Problem 1. Let $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$.

(a) The Lagrange polynomial of degree 2 is given by (Eqn. 3.1),

$$\begin{aligned}
P_2(x) &= y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} \\
&= 0 + 1 \frac{(x+1)(x-3)}{(2+1)(2-3)} + 1 \frac{(x+1)(x-2)}{(3+1)(3-2)} \\
&= -\frac{1}{3}(x+1)(x-3) + \frac{1}{4}(x+1)(x-2) \\
&= (x+1)(-\frac{1}{3}x + 1 + \frac{1}{4}x - \frac{1}{2}) \\
&= (x+1)(-\frac{1}{12}x + \frac{1}{2}) \\
&= -\frac{1}{12}(x+1)(x-6).
\end{aligned}$$

(b) By Newton's divided differences,

$$\begin{aligned}
\begin{bmatrix} x_1 & f[x_1] & & \\ x_2 & f[x_2] & f\begin{bmatrix} x_1 & x_2 \end{bmatrix} & \\ x_3 & f[x_3] & f\begin{bmatrix} x_2 & x_3 \end{bmatrix} & f\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \end{bmatrix} &= \begin{bmatrix} -1 & 0 & & \\ 2 & 1 & \frac{1-0}{2+1} & \\ 3 & 1 & \frac{1-1}{3-2} & f\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & & \\ 2 & 1 & 1/3 & \\ 3 & 1 & 0 & \frac{0-1/3}{3+1} \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & & \\ 2 & 1 & 1/3 & \\ 3 & 1 & 0 & -1/12 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_2(x) &= 0 + \frac{1}{3}(x+1) + \frac{1}{12}(x+1)(x-2) \\
&= (x+1)(\frac{1}{3} - \frac{1}{12}(x-2)) \\
&= -\frac{1}{12}(x+1)(x-6).
\end{aligned}$$

(c) Suppose $(x_4, y_4) = (5, 2)$. Then by Newton's divided differences, $f\begin{bmatrix} x_3 & x_4 \end{bmatrix} = (2 - 1)/(5-2) = 1/2$, $f\begin{bmatrix} x_2 & x_3 & x_4 \end{bmatrix} = (1/2 - 0)/(5-2) = 1/6$, $f\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = (1/2 - (-1/12))/(5 - (-1)) = 1/24$. Then the leading coefficient on the interpolating polynomial $P_3(x) = P_2(x) + 1/24(x+1)(x-2)(x-3)(x-5)$ of all four points must be $1/24$.

Problem 2. Let $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 5 \end{bmatrix}$.

(a)There is exactly one interpolating polynomial $P_1(x) \in \mathbb{R}[x]$ of degree 1 if and only if all three (x_i, y_i) are collinear; otherwise, the unique interpolating polynomial of degree $3-1=2$ or less by (Theorem 3.2) has degree 2 and there exists no interpolating polynomial of degree 1. The three points are not collinear if and only if $z_i = (x_i, y_i)$ are linearly independent among all i. The following matrix row operations arrive to a homogeneous linear combination:

$$\begin{bmatrix} 0 & 1 & z_1 \\ 1 & 3 & z_2 \\ 2 & 5 & z_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & z_2 \\ 2 & 5 & z_3 \\ 0 & 1 & z_1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & z_2 \\ 0 & -1 & z_3 - z_2 \\ 0 & 1 & z_1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & z_2 \\ 0 & 1 & z_2 - z_3 \\ 0 & 0 & z_1 - z_2 - z_3 \end{bmatrix}.$$

Thus, $z_1 - z_2 - z_3 = 0$ and the z_i are linearly dependent, hence collinear. Therefore, the degree-1 polynomial $P_1(x) = 2x + 1$ interpolates the three points.
(b) By (a) and (Theorem 3.2), there is a unique interpolating polynomial of degree 2 or less. It is shown that a polynomial of degree 1 interpolates the three points, therefore there exists no interpolating polynomial of degree 2.
(c) By (Theorem 3.2), there is an infinite family of interpolating polynomials of degree 3, namely $\left\{ P_3(x) = P_2(x) + f\begin{bmatrix} 0 & 1 & 2 & c \end{bmatrix}(x-0)(x-1)(x-2)(x-c) \,|\, c \in \mathbb{R} \right\} \subseteq \mathbb{R}[x]$.

Problem 3. A degree 5 polynomial cannot intersect a degree 8 polynomial in exactly 9 points.

Proof. Let $P(x), Q(x) \in \mathbb{R}[x]$ such that $\deg P = 9$ and $\deg Q = 5$. Suppose there are exactly 9 intersection points, by which $P(z_i) = Q(z_i)$ for all z_i, $i \in [1, 9] \cap \mathbb{N}$. Then P interpolates 9 points and has degree $9 - 1 = 8$, thus is unique by (Theorem 3.2). Since Q also interpolates 9 points and has degree ≤ 8, by uniqueness $Q = P$. But $\deg P \neq \deg Q$, a contradiction. Since P and Q were arbitrary polynomials of degree 8 and 5 respectively, it is impossible for any two such polynomials to intersect in exactly 9 points. $\square$

Problem 4. Let $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & -1 & 5 \\ 0 & 0 & 0 & 0 & 5 & 7 \end{bmatrix}^T$. There are 6 data points, (Theorem 3.2) asserts the existance and uniqueness of an interpolating polynomial of degree 5 or less. The first four terms of the Lagrange interpolating polynomial formula vanish; one need only calculate the seventh row of the table of Newton's divided differences.

$$\begin{bmatrix} 1 & 0 & & & & & \\ 2 & 0 & 0 & & & & \\ 3 & 0 & 0 & 0 & & & \\ 4 & 0 & 0 & 0 & 0 & & \\ -1 & 5 & -1 & 1/4 & -1/12 & 1/24 & \\ 5 & 7 & 4/3 & 13/24 & 5/13 & 5/24 & 1/24 \end{bmatrix}$$

Then

$$\begin{aligned} P_6(x) &= 1/24(x-1)(x-2)(x-3)(x-4) + 1/24(x-1)(x-2)(x-3)(x-4)(x+1) \\ &= 1/24(x-1)(x-2)(x-3)(x-4)(x+2) \end{aligned}$$

is a polynomial of degree 6 which interpolates all 7 points.

Problem 5. Let $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ -1 & -2 & -3 & -4 & -5 & -6 & 6 \end{bmatrix}^T$. Using Newton's divided differences,

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ 4 & -4 \\ 5 & -5 \\ 6 & -6 \\ 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & \\ 2 & -2 & \frac{-2+1}{2-1} = -1 \\ 3 & -3 & \frac{-3+1}{3-1} = -1 \\ 4 & -4 & \frac{-4+1}{4-1} = -1 \\ 5 & -5 & \frac{-5+1}{5-1} = -1 \\ 6 & -6 & \frac{-6+1}{6-1} = -1 \\ 0 & 6 & \frac{6-(-6)}{0-6} = -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & \\ 2 & -2 & \frac{-2+1}{2-1} = -1 \\ 3 & -3 & \frac{-3+1}{3-1} = -1 \\ 4 & -4 & \frac{-4+1}{4-1} = -1 \\ 5 & -5 & \frac{-5+1}{5-1} = -1 \\ 6 & -6 & \frac{-6+1}{6-1} = -1 \\ 0 & 6 & \frac{6-(-6)}{0-6} = -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & & & & & & \\ 2 & -2 & -1 & & & & & \\ 3 & -3 & -1 & 0 & & & & \\ 4 & -4 & -1 & 0 & 0 & & & \\ 5 & -5 & -1 & 0 & 0 & 0 & & \\ 6 & -6 & -1 & 0 & 0 & 0 & 0 & \\ 0 & 6 & -2 & 1/5 & -1/20 & 1/60 & -1/120 & 1/120 \end{bmatrix}.$$

Then the interpolating polynomial is found

$$P_6(x) = -1 - 1(x-1) + 1/120(x-6)(x-5)(x-4)(x-3)(x-2)(x-1).$$