Problem 1. When speaking of rings of polynomials over an infinite field k, it is useful to recall the fact that $\forall x \in k : f(x) = 0 \Leftrightarrow f = 0 \in k[x]$. This fact also applies to rings of polynomials in multiple variables. The Division Algorithm for polynomials in a single variable is explained in (Proposition 1.5.2), analogous to the Euclidean algorithm for integers, based on the fact that for any $f(x), g(x) \neq 0 \in k[x]$ there exist unique $q(x), r(x) \in k[x]$ such that $f = qg + r$, and $r = 0$ or $\deg r < \deg g$, a corollary of which states that every ideal $I \lhd k[x]$ is principally generated, i.e. $I = \langle f(x) \rangle$ for some $f(x) \in k[x]$. This does not hold in general over the ring of polynomials in multiple variables, $k[x_1, ..., x_n]$ where $n \geq 2$: for example, consider the ideal $I = \langle x, y \rangle \in k[x, y]$. Suppose $I = \langle f(x, y) \rangle$ for some $f(x, y) \in k[x, y]$. Then by rearranging terms, $f(x, y) = \sum_{i=1}^{m} g_i(y) x^i$, this is for fixed $a \in k$, $g(x) = f(x, a) \in k[x]$. Then $\deg g \leq 1$ by additivity of degree since $x \in I$. Similarly, $\deg h \leq 1$ where $h(y) = f(a, y) \in k[y]$. It follows that $f = ax + by + c$ for some $a, b, c \in k$. Then $x = (ax + by + c)p(x) \Rightarrow p(x) = 1, a = 1, b = c = 0$, but $y = (ax + by + c)q(x) \Rightarrow q(x) = 1, b = 1, a = c = 0$, a contradiction.

Moreover, the same division algorithm does not hold in general for such rings, in part because the notion of degree of a multivariate polynomial is not defined. For such rings however, there exists a similar notion of multidegree but this relies on a fixed monomial ordering. Thus the results of the division algorithm for polynomials over multiple variables must in general be expressed in terms of the corresponding monomial ordering. The second statement, for $I = \langle g \rangle \subseteq k[x_1, ..., x_n]$, $f \in k[x_1, ..., x_n]$ then $f \in I \Leftrightarrow \exists h \in k[x_1, ..., x_n] : f = gh$ holds for such principally generated ideals of polynomial rings of multiple variables. The ($\Leftarrow$) implication follows directly from (Definition 1.4.1.iii). The ($\Rightarrow$) implication follows from (Definition 1.4.2) in the case where $s = 1$.

Problem 2. Let $I = \langle f_1, f_2 \rangle \lhd k[x, y, z]$ where $f_1 = yz - x$, $f_2 = xy + 2z^2$.

(a) Using Buchberger's Algorithm (Theorem 2.7.2) with graded lex monomial ordering. As in (Definition 2.6.4), the S-polynomial

$$S(f_1, f_2) = \frac{x^{\gamma_1} y^{\gamma_2} z^{\gamma_3}}{\mathrm{LT}(f_1)} \cdot f_1 - \frac{x^{\gamma_1} y^{\gamma_2} z^{\gamma_3}}{\mathrm{LT}(f_2)} \cdot f_2$$

where $\gamma_i = \max \{\text{multideg}(f_1)_i, \text{multideg}(f_2)_i\}$; $\text{multideg}(f_1) = (0, 1, 1)$ and $\text{multideg}(f_2) = (1, 1, 0)$ hence $\gamma = (1, 1, 1)$ so

$$\begin{aligned} f_3 = S(f_1, f_2) &= xyz \left(\frac{yz - x}{yz} - \frac{xy + 2z^2}{xy} \right) = xyz \left(1 - \frac{x}{yz} - 1 - \frac{2z^2}{xy} \right) = xyz \frac{-x^2 - 2z^3}{xyz} \\ &= -x^2 - 2z^3. \end{aligned}$$

Using the division algorithm, $xy, yz \nmid -x^2, -z^3$ hence $\overline{S(f_1, f_2)}^F = f_3 \neq 0$. Here, F is the current list of generators. Thus, add f_3 to the list of generators. Repeating this process with the new list of generators,

$$S(f_1, f_3) = x^2 y \left(1 + \frac{2z^2}{xy} - 1 + \frac{2z^3}{x^2} \right) = 2xz^2 + 2yz^3,$$

$$S(f_2, f_3) = x^2yz\left(1 - \frac{x}{yz} - 1 + \frac{2z^3}{x^2}\right) = 2yz^4 - x^3.$$

Using the division algorithm, $2yz^3 + 2z^2x = 2z^2(yz - x) \Rightarrow \overline{S(f_1, f_3)}^F = 0$, $2yz^4 - x^3 = 2z^3(yz - x) + x(2z^3 - x^3) = 2z^3f_2 + xf_3 \Rightarrow \overline{S(f_2, f_3)}^F = 0$. Since there are no other combintions, the algorithm terminates. For reduction, multiply f_3 by the nonzero constant -1. There are no other reductions since every generator is irreducible in $k[x, y, z]$ (by inspection).
(b) By (Definition 2.5.5), since

$$G = \{g_1, g_2, g_3\} = \left\{yz - x, xy + 2z^2, 2z^3 + x^2\right\}$$

is a Grobner basis, $\langle \text{LT}(g_1), \text{LT}(g_2), \text{LT}(g_3)\rangle = \langle yz, xy, 2z^3\rangle = \langle LT(I)\rangle$ defined with graded lex ordering. Suppose x^3z^2 is the leading monomial of some polynomial $f(x, y, z) \in I$. Then for some $c \in k$, $cx^3z^2 = \text{LT}(f(x, y, z))$ for some $f(x, y, z) \in k[x, y, z]$. It follows that at least one of $yz, xy, 2z^3$ divides x^3z^2. But of course, $y \nmid x^3z^2 \Rightarrow x^3z^2 \Rightarrow yz, xy \nmid x^3z^2$, and $z^3 \nmid x^3z^2$, a contradiction. Therefore, x^3z^2 is not the leading monomial of any element of I.

Problem 3. Let $f_1 = y - xz$, $f_2 = xy - 2z^2$, $f_3 = x - yz$, $I = \langle f_1, f_2, f_3\rangle$.
(a) By Macaulay calculation, I has Grobner basis $G = \{g_1, g_2, g_3, g_4\}$ where $g_1 = z^4 - x^2$, $g_2 = yz^2 - y$, $g_3 = y^2 - 2z^3$, $g_4 = x - yz$. Then in reference to the Elimination Theorem, $I_1 = I \cap k[y, z] = \langle g_1, g_2, g_3\rangle$ and $I_2 = I \cap k[z] = \langle g_1\rangle$.
(b) Suppose $(x, y, z) \in \mathbf{V}(I)$ in $\mathbb{R}^3$. Then in particular, (x, y, z) solves g_1 so $z \in \{0, \pm1\}$. If $z = 0$ then $y = 0$ by f_1 and $x = 0$ by f_3. If $z = 1$ then $x = y\,(\because f_3) \Rightarrow x^2 = 2z^2 = 2 \Rightarrow x = y = \pm\sqrt{2}$. If $z = -1$ then $x = -y\,(\because f_3) \Rightarrow -x^2 = -2z^2 = -2 \Rightarrow x = y = \pm\sqrt{2}$. Therefore the five solutions in $\mathbb{R}^3$ are $(0, 0, 0), (1, \sqrt{2}, \sqrt{2}), (1, -\sqrt{2}, -\sqrt{2}), (-1, \sqrt{2}, \sqrt{2}), (-1, -\sqrt{2}, -\sqrt{2})$.

Problem 4. Let Let $F : k^2 \to k^3$ defined $F(u, v) = (f_1(u, v), f_2(u, v), f_2(u, v))$, $x = f_1(u, v) = uv$, $y = f_2(u, v) = u + v^2$, $z = f_3(u, v) = u^2$. We seek the implicitization problem of (Theorem 3.1.1). Consider the ideal $I = \langle x - f_1, y - f_2, z - f_3\rangle$, which by Macaulay calculation using lex ordering with $u > v > x > y > z$ has Grobnem basis $\{g_1, ...g_8\}$ where

$$\begin{aligned}
g_1 &= x^4 - 2x^2yz + y^2z^2 - z^3,\\
g_2 &= vz^2 + x^3 - xyz,\\
g_3 &= vxz + x^2y - y^2z + z^2,\\
g_4 &= vx^2 - vyz + xyz,\\
g_5 &= v^2z - x^2,\\
g_6 &= v^2y + vx - y^2 - xy,\\
g_7 &= v^2x + vz - xy,\\
g_8 &= v^3 - vy + x.
\end{aligned}$$

Then by the Elimination Theorem, $I_2 = I \cap k[x, y, z] = \langle g_1\rangle$ thus $V = \mathbf{V}(I_2)$ is the smallest variety containing S.
(b) Suppose (x, y, z) solves g_1 over $\mathbb{C}$. It suffices to check whether this partial solutions

extends to $\mathbf{V}(I)$. Note $I_1 = I \cap k[v, x, y, z] = I$ and I contains a generator g_8 with a constant leading coefficient not divisible by x, y, or z and there are no terms involving u in the generators of I, thus every solution extends over $\mathbb{C}$. However, over $\mathbb{R}$ this is not the case: for example $(0, i, -1) \in S$ over $\mathbb{C}$ but $(0, i, -1) \in V \setminus S$ over $\mathbb{R}$.