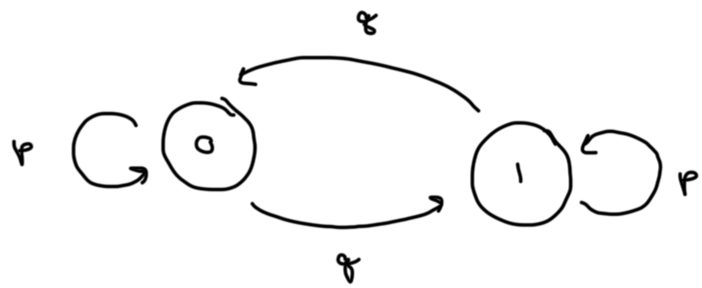
1a.



This chain is irreducible because I can start from one state and move to any other state in a finite number of steps. Both state 0 and state 1 have transitions to reach each other (ie. $P(0,1) = q$, $P(1,0) = q$) with positive probability. This chain is also aperiodic because there is no cyclical pattern in which the chain starts at one state and can return to itself. For example, if I start at state 0, I can return to state 0 in 2-steps ($P(0,1) \rightarrow P(1,0)$) or in 1-step ($P(0,0)$). The greatest common divisor of the length of all cycles is 1, which makes the chain aperiodic.

1b. $C_n \sim$ # of switches / state changes

ie. 0001000<u>1</u><u>1</u>

3 successes

$\therefore C_8 = 3$

$\sim \text{Binomial}(n, q)$

Each step is a Bernoulli trial with probability q when switching states. Since we perform n trials, each with the same probability of success (switching states), C_n follows the binomial distribution.

1c. 010100

C_5 : $0 \to 1 \to 0 \to 1 \to 0 \to 0$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$\therefore C_5 = 4$

↳ chain switches states 4 times for a $0 \to 0$ transition

C_2 : $0 \to 1 \to 0$

$0 \quad 1 \quad 2$

$\therefore C_2 = 2$

↳ chain switches states 2 times for a $0 \to 0$ transition

C_4 : $0 \to 1 \to 0 \to 1 \to 0$

$0 \quad 1 \quad 2 \quad 3 \quad 4$

$\therefore C_4 : 4$

↳ chain switches states 4 times for a $0 \to 0$ transition

$\therefore P_n(0,0) = P(C_n \text{ is even})$

1d. $C_n \sim \text{Binomial}(n, q)$

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} q^k (p)^{n-k}$$

$$(p-q)^n = \sum_{k=0}^{n} \binom{n}{k} (-q)^k (p)^{n-k}$$

$$(p+q)^n + (p-q)^n = \sum_{k=0}^{n} \binom{n}{k} q^k (p)^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (-q)^k (p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (p)^{n-k} \left(q^k + (-q)^k\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (p)^{n-k} \cdot \begin{cases} 2q^k & k = \text{even} \\ 0 & k = \text{odd} \end{cases}$$

$$= 2 \sum_{\substack{k=0 \\ (k \text{ is even})}}^{n} \binom{n}{k} q^k (p)^{n-k}$$

$$(p+q)^n + (p-q)^n = 2P(C_n \text{ is even}) = 2P_n(0,0)$$

$$P(C_n \text{ is even}) = \frac{(p+q)^n + (p-q)^n}{2}$$

success + failure probabilities $= 1$

$p + q = 1$

$$= \frac{1^n + (p-q)^n}{2}$$

$$= \frac{1 + (p-q)^n}{2}$$

1e. $P_n(0,0) = \dfrac{1 + (p-q)^n}{2}$

$$\lim_{n\to\infty} P_n(0,0) = \lim_{n\to\infty} \frac{1 + (p-q)^n}{2}$$

$$= \frac{1 + 0}{2}$$

$$\pi(0) = \frac{1}{2}$$

$|p - q| < 1$

$\therefore (p-q)^n$, as $n \to \infty$

$= 0$

$\therefore$ symmetric $\rightarrow$ same probabilities when transitioning from state 0 to state 1 and from state 1 to state 0 (q), and staying at the same state (p)

$\therefore$ by symmetry $\pi(1) = 1 - \pi(0)$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\therefore \pi = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

2a. $f_j \le f_i$, move to j

↳ f_j has fewer / same # friends as f_i, move to j

$f_j > f_i$

- lands heads, move to j
- lands tails, stay at i

↳ f_j has more # friends than f_i

- uniformly, at random

↳ each friend of Member i has equal chance of being selected

$\therefore \quad \frac{1}{f_i}$

$$P(i,j) = \frac{1}{f_i} \times \begin{cases} 1 & \text{if } f_j \le f_i \\ \frac{f_i}{f_j} & \text{if } f_j > f_i \\ 0 & \text{if } f_j \text{ NOT friends with } f_i \end{cases}$$

$$= \begin{cases} \frac{1}{f_i} & \text{if } f_j \le f_i \\ \frac{1}{f_j} & \text{if } f_j > f_i \\ 0 & \text{if } f_j \text{ NOT friends with } f_i \end{cases}$$

2b. - irreducible → able to move from one state to another state and return back to its initial start in finite # steps

∴ every member is linked to every other member

- aperiodic → able to stay at same state, no cyclical pattern

∴ stay at i if coin lands on tails

∴ can use detailed balance equation!

$$\pi(i)P(i,j) = \pi(j)P(j,i)$$

$$P(j,i) = \begin{cases} \frac{1}{f_j} & \text{if } f_j \leq f_i \\ \frac{1}{f_i} & \text{if } f_j > f_i \\ 0 & \text{if } f_i \text{ NOT friends with } f_j \end{cases}$$

∴ symmetric + doubly stochastic

↳ uniform distribution

$f_j \leq f_i$

$$\pi(i) \cdot \frac{1}{f_i} = \pi(j) \cdot \frac{1}{f_j} \cdot \frac{f_j}{f_i}$$

$$\pi(i) \cdot \frac{1}{f_i} = \pi(j) \cdot \frac{1}{f_i}$$

$$\pi(i) = \pi(j)$$

$\therefore$ uniform distribution

$f_j > f_i$

$$\pi(i) \cdot \frac{1}{f_j} = \pi(j) \cdot \frac{1}{f_i} \cdot \frac{f_i}{f_j}$$

$$\pi(i) \cdot \frac{1}{f_j} = \pi(j) \cdot \frac{1}{f_j}$$

$$\pi(i) = \pi(j)$$

$\therefore$ uniform distribution

$\therefore$ uniform distribution

$$\pi(i) = \frac{1}{m}, \quad 1 \leq i \leq m$$

Chebyshev's Inequality

$$P(|X - E[X]| \geq c) \leq \frac{var(X)}{c^2}$$

3a. $P(0 < X < 40)$

$\therefore P(-20 < X - 20 < 20)$

$P(|X - 20| < 20)$

$1 - \underbrace{P(|X - 20| \geq 20)}_{\leq \frac{4^2}{20^2}}$

take complement to find inside

$\therefore \geq 1 - \frac{1}{25}$

≥ 0.96

$\therefore P(0 < X < 40) \leq 1$

↳ upperbound for X must lie $0.96 \leq P(0 < X < 40) \leq 1$

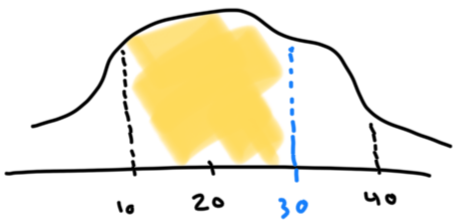
$\therefore$ lowerbound ≥ 0.96

upperbound $= 1$

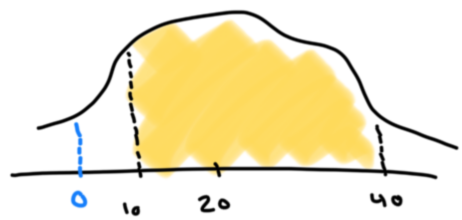
36. $P(10 < X < 40)$

$c = 20 - 10$
$= 10$

$c = 40 - 20$
$= 20$



$c = 10 \rightarrow P(10 < x < 30)$
∴ bounding a smaller region
↳ lowerbound



$c = 20 \rightarrow P(0 < x < 40)$
∴ bounding a larger outside region than necessary

lower bound :

$$\_\_\_\_\_\_ \leq P(10 < x < 30) < P(10 < x < 40) < 1$$

$$\therefore P(10 < x < 30)$$

$$P(-10 < X - 20 < 10)$$

$$P(|x - 20| < 10)$$

$$1 - \underbrace{P(|X - 20| \geq 10)}_{\frac{4^2}{10^2}}$$

$$\geq 1 - \frac{4}{25}$$

$$\geq 0.84$$

$\therefore P(10 < X < 40) \leq 1$

$\hookrightarrow$ upperbound for X must lie $0.84 \leq P(10 < X < 40) \leq 1$

$\therefore$ lower bound ≥ 0.84

upperbound : 1

Markov's Inequality

$$P(X \geq c) \leq \frac{E[X]}{c}$$

3c. $P(X \geq 40)$

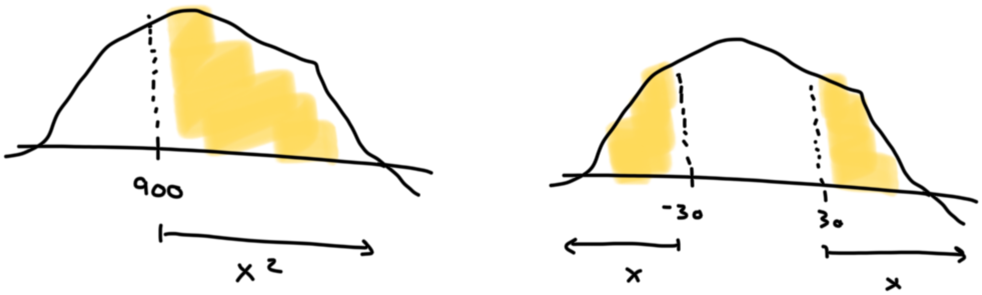
$$\therefore P(X \geq 40) \leq \frac{20}{40}$$

$$\leq 0.5$$

upperbound ≤ 0.5

3d. $P(X^2 \geq 900) = P(X \geq 30 \text{ or } X \leq -30)$

$= P(|X| \geq 30)$



$\therefore$ can't use Markov's Inequality

↳ X has to be non-negative

$\therefore P(X - 20 \geq 10)$

$$P(|X - 20| \geq 10) \leq \frac{4^2}{10^2}$$

$$\leq \frac{16}{100}$$

$\therefore P(X^2 \geq 900) \leq 0.16$

upperbound ≤ 0.16

4. $$MSE_\theta(T) = E_\theta\left[(T-\theta)^2\right]$$

$$= E_\theta\left[\left((T - E_\theta(T)) + (E_\theta(T) - \theta)\right)^2\right]$$

$$= E_\theta\left[(T - E_\theta(T))^2\right] + E_\theta\left[2(T - E_\theta(T))\underbrace{(E_\theta(T) - \theta)}_{0}\right] + E_\theta\left[(E_\theta(T) - \theta)^2\right]$$

$$= \underbrace{E_\theta\left[(T - E_\theta(T))^2\right]}_{Var(T)} + \underbrace{E_\theta\left[(E_\theta(T) - \theta)^2\right]}_{B_\theta^2(T)}$$

$$MSE_\theta(T) = Var(T) + B_\theta^2(T)$$