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MATH 419: Assignment 6

The Development of a Cubic Solution

According to legend, in 430 B.C.E. the city of Athens was beset by a deadly plague. Desperate, the citizens entreated the wisdom of an oracle at Delos, who pronounced that the god Apollo required a unique offering to end their suffering: an altar in the shape of a cube, reconstructed with exactly twice the volume of an existing cube. In some versions of the legend, at their inability to accomplish this task, Plato informed the Athenians that the problem could certainly be solved, but that the plague was a punishment for neglecting geometry (Sathaye 6).

Such "doubling of the cube" is one of several ancient problems implicitly requiring the solution of a cubic equation: here, one essentially requires a solution of the equation $x^3 = 2a^3$ for an arbitrary number a. Moreover, in the style of Greek mathematics, this solution must be obtained by geometrical means. Other such problems include the trisection of an arbitrary angle (Burton 126).

A general cubic equation has the form

$$ax^3 + bx^2 + cx + d = 0.$$

Here, a, b, c, d are assumed to be arbitrary rational numbers, as would be required at the time. It is assumed that this is a true cubic of degree 3, which is to say $a \neq 0$. Hence, one can normalize the equation by the leading coefficient to obtain the form

$$x^3 + ax^2 + bx + c = 0.$$

Many mathematicians throughout antiquity pursued such a solution by radicals. Its eventual discovery was contested and fraught with scandal. Several special cases were known before the general case. Italian mathematician Scipione del Ferro (1465-1526) first found a solution to the equation $x^3 + px + q = 0$ when p and q are positive, and passed this solution to his pupil Antonio Maria Fiore. Another Italian mathematician Nicolo Tartaglia (1500-1557) found a method for solving cubics of the form $x^3 + px^2 + q = 0$, and, at the challenge of Fiore, independently developed a formula for the above case. However, credit for the general case ultimately belongs to Girolamo Cardano (1501-1576). Known as a plagarist, a cheat, and a scoundrel to his contemporaries, Cardano nevertheless convinced Tartaglia to divulge his methods, was the first to consider imaginary numbers as potentially valid, broke the first ground in the theory of probability, and developed a cubic reduction technique that brought the general cubic level with known methods. This reduction technique uses the substitution $x = y - a/3$, hence

$$\begin{aligned} 0 &= \left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c \\ &= y^3 + \left(b - \frac{a^2}{3}\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right). \end{aligned}$$

Further substituting $p = b - a^2/3$ and $q = -(2a/27 - ab/3 + c)$ is becoming of the familiar form $y^3 + py + q = 0$.

Notably, Cardano's proofs were presented as geometrical arguments, in direct reference to the works of Euclid. The "liberation of algebra from the necessity of dealing only with concrete examples" would not come until the French mathematician François Viéte (1540–1603), who represented known quantities with consonants and variables with vowels (Burton 315). Though this system was used by Jordanus de Nemore some 350 years earlier, the practice was lost to time until Viéte, possibly due to the overwhelming primacy of geometry in Greek schools of thought. (Burton 283). As such, Cardano asserted the required identity

$$(a - b)^3 + 3ab(a - b) = a^3 - b^3$$

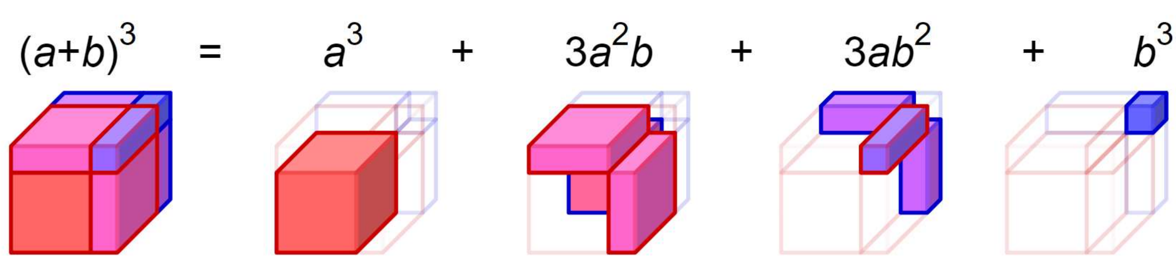

Figure 1: Geometric visualization of the binomial theorem in the cubic case by deconstruction of the volume of a cube (Cmglee).

by a known deconstruction of the volume of a cube (Figure 1)(Burton 323). Thus by the substitution $x = a - b$ into the form $x^3 + px^2 + q = 0$, Cardano brought the general cubic within reach of Tartaglia's formula

$$x = a - b = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

An Alternate Substitution

In his proof of the general cubic solution, Cardano might have used another substitution, which might have been equally valid as his binomial expansion and was also probably accessible to him at the time. This substitution involves cosine identities. Instead, Albert Girard (1595–1632) was the first to use it to solve cubic equations (Sathaye 1). This identity is derived from the usual angle-sum formula,

$$\begin{aligned}
\cos(3t) &= \cos(2t + t) = \cos(2t)\cos(t) - \sin(2t)\sin(t) \\
&= (2\cos^2 t - 1)\cos(t) - (2\sin(t)\cos(t))\sin(t) \\
&= 2\cos^3(t) - \cos(t) - 2(\cos(t))(1 - \cos^2(t)) \\
&= 4\cos^3(t) - 3\cos(t).
\end{aligned}$$

In this way, the same procedure may be applied to cosine identities in higher powers by recursive substitution to the previous polynomial. As Avinash Sathaye of the University of Kentucky writes, "[such] related polynomials were first studied by Viéte, so naturally they are called the Chebyshev polynomials, after the nineteenth-century Russian mathematician" (Sathaye 2).

By an appropriate change of variables, any reduced cubic equation can be converted into a form making use of the third Chebyshev polynomial. In light of the above equation, let $z = \cos(t)$, $p = -3k^2$ and $y = 2kz$. Then

$$\begin{aligned}
&y^3 + py + q = 0 \\
&\Leftrightarrow (2kz)^3 - 3k^2(2kz) + q = 0 \\
&\Leftrightarrow 8k^3z^3 - 6k^3z + q = 0 \\
&\Leftrightarrow 4z^3 - 3z + \frac{q}{2k^3} = 0.
\end{aligned}$$

As an example, consider the reduced cubic equation $x^3 = 3x + 1$. Here, $p = -3$ and $q = -1$. As the discriminant $-4p^3 - 27q^2 = 81$ is positive, the formula may apply to the attainment of three real roots. Thus, let $k = \sqrt{p/-3} = 1$ and $z = x/2k$. Then

$$4z^3 - 3z = 4\cos^3(t) - 3\cos(t) = \cos(3t) = \frac{q}{2k^3} = -\frac{1}{2},$$

where $z = \cos(t)$. This is a well known special cosine. Here,

$$3t = 60° \Rightarrow t = 20°,$$

from which it follows that $x = 2z = 2\cos(20°) \approx 1.8794$. This is the same approximation found via Cardano's formula.

As another example, suppose instead $x^3 = 4x + 1$, so that $p = -4$ and $q = -1$. Again, as the discriminant $-4p^3 - 27q^2 = 229$ is positive, we can proceed. This solution is more complicated. Again, let $k = \sqrt{p/-3} = 2/\sqrt{3}$, $\cos(t) = z = x/2k = \sqrt{3}x$. As such,

$$4z^3 - 3z = 4\cos^3(t) - 3\cos(t) = \cos(3t) = \frac{q}{2k^3} = -\frac{1}{2\left(\frac{2}{\sqrt{3}}\right)^3} = -\frac{3\sqrt{3}}{16}.$$

By calculator, one obtains $3t \approx 0.947727$, hence $t \approx 0.315909$, hence $x = 2kz = 4\cos(t)/\sqrt{3} \approx 2.195119$. The same approximation can be found by Cardano's formula. Of course, the remaining solutions $x \approx -1.8608$ and $x \approx -0.25410$ are found by rotating t by $120°$ and $240°$, respectively.

As may be evident by this method, the Chebyshev identity is intimately connected to the classical geometrical problem of trisecting an arbitrary angle. Such matters of construction call attention to another approach to solving cubic equations, which mirrors that of classical geometry.

Other Geometrical Approaches to Cubic Problems

According to renowned origamist Thomas Hull, "ancient Japanese sangaku (mathematical problems painted on wooden tablets and hung in shrines, circa 1600–1890) have been found that depict paper-folding geometry problems, indicating that the Japanese have mathematical as well as religious and artistic traditions in paper folding" (Hull 313). Geometric constructions using origami were codified into writing by T. Sundra Row in his 1893 treatise *Geometric Exercises in Paper Folding* (Hull 313)(Row). Cubic equations were not featured in this book; in fact, Row mistakenly claims that origami constructions were unable to address anything beyond quadratic problems. But in 1936 algebraic geometer Margherita Piazolla Beloch disproved this claim, showing that origami constructions can indeed solve general cubic equations by finding common tangents to two parabolas. She is the first person known to have demonstrated this rigorously, going so far as to provide a constructive proof (Figure 2).

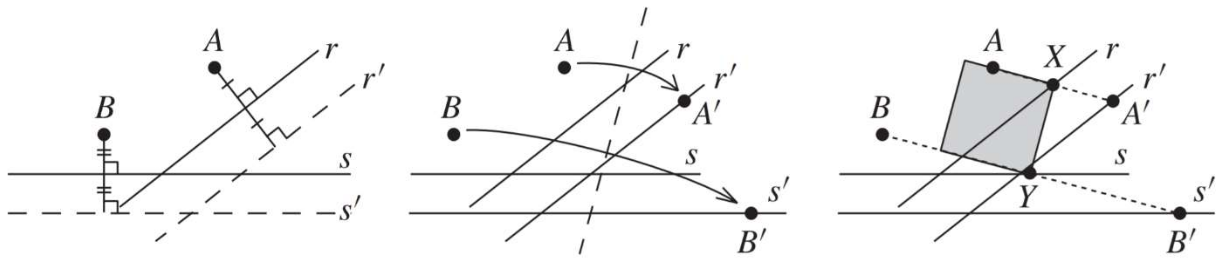

Figure 2: Origami construction of a Beloch square used to solve a cubic equation (Hull 310).

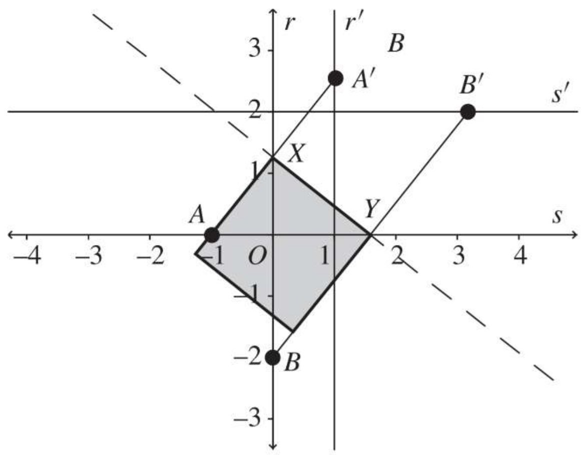

Figure 3: Beloch's origami construction of $\sqrt[3]{2}$, effectively "doubling the cube" (Hull 310).

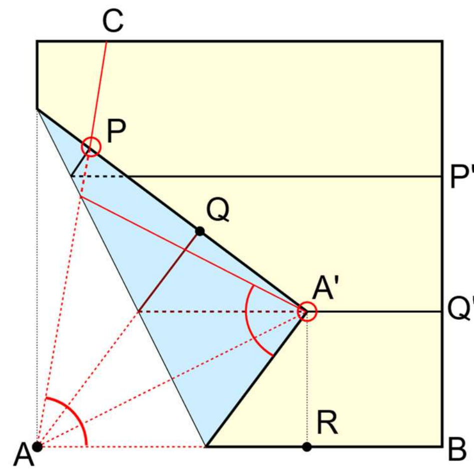

Figure 4: Origami trisection of an arbitrary angle (Dmcq).

Beloch's key insight, the construction of a so-called Beloch square, relies on Huzita's 6th axiom of origami construction, which states that one can always construct a fold which aligns a point to a line while simultaneously aligning another point to another line. Thereby, one can trisect a given angle (Figure 4), double the cube (Figure 3), and, given any cubic equation, construct by origami the solution (Dancso). While this is considered a reasonable folding mechanism by human standards, the accuracy of the fold still relies practically on the folder's skill, and is thus limited by human error. Thus, any numerical solution is bound to be more efficinently calculated to an arbitrary degree of accuracy by some variant of Newton's method, without requiring a significant upgrade in one's technology. On the other hand, one could say the same about locating points of intersection between two arbitrary circles, which Plato and Euclid similarly took for granted. Thus, in an abstract sense Beloch's discovery elevated the system of origami construction beyond even that of Platonic classical construction, and set apart origami as an interesting and valid axiomatic system of geometry.

Many other means of constructing solutions to cubic equations, and thereby taking on the vexing problems of antiquity, have been developed by means other than a compass and unmarked straightedge. These, including origami, are collectively known as *Neusis* constructions. Valid arbitrary angle trisections, for example, have been accomplished using a marked straightedge (Archimedes), mechanical linkages (Yates, Sylvester), a marked string (Hutcheson), a carpenter's square, and interconnected compasses. Even in ancient times, Hippias of Elis (circa 425 B.C.E.) invented the quadratix, a new curve which he used to square the circle. According to tradition however, Plato rejected this attempt on the grounds that it was "mechanical rather than mathematical", insisting that the task be performed with an unmarked straightedge and compass only. To this, Plutarch wrote in *Convivial Questions* "for in this way the whole good of geometry is set aside and destroyed, since it is reduced to things of the sense and prevented from soaring among eternal images of thought" (Plutarch)(Burton 122). Indeed, it would take until 1837 for French mathematician Pierre Wantzel to supply the first rigorous proof that the task under Plato's restrictions is ultimately futile (Burton 126). Alas, much to the misfortune of the Athenians, it would appear that the mercurial god Apollo could never have been persuaded.

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