1a. $x = p(\text{first draw red})(1) + p(\text{first draw blue})(0)$
$+ p(\text{first draw green})(x)$

$$x = p_r + p_g x$$

$$x - p_g x = p_r$$

$$x(1 - p_g) = p_r$$

$$x = \frac{p_r}{1 - \underline{p_g}}$$

$$p_r + p_b + p_g = 1$$

$$\underline{p_g} = 1 - p_r - p_b$$

$$x = \frac{p_r}{1 - (1 - p_r - p_b)}$$

$$= \frac{p_r}{p_r + p_b}$$

1b. $x = \dfrac{p_r}{p_r + p_b}$

$p_1(\text{face with one spot}) = 1/6$

$p_{even}(\text{face with even \# spots}) = 3/6$

$$x = \frac{p_1}{p_1 + p_{even}}$$

$$x = \frac{1/6}{1/6 + 3/6}$$

$$x = \frac{1/6}{4/6} \cdot 6$$

$$x = 1/4$$

1c. idea: Alan needs to draw red before Katherine. If he draws either blue or green, Katherine draws. If Katherine doesn't draw blue, game resets and it's Alan's turn again!

$$x = P(\text{Alan wins}) + P\begin{pmatrix}\text{Alan loses turn AND} \\ \text{Katherine loses turn}\end{pmatrix} x$$

$$P\begin{pmatrix}\text{Alan loses} \\ \text{turn}\end{pmatrix} = P(\text{draw blue}) + P(\text{draw green})$$

$$= p_b + p_g$$

$$= 1 - p_r$$

$$= q_r$$

$$P\begin{pmatrix}\text{Katherine loses} \\ \text{turn}\end{pmatrix} = P(\text{draw red}) + P(\text{draw green})$$

$$= p_r + p_g$$

$$= 1 - p_b$$

$$= q_b$$

$$\therefore \quad x = p_r + (q_r)(q_b)x$$

$$x = p_r + q_r q_b x$$

$$x - q_r q_b x = p_r$$

$$x(1 - q_r q_b) = p_r$$

$$x = \frac{p_r}{1 - q_r q_b}$$

1d. $D \sim$ expected # draws till someone wins

Possible Outcomes

1. Alan draws first + draws red $= p_r$
   ↳ game ends (draw 1)

2. Alan doesn't draw red + Katherine draws blue $= q_r \times p_b$
   ↳ game ends (draw 2)

3. Neither Alan draws red or Katherine draws blue $= q_r \times q_b \times (2 + D)$
   ↳ back to Alan (draw $2 + D$)

$$\therefore D = p_r(1) + q_r(p_b)(2) + q_r(q_b)(2+D)$$

$$D = p_r + 2q_r p_b + 2q_r q_b + D q_r q_b$$

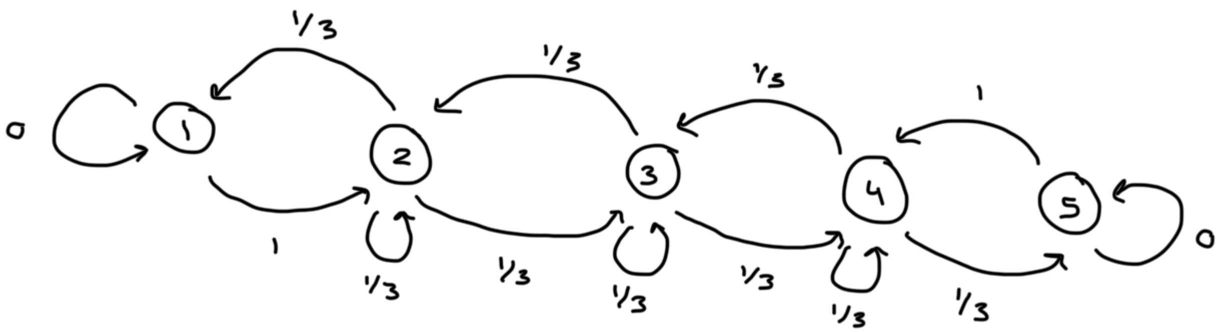
$$D - D q_r q_b = p_r + 2q_r p_b + 2q_r q_b$$

$$D(1 - q_r q_b) = p_r + 2q_r p_b + 2q_r q_b$$

$$D = \frac{p_r + 2q_r p_b + 2q_r q_b}{1 - q_r q_b}$$

$$\therefore E[D] = \frac{p_r + 2q_r p_b + 2q_r q_b}{1 - q_r q_b}$$

2a.



detailed balance

$\therefore \pi(i) p(i, j) = \pi(j) p(j, i)$ for all states $i \neq j$

$$\pi(1) p(1,2) = \pi(2) p(2,1)$$
$$\pi(1)(1) = \pi(2)\left(\frac{1}{3}\right)$$
$$\pi(1) \neq \frac{1}{3}\pi(2)$$

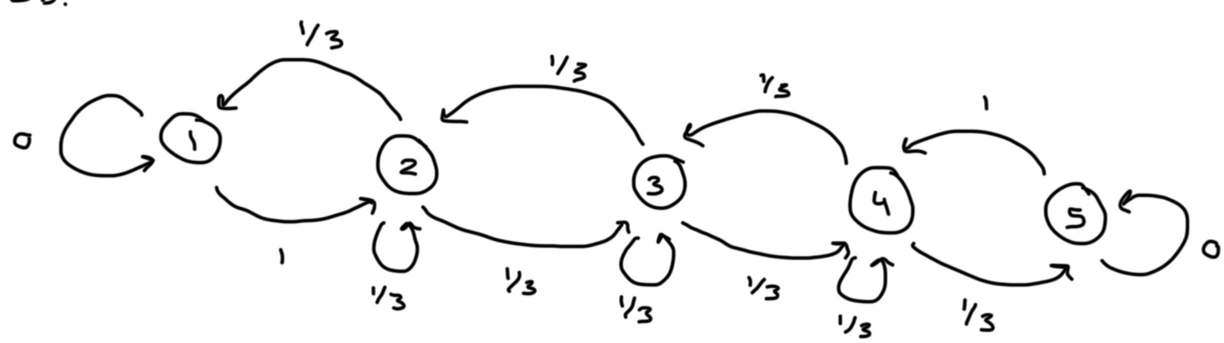
$$\pi(2) p(2,3) = \pi(3) p(3,2)$$
$$\pi(2)(1/3) = \pi(3)(1/3)$$
$$\frac{1}{3}\pi(2) = \frac{1}{3}\pi(3)$$

$$\pi(3) p(3,4) = \pi(4) p(4,3)$$
$$\pi(3)\left(\frac{1}{3}\right) = \pi(4)\left(\frac{1}{3}\right)$$
$$\frac{1}{3}\pi(3) = \frac{1}{3}\pi(4)$$

$$\pi(4)P(4,5) = \pi(5)P(5,4)$$
$$\pi(4)(1/3) = \pi(5)(1)$$
$$\frac{1}{3}\pi(4) \neq \pi(5)$$

∴ No probability distribution satisfies the detailed balance equations for this chain because the transition probabilities for some state pairs are NOT the same (ie. $P(1,2) \neq P(2,1)$ and $P(4,5) \neq P(5,4)$), meaning the chain is NOT reversible.

2b.



$$\pi(1) = \pi(1)P(1,1) + \pi(2)P(2,1)$$
$$\pi(1) = \pi(1)(0) + \pi(2)(\tfrac{1}{3})$$
$$\pi(1) = \frac{1}{3}\pi(2)$$
$$\hookrightarrow \pi(2) = 3\pi(1)$$

$$\pi(2) = \pi(1)P(1,2) + \pi(2)P(2,2) + \pi(3)P(3,2)$$
$$\pi(2) = \pi(1)(1) + \pi(2)(\tfrac{1}{3}) + \pi(3)(\tfrac{1}{3})$$
$$\pi(2) = \pi(1) + \tfrac{1}{3}\pi(2) + \tfrac{1}{3}\pi(3)$$
$$\pi(2) = \tfrac{1}{3}\pi(2) + \tfrac{1}{3}\pi(2) + \tfrac{1}{3}\pi(3)$$
$$\tfrac{1}{3}\pi(2) = \tfrac{1}{3}\pi(3)$$
$$\pi(3) = \pi(2)$$
$$\hookrightarrow \pi(3) = 3\pi(1)$$

$$\pi(3) = \pi(2)P(2,3) + \pi(3)P(3,3) + \pi(4)P(4,3)$$
$$\pi(3) = \pi(2)(\tfrac{1}{3}) + \pi(3)(\tfrac{1}{3}) + \pi(4)(\tfrac{1}{3})$$
$$\pi(3) = \tfrac{1}{3}\pi(2) + \tfrac{1}{3}\pi(3) + \tfrac{1}{3}\pi(4)$$
$$\tfrac{2}{3}\pi(3) = \pi(1) + \tfrac{1}{3}\pi(4)$$
$$2\pi(3) = 3\pi(1) + \pi(4)$$
$$6\pi(1) - 3\pi(1) = \pi(4)$$
$$3\pi(1) = \pi(4)$$
$$\hookrightarrow \pi(4) = 3\pi(1)$$

$$\pi(4) = \pi(3)P(3,4) + \pi(4)P(4,4) + \pi(5)P(5,4)$$
$$\pi(4) = \pi(3)(\tfrac{1}{3}) + \pi(4)(\tfrac{1}{3}) + \pi(5)(1)$$
$$\pi(4) = \tfrac{1}{3}\pi(3) + \tfrac{1}{3}\pi(4) + \pi(5)$$
$$\tfrac{2}{3}\pi(4) = \pi(1) + \pi(5)$$

$$2\pi(1) - \pi(1) = \pi(5)$$
$$\pi(1) = \pi(5)$$
$$\hookrightarrow \pi(5) = \pi(1)$$

$$\therefore \pi = [\pi(1) \quad \pi(2) \quad \pi(3) \quad \pi(4) \quad \pi(5)]$$
$$= [\pi(1) \quad 3\pi(1) \quad 3\pi(1) \quad 3\pi(1) \quad \pi(1)]$$

$$\therefore 1 = (1 + 3 + 3 + 3 + 1)\pi(1)$$
$$1 = 11\pi(1)$$
$$\pi(1) = \frac{1}{11}$$

$$\therefore \pi = \left[\frac{1}{11} \quad \frac{3}{11} \quad \frac{3}{11} \quad \frac{3}{11} \quad \frac{1}{11}\right]$$

2c. $P(X_n = j \mid X_0 = i) = P_n(i,j)$ textbook 10.1.6

$\hookrightarrow \pi(j)$ textbook 10.3.1

as n approaches ∞
(or large n)
$\therefore$ converges to stationary distribution

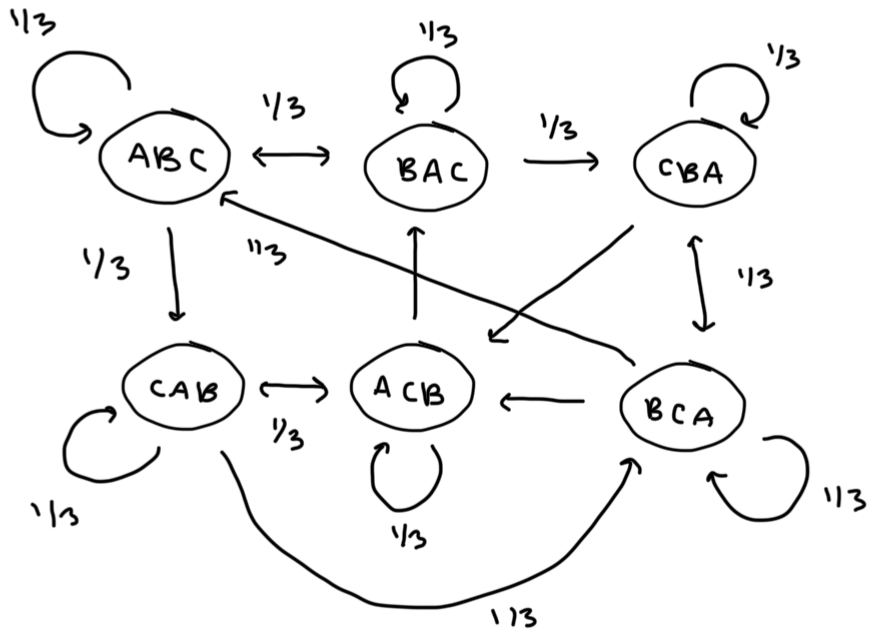
$$\therefore P(X_{1000} = 4 \mid X_0 = 3) = P_{1000}(3,4)$$
$$\hookrightarrow \pi(4)$$
$$= \frac{3}{11}$$
$$\approx 0.273$$

3a. The stationary distribution π of a MC satisfies the balance equation $\pi P = \pi$, meaning that over time, the probability of being in each state remains constant. This occurs because the total probability flowing into a state equals the total probability flowing out, ensuring equilibrium. If the transition matrix is doubly stochastic (both rows / columns sum to 1), then no state is treated differently in terms of how much probability it receives or gives out. This forces all states to have the same stationary probability. Since probabilities sum to 1, it follows that the only possible solution is $\pi_i = \frac{1}{N}$ for all i, meaning the stationary distribution is uniform. The MC must also be irreducible (can go to ALL states) and aperiodic (no cycles). These ensure the stationary distribution is unique and the chain converges to it. if a matrix is only stochastic (rows sum to 1), then states can be treated unequally, leading to non-uniform stationary distribution.

3b. A, B, C $\rightarrow$ 3 possible moves

$\therefore$ either A, B, or C ends up at the top

$\hookrightarrow$ $3! = 6$ possible permutations



$\therefore$ each permutation has 1/3 chance of picking a card at random and moving it to the front

$P(ABC, ABC) = 1/3$

$P(ABC, BAC) = 1/3$

$P(ABC, CAB) = 1/3$

$P(ABC, ACB) = 0$

$P(ABC, BCA) = 0$

$P(ABC, CBA) = 0$

$$\therefore P(ABC) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$$

$P(BAC, ABC) = 1/3$

$P(BAC, BAC) = 1/3$

$P(BAC, CAB) = 0$

$P(BAC, ACB) = 0$

$P(BAC, BCA) = 0$

$P(BAC, CBA) = 1/3$

$$\therefore P(ABC) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$P(CAB, ABC) = 0$

$P(CAB, BAC) = 0$

$P(CAB, CAB) = 1/3$

$P(CAB, ACB) = 1/3$

$P(CAB, BCA) = 1/3$

$P(CAB, CBA) = 0$

$$\therefore P(ABC) = \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

$P(ACB, ABC) = 0$

$P(ACB, BAC) = 1/3$

$P(ACB, CAB) = 1/3$

$P(ACB, ACB) = 1/3$

$P(ACB, BCA) = 0$

$P(ACB, CBA) = 0$

$$\therefore P(ABC) = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}$$

$P(BCA, ABC) = 1/3$

$P(BCA, BAC) = 0$

$P(BCA, CAB) = 0$

$P(BCA, ACB) = 0$

$P(BCA, BCA) = 1/3$

$P(BCA, CBA) = 1/3$

$$\therefore P(ABC) = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$P(CBA, ABC) = 0$

$P(CBA, BAC) = 0$

$P(CBA, CAB) = 0$

$P(CBA, ACB) = 1/3$

$P(CBA, BCA) = 1/3$

$P(CBA, CBA) = 1/3$

$$\therefore P(ABC) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$P = \begin{array}{c} \\ ABC \\ BAC \\ CAB \\ ACB \\ BCA \\ CBA \end{array} \begin{array}{c} \begin{array}{cccccc} ABC & BAC & CAB & ACB & BCA & CBA \end{array} \\ \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

$\therefore$ sum up row + columns
doubly stochastic

$$\pi = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

deck = 52 cards

- repeat! $\rightarrow$ think permutations

$\therefore$ 52!

$$\therefore \pi = \begin{bmatrix} \frac{1}{52} & \frac{1}{52} & \cdots & \frac{1}{52} \end{bmatrix}$$

From the example (ie. ABC) above + answer to 3a, we can apply its reasoning to a deck of cards. A random to front shuffle allows each card to have an equal $\frac{1}{52}$ chance of being chosen and moved to the front. This means that the transition matrix is doubly stochastic, as each row/column sum to 1. From answer 3a, the stationary distribution must be uniform, meaning all deck permutations will be equally likely in the long run. The MC is both irreducible and aperiodic, so any permutation can be achieved and it's possible to stay at the same permutation indefinitely. These conditions allow the chain to converge to its unique stationary distribution. So, performing this shuffling repeatedly results in a well-shuffled deck where all permutations are equally likely.