This course has introduced several knot and link invariants, among them the Alexander polynomial. This paper will introduce three more polynomial link invariants: the Conway-normalized Alexander polynomial, the Kauffman polynomial, and the HOMFLY polynomial. Some relationships between these and other known invariants will be discussed, and some patterns pertaining to their definitions established, particularly in the interest of so-called skein relations.

The Alexander polynomial $\Delta_L(t)$ of a link L has been defined according to Seifert matrices and the Alexander matrix. Now another definition may be explored in the form of skein relations, and in the process develop a canonical form of the Alexander polynomial. Alexander wrote down the relationship concerning diagrams with only a single change in crossing. Suppose diagrams D_+ and D_- differ from a link diagram D_s only in the neighborhood of a single crossing, wherein they resemble (Figure1). Then there exists a relation

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (1 - t)\Delta_{L_s}(t).$$

This is known as a skein relation. Intuitively, these coefficients are the familiar entries of a

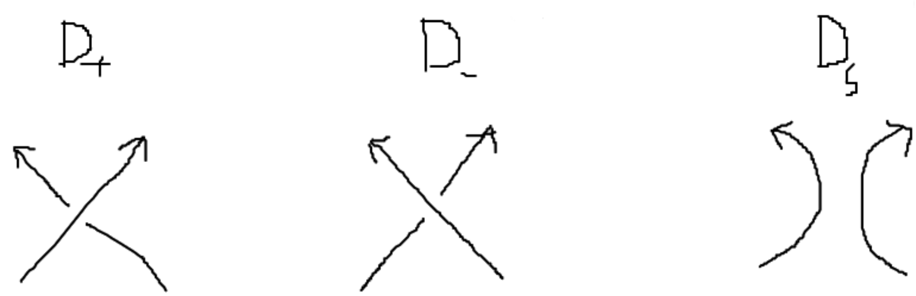

Figure 1: Conway-HOMFLY smoothing

single row of the Alexander matrix of D_+; this relation corresponds to the expression of the Alexander matrix determinant as some linear combination of the minor determinants in the entries of the row. These minor determinants are the results of the computation of one of these changed crossing diagrams D_- and D_s. Note that there is no other means of changing any one crossing which maintains the same orientation as the original diagram.

This observation is motivational: as some recurrence relation exists, it might indicate some recursive algorithm for computing the Alexander polynomial. It would be necessary to show that this does never result in an infinite loop, which is to say there is some strictly decreasing computational complexity with each iteration. This problem motivates the following definitions.

Definition 0.1. For any link L denote the number of components of L by $\#L$.

Definition 0.2. For any link diagram D, define $c^*(D)$ the *crossing number* of D the number of crossings of D.

Definition 0.3. For any link diagram D, define $u^*(D)$ the *uncrossing number* of D the number of crossings needed to change for an D to become an unlink diagram.

Definition 0.4. For any link diagram D, define the *complexity* $\chi(D) = (u^*(D), c^*(D))$. Define the relation $(<)$ on the set of complexities of link diagrams by lexicographic ordering.

The following Proposition makes this argument rigorous.

Proposition 1. For any link diagram, both changing and smoothing a single crossing results in a diagram of smaller complexity. Moreover, there exists no infinite sequence $\chi_1 > \chi_2 > \chi_3 > ...$ of complexities.

Proof. Let L be a link. Recall the crossing index $c(L)$ the minimum number of crossings in a diagram of L taken over all possible link diagrams, and the uncrossing number $u(L)$ the minimum number of crossing changes for a diagram of L to be changed into an unlink diagram, taken over all possible link diagrams, are finite. Thus by definition, there exists a diagram D of L with complexity $\chi(D) = (c^*(D), u^*(D)) = (c(L), u(L))$. Assume $u(L) \neq 0$. By some labeling of crossings, D may be changed into an unlink diagram by changing each of the crossings $U_D = \left\{x_1, ..., x_{u^*(D)}\right\} \subseteq \left\{x_1, ..., x_{c^*(D)}\right\} = C_D$ exactly once. Consider a single crossing x_1 of D. Let D_- and D_s be diagrams differing from D only in the neighborhood of this single crossing, wherein they look like (Figure1). Then by inspection, $u^*(D_-) = u^*(D)$ but $c^*(D_-) = c^*(D) - 1$. On the other hand, $c^*(D_s) = c^*(D)$ but $u^*(D_s) = u^*(D) - 1$. It follows that $\chi(D_-), \chi(D_s) < \chi(D)$. Since $x_1 \in U \subseteq C$ was arbitrary, and thus $|U_{D_s}| = |U_D \setminus \{x_1\}| = |U_D| - 1$ and $\left|U_{D_-}\right| = |U_D \setminus \{x_1\}| = |U_D| - 1$, this process may be repeated at most $c^*(D) + u^*(D)$ times. Since $c^*(D) = c(L)$ and $u^*(D) = u(L)$ are finite for arbitrary L, it follows that there exists no infinite sequence of decreasing complexitites. $\square$

It is shown that such an algorithm eventually terminates, if it exists. However, there is one more condition which is necessary for consistent results. Until now, the Alexander polynomial $\Delta_K(t)$ has only been defined up to a unit multiple of a power of t. Thus, the skein relation is not defined without specific consideration of the degree of t. A convenient solution is found in the Conway-normalized Alexander polynomial.

Definition 0.5. Recall for any link L the Alexander polynomial $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1}]$ is expressible in the form $\sum_{k=1}^{n} a_k(t^k + t^{-k})$. Define the *Conway polynomial*, *Conway-normalized Alexander polynomial*, or *potential function* $\nabla_K(z)$ of L to be the Alexander polynomial in this form with the substitution $z = t^{1/2} - t^{-1/2}$.

One may observe in special cases why this normalization simplifies the problem. Consider the following examples of Alexander polynomials and their respective Conway-normalizations.

Example 1. (Problem 10.1.1) The Conway-normalized Alexander polynomial $\nabla_K(z)$ of a knot K may be obtained from the Alexander polynomial $\Delta_K(t)$ by the substitution $z =$

$t^{1/2} - t^{-1/2}$. For various such K,
(a)

$$\begin{aligned}\Delta_{6_1}(t) &= -2t - 2t^{-1} + 5\\ &= 1 - 2t + 4 - 2t^{-1}\\ &= 1 - 2(t - 2 + t^{-1})\\ &= 1 - 2(t^{1/2} - t^{-1/2})^2\\ &= 1 - 2z^2\\ &= \nabla_{6_1}(z).\end{aligned}$$

(b)

$$\begin{aligned}\Delta_{8_3}(t) &= -4t - 4t^{-1} + 9\\ &= 1 - 4t + 8 - 4t^{-1}\\ &= 1 - 4(t - 2 + t^{-1})\\ &= 1 - 4(t^{1/2} - t^{-1/2})^2\\ &= 1 - 4z^2\\ &= \nabla_{8_3}(z).\end{aligned}$$

(c)

$$\begin{aligned}\Delta_{8_{18}}(t) &= t^2 - t^{-3} + 5t^2 + 5t^{-2} + 10t - 10t^{-1} + 13\\ &= t^3 - t^{-3} - 6t^2 - 6t^{-2} + 9t - 9t^{-1} + 4 - t^2 - t^{-2} + 4t - 4t^{-1} + t - 2 + t^{-1} + 1\\ &= (t - 2 - t^{-1})^3 - (t - 2 - t^{-1})^2 + (t - 2 + t^{-1})^2 + 1\\ &= (t^{1/2} - t^{-1/2})^6 - (t^{1/2} - t^{-1/2})^4 + (t^{1/2} - t^{-1/2}) + 1\\ &= z^6 - z^4 + z^2 + 1\\ &= \nabla_{8_{18}}(z).\end{aligned}$$

In each case, the terms of these $\Delta_K(t)$ may be grouped in pairs of powers of t. Excluding the constant term, a given $\Delta_K(t)$ has coefficients of t and t^{-1} equal up to a unit multiple–this being for some suitable multiplier $\pm t^p$ where p is an integer. This choice is suitable precisely when $\Delta_K(t) = \Delta_K(t^{-1})$. Then after such normalization, the skein relation under the same change of variables requires

$$\begin{aligned}&\Delta_{L_+}(t) - \Delta_{L_-}(t) = (1 - t)\Delta_{L_s}(t)\\ \Leftrightarrow\ &\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_s}(t)\\ \Leftrightarrow\ &\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_s}(z).\end{aligned}$$

This is the skein relation for the Conway-normalized Alexander polynomial. In computational practice, one may observe that the unknot U has known Alexander polynomial 1, which does not change under Conway-normalization except to fix the sign and the degree. The computation process is made simpler in light of the following Proposition.

Proposition 2. For an oriented link L, the Conway-normalized Alexander polynomial may be written in the form $\nabla_L(z) = \sum_{t\geq 0} a_i(L)z^i$ and has the following properties.
(1) If L is a split link, then $\nabla_L(z) = 0$.
(2) $a_i(L) = 0$ for all $i \equiv \#L \bmod (2)$ and also for $i < \#L - 1$.
(3) $\#L = 1 \Rightarrow a_0(L) = 1$.
(4) If $\#L = 2$ then $a_1(L) = \ell k(L)$, where $\ell k(L)$ is the linking number of the two components of L.
(5) If L_+, L_-, and L_0 have diagrams which are related in the manner of (Figure1) and $\#L = \#L_- = 1$, then $a_2(L_+) - a_2(L_-) = \ell k(L_0)$.

Proof of this proposition is given in (Lickorish 83). In general, it is not necessarily more efficient to calculate the Alexander polyonmial in this manner: the naive approach requires the computational complexity of evaluating $\mathcal{O}(2^n)$ state diagrams, each of which are a nontrivial storage task. However, as will be seen there is theoretical use in considering skein relations such as these in coming up with further knot invariants.
The Kauffman polynomial is one such invariant. Note that the Kauffman polynomial is only defined on framed links, and not on general links. In order to define the Kauffman polynomial, it is first necessary to define the Λ polynomial, which is defined for link diagrams rather than links.

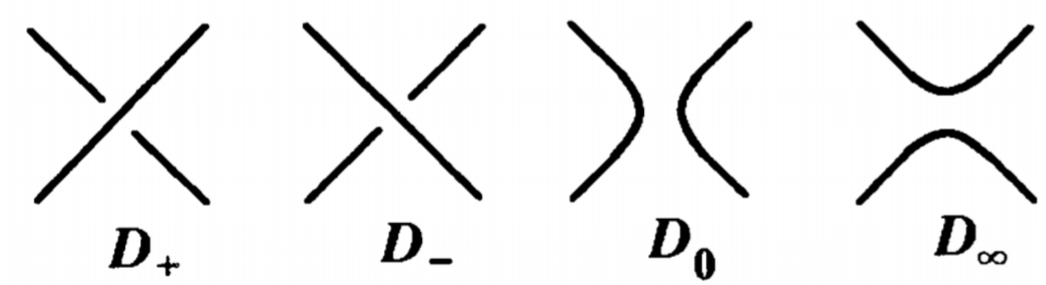

Figure 2: Kauffman smoothing

Theorem 1. *There exists a function*

$$\Lambda : \left\{\text{oriented link diagrams in } \mathbb{R}^2\right\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

that is uniquely defined by the following.
(1) $\Lambda(U) = 1$ *where* U *is the zero-crossing diagram of the unknot.*
(2) $\Lambda(D') = a\Lambda(D)$ *where* D' *differs from* D *only by a Reidemeister I move, where* $c^*(D') = c^*(D) + 1$.
(3) $\Lambda(D') = \Lambda(D)$ *where* D' *differs from* D *only by a Reidemeister move II or III.*

(4) If D_+, D_-, D_0 and D_∞ are four diagrams which are exactly the same except in the neighborhood of a single crossing, wherein they look like (Figure2), then

$$\Lambda(D_+) + \Lambda(D_-) = z(\Lambda(D_0) + \Lambda(D_\infty)).$$

A proof of this theorem is given by (Lickorish 174). Note that this Λ function is not a knot invariant, as it is not invariant under Reidemeister-I moves. However, it has been shown to react to such Reidemeister-I moves in a predictable way, and thus is still used in the definition of an actual knot invariant. To show this, it is also helpful to define the following terms.

Definition 0.6. The *writhe* of an oriented diagram D of a link L is $w(D) = \sum_i^{c^*(D)} \mathrm{sgn}(x_i)$ where $x_1, ..., x_{c^*(D)}$ are the crossings of D.

Definition 0.7. The *self writhe* of a diagram D of a link L is $\overline{w}(D) = w(D) - 2\sum_{i,j}^{\#L} \ell k(x_i, x_j)$ where all $x_1, ..., x_{\#L}$ are the link components of L.

These allow to define the Kauffman function.

Definition 0.8. The Kauffman polynomial is the function

$$F : \left\{\text{oriented links in } \mathbb{R}^3\right\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

defined by $F(L) = a^{-w(D)}\Lambda(D)$ where D is any diagram of the oriented link L.

As a note, this polynomial may well have been defined in terms of the self-writhe, as $a^{-\overline{w}(D)}\Lambda(D)$, which also holds for unoriented links. This polynomial may also be calculated recursively using the skein relation in much the same way as the Conway-normalized Alexander polynomial. An example is shown in (Figure6) which also employs the propositions concerning the Kauffman polynomial of a disjoint union of link components.
The HOMFLY polynomial is named after the initials of some of its discoverers (Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter). This link invariant also employs a skein relation in its definition.

Theorem 2. *There exists an unique function*

$$P : \left\{\text{oriented links in } \mathbb{R}^3\right\} \to \mathbb{Z}[\ell^{\pm 1}, m^{\pm 1}]$$

such that $P_U(m, \ell) = 1$ and if L_+, L_-, L_0 are links which have diagrams D_+, D_- and D_0 that are exactly the same except within the neighborhood of a single crossing, wherein they look like (Figure2), then

$$\ell P(L_+) + \ell^{-1} P(L_-) + mP(L_0) = 0.$$

A proof of this theorem is given by (Lickorish 168).

Definition 0.9. The function of the previous theorem is defined the *HOMFLY polynomial.*

In computational practice, it is more simple with the aid of the following propositions. Such a computation is shown as an example in (Figure7).

Proposition 3. If L is an oriented link and $\overline{L}$ is its reflection, then
(1) changing the signs of both variables leaves $P(L)$ and $F(L)$ unchanged,;
(2) $\overline{P(L)} = P(\overline{L})$ where $\overline{\ell} = \ell^{-1}$ and $\overline{m} = m$;
(3) $\overline{F(L)} = F(\overline{L})$ where $\overline{a} = a^{-1}$ and $\overline{z} = z$.

Proposition 4. If L_1 and L_2 are oriented links, then
(1) $P(L_1 + L_2) = P(L_1)P(L_2)$;
(2) $F(L_1 + L_2) = F(L_1)F(L_2)$;
(3) $P(L_1 \cup L_2) = -(\ell + \ell^{-1})m^{-1}P(L_1)P(L_2)$;
(4) $F(L_1 \cup L_2) = ((a + a^{-1})z^{-1} - 1)F(L_1)F(L_2)$. Note that the sum of two links is not well defined; it depends on which components are taken as a sum. However, the above results always hold whenever L_1 and L_2 are knots. Here "$\cup$" denotes the disjoint union of two split links.

The following proof demonstrates that (3) and (4) follow from (1) and (2) respectively.

Proof. Suppose L_1 and L_2 are split links.

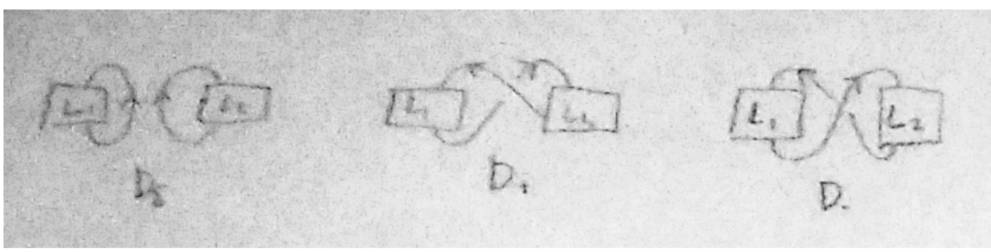

Figure 3: HOMFLY resolution

$(1 \Rightarrow 3)$ Without loss of generality, let D_s be an oriented diagram of $L_1 \cup L_2$, and let D_+ and D_- be diagrams which differ from D_s only in the neighborhood of a single crossing, as in (Figure3), and let L_+ and L_- be links corresponding to these diagrams respectively. Then it is visible that D_+ and D_- differ from a diagram of $L_- \# L_+$ by a single Reidemeister-I move. Since the HOMFLY polynomial is invariant under Reiemeister moves, it follows by the skein relation of the HOMFLY polynomial that

$$\begin{aligned} & P(L_+) = P(L_-) = P(L_1 \# L_2) = P(L_1)P(L_2) \\ \Rightarrow & -mP(L_1 \cup L_2) = \ell P(L_+) + \ell^{-1}P(L_-) \\ & = \ell P(L_1)P(L_2) + \ell^{-1}P(L_1)(L_2) \\ & = (\ell + \ell^{-1})P(L_1)P(L_2). \end{aligned}$$

$(2 \Rightarrow 4)$ Now let D_0 be a diagram of $L_1 \cup L_2$, and let D_+, D_-, D_0 and D_∞ be diagrams

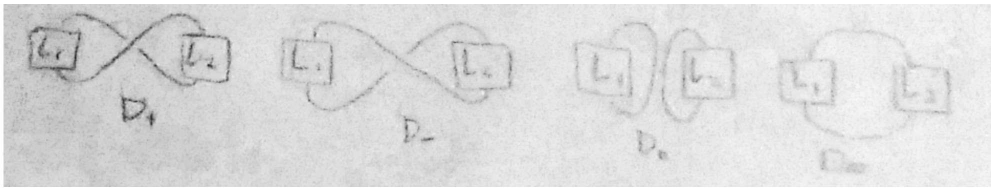

Figure 4: Kauffman resolution

which differ from D_0 only in the neighborhood of a single crossing, as in (Figure4), and let L_+, L_- and L_∞ be links corresponding to these diagrams respectively. Then it is visible that $L_\infty = L_1 \# L_2$ and D_1 and D_+ differ from D_∞ by a single Reidemeister-I move. Since the Kauffman polynomial is invariant under Reidemeister moves, it follows by the skein relation of the Kauffman polynomial that

$$\begin{aligned}
&F(L_1 \cup L_2) = a^{-w(D_0)}\Lambda(D_0) \\
&= a^{-w(D_0)}(z^{-1}\Lambda(D_+) + z^{-1}\Lambda(D_-) - \Lambda(D_\infty)) \\
&= a^{-w(D_0)}(z^{-1}a^{w(D_+)}F(L_+) + z^{-1}a^{w(D_-)}\Lambda(L_-) - a^{w(D_\infty)}\Lambda(L_\infty)) \\
&= a^{-w(D_0)}(z^{-1}a^{w(D_+)} + z^{-1}a^{w(D_-)} - a^{w(D_\infty)})F(L_1)F(L_2) \\
&= a^{-w(D_0)}(z^{-1}(a + a^{-1})a^{w(D_0)} - a^{w(D_0)})F(L_1)F(L_2) \\
&= (z^{-1}(a + a^{-1}) - 1)F(L_1)F(L_2).
\end{aligned}$$

□

Calculating the HOMFLY polynomial may also require the evaluation of up to $\mathcal{O}(2^n)$ state diagrams, and thus is not often efficient for knots with large crossing number. However, there are theoretical applications to this invariant which provide useful insight. For one, knowledge of the HOMFLY polynomial $P(K)$ for some knot (or link) K is sufficient information to recover the Conway polynomial $\nabla_K(z)$ and the Jones polynomial $V(K)$. The following Proposition is proof in the case of the Conway polynomial.

Proposition 5. For any link L, $P_L(i, iz) = \nabla_L(z)$.

Proof. Let L be an arbitrary link which has some diagram D. Suppose that D has complexity $\chi(D) = (c^*(D), u^*(D))$, defined as in (Problem ...). If $\chi(D) = (0, 0)$ then L is the unlink, hence by definition $P_L(i, iz) = P_U(i, iz) = 1 = \nabla_U(z) = \nabla_L(z)$. Suppose $P(i, iz) = \nabla(z)$ for all links with diagrams with complexity less than $\chi(D)$, by lex ordering. Consider a single crossing of D; without loss of generality, fix an orientation of D such that this crossing is positive. Let D_- and D_s be diagrams differing from D only in the neighborhood of this single crossing, wherein they look like (Figure1). Then by the above Proposition $\chi(D_-), \chi(D_s) < \chi(D)$, hence

$$\begin{aligned}
P_L(i, iz) &= -\frac{iz}{i}P_{L_s}(i, iz) - \frac{1}{i^2}P_{L_-}(i, iz) = -zP_{L_s}(i, iz) + P_{L_-}(i, iz) \\
&= -z\nabla_{L_s}(z) + \nabla_{L_-}(z) = \nabla_L(z).
\end{aligned}$$

As there exists no infinite sequence of decreasing complexities, it follows by induction that $P_L(i, iz) = \nabla_L(z)$. $\square$

For a given link L from the HOMFLY polynomial $P_L(a, z)$ the Jones polynomial $V_L(t)$ may be recovered by a similar method using the substitution $a = it^{-1}$ and $z = i(t^{-1/2} - t^{1/2})$ (Lickorish 180).
The Kauffman and HOMFLY polynomials are not redundant; they may be used to distinguish different pairs of links. For example, knots 8_8 and 10_{129} have the same HOMFLY polynomials but distinct Kauffman polynomials, whereas knots 11_{255} and 11_{257} have the same Kauffman polynomial but distinct HOMFLY polynomials (Figure5), (Lickorish 180).

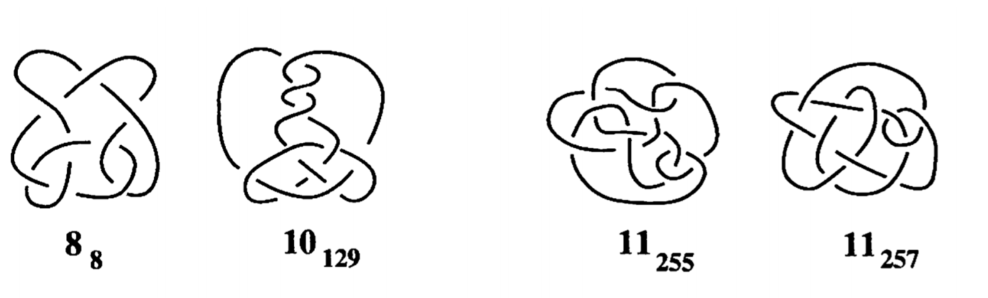

Figure 5: Distinguishable knots

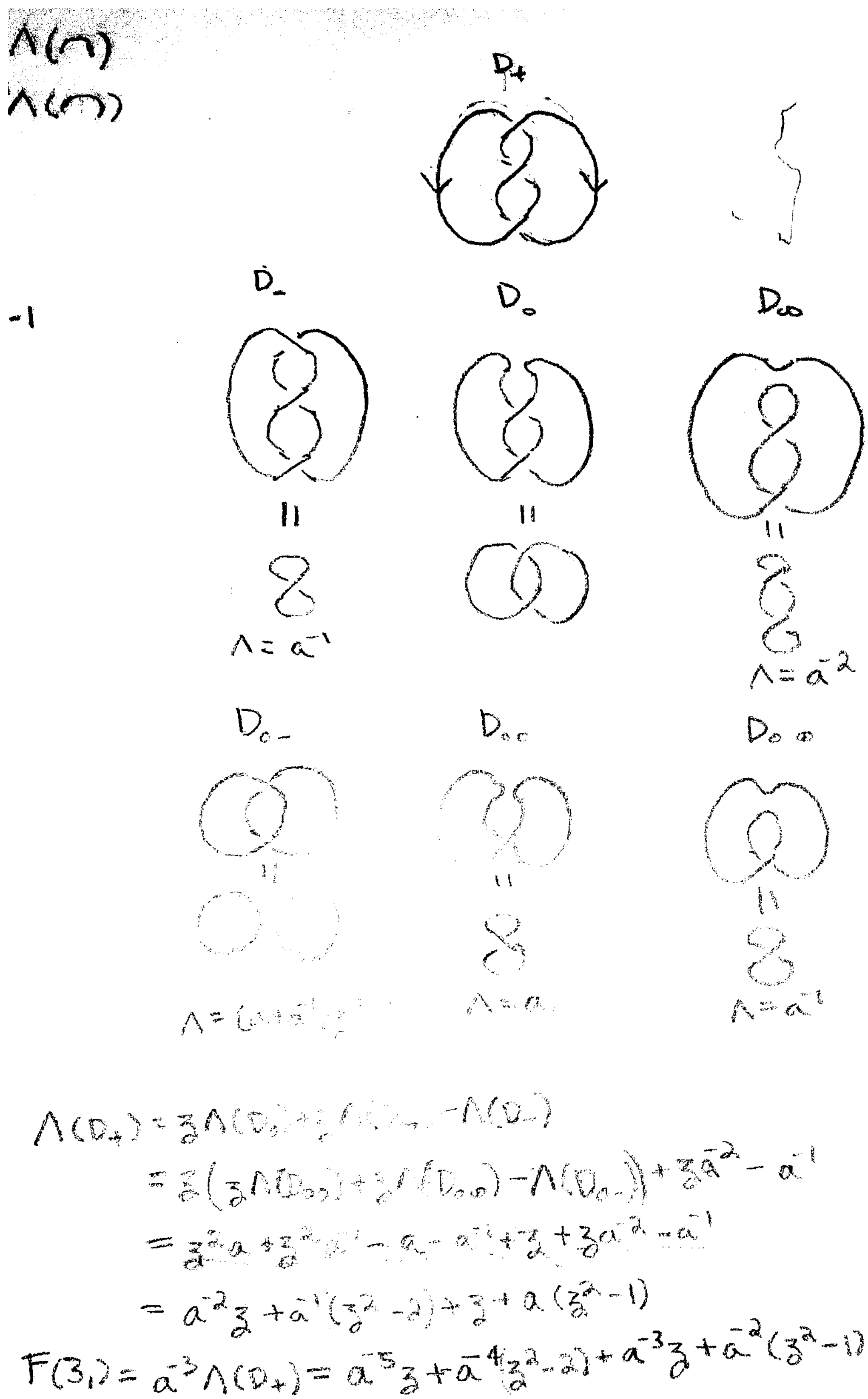

Figure 6: Example Kauffman polynomial computation

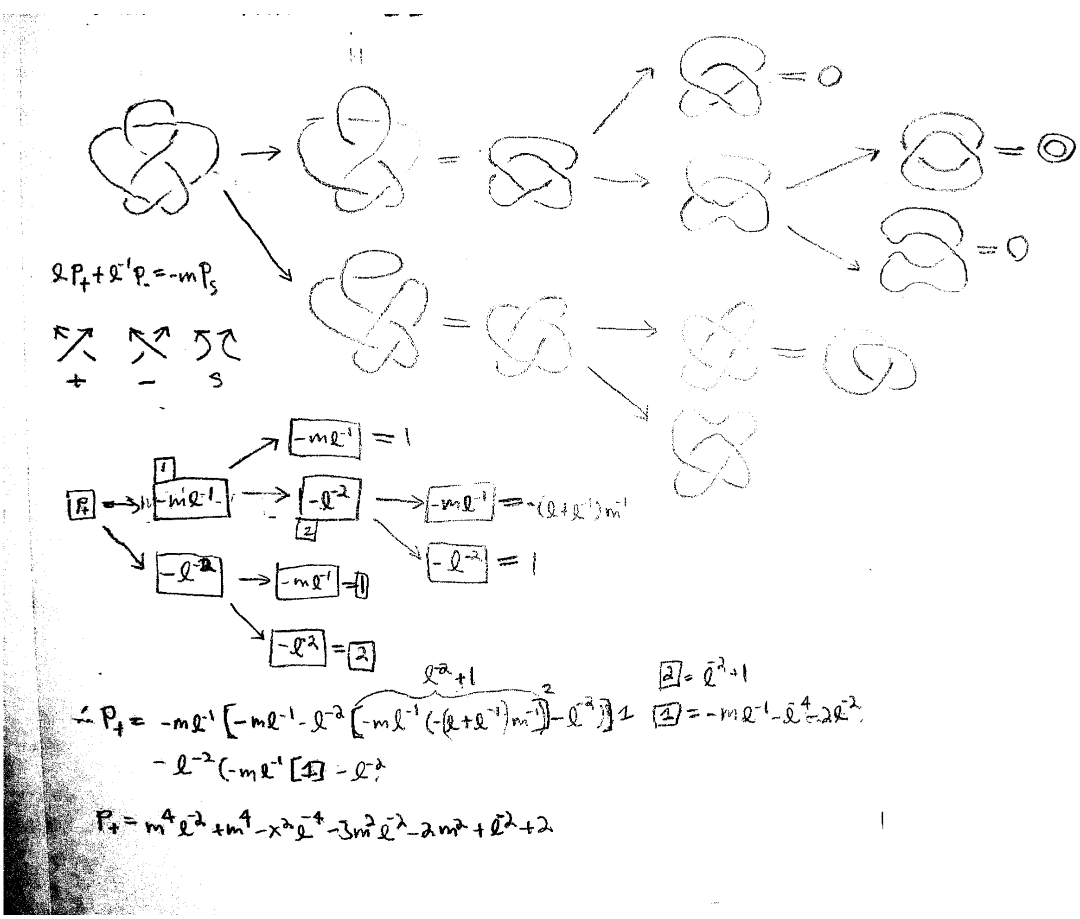

Figure 7: Example HOMFLY polynomial computation