Two-Population Model $\mathcal{R}_0$ using Next Generation Matrix (NGM) Method

$\mathcal{R}_0$ is the dominant eigen value of $K = -\mathcal{F}\mathcal{V}^{-1}$

Find the disease free equilibria (DFE)

In[1]:=
```
DFE = {Si → Ni, Sj → Nj, Ei → 0, Ej → 0, Ii → 0, Ij → 0};
```

Build vectors $\mathcal{F}$ and $\mathcal{V}$ to linearize the *infection subsystem*

$\mathcal{F}$ represents the rate of appearance of new infections in compartment i, $\mathcal{V}_i^+(x)$ represents the rate of transfer of individuals into compartment i by all other means, and $\mathcal{V}_i^-(x)$ represents the rate of transfer of individuals out of compartment i, where $\mathcal{V}_i$(x)=$\mathcal{V}_i^-(x)$ -$\mathcal{V}_i^+(x)$. The infected compartments x={E,I} and the non-infected compartments are y={S,R}. Thus, this can be written as $\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{V}(x)$.
https://www.ncbi.nlm.nih.gov/pmc/articles/PMC2871801/

In[2]:=
```
𝓕 = {βi Si Ii + βij Si Ij, βj Sj Ij + βij Sj Ii}
```

Out[2]= {Ii Si βi + Ij Si βij, Ii Sj βij + Ij Sj βj}

In[3]:=
```
MatrixForm[𝓕]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} \text{Ii Si } \beta\text{i} + \text{Ij Si } \beta\text{ij} \\ \text{Ii Sj } \beta\text{ij} + \text{Ij Sj } \beta\text{j} \end{pmatrix}$$

In[4]:=
```
𝓥 = {-Ii γ, -Ij γ}
```

Out[4]= {-Ii γ, -Ij γ}

In[5]:=
```
MatrixForm[𝓥]
```

Out[5]//MatrixForm=

$$\begin{pmatrix} -\text{Ii } \gamma \\ -\text{Ij } \gamma \end{pmatrix}$$

Find the Jacobian of $\mathcal{F}$ and $\mathcal{V}$ around the DFE

In[6]:=
```
F = Grad[𝓕, {Ii , Ij}]
```
Out[6]= $\{\{Si\,\beta i, Si\,\beta ij\}, \{Sj\,\beta ij, Sj\,\beta j\}\}$

In[7]:=
```
Fdfe = F /. DFE
```
Out[7]= $\{\{Ni\,\beta i, Ni\,\beta ij\}, \{Nj\,\beta ij, Nj\,\beta j\}\}$

In[8]:=
```
MatrixForm[Fdfe]
```
Out[8]//MatrixForm=
$$\begin{pmatrix} Ni\,\beta i & Ni\,\beta ij \\ Nj\,\beta ij & Nj\,\beta j \end{pmatrix}$$

In[9]:=
```
V = Grad[𝓥, {Ii , Ij}]
```
Out[9]= $\{\{-\gamma, 0\}, \{0, -\gamma\}\}$

In[10]:=
```
MatrixForm[V]
```
Out[10]//MatrixForm=
$$\begin{pmatrix} -\gamma & 0 \\ 0 & -\gamma \end{pmatrix}$$

In[11]:=
```
Vdfe = V /. DFE
```
Out[11]=
$\{\{-\gamma, 0\}, \{0, -\gamma\}\}$

In[12]:=
```
MatrixForm[Vdfe]
```
Out[12]//MatrixForm=
$$\begin{pmatrix} -\gamma & 0 \\ 0 & -\gamma \end{pmatrix}$$

Take the inverse of V

In[13]:=
```
Vinverse = Inverse[Vdfe]
```
Out[13]=
$$\left\{\left\{-\frac{1}{\gamma}, 0\right\}, \left\{0, -\frac{1}{\gamma}\right\}\right\}$$

In[14]:=
```
MatrixForm[Vinverse]
```
Out[14]//MatrixForm=
$$\begin{pmatrix} -\frac{1}{\gamma} & 0 \\ 0 & -\frac{1}{\gamma} \end{pmatrix}$$

In[15]:=
```
K = -Fdfe.Vinverse
```
Out[15]=
$$\left\{\left\{\frac{Ni\,\beta i}{\gamma}, \frac{Ni\,\beta ij}{\gamma}\right\}, \left\{\frac{Nj\,\beta ij}{\gamma}, \frac{Nj\,\beta j}{\gamma}\right\}\right\}$$

In[16]:= `MatrixForm[K]`

Out[16]//MatrixForm=

$$\begin{pmatrix} \frac{\text{Ni}\,\beta\text{i}}{\gamma} & \frac{\text{Ni}\,\beta\text{ij}}{\gamma} \\ \frac{\text{Nj}\,\beta\text{ij}}{\gamma} & \frac{\text{Nj}\,\beta\text{j}}{\gamma} \end{pmatrix}$$

In[17]:= `Det[K]`

Out[17]=

$$-\frac{\text{Ni}\,\text{Nj}\,\beta\text{ij}^2}{\gamma^2}+\frac{\text{Ni}\,\text{Nj}\,\beta\text{i}\,\beta\text{j}}{\gamma^2}$$

In[18]:= `Evalues = Eigenvalues[K]`

Out[18]=

$$\left\{\frac{\text{Ni}\,\beta\text{i}+\text{Nj}\,\beta\text{j}-\sqrt{\text{Ni}^2\,\beta\text{i}^2+4\,\text{Ni}\,\text{Nj}\,\beta\text{ij}^2-2\,\text{Ni}\,\text{Nj}\,\beta\text{i}\,\beta\text{j}+\text{Nj}^2\,\beta\text{j}^2}}{2\,\gamma},\right.$$
$$\left.\frac{\text{Ni}\,\beta\text{i}+\text{Nj}\,\beta\text{j}+\sqrt{\text{Ni}^2\,\beta\text{i}^2+4\,\text{Ni}\,\text{Nj}\,\beta\text{ij}^2-2\,\text{Ni}\,\text{Nj}\,\beta\text{i}\,\beta\text{j}+\text{Nj}^2\,\beta\text{j}^2}}{2\,\gamma}\right\}$$

In[19]:= `R0 = FullSimplify[Evalues[[2]]]`

Out[19]=

$$\frac{\text{Ni}\,\beta\text{i}+\text{Nj}\,\beta\text{j}+\sqrt{\text{Ni}^2\,\beta\text{i}^2+4\,\text{Ni}\,\text{Nj}\,\beta\text{ij}^2-2\,\text{Ni}\,\text{Nj}\,\beta\text{i}\,\beta\text{j}+\text{Nj}^2\,\beta\text{j}^2}}{2\,\gamma}$$