EE123
Digital Signal Processing

Lecture 15
Sampling II

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Topics

- Last time
  - Ideal Sampling model C/D
  - Impulse sampling $x_c(t) \Rightarrow x_s(t)$
  - Impulses to discrete samples $x_s(nT) \Rightarrow x[n]$
  - Relationship $X_c(j\Omega) \Leftrightarrow X_s(j\Omega) \Leftrightarrow X(e^{j\omega})$
  - Ideal reconstruction D/C
- Today
  - D.T processing of C.T signals
  - C.T processing of D.T signals (ha?????)
  - Downsampling

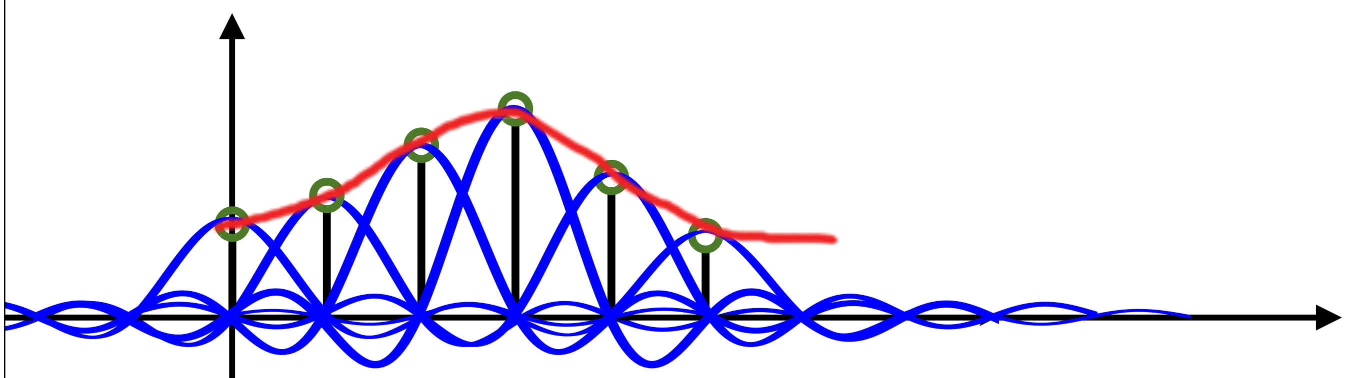
Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) \quad = \quad \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t)$$

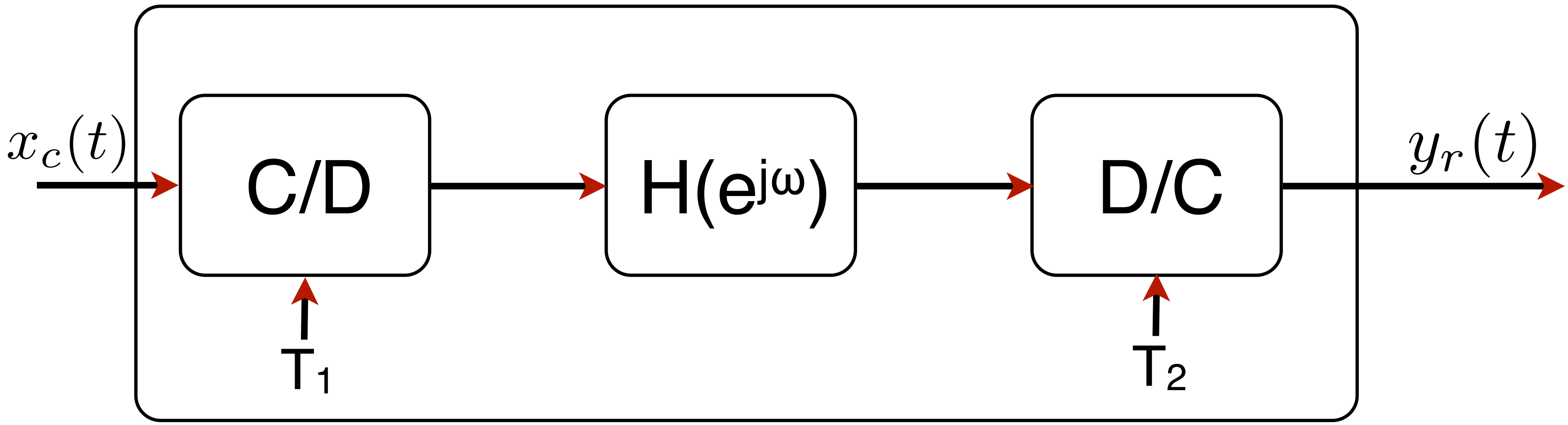
$$= \quad \sum_n x[n]h(t - nT)$$

The sum of sincs gives $x_r(t)$ ⇒ Unique signal

bandlimited by Ω_s

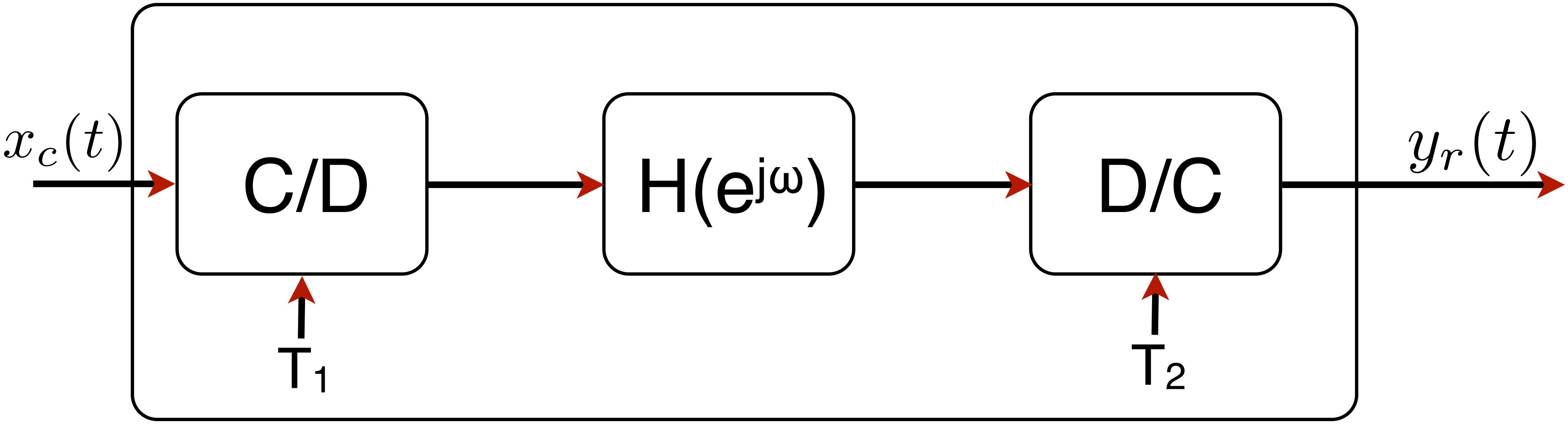


Discrete-Time Processing of C-T Signals



- Q: If *h[n]* is LTI, $H(e^{j\omega})$ exists,
  Is the whole system LTI?

Discrete-Time Processing of C-T Signals

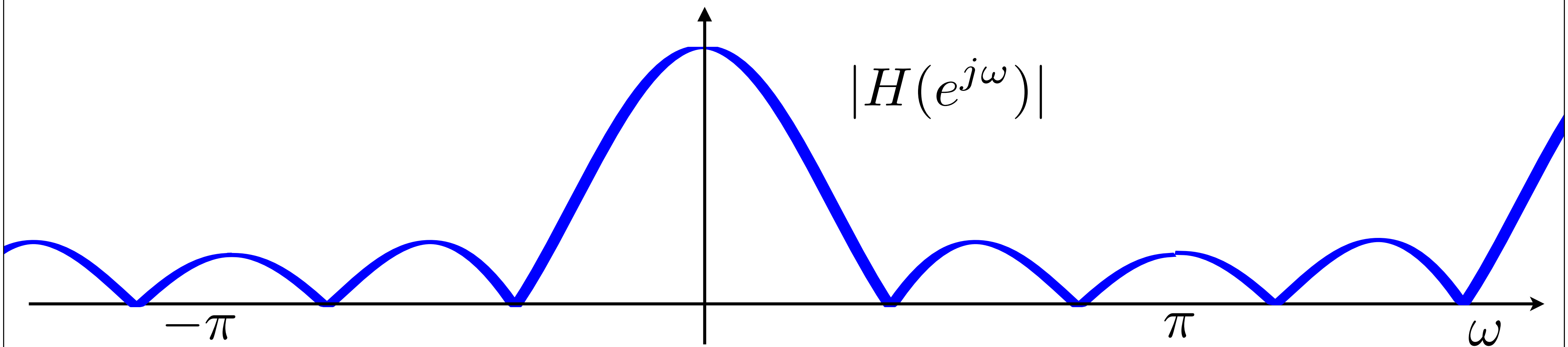


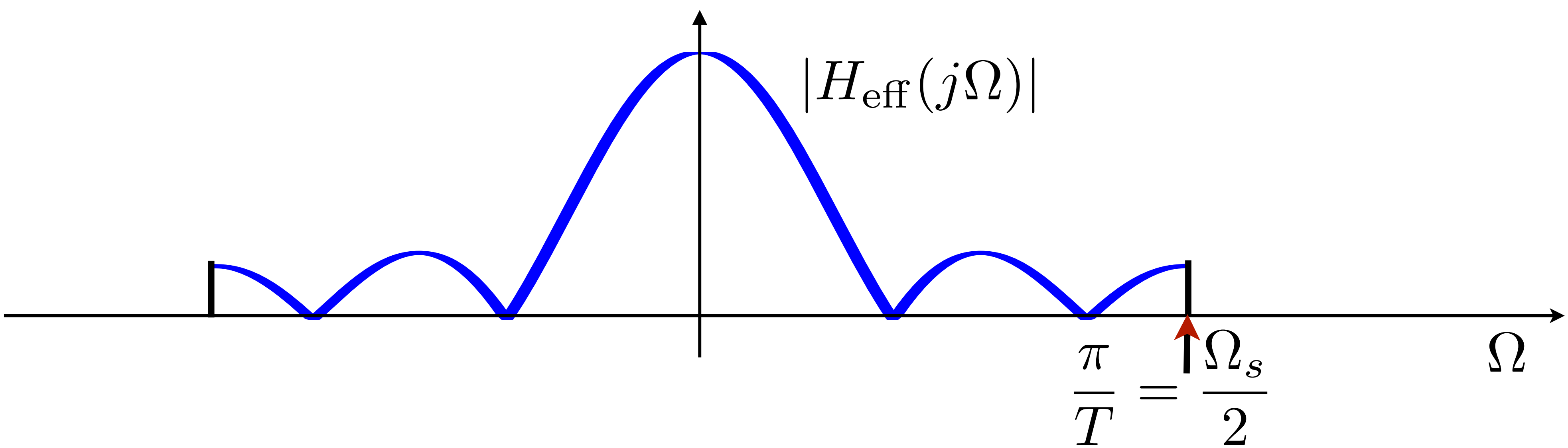
- Q: If *h[n]* is LTI, $H(e^{j\omega})$ exists,
  Is the whole system LTI?
- A: If xc(t) is bandlimited by $\frac{\Omega_s}{2} = \frac{\pi}{T}$ then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\omega})\Big|_{\omega=\Omega T} & |\Omega| < \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

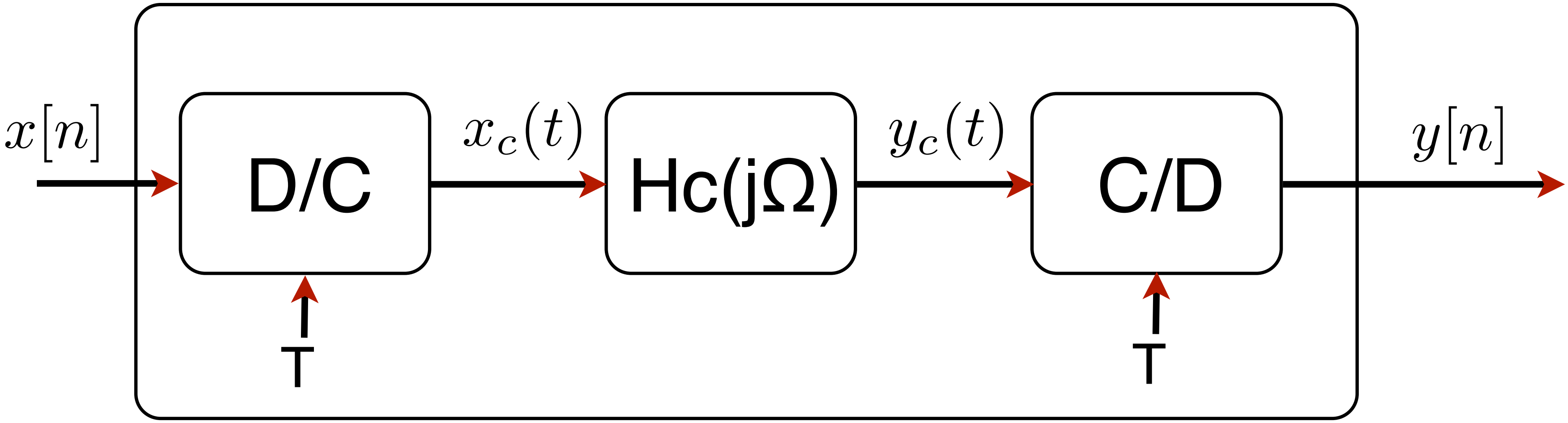
Example:

- Length 5 moving average





C.T Processing of D.T Signals



LTI - has $H(e^{jw})$!

- Useful to interpret D.T. systems with no simple interpretation in discrete domain.

- <u>Tool:</u> recall: $$x_c(t) = \sum_{n=-\infty}^{\infty} x[n]\operatorname{sinc}\left(\frac{t-nT}{T}\right)$$

Derivation

$$X_c(j\Omega) = \begin{cases} TX(e^{j\omega})\Big|_{\omega=\Omega T} & |\Omega| \le \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$ ⇒ also bandlimited

so,

$$Y(e^{j\omega}) = \frac{1}{T}\sum_k Y_c(j(\Omega - k\Omega_s))\Bigg|_{\Omega=\frac{\omega}{T}} = \frac{1}{T}Y_c(j\Omega)\Bigg|_{\Omega=\frac{\omega}{T}}$$

no aliasing!

Derivation

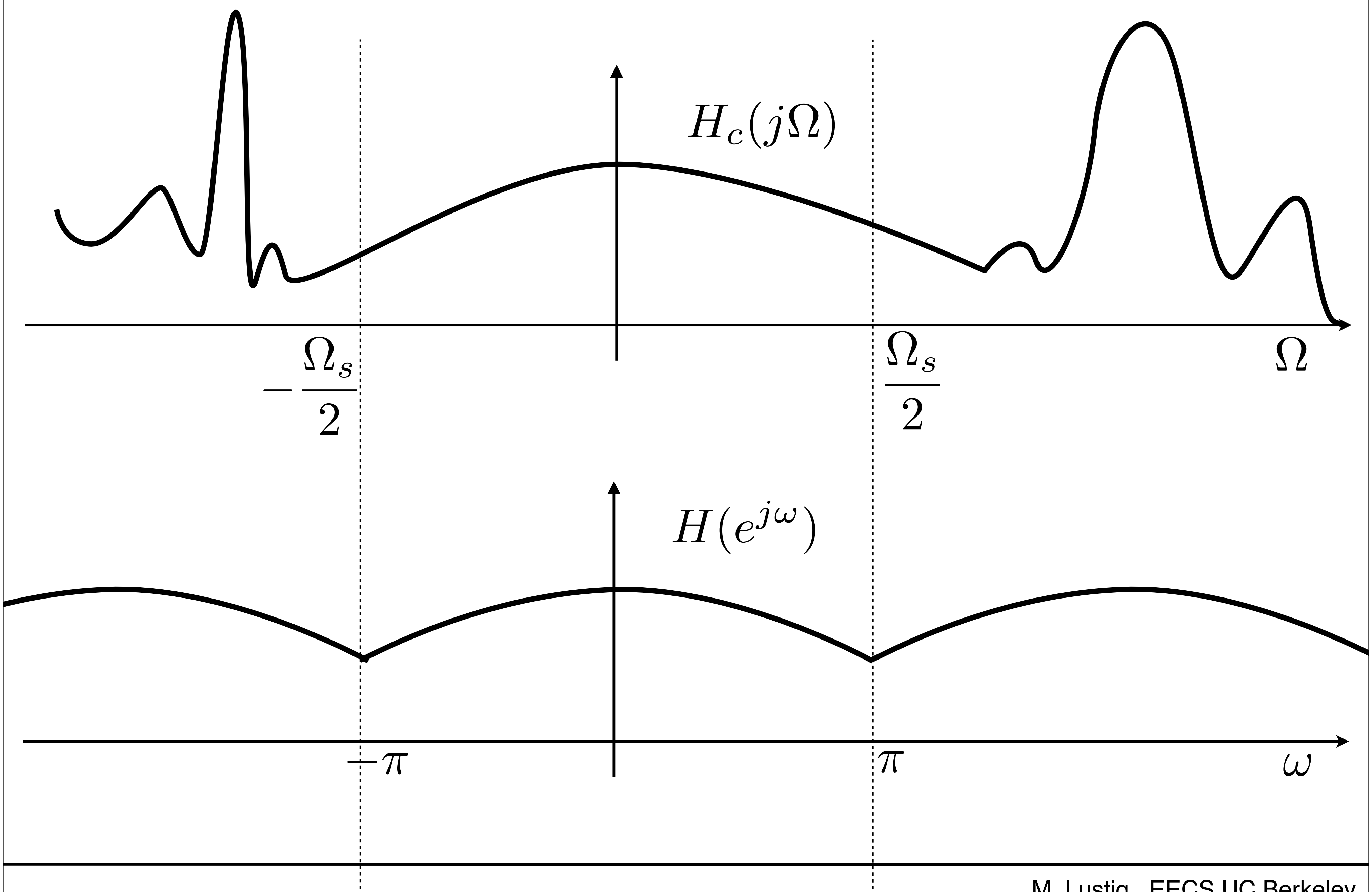
$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_k Y_c(j(\Omega - k\Omega_s))\Big|_{\Omega=\frac{\omega}{T}} = \frac{1}{T}Y_c(j\Omega)\Big|_{\Omega=\frac{\omega}{T}}$$

Combining the result:

$$Y(e^{j\omega}) = \underbrace{H_c(j\Omega)|_{\Omega=\frac{\omega}{T}}}_{H(e^{j\omega})} X(e^{j\omega}) \qquad |\omega| < \pi$$

Example:



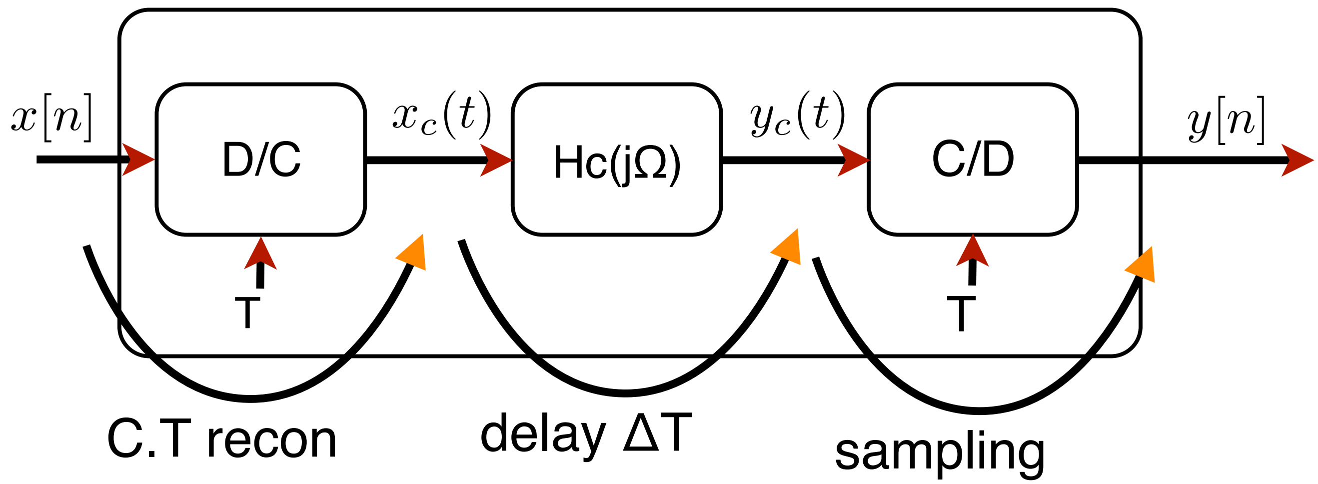
$H_c(j\Omega)$
$-\frac{\Omega_s}{2}$
$\frac{\Omega_s}{2}$
Ω
$H(e^{j\omega})$
$-\pi$
π
ω

Example:

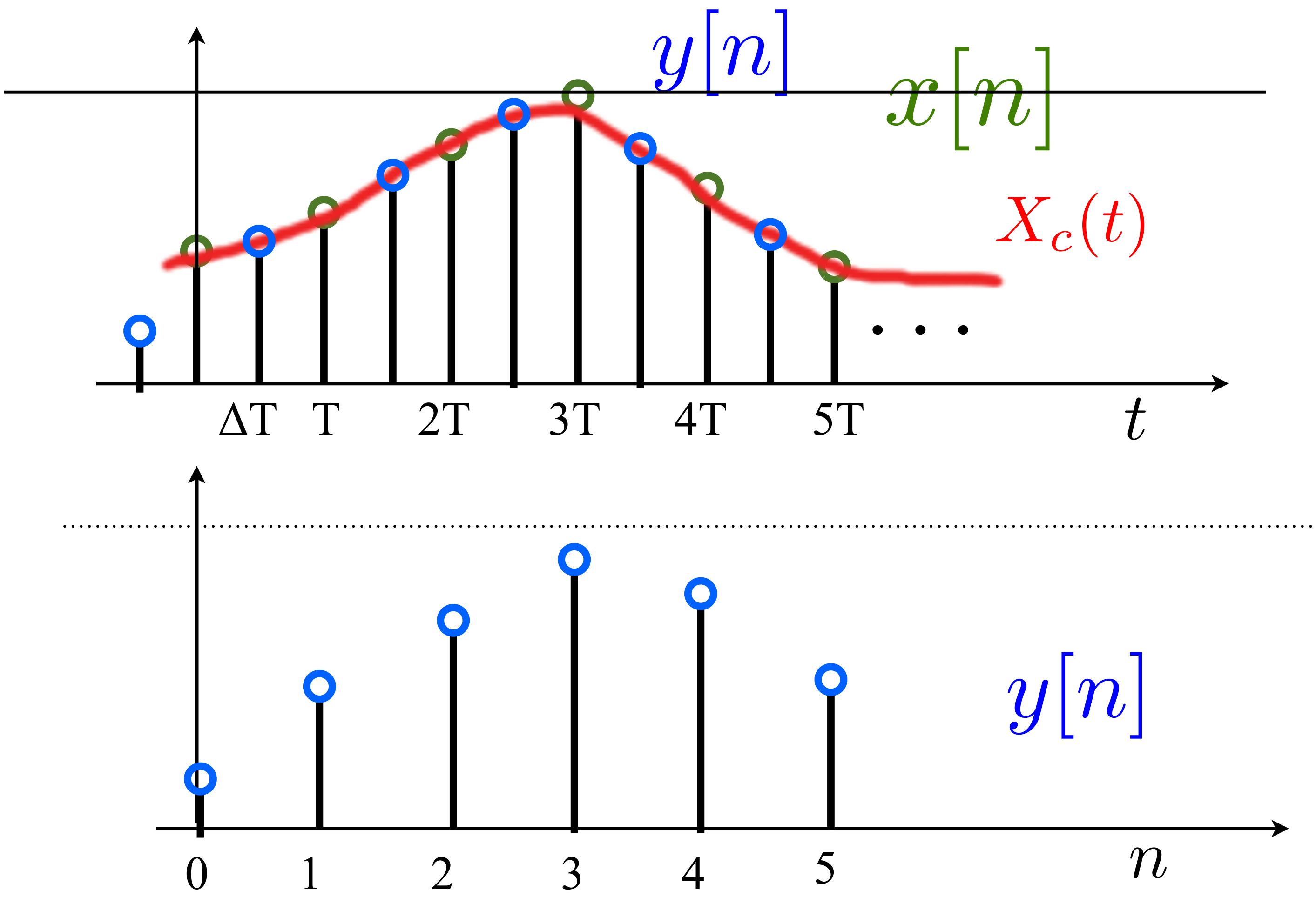
Non-integer delay: $$H(e^{j\omega}) = e^{-j\omega\Delta}$$

- What is the time-domain operation when Δ is not an integer ($\Delta=1/2$)?

Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ <u>delay of ΔT in time</u>



Example: Non Integer Delay



Example: Non Integer Delay

- The block diagram is only for interpretation!

$$y_c(t) = x_c(t - T\Delta)$$

$$\begin{aligned} y[n] &= y_c(nT) = x_c(nT - T\Delta) \\ &= \sum_k x[k]\mathrm{sinc}\left(\frac{t - kT - T\Delta}{T}\right)\Bigg|_{t=nT} \end{aligned}$$

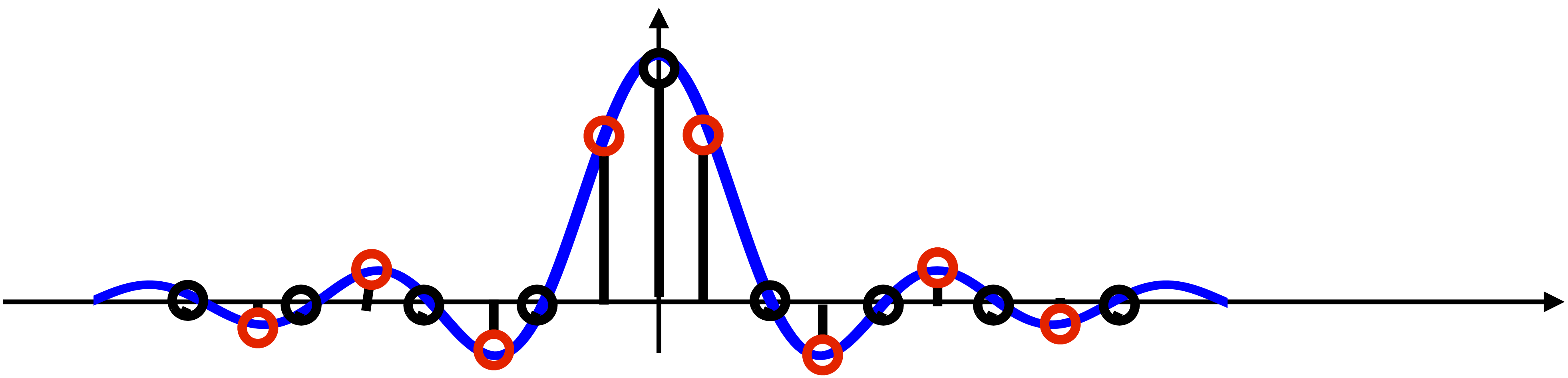
T's cancel!

$$= \sum_k x[k]\mathrm{sinc}(n - k - \Delta)$$

Example: Non Integer Delay

$$h[n] = \mathrm{sinc}(n - \Delta)$$

Example: a discrete delta is a representation of a sampled sinc



shifted by partial samples results in many coefficients!

DownSampling

- Much like C/D conversion
- Expect similar effects:
  - –Aliasing
  - –mitigate by antialiasing filter
- Finely sampled signal $\Rightarrow$ almost continuous
  - –Downsample in that case is like sampling!

Changing Sampling-rate via D.T Processing

Downsampling:

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT) \qquad\qquad x_d[n] = x_c(n\underbrace{MT}_{T'})$$

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

Changing Sampling-rate via D.T Processing

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

we would like to bypass X_c and go from $X(e^{j\omega}) \Rightarrow X_d(e^{j\omega})$

substitute r = kM + i   i=0,1,...,M-1

k=-∞,...,∞

two counters

e.g., k: hours, i: minutes

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}k\right)\right)}_{X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}} - \underbrace{\frac{2\pi}{T}}k\right)\right)$$

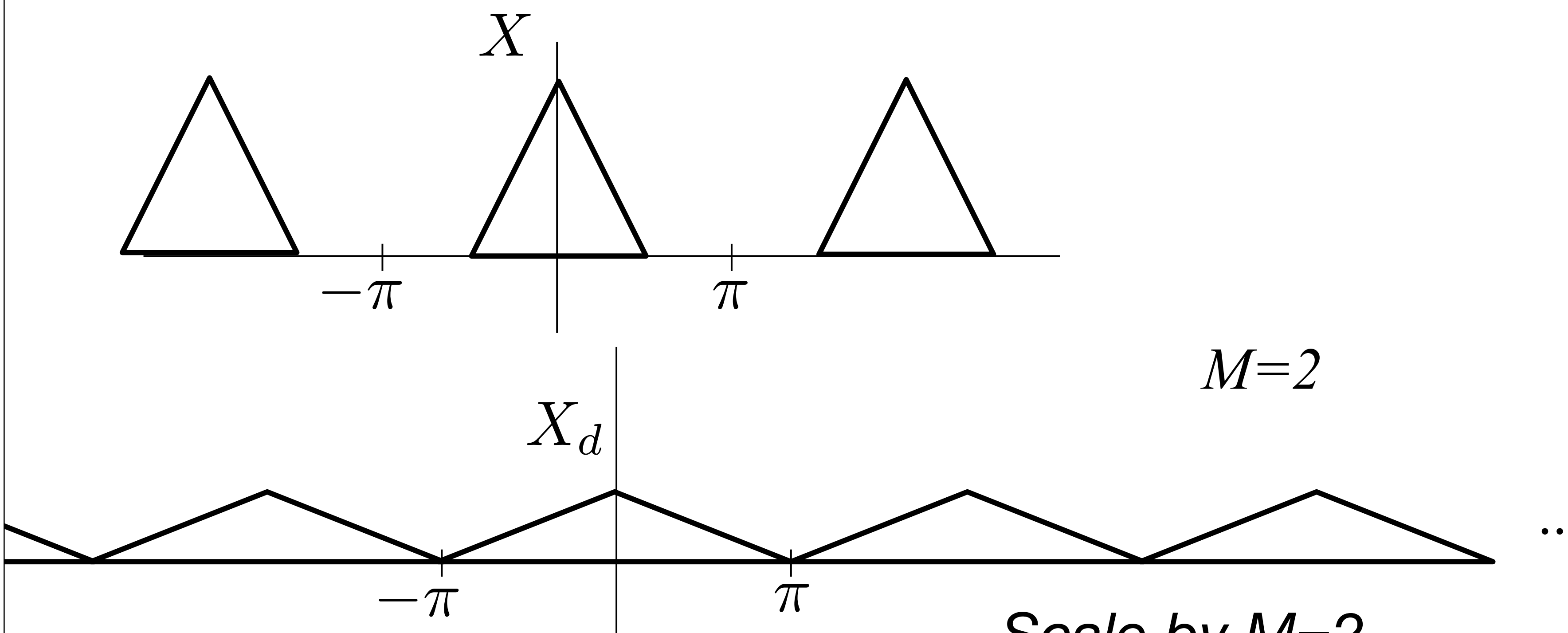
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

stretch by M

replicate

Changing Sampling-rate via D.T Processing

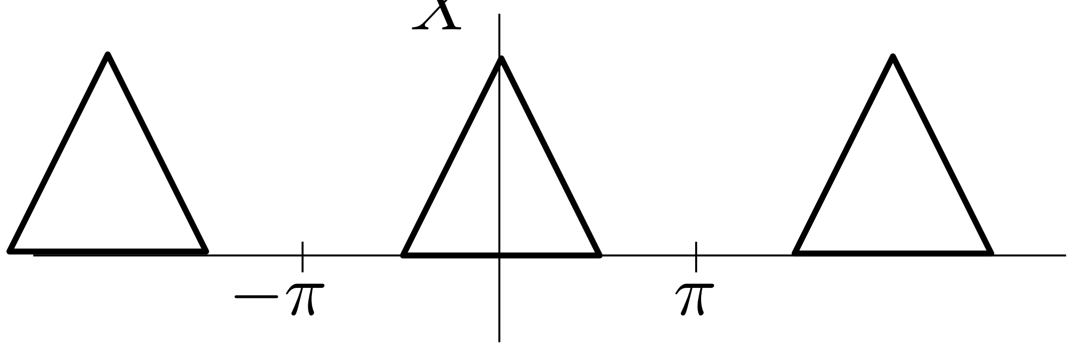
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j({\color{blue}w/M}-{\color{red}2\pi i/M})}\right)$$



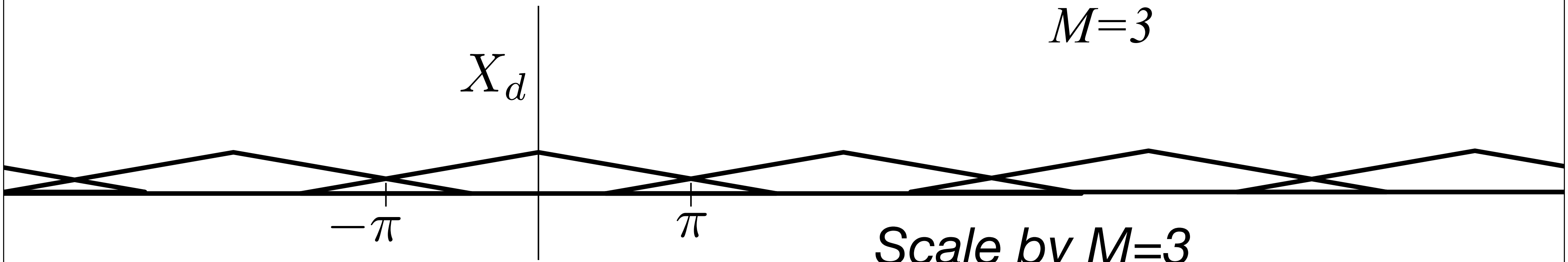
Scale by M=2
*Shift by (i=1)*2π/(M=2)*

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j(w/M - 2\pi i/M)}\right)$$



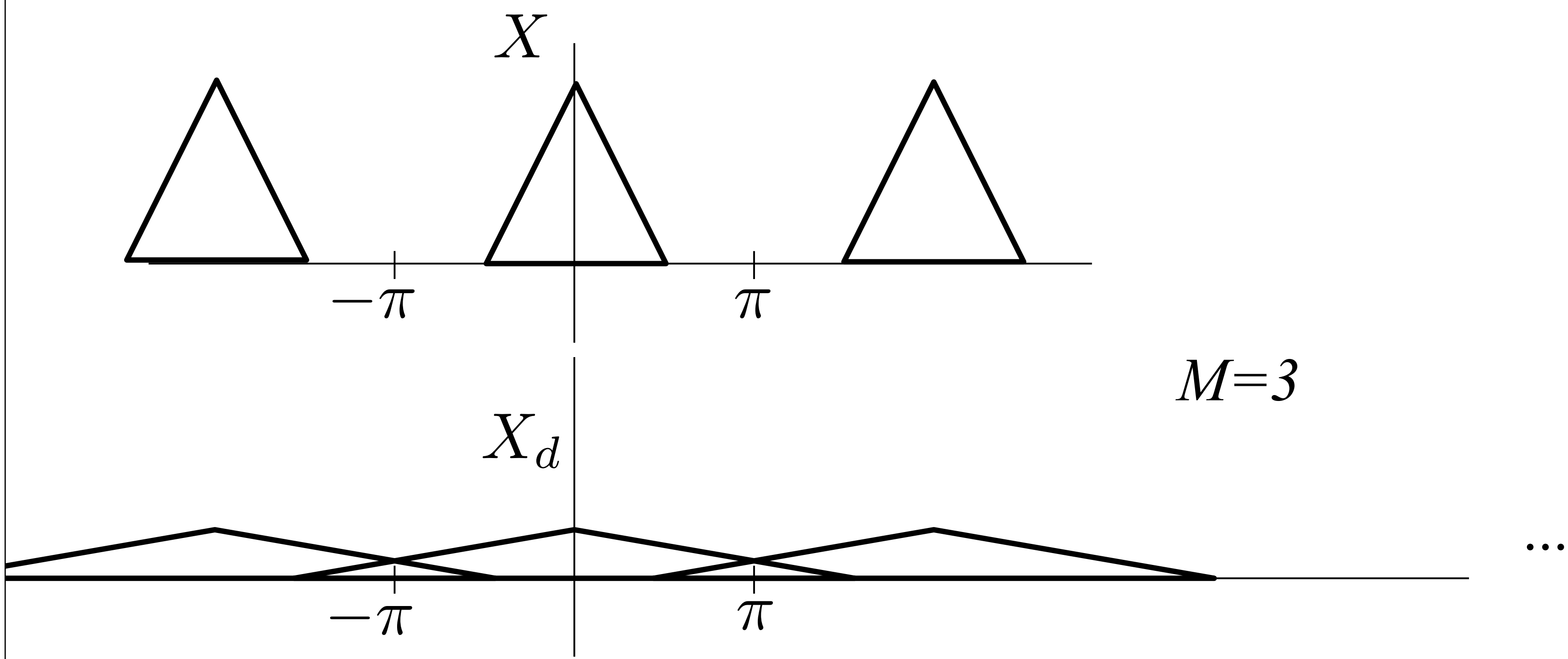
M=3



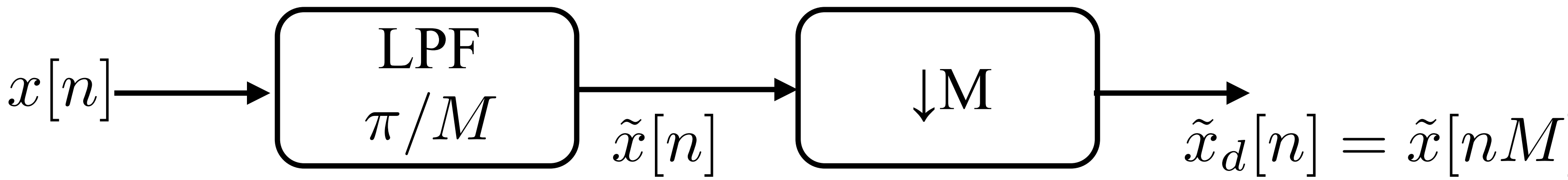
Scale by M=3
*Shift by (i=1)*2π/(M=3)*
*Shift by (i=2)*2π/(M=3)*

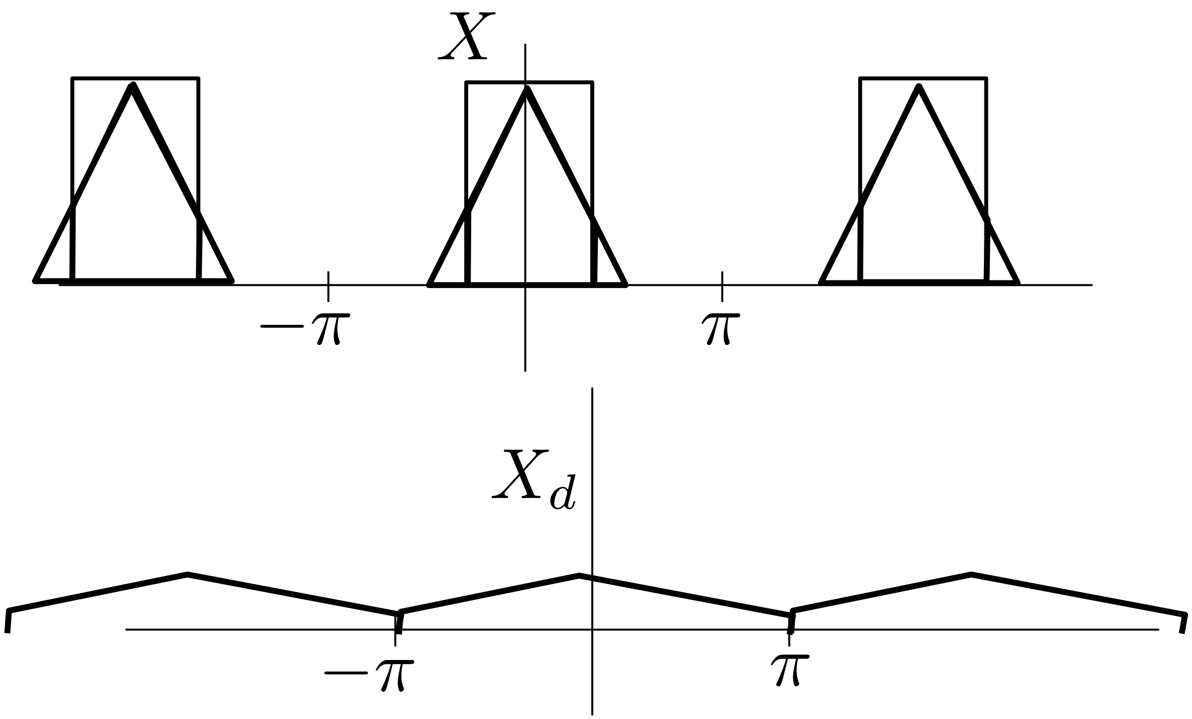
Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j({\color{blue}w/M}-{\color{red}2\pi i/M)}}\right)$$



Anti-Aliasing





M=3