\gamma}$$

In[60]:=
```
Plot[{ιhatD /. {C1 → 0.5, C2 → 0.5, γ → 1 / 10,
    f → 2 / 365, ψ → 0, s → 0.5, t → 139.0031905514053`}}, {ωA, 0, 1 / 90}]
```

Out[60]=



In[61]:=
```
ιhat' = D[ιhat, t] /. {C1 → 0.5, C2 → 0.5, γ → 1 / 10, f → 2 / 365, ψ → 0, b → 0}
```

Out[61]=

$$10\left(-0.000296329\, e^{-s\operatorname{Cos}\left[\frac{2\pi t}{365}\right]^2} s\operatorname{Cos}\left[\frac{2\pi t}{365}\right]^2 + 0.000296329\, e^{-s\operatorname{Cos}\left[\frac{2\pi t}{365}\right]^2} s\operatorname{Sin}\left[\frac{2\pi t}{365}\right]^2 - 0.000592658\, e^{-s\operatorname{Cos}\left[\frac{2\pi t}{365}\right]^2} s^2\operatorname{Cos}\left[\frac{2\pi t}{365}\right]^2\operatorname{Sin}\left[\frac{2\pi t}{365}\right]^2 - 0.0172142\, e^{-s\operatorname{Cos}\left[\frac{2\pi t}{365}\right]^2} s\operatorname{Cos}\left[\frac{2\pi t}{365}\right]\operatorname{Sin}\left[\frac{2\pi t}{365}\right]\omega_A\right)$$

In[62]:=
```
FullSimplify[Solve[ιhat' == 0, t]]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[62]=

$$\{\{t \to 116.183\operatorname{ArcTan}[\operatorname{Root}[5.70051\times10^{23} + (-2.2802\times10^{24} + 4.56041\times10^{24}\, s)\, \#1^2 + (-5.70051\times10^{24} - 9.12082\times10^{24}$$
$$\#1^4 + (-2.2802\times10^{24} + 4.56041\times10^{24}\, s)\, \#1^6 + 5.70051\times10^{23}\, \#1^8 + 6.62303\times10^{25}\, \#1\, \omega_A$$
$$6.62303\times10^{25}\, \#1^3\, \omega_A - 6.62303\times10^{25}\, \#1^5\, \omega_A - 6.62303\times10^{25}\, \#1^7\, \omega_A \,\&,\, 1]] + 365.\, c_1 \text{ if } c_1$$

In[63]:=
```
fourthsolfree = 116.1831084570836` ArcTan[
   Root[5.7005122040658496`*^23 + (-2.2802048816263398`*^24 + 4.56040976325268`*^24 s)
      #1^2 + (-5.70051220406585`*^24 - 9.12081952650536`*^24 s) #1^4 +
     (-2.2802048816263398`*^24 + 4.56040976325268`*^24 s) #1^6 +
     5.7005122040658496`*^23 #1^8 + 6.623032276659112`*^25 #1 ωA +
     6.623032276659112`*^25 #1^3 ωA - 6.623032276659112`*^25 #1^5 ωA -
     6.623032276659112`*^25 #1^7 ωA &, 4]] + 365.` c1 /. {c1 → 0, s → 0.5}
```

Out[63]=

$$0. + 116.183\operatorname{ArcTan}[\operatorname{Root}[5.70051\times10^{23} + 2.68435\times10^{8}\, \#1^2 - 1.02609\times10^{25}\, \#1^4 + 2.68435\times10^{8}\, \#1^6 + 5.70051\times10^{23}\, \#1^8 + 6.62303\times10^{25}\, \#1\, \omega_A + 6.62303\times10^{25}\, \#1^3\, \omega_A - 6.62303\times10^{25}\, \#1^5\, \omega_A - 6.62303\times10^{25}\, \#1^7\, \omega_A \,\&,\, 4]]$$

In[64]:=

```
Plot[fourthsolfree, {ωA, 0, 1 / 30}]
```

Out[64]=