Overall good work. Fix 158. A-, 43/50.

I enjoyed reading your work this quarter :)

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Homework 9

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Definition 152. A subset K of $\mathbb{R}^l$, $l \in \{1,2\}$, is *compact* if every sequence $\{x_n\}_{n=d}^{\infty}$ of elements K has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to an element of K.

Definition 165. A function $f : D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^l$, is *bounded* if range$f \subseteq \mathbb{R}^m$ is bounded. In particular, $f : D \to \mathbb{R}$ is bounded if $\exists M > 0 : \forall x \in D : |f(x)| \leq M$, $f : D \to \mathbb{R}^2$ is bounded if $\exists r > 0 : \forall x \in D : x \in \overline{B_r(0)} \setminus 0$.

Problem 154. Every finite subset of $\mathbb{R}^l$, $l \in \{1,2\}$ is compact.

Proof. Let $A \subseteq \mathbb{R}^l$, $l \in \{1,2\}$. Assume A is finite, which is to say $A = \{a_1, ..., a_s\}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with domain $E \subseteq A$. Then there must be some element $a_i \in A$ which features an infinite number of times in $\{x_n\}_{n=1}^{\infty}$: suppose every element of E features a finite number of times in $\{x_n\}_{n=1}^{\infty}$, say each $a_i \in A$ features c_i times, Then the number of terms in the sequence $\{x_n\}_{n=1}^{\infty}$ is $\sum_{i=1}^{s} c_i$ which is finite, which is a contradiction. Thus, let a_i be such an element which features infinitely many times in $\{x_n\}_{n=1}^{\infty}$. Then define $\{a_{n_k}\}_{k=1}^{\infty}$ a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that range$\{n_k\}_{k=1}^{\infty} \subseteq \{n \in \mathbb{N}^+ \mid x_n = a_i\}$ which converges to $a_i \in A$, from which it follows by (Definition 152) that $\{x_n\}_{n=1}^{\infty}$ is compact. $\square$

Problem 156. Every closed, bounded subset of $\mathbb{R}$ is compact.

Proof. Let $A \subseteq \mathbb{R}$ be a closed set. Let $\{x_n\}_{n=d}^{\infty}$ be a sequence with range $E \subseteq A$. Then the upper and lower bound for A are upper and lower bound for E respectively, thus E is bounded. Then by (Problem 150), there exists a monotonic subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which is shown to converge to the limit $\sup E$. If $\sup E \in E \subseteq A$ done, thus assume $\sup E \notin E$. Then by (Problem 110, Definition 62),

does not have to converge to the sup.

$$\lim_{k\to\infty} x_{n_k} = \sup E \in \operatorname{Cl} E \subseteq \operatorname{Cl} A \subseteq A$$

since A is a closed set. Therefore by (Definition 152), A is compact. $\square$

Problem 158. Every closed, bounded subset of $\mathbb{R}^2$ is compact.

Proof. Let $E \subseteq \mathbb{R}^2$ be a closed bounded set. As such, by (Problem 138) let $M > 0$ such that $\forall x \in E : d(x,(0,0)) \leq M$. Suppose $\{(x_n, y_n)\}_{n=d}^{\infty}$ is a sequence with range $A \subseteq E$. Let $A_1 = \{x \in \mathbb{R} \mid (x,0) \in A\}$. Then A_1 is bounded, since for any $x \in A_1$, $|x| = \sqrt{(x-0)^2 + (0-0)^2} = d((x,0),(0,0)) \leq M$. For closed, suppose $x_0 \in \operatorname{Cl} A_1$. Let $r > 0$ then $B_r(x_0) \cap A_1$ is an infinite set. Consider the bijection $\phi : A_1 \to A$ defined $\phi(x) = (x,0)$, from which it follows $B_r(\phi(x_0)) \cap \phi(A_1) = B_r((x_0,0)) \cap A$ is an infinite set. Hence, $(x_0,0) \in \operatorname{Cl} A \subseteq A$ by (Definition 62) since A is a closed set. Thus, $x_0 \in A_1$ from which it follows A_1 is closed. Then by (Problem 156) A_1 is compact, thus there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=d}^{\infty}$ which converges to some $x \in A_1$. Therefore by (Problem 135), the subsequence $\{(x_{n_k},0)\}_{k=1}^{\infty}$ of $\{(x_n,y_n)\}_{n=d}^{\infty}$ converges to $(x,0)$ in $A \subseteq E$. Therefore by (Definition 152), E is compact. $\square$

Problem 159. If $K \subseteq R^l$, is compact, $l \in \{1,2\}$ then K is closed.

{(x_{n_k},0)} may not be in the original set... Check the posted solution. This problem is important.

To fix, once you found your {x_{n_k}, consider those corresponding {y_{n_k}} s.t. {(x_{n_k}, y_{n_k})} are in the original set...

THE OPPOSITE OF 'CLOSED' IS NOT 'OPEN'.

Proof. Let $K \subseteq \mathbb{R}^l$ be compact, $l \in \{1,2\}$. Suppose K is open, which is to say by (Definition 62) there exists $x \in \mathrm{Cl}\, K \setminus K$. Then for any $n \in \mathbb{N}^+$, define $x_n \in B_{1/n}(x) \cap K$ any element, which exists by (Definition 52). Thus there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of K such that for any $N \in \mathbb{N}^+$, $\forall n \geq N : x_n \in B_{1/n}(x) \subseteq B_{1/N}(x)$, which is to say $\lim_{n\to\infty} x_n = x$. Then by (Problem 148), every subsequence of $\{x_n\}_{n=1}^{\infty}$ also converges to x, which contradicts to K is compact. Therefore, K is closed. *The proof is consistent if K is open...* □

Problem 160. Let $K \subseteq \mathbb{R}$ be compact. Then K is bounded.

Proof. Suppose K is unbounded, without loss of generality assume K is not bounded above. By (Problem 141), there exists an increasing sequence $\{x_n\}_{n=1}^{\infty}$ of elements of K such that $\forall n \in \mathbb{N}^+ : x_n > n$. Then for any subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, for any $k \in \mathbb{N}^+$ $n_k \geq n \Rightarrow x_{n_k} \geq x_n > n$. Thus $\{x_{n_k}\}_{k=1}^{\infty}$ is unbounded, hence diverges by (Problem 139), which contradicts to K is compact. Therefore, K is bounded. □

Problem 161. Let $K \subseteq \mathbb{R}^l$ be compact, $l \in \{1,2\}$, $A \subseteq K$ be an infinite set. Then $\exists x \in \mathrm{Cl}\, A \cap K$.

Proof. Since K is compact, K is closed by (Problem 159). Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of A. Let $E = \mathrm{range}\{x_n\}_{n=1}^{\infty}$. Note that A is bounded since the upper and lower bound of K is an upper and lower bound of A, respectively. As such, by (Problem 150) let $\{x_{n_k}\}_{k=1}^{\infty}$ be an increasing subsequence of $\{x_n\}_{n=1}^{\infty}$ which converges to $x = \sup E \in \mathrm{Cl}\, E$ ($\because$ Problem 143) $\subseteq \mathrm{Cl}\, A \subseteq \mathrm{Cl}\, K \subseteq K$. □

Problem 162. Let $K \subseteq \mathbb{R}^l$, $l \in \{1,2\}$, $(A \subseteq K \wedge |A| = \infty) \Rightarrow \exists x \in \mathrm{Cl}\, A \cap K$. Then K is compact.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ such that $E = \mathrm{range}\,\{x_n\}_{n=1}^{\infty} \subseteq K$. Then if $|E| \neq \infty$ then $\lim_{n\to\infty} x_n = x$ for some $x \in K$ by (Problem 154). Assume $|E| = \infty$ then $\exists x \in \mathrm{Cl}\, E \subseteq K$, thus by (Problem 142) there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} = x$. Therefore by (Definition 152), K is compact. □

Problem 166. (a) $f(x) = x$ is continuous on $\mathbb{R}$ but not bounded.
(b) $f(x) = 1/x^2$ is continuous on $(0,1]$ but not bounded.
(c) $f(x) = 1/(x-a)^2$ is continuous on $(a,b]$ but not bounded.

Problem 167. Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^l$ be a continuous function, $K \subseteq D$ compact. Then $f(K) = \{f(x) \,|\, x \in K\}$ is compact.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ such that $E = \mathrm{range}\,\{x_n\}_{n=1}^{\infty} \subseteq K$. Then since K is compact, by (Definition 152) there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in K$. Suppose $|E| = \infty$. Then for the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of elements of $f(E)$, there exists a subsequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} f(x_{n_k}) = f(x)$ by (Problem 133), hence by (Problem 143) $f(x) \in \mathrm{Cl}\, f(E) \subseteq \mathrm{Cl}\, f(K)$. For containment let $r > 0$ then

Correct but better say {y_n = f(x_n)} is an arbitrary sequence in f(K)...

$\exists N \geq 1 : \forall n \geq N : f(x_n) \in B_r(f(x))$ by (Definition 124), hence $B_r(f(x)) \cap E$ is infinite since $N \neq \infty = |E|$, from which it follows $B_r(x)$ is infinite since f is a function. Therefore by (Definition 152) $f(x) \in f(K)$ and $f(K)$ is compact. □

Problem 168. In particular, it follows from (Problem 167) that a continuous function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, where D is closed and bounded, hence compact, must be compact, hence bounded.

Problem 169. Let $f : D \to \mathbb{R}$ be continuous, $D \subseteq \mathbb{R}$ compact. Then $\exists x_1, x_2 \in D : \forall x \in D : f(x_1) \leq f(x) \leq f(x_2)$.

Proof. Note $f(D)$ is bounded by (Problem 160), hence by assumption of (Axiom 103) let $u = \sup f(D)$, $w = \inf f(D)$. Then by (Problem 110), if $u \notin f(D)$ then $u \in \operatorname{Cl} f(D) \subseteq f(D)$ since $f(D)$ is closed (Problem 166). By a similar argument, $w \in f(D)$. Thus, $u = f(x_1)$ and $w = f(x_2)$ for $x_1, x_2 \in D$. Of course, by (Definition 100) for any $x \in D$, $w \leq f(x) \leq u$. □

Problem 170. Let $f : [a, b] \to \mathbb{R}$ be continuous, nonconstant. Then range$f$ is a closed, bounded interval.

Proof. In light of (Problem 169), it is sufficient to show that range$f$ is an interval. As before, let $u = \sup f(D)$, $w = \inf f(D)$. Since f is nonconstant, $\{w\} \subset [w, u]$. By (Problem 113) and by assumption of (Axiom 103), $\forall y \in [w, u] : \exists c \in [a, b] : f(c) = y$. Therefore by (Definition 94), range$f(D) = [w, u]$ is a closed bounded interval. □