Assignment 4—by Erin Grimes, Dec. 2, 2019—MATH 451 Fall 2019—Western Washington University

1. Let $P = \{x, y, z\}$ be a Pythagorean triple, which is to say $x^2 + y^2 = z^2$ and $x, y, z \in \mathbb{N}$. Suppose P forms an arithmetic progression, which is to say, assuming $z > y > x, k \in \mathbb{N}$,
   $x^2 + (x + k)^2 = (x + 2k)^2$
   $\Leftrightarrow x^2 + x^2 + 2kx + k^2 = x^2 + 4kx + 4k^2$
   $\Leftrightarrow x^2 - 2kx - 3k^2 = 0$
   $\Leftrightarrow (x - 3k)(x + k) = 0$
   $\Leftrightarrow x = 3k \vee x = -k.$
   Since $x = -k$ contradicts to $x \in \mathbb{N}$, $x^2 + (x + k)^2 = (x + 2k)^2 \Leftrightarrow x = 3k$. If $k = 1$ then $(x, y, z) = (3, 4, 5)$. Therefore, P forms an arithmetic progression if and only if P is a positive multiple of $(3, 4, 5)$.■
2. Let $\mathbb{T} = \{(x, y, z) \in \mathbb{N}^3 : x^2 + y^2 = z^2\}$ and $r: \mathbb{T} \to \mathbb{N}, r(x, y, z) = r_{x,y,z}$ be the radius of the incircle of the Pythagorean triangle formed of side lengths x, y, z.
   a. Let $(a, b, c) \in \mathbb{T}$ and let $\triangle ABC$ be with side lengths a, b, c. By the converse of the Pythagorean theoem, $\triangle ABC$ is a right triangle; hence, let $A, B, C \in \mathbb{N}^2$, for $A = (b, 0), B = (0, a), C = (0, 0)$, where $c = \mathcal{L}(\overline{BC}) = a^2 + b^2$. Let $\mathcal{C}$ be the incircle of $\triangle ABC$ with radius $r_{a,b,c}$ and center O at the interior incenter of $\triangle ABC$, *i.e.* the intersection of the angle bisectors of the interior angles of $\triangle ABC$. By definition, $\mathcal{C}$ is tangent to each side of $\triangle ABC$. If $X = \mathcal{C} \cap \overline{AC}$ and $Y = \mathcal{C} \cap \overline{BC}$, then $OX \perp AC$ and $OY \perp BC$; hence, $OX \perp OY$. Then $XCYO$ is a square and $O = (r, r)$.
      The area $\mathcal{A}(\triangle ABC) = \mathcal{A}(\triangle AOC) + \mathcal{A}(\triangle BOC) + \mathcal{A}(\triangle AOB)$, hence $\frac{ab}{2} = \frac{br}{2} + \frac{ar}{2} + \frac{cr}{2}$, or equivalently,
      $$r_{a,b,c} = \frac{ab}{a + b + c}.$$
      It remains to be shown that $a + b + c \mid ab$. Suppose first that (a, b, c) is a primitive Pythagorean triple. Then there exist $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1, m > n$ and $(a, b, c) = (m^2 - n^2, 2mn, n^2 + m^2)$. The formula then becomes
      $$r_{a,b,c} = \frac{2mn(m^2 - n^2)}{2m^2 + 2mn} = n(m - n).$$
      Clearly, $r_{a,b,c} \in \mathbb{Z}$ and $m > n$ assures $r_{a,b,c} > 0$. Every Pythagorean triple is expressible as a multiple of a primitive Pythagorean triple, so suppose $(x, y, z) = k(a, b, c)$ for $k \in \mathbb{N}$. Then $\triangle ABC$ is similar to $\triangle XYZ$ and $r_{x,y,z} = kr_{a,b,c} = kn(m - n)$ by similarity of corresponding parts of similar figures. Equivalently, by setting kn for n, this is reduced to the previous formula by releasing the constraint that $\gcd(m, n) = 1$. Thus, $r_{x,y,z} \in \mathbb{N}$.
   b. It is shown that $r: \mathbb{T} \to \mathbb{N}$ is well defined. Since (x, y, z) in (a) was arbitrary, it follows that $r(x, y, z) \in \mathbb{N}$ for all $(x, y, z) \in \mathbb{T}$. Thus, r is a surjection from $\mathbb{T}$ to $\mathbb{N}$.
   c. Suppose $r_{x,y,z} = 2$, which is to say $\frac{xy}{x+y+z} = n(m - n) = 2$ for $m, n \in \mathbb{N}$ and (m, n) defined as usual. Since $2 = 1 \cdot 2$ is prime, exactly one of $n, m - n$ is equal to 2. There are two cases:
      Case $n = 2$: Then $m - 2 = 1 \Leftrightarrow m = 3$, which implies $(x, y, z) = (5, 12, 13)$.
      Case $n = 1$: Then $m - 1 = 2 \Leftrightarrow m = 3$, which implies $(x, y, z) = (8, 6, 10)$.
      Therefore, up to labeling, $r^{-1}(\{2\}) = \{(6, 8, 10), (5, 12, 13)\}$.

For a given positive integer $k \geq 2$, every element in the solution set $r^{-1}(\{k\})$ has in equal proportion the area and semiperimeter of its corresponding Pythagorean triangle. Moreover, this ratio is equal to k.

d. As seen from (b), the function T is onto. However, as seen from (c) the function T is not one to one, since two distinct elements of $\mathbb{T}$ map to the same element in $\mathbb{N}$. Therefore, T is a surjection but is not a bijection.■

3. Let $(x, y, y+1)$ be a primitive Pythagorean triple. This is to say $(x, y, y+1) = (m^2 - n^2, 2mn, m^2 + n^2)$. Then $m^2 + n^2 = 2mn + 1 \Leftrightarrow (m+n)^2 = 1 \Rightarrow m = n \pm 1$. If $m = n - 1$ then $x = (n-1)^2 - n^2 = -2n + 1$, which contradicts to $x \in \mathbb{N}$; hence, $m = n + 1$. To check, $(2n+1)^2 + \left(2n(n+1)\right)^2 = 4n^2 + 4n + 1 + 4n^2(n+1)^2 = 4n^2(n+1)^2 + 4n(n+1) + 1 = (2n(n+1)+1)^2$. Therefore, $(x, y, y+1) \in \{(2n+1, 2n(n+1), 2n(n+1)+1) : n \in \mathbb{N}\}$ if and only if $(x, y, y+1)$ is a primitive Pythagorean triple.■

4. Let $x^4 - y^4 = 2z^2$ and assume $(x, y, z) \in \mathbb{N}^3$ is least such.
As a preliminary, $x \not\equiv y \bmod 2 \Rightarrow x^4 - y^4 \equiv 1 \bmod 2$, which contradicts to $2z^2$ is even. Also, $x \equiv y \equiv 0 \pmod 2 \Rightarrow x^4 \equiv y^4 \equiv x^4 - y^4 = 2z^2 \equiv 0 \pmod{16} \Rightarrow 8 \mid z^2 \Rightarrow z$ is even, which means $(x/2, y/2, z/2)$ is also a less solution, a contradiction. Thus, x, y are odd.
Now let us suppose $\gcd(x, y) = d > 1$. For $x = dx'$ and $y = dy'$, $(dx')^4 - (dy')^4 = d^4((x')^4 - (y')^4) = 2z^2 \Rightarrow d^4 \mid 2z^2 \Rightarrow d^4 \mid z^2$ since d is odd, hence $d \mid z$. So for $z = dz'$, $(dx')^4 - (dy')^4 = 2(dz')^4 \Rightarrow (x', y', z')$ is a solution and $z' < z$, which contradicts to (x, y, z) is least solution. Thus, consider only when $\gcd(x, y) = 1$.

Lemma: $\gcd(x, y) = 1 \wedge x, y$ odd $\Rightarrow \gcd(x^2 + y^2, x^2 - y^2) = 2$. *Proof*: Suppose $\gcd(x, y) = 1$ and let $d = \gcd(x^2 + y^2, x^2 - y^2)$. Assume x, y odd so let $d = 2c$ for $c \in \mathbb{N}$. Then $x^2 + y^2 \equiv x^2 - y^2 \bmod d \Leftrightarrow 2y^2 \equiv 0 \bmod d \Leftrightarrow y^2 \equiv 0 \bmod c \Rightarrow x^2 + y^2 \equiv x^2 \equiv 0 \bmod c \Rightarrow c \mid x, y \Rightarrow c = 1$.

Now $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = 2k(x^2 - y^2) = 2z^2 \Leftrightarrow k(x^2 - y^2) = z^2 \Rightarrow k, x - y^2$ are perfect squares since they are relatively prime, by the lemma. In particular, let $x^2 - y^2 = c^2$, which is to say (y, c, x) are a primitive Pythagorean triple with c even, since $\gcd(x, y) = \gcd(x, y, c) = 1$, which is to say $x = m^2 + n^2$ and $y = m^2 - n^2$ for $m, n \in \mathbb{N}$. Then by substitution, $x^2 + y^2 = (m^2 + n^2) + (m^2 - n^2) = 2(m^4 + n^4)$ and $k = m^4 + n^4$, which is not a perfect square by Fermat's theorem. Therefore, there are no integer solutions.■