EE123
Digital Signal Processing

Lecture 23
Phase Response
All-Pass and Minimum Phase

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* Beautiful handwritten figures by Prof. Murat Arcak

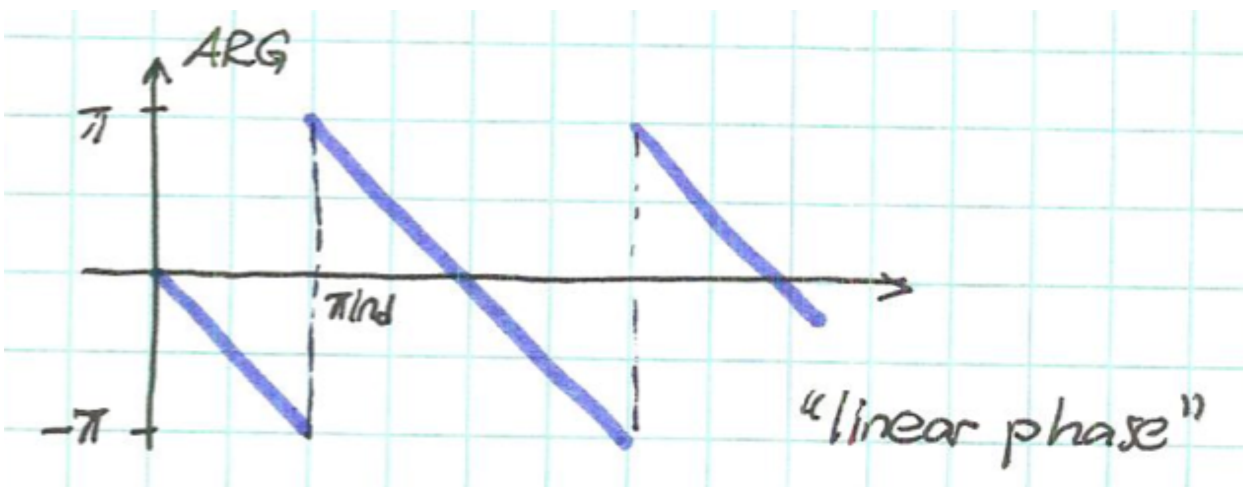
Phase response

Example: $H(e^{j\omega}) = e^{-j\omega n_d} \quad \leftrightarrow \quad h[n] = \delta[n - n_d]$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

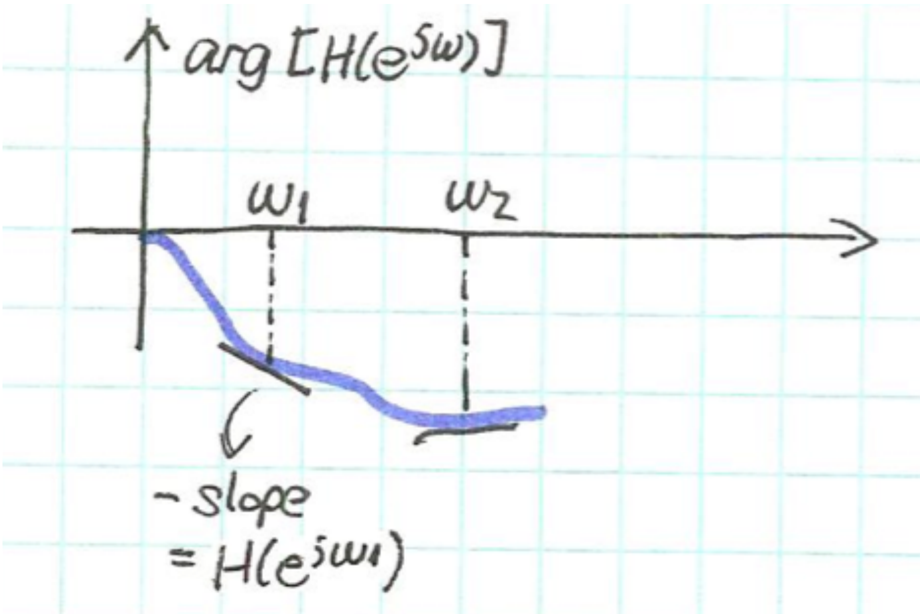
ARG is the wrapped phase
arg is the unwrapped phase



Group delay

To characterize general phase response, look at the group delay:

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

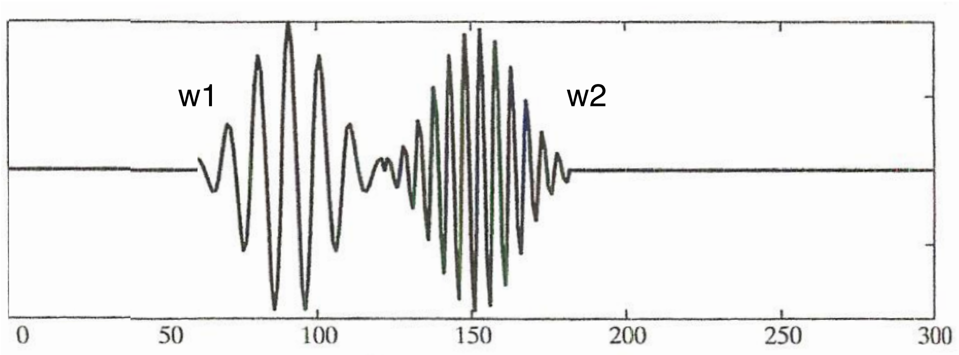


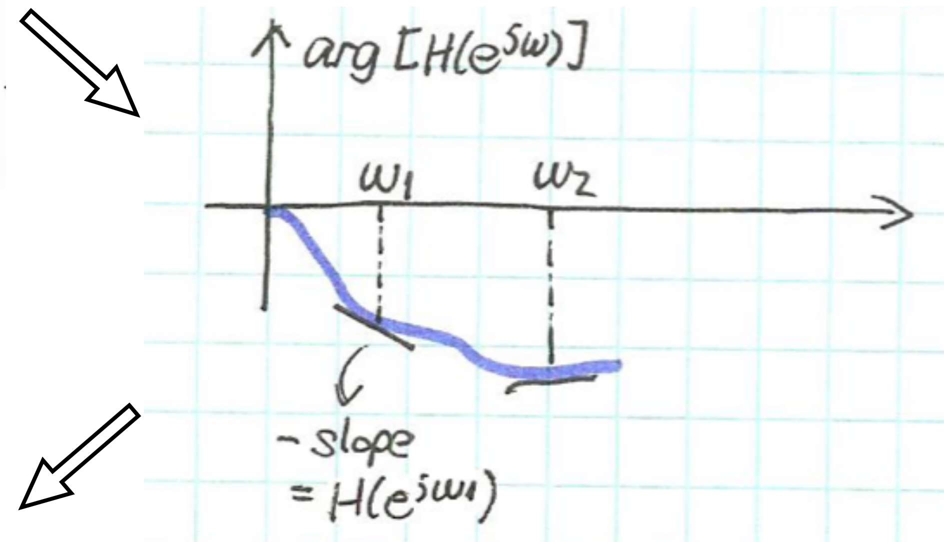
For linear phase system, the group delay is n_d

Group delay

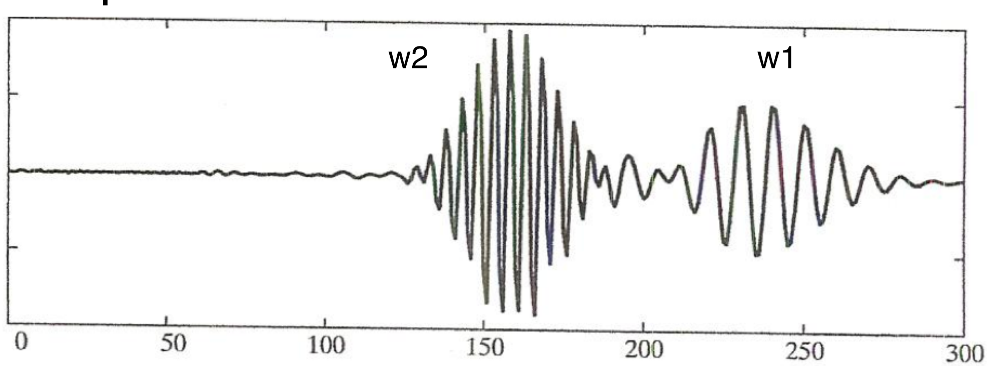
$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Input





Output



For narrowband signals, phase response looks like a linear phase

Group delay math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

arg of products is sum of args

$$\arg[H(e^{j\omega})] = -\sum_{k=1}^{N} \arg[1 - d_k e^{-j\omega}] + \sum_{k=1}^{M} \arg[1 - c_k e^{-j\omega}]$$

$$\mathrm{grd}[H(e^{j\omega})] = -\sum_{k=1}^{N} \mathrm{grd}[1 - d_k e^{-j\omega}] + \sum_{k=1}^{M} \mathrm{grd}[1 - c_k e^{-j\omega}]$$

Group delay math

$$\mathrm{grd}[H(e^{j\omega})] = -\sum_{k=1}^{N} \mathrm{grd}[1-d_k e^{-j\omega}] + \sum_{k=1}^{M} \mathrm{grd}[1-c_k e^{-j\omega}]$$

Look at each factor:

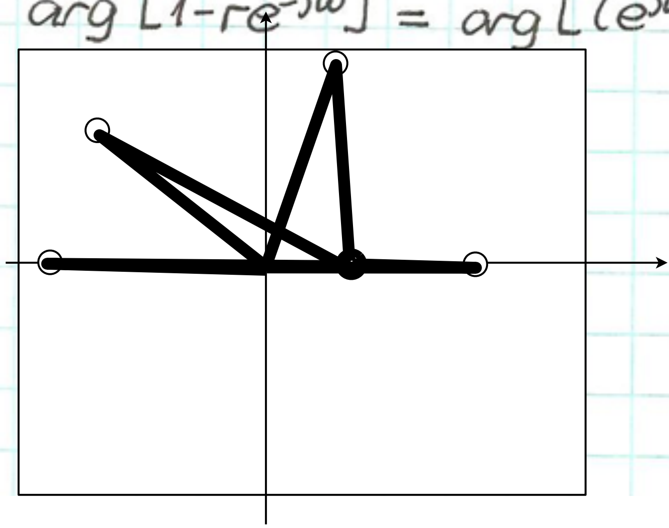
$$\arg[1-\underbrace{r e^{j\theta}}_{c_k \text{ or } d_k} e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

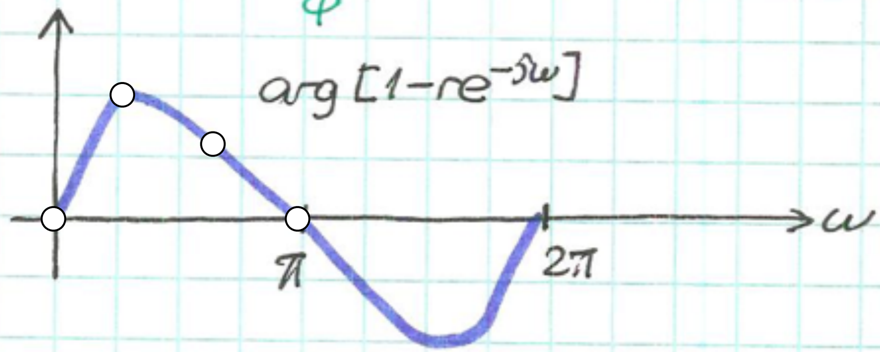
$$\mathrm{grd}[1-re^{j\theta}e^{-j\omega}] = \frac{r^2-r\cos(\omega-\theta)}{|1-re^{j\theta}e^{-j\omega}|^2}$$

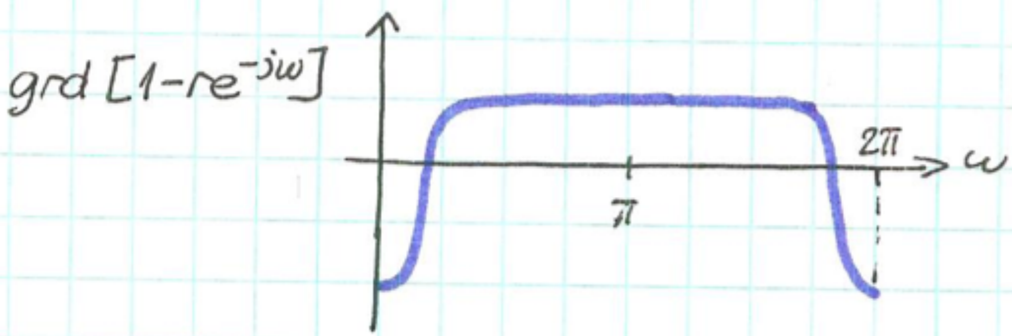
Look at a zero lying on the real axis

Geometric Interpretation (for $\theta = 0$)

$$\arg[1-re^{-j\omega}] = \arg[(e^{j\omega}-r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega}-r]}_{\phi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

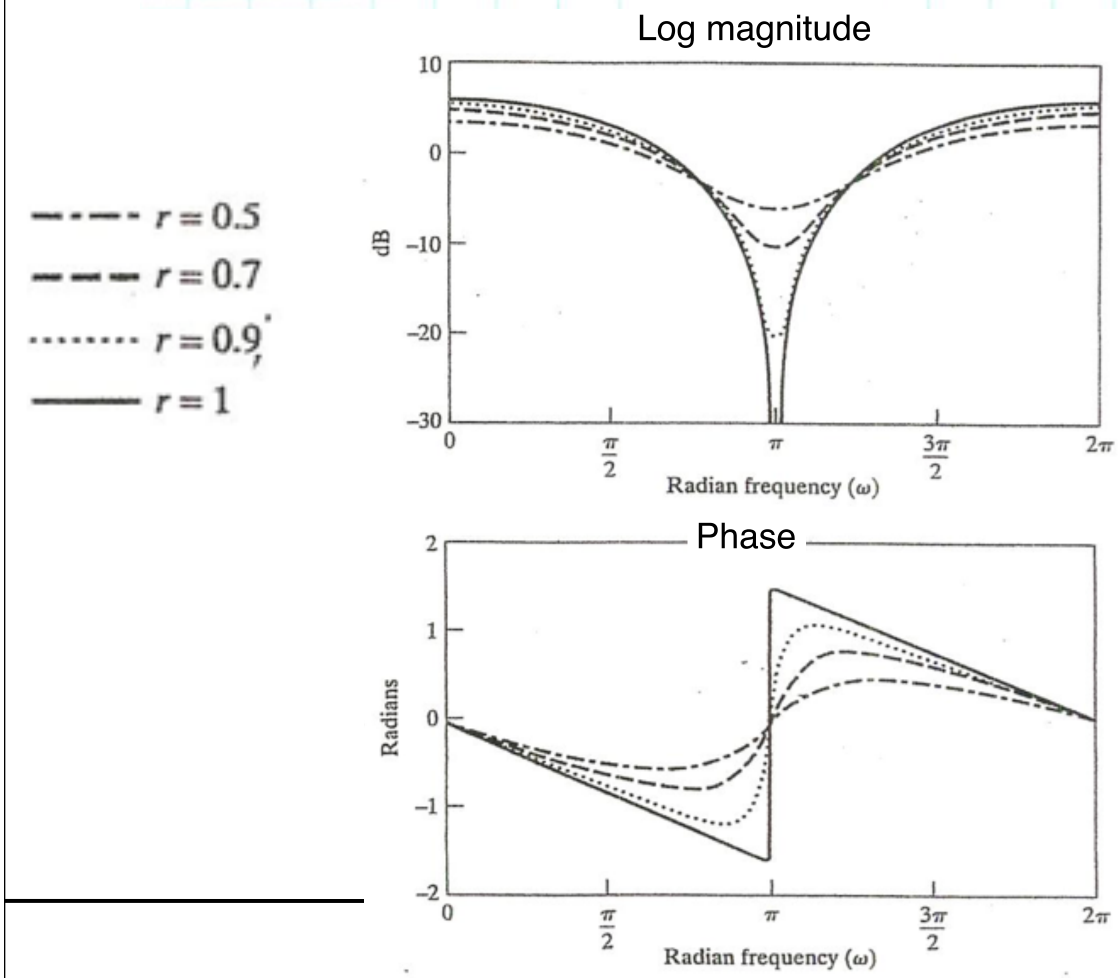


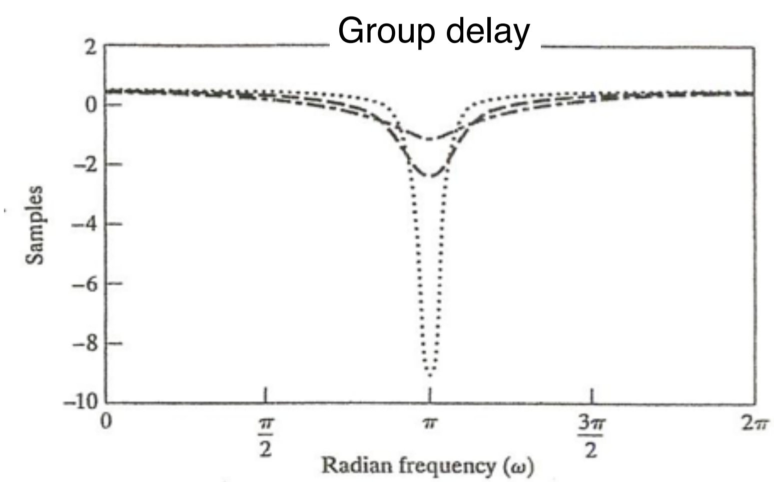




$\theta \neq 0 \Rightarrow$ shift to the right by θ

* Poles increase magnitude, but introduce phase lag and group delay.
* Zeros do the opposite.
* These effects are more marked when $r \to 1$.

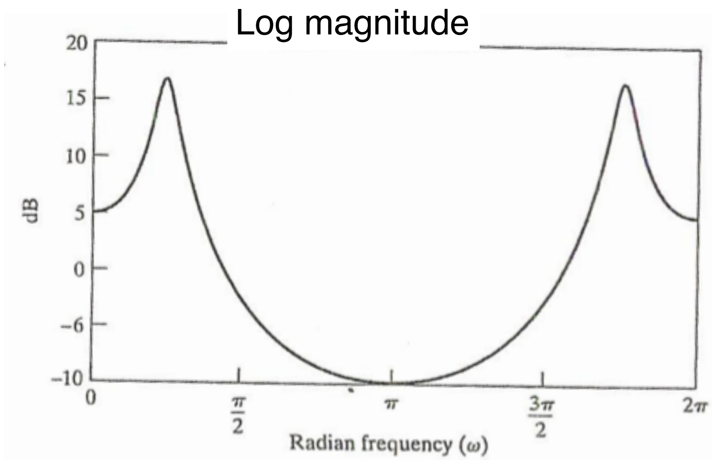


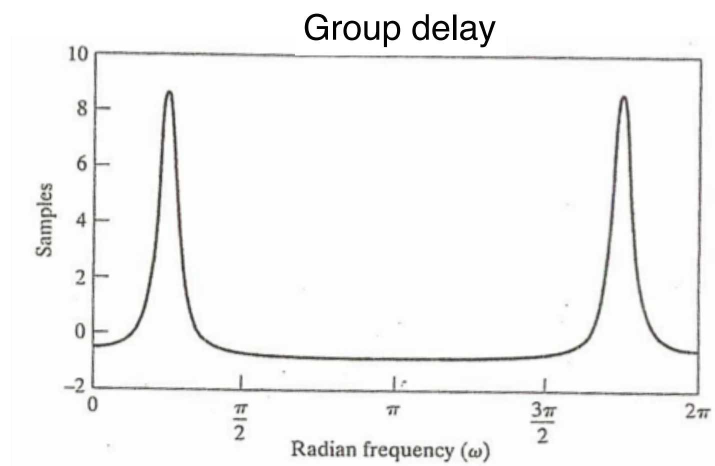


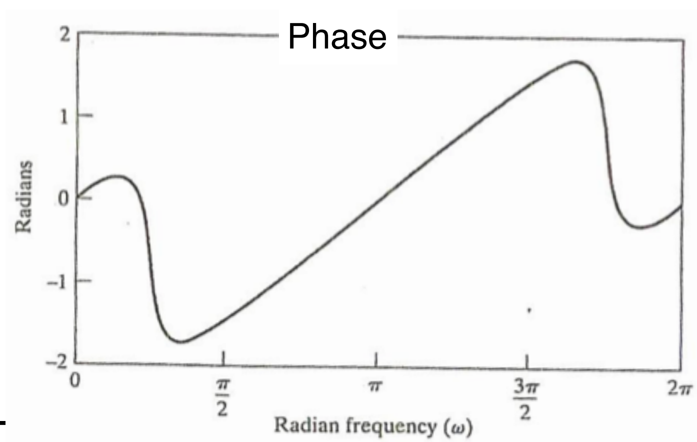
2nd order IIR example

Example: 2nd order IIR with complex poles

$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

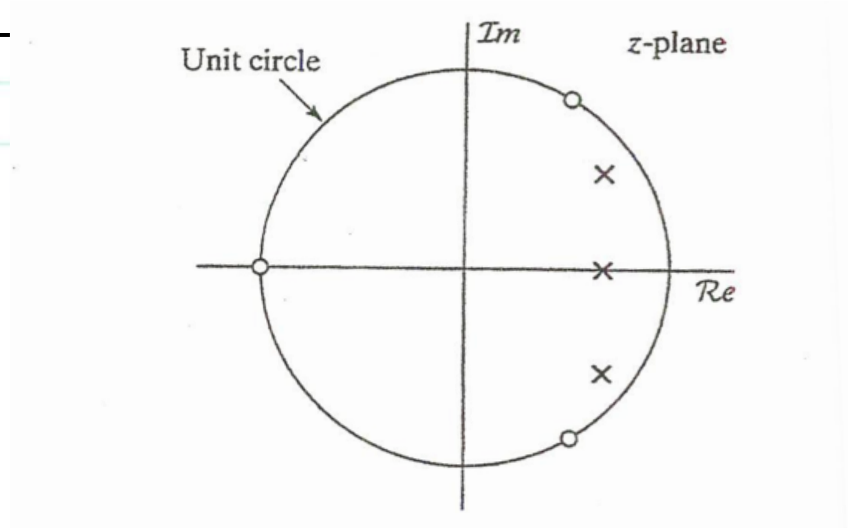




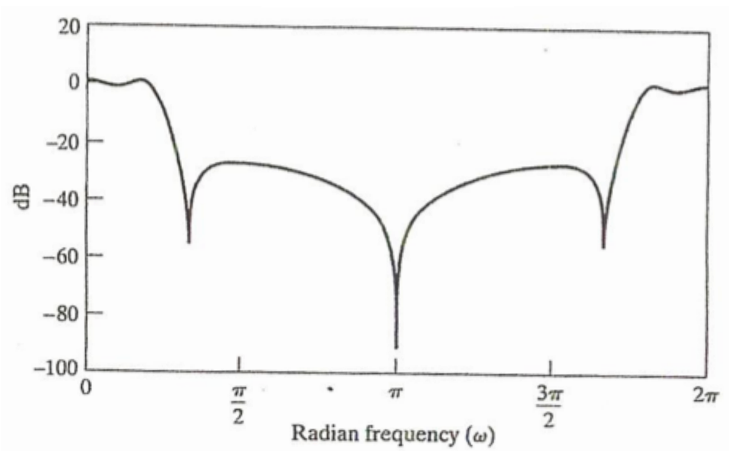


3nd order IIR example

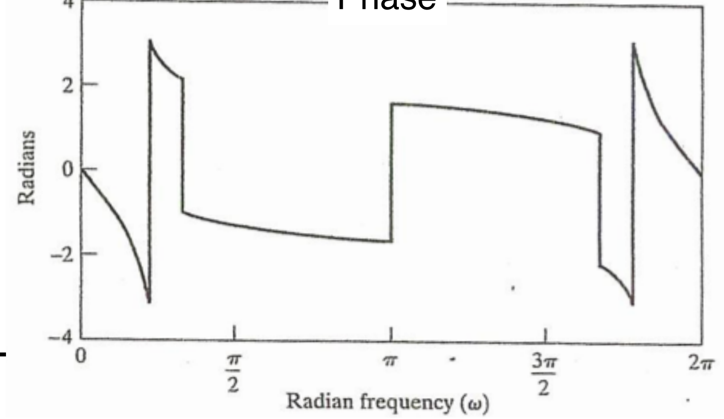
Example: 3rd order IIR



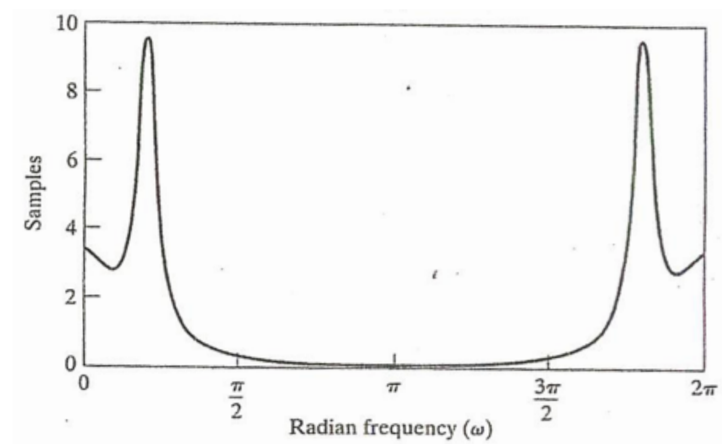
Log magnitude



Phase



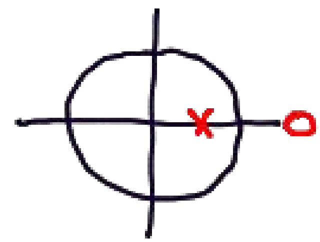
Group delay



All-Pass Systems

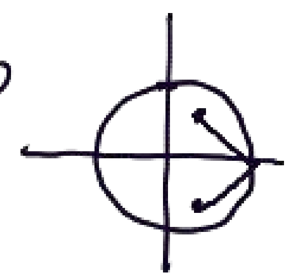
what is the magnitude response of

$$H(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$$



$$|H(e^{j\omega})| = \frac{|e^{-j\omega} - a^*|}{|1 - a e^{-j\omega}|} = \frac{|e^{-j\omega}(1 - a^* e^{j\omega})|}{|1 - a e^{-j\omega}|} =$$

$$= \frac{|1 - a^* e^{j\omega}|}{|1 - (a^* e^{j\omega})^*|} = 1 \quad \forall \omega$$



A generall all-pass system:

Real complex

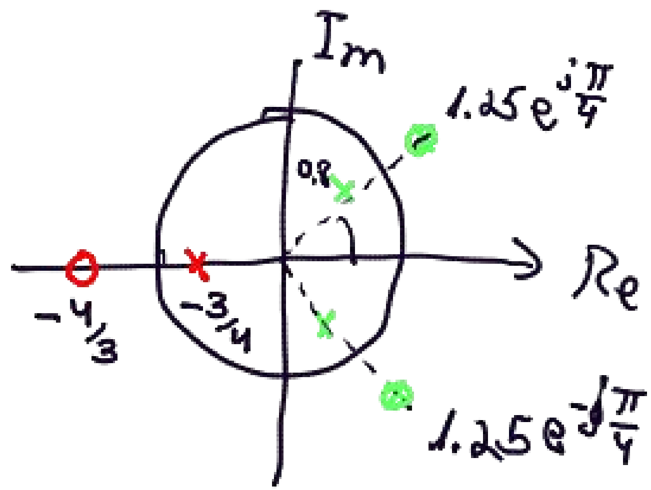
$$H_{ap}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \cdot \prod_{k=1}^{M_c} \frac{z^{-1} - e_k^*}{1 - e_k z^{-1}} \cdot \frac{z^{-1} - e_k}{1 - e_k^* z^{-1}}$$

d_k: real Poles

e_k: complex poles paired w/ conjugate e_k^*

$$|H_{ap}(e^{j\omega})| \equiv 1$$

Example



phase response of an all-pass:

(4)

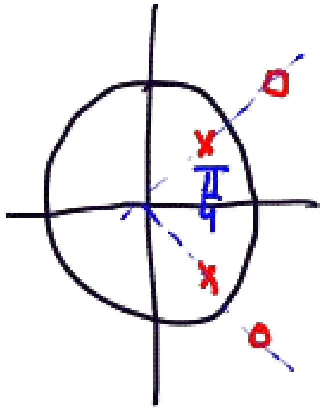
$$\arg\left[\frac{e^{-j\omega} - \overbrace{re^{-j\theta}}^{a^*}}{1 - \underbrace{re^{j\theta}}_{a} e^{-j\omega}}\right] = \arg\left[\frac{e^{-j\omega}\left(1 - re^{-j\theta}e^{-j\omega}\right)}{1 - re^{j\theta}e^{-j\omega}}\right] =$$

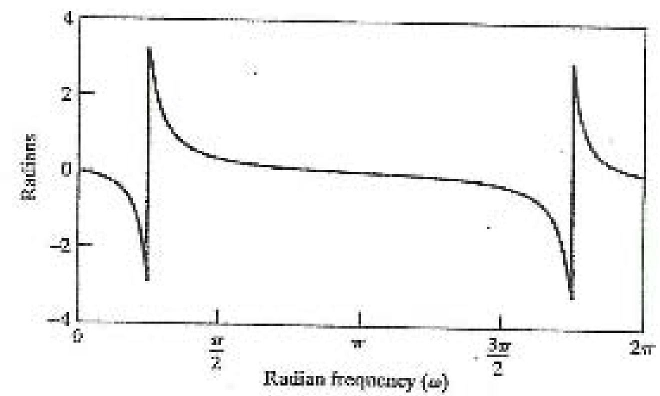
$$= \underbrace{\arg\left[e^{-j\omega}\right]}_{-\omega} - 2\arg\left[1 - re^{j\theta}e^{-j\omega}\right]$$

$$\mathrm{grd}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = 1 - 2\,\mathrm{grd}\left[1 - re^{j\theta}e^{-j\omega}\right]$$

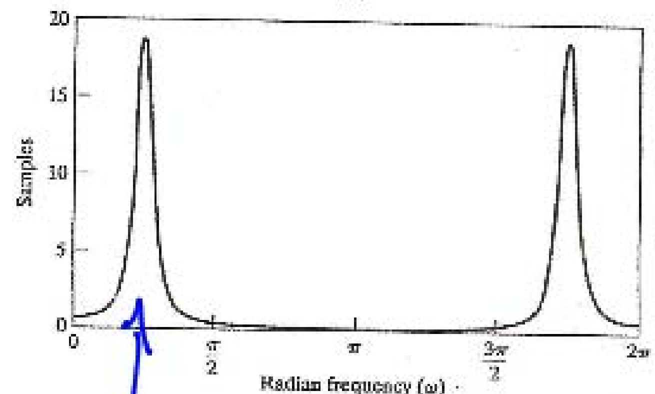
< figure 5.20 >

Example:





(b)



(c)

can be used to compensate phase distortion.

Claim: for a stable ap system $H_{ap}(z)$: 6

(i) $\text{grd}\left[H_{ap}(e^{j\omega})\right] > 0$

(ii) $\arg\left[H_{ap}(e^{j\omega})\right] \leq 0$

Delay positive → causal
phase negative → phase lag.

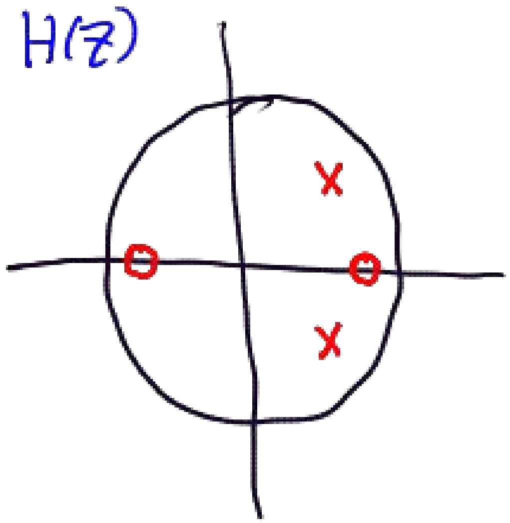
proof in book.

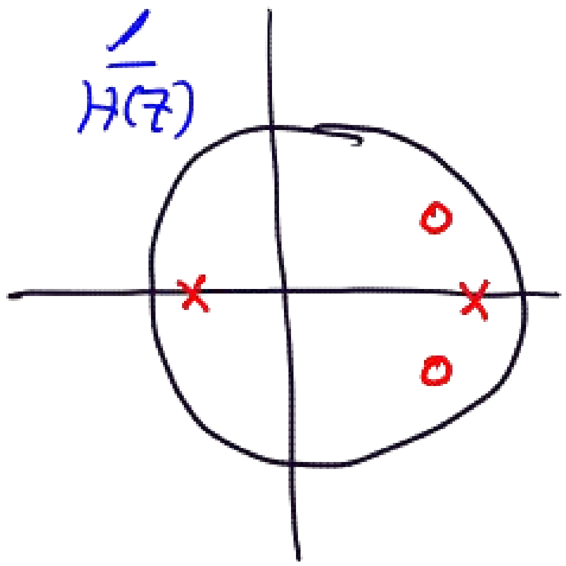
Minimum-Phase Systems

Definition: a $\underbrace{\text{stable and causal}}_{\text{poles inside unit circle}}$ system $H(z)$

who'se inverse $\frac{1}{H(z)}$ is also stable & causal

zeros are inside unit circle.





AP-Min-Phase decomposition:

stable, causal system can be decomposed to:

$$H(z) = \underbrace{H_{min}(z)}_{\text{min phase}} \cdot \underbrace{H_{ap}(z)}_{\text{all pass}}$$

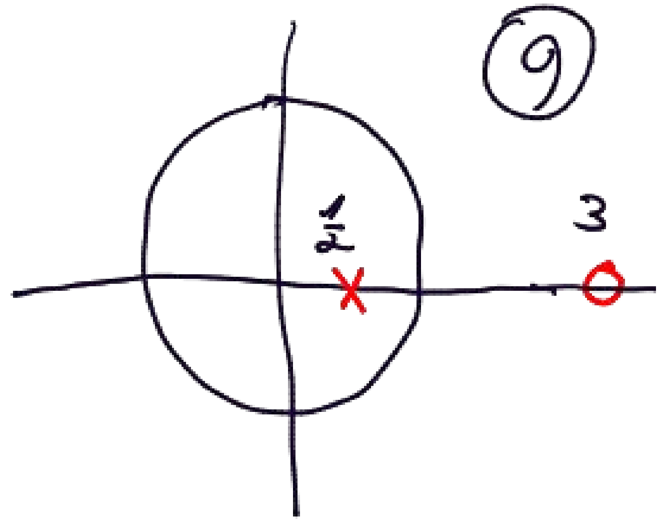
Approach: ① first construct H_{ap} with all zeros outside unit circle

② compute

$$H_{min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$



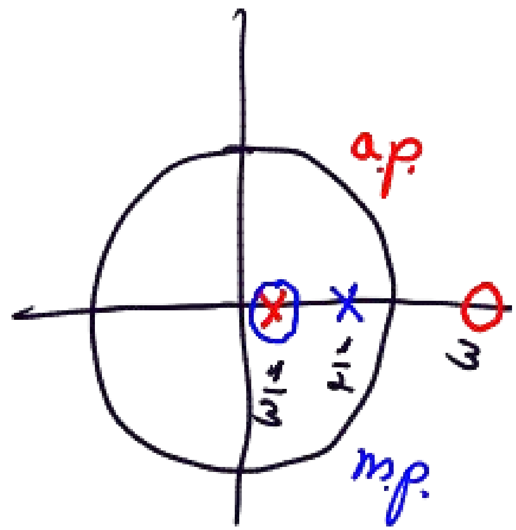
Set:

$$H_{ap} = \frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$$

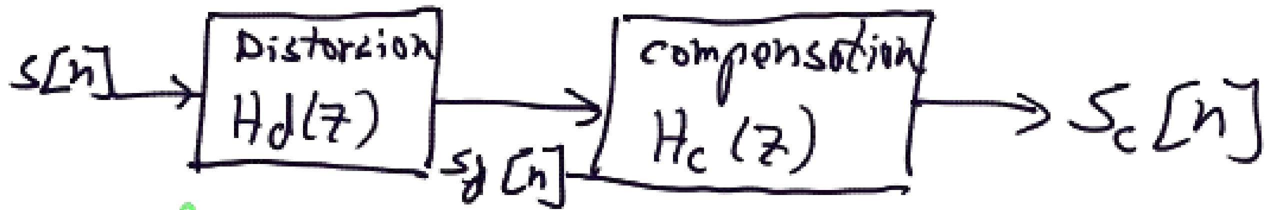
$-3 =$

$$H_{min}(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}} \cdot \frac{1-\frac{1}{3}z^{-1}}{z^{-1}-\frac{1}{3}} =$$

$$= -3\,\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$



why m.p. property important?



for ex. communication chan.

If $H_d(z)$ is minimum phase, design

$$H_c(z) = \frac{1}{H_d(z)} \quad \text{(stable!)}$$

If not m.p., decompose: $H_d(z) = H_{d,mp}(z) \cdot H_{d,ap}(z)$

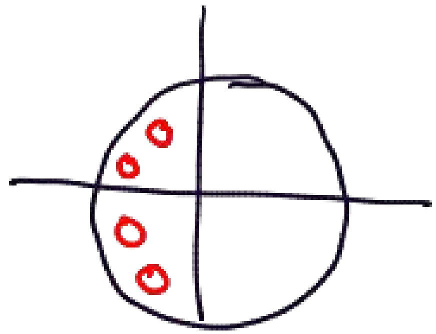
$$H_c(z) = \frac{1}{H_{d,min}(z)} \Rightarrow H_d H_c = H_{d,ap}(z)$$

only compensate for mag.

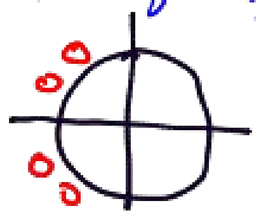
Why called "minimum phase"?

Different systems can have same mag. response.

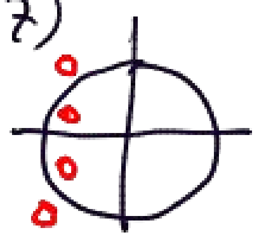
$H_1(z)$ min phase:



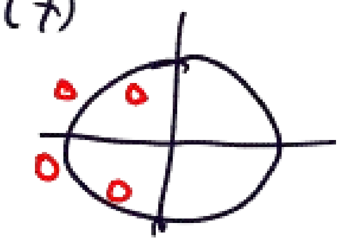
$H_2(z)$ (max phase)



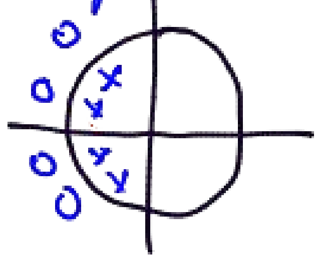
$H_3(z)$



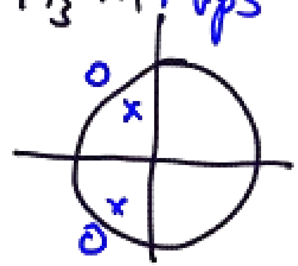
$H_4(z)$



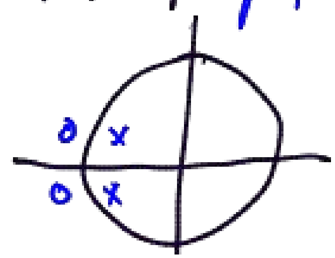
$H_2 = H_1 H_{ap,2}$



$H_3 = H_1 H_{ap3}$



$H_4 = H_1 \cdot H_{ap4}$



of all, $H_1(z)$ has minimum phase lag (12)

because:

$$\arg[H_i(e^{j\omega})] = \arg[H_1(e^{j\omega})] + \arg[H_{ap_i}]$$

$$\geq 0$$

other properties:

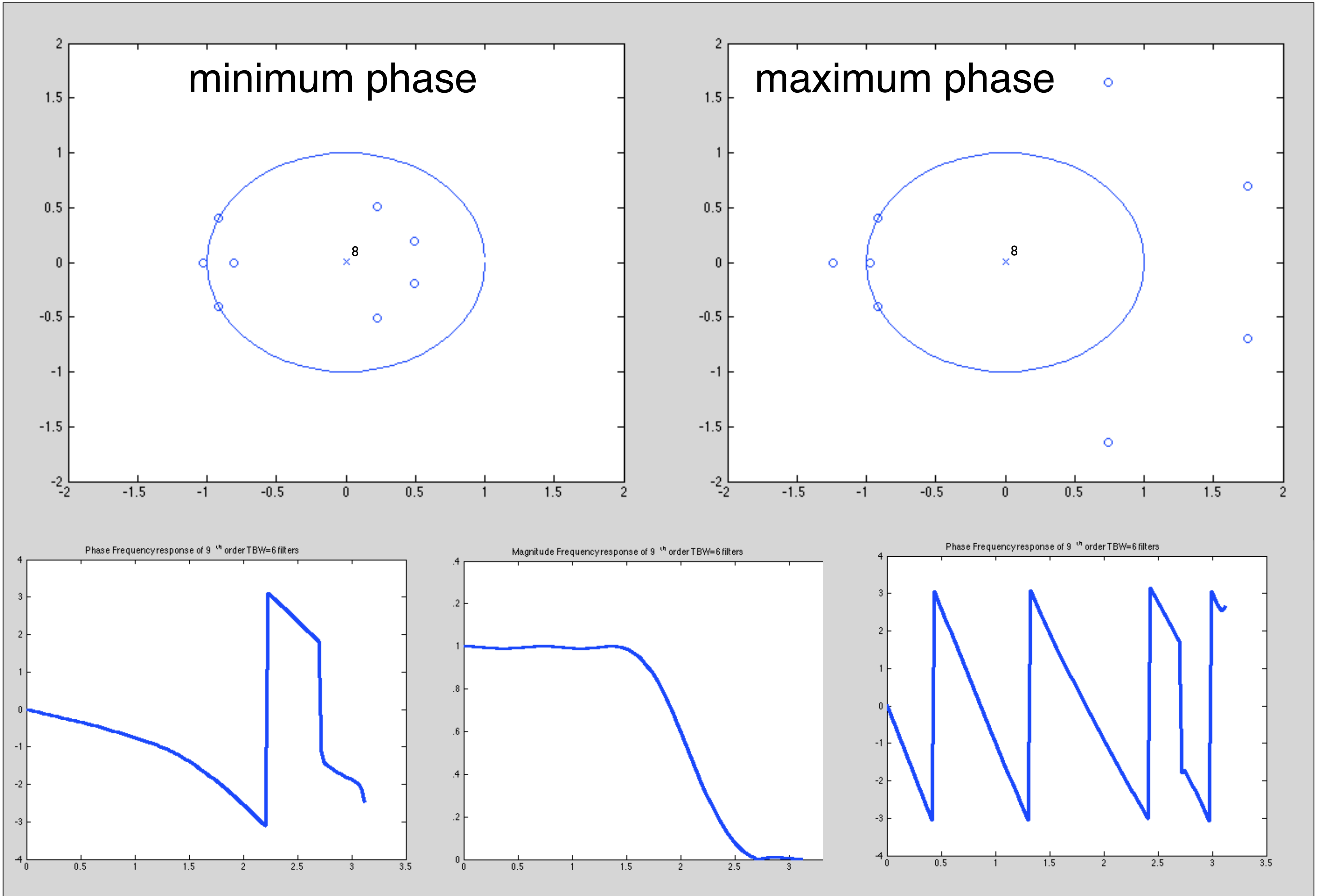
<u>minimum group delay:</u>

$$\mathrm{grd}[H(e^{j\omega})] = \mathrm{grd}[H_{min}] + \mathrm{grd}[H_{ap}]$$

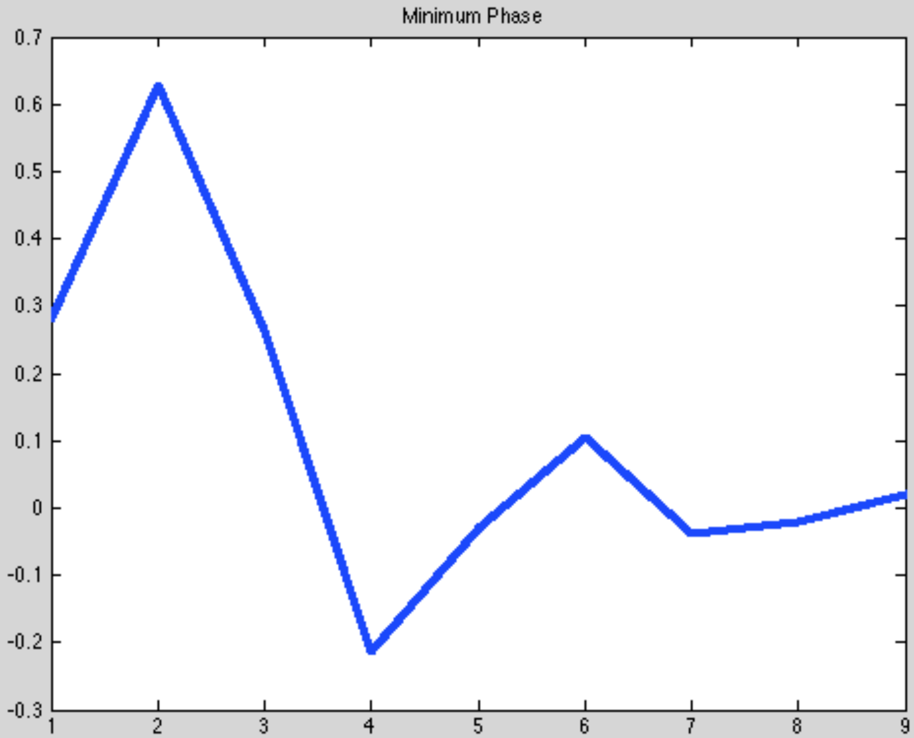
$$\geq 0$$

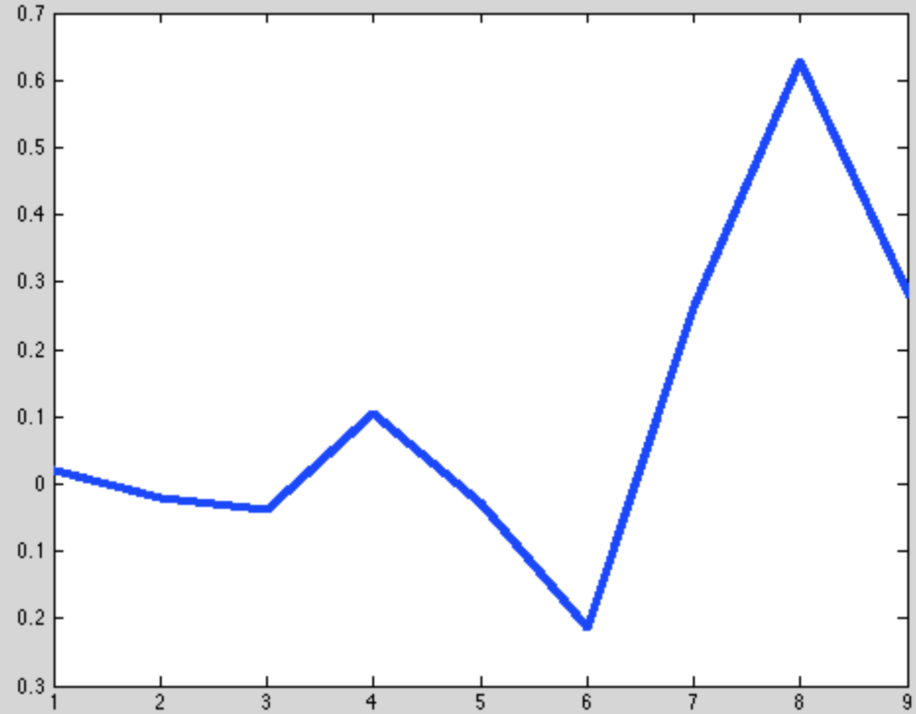
<u>minimum energy delay:</u>

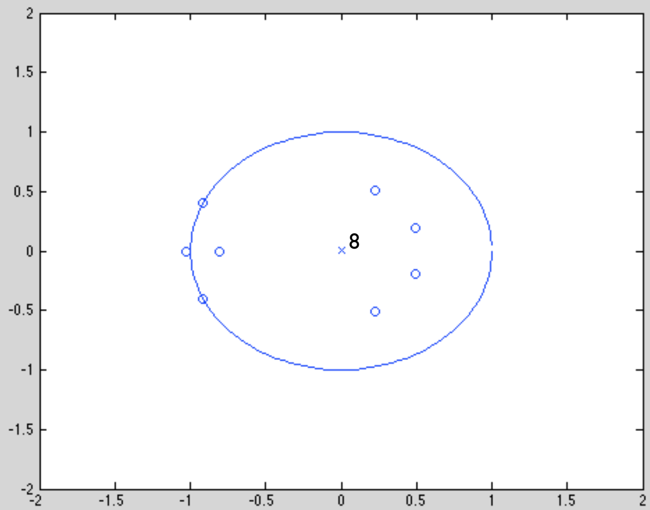
Problem 5.72

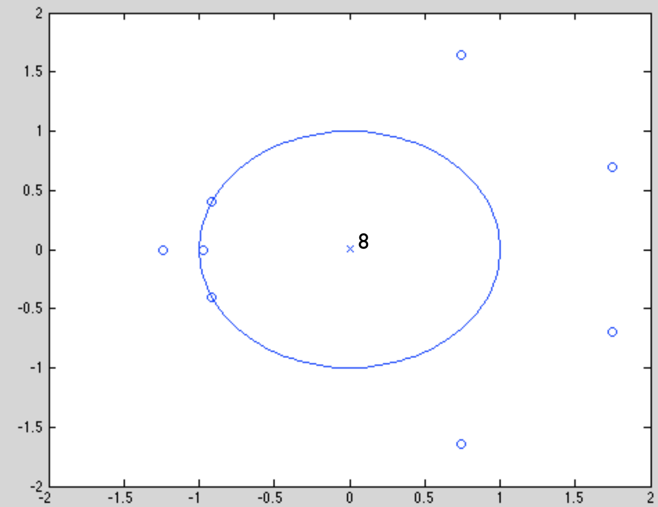

minimum phase
maximum phase
8
8
Phase Frequency response of 9 th order TBW=6 filters
Magnitude Frequency response of 9 th order TBW=6 filters
Phase Frequency response of 9 th order TBW=6 filters

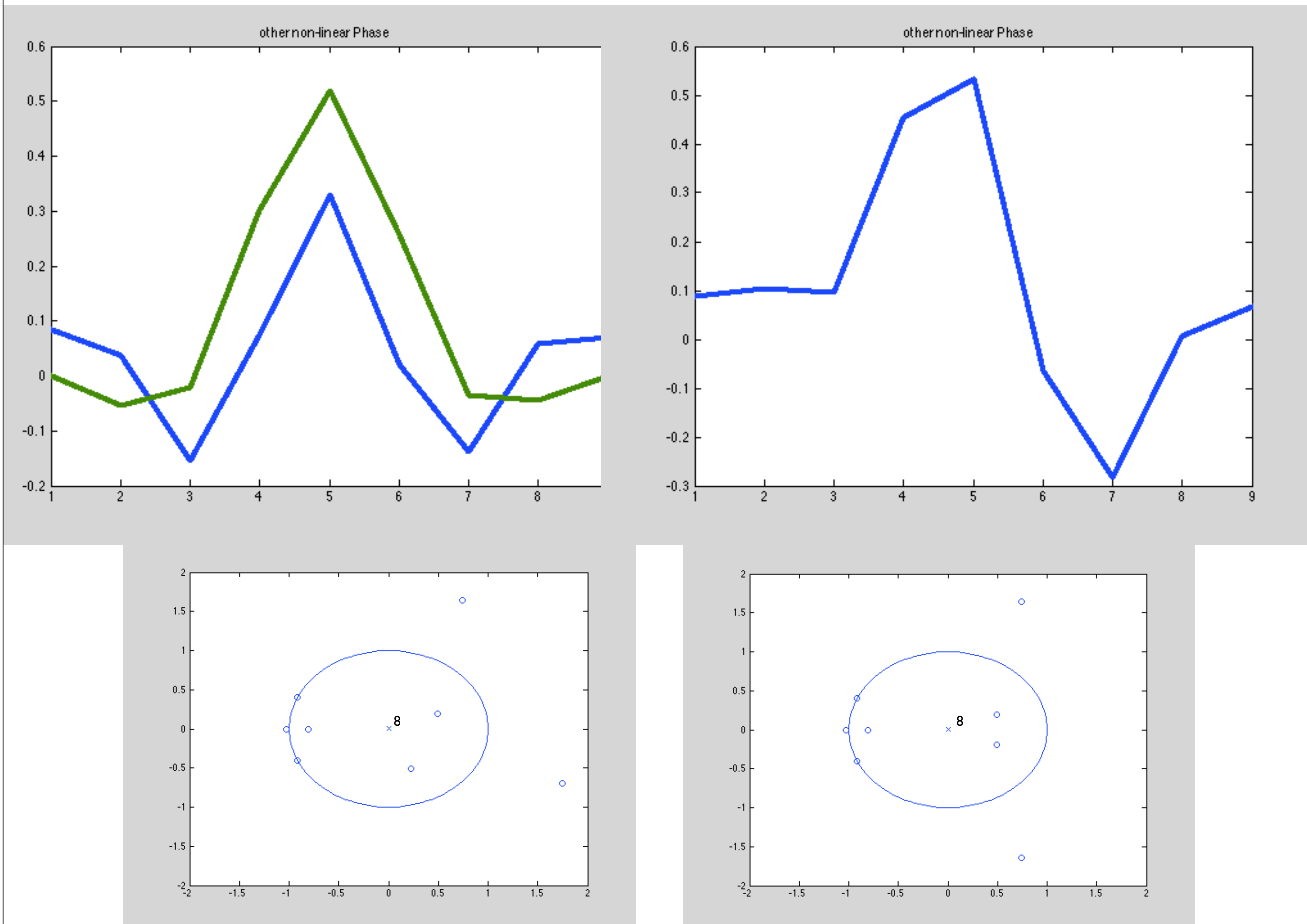
minimum phase	maximum phase







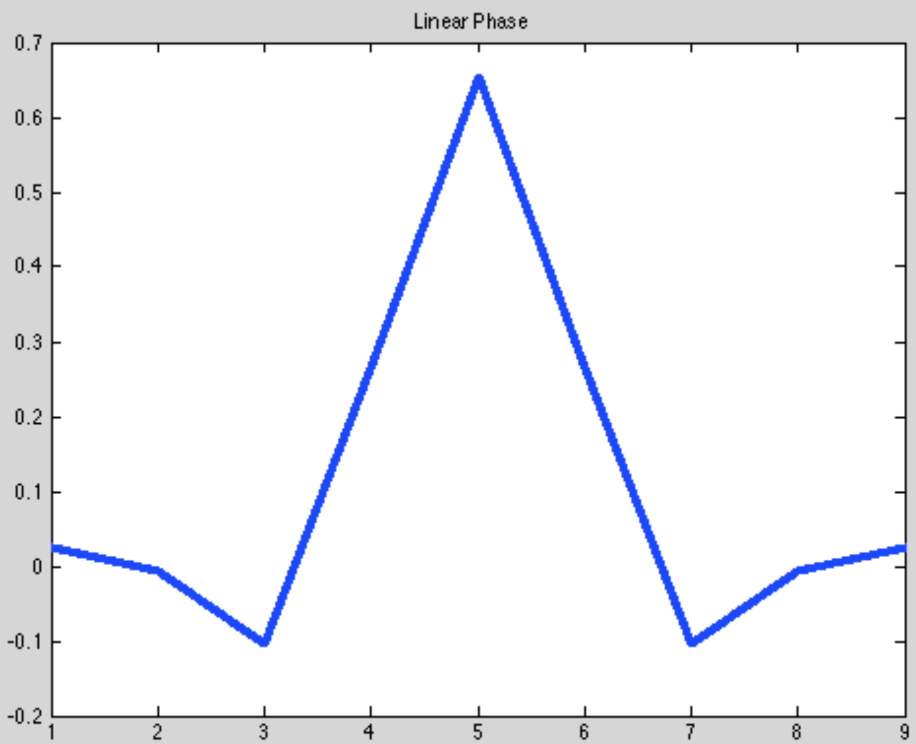


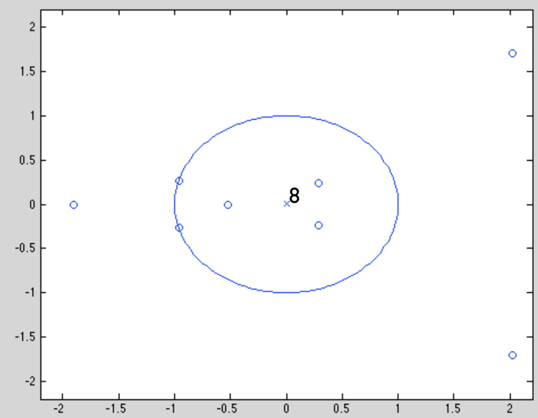


othernon-linear Phase
othernon-linear Phase
8
8

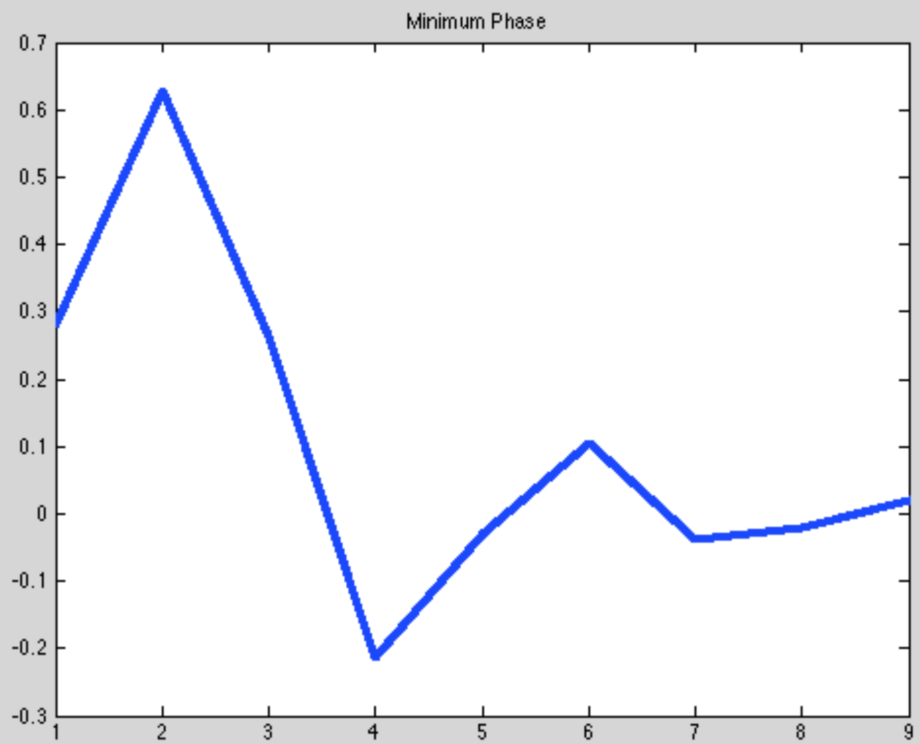
Minimum-phase VS Linear Phase

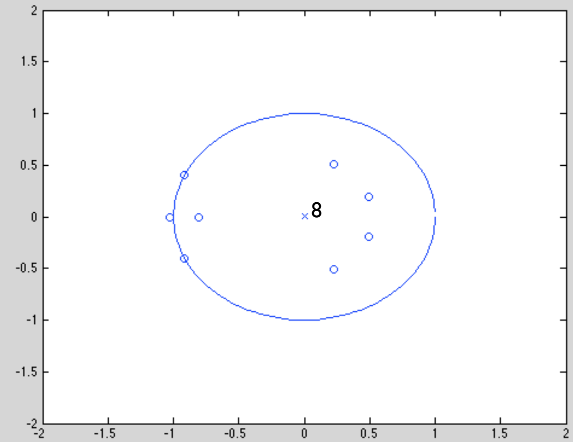
linear phase

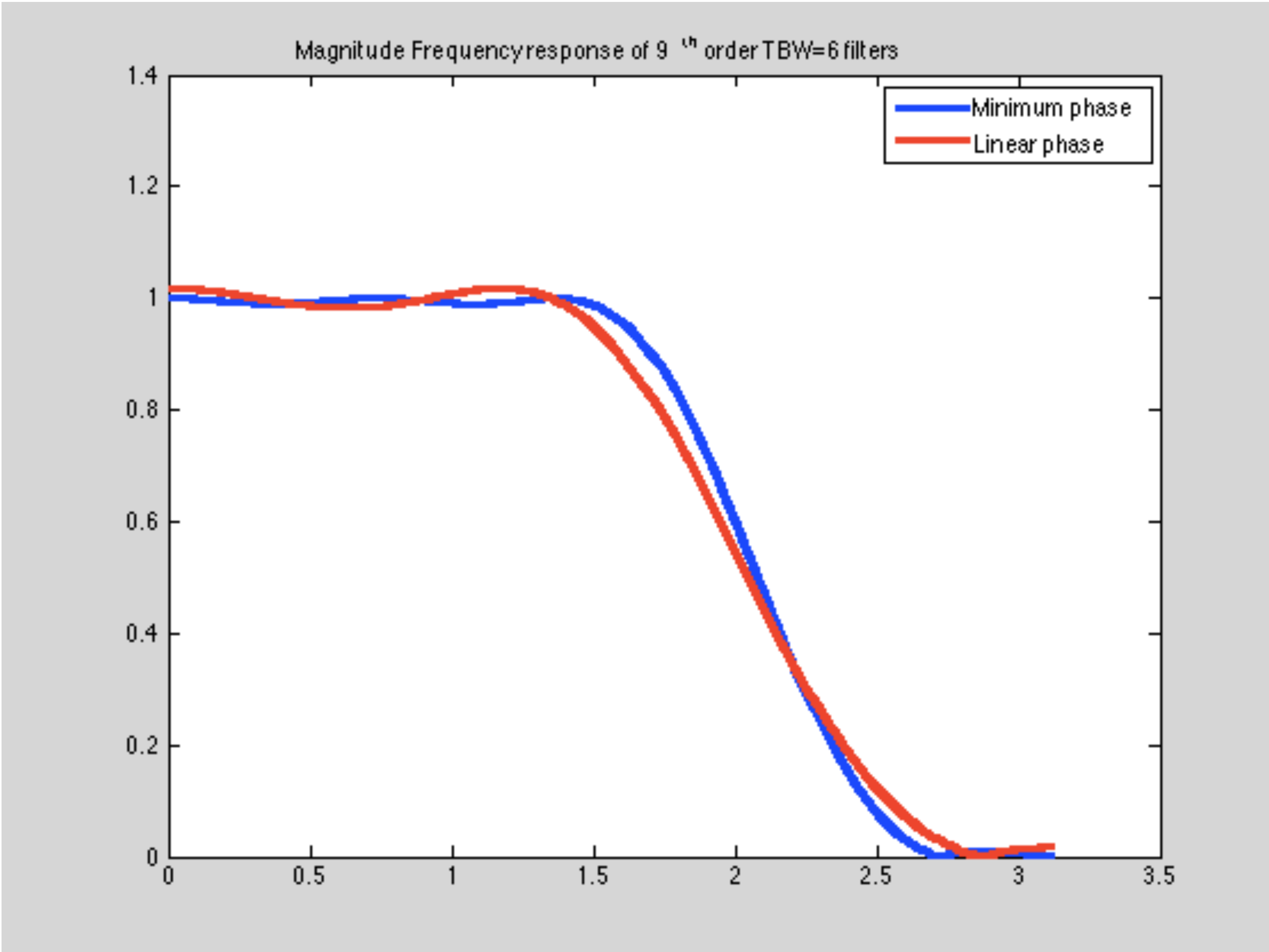




minimum phase






Magnitude Frequency response of 9 th order TBW=6 filters
Minimum phase
Linear phase
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5