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Discussion 1

1. Consider the system
$$y[n] = \alpha x[n].$$
Is the system linear? Time invariant? Causal? BIBO stable?

Solution: Yes, to all.

2. Consider the system
$$y[n] = \begin{cases} \alpha x[n], & x[n] \leq 1, \\ \alpha, & x[n] > 1 \end{cases}.$$
Is the system linear? Time invariant? Causal? BIBO stable?

Solution: Not linear: whenever $x[n] > 1, y[n] = \alpha$, this is not homogeneous.

Time invariant: Plug in $x[n-1]$, the output is $y[n-1]$.

Causal: It only depends on current $x[n]$, which is memoryless/causal.

Stable: both α and $x[n]$ are bounded then output $y[n]$ is bounded.

3. A discrete-time system H produces an output signal y that is the symmetric part of the input:
$$y[n] = \frac{x[n] + x[-n]}{2}$$
Which of the following is **true**?

- The system must be LTI
- The system cannot be LTI

Solution: Not time invariant:

For $x_1[n] = \delta[n]$, then $y_1[n] = \delta[n]$.

For $x_2[n] = \delta[n-1]$, then $y_2[n] = \frac{\delta[n-1]+\delta[n+1]}{2}$.

$y_1[0] = 1$, but $y_2[1] = \frac{1}{2}$.

$\therefore$ the system is not time invariant (however, it is linear).

4. Consider an LTI system with input $x[n]$ and output $y[n]$. When we input a signal $(1/3)^n u[n]$, where $u[n]$ is the unit step function, we observe an output $g[n]$. Can we express $y[n]$ in terms of $x[n]$ and $g[n]$?

Solution: The key is to massage the input into $\delta[n]$.

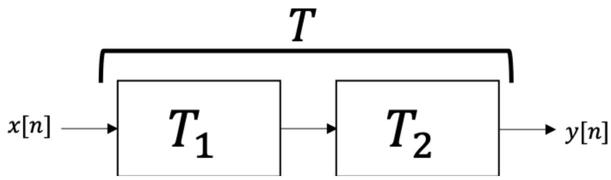
We know $u[n] - u[n-1] = \delta[n]$.

By time-invariance: input $(1/3)^{n-1} u[n-1]$ gives output $g[n-1]$

By linearity: input $(1/3)^n (u[n] - u[n-1]) = \delta[n]$ gives output $g[n] - 1/3 g[n-1]$ (impulse response).

$$y[n] = x[n] * \left(g[n] - \frac{1}{3} g[n-1] \right)$$

5. Consider the following system:



Let T_1 and T_2 be separate systems and T be the cascaded system. **True or False**:

(a) If T_1 is LTI and T_2 is not LTI, then T cannot be LTI

(b) If T_1 is not LTI and T_2 is not LTI, then T cannot be LTI

Solution:

(a) False. Consider the system $T_1 = 0$. Then $T = 0$.

(b) False. Let $T_1\{x[n]\} = x[n]^3$ and $T_2\{x[n]\} = x[n]^{1/3}$. Then T is the identity.

6. Find $\beta \in \mathbb{R}^2$ which minimizes the mean squared error (MSE):

$$\frac{1}{2}\|\mathbf{x} - \mathbf{K}\beta\|_2^2$$

for known $\mathbf{x} \in \mathbb{R}^d$ and

$$\mathbf{K} = \begin{bmatrix} n & 1 \\ n-1 & 1 \\ \vdots & \vdots \\ -n & 1 \end{bmatrix}.$$

Solution: For each value of k, we have a linear equation for our model:

Example, $k = 2 : x[2] = 2m + b$.

We also have a squared error for the data:

Example, $k = 2 : (x[2] - (2m + b))^2$.

The sum of squared errors is

$$\sum_k (x[k] - (mk + b))^2,$$

or in matrix form

$$\|\mathbf{x} - \mathbf{K}\beta\|_2^2$$

$$\text{Error} = \left\| \begin{pmatrix} x_2 \\ x_1 \\ x_0 \\ x_{-1} \\ x_{-2} \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} \right\|_2^2$$

To find the best fit from a least squares sense, minimize the sum of squared errors:

$$\min_\beta \|\mathbf{x} - \mathbf{K}\beta\|_2^2$$

To solve for b and m, take the derivative (gradient) with respect to b and to m, and set to zero:

$$\mathbf{K}^\top \mathbf{K}\beta - \mathbf{K}^\top \mathbf{x} = \mathbf{0}$$
$$\left(\mathbf{K}^\top \mathbf{K}\right)^{-1} \mathbf{K}^\top \mathbf{x} = \beta$$

In python:

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K = np.array( [...]  )
x = np.array( [...]  )
beta = np.linalg.solve(K, x)
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