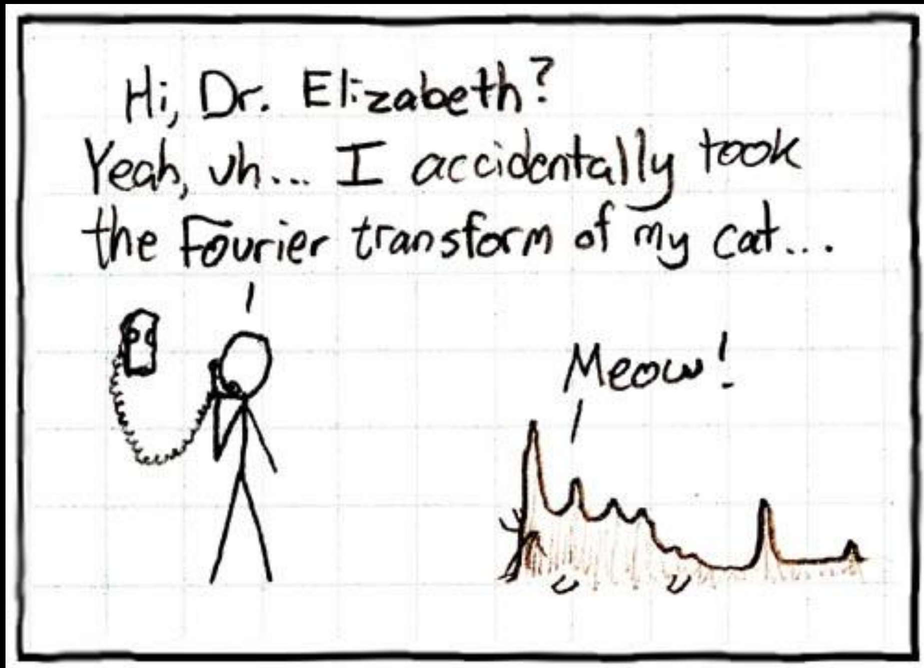

https://xkcd.com/26/

EE 123 Discussion Section 1

Jan 20, 2023

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Based on notes by Suma Anand, Josh Sanz, Li-Hao Yeh, Jon Tamir, Giulia Fanti and Frank Ong

Announcements

- HW 1 – due Friday, Jan 27
- Check website for OH calendar
- Lab 0 due next week, Lab 1 in 3 weeks
- Discussion guidelines:
  - Respect fellow students and staff
  - We are here to learn
    - Wrong answers ok
    - Questions encouraged!

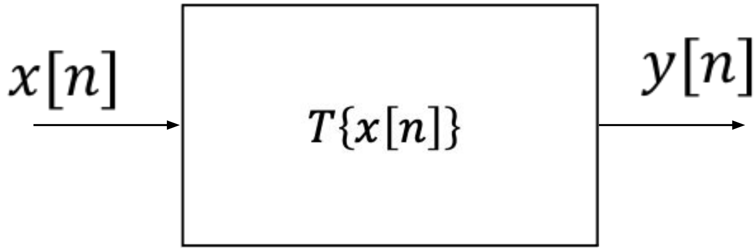
Today

- Properties of systems
- Review of linear regression (least-squares)

Concept Review

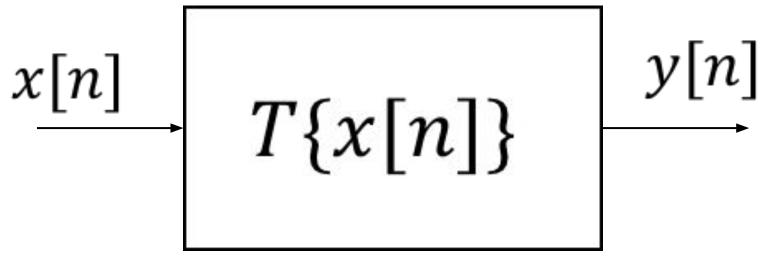
- Causal
  - $y[n_0] = f(x[n], x[n-1], \dots),\ -\infty < n \le n_0$
- Memoryless
  - $y[n] = f(x[n])\ \forall\, n$
- Linear
  - $T\{x_i[n]\} = y_i[n]$
  - $T\{\alpha x_1[n] + \beta x_2[n]\} = \alpha y_1[n] + \beta y_2[n]$
- Time Invariant
  - $T\{x[n-n_0]\} = y[n-n_0]$
- BIBO Stable
  - $|x[n]| \le B_x < \infty \Rightarrow |y[n]| \le B_y < \infty\ \forall\, n$
  - LTI: stable IFF $\sum |h[k]| < \infty$

Concept Review



- Impulse response
  - $h[n] = T\{\delta[n]\}$
- Convolution
  - $y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

Simple example



Consider a system below:

$$y[n] = \alpha x[n]$$

Is this system Linear/Time-invariant/Causal/BIBO stable?

Simple example

$$x[n] \longrightarrow \boxed{T\{x[n]\}} \longrightarrow y[n]$$

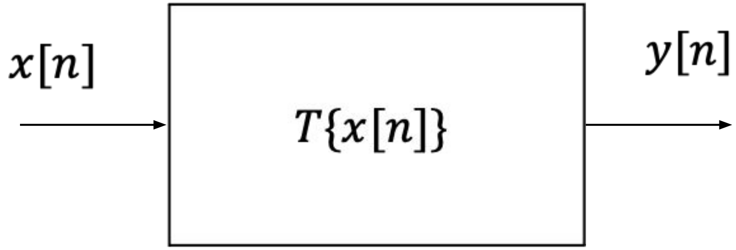
Consider a system below:

$$y[n] = \alpha x[n]$$

Is this system Linear/Time-invariant/Causal/BIBO stable?

YES to all

Simple example

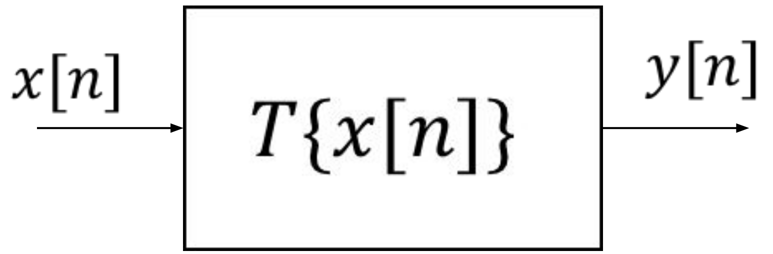


What about this modified system:

$$y[n] = \begin{cases} \alpha x[n], & x[n] \leq 1 \\ \alpha, & x[n] > 1 \end{cases}$$

Is this system Linear/Time-invariant/Causal/BIBO stable?

Simple example



What about this modified system:

$$y[n] = \begin{cases} \alpha x[n], & x[n] \le 1 \\ \alpha, & x[n] > 1 \end{cases}$$

Is this system Linear/Time-invariant/Causal/BIBO stable?

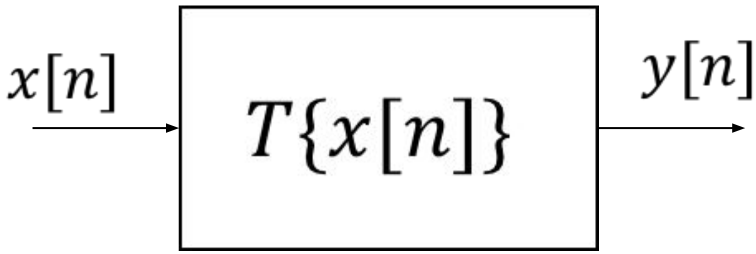
Not linear → whenever $x[n] > 1, y[n] = \alpha$, this is not homogeneous.

Time invariant → Plug in $x[n-1]$, the output is $y[n-1]$

Causal → It only depends on current $x[n]$, which is memoryless/causal

Stable → both α and $x[n]$ are bounded then output $y[n]$ is bounded

Simple example



What about this modified system:

$$y[n] = \begin{cases} \alpha x[n], & x[n] \leq 1 \\ \alpha, & x[n] > 1 \end{cases}$$

Do real systems act this way?

1. MOS/BJT amplifier input/output
2. Hooke's law (elastic materials)
3. Hysteresis

They all have a limited linear region

Another system (from old exam)

A discrete-time system H produces an output signal y that is the symmetric part of the input:

$$y[n] = \frac{x[n] + x[-n]}{2}$$

Which of the following are true?

- The system must be LTI
- The system cannot be LTI

Another system (from old exam)

A discrete-time system H produces an output signal y that is the symmetric part of the input:

$$y[n] = \frac{x[n] + x[-n]}{2}$$

Which of the following are true?

- The system must be LTI
- **The system cannot be LTI**

Another system (from old exam)

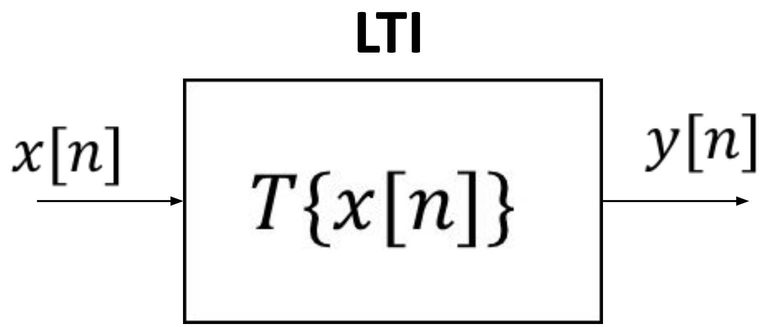
Not time invariant:

- For $x_1[n] = \delta[n]$, then $y_1[n] = \delta[n]$
- For $x_2[n] = \delta[n-1]$, then $y_2[n] = \frac{\delta[n-1]+\delta[n+1]}{2}$
- $y_1[0] = 1$ but $y_2[1] = \frac{1}{2}$

→Not time invariant

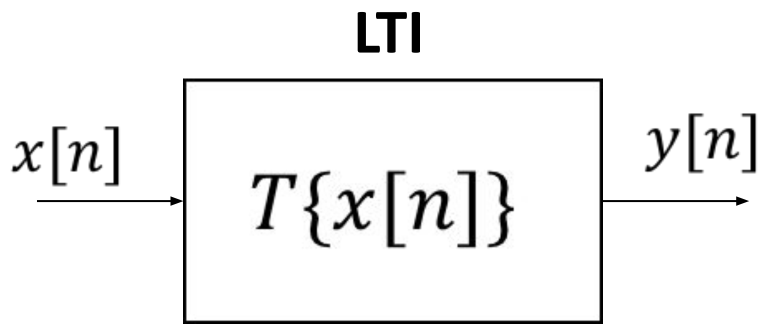
(however, the system is linear)

LTI system example (Problem 2.43)



Consider an LTI system with input $x[n]$ and output $y[n]$. When we input a signal $\left(\frac{1}{3}\right)^n u[n]$, where $u[n]$ is unit step function, we observe an output $g[n]$. Can we express $y[n]$ in terms of $x[n]$ and $g[n]$?

LTI system example (Problem 2.43)



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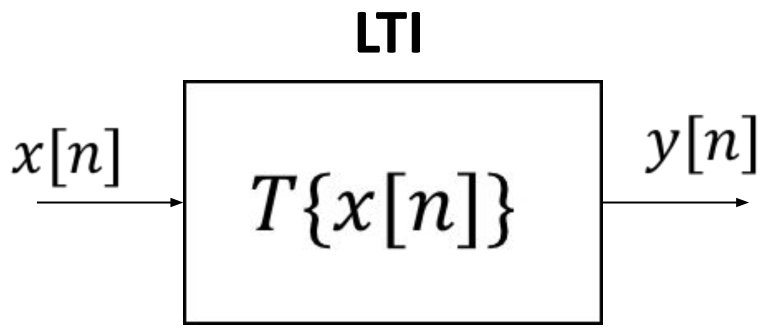
The key is to massage the input into $\delta[n]$

We know $u[n] - u[n-1] = \delta[n]$

LTI system example (Problem 2.43)



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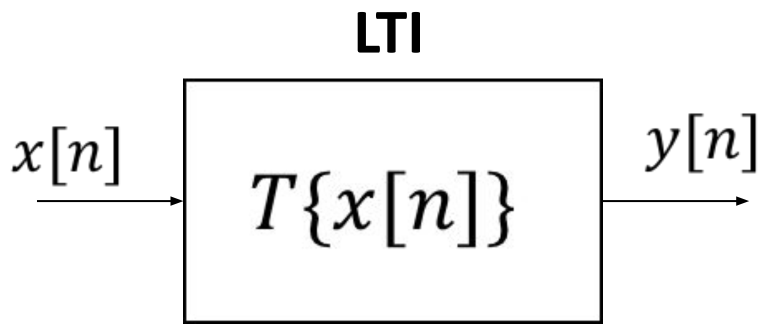
By time-invariance: input $\left(\frac{1}{3}\right)^{n-1} u[n-1]$, output $g[n-1]$

By linearity: input $\left(\frac{1}{3}\right)^{n} (u[n] - u[n-1]) = \delta[n]$,

output $g[n] - \frac{1}{3} g[n-1]$ (impulse response)

$$y[n] = x[n] * \left(g[n] - \frac{1}{3} g[n-1]\right)$$

LTI system example (Problem 2.43)



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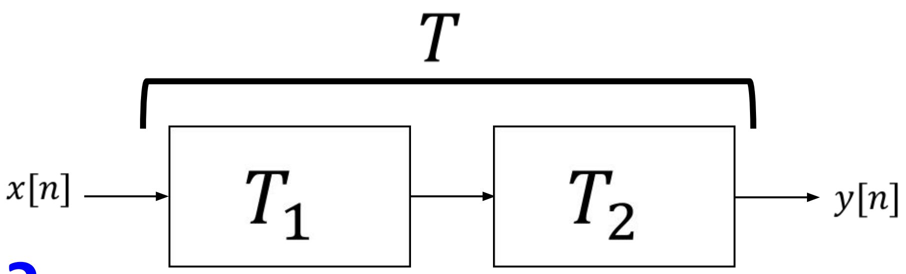
Cascaded system problem

Let T1 and T2 be two separate systems and T be the cascaded system:



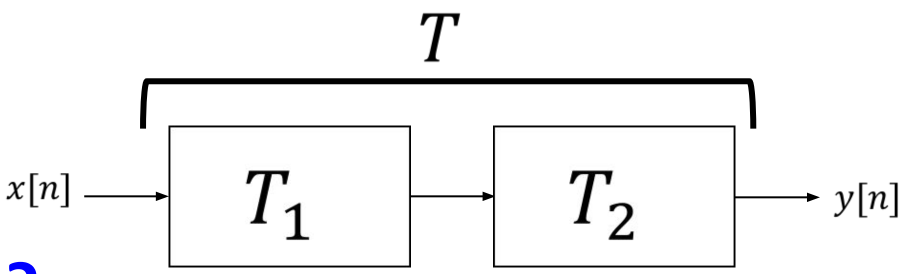
True or False?

- If T1 is LTI and T2 is not LTI, then T cannot be LTI

- If T1 is not LTI and T2 is not LTI, then T cannot be LTI

Cascaded system problem

Let T1 and T2 be two separate systems and T be the cascaded system:



True or False?

- If T1 is LTI and T2 is not LTI, then T cannot be LTI

False

Consider the system T1=0. Then T=0

- If T1 is not LTI and T2 is not LTI, then T cannot be LTI

False

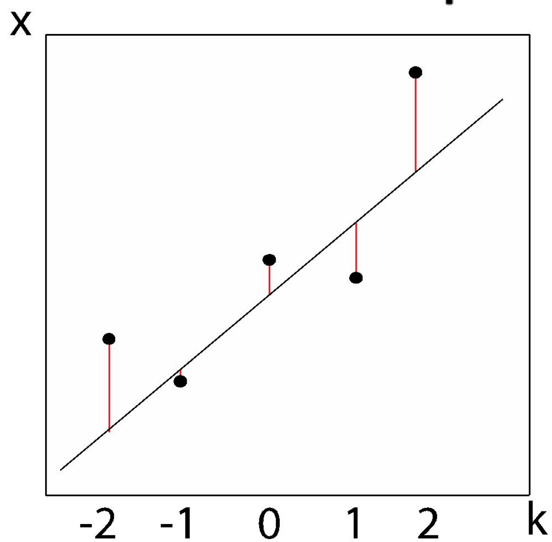
Consider the system $T_1\{x\} = x^3$ and $T_2\{x\} = x^{\frac{1}{3}}$. Then $T\{x\} = x$

Linear regression primer

Many signal processing problems can be formulated as a **least squares** problem, where we try to find model parameters that best fit the observed data. We will see this many, many times

Linear regression primer

<u>Example:</u> Linear regression. Suppose we observe five data points $x[k]$, where $k = \{-2, -1, 0, 1, 2\}$. We want to fit a line $x = mk + b$ by minimizing the squared distance between the line and the data points:



1. We want to write the squared distance in the form $\frac{1}{2}\|\mathbf{x} - \mathbf{K}\boldsymbol{\beta}\|_2^2$, where $\mathbf{x} = [x[-2], x[-1], \dots, x[2]]^T$ and $\boldsymbol{\beta} = [m\ b]^T$. What is K?
2. Solve for m and b in terms of K and x.

Linear regression primer

For each value of k, we have a linear equation for our model:

Example, $k = 2$: $x[2] = 2m + b$

And we have a squared error with our data:

Example, $k = 2$: $\left(x[2] - (b + 2m)\right)^2$

Sum of squared errors: $\sum_k \left(x[k] - (mk + b)\right)^2$

→ In matrix form, Error = $\frac{1}{2}\|\mathbf{x} - \mathbf{K}\boldsymbol{\beta}\|_2^2$

$$\text{Error} = \frac{1}{2}\left\|\begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \end{pmatrix} - \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{pmatrix} m \\ b \end{pmatrix}\right\|_2^2$$

Linear regression primer

To find the best fit from a least squares sense, minimize the sum of squared errors:

$$\underset{m,b}{\text{minimize}}\frac{1}{2}\left\|\begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \end{pmatrix} - \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{pmatrix} m \\ b \end{pmatrix}\right\|_2^2 = \underset{\boldsymbol{\beta}}{\text{minimize}}\frac{1}{2}\|\mathbf{x} - \mathbf{K}\boldsymbol{\beta}\|_2^2$$

Linear regression primer

To solve for b and m, take the derivative (gradient) with respect to b and to m, and set to zero:

$$\underset{\boldsymbol{\beta}}{\text{minimize}} \frac{1}{2} \|\mathbf{x} - \mathbf{K}\boldsymbol{\beta}\|_2^2$$

$$\mathbf{K}^{\mathbf{T}}\mathbf{K}\boldsymbol{\beta} - \mathbf{K}^{\mathbf{T}}\mathbf{x} = 0 \quad \Rightarrow \boldsymbol{\beta} = \left(\mathbf{K}^{\mathbf{T}}\mathbf{K}\right)^{-1}\mathbf{K}^{\mathbf{T}}\mathbf{x}$$

In Python,

```
K = np.array( […] )
x = np.array( […])
beta = np.linalg.solve(K, x)
```