Part 2

November 23, 2015

The Jacobi and Gauss-Seidel methods are two ways in which one can calculate an approximate solution $\vec{x}$ to the matrix equation $A\vec{x} = \vec{b}$. For this particular assignment, matrix A and vector $\vec{b}$ were already chosen to be:

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1 & 1/4 \\ 1/3 & 1/4 & 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}, \text{ with exact solution } \vec{x}_{exact} = \begin{pmatrix} 9/190 \\ 28/475 \\ 33/475 \end{pmatrix}$$

To solve this equation using either method, a vector $\vec{x}_0$ must be chosen to begin the initial iterations. Each element in $\vec{x}_0$ was randomly generated within the range of [-1, 1]. Gauss-Seidel in general required fewer iterations to find an approximate answer when elements in $\vec{x}_0$ were very close to the real answer, and it required more iterations when elements were very close to the bounds of the range. This often held true for the Jacobi method as well, although there were several cases in which vectors with elements close to the bounds of the range converged relatively much faster using Jacobi than usual. Some examples include:

$$\begin{pmatrix} 0.49538 \\ 0.41532 \\ -0.93904 \end{pmatrix}, \text{ 14 iterations using Jacobi but 9 iterations using Gauss-Seidel}$$

$$\begin{pmatrix} 0.33392 \\ -0.90233 \\ 0.87071 \end{pmatrix}, \text{ 15 iterations using Jacobi but 9 iterations using Gauss-Seidel}$$

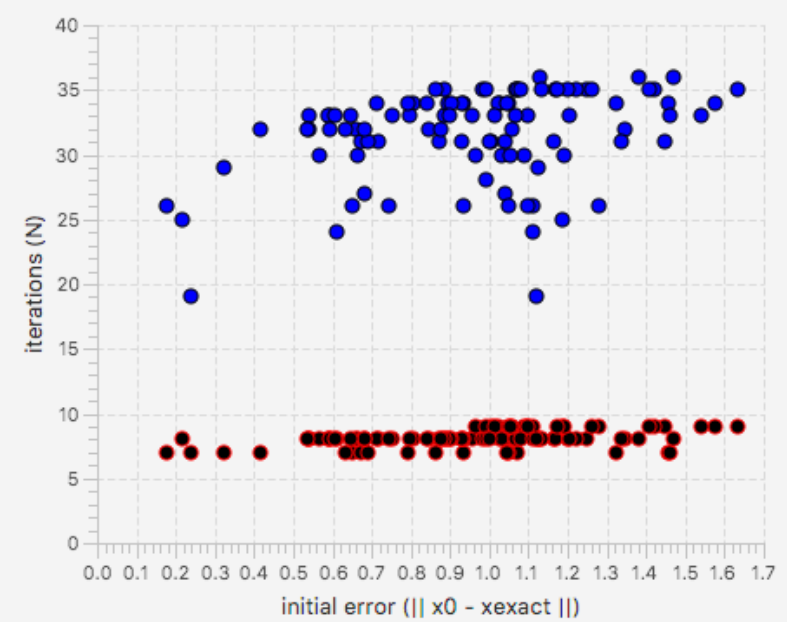
$$\begin{pmatrix} 0.82679 \\ -0.89843 \\ 0.23243 \end{pmatrix}, \text{ 15 iterations using Jacobi but 9 iterations using Gauss-Seidel}$$

Both methods took relatively longer to converge when elements of $\vec{x}_0$ were both all close to the bounds of the range and all of the same sign (i.e., all positive or all negative). The methods converged quicker when at least one of the three elements was of opposite sign.

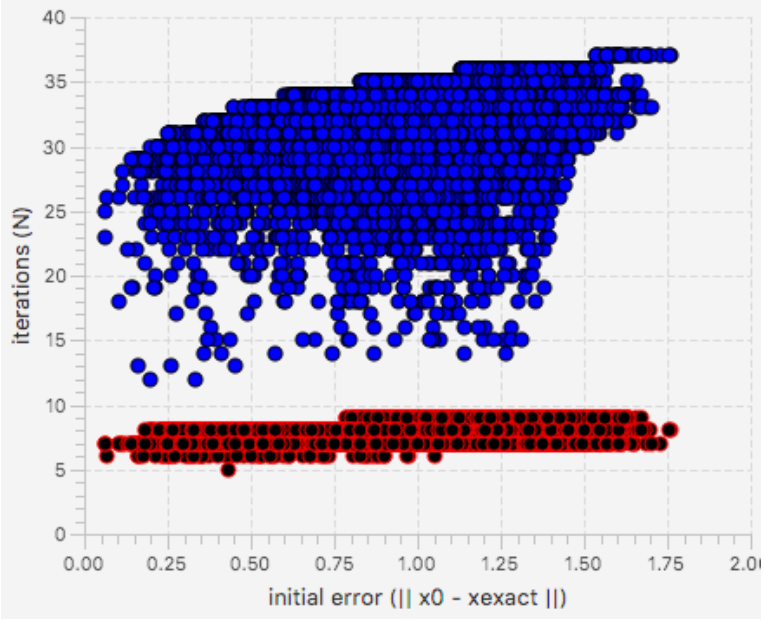
$$\vec{x}_{k+1} = S^{-1}(T\vec{x}_k + \vec{b})$$

Both Jacobi and Gauss-Seidel methods use the same general formula above to iteratively approximate $\vec{x}_0$, but the values for matrices S and T for the methods differ. For both methods, the matrix A can be divided into three submatrices L, U, and D such that A = L + U + D; where L is the lower triangular matrix of A, D is the diagonal along A, and U is the upper triangular matrix of A. For the Jacobi method, S = D and T = −(L + U). For Gauss-Seidel, S = L + D and T = −U. Each method will then continue to iterate until the difference between x_{k+1} and x_k is less than or equal to a selected epsilon ε, which was 0.00005 for this assignment. For this assignment, the methods will additionally quit in failure if, after M = 100 iterations, an approximate solution was unable to be found.

The ratio of number of iterations for Jacobi to number of iterations for Gauss-Seidel for this particular equation is approximately 3.5. This means that, at least for this equation, Gauss-Seidel converges onto an approximate solution about 3.5 times faster than Jacobi does. The speed of convergence is relative to the spectral radius of each method, which equals the maximum eigenvalue in absolute value.



The above is a sample scatter plot of 100 points of data for each iterative method graphing its number of iterations by the difference between $\vec{x}_0$ and $\vec{x}_{exact}$. Data for Jacobi is plotted using blue circles with black outlines, and data for Gauss-Seidel is plotted using black circles with red outlines. Here, it is very clear that Gauss-Seidel converged on an approximate solution much faster than Jacobi. Because there is such little variation in the number of iterations for Gauss-Seidel (i.e., all approximations using Gauss-Seidel were found within 7-9 iterations), it is easy to see that having an initial vector $\vec{x}_0$ further away from $\vec{x}_{exact}$ in magnitude increases the number of iterations taken to find an approximate solution. Because the data for Jacobi is less condensed, it is harder to tell whether it too follows the same pattern.



The above plot contains 10,000 points of data for both Jacobi and Gauss-Seidel. Here, points with higher iteration numbers for both methods only occurred when the difference between $\vec{x}_0$ and $\vec{x}_{exact}$ was relatively high. Additionally, points with lower iteration numbers for Gauss-Seidel occurred only when the difference between $\vec{x}_0$ and $\vec{x}_{exact}$ was relatively low. For Jacobi, however, very low iteration numbers (e.g., between 13

and 22 iterations) are not clustered around a low initial error, although they never originated from too high of an initial error.