EE123
Digital Signal Processing

Lecture 11
Introduction to Wavelets

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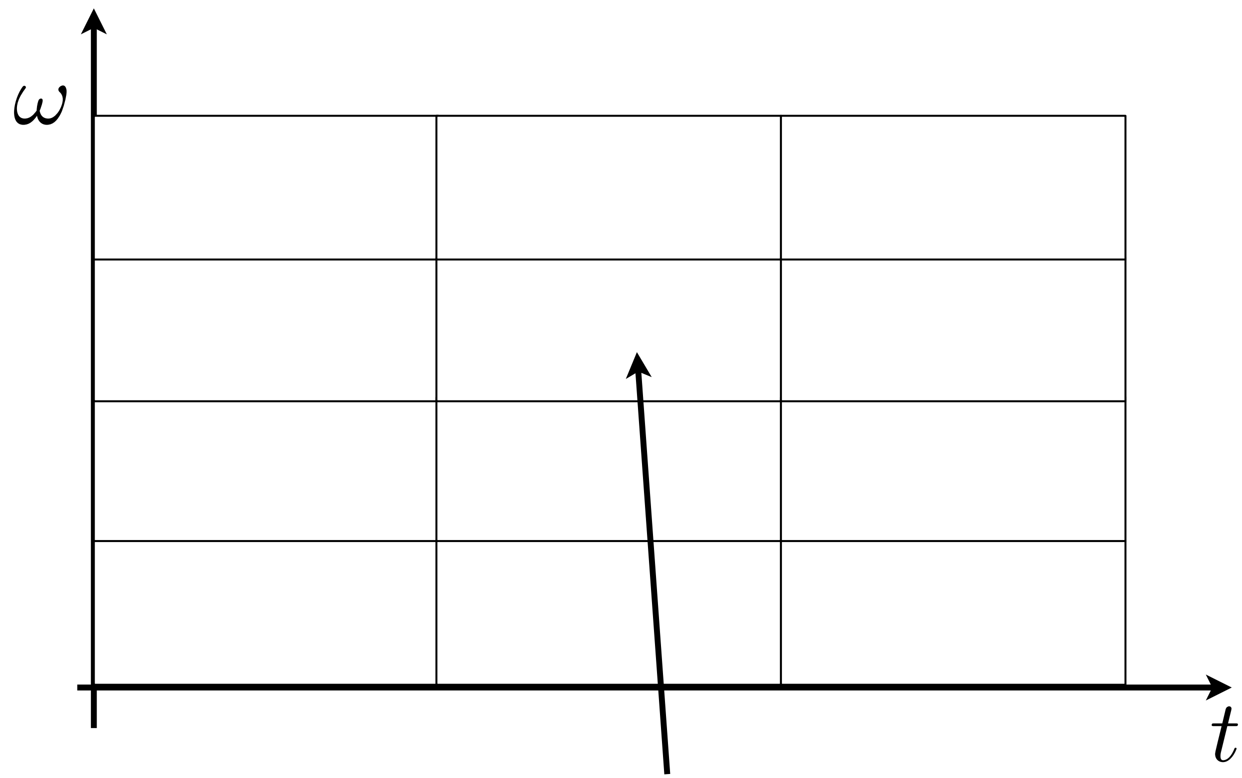
Discrete STFT

$$X[r,k] = \sum_{m=0}^{L-1} x[r\overset{\text{optional}}{\overset{\downarrow}{R}} + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



one STFT coefficient

Limitations of Discrete STFT

- Need overlapping $\Rightarrow$ Not orthogonal

- Computationally intensive $O(MN \log N)$

- Same size Heisenberg boxes

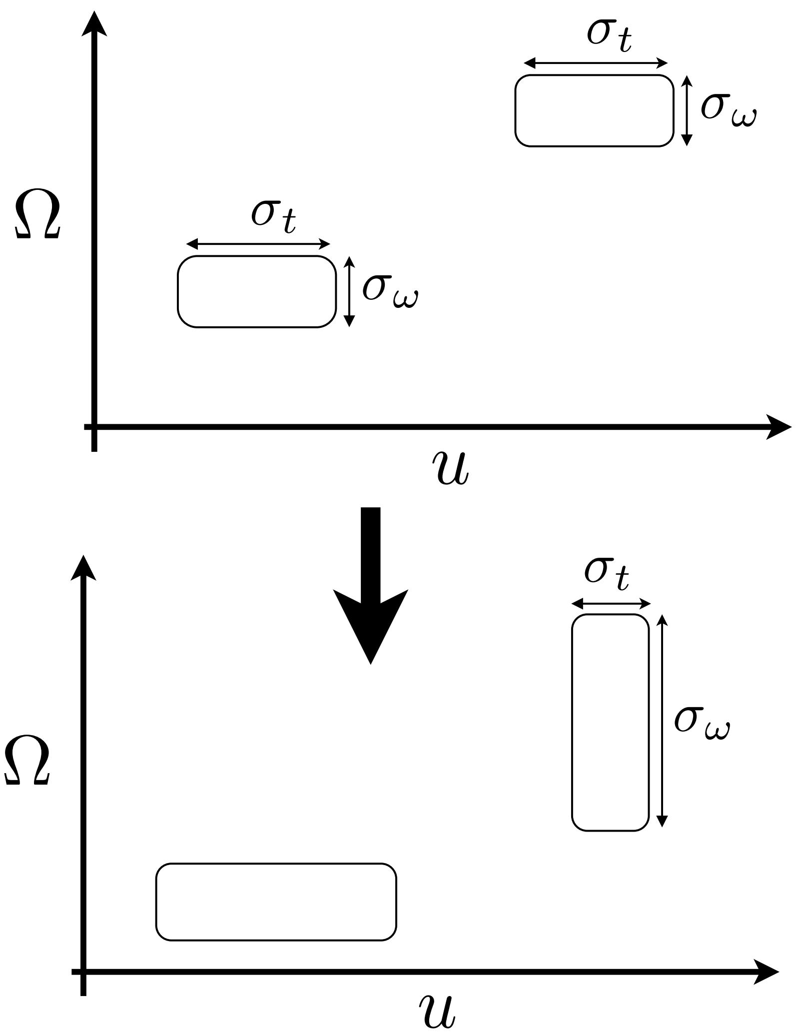
From STFT to Wavelets

- Basic Idea:
  - –low-freq changes slowly - fast tracking unimportant
  - –Fast tracking of high-freq is important in many apps.
  - –Must adapt Heisenberg box to frequency

- Back to continuous time for a bit.....

From STFT to Wavelets

- Continuous time



$$Sf(u,\Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t}dt$$

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t-u}{s}) dt$$

- The function Ψ is called a mother wavelet
  - Must satisfy:

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1$$ ⇒ unit norm

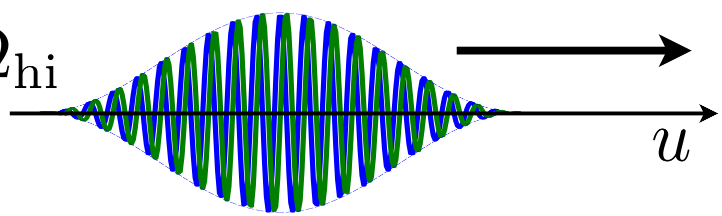
$$\int_{-\infty}^{\infty} \Psi(t) dt = 0$$ ⇒ Band-Pass

STFT and Wavelets “Atoms”

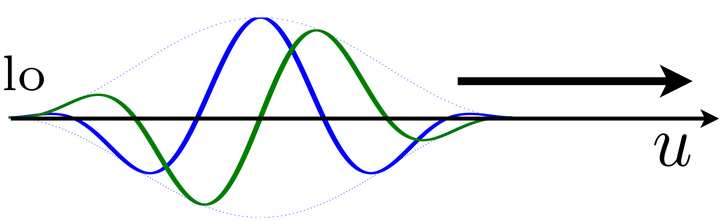
STFT Atoms
(with hamming window)

$$w(t-u)e^{j\Omega t}$$

Ω_{hi}



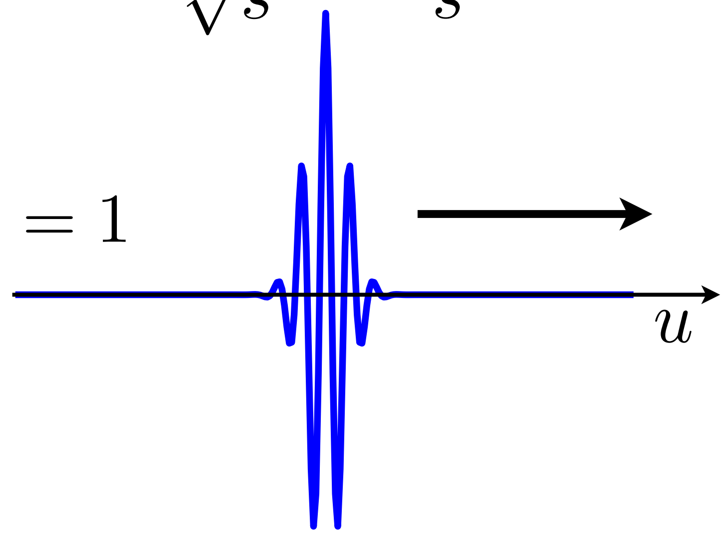
Ω_{lo}



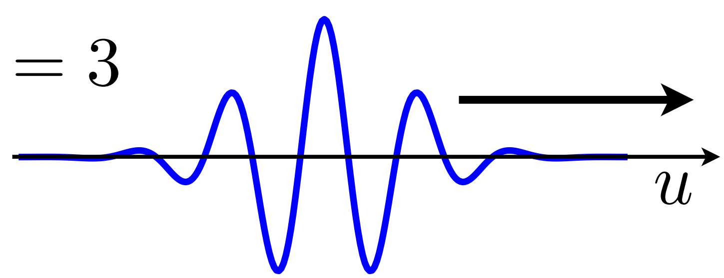
Wavelet Atoms

$$\frac{1}{\sqrt{s}}\Psi\left(\frac{t-u}{s}\right)$$

$s = 1$



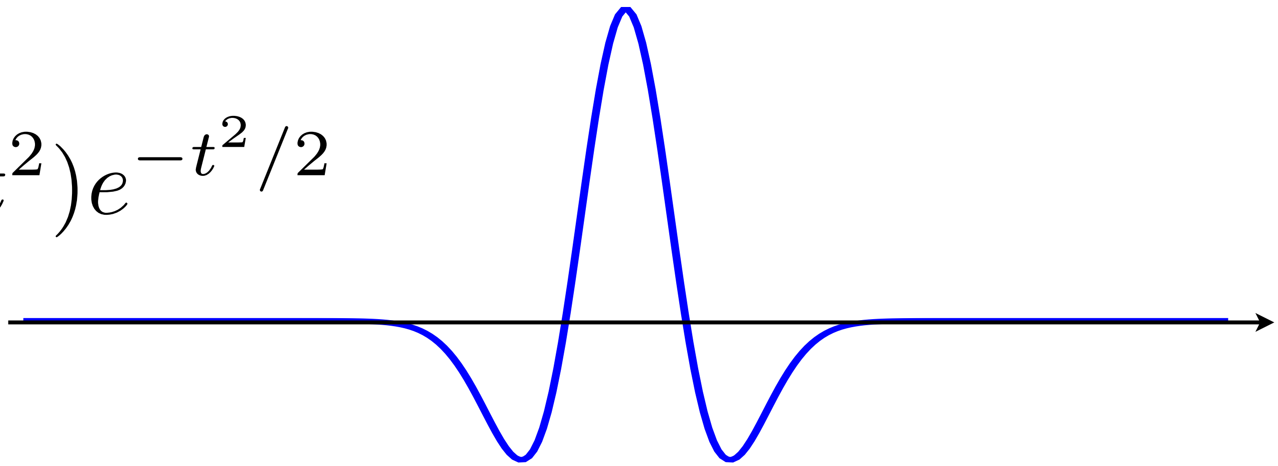
$s = 3$



Examples of Wavelets

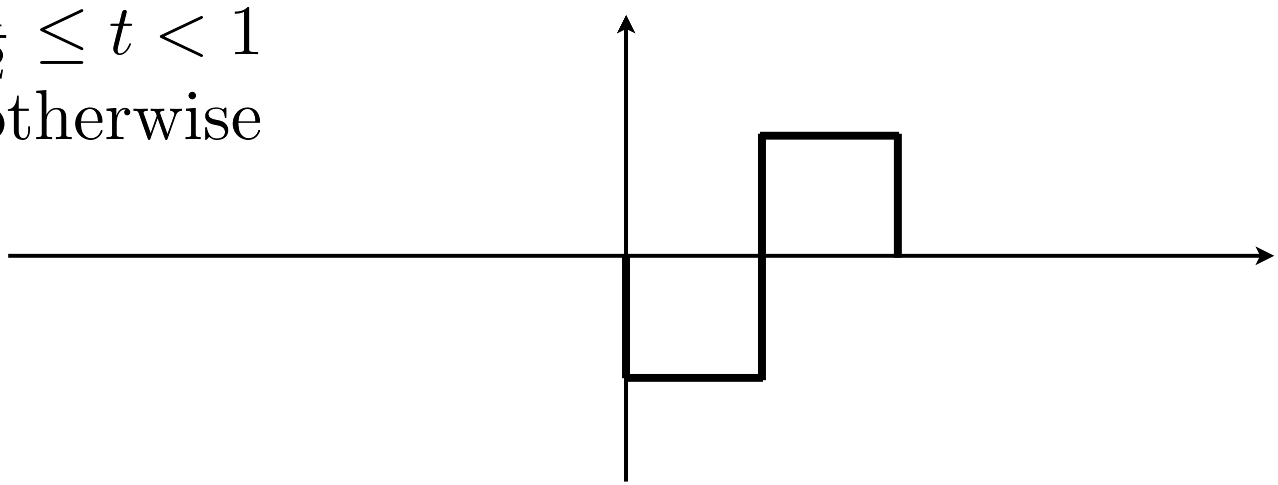
- Mexican Hat

$$\Psi(t) = (1 - t^2)e^{-t^2/2}$$

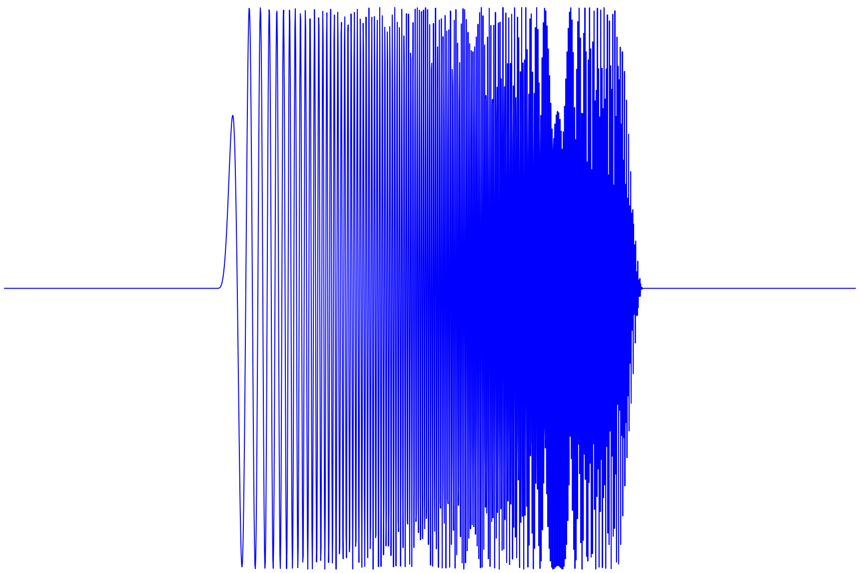


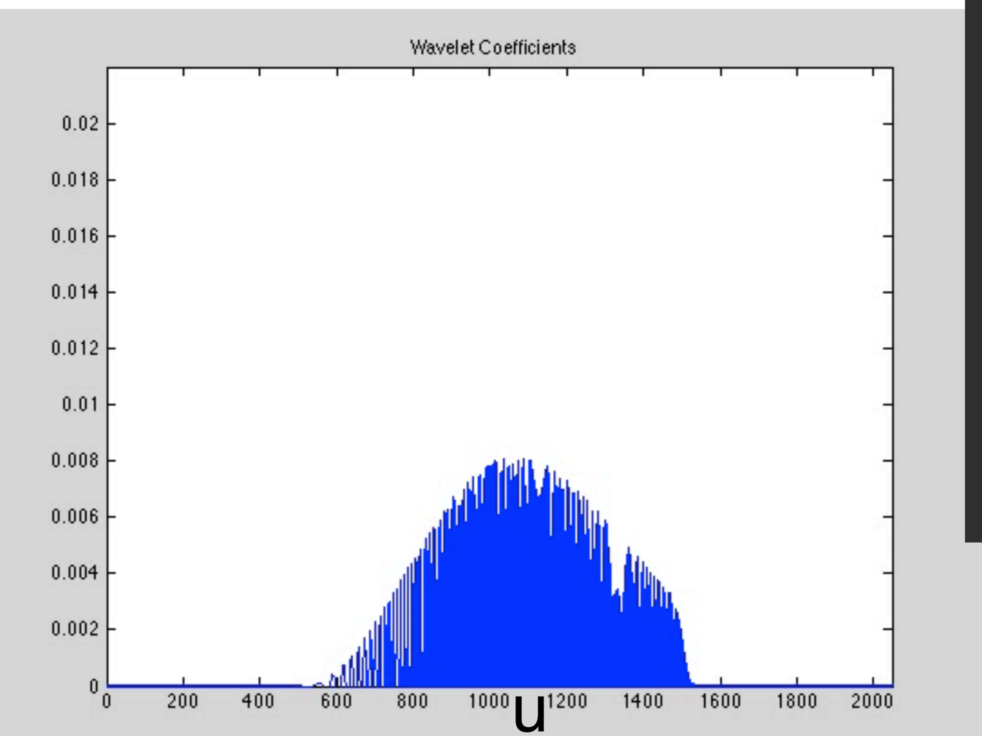
- Haar

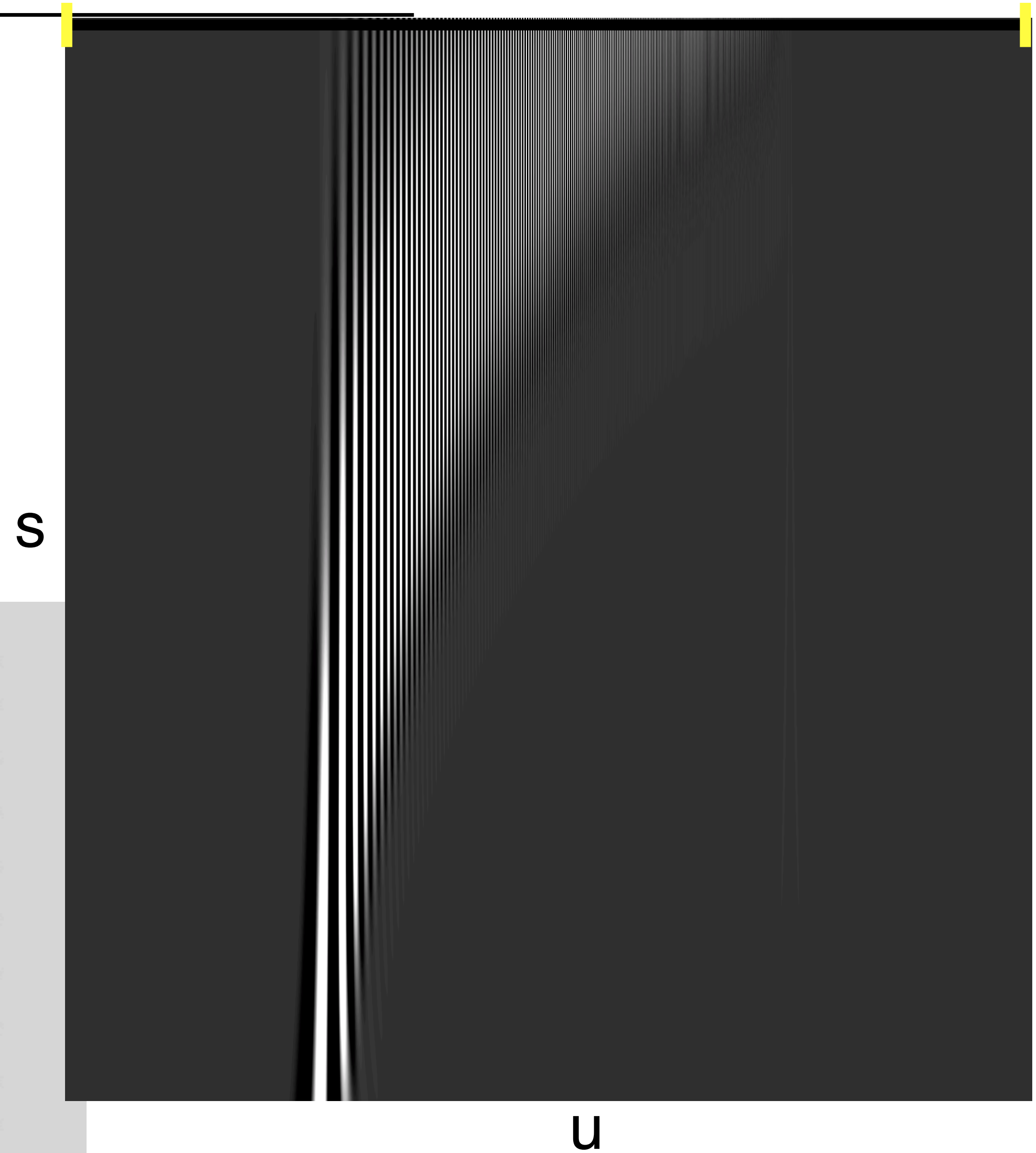
$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



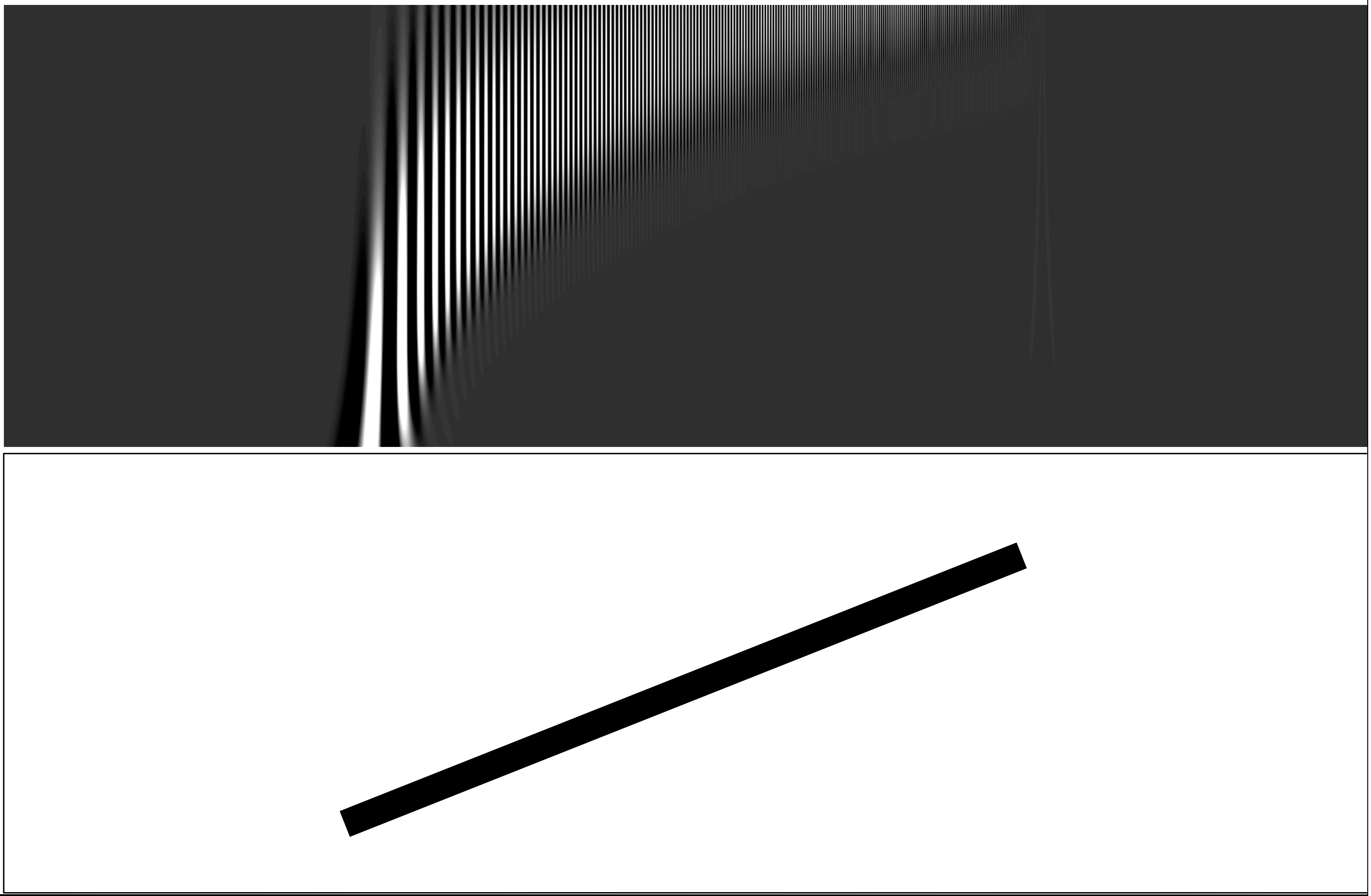
Example: Wavelet of Chirp



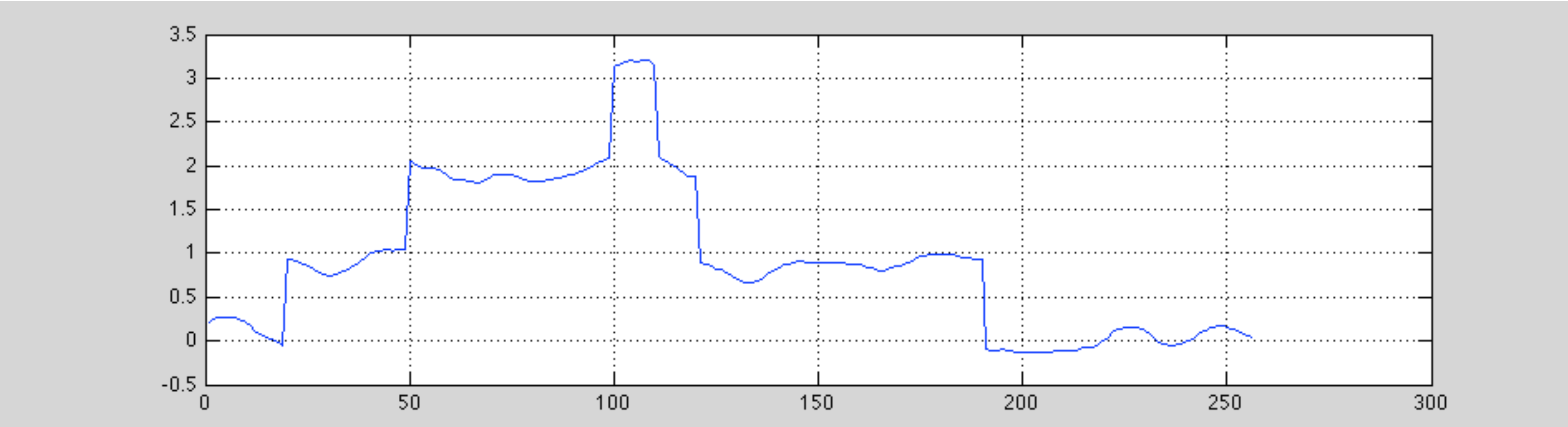




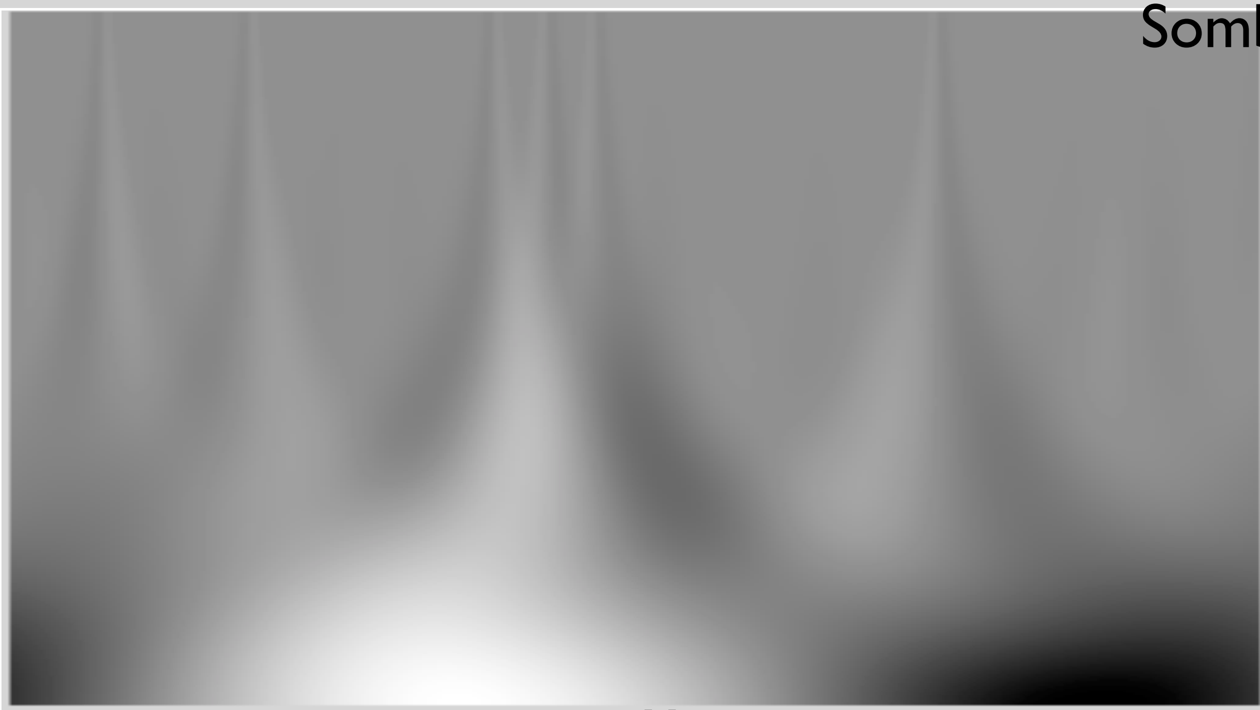
Wavelets VS STFT



Example 2: “Bumpy” Signal



SombreroWavelet



Wavelets Transform

- Can be written as linear filtering

$$
\begin{aligned}
Wf(u,s) &= \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} f(t)\Psi^*(\frac{t-u}{s})dt \\
&= \left\{f(t) * \overline{\Psi}_s(t)\right\}(u)
\end{aligned}
$$

$$\overline{\Psi}_s = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

- Wavelet coefficients are a result of bandpass filtering

Wavelet Transform

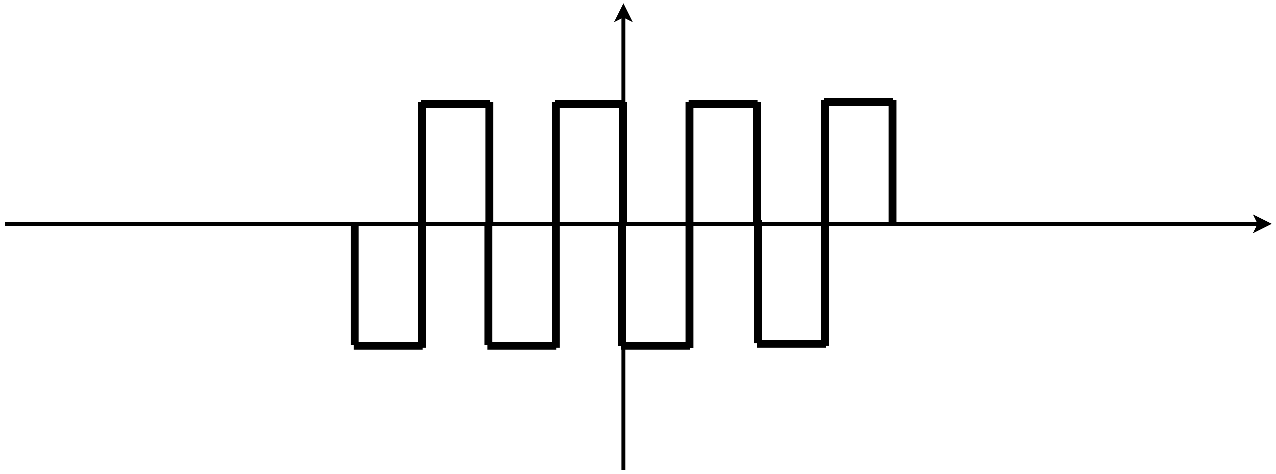
- Many different constructions for different signals
  - –Haar good for piece-wise constant signals
  - –Battle-Lemarie' : Spline polynomials

- Can construct Orthogonal wavelets
  - – For example: dyadic Haar is orthonormal

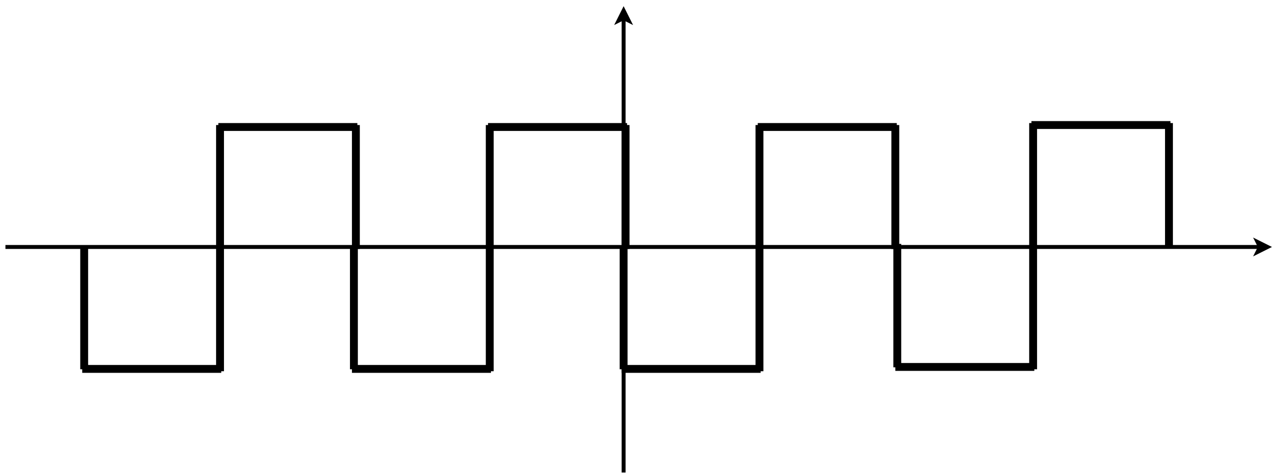
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$

$$i = [1,\ 2,\ 3, \cdots]$$

Orthonormal Haar



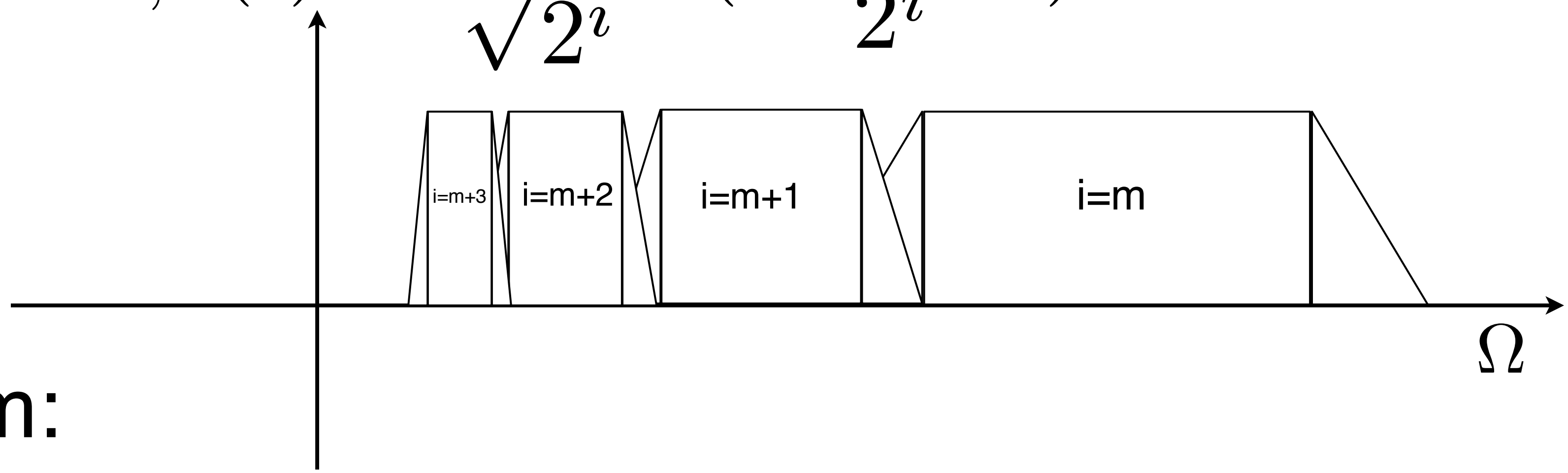
Same scale
non-overlapping



Orthogonal
between scales

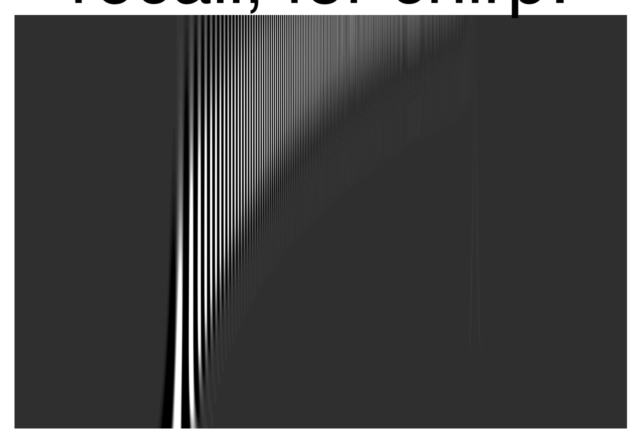
Scaling function

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$



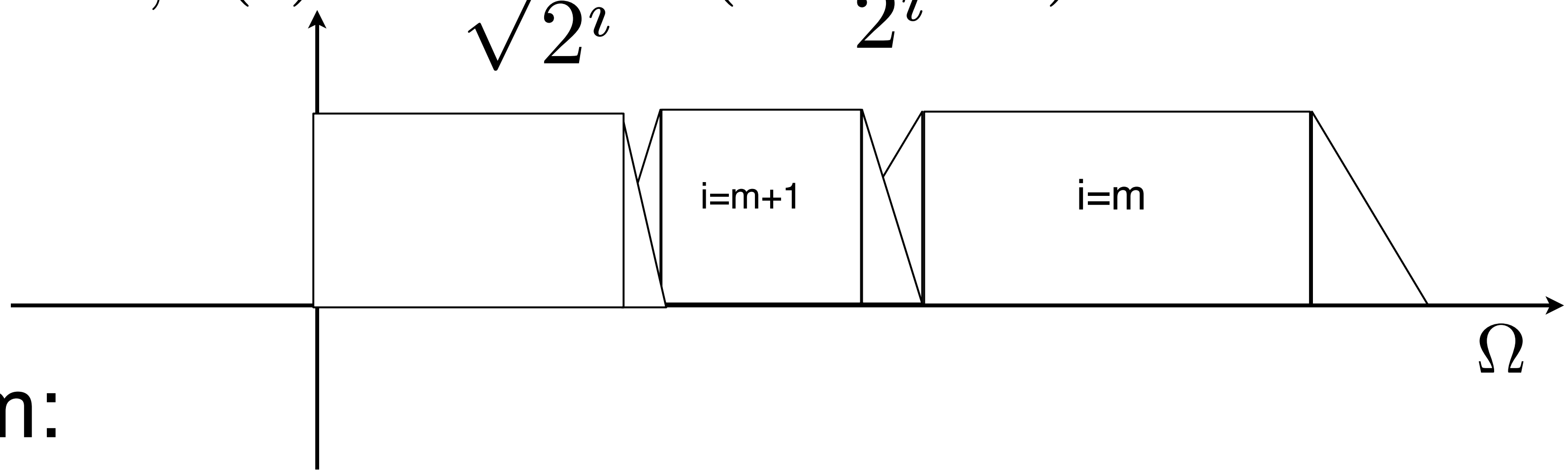
- Problem:
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions

recall, for chirp:



Scaling function

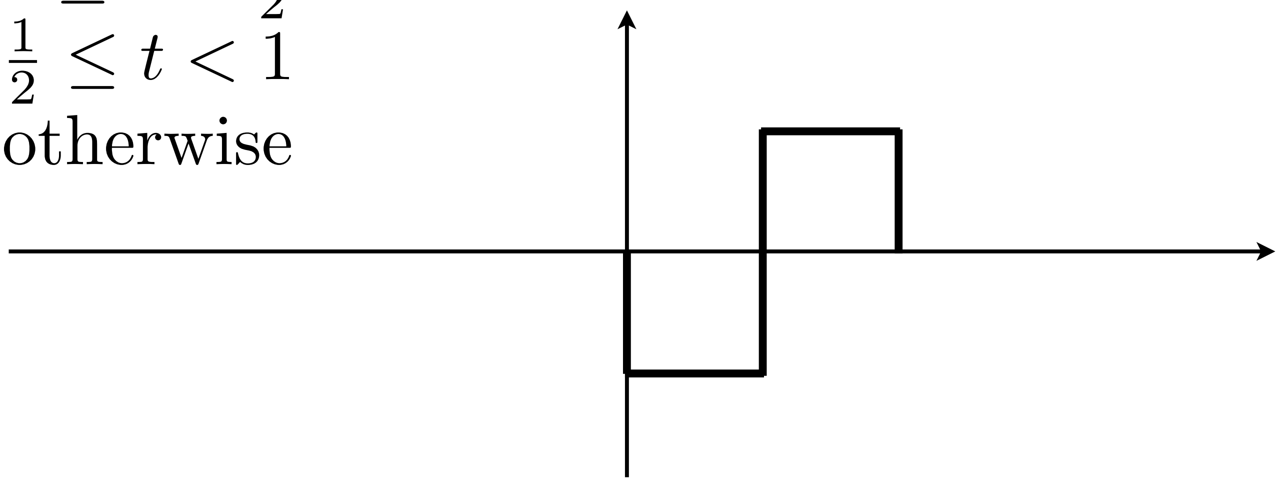
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$



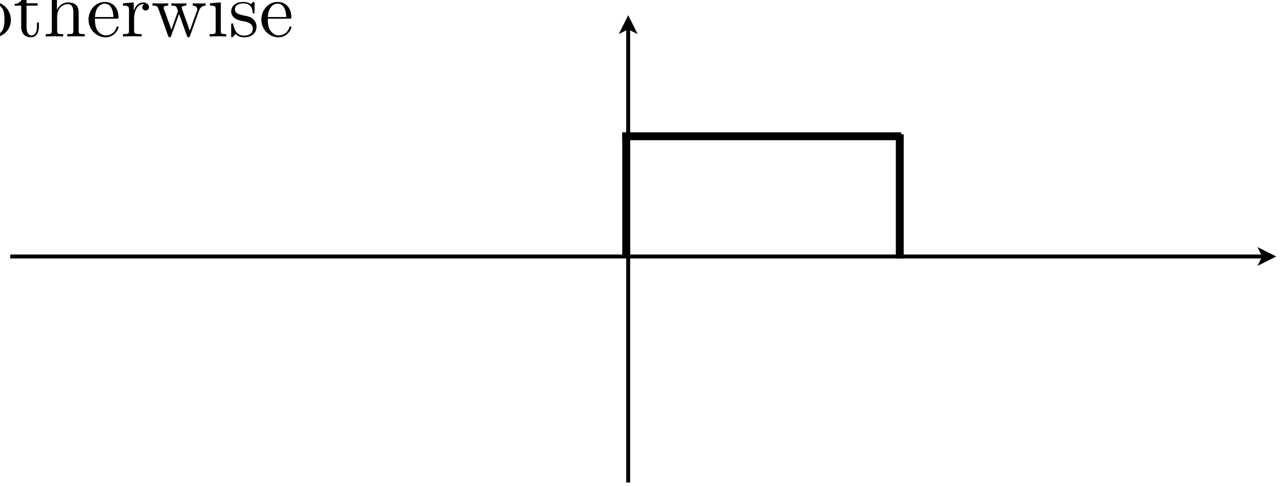
- Problem:
  - –Every stretch only covers half remaining bandwidth
  - –Need Infinite functions
- Solution:
  - –Plug low-pass spectrum with a scaling function $\overline{\Phi}$

Haar Scaling function

$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\Phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Back to Discrete

- Early 80's, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80's link to DSP by Daubechies and Mallat.

- From CWT to DWT not so trivial!
- Must take care to maintain properties