EE123
Digital Signal Processing

z-Transform

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Today

- Last time:
  - DTFT - Ch 2
- Today:
  - finish DTFT
  - Z-Transform briefly!
  - Ch. 3

- Don't forget -- ham lectures 6:30pm!

Somthing Fun

- goTenna
  - Text messaging radio
  - Bluetooth phone interface
  - MURS VHF radio (5chnnels)
  - 2W
  - 0.5-5 mile range
  - encryption
  - 2x100$

- Lab 6 implements a similar approach -- but without the slick system integration

Frequency Response of LTI Systems

Check response to a pure frequency:

$$e^{i\omega_0 n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \\ &= \left(\underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}}_{H\left(e^{j\omega}\right)\big|_{\omega=\omega_0}} \right) e^{j\omega_0 n} \end{aligned}$$

Frequency Response of LTI Systems

Check response to a pure frequency:

$$e^{i\omega_0 n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$H(e^{j\omega}) = \mathrm{DTFT}\{h[n]\}$$

$$y[n] = H\left(e^{j\omega}\right)\Big|_{\omega=\omega_0} e^{j\omega_0 n}$$

Output is the same pure frequency, scaled and phase-shifted!

$e^{j\omega_0 n}$ is an eigen function of LTI systems

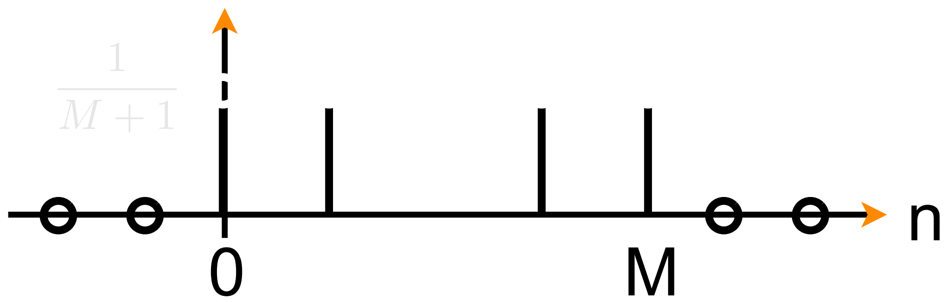
Recall eigen vectors satisfy: $A\nu = \lambda\nu$

Example 3

Frequency response of a causal moving average filter

$$y[n] = \frac{x[n-M] + \cdots + x[n]}{M+1}$$

Q: What type of filter is it? A: Low-Pass



$$h[n] = \frac{1}{M+1} w[n - \frac{M}{2}]$$

Example 3 Cont.

Frequency response of a causal moving average filter

$$h[n] = \frac{1}{M+1} w[n - \frac{M}{2}]$$

Same as example 1, only: Shifted by N, divided by M+1, M=2N

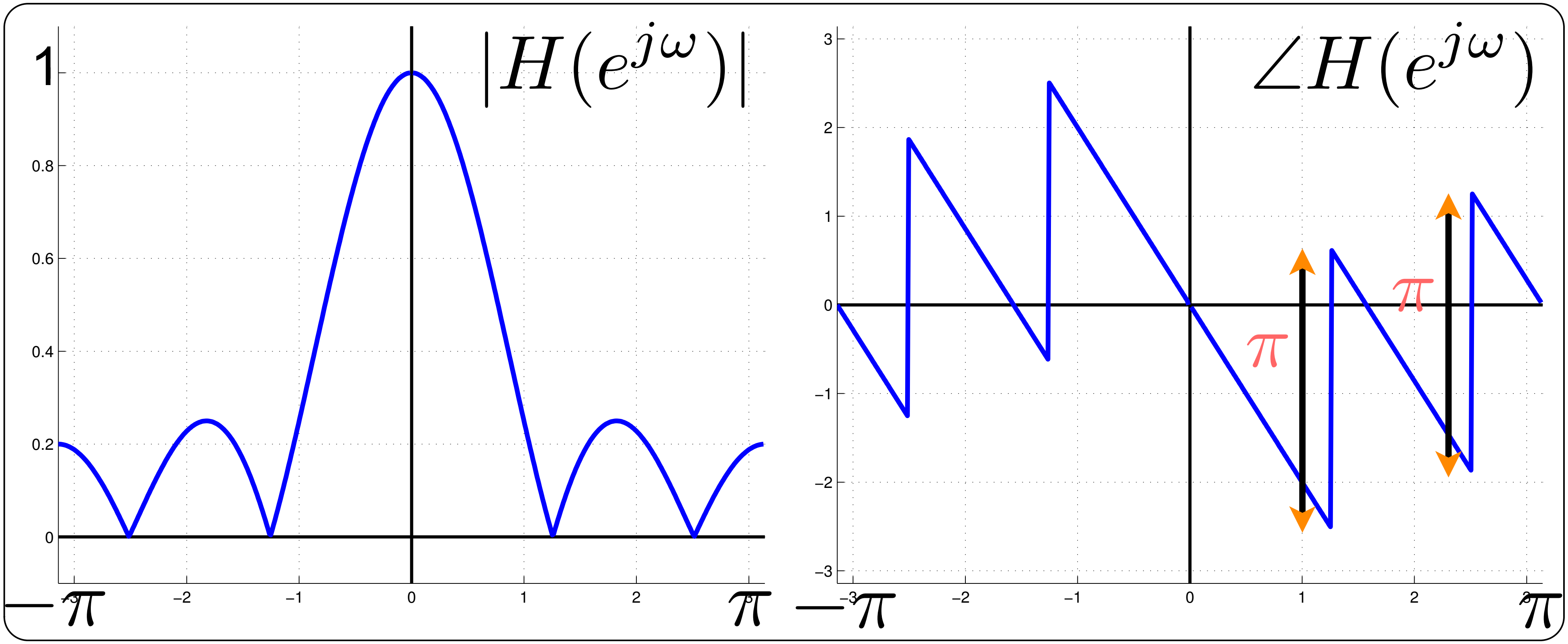
$$H(e^{j\omega}) = \frac{e^{-j\omega\frac{M}{2}}}{M+1} \cdot \frac{\sin\left((\frac{M}{2}+\frac{1}{2})\omega\right)}{\sin(\frac{\omega}{2})}$$

Example 3 Cont.

Frequency response of a causal moving average filter

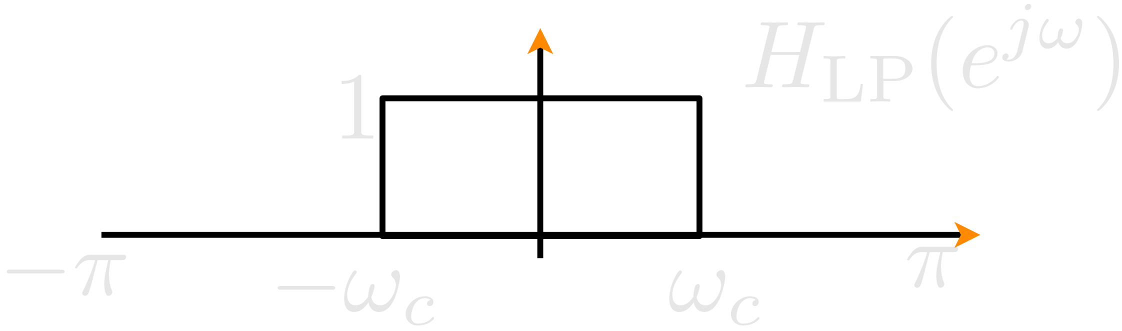
$$H(e^{j\omega}) = \frac{e^{-j\omega\frac{M}{2}}}{M+1} \cdot \frac{\sin\left((\frac{M}{2}+1)\omega\right)}{\sin(\frac{\omega}{2})}$$

Not a sinc!



Example 4:

Impulse Response of an Ideal Low-Pass Filter



$$
\begin{aligned}
h_{\mathrm{LP}}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{jw})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega
\end{aligned}
$$

Example 4

Impulse Response of an Ideal Low-Pass Filter

$$h_{\mathrm{LP}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{jw})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi jn}\boxed{e^{j\omega n}\Big|_{-\omega_c}^{\omega_c}} = 2j\sin(w_c n)$$

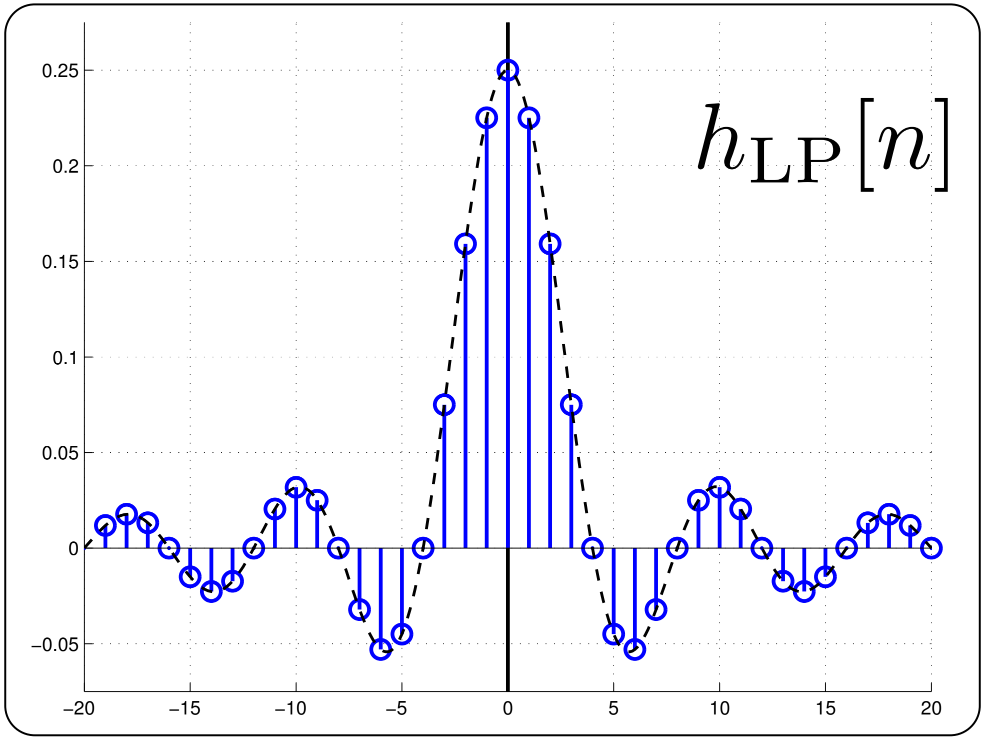
$$= \frac{\sin(w_c n)}{\pi n}$$

Example 4

Impulse Response of an Ideal Low-Pass Filter

$$h_{\mathrm{LP}}[n] = \frac{\sin(w_c n)}{\pi n}$$

sampled “sinc”



Non causal! <u>Truncate</u> and <u>shift right</u> to make causal

Example 4

Impulse Response of an Ideal Low-Pass Filter

Non causal! Truncate and shift right to make causal

How does it changes the frequency response?

Truncation:

$$\tilde{h}_{\mathrm{LP}}[n] = w_N[n] \cdot h_{\mathrm{LP}}[n]$$

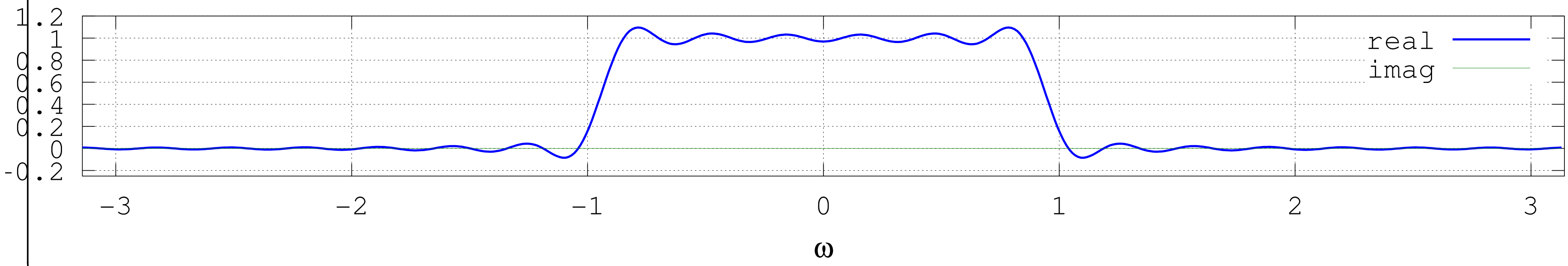
property 2.9.7:

$$\tilde{H}_{\mathrm{LP}}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

Periodic convolution

Example 4

We get "smearing" of the frequency response
We get rippling

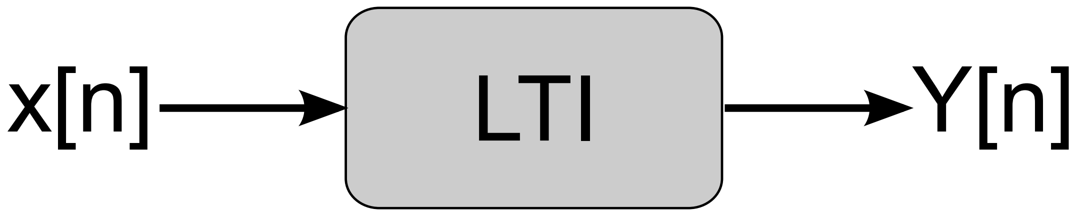


The z-Transform

- Used for:
  - Analysis of LTI systems
  - Solving difference equations
  - Determining system stability
  - Finding frequency response of stable systems

Eigen Functions of LTI Systems

- Consider an LTI system with impulse response h[n]:



- We already showed that $x[n] = e^{j\omega n}$ are eigen-functions
- What if $x[n] = z^n = re^{j\omega n}$

Eigen Functions of LTI Systems

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \\
&= \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n = H(z) z^n
\end{aligned}
$$

- $x[n] = z^n$ are also eigen-functions of LTI Systems
- $H(z)$ is called a transfer function
- $H(z)$ exists for larger class of $h[n]$ than $H(e^{j\omega})$

The z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Since $z = re^{j\omega}$

$$X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \mathcal{DTFT}\{x[n]\}$$

Region of Convergence (ROC)

- The ROC is a set of values of z for which the sum

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Converges.

Region of Convergence (ROC)

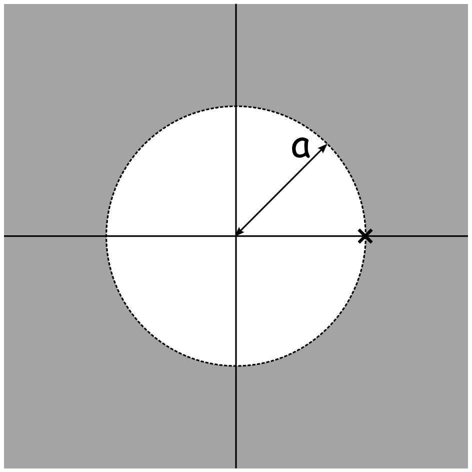
- Example 1: Right-sided sequence $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

recall:

$$1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{if} |x| < 1$$

So: $$X(z) = \frac{1}{1 - az^{-1}}, \qquad \text{ROC} = \{z : \ |z| > |a|\}$$

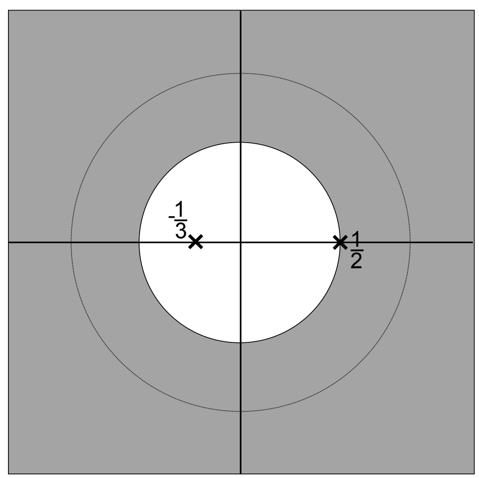


Region of Convergence (ROC)

- Example 2: $x[n] \;=\; \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$\begin{aligned} \mathrm{ROC} \quad &= \quad \{z: \;\; |z| > \frac{1}{2}\} \;\; \cap \;\; \{z: \;\; |z| > \frac{1}{3}\} \\ &= \quad \{z: \;\; |z| > \frac{1}{2}\} \end{aligned}$$



Region of Convergence (ROC)

- Example 3: Left sided sequence $x[n] \;=\; -a^n u[-n-1]$

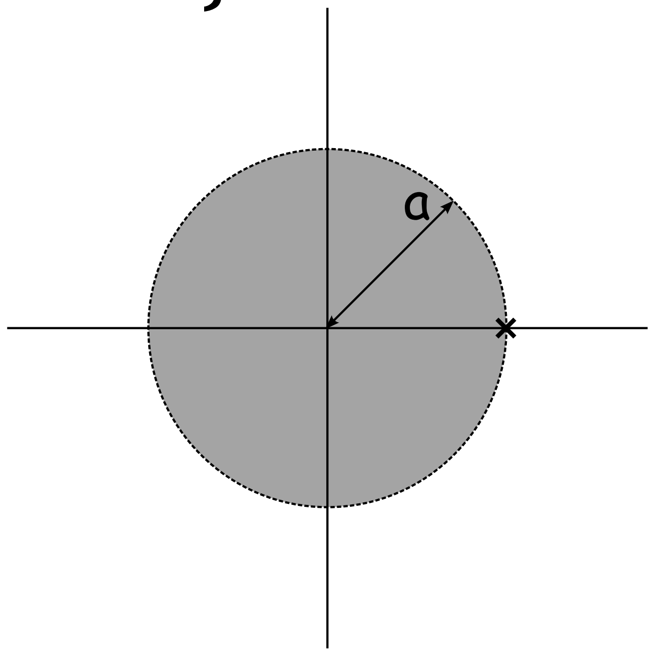
$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{m=1}^{\infty} -a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$

if $|a^{-1}z| < 1$, *i.e,* $|z| < |a|$ then,

$$\begin{aligned} X(z) &= 1 - \frac{1}{1-a^{-1}z} \\ &= \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}} \end{aligned}$$

Region of Convergence (ROC)

- Expression is the same as Example 1!
- ROC = $\{z: |z| < |a|\}$ is different



- The z-transform without ROC does not uniquely define a sequence!

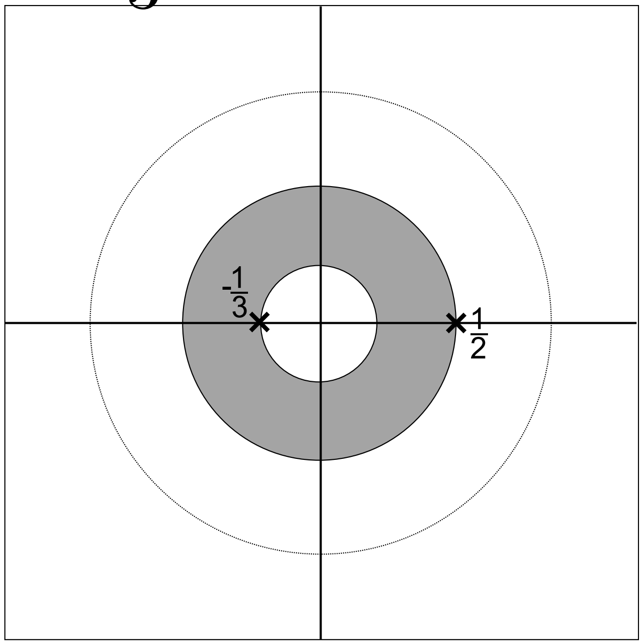
Region of Convergence (ROC)

- Example 4: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$

$$X(z) \quad = \quad \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$$

Same as example 2

$$\begin{aligned} \text{ROC} \quad &= \quad \{z: \quad |z| < \frac{1}{2}\} \quad \cap \quad \{z: \quad |z| > \frac{1}{3}\} \\ &= \quad \{z: \quad \frac{1}{3} < |z| < \frac{1}{2}\} \end{aligned}$$



Region of Convergence (ROC)

- Example 5: $x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$

$$\begin{aligned} \text{ROC} &= \{z: \ |z| > \frac{1}{2}\} \ \cap \ \{z: \ |z| < \frac{1}{3}\} \\ &= 0 \end{aligned}$$

- Example 6: $x[n] = a^n, \quad \text{two sided } a \neq 0$

$$\begin{aligned} \text{ROC} &= \{z: \ |z| > a\} \ \cap \ \{z: \ |z| < a\} \\ &= 0 \end{aligned}$$

Region of Convergence (ROC)

- Example 7: Finite sequence $x[n] = a^n u[n] u[-n+M-1]$

$$X[z] = \sum_{n=0}^{M-1} a^n z^{-n}$$ Finite, always converges

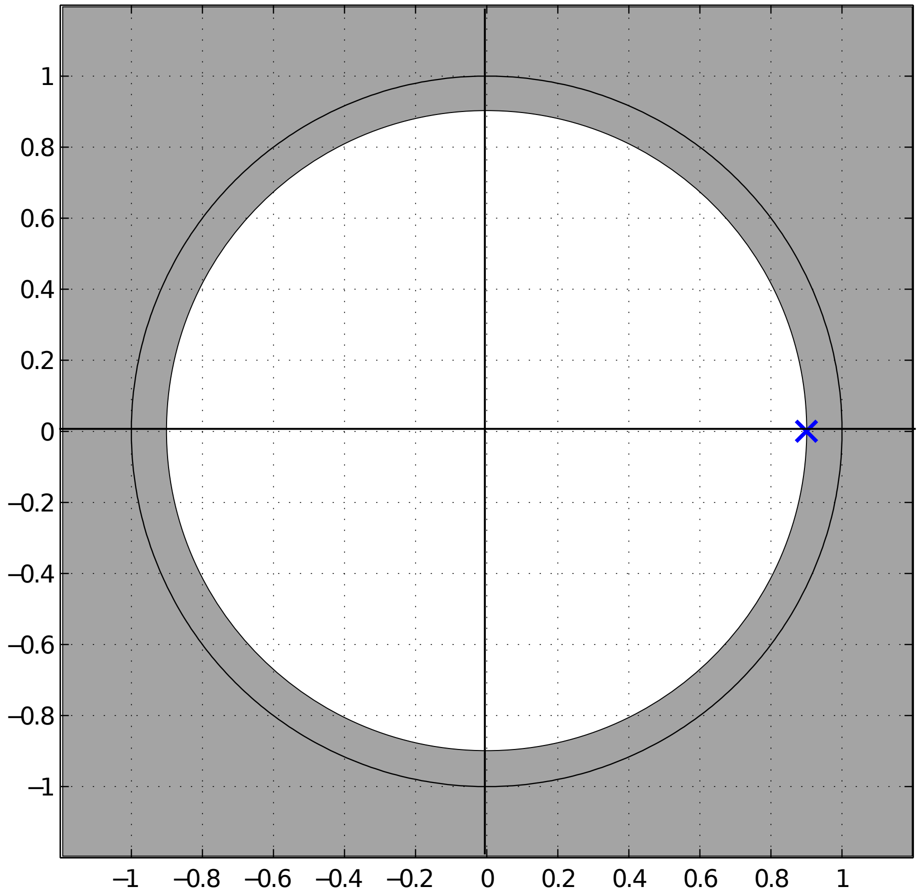
$$= \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$$ Zero cancels pole

$$= \prod_{k=1}^{M-1} \left(1 - a e^{j\frac{2\pi k}{M}} z^{-1}\right)$$

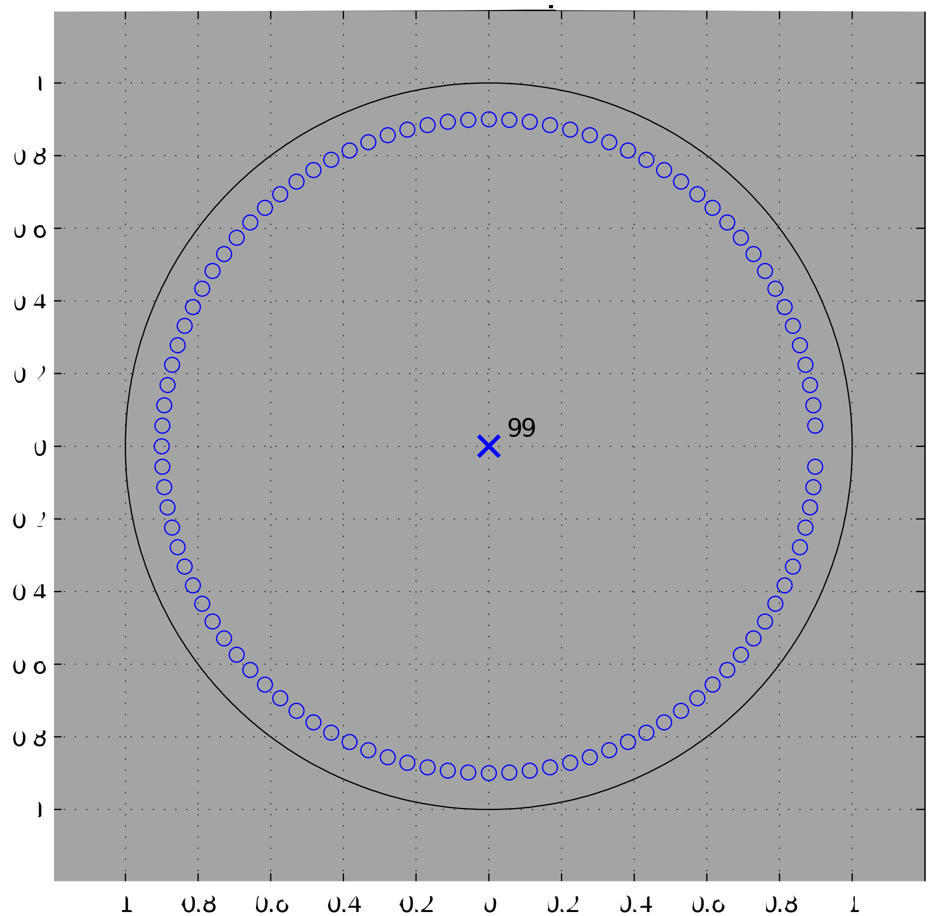
$$\mathrm{ROC} = \{z : \ |z| > 0\}$$

Region of Convergence (ROC)

Infinite sequence



finite sequence



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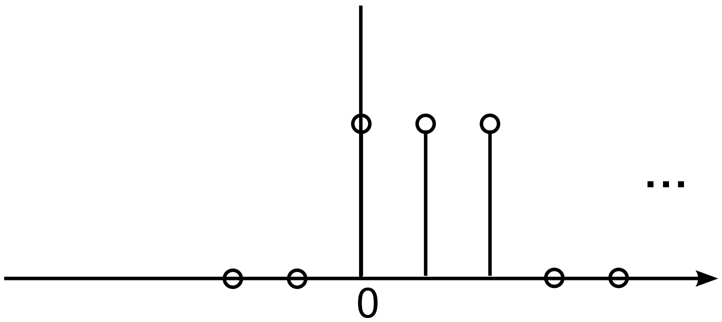
Properties of ROC

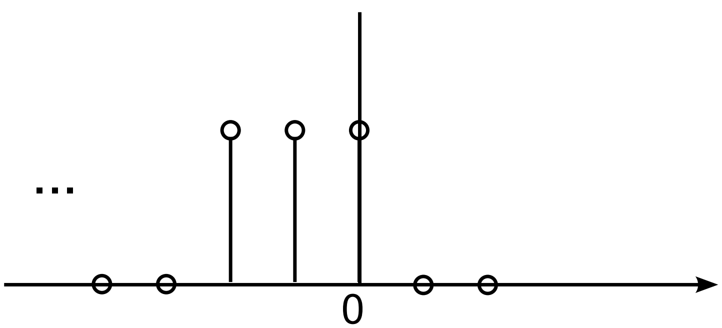
- A ring or a disk in Z-plane, centered at the origin
- DTFT converges iff ROC includes the unit circle
- ROC can’t contain poles

Properties of ROC

- For finite duration sequences, ROC is the entire z-plane, except possibly z=0, z=∞



$$X(z) = 1 + z^{-1} + z^{-2} \quad \text{ROC excludes } z = 0$$



$$X(z) = 1 + z^{1} + z^{2} \quad \text{ROC excludes } z = \infty$$

Properties of the ROC

- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  Examples 1,2
- For left-sided: inwards from inner most pole to zero
  Example 3
- For two-sided, ROC is a ring - or do not exist
  Examples 4,5,6

Several Properties of the Z-transform

$$x[n-n_d] \quad \leftrightarrow \quad z^{-n_d}X(z)$$

$$z_0^n x[n] \quad \leftrightarrow \quad X(\frac{z}{z_0})$$

$$nx[n] \quad \leftrightarrow \quad -z\frac{dX(z)}{dz}$$

$$x[-n] \quad \leftrightarrow \quad X(z^{-1})$$

$$x[n]*y[n] \quad \leftrightarrow \quad X(z)Y(z)$$

ROC at least ROCx ∩ ROCy

Inversion of the z-Transform

- In general, by contour integration within the ROC

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}$$

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Power series expansion
  - Partial fraction expansion
  - Residue theorem

- Most useful is the inverse of rational polynomials

$$X(z) = \frac{B(z)}{A(z)}$$

Why?