EE123
Digital Signal Processing

Lecture 16
Resampling

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Topics

- http://rtl-sdr.com
- Did you sign up for the ham exam?
- Last time
  - D.T processing of C.T signals
  - C.T processing of D.T signals (ha?????)
- Today
  - Downsampling
  - Changing Sampling Rate via DSP
  - Upsampling
  - Rational resampling

DownSampling

- Much like C/D conversion
- Expect similar effects:
  - –Aliasing
  - –mitigate by antialiasing filter

- Finely sampled signal ⇒ almost continuous
  - –Downsample in that case is like sampling!

Changing Sampling-rate via D.T Processing

Downsampling:

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT)$$

$$x_d[n] = x_c(n\underbrace{MT}_{T'})$$

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

Changing Sampling-rate via D.T Processing

The DTFT:

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$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

we would like to bypass X_c and go from $X(e^{j\omega}) \Rightarrow X_d(e^{j\omega})$

substitute counter to
$k = rM + i$

$i=0,1,\ldots,M-1$
$r=-\infty,\ldots,\infty$

two counters
e.g., r: hours, i: minutes

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) \quad = \quad \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

$$= \quad \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}r\right)\right)}_{X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi}{M}i\right)}\right)}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{} - \underbrace{\frac{2\pi}{T}}_{}k\right)\right)$$

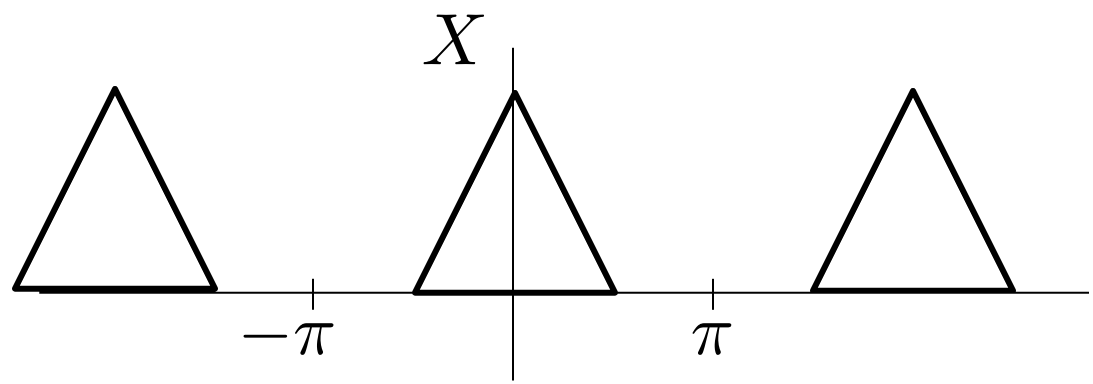
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi}{M}i\right)}\right)$$

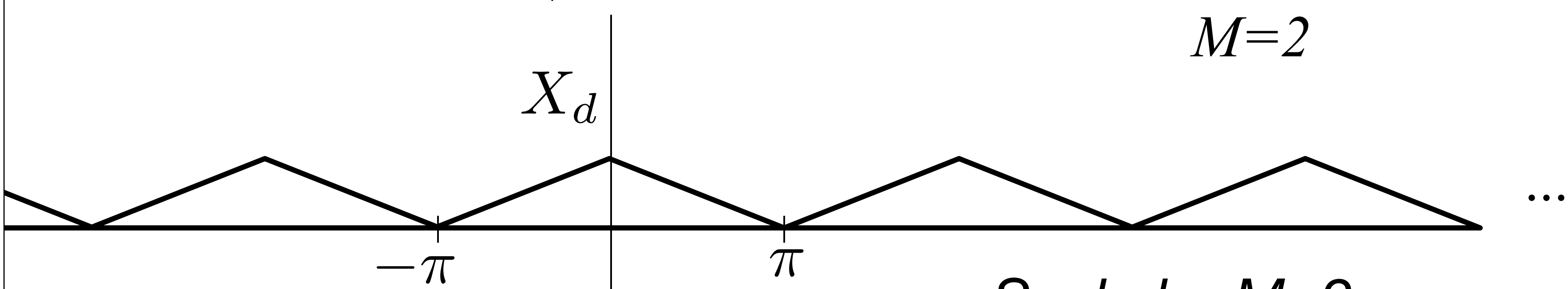
stretch by M

replicate

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M} i\right)}\right)$$

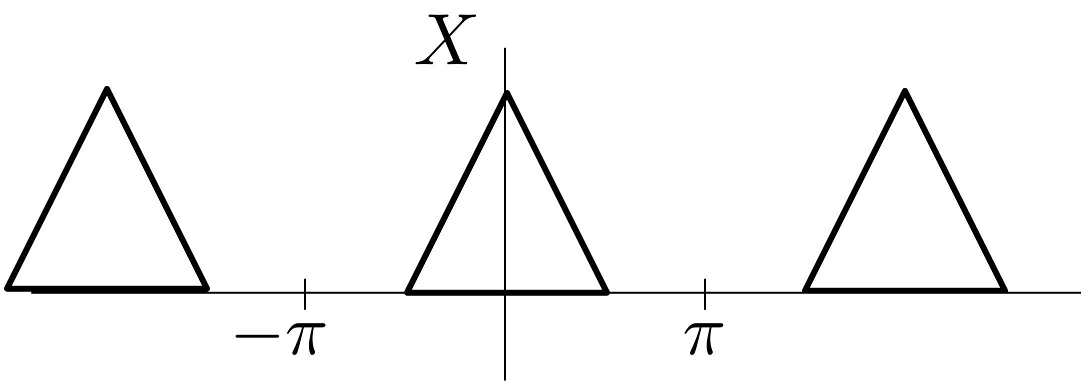


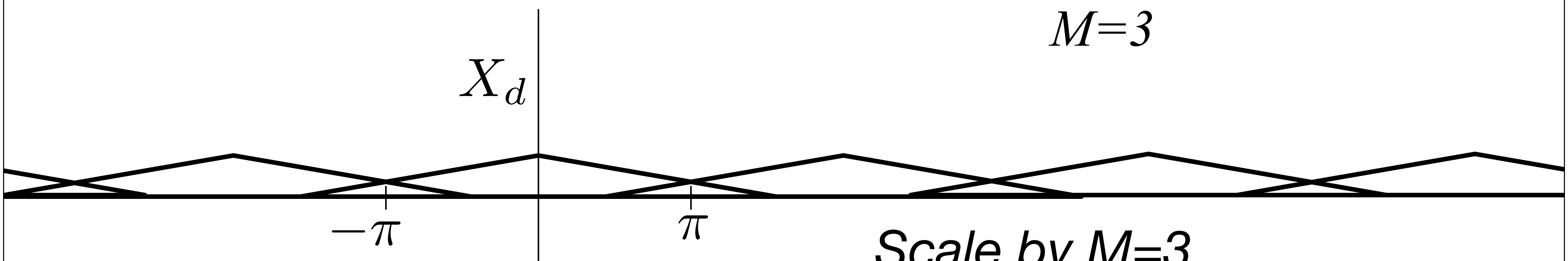


Scale by M=2
*Shift by (i=1)*2π/(M=2)*

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M} i\right)}\right)$$



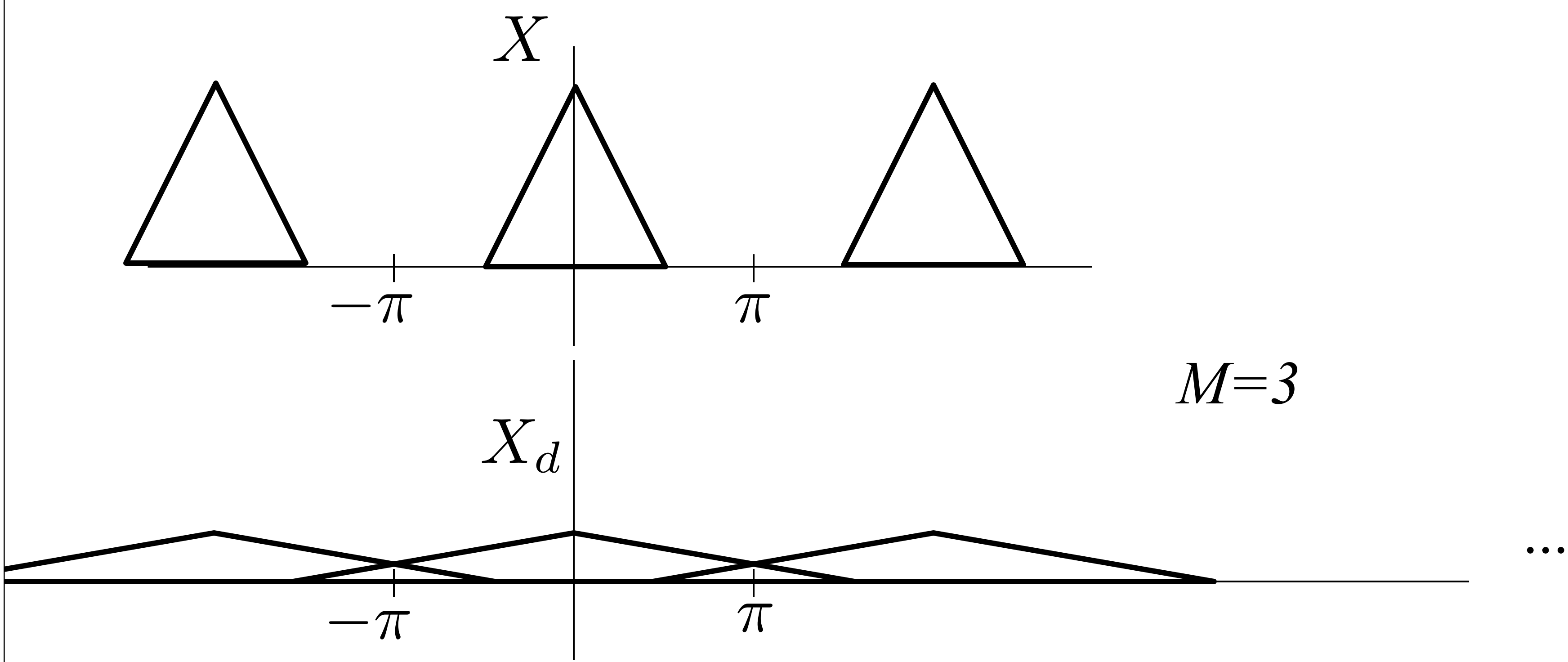


M=3

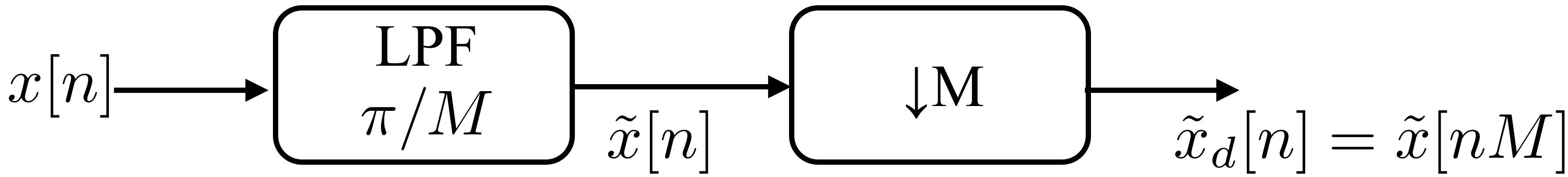
Scale by M=3
*Shift by (i=1)*2π/(M=3)*
*Shift by (i=2)*2π/(M=3)*

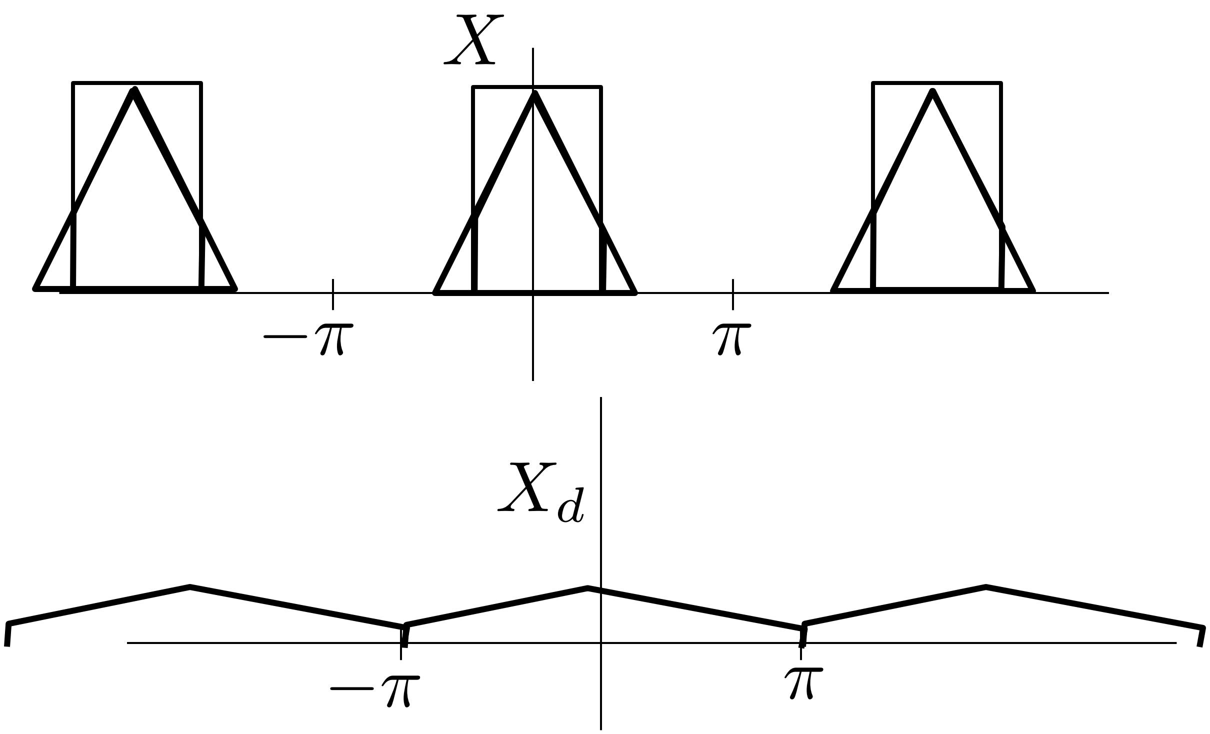
Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M} i\right)}\right)$$



Anti-Aliasing





M=3

UpSampling

- Much like D/C converter
- Upsample by A LOT ⇒ almost continuous
- Intuition:
  - Recall our D/C model: $x[n] \Rightarrow x_s(t) \Rightarrow x_c(t)$
  - Approximate "$x_s(t)$" by placing zeros between samples
  - Convolve with a sinc to obtain "$x_c(t)$"

Up-sampling

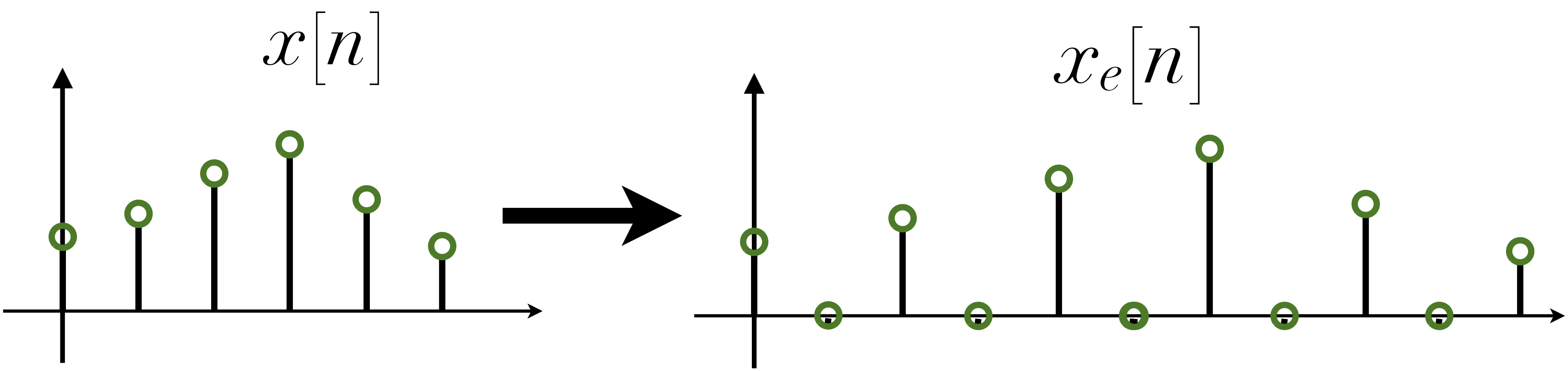
lower x

$$x[n] = X_c(nT)$$

$$x_i[n] = X_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

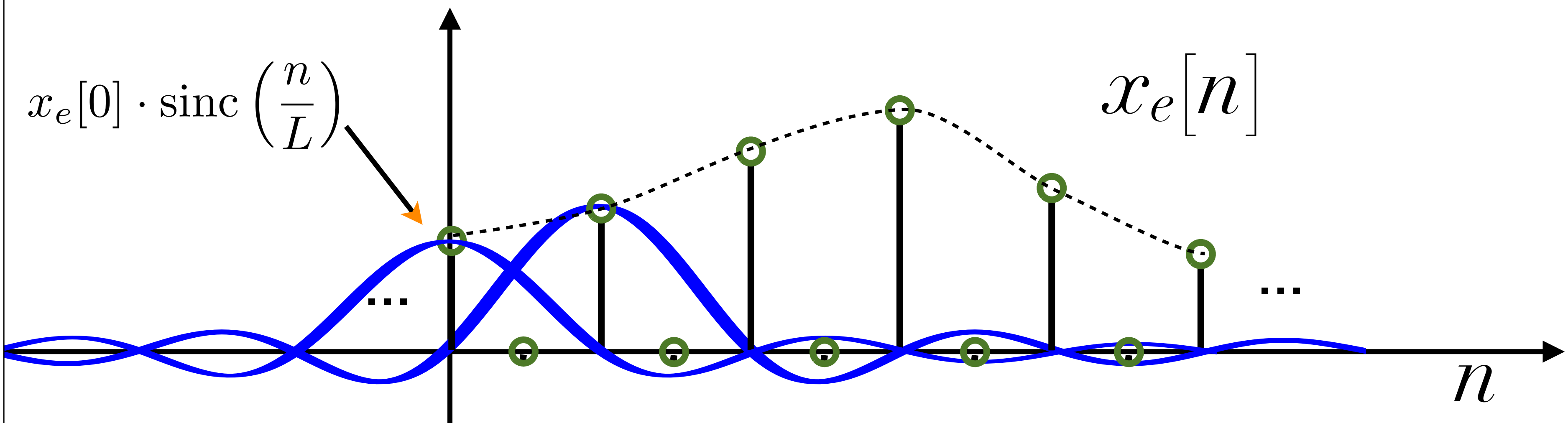
Obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate: $x_e = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

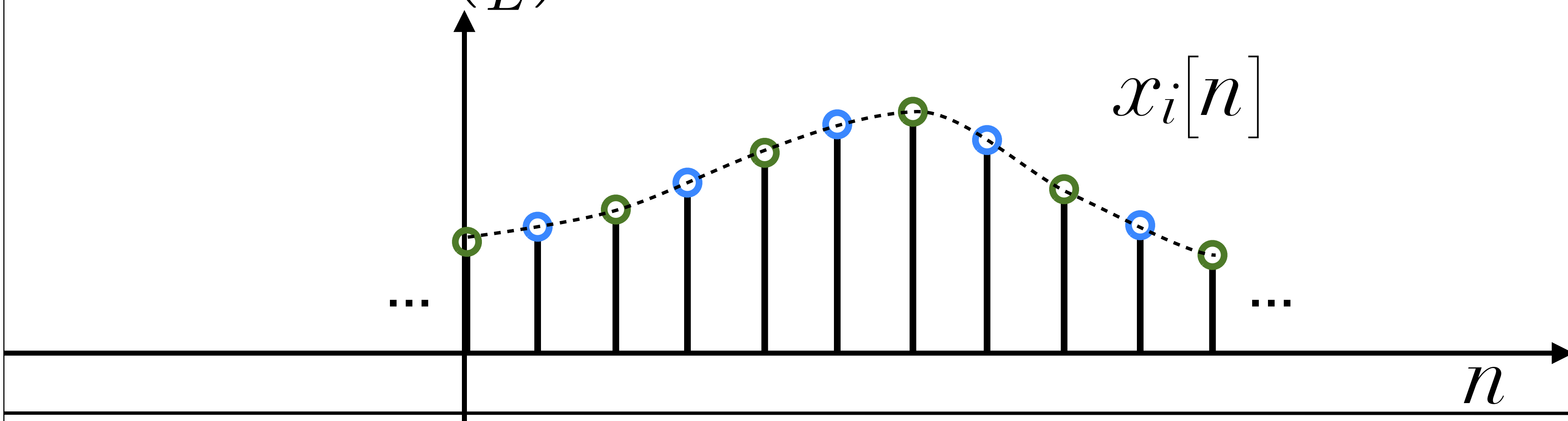


Up-Sampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



$x_i = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$



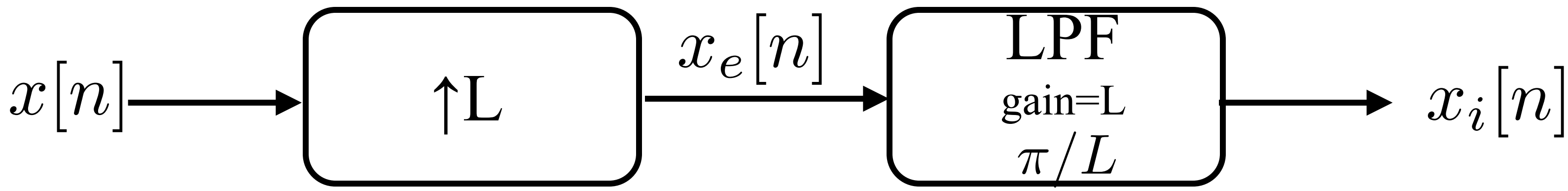
Up-Sampling

$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$

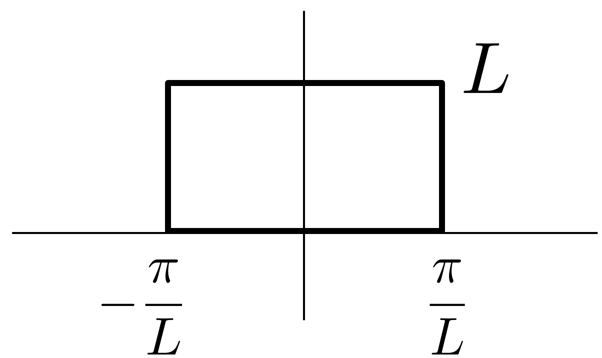
$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}(\frac{n-kL}{L})$$

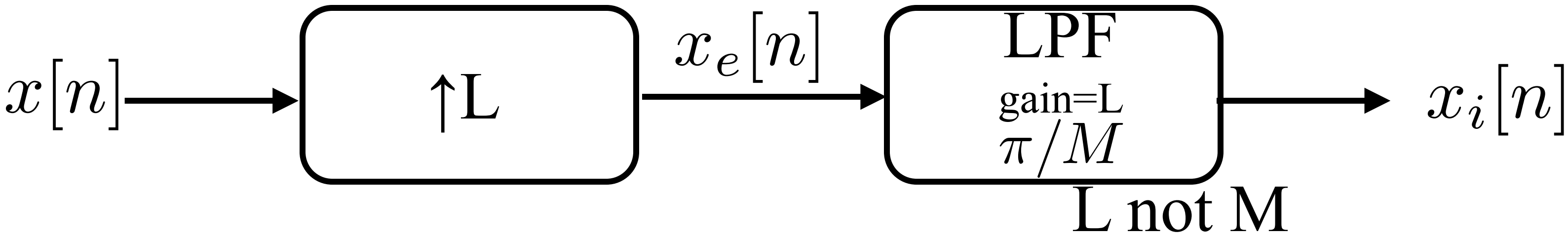
Frequency Domain Interpretation



$\mathrm{sinc}(n/L)$ DTFT $\Rightarrow$



Frequency Domain Interpretation

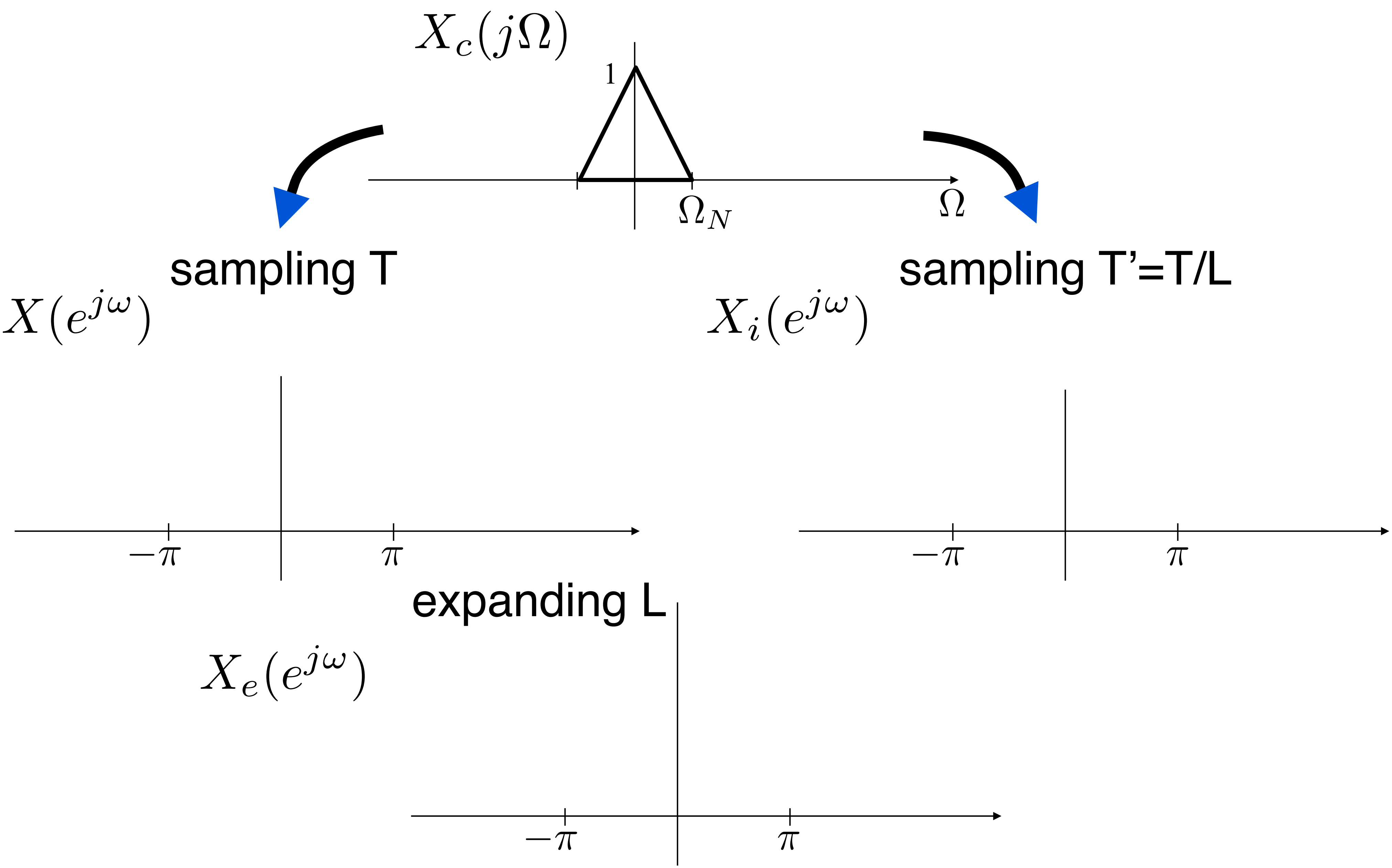


$$X_e(e^{j\omega}) = \sum_{\mathrm{n}=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} \quad = X(e^{j\omega L})$$

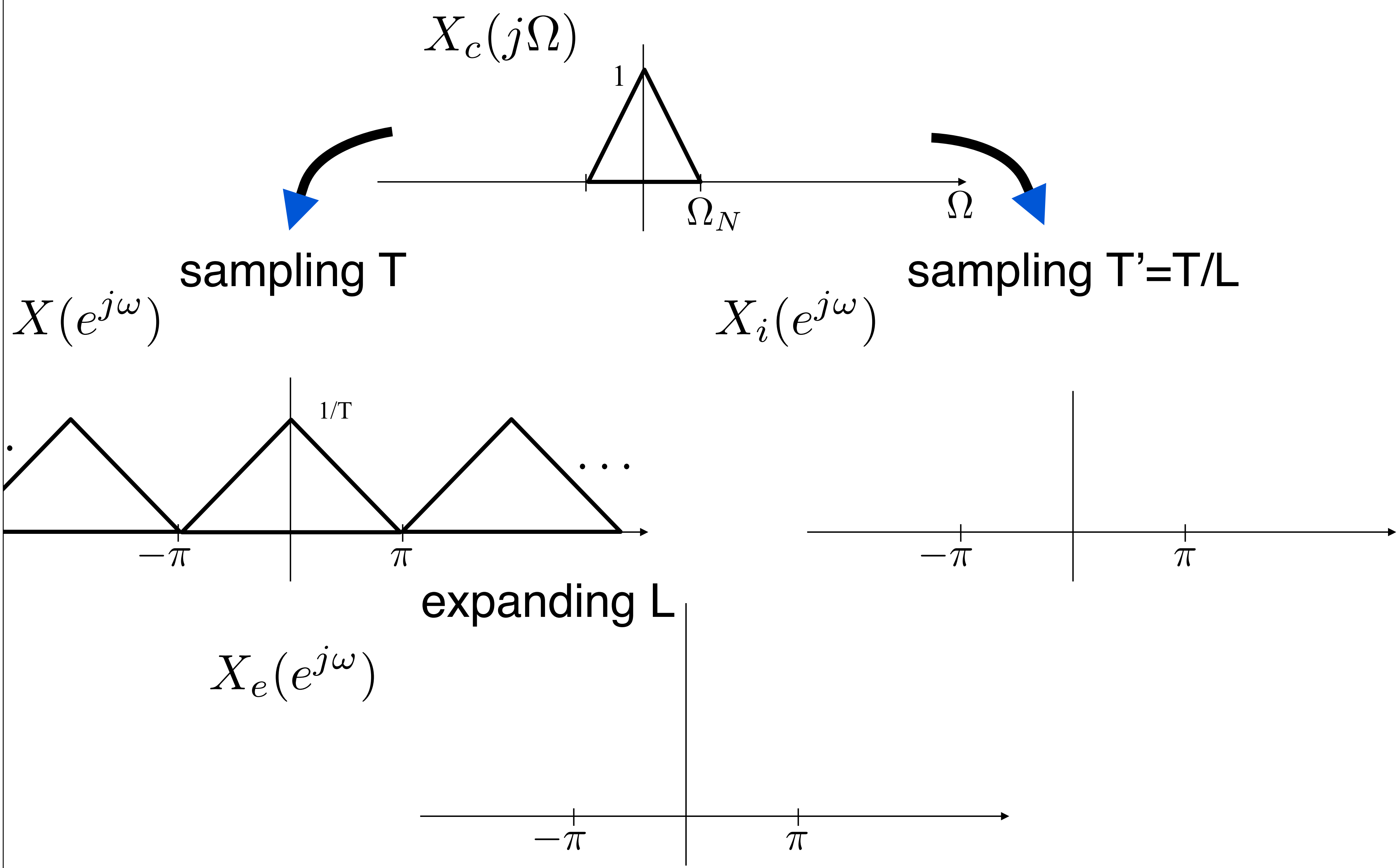
Compress DTFT by a factor of L!

Example:



$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
$X_i(e^{j\omega})$
$-\pi$
π
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example:


$X_c(j\Omega)$
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Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
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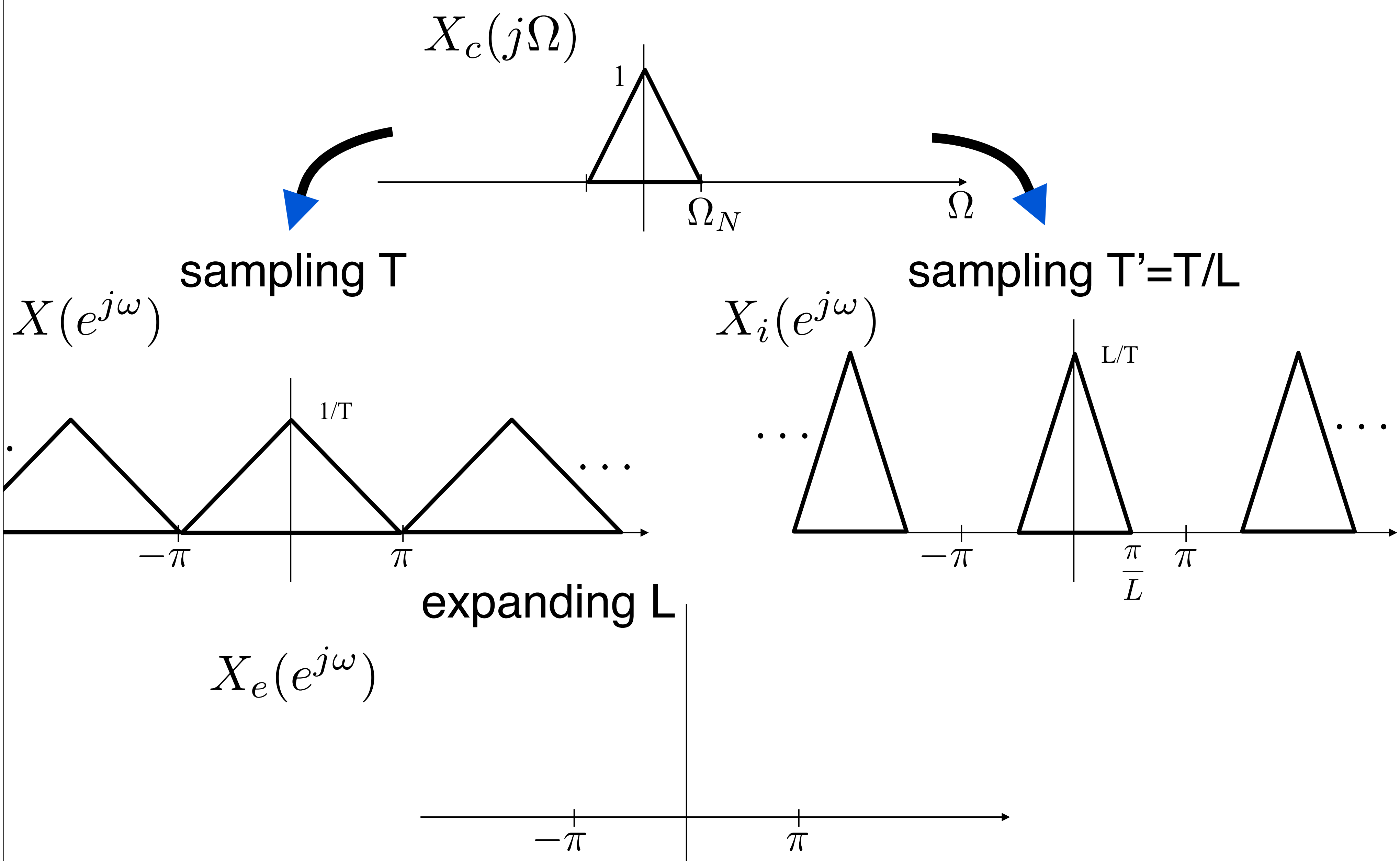
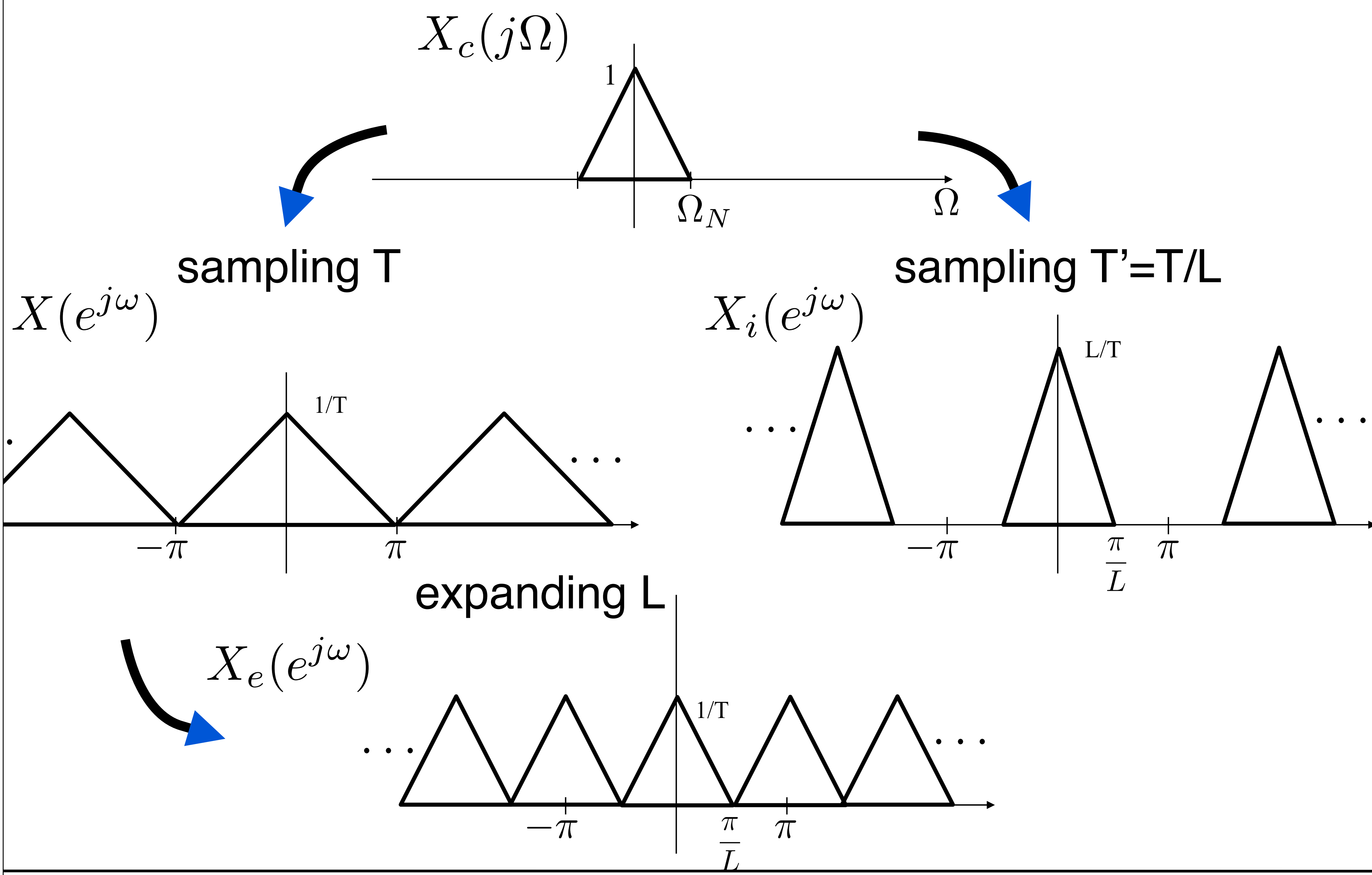
π
$X_i(e^{j\omega})$
expanding L
$X_e(e^{j\omega})$

Example:



$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$\frac{\pi}{L}$
expanding L
$X_e(e^{j\omega})$

Example:



$X_c(j\Omega)$
1
Ω_N
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sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
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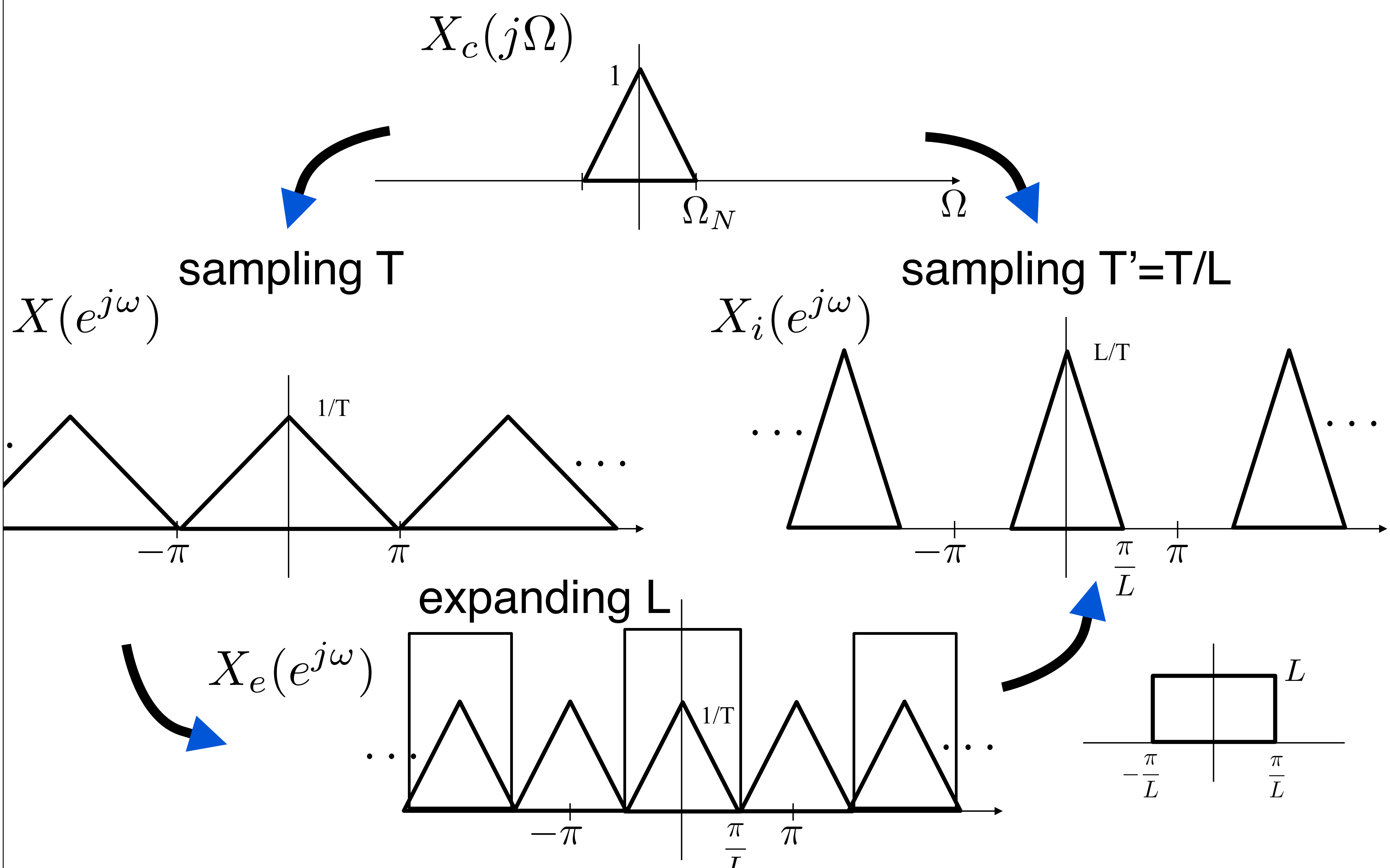
$X_e(e^{j\omega})$

Example:



$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
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1/T
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π
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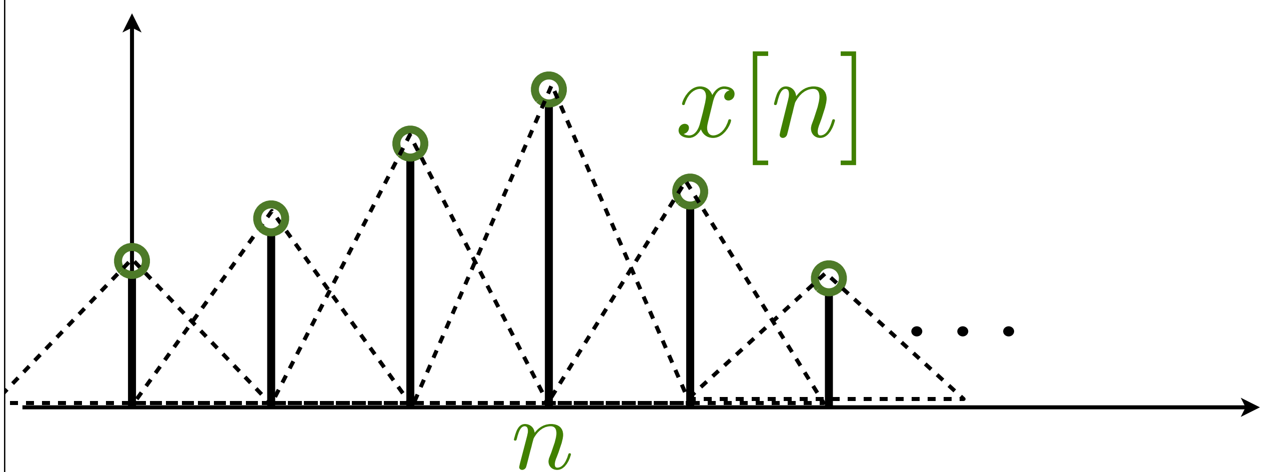
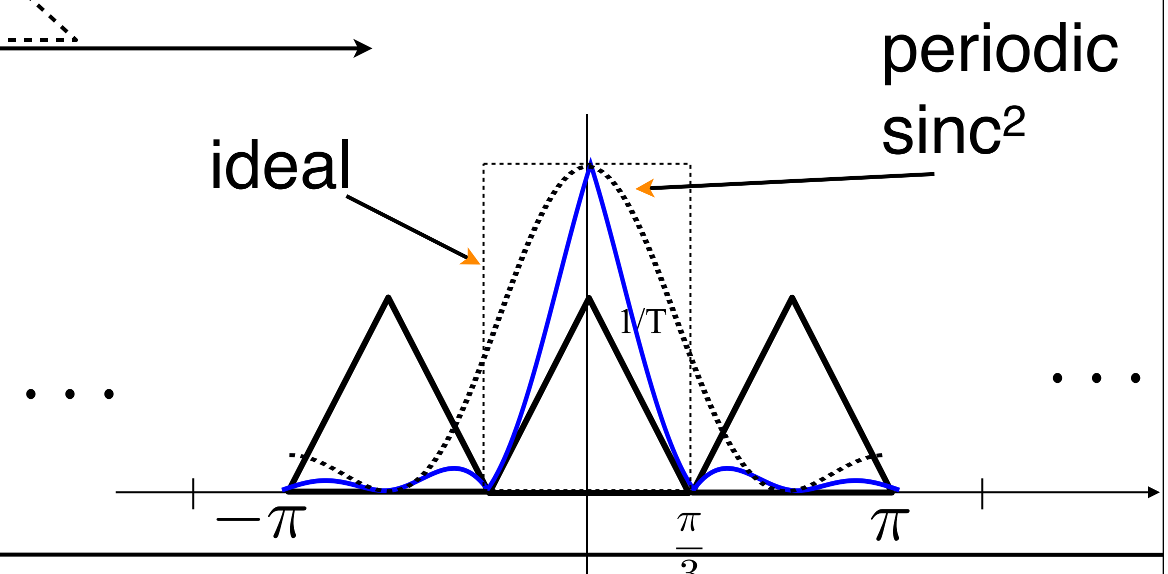
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Practical Upsampling

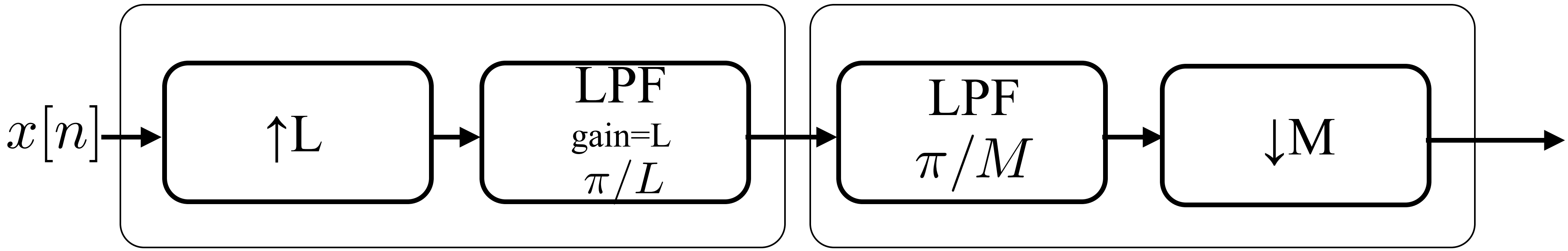
- Can interpolate with simple, practical filters. What's happening?
- Example: L=3, linear interpolation - convolve with triangle



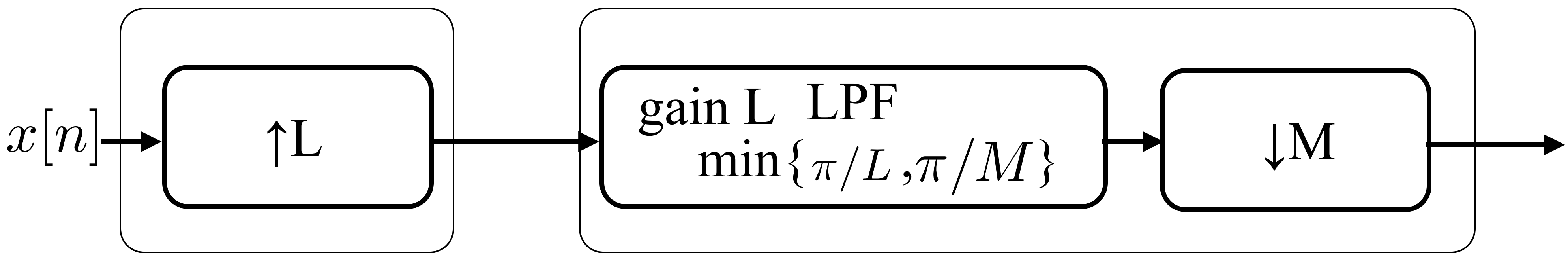


Resampling by non-integer

- T' = TM/L (upsample L, downsample M)



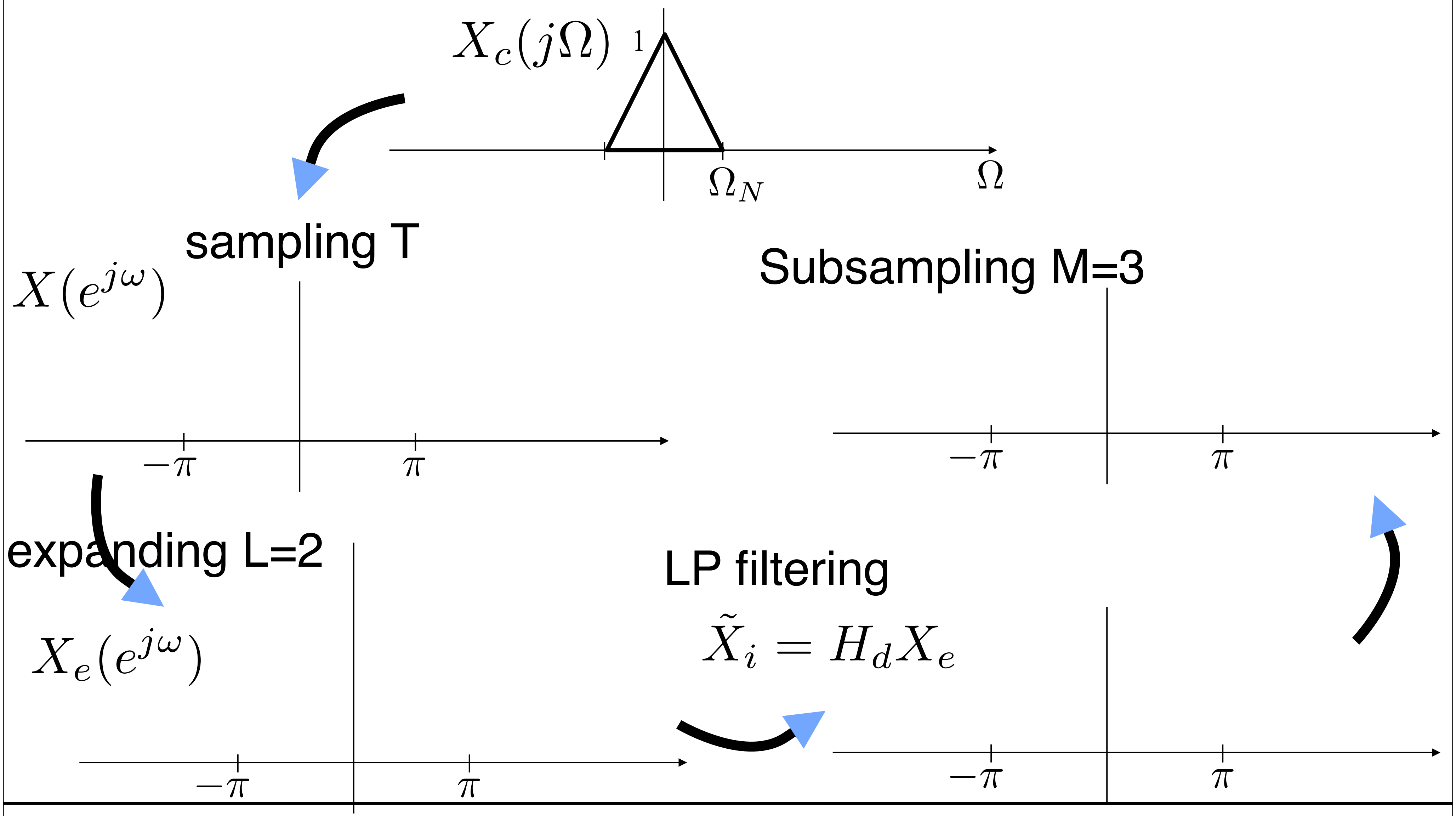
Or,



- What would happen if change order?

Example:

- L = 2, M=3,   T'=3/2T   (fig 4.30)



Example:

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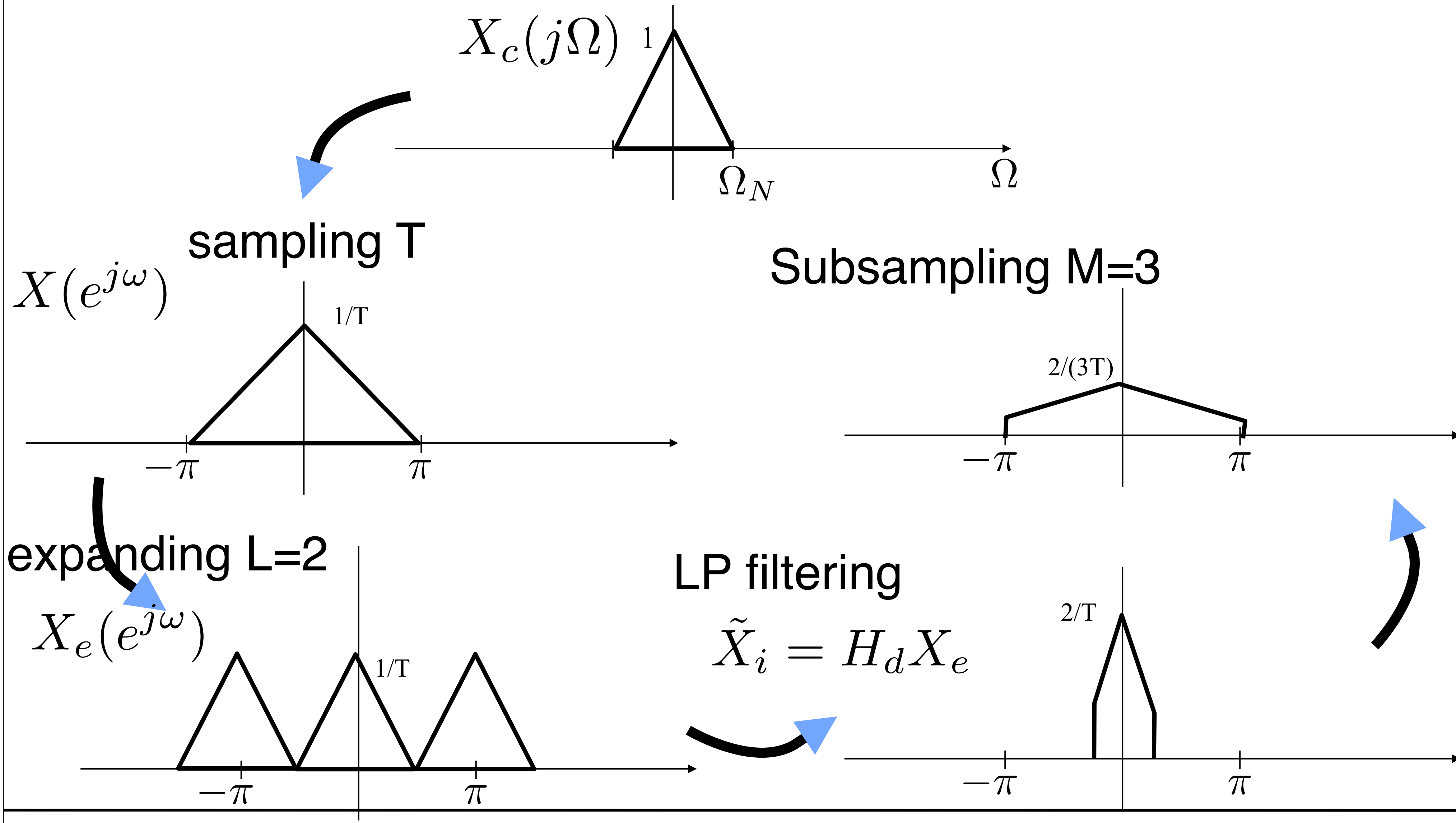

Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π/101 ($$$$$)
  - Downsample by M=101
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering