The Erlangen Program: A Script

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Introduction

In general, complex numbers are a field extension of the reals of the form $a+bi$, where $a, b \in \mathbb{R}$, and the symbol i is defined such that $i^2 = -1$. The field of complex numbers is denoted by the symbol $\mathbb{C}$. According to Western history, they were first conceived by Gerolamo Cardano in 1545 to accommodate roots of those polynomials with one or more irreducible quadratic factors. Notably, the Fundamental Theorem of Algebra relies on such a definition of such complex numbers, which form an algebraically closed field in which every polynomial of degree n has exactly n roots.

We will define the usual arithmetic operations of complex numbers as developed by Rafael Bombelli. For the following, suppose z_1 and z_2 are complex numbers:

$$z_1 = x_1 + iy_1, x_1, y_1 \in \mathbb{R}$$
$$z_2 = x_2 + iy_2, x_2, y_2 \in \mathbb{R}$$

$$
\begin{aligned}
z_1 \pm z_2 &= x_1 \pm x_2 + i(y_1 \pm y_2) \\
z_1 z_2 &= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \\
\bar{x + iy} &= x - iy
\end{aligned}
$$

Euler's formula states that for any real number θ,

$$e^{i\theta} = \sin\theta + i\cos\theta$$

In light of this result, it is clear that any complex number is representable in the form

$$z = x + iy = |z|\, e^{i\theta} = |z| \sin\theta + i\cos\theta$$

where $|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$ is defined as the modulus of z. The argument, or angle, function is defined as:

$$\arg(re^{i\theta}) = \theta$$

taking the principle argument as θ: in other words, $-\pi < \theta \leq \pi$. The usual exponent operations are defined for $z = re^{i\theta} \in \mathbb{C}, a \in \mathbb{R}$:

$$z^a = r^a e^{ia\theta}$$

In particular, $\sqrt{z} = z^{\frac{1}{2}} = \sqrt{r}e^{i\frac{\theta}{2}}$, and $\frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{z}{\bar{z}}$.

A complex number is formed of two real-numbered components, and thus is envisioned naturally in the context of a plane. The canonical form of this plane, called the Complex Plane, corresponds the dimensions of $\mathbb{C}$ with the axes of the usual Cartesian plane. We will show that the geometry of the complex numbers, together with a particular form of Möbius transformation, define a geometry which is more general than Euclidean and less general than Affine geometry; informally, that which is Euclidean with the exclusion of general congruent scalene triangles. We will present definitions and proofs of the affine axioms and define the notions of length and angle measure. Additionally, we will present a selection of Euclidean theorems which hold in our "Figginsian" geometry.

1 Euclidean Geometry

In Euclidean Geometry, we take the points to be $\mathcal{P} = \mathbb{C}$, and the translations to be $\mathcal{T} = \mathcal{F} = \{T : T(z) = e^{i\theta}z + b\}$; in other words, rotations and translations.

Definition 1.1. A *figure* is a subset $f \subseteq \mathcal{P}$.

Definition 1.2. Two figures f_1, f_2 are *congruent*, written $f_1 \cong f_2$, when there exists a transformation $T \in \mathcal{F}$ such that $T(f_1) = f_2$.

Definition 1.3. Given two congruent figures f_1, f_2 under the transformation $T(f_1) = f_2$, and another figure $g_1 \subseteq f_1$, the *corresponding part* in f_2 of g_1 is $g_2 = T(g_1)$.

Note that $g_1 \cong g_2$ trivially.

We then define a few basic figures and notations:

- A line segment $AB = \{A, B\}$
- A ray $\overrightarrow{AB} = \{A + (B - A) \cdot k : k \in \mathbb{R}, k \geq 0\}$
- A line $\overleftrightarrow{AB} = \{A + (B - A) \cdot k : k \in \mathbb{R}\}$
- A triangle $\triangle ABC = \{A, B, C\}$
- An angle $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$; note that A, B, and C cannot be collinear

Then, we may proceed by proving the axioms of Figginsian Geometry!

Theorem 1.4. *Given any two points A, B, there exists exactly one line ℓ such that $A \in \ell, B \in \ell$, and $\ell = \overleftrightarrow{AB}$.*

Definition 1.5. Two lines ℓ, m are parallel, denoted $\ell \parallel m$, when $\ell \cap m = \emptyset$.

Before we prove the parallel lines theorem to show that this geometry is affine, we should first prove a lemma which will make the parallel lines theorem much easier to prove.

Definition 1.6. We define the slope of a line $\arg(\ell)$ as follows:

With $x_1 + iy_1, x_2 + iy_2 \in \ell$ distinct points,

$$\arg(\ell) = \begin{cases} \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right) & x_1 \neq x_2 \\ \frac{\pi}{2} & x_1 = x_2 \end{cases}$$

You'll notice that $\arg(\ell) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Now, a simple lemma:

Lemma 1.7. *Given two lines ℓ, m,* $\arg(\ell) = \arg(m)$ *if and only if $\ell = m$, or $\ell \parallel m$.*

Theorem 1.8. *Given a line ℓ, and a point $P \notin \ell$, there exists one and only one line m parallel to ℓ with $P \in m$.*

We now wish to define the length of a line segment, in order to prove the third axiom. Take the length of AB to be

$$\mathcal{L}(AB) = |A - B|$$

Theorem 1.9. *The following properties hold:*

1. *The measure is well-formed; $\mathcal{L}(AB) = \mathcal{L}(BA)$.*
2. *Two segments are congruent if and only if they have the same length.*
3. *If A, B, C are three points with B between A and C, then $\mathcal{L}(AC) = \mathcal{L}(AB) + \mathcal{L}(BC)$.*

Definition 1.10. An angle is *right* if it is congruent to its supplement.

Theorem 1.11. *There is an angle measure $\mathcal{D}(\angle ABC)$, defined by taking taking the argument of the rotation around B of C onto A:* $\left|\arg\left(\frac{C-B}{A-B}\right)\right|$

1. *The measure is well defined: $\mathcal{D}(ABC) = \mathcal{D}(CBA)$.*
2. *Two angles are congruent if and only if they have the same degree measure.*
3. *A right angle has degree measure $\frac{\pi}{2}$.*
4. *Given an angle $\angle ABD$, with point C interior to $\angle ABD$, $\mathcal{D}(ABC) + \mathcal{D}(CBD) = \mathcal{D}(ABD)$.*

Theorem 1.12. *1.9.3: If a point B is between points A and C, then $\mathcal{L}AB + \mathcal{L}BC = \mathcal{L}AC$.*

Proof. Assume the point B is between the points A and C. Then $\mathcal{L}AB + \mathcal{L}BC = |B - A| + |C - B|$. A, B, C are collinear, which is to say $C \in \overrightarrow{AB}$, which is to say $C = A + (B - A)k$, $k \in \mathbb{R}$. Then $|C - B| = |A + (B - A)k| = |(B - A)(k - 1)| = |k - 1||B - A|$. Since B is between A and C, $k - 1 > 0$ Therefore, $\mathcal{L}AB + \mathcal{L}BC = (k - 1 + 1)|B - A| = k|B - A| = A + (B - A)k - A = |C - A| = \mathcal{L}(AC)$. $\square$