EE123
Digital Signal Processing

Discrete Time Fourier Transform

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A couple of things

- Read Ch 2 2.0-2.9
- It's OK to use 2nd edition
- Class webcast in bcourses.berkeley.edu or linked from our website
- My office hours: posted on-line
  - W 4-5pm (EE123 priority), 5pm-6pm (ham-shack)
    Th 2p-3p (EE225E Priority) Cory 506 / 504
- Reward: 2$ for every typo/errors in my slides/slide
- ham radio lectures. Wednesday 6:30-8:30pm Cory 521

Discrete Time Systems

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n]$$

- Causality
- Memoryless
- Linearity
- Time Invariance
- BIBO stability

Discrete-Time LTI Systems

- The impulse response h[n] completely characterizes an LTI system “DNA of LTI”

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

discrete convolution

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = h[n] * x[n]$$

$$y[n] = \sum_{\substack{N=-\infty \\ \mathrm{m}}}^{\infty} h[m]x[n-m]$$

Sum of weighted, delayed impulse responses!

BIBO Stability of LTI Systems

- An LTI system is BIBO stable iff h[n] is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

BIBO Stability of LTI Systems

- Proof: "if"

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot \underbrace{|x[n-k]|}_{\leq B_x}$$

$$\leq B_x \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

BIBO Stability of LTI Systems

- Proof: “only if”
  - suppose $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$
    show that there exists bounded x[n] that gives unbounded y[n]
  - Let:

$$x[n] = \frac{h[-n]}{|h[-n]|} = \text{Sign}\{h[-n]\}$$

$$y[n] = \sum h[k]x[n-k]$$

$$y[0] = \sum h[k]x[-k] = \sum h[k]h[k]/|h[k]| = \sum |h[k]| = \infty$$

Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Why one is sum and the other integral?

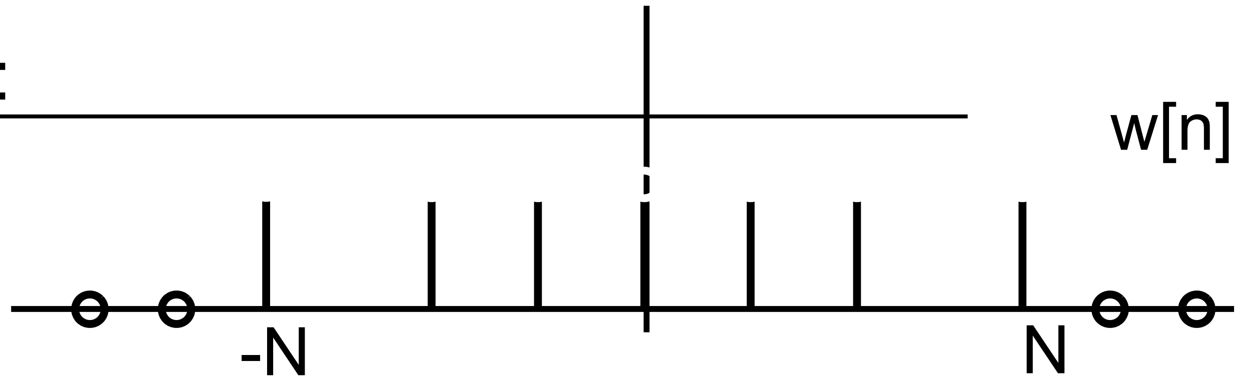
Why use one over the other?

Alternative

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi f k}$$

$$x[n] = \int_{-0.5}^{0.5} X(f)e^{j2\pi f n}df$$

Example 1:



w[n] “window”

DTFT:

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{-j\omega N}\left(1 + e^{j\omega} + \cdots + e^{j\omega 2N}\right)$$

Recall: $1 + p + p^2 + \cdots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

$$p = e^{j\omega}$$
$$M = 2N$$

Example 1 cont.

DTFT:

Example 1 cont.

DTFT:

$$W(e^{j\omega}) = e^{-j\omega N}\left(1+e^{j\omega}+\cdots+e^{j\omega 2N}\right)$$

$$\frac{\cdots -j\frac{\omega}{2}}{\cdots -j\frac{\omega}{2}}$$

nc

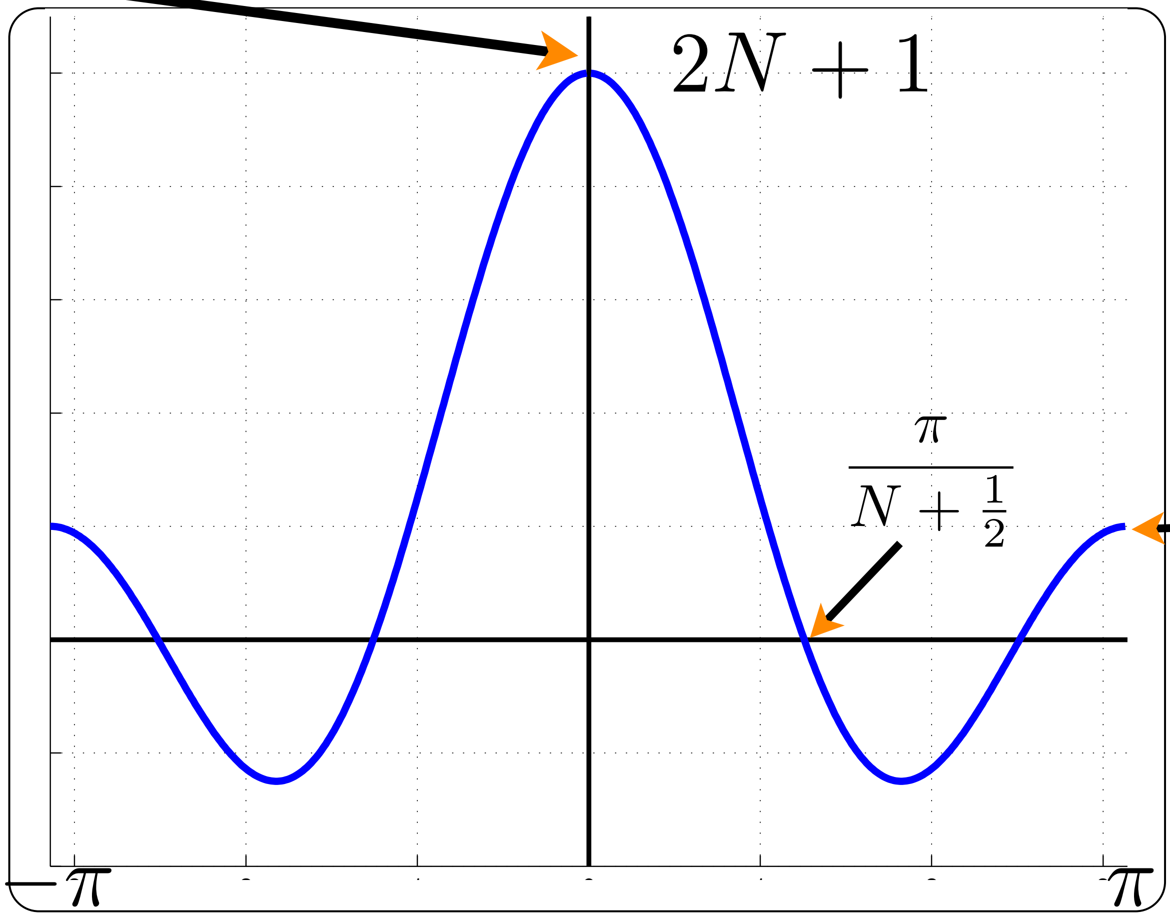
Example 1 cont.

$$W(e^{j\omega}) = \frac{\sin[(N+\frac{1}{2})\omega]}{\sin(\frac{\omega}{2})}$$

$$\rightarrow (2N+1) \quad \text{as} \quad \omega \rightarrow 0$$

from l'Hôpital

also, Σx[n]



=1, why?

Properties of the DTFT

Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$ if x[n] is real

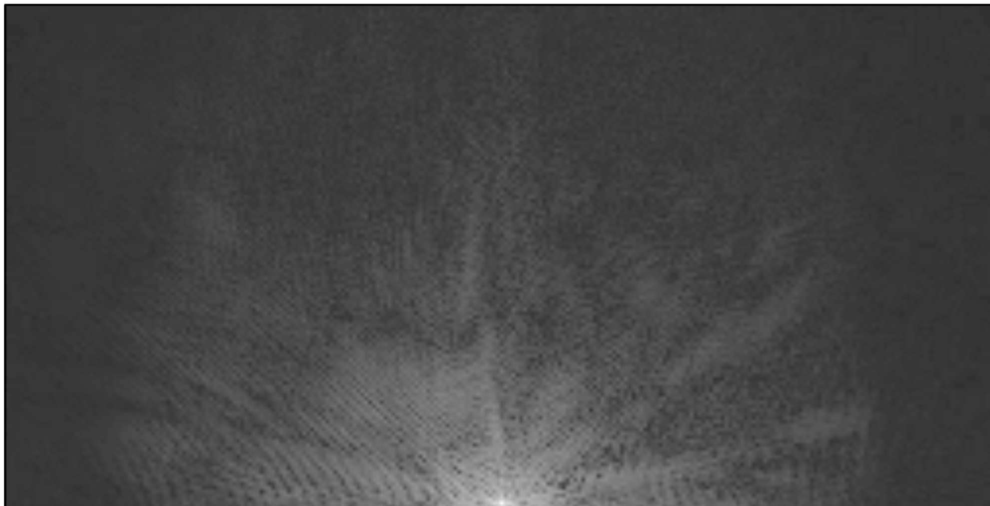
$$\mathcal{R}e\left\{X(e^{-j\omega})\right\} = \mathcal{R}e\left\{X(e^{j\omega})\right\}$$

$$\mathcal{I}m\left\{X(e^{-j\omega})\right\} = -\mathcal{I}m\left\{X(e^{j\omega})\right\}$$

Big deal for: MRI, Communications, more....

Half Fourier Imaging in MR

k-space (Raw Data)

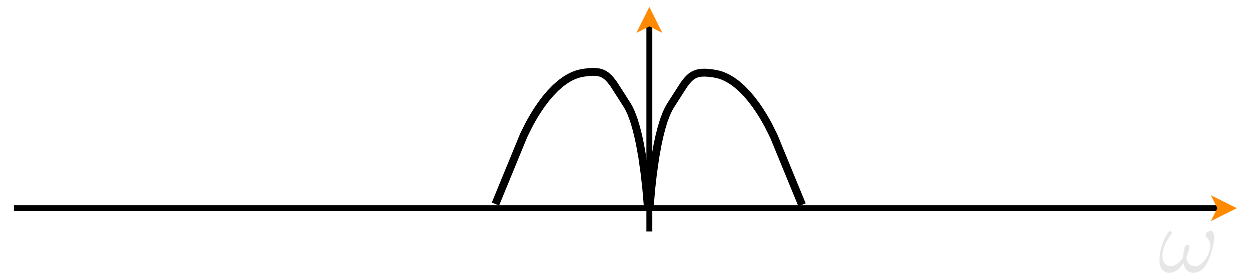


Image

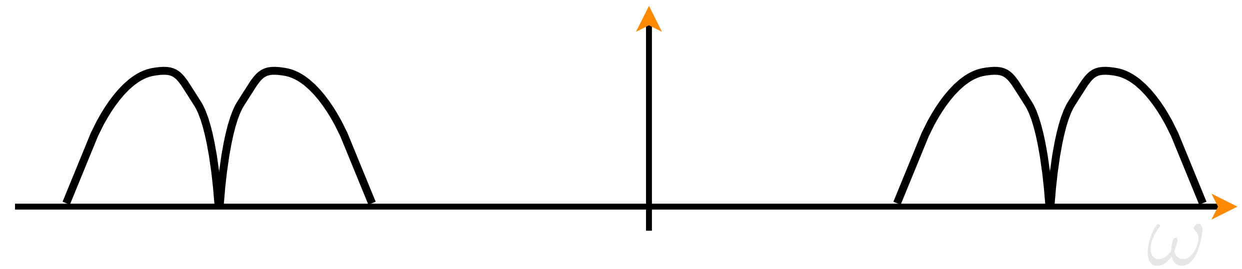
Complete based on conjugate symmetry
<u>Half the Scan time!</u>

SSB Modulation

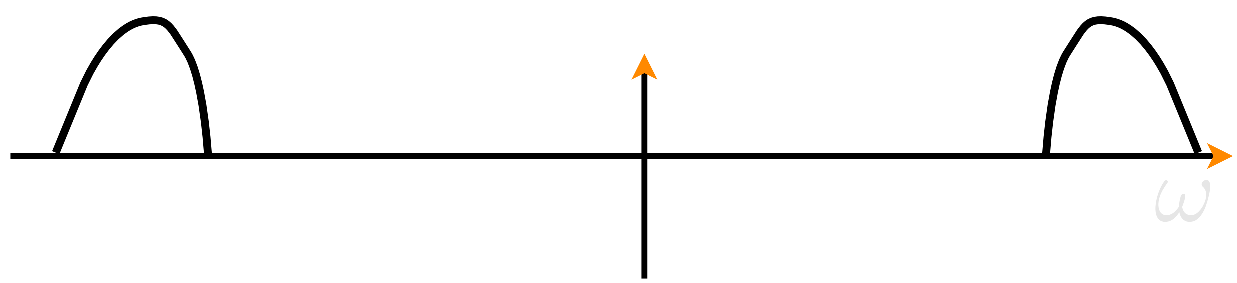
Real Baseband signal has conjugate symmetric spectrum



AM modulation (DSB-SC)



Single sideband (USB) half bandwidth



SSB

Amateur radio on shortwaves often use SSB modulation

Example: Websdr

http://websdr.org

http://100.1.108.103:8902

Properties of the DTFT cont.

Time-Reversal

$$x[n] \leftrightarrow X(e^{i\omega})$$

$$x[-n] \leftrightarrow X(e^{-i\omega})$$

$$= X^*(e^{j\omega}) \;\; \text{if } x[n] \in \mathcal{R}eal$$

If x[n] = x[-n] and x[n] is real, then:

$$X(e^{j\omega}) = X^*(e^{j\omega})$$

$$\rightarrow X(e^{j\omega}) \in \mathcal{R}eal$$

Q: Suppose:

$$x[n] \quad \leftrightarrow \quad X(e^{j\omega})$$

$$? \quad \leftrightarrow \quad \mathcal{R}e\left\{X(e^{j\omega})\right\}$$

A: Decompose x[n] to even and odd functions

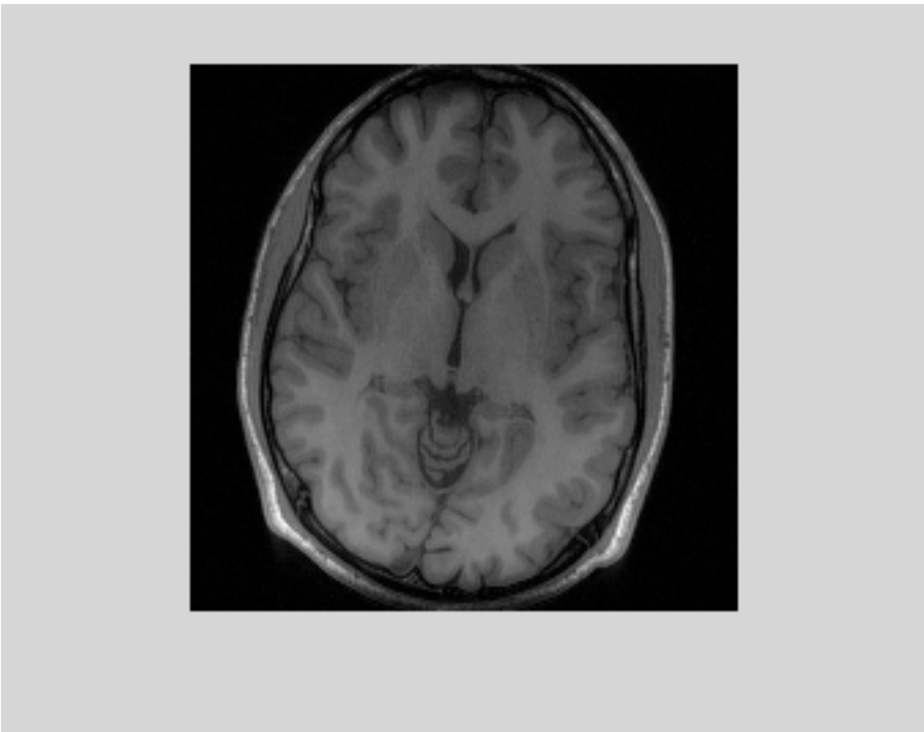
$$x[n] = x_e[n] + x_o[n]$$

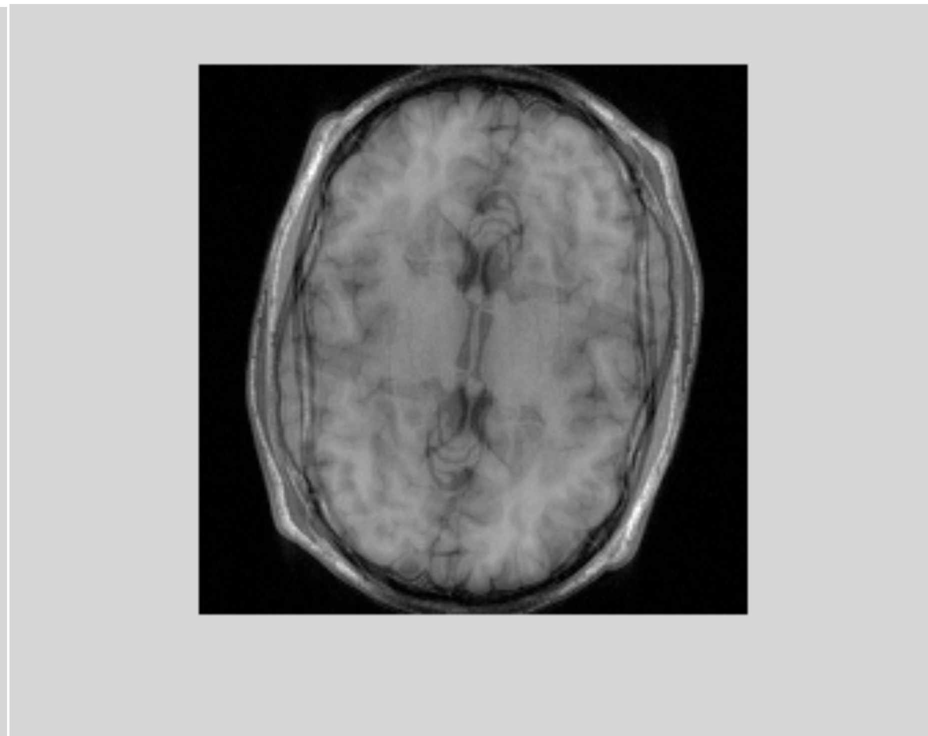
$$x_e[n] \quad := \quad \frac{1}{2}(x[n] + x[-n])$$

$$x_o[n] \quad := \quad \frac{1}{2}(x[n] - x[-n])$$

$$\boxed{x_e[n]} + x_o[n] \rightarrow \mathcal{R}e\left\{X(e^{j\omega})\right\} + j\mathcal{I}m\left\{X(e^{j\omega})\right\}$$

Oops!





Properties of the DTFT cont.

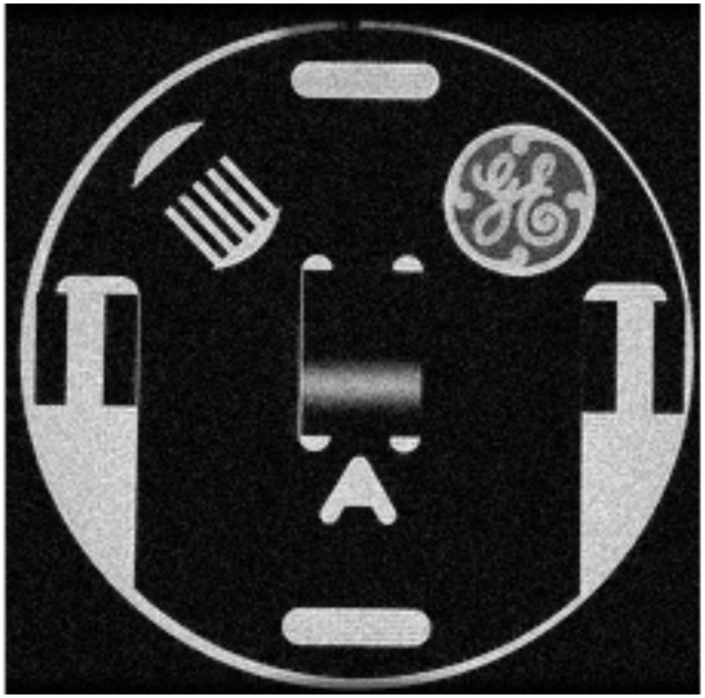
Time-Freq Shifting/modulation:

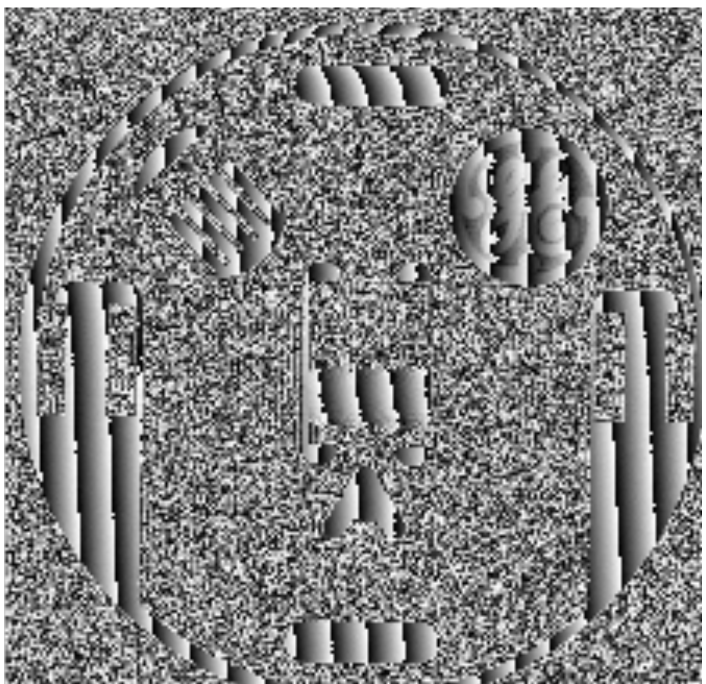
$$x[n] \quad \leftrightarrow \quad X(e^{j\omega})$$

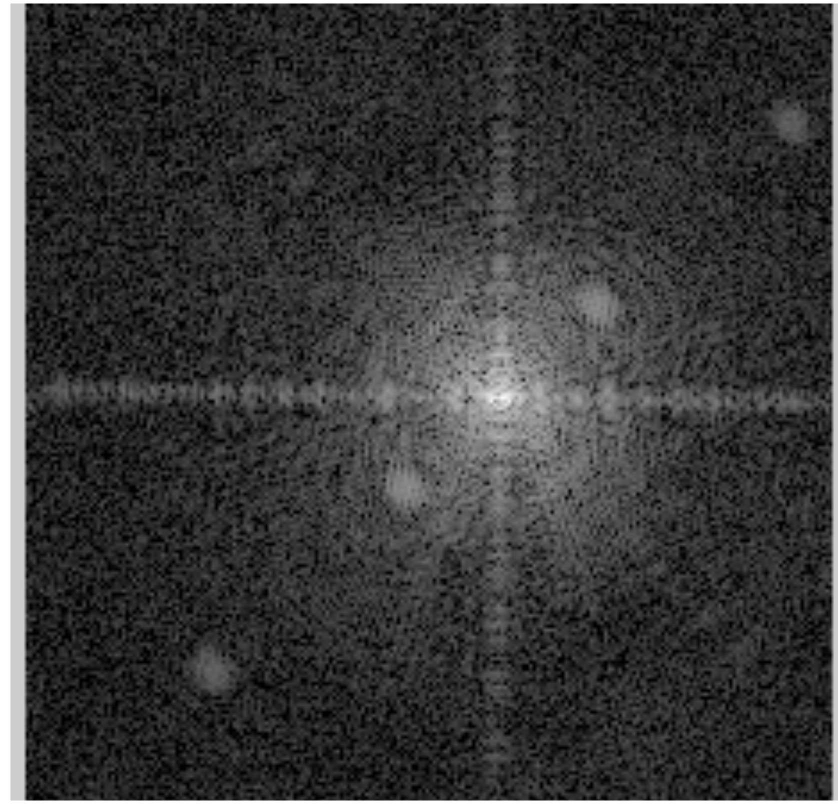
Good for MRI! Why

$$\boxed{x[n-n_d] \quad \leftrightarrow \quad e^{-j\omega n_d} X(e^{j\omega})}$$

$$e^{j\omega_0 n} x[n] \quad \leftrightarrow \quad X(e^{j(\omega-\omega_o)})$$

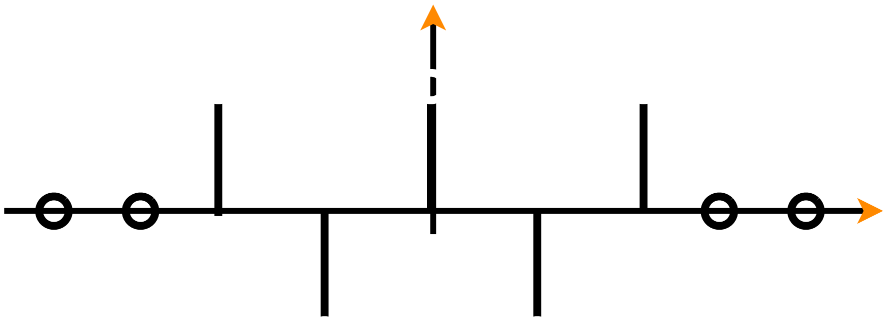




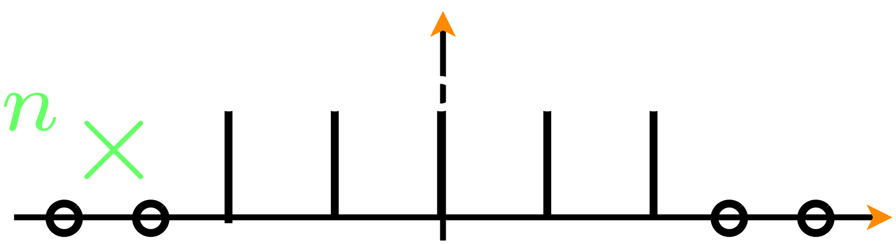


Example 2

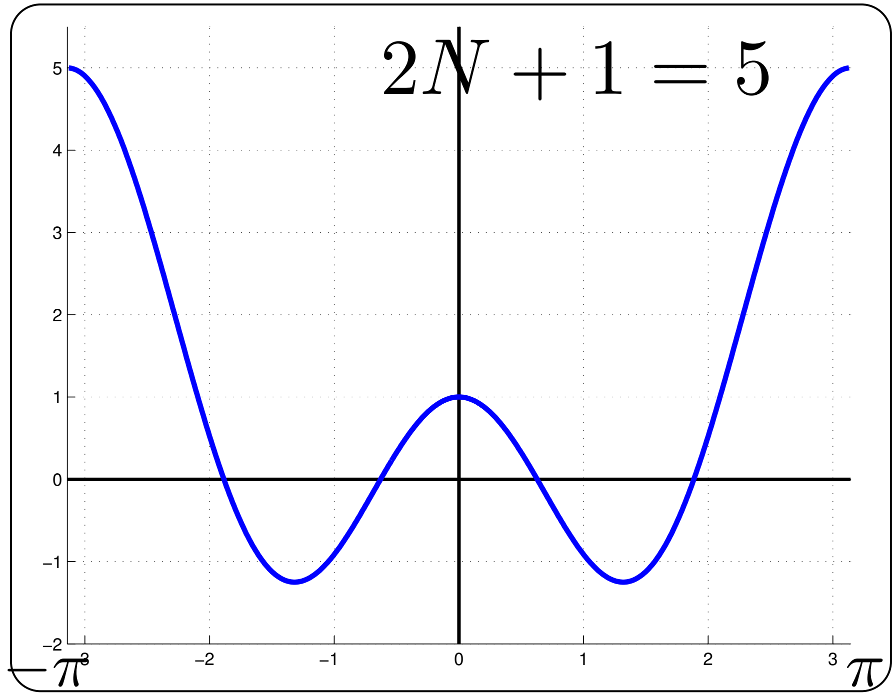
What is the DTFT of:



$e^{j\pi n} \times$



High Pass Filter



See 2.9 for more properties

Frequency Response of LTI Systems

Check response to a pure frequency:

$$e^{i\omega_0 n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \\ &= \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \right)}_{H\left(e^{j\omega}\right)\big|_{\omega=\omega_0}} e^{j\omega_0 n} \end{aligned}$$

Frequency Response of LTI Systems

Check response to a pure frequency:

$$e^{i\omega_0 n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\}$$

$$y[n] = H\left(e^{j\omega}\right)\Big|_{\omega=\omega_0} e^{j\omega_0 n}$$

Output is the same pure frequency, scaled and phase-shifted!

$e^{j\omega_0 n}$ is an eigen function of LTI systems

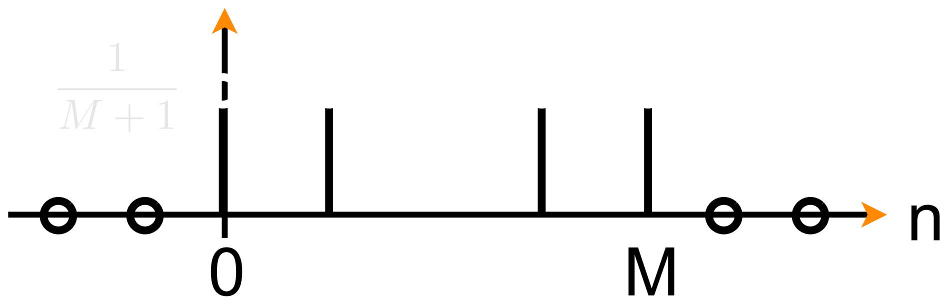
Recall eigen vectors satisfy: $A\nu = \lambda\nu$

Example 3

Frequency response of a causal moving average filter

$$y[n] = \frac{x[n-M] + \cdots + x[n]}{M+1}$$

Q: What type of filter is it?   A: Low-Pass



$$h[n] = \frac{1}{M+1} w[n - \frac{M}{2}]$$

Example 3 Cont.

Frequency response of a causal moving average filter

$$h[n] = \frac{1}{M+1} w[n - \frac{M}{2}]$$

Same as example 1, only: Shifted by N, divided by M+1, M=2N

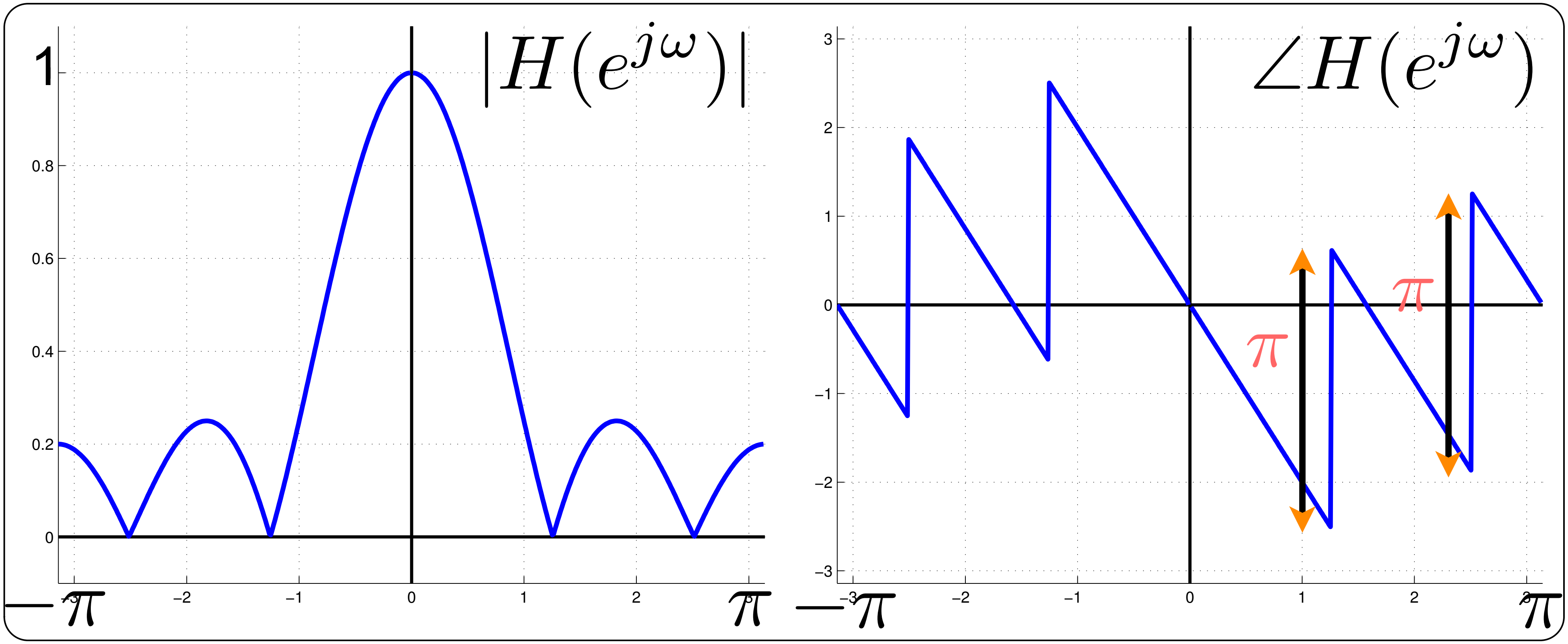
$$H(e^{j\omega}) = \frac{e^{-j\omega\frac{M}{2}}}{M+1} \cdot \frac{\sin\left((\frac{M}{2}+\frac{1}{2})\omega\right)}{\sin(\frac{\omega}{2})}$$

Example 3 Cont.

Frequency response of a causal moving average filter

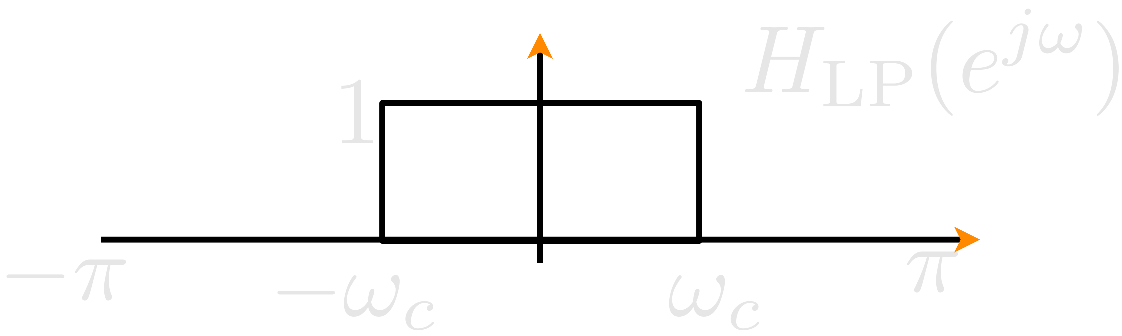
$$H(e^{j\omega}) = \frac{e^{-j\omega\frac{M}{2}}}{M+1} \cdot \frac{\sin\left((\frac{M}{2}+1)\omega\right)}{\sin(\frac{\omega}{2})}$$

Not a sinc!



Example 4:

Impulse Response of an Ideal Low-Pass Filter



$$
\begin{aligned}
h_{\mathrm{LP}}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{jw})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega
\end{aligned}
$$

Example 4

Impulse Response of an Ideal Low-Pass Filter

$$h_{\mathrm{LP}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{jw})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi jn}\boxed{e^{j\omega n}\Big|_{-\omega_c}^{\omega_c}} = 2j\sin(w_c n)$$

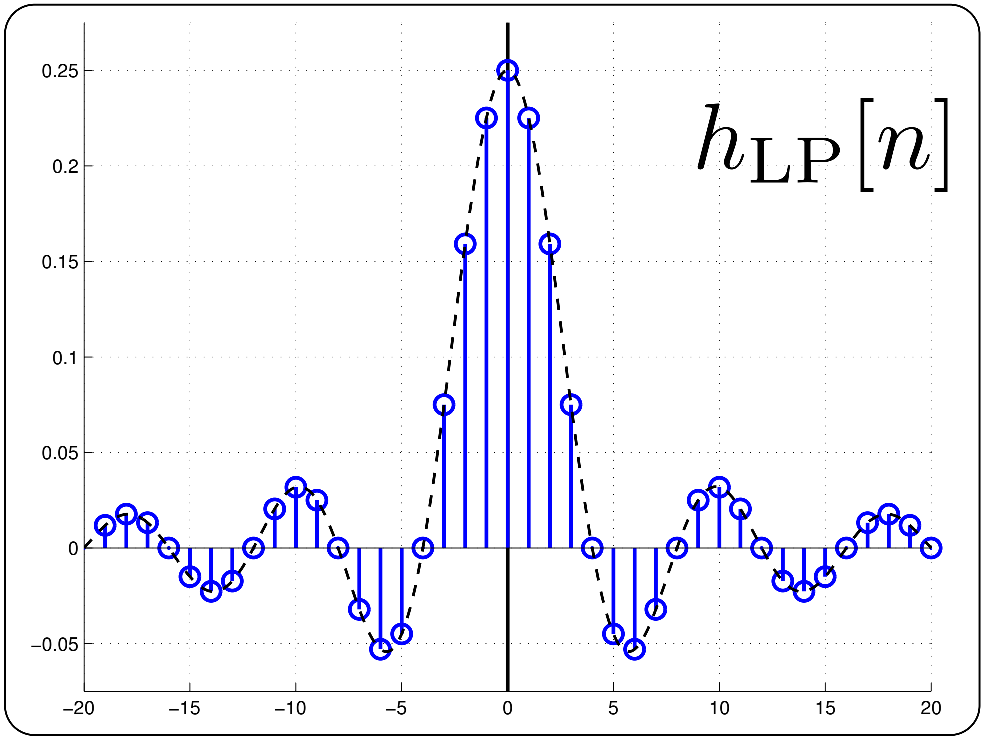
$$= \frac{\sin(w_c n)}{\pi n}$$

Example 4

Impulse Response of an Ideal Low-Pass Filter

$$h_{\mathrm{LP}}[n] = \frac{\sin(w_c n)}{\pi n}$$

sampled "sinc"



Non causal! Truncate and shift right to make causal

Example 4

Impulse Response of an Ideal Low-Pass Filter

Non causal! Truncate and shift right to make causal

How does it changes the frequency response?

Truncation:

$$\tilde{h}_{\mathrm{LP}}[n] = w_N[n] \cdot h_{\mathrm{LP}}[n]$$

property 2.9.7:

$$\tilde{H}_{\mathrm{LP}}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

Periodic convolution

Example 4

We get “smearing” of the frequency response
We get rippling