CS 161 Fundamentals of Artificial Intelligence
Lecture 4
Informed Search Algorithms

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Outline

- Best-first search
- A^* search
- Heuristics

Review: Tree search

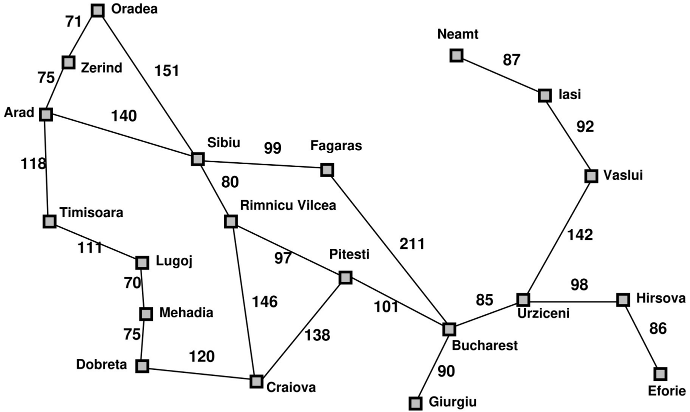
```
function TREE-SEARCH( problem, fringe) returns a solution, or failure
fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
loop do
if fringe is empty then return failure
node ← REMOVE-FRONT(fringe)
if GOAL-TEST[problem] applied to STATE(node) succeeds return node
fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

A strategy is defined by picking the **order of node expansion**

Best-first search

Idea: use an **evaluation function** for each node
– estimate of "desirability"
$\Rightarrow$ Expand most desirable unexpanded node
Implementation:
fringe is a queue sorted in decreasing order of desirability
Special cases:
greedy search
A^* search

Romania with step costs in km



Straight−line distance to Bucharest	
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	176
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
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Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Greedy search

Evaluation function $h(n)$ (**h**euristic)
$\quad = $ estimate of cost from n to the closest goal
E.g., $h_{\mathrm{SLD}}(n) =$ straight-line distance from node n to Bucharest
Greedy search expands the node that **appears** to be closest to goal

Greedy search example

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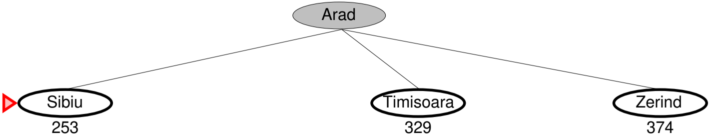
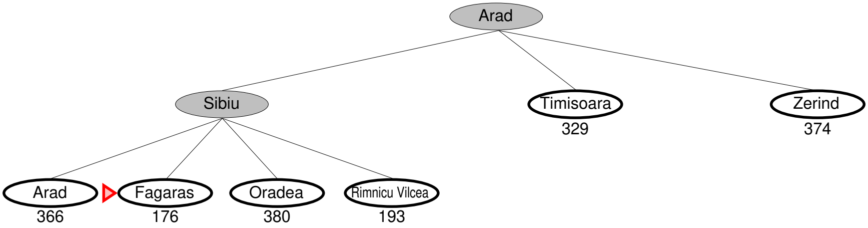


Greedy search example



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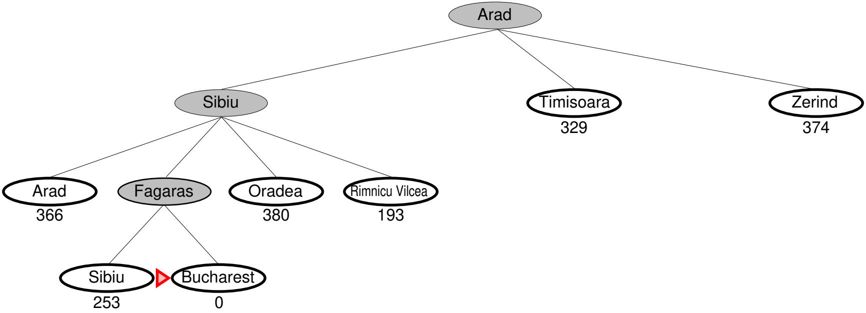

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Space?? $O(b^m)$—keeps all nodes in memory
Optimal?? No

A* search

	$f(n)$
Uniform cost	$g(n)$
Greedy	$h(n)$
A^*	$g(n) + h(n)$

Idea: avoid expanding paths that are already expensive

Evaluation function $f(n) = g(n) + h(n)$

$g(n)$ = cost so far to reach node n

$h(n)$ = estimated cost to goal from n

$f(n)$ = estimated total cost of path through n to goal

A* search uses an **admissible** heuristic

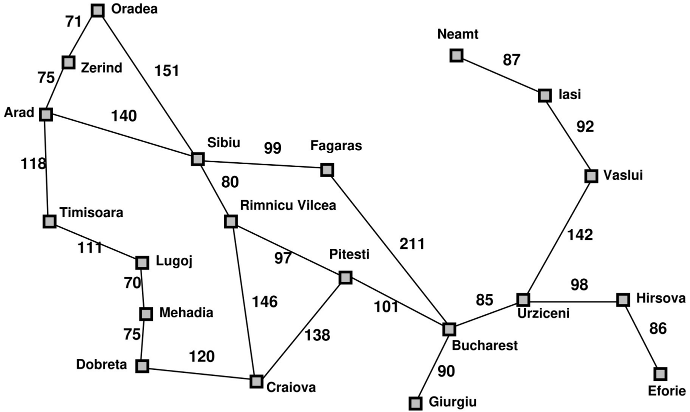
i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the **true** cost from n.

(Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G.)

E.g., $h_{\mathrm{SLD}}(n)$ never overestimates the actual road distance

Theorem: A* search is optimal

Romania with step costs in km



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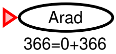

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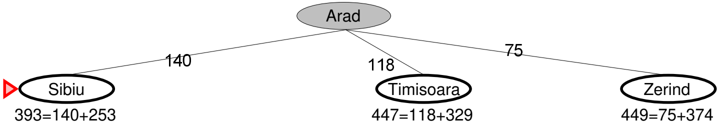

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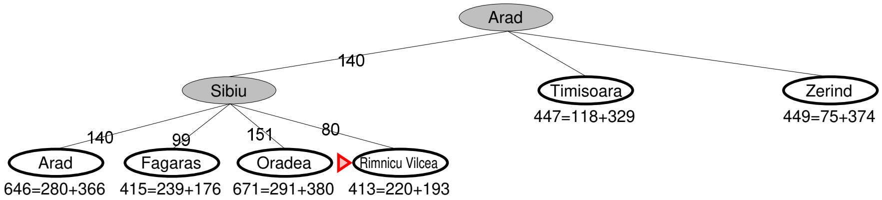

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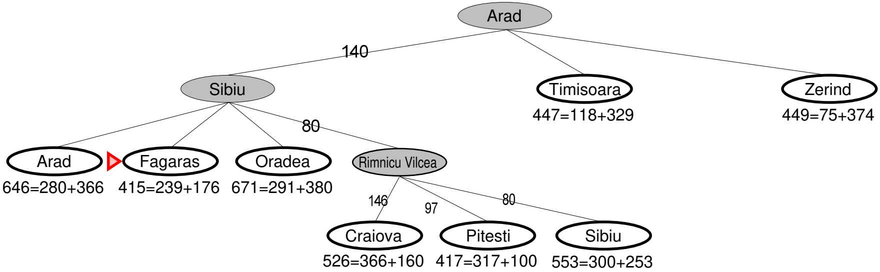

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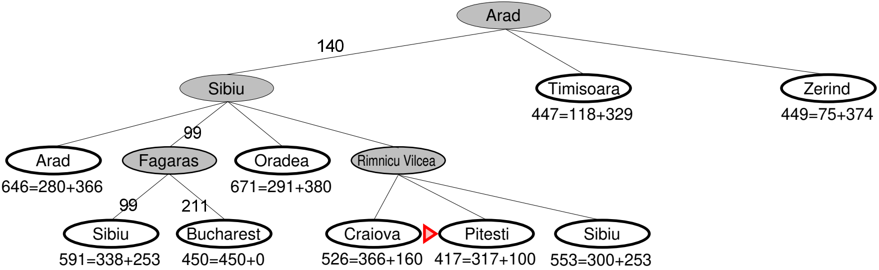

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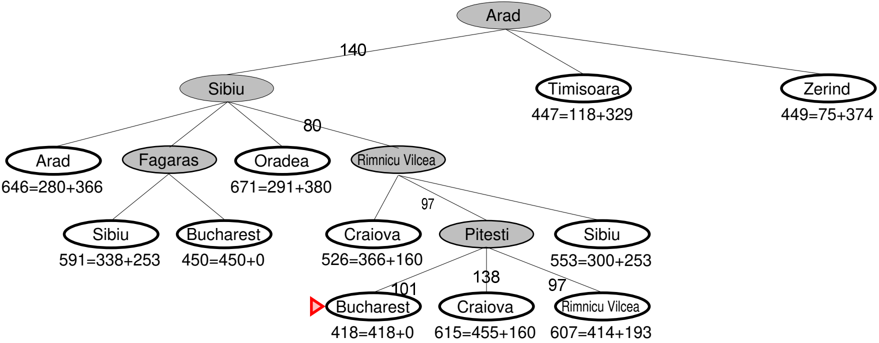
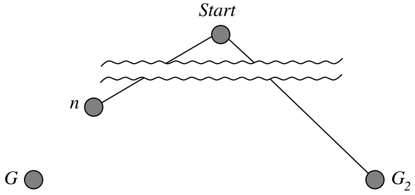


Optimality of A* (standard proof)

Suppose some suboptimal goal G_2 has been generated and is in the queue. Let n be an unexpanded node on a shortest path to an optimal goal G.



$+h(G_2)$

$$
\begin{aligned}
f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\
&> g(G) && \text{since } G_2 \text{ is suboptimal} \\
&= g(n) + h^*(n) && \text{By the definition of } h^*(n) \\
&\geq g(n) + h(n) && \text{since } h \text{ is admissible} \\
&= f(n)
\end{aligned}
$$

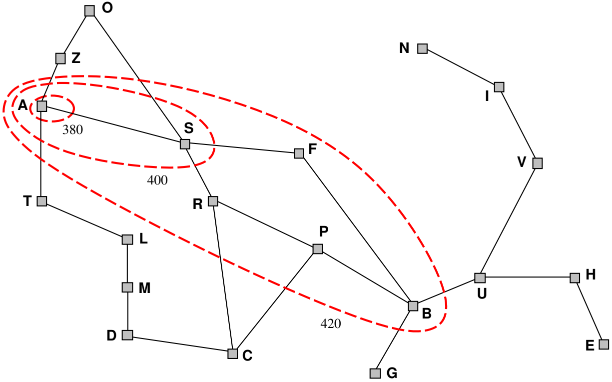
Since $f(G_2) > f(n)$, A* will never select G_2 for expansion

Optimality of A* (more useful)

Lemma: A* expands nodes in order of increasing f value*

Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)

Contour i has all nodes with $f \leq f_i$, where $f_i < f_{i+1}$



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Time?? $O(b^{\Delta})$, $\Delta = h^* - h$, h^*: actual cost from the root to the goal; h: estimated cost

Proof idea: $f(G) - f(S) = \Delta$, S: starting node. f increase along path $\rightarrow O(b^{\Delta})$(time complexity of BFS)

$$f(S) = g(S) + h(S) = h(S)$$

$$f(G) = h^*(S)$$

$$\Rightarrow f(G) - f(S) = h^*(S) - h(S) \stackrel{\triangle}{=} \triangle$$

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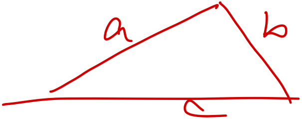
A* expands all nodes with $f(n) < C^*$, C^* is the cost of optimal solution path

A* expands some nodes with $f(n) = C^*$

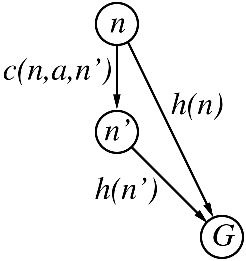
A* expands no nodes with $f(n) > C^*$

Proof of lemma: Consistency



A heuristic is **consistent** if

$$h(n) \leq c(n, a, n') + h(n')$$



where $c(n, a, n')$ is the cost of path from n to n' by choosing action a

If h is consistent, we have

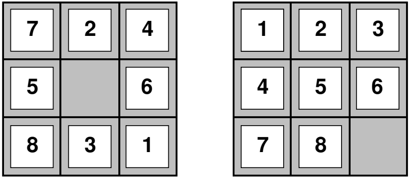
$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n, a, n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

I.e., $f(n)$ is nondecreasing along any path. **Thus, the goal state with the lowest f-cost will be found first**

Every consistent heuristic is admissible!

Admissible heuristics

E.g., for the 8-puzzle:
$h_1(n)$ = number of misplaced tiles
$h_2(n)$ = total **Manhattan** distance
(i.e., no. of squares from desired location of each tile)



Start State **Goal State**

$\underline{h_1(S) =}$??
$\underline{h_2(S) =}$??

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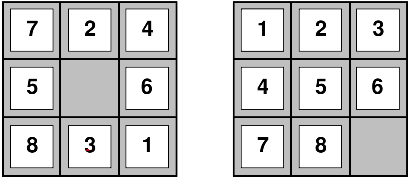


Start State Goal State

$h_1(S) =$?? 6 $= 1+1+1+1+1+1$
$h_2(S) =$?? 4+0+3+3+1+0+2+1 = 14

Relaxed problems

Admissible heuristics can be derived from the **exact** solution cost of a **relaxed** version of the problem

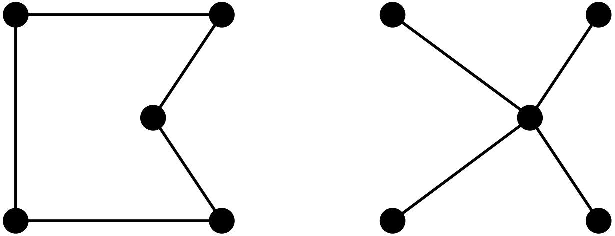
If the rules of the 8-puzzle are relaxed so that a tile can move **anywhere**, then $h_1(n)$ gives the shortest solution

If the rules are relaxed so that a tile can move to **any adjacent square**, then $h_2(n)$ gives the shortest solution

Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Relaxed problems contd.

Well-known example: **travelling salesperson problem** (TSP)
Find the shortest tour visiting all cities exactly once



Minimum spanning tree can be computed in $O(n^2)$
and is a lower bound on the shortest (open) tour

Summary

Heuristic functions estimate costs of shortest paths
Good heuristics can dramatically reduce search cost
Greedy best-first search expands lowest h
– incomplete and not always optimal
A* search expands lowest $g + h$
– complete and optimal
– also optimally efficient (up to tie-breaks, for forward search)
Admissible heuristics can be derived from exact solution of relaxed problems

Acknowledgment

The slides are adapted from Stuart Russell, Guy Van den Broeck et al.