Problem 1. (a) The union of a chain $I_1 \subset I_2 \subset \cdots$ of ideals of a commutative ring R is an ideal of R.

Proof. Let $a, b \in \cup I_j$. Then $a \in I_i$, $b \in I_j$ for some $i, j \in \mathbb{N}$. Without loss of generality, and by the well ordering principle of $\mathbb{N}$ assume $j \leq i$, thus $a, b \in I_i$. Then $a - b \in I_i \subset \cup I_j$ by the closure of the ideal I_i. By the absorption of the ideal I_i of R, for any $r \in R$ $ar = ra \in I_i \subseteq \cup I_j$. Therefore by the ideal test, $\cup I_j \lhd R$. □

(b) In a prinipal ideal domain, any such strictly increasing chain of ideals must be finite in length (Ascending Chain Condition).

Proof. Let R be a principal ideal domain. Note that any principal ideal domain is a commutative ring by definition, thus part (a) applies and the union $\cup I_j$ of ideals in a strictly ascending chain is an ideal of R. Suppose this chain is infinite. Then since R is a principal ideal domain, $\cup_{i=1}^{\infty} I_i = \langle a \rangle$ for some $a \in \cup_{i=1}^{\infty} I_i$. Then $a \in I_i$ for some $i \in \mathbb{N}$, as a union, thus by the closure of I_i as an ideal , $\cup_{i=1}^{\infty} I_i = \langle a \rangle \subseteq I_i$, a contradiction to $I_i \subset \cup_{i=1}^{\infty} I_i$. Therefore, any strictly ascending chain is finite in length. □

Problem 2. As rings, $\mathbb{Q}(\sqrt{3}) \ncong \mathbb{Q}(\sqrt{-3})$.

Proof. Suppose $\phi : \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{-3})$ is an isomorphism. Then by (Theorem 15.1), note $(\phi(\sqrt{3}))^2 = (\phi(\sqrt{3}))(\phi(\sqrt{3})) = \phi((\sqrt{3})^2) = \phi(3) = 3$. Hence, $\phi(1 + \sqrt{3}) = \phi(1) + \phi(\sqrt{3}) = 1 + \sqrt{3} = a + b\sqrt{-3}$ for some $a, b \in \mathbb{Q}$, which implies $b = 0$ since $1 + \sqrt{3} \in \mathbb{R}$. But then since $\ker \phi = \{0\}$, and for whatever integers n, m, $\phi(a) = \phi((n)(m^{-1})) = \phi(n)\phi(m^{-1}) = nm^{-1} = n/m = a$, $\phi(1 - a + \sqrt{3}) = 1 - a + \sqrt{3} = 0 \Rightarrow a = 1 + \sqrt{3} \in \mathbb{Q}(\sqrt{-3})$, a contradiction. Therefore, there is no such isomorphism. □

Problem 3. Suppose $f(x)$ and $g(x)$ are irreducible over a field F and that $(\deg f, \deg g) = (n, m) = 1$. If $f(a) = 0$ in some extension $F(a)$ of F, then $g(x)$ is irreducible over $F(a)$.

Proof. By (Theorem 20.2), let K be the splitting field for $f(x)g(x)$ over F. Then let $a, b \in K$ such that $f(a) = g(b) = 0$. Then $[F(a) : F] = n$ and $[F(b) : F] = m$ since as vector spaces over F, $F(a)$ and $F(b)$ has a bases $\{1, a, ..., a^n\}$ and $\{1, b, ..., b^n\}$, respectively. Then by (Theorem 21.5), $[F(a, b) : F] = [F(a, b) : F(b)][F(b) : F] \Rightarrow [F(b) : F] = m \mid [F(a, b) : F]$. Thus for some $q \in \mathbb{N}$, $m = q[F(a, b) : F] \Rightarrow mn = q[F(a, b) : F][F(a) : F] \Rightarrow m \mid [F(a, b) : F(a)]$, from which it follows that $m \mid \deg p(x)$ where $p(x)$ is the minimal polynomial for b over $F(a)$. But certainly $m \geq p(x)$ by (Theorem 17.5.1), from which it follows that $g(x)$ is the minimal polynomial for b, thus $g(x)$ is irreducible over $F(a)$. □

Problem 4. $\mathbb{Z}_2[x]/\langle x^3 + x^2 + 1 \rangle$ is a field with 8 elements.

Proof. Let $p(x) = x^3 + x^2 + 1$. To show $\mathbb{Z}_2[x]/\langle p(x) \rangle$ is a field, observe that $p(x)$ is irreducible over $\mathbb{Z}_2$: $p(1) = 1^3 + 1^2 + 1 = 1$ and $p(0) = 0^3 + 0^2 + 1 = 1$, thus by (Theorem 17.1), $p(x)$ is irreducible. It follows that $\mathbb{Z}_2[x]/\langle p(x) \rangle$ is a field by (Theorem 17.5.1). To show $\mathbb{Z}_2[x]/\langle p(x) \rangle$ has 8 elements, for any element $f(x) + \langle p(x) \rangle = g(x) + q(x)p(x) + \langle p(x) \rangle = g(x)$, where

$g(x) = 0$ or $g(x)$ has degree not more than 2, by the division algorithm for polynomials over a field (Theorem 16.2). The 8 coset representatives are exhaustively,

$$\left\{0, 1, x, x^2, 1+x, 1+x^2, x+x^2, 1+x+x^2\right\}.$$

$\square$