EE 123 Discussion Section 5
Discrete wavelet transform

March 6th, 2019

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Based on slides by Frank Ong, Nick Antipa

Announcements

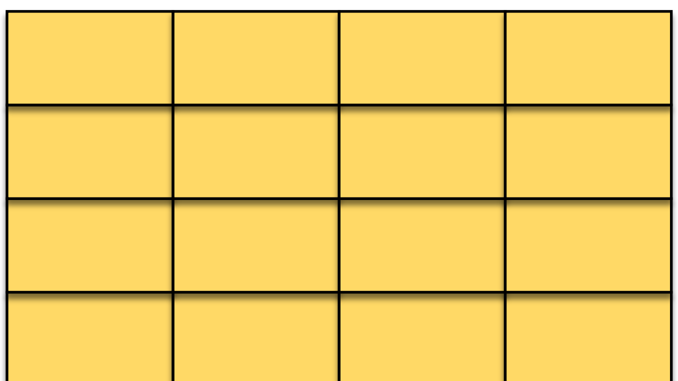
- Lab 2 – due Thursday March 7th (real-time is optional).
- HW 6 – due next Monday March 11th.
- Questions?

Wavelets

$$W\{f\}(u,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1}{\sqrt{s}}\,\Psi^*\left(\frac{t-u}{s}\right)dt$$

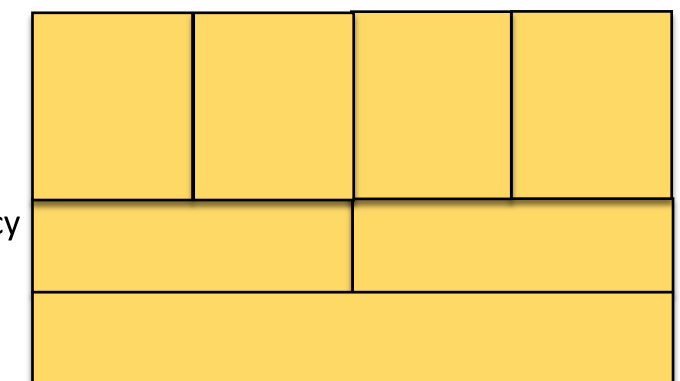
Is this shift invariant?

STFT



Wavelets

S
frequency



u
(time)

Discrete wavelets

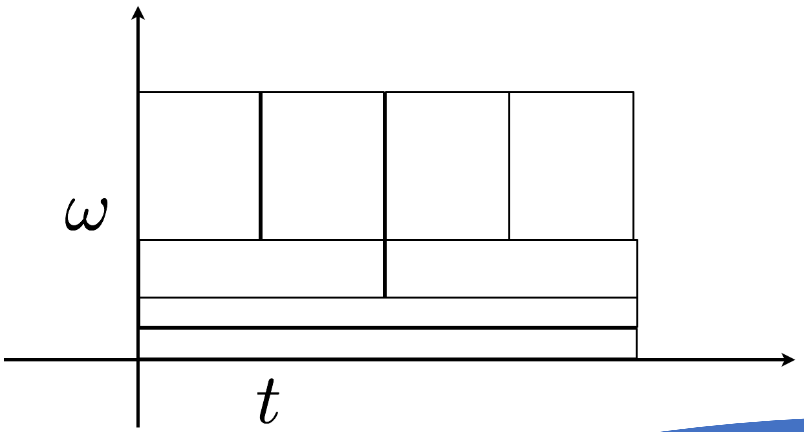
$$W\{f\}[s,u] = \sum_{n=0}^{N-1} x[n]\Psi_{s,u}[n]$$

$$= \sum_{n=0}^{N-1} x[n]\frac{1}{\sqrt{2^s}}\Psi\left(\frac{1}{2^s}(n-2^s u)\right)$$

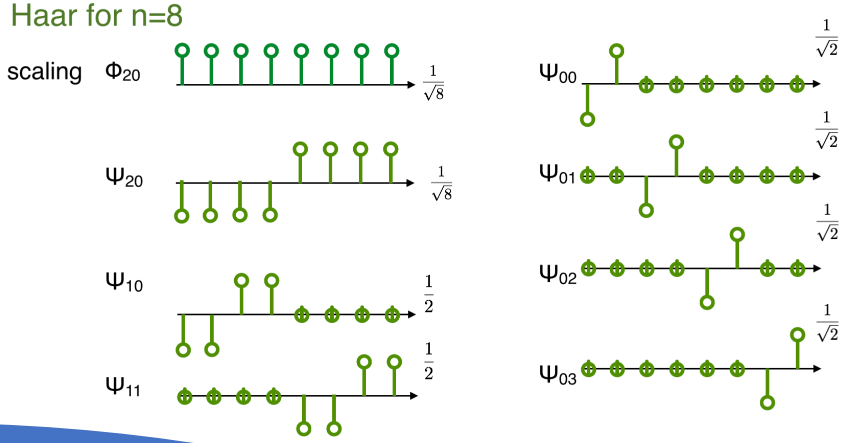
Is this shift invariant?

Three Views of the Wavelet Transform

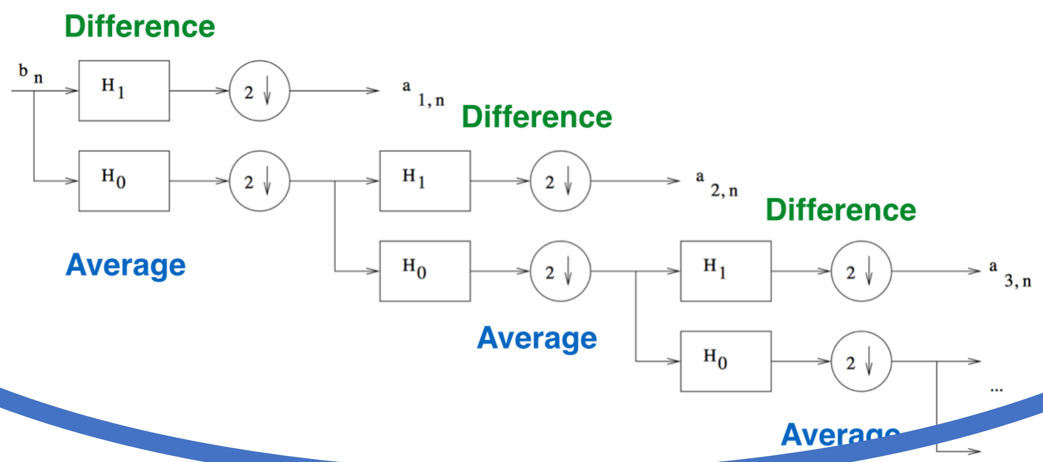
Multi-scale Time-Frequency Tiling



Wavelet Basis functions



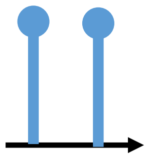
Fast Wavelet Transform



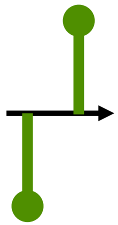
Haar filters

- For the Haar wavelet, the filters are:

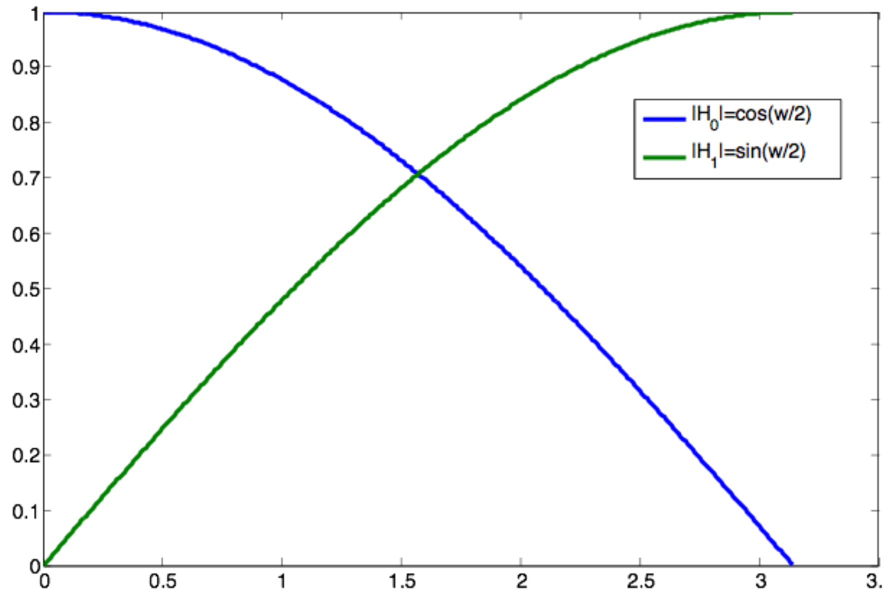
Average filter h_0



Difference filter h_1

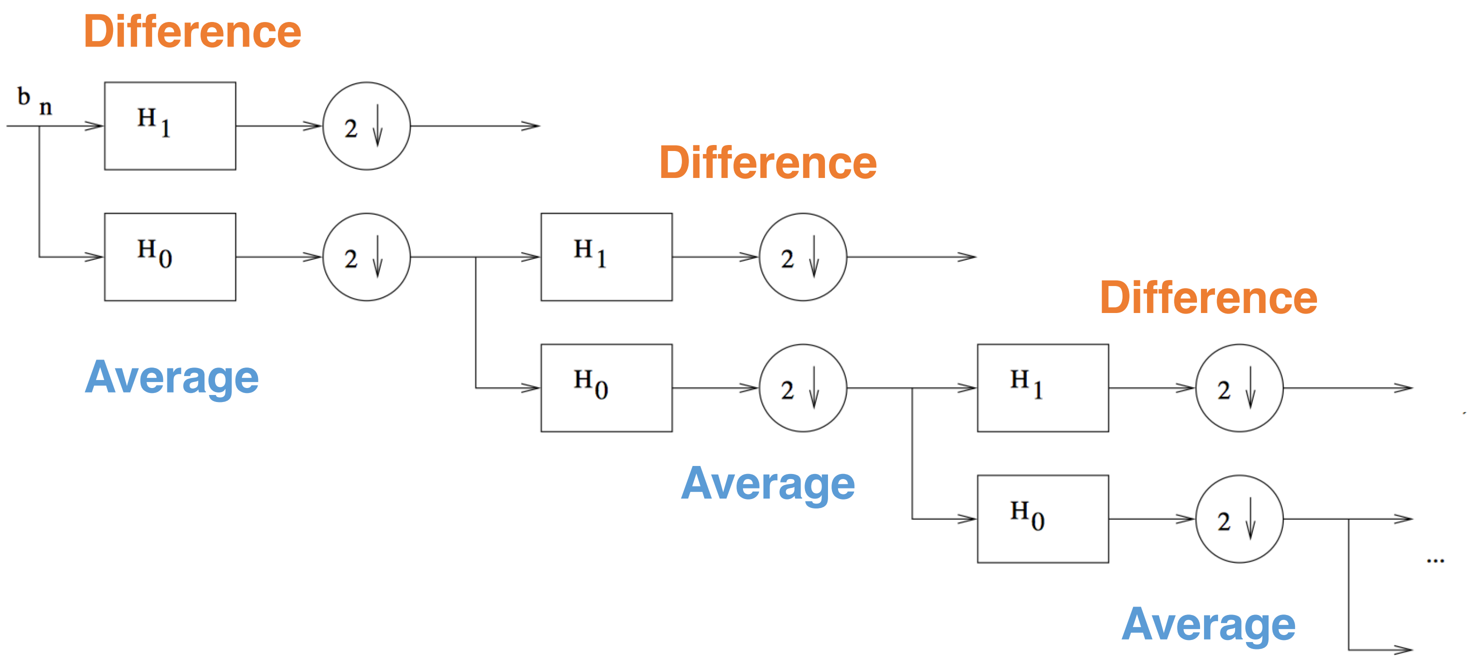


- And their magnitude responses are:



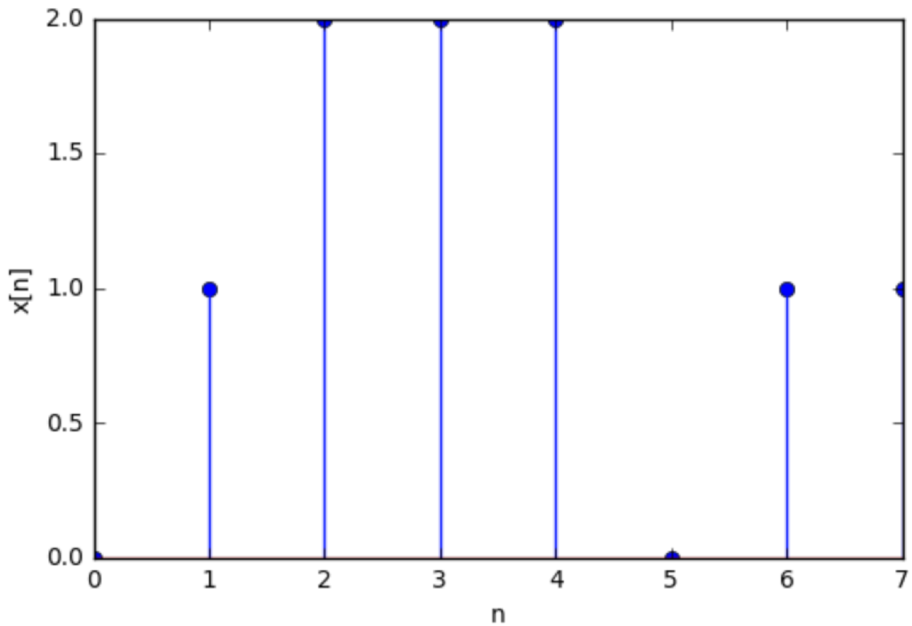
The filters have complementary frequency responses

Fast Wavelet Transform



- Once you have the 2-channel filter bank
- Simply iterate on the average coefficients to obtain the wavelet transform

Haar decomposition example



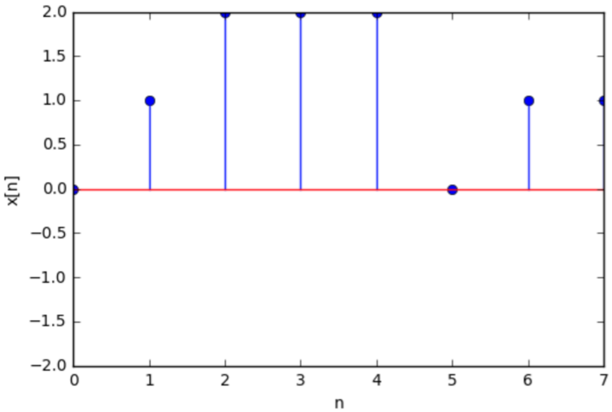
$$x[n] = \{0,1,2,2,2,0,1,1\}$$

$$\boxed{d_{0,u} = \{1, 0, -2, 0\} \cdot \frac{1}{\sqrt{2}}} \qquad \boxed{d_{1,u} = \{3, 0\} \cdot \frac{1}{2}} \qquad \boxed{d_{2,u} = \{-1\} \cdot \frac{1}{2\sqrt{2}}}$$

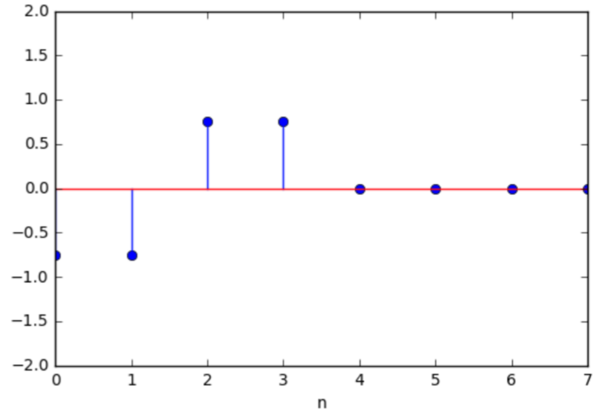
$$a_{0,u} = \{1, 4, 2, 2\} \cdot \frac{1}{\sqrt{2}} \qquad a_{1,u} = \{5, 4\} \cdot \frac{1}{2} \qquad \boxed{a_{2,u} = \{9\} \cdot \frac{1}{2\sqrt{2}}}$$

Haar representation

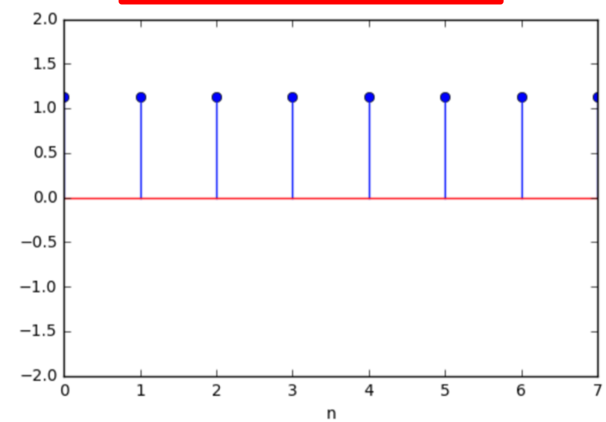
$$x[n] = \{0,1,2,2,2,0,1,1\}$$



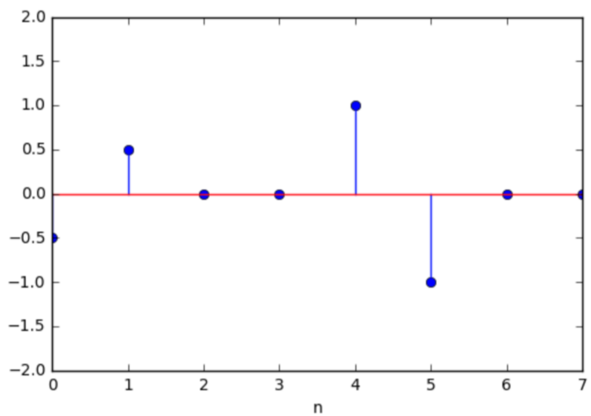
$$\boxed{d_{1,u} = \{3, 0\} \cdot \frac{1}{2}}$$



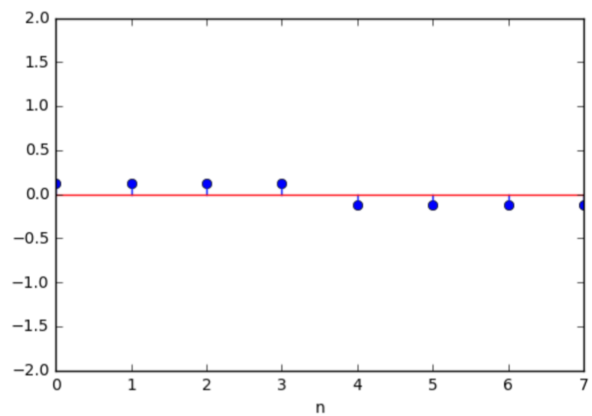
$$\boxed{a_{2,u} = \{9\} \cdot \frac{1}{2\sqrt{2}}}$$



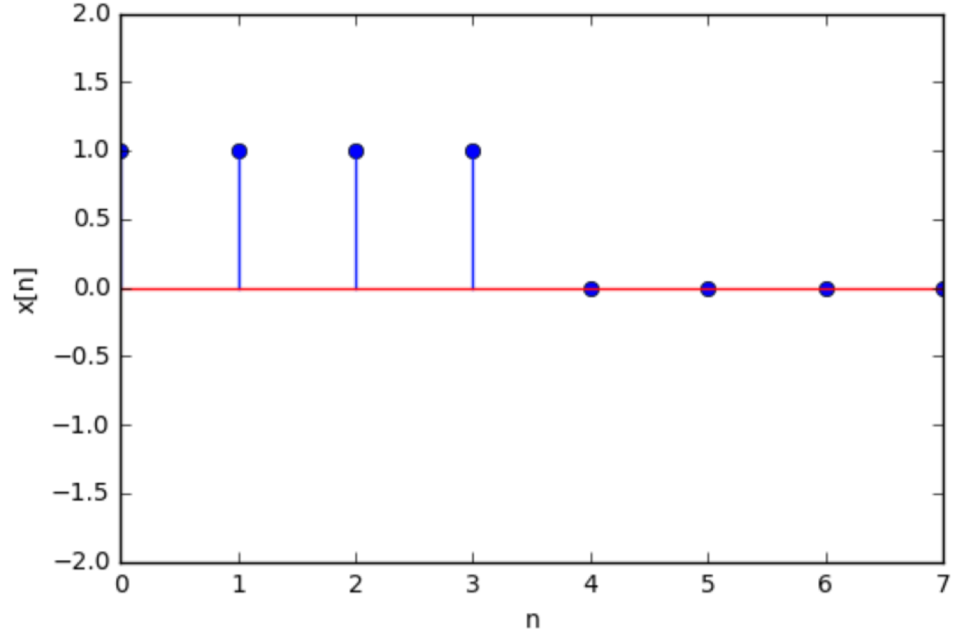
$$\boxed{d_{0,u} = \{1, 0, -2, 0\} \cdot \frac{1}{\sqrt{2}}}$$



$$\boxed{d_{2,u} = \{-1\} \cdot \frac{1}{2\sqrt{2}}}$$



Haar denoising example



$x[n] = \{1,1,1,1,0,0,0,0\}$ DFT?

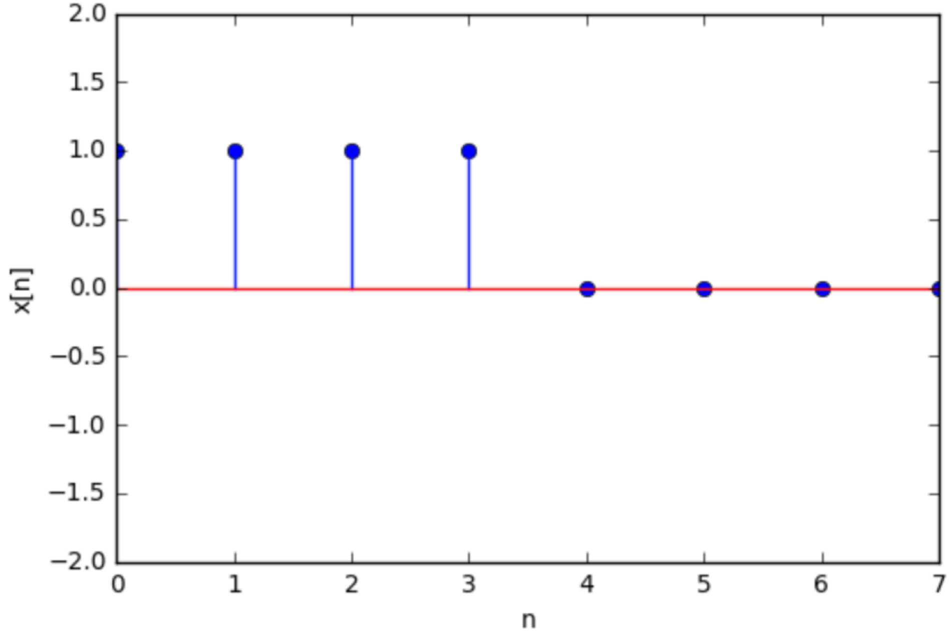
$$X[k] = \left(1 + e^{-\frac{j2\pi k}{8}} + e^{-\frac{j2\pi\cdot 2k}{8}} + e^{-\frac{j2\pi\cdot 3k}{8}}\right) \cdot \frac{1}{2\sqrt{2}}$$

$$= \left[e^{-\frac{j\pi k}{8}}\left(e^{\frac{j\pi k}{8}} + e^{-\frac{j\pi k}{8}}\right) + e^{-\frac{j5\pi k}{8}}\left(e^{\frac{j\pi k}{8}} + e^{-\frac{j\pi k}{8}}\right)\right] \cdot \frac{1}{2\sqrt{2}}$$

$$= \sqrt{2}e^{-\frac{j3\pi k}{8}} \cos\left(\frac{2\pi k}{8}\right) \cos\left(\frac{\pi k}{8}\right)$$

3 zeros, 5 points to represent 4-point sequence

Haar denoising example



$x[n] = \{1,1,1,1,0,0,0,0\}$ DWT?

$$\boxed{d_{0,u} = \{0,0,0,0\} \cdot \frac{1}{\sqrt{2}}}$$

$$a_{0,u} = \{2,2,0,0\} \cdot \frac{1}{\sqrt{2}}$$

$$\boxed{d_{1,u} = \{0,0\} \cdot \frac{1}{2}}$$

$$a_{1,u} = \{4,0\} \cdot \frac{1}{2}$$

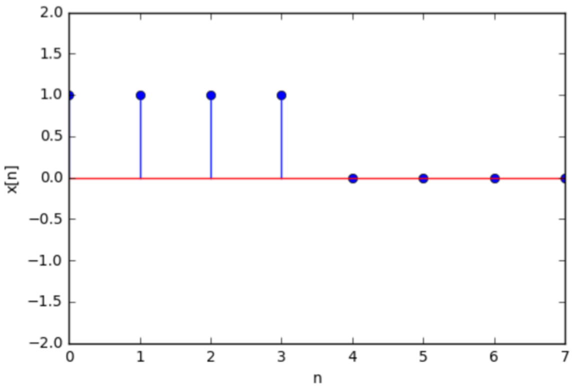
$$\boxed{d_{2,u} = \{-4\} \cdot \frac{1}{2\sqrt{2}}}$$

$$\boxed{a_{2,u} = \{4\} \cdot \frac{1}{2\sqrt{2}}}$$

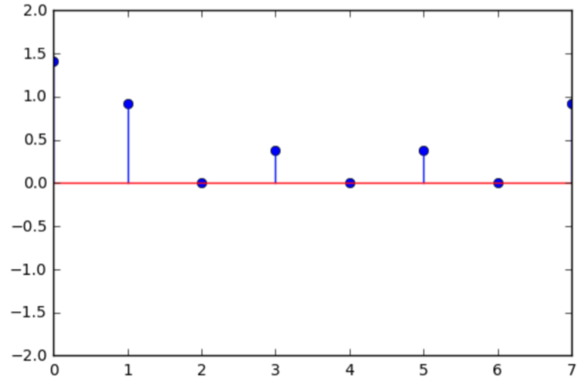
2 points to represent 4-point sequence → more efficient

Haar denoising example

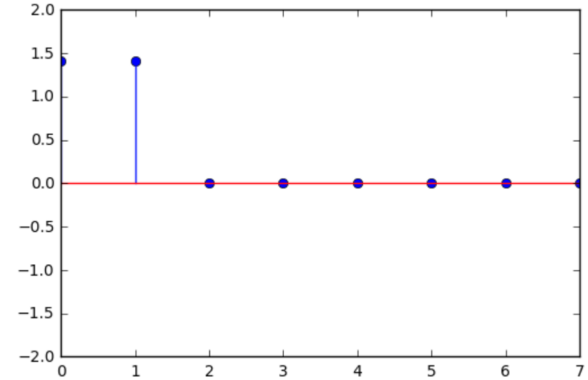
$x[n]$



DFT



DWT



White Gaussian noise $(0, \sigma)$

$x'[n] = x[n] + v[n]$ ⟶ For every point with signal $SNR = \frac{1}{\sigma^2}$

$x'_F[n] = x_F[n] + v_F[n]$ ⟶ Max $SNR = \frac{2}{\sigma^2}$ Min $SNR = \frac{0.146}{\sigma^2}$

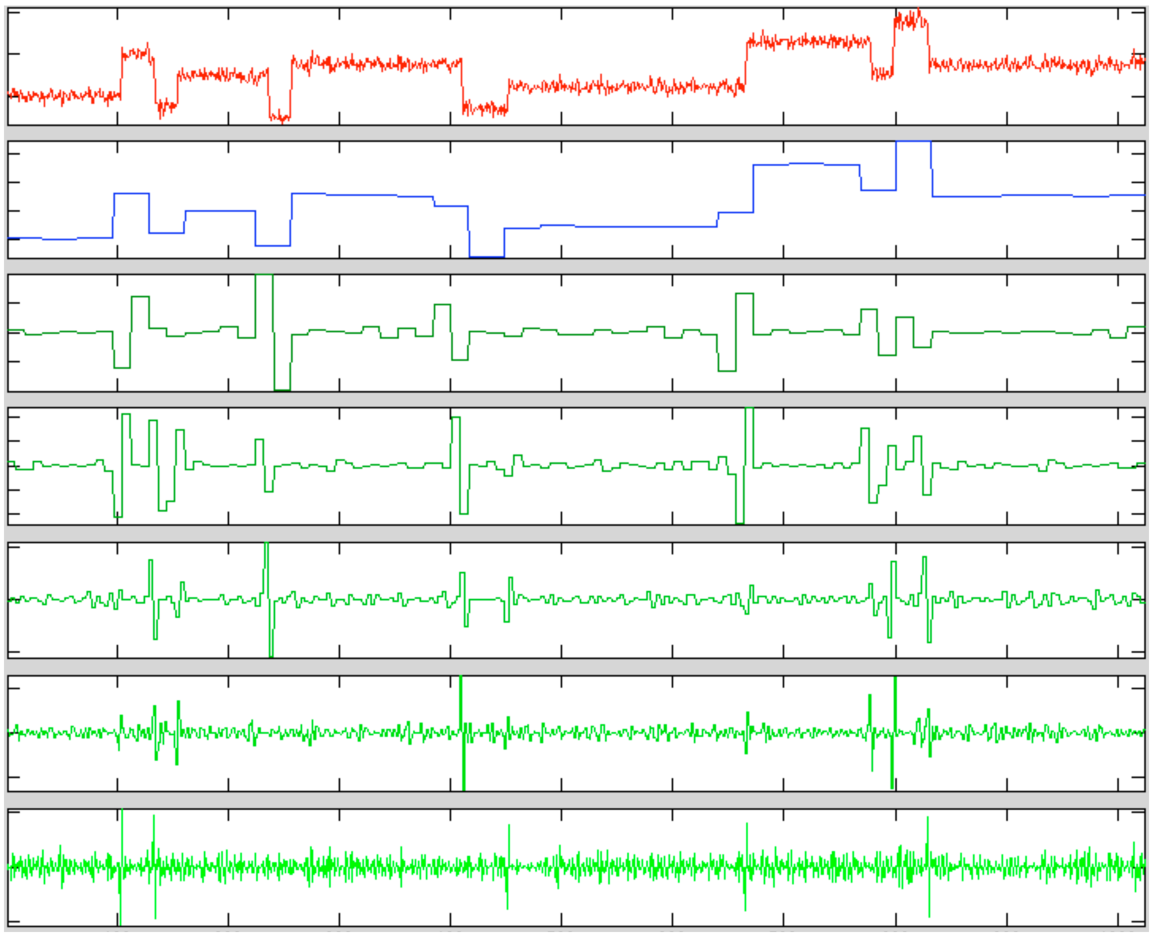
$\boxed{x'_W[n] = x_W[n] + v_W[n] \longrightarrow \text{For every point with signal } SNR = \frac{2}{\sigma^2}}$

Energy more concentrated, easier to denoise through thresholding

Haar Wavelet Example

Input

Coarse



Detail

5 level decomposition

(each subband is normalized separately)