

# Essential Demographic Methods

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## Preface

This book has grown out of courses in demographic methods which I have taught at Berkeley over the last twenty-five years. The students range from freshmen to graduate students, and they bring a wide range of backgrounds to these courses. I have tried to make the book accessible to those who are new to quantitative analysis, but also to keep it interesting to those with prior training in mathematics and social sciences. The advanced, graduate-level material (and any use of calculus) is confined to sections marked with a star (\*).

For many students, one of the attractions of a course in demographic methods is the opportunity to develop or refresh quantitative skills and overcome math anxiety while studying subjects of direct human and personal interest. There are no official prerequisites for the courses I teach. A high degree of fluency with the simpler parts of high-school mathematics is an advantage. A little calculus is used in the starred sections, but I have tried to explain it along the way and make the presentation self-contained.

Studying demographic methods means, above all, drawing analogies and exploring examples. It thus provides opportunities to enjoy snippets from our cultural inheritance along the way. In this book, the reader will come across a personal sampling of some of my favorite passages and occasional excursions into metaphor.

I am grateful to the many students who have helped me develop the approach presented in this book. I am especially grateful to my outstanding Graduate Student Instructors, Robert Chung, Debbie Blackwell, Bill Chu, Josh Goldstein, Laura Hill, Jie Huang, Aaron Gullickson, and Bryan Sykes, and to Mike Anderson as Visiting Professor, who have given great support and advice. Comments and suggestions for further improvements would be most welcome.

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# Introduction

## Why study demography?

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As children, our earliest steps forward in awareness take us into demography, as we number our birthdays, put numbers on the ages of people around us, number the others in our family, our town, our country, and our planet, as we learn that the years we will have to live are numbered.

As adults, we cope with a world whose problems and opportunities are shaped by demography. Low growth rates and high growth rates form the backdrop to the contrasts between rich nations and poor nations, peace and conflict, environmental protection and degradation that form the stuff of each evening's news. The expanding proportion of elderly in developed societies challenges the social insurance and welfare systems. Reduced fertility is part and parcel of the transformation of gender roles reshaping social consciousness. Migratory movement puts strains on social cohesion. The transformations of marriage and family come to dominate political debate. Longer average lifespans give many of us new flexibility in planning out our life's fulfillments.

It is one thing to speculate freely about such trends, as all of us do. It is another to gain precise information about their extent so that definite numerical comparisons become possible. The science of demography has grown up over the last several centuries out

of efforts to make measurements of population and lifecourse in consistent and meaningful ways. As such, it is an example, one of the best, of mathematics at work on the world.

Some of the readers of this book may, conceivably, have been born to be demographers. Demography is an attractive career, and this book provides a foundation for a professional career in demography. But others come to this subject, not because they plan to stay with it, but because of its intrinsic intellectual appeal. Demography is the quintessential quantitative social science. It bears something of the same relationship to other social sciences that physics bears to other natural sciences. Economics, statistics, and sociology have all grown up in tandem with demography. Of all the sciences of human behavior, demography is the one where it is easiest to count. The earliest statistics, in the *Book of Numbers* in the *Bible* and the catalogue of ships in the *Iliad*, were demographic statistics.

There are three main reasons for the centrality of demography. First is the simplicity of the units of analysis. In sociology and psychology and many other fields it is difficult to settle on the right conceptual units for quantification. In demography, what to count, in the first instance, is obvious – men and women, households, families, old folks, young folks, babies, corpses. There is little fuzziness about the basic units. Second, demographic processes, especially fertility and mortality, are fundamentally more regular and law-like than many other aspects of behavior. Demographic models are closer to the phenomena themselves than models in many other disciplines, with more hope of real predictive content. Third, demography relies heavily on all the other social sciences which bear on human motivation and response, and it reaches out to them, having an area of study in common with each of them. It is interdisciplinary in fact, not merely in aspiration.

## Measurements and models

In this book we shall switch back and forth constantly between



measures and models. When we study a calculation to measure something, we shall always have a model in mind or in the background. A measurement is more than just a number computed in a specified way. As we use the term, a measurement is a number which has a particular role in a predictive or explanatory model. Consider some examples of things people measure – the probability of rain, the population growth rate, the distance from the earth to the sun. All these are concepts which look natural enough on the surface, but, on closer examination, it becomes less than obvious what one means or what one wants to measure. Consider the distance to the sun, from here to up there. If we want a precise measurement, problems come up. The sun is farther away in summer than in winter. Do we want to measure from the surface or the middle of the earth? What is the middle of our slightly pear-shaped planet? However, suppose we want to define the distance from earth to sun in the context of Kepler’s Laws for planetary motion. Then the right thing to measure is the maximum distance from the center of gravity of the earth to the center of gravity of the sun, because this is the distance which is involved in simple formulas for the motion of the earth around the sun. To know what we want to measure, we need a model which shows us how we are going to make use of the measurements for prediction or explanation.

Just the same is true of our first concept, the concept of a “growth rate” for a population. To know what to measure, we need a model within which our measurement has sense. The model for “exponential population growth” found in Chapter One is the simplest demographic model. It is a model introduced for defining population growth rates in a meaningful way.

## Sources of demographic knowledge

Defining appropriate measures is important, but having sources of information from which to calculate them is just as important. Demographic information comes principally from four main kinds

of sources – censuses, vital records, surveys, and ethnographic observations.

Censuses are the oldest, most demanding, and most important source of demographic information. They go back to Biblical times; King David took a Census, and, the Book of Kings tells us, was punished by God for doing so. The Roman and Chinese Empires both took periodic censuses which today supply some of our best knowledge about ancient times. In the United States the Census is established by Article I of the Constitution, and censuses are taken every ten years by law.

Vital records are the second main source for demographers. All modern governments register births, deaths, and marriages and collect and tabulate totals from these records.

Sample surveys have become more and more important as statistical science has developed. The U.S. has a highly developed Federal Statistical System, including regular periodic surveys like the Current Population Survey, and special surveys like the National Survey of Families and Households or the Health and Retirement Survey. In the last two decades, coordinated demographic surveys have been taken around the world through the World Fertility Survey and the Demographic and Health Surveys of the Agency for International Development.

Ethnographic observations are the fourth source, for example, information gathered from open-ended interviews or from systematic notes from “participant observers”. Demographers are coming to recognize the importance of going behind the standardized information that censuses, vital records, and surveys provide and probing the complexity of personal strategies and motivations.

## Structure of this book

We begin with two chapters on the most basic concepts of demography, exponential growth and the distinction between cohorts

and periods. A cohort is a group of individuals, for example, those born in a certain year, followed through time as they age together. All members of a cohort have the same age at the same time. If we look at a population in a period or interval of time, different members of the population will have different ages at the same time. Demographic measures are conceptually simpler when they pertain to cohorts. But they are more useful for practical purposes when they pertain to periods. In this book cohort measures come first, for mortality in Chapter 3 and fertility in Chapter 4. There is a discussion of population projection in Chapter 5, and then we return to period measures, this time taking fertility first in Chapter 6 and mortality next in Chapters 7 and 8. Measures of marital status are described in Chapter 9. The mathematical theory called Stable Population Theory which helps demographers think about the distribution of population by age, with all its economic, political, and social implications, is the theme of Chapter 10. We return to fertility in Chapter 11 to present specialized measures of family limitation, and conclude in Chapter 12. The appendix collects useful formulas for ready reference.



# Chapter 1

## Exponential Growth

### 1.1 The Balancing Equation

The most basic demographic equation is the “Balancing Equation”. The Balancing Equation for the world as a whole from 2000 to 2001 takes the form

$$K(2000) + B(2000) - D(2000) = K(2001)$$

Here  $K(2000)$  is the world population at the start of 2000,  $B(2000)$  are the births *during* 2000,  $D(2000)$  are the deaths during 2000, and  $K(2001)$  is the population at the start of 2001. The letter “K” is traditionally used for population, instead of “P”, to avoid confusion with probability which also starts with p. Estimated values for the numbers that go into the balancing equation for the millenium year 2000, taken from the U.S. Census Bureau’s International Data Base on the World Wide Web, are shown in Table 1.1

The world population is a “closed population”. The only way to enter is by birth. The only way to exit is by death. The general form of the Balancing Equation for a closed population is

$$K(t) + B(t) - D(t) = K(t + n)$$

Table 1.1: The World Population 2000 to 2001.

	Population 1 January 2000	:	6,048 million
+	Births 2000	:	+130 million
-	Deaths 2000	:	-55 million
<hr/>			
=	Population 1 January 2001	:	6,123 million

Here  $K(t)$  is the size of the population at time  $t$ ,  $n$  is the length of a period like one year or ten years, and  $B(t)$  and  $D(t)$  are the births and deaths *during* the period from  $t$  to  $t + n$ .

The Balancing Equation for national or regional populations is more complicated, because such populations also change through migration. The U.S. Census Bureau keeps track of the nation's population by age, sex, and race from decade to decade with a refinement of the Balancing Equation called "Demographic Analysis" with a capital "D" and capital "A". The Bureau's Balancing Equation includes several kinds of migration. In much of this book, however, we shall be studying models for closed populations. This is a simplification which will help us to understand basic concepts more easily. But no one should imagine that most national or local populations are at all close to being closed.

Six billion, in American English, is six thousand million, It is written with a 6 followed by nine zeros, or  $6 * 10^9$ . Computers have popularized words for large numbers based on Greek prefixes: *kilo* for thousands ( $10^3$ ), *mega* for millions ( $10^6$ ), *giga* for billions ( $10^9$ ), and *tera* for trillions ( $10^{12}$ ). In this terminology, the world contains six gigapersons.

An important pattern can be seen when we consider a closed population and combine the Balancing Equation for this year with the Balancing Equation for next year. We let the present year correspond to  $t = 0$  and let our interval be  $n = 1$  year long. The assumption of closure gives us

$$K(1) = K(0) + [B(0) - D(0)]$$

The square parentheses are inserted to point out that we are decomposing next year's population "stock" into this year's "stock" plus this year's "flow". Of course the sizes of the flows are very much dependent on the size of the stock. More people mean more births and deaths. We rewrite the same equation in a form which separates out the elements which vary a lot from case to case and time to time from those that vary less. We do so by multiplying and dividing the right-hand side by the starting population  $K(0)$ . Multiplying and dividing by the same thing leaves the equation unchanged.

$$K(1) = K(0) \left( 1 + \frac{B(0)}{K(0)} - \frac{D(0)}{K(0)} \right)$$

The same equation for the following year is

$$K(2) = K(1) \left( 1 + \frac{B(1)}{K(1)} - \frac{D(1)}{K(1)} \right)$$

Here  $K(1)$  means the population at the start of year 1, but  $B(1)$  and  $D(1)$  mean the births and deaths in the whole of year 1, from time  $t = 1.0000$  to  $t = 1.999999....$  Substituting for  $K(1)$  in this equation gives

$$K(2) = \left( 1 + \frac{B(1)}{K(1)} - \frac{D(1)}{K(1)} \right) \left( 1 + \frac{B(0)}{K(0)} - \frac{D(0)}{K(0)} \right) K(0)$$

The important thing about this equation is that we go from a starting population to a later population by using **multiplication**. The obvious fact that  $B/K$  and  $D/K$  are less directly dependent on  $K$  than  $B$  and  $D$  themselves makes population growth appear as a **multiplicative** process. *Multiplicative* growth is sometimes called "*geometric*" growth; geo and metric come from Greek roots for earth and measure. Ancient Greeks and Egyptians multiplied by laying out two lengths as the sides of a rectangle and measuring the area of ground. The name "geometric growth" applies for growth through whole time intervals. When fractions of intervals are also involved, we need to make use of the exponential function, as we shall see, and geometric growth is then called "*exponential*" growth.

Our equations are most interesting in the particularly simple case when the ratios  $B/K$  and  $D/K$  are not changing enough to make any difference. Then we can drop the time labels on the ratios. Let

$$A = 1 + \frac{B}{K} - \frac{D}{K}$$

Our equations take the form

$$\begin{aligned} K(1) &= A K(0) \\ K(2) &= A^2 K(0) \\ K(3) &= A^3 K(0) \\ &\vdots \\ K(T) &= A^T K(0). \end{aligned}$$

During July 2000 newspapers announced that world population had just passed the six-billion mark, but revised estimates now indicate that there were already six billion people by the middle of 1999. Population at the start of the new millenium was about 6.048 billion, with births exceeding deaths by about 75 million during the year. These estimates imply a value for  $A$  equal to  $1 + B/K - D/K = 1 + 75/6048$  which is about 1.0124. Letting  $t = 0$  stand for the year 2000, and assuming roughly constant levels for the ratios of births and deaths to total population, our formula predicts the following populations in billions:

$$\begin{aligned} K(0) &= 1.0124^0 * 6.048 = 6.048 \\ K(1) &= 1.0124^1 * 6.048 = 6.123 \\ K(2) &= 1.0124^2 * 6.048 = 6.199 \\ K(3) &= 1.0124^3 * 6.048 = 6.276 \\ K(4) &= 1.0124^4 * 6.048 = 6.354 \\ K(5) &= 1.0124^5 * 6.048 = 6.432 \end{aligned}$$

Naturally, imprecision in the estimates, uncertainties about the future, and rounding errors in the calculations make our answers only approximations to the truth. Nonetheless, they give a useful



idea of the pace of current population growth. A short five years from the turn of the millenium, there are likely to be nearly 400 million more people in the world.

## 1.2 The Growth Rate R

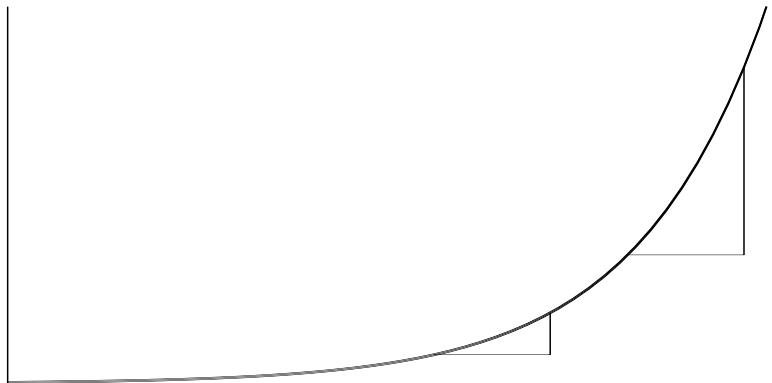
The Balancing Equation for a closed population led directly to an equation for population growth,

$$K(T) = A^T K(0).$$

Built into this equation is the assumption that the ratio of births to population size  $B(t)/K(t)$  and the ratio of deaths to population size  $D(t)/K(t)$  are not changing enough to matter over the period from  $t = 0$  to  $t = T$ . (We generally use small “t” to stand for time in general, and capital T to stand for a particular time like the end of a period.).

When births exceed deaths,  $A$  is bigger than 1, and the population is increasing. Keeping the same value of  $A$  from year to year, a graph of  $K(t)$  as a function of  $t$ , would look like Figure 1.1.

Figure 1.1:  $K(t)$  with Ever-Changing Slope

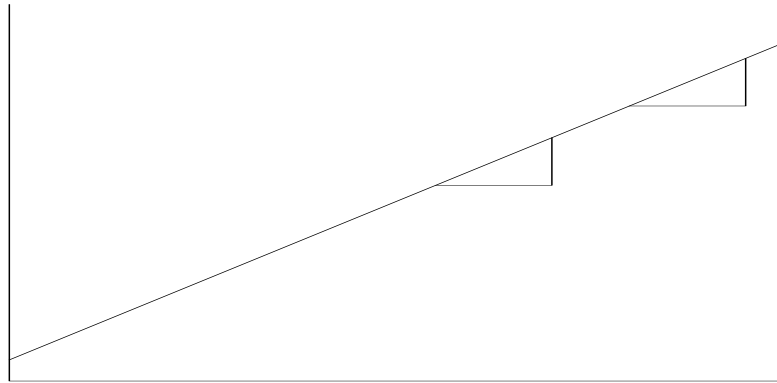


Not only is the population  $K(t)$  increasing with time. The amount of increase in population per unit time is also increasing;

the graph gets steeper as it gets higher. The two triangles in the picture highlight this steepening. They have the same base, but the right-hand triangle has much greater height. The amount of increase per unit time is the slope of the  $i$  graph. While we often measure a rate by the slope of a graph, we cannot very well measure the growth rate of the population by the slope of the graph of  $K(t)$  versus  $t$ , because that slope does not stay fixed. It keeps changing even when the demographic quantities in our model  $B/K$  and  $D/K$  are assumed unchanging. We need a measure of growth which stays fixed when  $B/K$  and  $D/K$  are fixed.

The solution is to take logarithms of  $K(t)$ . Taking logarithms is the usual way of converting multiplication into addition. When  $B/K$  and  $D/K$  are unchanging, the graph of  $\log K(t)$  versus  $t$  has constant slope. In other words, the graph is a straight line. Taking logarithms converts the curve of Figure 1.1 into the straight line in Figure 1.2

Figure 1.2:  $\log K(t)$  with Constant Slope



In symbols, taking logarithms converts the equation

$$K(T) = A^T K(0)$$

into the equation

$$\log(K(T)) = \log(K(0)) + \log(A)T$$

This is a *linear* equation.. It has the form  $y = a + bx$  if we put  $y$  in for  $\log(K(T))$  and  $x$  for time  $t$ . The intercept  $a$  where the graph

crosses the vertical axis is  $\log(K(0))$ , and the slope  $b$  is  $\log(A)$ . We shall call the slope  $R$ . It is our measure of population growth.

The value of  $R$  for world population in 2000, based on Table 1.1, would be  $\log(1 + (130 - 55)/6048) = 0.012324$ , or about 12 per thousand in round numbers. It is a statistic worth keeping in mind.

World population grew at a rate of about 12 per thousand per year at the turn of the millenium in 2000.

We always use logarithms to the base  $e = 2.71828\dots$ . These are “natural logarithms”. For consistency with computer languages, we write “log” for natural logarithm, not “ln” which is the older notation. When working with hand calculators, it is important to check whether a button labeled “log” or a button labeled “ln” is the correct button to use for the natural logarithm. The number “e” is the choice for  $A$  which makes the slope of the graph of  $K(t)$  equal 1 when  $t = 0$  and  $K(0) = 1$ ; this is the mathematical definition of “e”.

The key concept here is to measure growth on a **logarithmic** scale. We rephrase the idea in the form of a definition:

The population **growth rate**  $R$  is the slope of the graph of the logarithm of population size over time.

This definition gives us a measure of growth which is unchanging when the ratios of births and deaths to population size are unchanging, and only changes when the ratios change.

Now the slope of any graph is equal to the “rise” over the “run”. The “rise” is the vertical increase corresponding to a given “run” along the horizontal axis. When we calculate the growth rate from time  $t = 0$  to time  $t = T$ , the run is the length of the

interval,  $T - 0$  or  $T$ , and the rise is the increase in  $\log(K(t))$ . The slope is the quotient:

$$R = \frac{\log(K(T)) - \log(K(0))}{T - 0}$$

Using rules for differences of logarithms, we see that this expression is the same as the expression

$$R = \frac{1}{T} \log \left( \frac{K(T)}{K(0)} \right)$$

If we are measuring growth between two times  $t_1$  and  $t_2$  we have another version of the same formula:

$$R = \frac{\log(K(t_2)) - \log(K(t_1))}{t_2 - t_1}$$

Any one of these versions of the formula can be used when we need to calculate a rate of population growth.

In contrast to present-day growth rates, it is interesting to calculate  $R$  for earlier periods, for example, for the population of the world between the start of agriculture around 8000 B.C. and the birth of Christ. Guesses at populations for these epochs, 5 million and 250 million, are given in Table 1.3. We calculate the growth rate  $R$  to be

$$R = (1/8000) \log(250/5) = 0.000489.$$

This value is less than 1 per thousand per year. The units (millions of people) on 250 and 5 cancel when we take their ratio, while the units on 8000 – that is, years – in the denominator give us a measure *per year*.

If we average the growth rates for two successive years, say this year ( $t$  goes from 0 to 1) and next year ( $t$  goes from 1 to 2), we end up with the overall growth rate for the two year period from  $t = 0$  to  $t = 2$ :

$$\frac{R_0 + R_1}{2} = \frac{1}{2} \left( \log \left( \frac{K(1)}{K(0)} \right) + \log \left( \frac{K(2)}{K(1)} \right) \right)$$

$$\begin{aligned}
&= \frac{1}{2} \log \left( \frac{K(1)}{K(0)} \frac{K(2)}{K(1)} \right) \\
&= \frac{1}{2} \log \left( \frac{K(2)}{K(0)} \right)
\end{aligned}$$

This property of our formula for growth is convenient. Knowledge of intermediate populations give us no extra information when we seek the average growth rate between the start and the end of a long period.

The world's growth rate is believed to have peaked between midyear 1962 and midyear 1964 when the world population as registered on the "World Clock" maintained by the U.S. Census Bureau grew from 3,136,556,092 to 3,277,192,099. This peak growth rate is calculated in the same fashion:

$$R = (1/2) \log(3,277,192,099/3,136,556,092) = 0.021931$$

Of course no one knows the population down to the nearest person, or even down to the nearest ten million. Population totals for countries are thought to be accurate only within 2% or 3% at best. It is sometimes useful, however, to cite totals down to the last unit, to make it easier to pinpoint the numbers' source. As for growth rates, demographers usually quote growth rates in parts per thousand per year. A demographer who says that the world's growth rate peaked at "22" means it peaked at "22 per thousand per year".

## 1.3 The Exponential Curve

We return now from our discussion of growth rates to our formula for the population over time in the case when the ratios of births and deaths to population remain constant.

$$K(t) = A^t K(0)$$

We rewrite this formula in the form

$$K(t) = e^{Rt} K(0) = \exp(Rt) K(0)$$

Here  $Rt$  is the “exponent” of the number “e”. We are raising the number “e” to the power  $Rt$ , and another notation for this quantity is  $\exp(Rt)$ , the “exponential function” of  $Rt$ . The exponential function is the inverse function for natural logarithms. That is to say, for any  $x$  or  $y$ ,

$$e^{\log(x)} = \exp(\log(x)) = x$$

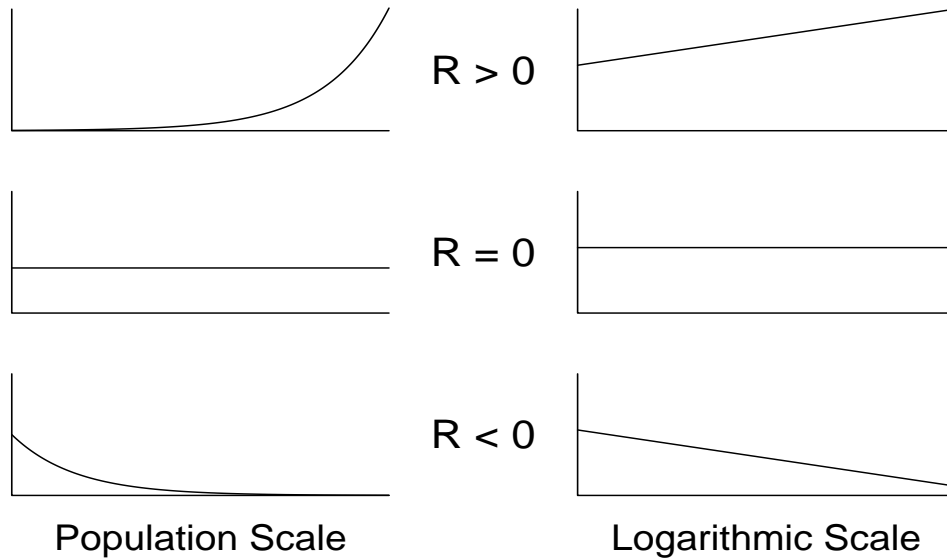
and

$$\log(e^y) = \log(\exp(y)) = y.$$

We know from our linear equation that  $\log(A)$  is  $R$ , so we have

$$A = e^{\log(A)} = e^R \text{ and } A^t = (e^R)^t = e^{Rt} = \exp(Rt).$$

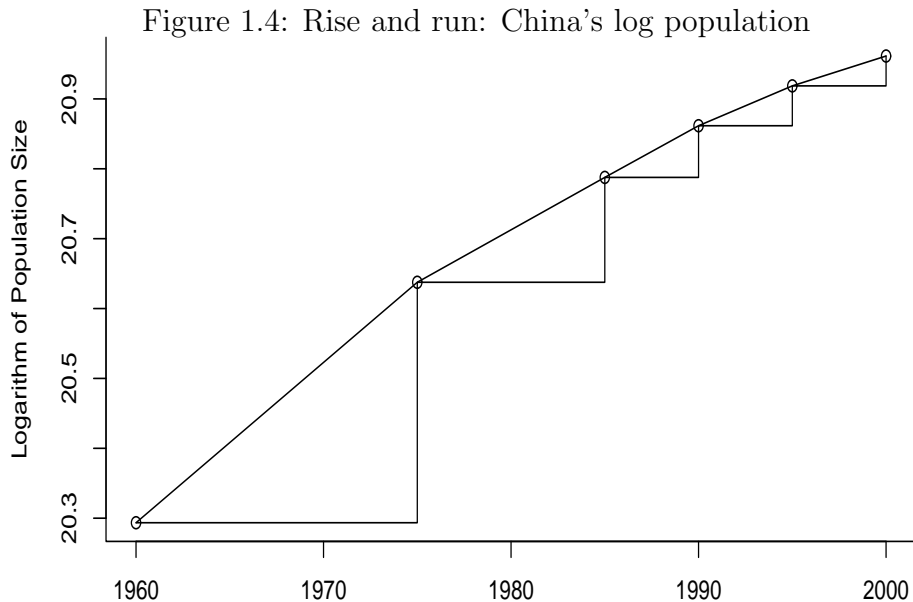
Figure 1.3: Trajectories of Exponential Growth



These facts about logarithms show us that our equation with  $A$  and our equation with  $e$  are the same equation. The graph of  $\exp(Rt)$  as a function of  $t$  is called the “exponential curve”, the continuous-time version of the curve for geometric growth. It is important to become familiar with its shape. The shape depends on whether  $R$  is greater than, equal to, or less than zero, as shown

in Figure 1.3. If the ratios  $B/K$  and  $D/K$  remain unchanged, the population either grows faster and faster beyond all bounds, stays the same, or heads toward zero steeply at first and more gradually later. The most common case is the case in the top panel of Figure 1.3, the case when births exceed deaths and when  $R > 0$ . This is the case of exponential growth that leads so many people to feel alarm at the future prospects for humanity on the planet.

Our formula for exponential growth treats the growth rate  $R$  as a constant. To apply it in practice, we break up any long period of time into shorter intervals within which the growth rate does stay more or less constant and proceed step by step. The graph of the logarithm of population size becomes a series of straight-line pieces. In each step we “run” forward in time by some amount  $n$ , and our graph has a corresponding “rise” equal to  $n$  times  $R$ . The value of  $R$  at the step is the growth rate at that step, the current slope of the logarithm of  $K(t)$ , which we know has to equal the rise over the run.



The process is like going up a staircase. Figure 1.4 shows such a “staircase plot” for China, the country with the largest

population in the world. Intervals and growth rates are found in Table 1.2. The population estimate at the start in 1960 is  $K(0) = 650,660,513$ . Its logarithm is  $\log K(0) = 20.2935$ .

To go from 1960 to 1975, we have a run of  $n = 15$  years and a rise of  $Rn = 0.0229 * 15 = 0.3435$ . Log population goes up from 20.2935 to  $20.2935 + 0.3435 = 20.6370$ . At the next step, from 1975 to 1985, we go up by  $0.0150 * 10 = 0.1500$ , as in the second row of Table 1.2. In five steps from 1960 to 2000, log population rises by the amounts shown in the last column of the table, which sum to 0.6670 and bring log population up to  $20.2935 + 0.6670 = 20.9605$ . The population estimate for 2000 is the exponential of this logarithmic value,  $\exp(20.9605)$ , which is 1.268 billion.

Table 1.2: Growth Rates in China.

date	t	n	R	R n
1960	0	15	0.0229	.3435
1975	15	10	0.0150	.1500
1985	25	5	0.0148	.0740
1990	30	5	0.0114	.0570
1995	35	5	0.0085	.0425
2000	40	4	0.0058	

Since growth rates are rounded to four decimal places, these estimates have limited precision, but they come within ten million or so of official figures. We see how  $R$  has decreased interval by interval, moderating the slope, in the wake of China's energetic program of family limitation. We also see how even moderate slopes continue to drive substantial multiplicative growth.

This example for China illustrates how we go from information about growth rates to estimates for population. The formula for each step involves simple addition:

$$\log K(t + n) = \log K(t) + Rn$$



This addition corresponds to a multiplication of population size itself by an exponential factor:

$$K(t+n) = K(t)e^{Rn}$$

. A handy way to perform such staircase calculations is to adjoin columns to Table 1.2 in which we keep track of the sum of the  $Rn$  values and the implied value for log population and its exponential, our estimate of population itself. We can also work backward from our staircase plot. Suppose we want to find out when China's population surpassed a billion people. The logarithm of a billion is 20.7233. If we draw a flat line across Figure 1.4 at this height, we see it crosses the graph between 1975 and 1985. Beyond 1975, we need a rise of  $20.7233 - 20.6376 = 0.0857$  to reach our target, so we need a run of  $0.0857/R = 0.0857/0.0150 = 5.713$  years, or about five years and eight months, yielding August 1980 as our estimate for the date at which China's population passed a billion.

## 1.4 \*Taylor Series

The multiplicative property of population growth can matter a great deal over the long term, but over the short term growth can look very nearly additive. The calculated populations  $K(0), K(1), \dots, K(6)$  in the example in Section 1.1 were obtained by multiplication, but if we had simply added 80 million each year, we would have come out with nearly the same populations. How short is "short-term"? How long does it take for the multiplicative property to become important?

We can find answers to this question and to other questions about exponentially growing populations using Differential Calculus to produce approximations known as Taylor Series. Our exponential function  $e^{Rt}$  has the following Taylor Series:

$$e^{Rt} = 1 + \frac{Rt}{1} + \frac{R^2t^2}{(2)(1)} + \frac{R^3t^3}{(3)(2)(1)} + \frac{R^4t^4}{(4)(3)(2)(1)} + \dots$$

The numerators contain powers of  $Rt$ , and the denominators contain factorials, that is, descending products of numbers, often written  $1!, 2!, 3!$  etc. When the product  $Rt$  is small, only the first few terms are big enough to matter.

The first term in the expansion is a constant. The second term is linear; if  $t$  goes up by 1, the term goes up by the addition of the amount  $R$ . The third term shows the first multiplicative effect. The third term will be as much as, say, 10% of the second term if  $(1/10)Rt \leq (1/2)R^2t^2$  or  $t \geq 1/(5R)$ . That gives us a rough rule of thumb. For a population like China today, if we care about 10% precision, the multiplicative effect becomes important within  $1/(5 * 0.010)$  or about 20 years.

For a second application of Taylor Series, consider the competing effects of multiplicative growth and declining growth rates. According to U.S. Census Bureau estimates, world population growth rates have declined in every year since 1986, on average by about  $s = 0.000372$  per year. Taking  $t = 0$  to be 1986, the estimate for  $R(0)$  is 0.017305. Using a linear approximation for the changes in  $R$ , we have  $R(t) = R(0) - st$  for the growth rate *in* year  $t$ , or  $R(0) - (1/2)st$  for the *average* growth rate between 0 and  $t$ . We have seen that the overall growth rate over a period is the same as the average growth rate in the period. So we can plug in  $(R(0) - (1/2)st)t$  in place of  $Rt$  in our Taylor Series for  $e^{Rt}$ :

$$\begin{aligned} e^{Rt} &= 1 + R(0)t - (1/2)st^2 + (1/2)(R(0)t - (1/2)st^2)^2 \dots \\ &\approx 1 + R(0)t - (1/2)(R(0)^2 - s)t^2 \dots \\ &= 1 + 0.017305 * t - 0.000036 * t^2 \dots \end{aligned}$$

We see that the declines in the growth rate are wiping out the effects of multiplicative growth and actually changing the term in  $t^2$  to a small negative. For most of history, changes in growth rates reinforced the multiplicative effects, but in recent years they have been counteracting them.

In this respect, the United States differs from the world as a whole. Changes in growth rates have been less important than

multiplicative effects for a long time. Between the last two censuses of 1990 and 2000, U.S. population grew at a rate of about  $(1/10) \log(281.4/248.7)$  or about 12.35 per thousand per year, which is close to its overall growth rate of 13.09 per thousand over the whole Twentieth Century. The U.S. population is far from closed; it has the highest net immigration of any large country in the world. (“Net” immigration means immigrants minus emigrants, those coming in minus those going out.) But, as long as the ratio of net immigrants to starting population in each period is relatively stable, like the ratios  $B/K$  and  $D/K$ , our argument leading to the multiplicative formula for population growth remains valid, as it does in the U.S. case. The U.S. census count in 1900 was nearly 76 million. Six terms from our Taylor Series with  $R = 0.013$  would give

$$76 * (1.000 + 1.300 + 0.845 + 0.366 + 0.119 + 0.031) = 278$$

That is an excellent approximation to the 281 million of the official 2000 Census count.

For understanding Taylor Series, it is helpful to go over some facts from calculus about the exponential function. We concentrate on the case when  $R$  equals 1. Any case can be transformed into this case by reckoning time in units of  $1/R$ . For instance, if  $R$  is 0.010 per year, as in China, then  $R$  is 1 per century. If  $R$  is 0.025 as in Nigeria, then  $R$  is 1 per forty-year-interval. With  $R = 1$ , the exponential function is  $\exp(t)$  which is the same as the number  $e$  raised to the  $t$ -th power,  $e^t$ .

What is the number  $e$ ? As we have mentioned,  $e$  is defined by a condition on slopes. We start at  $t = 0$ . The starting value for the function  $A^t$  is the same for any choice of  $A$ ; it is  $A^0 = 1$ . The starting slope depends on  $A$ . When  $A = 1$ , the starting slope is zero, since  $1^t$  is always 1. When  $A$  is as big as 10, the starting slope is well above 1, as can be checked by drawing the graph. Somewhere between  $A = 1$  and  $A = 10$  is a number for which the starting slope is exactly 1, and this number is  $e$ .

The slope of  $e^t$  between  $t = 0$  and  $t = n$  is the rise from  $e^0 = 1$  up to  $e^n$ , divided by the run from 0 over to  $n$ . The starting slope

is obtained by taking the limit as  $n$  drops toward zero:

$$\lim_{n \rightarrow 0} \frac{e^n - 1}{n} = 1$$

That is the mathematical definition of the number  $e$ .

This starting slope, at  $t = 0$ , determines the slope at every other value of  $t$ . A run of the same length  $n$  from  $t$  to  $t + n$  goes with a rise from  $e^t$  to  $e^{t+n}$ , a rise of size  $e^{t+n} - e^t$  which equals  $e^t e^n - e^t$  or  $e^t(e^n - 1)$ . When we let  $n$  go to zero, we are left with  $e^t$  itself:

$$\lim_{n \rightarrow 0} e^t(e^n - 1)/n = e^t$$

This equation tells us the slope of the function  $e^t$  is equal to the value of the function  $e^t$  for all  $t$ . The function  $e^t$  is the only function whose slope always equals its value. This property is useful. Since the slope of  $e^t$  is  $e^t$ , the slope of the slope of  $e^t$  is also  $e^t$ , and the slope of the slope of the slope is  $e^t$ .

We can understand the Taylor Series for  $e^t$  in terms of these slopes. We obtain the value at  $t$  by first taking the value at zero,  $e^0$ . Then we add the rise produced by a slope of  $e^0$  applied to a run of length  $t$ . Then we add an extra rise produced by increases in the slope averaged over the interval from 0 to  $t$ . Increases in the slope come from the slope of the slope, so we multiply  $e^0$  (in its role as slope of the slope) by the average of 0 and  $t$  (which is  $t/2$ ). That give us the extra slope, and we multiply the extra slope by our run of  $t$  to find the extra rise.

We now have  $1 + t + (1/2)t^2$ . Averaging the effect of the slope of the slope of the slope, a more complicated calculation, turns out to contribute  $(1/2)(1/3)t^3$ , and further effects build up the whole series. When we plug in  $t = 1$ , we obtain the numerical value of “e” to as many decimal places as we like:

$$\begin{aligned} e &= e^1 = 1 + 1 + (1/2) + (1/2)(1/3) + (1/2)(1/3)(1/4) \dots \\ &= 1 + 1 + .5000 + .1666 + .0416 + .0083 + \dots \\ &= 2.718\dots \end{aligned}$$

We plug in  $Rt$  in place of  $t$  for the Taylor Series for  $e^{Rt}$ . The slope of  $e^{Rt}$  is  $Re^{Rt}$  and the slope of the slope is  $R^2e^{Rt}$ . The formula for slope is worth highlighting:

The slope of $e^{Rt}$ is $Re^{Rt}$ .
--------------------------------------

## 1.5 Doubling Times

Obviously, there has been an enormous acceleration in average growth rates between the dawn of civilization and our own era. The differences between these numbers are not, however very easy to visualize. In order to put population growth rates on a more intuitive scale, demographers often convert them to doubling times. The doubling time  $T_{double}$  is the time it would take a population to double at a given growth rate if the exponential model were exactly true. We start with the model

$$K(t) = e^{Rt} K(0)$$

We seek  $T_{double}$  such that

$$\begin{aligned} K(T_{double}) &= 2K(0) = \exp(RT_{double})K(0) \\ 2 &= \exp(RT_{double}) \\ \log(2) &= RT_{double} \end{aligned}$$

We conclude that

$$T_{double} = \log(2)/R = 0.6931/R$$

Table 1.3 shows some guesses at total world populations for various epochs in the past. The growth rate and doubling time for each period is shown on the row for the *starting date* of the period. Thus the average growth rate from the origins of agriculture and

settled life around 8000 B.C. up to 1 A.D., estimated at  $R = 0.000489$  is shown in the row labeled “8000 B.C.” There was no ‘year “0”, since the number zero had not come into use when the calendar was established. Thus the first period in the table is 8000 years in length, and this growth rate equals  $(1/8000) \log(250/5)$ . The same formula can be applied to fill in the growth rates and doubling times for recent periods in the table.

Table 1.3: World Populations and Doubling Times over History.

Date	Population	Growth Rate	Doubling Time
8000 B.C.	5 million	.000489	1417 years
1 A.D.	250 million	.000628	1103 years
1750	750 million	.006020	115 years
1950	2,500 million	.018800	37 years
1975	4,000 million		
1999	6,000 million		
2005	6,451 million		

Source: Rounded numbers drawn from estimates reported by Cohen(1995:30,400-401) and, for mid-year 2005, from the International Data Base of the U.S. Census Bureau at [www.census.gov/idb](http://www.census.gov/idb).

If the growth rate  $R$  is a negative number, the doubling time according to our formula comes out to be a negative number too. How do we interpret it? A negative time from now is a time in the past; the population would have been twice as big at some time in the past. We can also think of a negative doubling time as telling us a “half-life”. If the doubling time is -100 years, then it takes 100 years for the population to decrease to half its size. Half-lives are very familiar for populations of atoms in radioactive rocks, but they are not so common for populations of humans.

How many doublings would there have been from a primordial couple, Adam and Eve, to the world population in 2000? This

question is a way of asking what value of  $n$  will give

$$6,000,000,000 = 2(2^n)$$

The answer is 31.5 doublings. However, the population may have passed through many spells of decrease as well as spells of increase through human prehistory. So this value of  $n$  should not be taken as the number of times the population has doubled but as the net number of times the population would have doubled, starting with a unique couple and progressing up till now.

Turning this calculation around, consider that each person has two biological parents, and, in the absence of incest, four biological grandparents, eight great-grandparents, and  $2^n$  positions in the ancestral tree at the  $n$ -th generation. If all these positions were filled by distinct people, then each one of us would have to have nearly 6 billion ancestors a mere 31 generations back, around the year 1000. Of course, this cannot be. The explanation of the paradox is that people inevitably marry their distant cousins, so that most positions far up in an ancestral tree are not filled by distinct people. A discussion of the mathematics of such calculations can be found in the essay “Ancestors at the Norman Conquest”, Wachter (1980). Rohde, Chang, and Olson (2004) have gone further to estimate dates when the most recent common ancestor of all currently living humans could have lived.

Doubling times have been growing shorter over human history; over the long-term the population has been increasing faster than exponentially, for the growth rate has itself been increasing.

The properties of exponential growth fascinated thinkers of the Eighteenth Century, especially the feature that late increases come so much more rapidly than early increases. The growth cumulates very suddenly at the end. A Malthusian world-view, which sees the future stalked by the consequences of continuing rates of growth, has, like Marxism, Freudianism, and Christianity, an explanation of why non-believers should non-believe: the cumulative growth is nearly invisible until it is upon us.

## Highlights of Chapter One:

The Balancing Equation

Closed populations.

Exponential Growth.

The Population Growth Rate.

Doubling Times.

## Key Formulas

$$\begin{aligned}K(t+n) &= K(t) + B(t) - D(t) \\ R &= (1/T) \log(K(T)/K(0)) \\ K(t) &= K(0)e^{Rt}\end{aligned}$$

## Further Reading

The new *Macmillan Encyclopedia of Population* edited by Paul Demeny and Geoffrey McNicoll (2003) has readable overviews of many of the topic treated in this book. Massimo Livi-Bacci's (1997) *A Concise History of World Population* shows demographic methods at work to understand broader issues of history. The textbook by Preston, Heuveline, and Guillot (2001) is a good companion volume with many details about methods presented here.

## Exercises for Chapter One

1. Between 1960 and 1975 the population of Indonesia grew from about 100.6 million to 137.5 million. Between 1990 and 2000 it grew from 187.7 million to 225.0 million. Find the growth rates between 1960 and 1975 and between 1990 and 2000. Have the growth rates increased or declined?



2. If the average growth rate for the Indonesian population between 2000 and 2020 turns out to be 18 per thousand per year, what will the population in 2020 be?
3. The population of Nigeria in 1960 was about 39.2 million. The average growth rate  $R$  during the 1960s was 23.99 per thousand per year. During the 1970s it was 28.41; during the 1980s, 29.41; during the 1990s, 35.169. It is predicted to be around 30 per thousand per year from 2000 to 2020. Draw a rough freehand graph of the logarithm of Nigerian population over time based on these data, label the axes, and predict the population in 2020.
4. How much longer would it take the population of Indonesia to double, as compared with Nigeria, under the rates predicted for 2000 to 2020?
5. Official U.S. census counts are shown in Table 1.4. What were the yearly growth rates and population doubling times between the following years: a) 1790 to 1900? b) 1900 to 1950? c) 1950 to 2000? d) 1790 to 2000?

Table 1.4: U.S. Census Counts in Millions

1790	3.929		
1800	5.308	1900	75.995
1810	7.240	1910	91.972
1820	9.638	1920	105.711
1830	12.866	1930	122.775
1840	17.069	1940	131.669
1850	23.192	1950	150.697
1860	31.443	1960	178.464
1870	39.818	1970	203.212
1880	50.156	1980	226.546
1890	62.948	1990	248.710
		2000	281.421

6. If the world's population continued to grow at its present rate

of 0.011374 per year, how long would it take for the population to triple?

7. Suppose the world's population began to decline at the rate of  $R = 0.007$  per year. How many years would it take for population size to return to its 1950 level?

8. If  $R = .035$  and  $t = 20$  years, what is  $e^{Rt}$  ? What is the natural log of the square root of  $\exp(Rt)$  ?

9. Pakistan, Bangladesh, Russia, and Nigeria had similar total populations at mid-year 2005, 162, 144, 143, and 132. Suppose the current growth rates of these four countries, namely 0.024, 0.019,  $-0.006$ , and 0.024 continue for the next seventeen years to 2022. Use the exponential model to project future populations, and find how these countries would rank in population size at the end of the period.

\*10. Suppose we care about 20% accuracy in predictions of proportional increases in population for countries like Egypt with growth rates around 21 per thousand. How soon would multiplicative effects be sure to become important?

\*11. From the U.S. Census counts in Table 1.4, find the growth rate between 1900 and 2000 to six decimal places. Use this growth rate along with the population in 1900 to predict populations throughout the century from an exponential model with constant growth. Draw a plot which shows the actual census counts as points, and the predicted census counts as a curve. Differences between observed and predicted values reflect the effects of changing growth rates over time. Are these differences more or less important than the multiplicative effects visible in the steepening of the predicted curve?

\*12. World population growth rates are estimated at .017305 in 1986, .015794 in 1990, .014283 in 1994, .012871 in 1998, and .011556 in 2002. Fit a straight line equation of the form  $Y = a + bX$  to these estimates by the method of "least-squares linear regression", Let  $Y$  be the growth rates and  $X$  be the times. Find  $a$  and

$b$ , and compare  $b$  with the rate used in Section 1.4. The formula for the slope  $b$  in linear regression is

$$\text{slope} = \frac{\text{mean}(X * Y) - \text{mean}(X) * \text{mean}(Y)}{\text{mean}(X * X) - \text{mean}(X) * \text{mean}(X)}.$$

The formula for the intercept  $a$  is

$$\text{intercept} = \text{mean}(Y) - (\text{slope})(\text{mean}(X))$$

\*13. What is the range of values of the growth rate  $R$  for which  $1 + Rt$  is within 5 percent of  $e^{Rt}$  when  $t = 10$  ?

\*14. Suppose the growth rate over a period from 0 to  $T$  is not constant but itself increasing exponentially, so that  $R(t) = R(0) \exp(\beta t)$ . Is the following expression for  $K(t)$  plausible?

$$K(t) = K(0) \exp((R(0)/\beta)(\exp(\beta t) - 1))$$

Check the value for  $t = 0$ . Use your Taylor expansion for both of the exponential functions in the expression to see if the formula behaves as it ought for small  $t$ . Try one or two cases to see if the answer is reasonable for larger  $t$ .



# Chapter 2

## Periods and Cohorts

### 2.1 Lexis Diagrams

We have managed to develop our model for exponential population growth with no reference to the most important word in demography, the word *age*. In effect, the exponential model treats all people as if they were alike. As we shall see, it can be expanded into a model formulated in the language of probability theory in which all members of a population have identical constant risks of dying or of producing offspring without reference to sex or age. But of course in any real population, even one for which the exponential model gives a good approximation of population growth, the people in the population are in fact aging.

Time enters demography in two distinct ways: through chronological time, measured by dates on the calendar, the same for everyone, and through personal time, measured by age for each person, unique for each person or, more exactly, to each set of people who share the same birthdate. The Lexis Diagram is a simple visual display which helps us keep track of the relationships between chronological time  $t$  and age  $x$ . The horizontal axis represents chronological time, that is, calendar time. The vertical axis represents age. Each person has a lifeline on a Lexis diagram, starting at the point with coordinates  $(t, 0)$  where  $t$  is the person's

birthdate and 0 is the person's age at birth. The line goes up to the right, with a slope equal to 1 (that is, at an angle of 45 degrees) because

*people age one year in one year.*

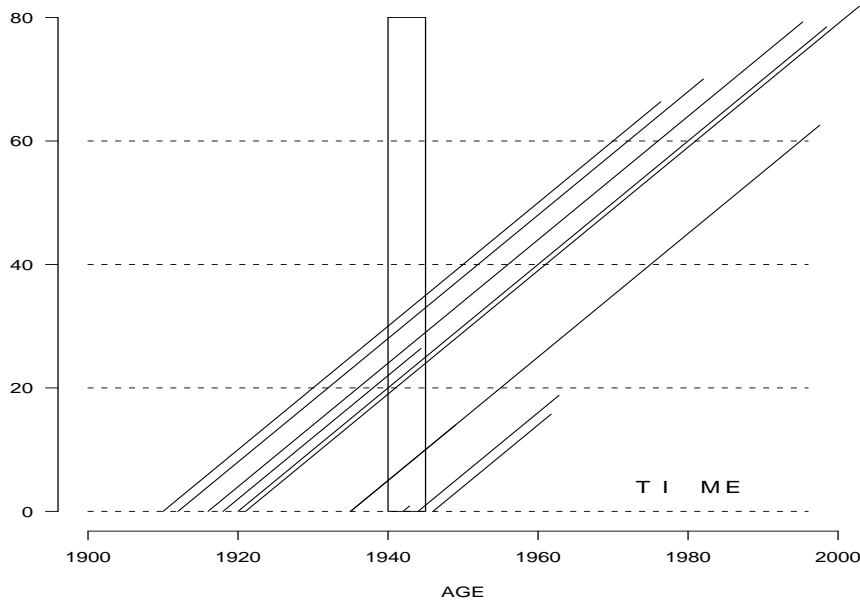
Each lifeline continues up and to the right until the time and age of the person's death.

To find the population at any instant, we draw a vertical line at the time on the horizontal axis representing the instant. Then we count how many lifelines cross that line. That is the population. To find the size of an age group, for example, those aged 20 to 25 in 1996, we draw two horizontal lines for ages 20 and 25 and one vertical line for 1996, and count the number of lifelines that cross the vertical line between the two horizontal lines. A period of time, for example, the five-year period between 1 January 1940 and 1 January 1945, corresponds to the rectangle between the vertical lines for 1940.0 and 1945.0, bounded by the horizontal axis on the bottom and by some maximum age like 120 on the top. Such a rectangle is shown on the Lexis diagram in Figure 2.1.

If we have a large population, some people share the same birthdate and their lifelines overlap at least at the start. We can imagine drawing our lifelines onto a board with paint that has some thickness to it. Then after we paint all the lifelines onto the board, the paint is thickest where there are a lot of people of a that age at that time and thin where there are few people. The thickness of the paint represents the density of people as a function of time and age. Such a 3-dimensional representation is called a Lexis Surface, which is a 3-D generalization of a Lexis Diagram.

Now let us focus on all the people sharing the same birthdate. For them, the relationship between time and age is always the same. We call this group of people a "cohort". They are born at the same time and go through life aging together. We can define cohorts narrowly or broadly. We might form a cohort only out of people who share the same hour of birth. Such a cohort would have about 16,000 members around the world or about 430

Figure 2.1: A Lexis Diagram



members in the United States. Or we might form a cohort out of members with the same day of birth. Often, we take a broader perspective and form a cohort out of people with the same year of birth or with births within the same five-year timespan, called a quinquennium from the Latin words for five and year.

A **cohort** is a group of individuals followed simultaneously through time and age.

The cohort is found on the diagonal of a Lexis diagram. The person-years lived by a cohort are found inside a parallelogram stretching up from the axis at 45 degrees. That is a contrast to the person-years lived by a population during a period, which are found in a vertical rectangle on the Lexis diagram.

We have been discussing the basic idea of a cohort, which is the idea of a group of people born together in time. Birth is the time of entry into the human population. We can extend the concept of cohort so it can also be used when considering other

sorts of entry. For example, members of the the Demography Department at Berkeley talk about “cohort” of students who enter the Ph.D. program together. Students entering in 2000 are the “twenty second cohort”.

We also talk of marriage cohorts. All those who got married in 1982 form the Marriage Cohort of 1982. In this picture, duration of marriage takes the place of age. After all, age itself is the same as duration of life.

The word cohort was originally a military term, referring to a group of about 600 Roman infantrymen who marched along together. Ten cohorts made a legion. In *Paradise Lost*, Milton writes about a cohort of angels, “the cohort bright of watchful cherubim”. We picture a birth cohort forming ranks along the horizontal axis of the Lexis Diagram and marching out along the diagonal through life together.

A cohort is special, because it shares experiences. It arrives at the same point of history at the same age. The cohort born at the start of the Twentieth Century were all teenagers during World War I and fought as mature men in World War II. The cohort born immediately after World War II, at the start of the Baby Boom, experienced President Kennedy’s assassination in High School, and saw one of its members enter the White House in 1992 at the age of 45. There is a fascinating book by William Strauss and Neil Howe (1991) called *Generations* which traces American history from 1584 into the future in terms of the overlapping cohorts taking different approaches to political and social problems at the same time.

## 2.2 Period Person-Years Lived

Demographers are constantly moving back and forth between thinking in terms of periods and thinking in terms of cohort. The



key tool for seeing relationships between formulas involving periods and formulas involving cohorts is the concept of *person-years lived*. We begin, in this section, by developing the concept of “person-years lived” in terms of periods. This concept helps us give a fuller description of our exponential model for population growth, which is essentially a model for the size of the period population. In later sections, we express the same concept of “person-years lived” in terms of cohorts. Our two ways of counting person-years, by period and by cohort, lead to an identity between period-based and cohort-based kinds of demographic measures, the “Stationary Population Identity” which we derive at the end of this chapter.

We are led to the concept of “person-years lived” when we seek to refine the basic insight that led us to the equation for exponential growth. We began in Chapter One with the insight that more people mean more births and deaths. We proceeded to divide births and deaths during an interval by population at the start of the interval to remove the most direct effects of population size. Now we ask, why take population at the start rather than at the end or in the middle? People who are present during part of the period can also have babies or become corpses. It is not just more people that mean more births and deaths, but more people present for more of the time.

Consider first an interval of time one decade long, say the period from 1980 to 1990. Each full year that a person is present, he or she contributes one “person-year” to the total of “Period Person-Years Lived” or “PPYL” for the population in the period. We make two new definitions:

**The Crude Birth Rate** or C.B.R. (also written as  $b$ ) is the number of births to members of the population in the period divided by the total period person-years lived.

**The Crude Death Rate** or C.D.R. (also written as  $d$ ) is the number of deaths to members of the population in the period divided by the total period person-years lived.

It is important to mention that we count up person-years in fractions of years as well as whole number of years. Each day a person is present in the population, he or she contributes one person-day or  $1/365$  person-year to the period person-years lived. Each hour a person is present, he or she contributes one person-hour or  $1/8,766$  person-years to the period person-years lived. Thus the total period person-years lived is a meaningful concept even if our interval is only a year or even a month in length.

Whenever we know the whole sequence of population sizes throughout a period, say, the population sizes on each day over the period of a year, we can add up the person-years day by day. We take the number of people present on the first day times  $1/365$  of a person-year for each of them, plus the number of people present on the second day for each of them times  $1/365$  person-years, and add up all the contributions. For an even more precise calculation, we could use hours or minutes or seconds. As we take smaller and smaller subintervals, we have more and more of them to add up. When our subintervals are small enough, our sum is virtually equal to the area under the curve of population as a function of time during the period. This outcome is a fact of calculus which we discuss further in the advanced (starred) sections later in this chapter.

Whenever the sequence of population sizes throughout a period are unknown, we use an approximation for PPYL, the period person-years lived. The most popular practice is to take the population in the *middle* of the period and multiply by the length of the period. For example, to calculate the period person-years lived in the United States in the period from 1985 to 1995, we take 253,335,742 people times 10 years and obtain 2,533,357,420 person-years in the period. Many countries like Great Britain estimate their populations in the middle of each year for just this purpose. If we do not know the mid-year population, then we can take the average of the starting and ending populations, and multiply by the length of the period.

## 2.3 The Crude Rate Model

The concept of period person-years lived allows us to give a fuller statement of the assumptions behind our model of exponential growth. We imagine a closed population in which each person, each instant is subject to a constant independent risk

- (i) of dying, and
- (ii) of contributing a baby to the population.

Then  $b$  and  $d$  are the expected numbers of births per person per year and of deaths per person per year. The expected size of the population displays exponential growth.

Even for childbirth and dying, the model makes assumptions which are extreme simplifications. We know that in fact the risks of childbirth and dying are not the same from person to person and day to day. Women are more likely to give birth than men! Adults are more likely to give birth than children. Old people and infants are likely to die sooner than students or professors. But for crude estimates, for populations which have a run-of-the-mill mix of different kinds of people, it is useful to lump all the people together, attribute the risk to all of them without distinction.

The model is **crude** because it treats all people equally, even though men don't give birth to babies, old people die more often than young people, summer days see more births than autumn days, etc. All person-days are treated the same; a person-day is  $1/365$  of a person-year.

What is more, it allows perfect tradeoff between people and days. People and days contribute on the same footing to the product. Recall the proverb, "You can fool some of the people all of the time, and all of the people some of the time." When counting person-years, all of the people some of the time is just as good as some of the people all of the time.

This model is appropriate if we do not know who the people are or what the days are. It assumes

- (a) a closed population
- (b) homogeneous risks among people and time
- (c) no measurement of change over time inside the period

We have been using the word “risk”. Our model as we have stated it in this section refers to “risks” or “chances” and is really a model of probability theory, where it is called a “Birth and Death Process” model. Another name for this model is the “Crude Rate Model”. If the Crude Rate Model is a true description of a closed population within the limits of accuracy that we care about for a particular purpose, then the Growth Rate  $R$  as we have defined it equals the Crude Birth Rate minus the Crude Death Rate. This fact is proved, using results from the calculus, in the later, starred sections of this chapter. Of course, most actual populations are not closed and risks are not homogeneous over time, so in most cases the Growth Rate  $R$  does not exactly equal the Crude Birth Rate minus the Crude Death Rate. Some demographers define the “Crude Growth Rate” or “CGR” to equal the Crude Birth Rate minus the Crude Death Rate plus a Crude Net Migration Rate. Other demographers use the term “Crude Growth Rate” more or less interchangeably with the Growth Rate  $R$  as we have defined it, the slope of the graph of the logarithm of population size over time.

Table 2.1 shows crude rates for the ten most populous countries in the world in 2004. The CBR, CDR and  $R$  have already been discussed. “MIG” is a rate of net immigration. The “IMR” is the “Infant Mortality Rate” and  $e_0$  is the “expectation of life at birth”, to be discussed shortly. The source is the *2004 World Population Data Sheet* of the Population Reference Bureau.

Table 2.1: The World's Ten Most Populous Countries  
Rates per thousand, population in millions for mid-year 2005

Rank	Country	Pop.	CBR	CDR	MIG	R	IMR	$e_0$
1	China	1,304	12	6	-0	6	27	72
2	India	1,104	25	8	-0	17	60	62
3	USA	296	14	8	4	6	7	78
4	Indonesia	222	22	6	0	16	46	68
5	Brazil	184	21	7	0	17	27	71
6	Pakistan	162	34	10	-1	24	85	62
7	Bangladesh	144	27	8	1	19	65	61
8	Russia	143	11	16	-1	-6	12	66
9	Nigeria	132	43	19	0	24	100	43
10	Japan	128	9	8	0	1	3	82
	world	6,477	21	9	0	12	54	67

## 2.4 The Infant Mortality Rate

The Crude Birth Rate and the Crude Death Rate are period measures. We can calculate them using nothing besides what we can find in the rectangle for the period on the Lexis Diagram. Most measures of mortality other than the C.D.R. are based on cohort data or modeled on cohort concepts, and we shall be studying them in Chapter Three. The Infant Mortality Rate, however, is another pure period measure, most easily understood by referring to the Lexis Diagram. It is defined as follows:

$$IMR = \frac{\text{the number of deaths under age 1 in the period}}{\text{the number of live births in the period}}$$

Take a square on the Lexis Diagram with a base of one year on the horizontal axis and a height of one year on the vertical axis. The deaths which go into the numerator of the I.M.R. are all the circles in the square. The births that go into the numerator of the

I.M.R. are the starts of all the lifelines that start out on the base of the square.

From a certain point of view, the I.M.R is an odd hybrid measure. The deaths in the numerator are not all happening to the babies whose births are found in the denominator. Babies from cohorts born before the start of the year whose births fall outside the square may die in the square, and babies born during the year may still die as infants outside the square. Because of this mismatch between numerator and denominator, it is mathematically possible for the I.M.R. to be greater than 1, if the number of births in the preceding year was much larger than the number in the current year. In practice, however, such a strange result does not happen.

Despite its theoretical oddity, the I.M.R. is a handy measure. It uses only current information from vital registration. Unlike the life table measures presented in Chapter Three, it can be computed for countries that do not have a reliable census or other source for a count of the population at risk by age.

Infant Mortality Rates for the world's ten largest countries range from 4 per thousand for Japan to 108 per thousand for Bangladesh. The highest rates in the 1990 United Nations world data sheet were 162 for Afghanistan, 159 for Mali, and 143 for Sierra Leone. Up to the last century, however, rates as high as 350 were not uncommon.

It is important to bear in mind that the infant mortality rate depends to some extent on the ages of mothers. Infants borne by teenagers and by older mothers are at higher risk. The popular press tends to forget this point when it features differences in Infant Mortality Rates between the U.S. and other countries or between subgroups of the population.

## 2.5 \* Person-Years and Areas

We have seen in Section 2.2 that the period number of person-

years lived in a population equals the area under the curve of population size over time. The “PPYL” between two times 0 and  $T$  is the area under the curve  $K(t)$  between 0 and  $T$ . In calculus, areas are written as integrals with the integral sign:

$$PPYL = \int_0^T K(t)dt$$

The symbol “dt” stands for an arbitrarily tiny piece of the whole time interval. We multiply the number of people alive in each piece by the length of the piece and add up over all the pieces. We take smaller and smaller pieces, hours, minutes, seconds, and in the limit we have a precise answer for the area.

We can find an exact formula for PPYL when the population is growing exactly exponentially, that is, when the growth rate is constant over the interval of interest. We use a property of the exponential function called “self-similarity”. Imagine an exponential curve stretching from the infinite past to the present, from  $t = -\infty$  to  $t = 0$  and onward into the infinite future. (The mathematical symbol for infinity is  $\infty$ , resembling an 8 fallen on its side.) Suppose we shift the curve  $n$  units to the left, so that the value at  $t + n$  becomes the value at  $t$ . The shifted curve is the same as a curve we would produce by multiplying every value by  $e^{Rn}$ .

$$e^{R(t+n)} = e^{Rn}e^{Rt}$$

Suppose we write  $A$  for the area under the curve all the way from  $t = -\infty$  to  $t = 0$ . To find the area from  $t = -\infty$  to some other time  $T$ , we shift the curve back by  $T$  units, so that the shifted curve ends at  $t = 0$ . Every value is now higher by a factor of  $e^{RT}$ , so the area is now  $Ae^{RT}$ . In other words, the area from the infinite past to the present is  $A$  and the area from the infinite past to  $T$  years in the future is  $Ae^{RT}$ . By subtraction, the area from time 0 to time  $T$  is  $A(e^{RT} - 1)$ .

What is  $A$ ? We find  $A$  by replacing  $T$  by a tiny interval  $n$ . We know the area from 0 to  $n$  has to be very close to the height at zero,  $e^0 = 1$  times the width  $n$ . Next we can use our Taylor Expansion from Chapter Two:

$$A(e^{Rn} - 1) \approx A(1 + Rn + \dots - 1) = ARn = n$$

Therefore  $A$  has to be the reciprocal of the growth rate,  $A = 1/R$  and our exact formula for the area has to be

$$Area = \frac{e^{RT} - 1}{R}$$

When we multiply by  $K(0)$  we obtain the full formula for the Period Person-Years Lived between 0 and  $T$  when the growth rate is constant:

$$PPYL = K(0) \frac{(e^{RT} - 1)}{R}$$

As a first application, let us calculate how many person-years would have been lived altogether by members of the human race if the growth rate had always been constant at its present value near 13 per thousand. The answer is

$$PPYL = 6 * 10^9 / 0.012 = 500 \text{ billion}$$

We come back to this answer in the next section, when we consider how many people have ever lived on earth.

As a second application, consider the CBR and CDR under the assumption that population size is growing exactly exponentially over the course of a year. Population increases by

$$K(1) - K(0) = K(0)(e^R - 1) = B - D$$

The difference between the CBR and the CDR, using our exact expression for the PPYL in the denominator, is

$$CBR - CDR = \frac{B - D}{K(0)(e^R - 1)(1/R)} = 1/(1/R) = R$$

Here is a sensible conclusion. The growth rate equals the difference between the Crude Birth Rate and the Crude Death Rate in a closed population subject to truly exponential growth.



As a third application, take the interval from 0 to  $T$  and substitute our Taylor Expansion for the exponential in the formula for PPYL:

$$\begin{aligned} PPYL &= K(0)(e^{RT} - 1)(1/R) \\ &\approx K(0)(1 + RT + R^2T^2/2 + \dots - 1)(1/R) \\ &= K(0)(1 + RT/2 + \dots)T \approx K(T/2)T \end{aligned}$$

We see that the approximation for PPYL in terms of the mid-period population  $K(T/2)$  agrees with the exact expression up to first order when the exponential model is true. It is a useful exercise to carry the expansions one step further, including the term  $R^3T^3$ . These expansions tell us the differences between the area formula, the mid-period approximation, and the average approximation  $(T/2)(K(0) + K(T))$  as estimates of Period Person-Years Lived.

## 2.6 Cohort Person-Years Lived

Up to this point we have been counting person-years lived by period and calling them Period Person-Years Lived, or PPYL. We have been taking all the person-years from the first day, adding all the person-years from the second day, and so forth, counting up by subsets of time.

But we could have proceeded differently. We could have taken a cohort, counted the person-years lived by the first member of the cohort, added the person-years lived by the second member of the cohort, and so forth for the whole cohort. This gives us a the total Cohort Person-Years Lived or CPYL. The person-years are the same things, but we are counting them up in two different ways.

For PPYL we count person-years in a rectangle of the Lexis Diagram. For CPYL, we count person-years along a diagonal stretch of the Lexis Diagram.

If we add up all the cohort person-years lived and divide by the total number of members of the cohort (counted at birth, when the cohort is formed), we obtain a number called  $e_0$ , the “expectation of life at birth.” This is the average number of person-years lived in their whole lifetimes by members of the cohort which will be discussed at length in Chapter Three.

To calculate  $e_0$  we can measure the lengths of all the lifelines of cohort members with a ruler and take the average. Notice, however, that if one year corresponds to one inch on the axes, one year corresponds to 1.41 (the square root of 2) inches on the diagonal, because of Pythagoras’ Theorem. In a right triangle, the square of the diagonal is equal to the sum of the squares of the other two sides, and one plus one equals two.

The quantity  $e_0$  is cohort measure of survival, or of avoidance of mortality. The CDR, on the other hand, is a period measure of mortality. We calculate it from the lifelines in the rectangle on the Lexis Diagram. The numerator for the rate is the number of lifelines which end in the box. (Often we mark the ends of lifelines with circles so we can see them.) Not every lifeline, of course, has a death on it in a particular period; for the numerator, we only count the circles. For the denominator, we are effectively adding up the length of all the lifelines in the box, by counting how many cross each vertical line in a dense grid of vertical lines.

Many people wonder how many human beings have lived on our planet. Our methods allow us to make some calculations which are relevant to this question. If we draw a curve of total population versus time over a long period, and if one year is one unit on the horizontal (time) axis, then the area under the curve is the total period person-years lived. If we know the average lifespan over the whole period, then the total person-years lived divided by the average lifespan gives us the number of people who lived those person-years.

In Section 2.5, using methods from the calculus, a formula for the area under an exponential curve from  $t = 0$  to  $t = T$  has been derived:

$$\text{Area} = (K(0)/R)(e^{RT} - 1)$$

Using the guesses at historical population sizes in Table 1.4, under the assumption of smooth exponential growth, the number of person-years between the origins of farming around 8000 B.C. and the birth of Christ would be

$$(5,000,000/0.000489)(250/5 - 1) \approx 501 \text{ billion person-years}$$

If average lifespans over this period were around 25 years, then a total of 22 billion people would have lived over this period of early human civilization.

For fully accurate results in such person-year calculations, we need to remember that people still alive at the end of the period still have some of their lifespans to live, and people already alive the beginning of the period have already lived some of their lifespans. To obtain the total person-years for all the cohorts born between two dates, which contribute to the average lifespan of the cohorts, we have to add in a contribution for remaining life at the end and subtract away a contribution for expended life at the beginning.

This effect of extra life is small for small growth rates like 0.000489 in our previous calculation. But it can be large. For example, in Section 2.5 we saw that under the unrealistic assumption that the population had always been growing at its present rate of about 13 per thousand per year we would have about 461 billion person-years lived from the indefinite past up to the present. At the present-day world-wide average lifespan of about 63 years, the formula we have been using would mislead us, suggesting  $661/63 = 7.317$  billion people, of whom 82% would be alive today. The fallacy is that the 6 billion people alive today have something like 30 years apiece still to live on average, contributing another 180 billion person-years which would have to be added into the calculation.

Under the exponential model, the slower the growth rate, the more people have lived in the past and the smaller the proportion who are still living in the present. This outcome is reasonable. The slower the growth rate looking forward in time, the slower

the population shrinks as we look back in time, and the more people we expect in the distant past.

When we ask how many humans have ever lived, some substantial uncertainty in the answer comes from prehistory. First, there are different choices for what counts as “humans” and what point in evolution should be taken as the starting point for the calculation. Second, estimates of prehistoric population sizes are speculative and estimates of prehistoric lifespans are controversial. Third, the exponential model is probably a poor guide to human prehistory. It may well be better to picture one or several periods of early rapid growth followed by long stretches of time in which populations remained roughly in balance with resources and predators or in which populations cycled through periods when they grew and periods when they shrunk.

Genetic studies with the part of our genetic inheritance known as mitochondrial DNA are thought to suggest that all humans now living are descended from one woman who lived in Africa in very roughly something like 180,000 B.C. If we are counting the descendants of this woman, the exponential model very likely gives a low lower bound on numbers living before the dawn of agriculture. With an upper bound of 25 years on average lifespan, the lower bound would be about 22 billion people. The popular view that half or more of all people who have ever lived are living today is obviously wrong.

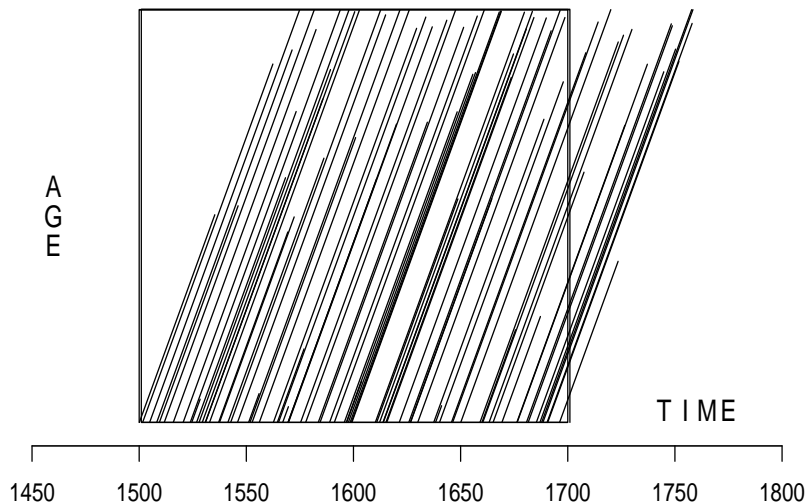
## 2.7 The Stationary Population Identity

Usually period person-years lived and cohort person-years lived are different, because, though we are counting the same kind of thing in both cases, we are counting it for different groups of people or stretches of time. In special circumstances, however, we can count nearly the same batch of person-years lived in two different ways, by period and by cohort, and then we should get nearly the same answer. The special circumstances are these: First, the

rates are unchanging. Second, the numbers are unchanging. This second condition means that total population is the same from year to year, and so are the number of births and the number of deaths. In this case the population is called “stationary”. It is important to distinguish the word “stationary”, as demographers use it, from the word “stable.” Populations which keep the same size are called stationary in demography, and the word stable means something else, which we shall study in Chapter Nine.

When the population is stationary, the population  $K(t)$  is the same number  $K$ , whatever the year  $t$ . Then the period count of person years equals the product  $(K)(1 \text{ year})$  for each year, or  $KT$  person years over  $T$  years. The curve of total population is a flat curve, and the total number of period person-years lived is the area under it, a rectangle with height  $K$  and base  $T$ .

Figure 2.2: Lexis Diagram for a Stationary Population



The cohort count of person-years lived gives, on average,  $e_0$  person years for each birth. Using the Crude Birth Rate  $b$ , we obtain an estimate of  $Kb$  births each year, or  $KTb$  births total. Now we are counting exactly the same person-years in two ways, except at the start and the end of the period. We can see this

on a Lexis Diagram like the one in Figure 2.3. We draw vertical lines at times  $t = 0$  and  $t = T$  and draw lifelines (sloping up at a diagonal) for all cohorts born between times 0 and  $T$ . There is a blank triangle on the upper left, because none of the cohorts born after time zero are old at times immediately after time zero. There is an extra triangle dark with lifelines at the right, because the cohorts born just before time  $T$  live their person-years at older ages after time  $T$  has passed. We have already encountered these contributions of years of life remaining from those still alive at the end and years of life expended by those already alive at the beginning in our discussion of the numbers of humans who have ever lived. Except for these contributions at the endpoints representing triangles on the Lexis diagram, the period person-years and the cohort person-years are the same:

$$KT = KTbe_0 + \text{left triangle} - \text{right triangle}$$

$$1 = be_0 + \frac{\text{left triangle} - \text{right triangle}}{KT}$$

The two triangles constitute an “endpoint correction”, a contribution that comes in only at the end points of the period. Now, there are two approaches to seeing that the endpoint correction does not matter. First we can argue that the two triangles cancel out. The person-years in the period missed by the cohort at the start are the same as the extra person-years lived by the cohort after the end of the period. Or, we can simply argue that the size of the triangles does not change as we make  $T$  larger and make the rectangle bigger and bigger. Thus as  $T$  gets big, the contribution of the triangles divided by  $T$  goes to zero. In either case, we obtain the “Stationary Population Identity”:

$1 = be_0 \quad \text{when} \quad R = 0$
--

This is a deep identity, relating a period fertility measure  $b$  to a cohort mortality measure  $e_0$  in a stationary population. It is

extremely useful for quick rough calculations. For France in 1600, which was close to a stationary population,  $e_0 = 30$ , so the Crude Birth Rate must have been around  $1/30$  or 0.033 per year. Since  $b$  is a rate, it has units of  $1/\text{time}$ , while  $e_0$  has units of time. Their product is the dimensionless constant 1.

## Highlights of Chapter Two:

The Lexis Diagram

Cohorts

Person-Years Lived

The Infant Mortality Rate

How many people have every lived.

The Stationary Population Identity

## Key Formulas

$$\begin{array}{lcl} PPYL & \approx & K(T/2)T \\ R = 0 & \text{implies} & be_0 = 1 \end{array}$$

## Further Reading

Joel Cohen's (1995) book *How Many People Can the Earth Support?* has a captivating overview of population sizes and changing growth rates over time. Nathan Keyfitz (1985:13) discusses how many people have ever lived.

## Exercises for Chapter Two

1. Of the world's ten most populous countries, which has the highest rate of immigration today? Which has the lowest Infant Mortality Rate? What fraction of the world's population is comprised by these ten most populous countries?
2. Mexico is the country with the eleventh-largest population in the world. Its Crude Birth Rate during 2003 is reported to have been 21.44 per thousand, and 2,223,714 are reported to have been born. What figure for the mid-year population would be consistent with these numbers?
3. Poland has a nearly stationary population with  $e_0 = 75$ . There were about 300,000 births last year. What is the approximate size of the population?
4. Study the data in Table 2.1 and determine whether  $be_0$  is typically greater than, equal to, or less than 1 in a growing population.
5. From an almanach or other source, find dates of birth and death for the presidents of the United States from Theodore Roosevelt to George W. Bush. Draw a freehand Lexis diagram with the lifelines of these presidents. Label the axes clearly.
  - a) Draw and label a line representing the age 30.
  - b) Draw and label a line representing the year 1945.
  - c) Draw and label the area containing person-years lived by people between the ages of 20 and 30 in the years between 1964 and 1968.
  - d) Draw and label the area representing the lifetime experience of the cohort aged 10 to 30 in 1917.
- \*6. Explain briefly what step in the derivation of the stationary population identity  $be_0 = 1$  fails if the population is growing rather than stationary.
- \*7. Apply Taylor Expansions to three expressions for the Period Person-Years Lived over an interval from 0 to  $T$ :



the mid-period approximation  $K(T/2)T$

the average approximation  $(K(0) + K(T))(T/2)$

the exponential area  $K(0)(e^{RT} - 1)/R$

Find formulas for the differences among these three expressions up to a term in  $T^3$ . Which is the largest and which is the smallest, or does the ordering depend on  $R$  and  $T$ ?

8. Work out an estimate of how many humans have ever lived based on the estimate for the time before the dawn of agriculture given in the text and the estimates of population sizes given in Table 1.4, under the assumption of constant rates of growth between the years cited in the table. For average lifespans, use 30 years before 1750, 40 years for 1750 to 1950, 50 years for 1950 to 1975, and 60 years for 1975 to 2000.



# Chapter 3

## Cohort Mortality

### 3.1 Cohort Survival by Analogy

We begin our study of mortality by focussing on cohorts. We look at the lifelines and the deaths that occur in the diagonal swath of the Lexis Diagram that represents a particular cohort's experience, and we introduce measures of survival and probabilities of dying as a function of age for the cohort. We could, instead, look at the rectangle on a Lexis Diagram that represents some period, and consider the lifelines that cross the rectangle and the deaths that fall inside it. That is more complicated, however, because in that case the people at risk of dying at different ages are different people. When we trace the experience of a cohort over time, the people at risk of dying at different ages are the same people. That makes cohort measures of mortality conceptually simpler than period measures of mortality and gives us an incentive to begin by studying cohort measures.

In Chapter Seven, we shall go on to study period mortality, and adapt our cohort-based calculations to use period data. The advantage of cohort measures is their conceptual simplicity. The disadvantage is their being out of date. To have complete measures of cohort mortality for all ages, we have to wait until all members of the cohort have died, and by then rates for young ages hark

back to the distant past. The most recent cohorts with complete mortality data are those born in the 1880s. Measures of period mortality are more complicated, but they use more recent data.

The basic measures of cohort mortality are extremely simple. All we have to do is to take our model for exponential population growth and apply it to a closed population consisting of the members of a single cohort. We change the symbols in the equations but keep the equations themselves.

If our population consists of a single cohort, then after the cohort is born, no one else enters the population. Babies born to cohort members belong to the babies' own, later cohorts, not to their parents' cohort. So, for this cohort "population", the only changes in population size come from deaths to members of the population. The measures we have studied in Chapter One reappear with new names. We have a tight analogy between population growth and cohort mortality illustrated in Table 3.1.

Table 3.1: An Analogy Between Populations and Cohorts

<b>Population Growth</b>	<b>Cohort Mortality</b>
Time $t$	Age $x$
Population Size $K(t)$	Cohort Survivors $l_x$
Multiplier $A = 1 - D/K$	Survival Probability $1 - q$
Growth Rate $R$	Hazard Rate $h$ (with minus sign)
Area Under $K(t)$ PPYL	Area Under $l_x$ CPYL $L$
Crude Rate $d$	Age-Specific Rates $m$

In a cohort, age  $x$  takes the place of time. The number of

surviving members of the cohort at age  $x$ ,  $l_x$ , takes the place of population size at time  $t$ ,  $K(t)$ . The initial cohort size  $l_0$ , taking the place of  $K(0)$ , is called the “radix”. Cohort deaths, with little  $d$ , take the place of population deaths with capital  $D$ . The multiplier  $A$  for calculating  $K(t + 1)$  as  $AK(t)$  has the form  $1 + (B - D)/K$  as we know from Chapter One. For a cohort, after age zero, births are out of the picture and  $A$  is  $1 - D/K$ .

We write the cohort counterpart of  $1 - D/K$  as  $1 - d/l$  or  $1 - q$ . It is a probability of survival. The letter  $q$  in this expression stands for the probability of dying. The probability of surviving is 1 minus the probability of dying. As a cohort ages across an interval from  $x$  to  $x + n$ , each member either dies or survives. There is no other alternative, so the probabilities have to add to 1.

The counterpart of the growth rate is called the “hazard rate”. Like the growth rate, the hazard rate is defined using logarithms, as we shall shortly see. Just as the area under the  $K(t)$  curve gives Period Person-Years Lived, so the area under the  $l_x$  curve gives Cohort Person-Years lived. Its symbol will be a capital  $L$ . Crude rates as functions of time, with period person-years in their denominators, correspond to “age-specific” rates for cohorts, with cohort person-years in their denominators. Each measure for population growth has a partner among the measures for cohort mortality.

The important step in Chapter One is to let us see growth as a process of multiplication. The same is true of cohort mortality. But we have to be careful. It is not the mortality rates that multiply. Nor is it the probabilities of dying. It is the probabilities of *surviving*.

We write the age  $x$  as a subscript on the cohort survivors  $l_x$ , whereas we write the time  $t$  inside parentheses by the population size  $K(t)$ . The notation is different but the idea is the same. The multiplicative rule for population growth

$$K(t + n) = AK(t)$$

carries over to a multiplicative rule for cohort survivorship:

$$l_{x+n} = (1 - {}_nq_x)l_x$$

The letter  $q$  in this formula has an extra subscript  $n$ , written *in front of* the letter. This left-side subscript specifies the length of the interval. Thus  ${}_nq_x$  is the probability of dying within an interval of length  $n$  that starts at age  $x$  and ends at age  $x + n$ . Similarly,  ${}_nd_x$  are cohort deaths between ages  $x$  and  $x + n$  and  ${}_nL_x$  are cohort person-years lived in this interval. This traditional notation dates from long before typewriters and computers.

The  $l_x$  cohort members alive at age  $x$  split into two groups, the  ${}_nd_x$  members who die before age  $x + n$  and the  $l_x - {}_nd_x = l_{x+n}$  members who survive to age  $x + n$ .

${}_nq_x$  is the **probability of dying** between ages  $x$  and  $x + n$  among cohort members alive at age  $x$ , equal to  ${}_nd_x/l_x$ .

$1 - {}_nq_x$  is the **probability of surviving** from age  $x$  to age  $x + n$  among cohort members alive at age  $x$ , equal to  $l_{x+n}/l_x$ .

The cohort born in 1984 reached age 18 in 2002, and about 1,767,644 were alive at their birthdays. Only about 724 of them died before age 19. We see that

$${}_1q_{18} = 724/1,767,644 = 0.000410$$

is their probability of dying and

$$1 - {}_1q_{18} = 1,766,919/1,767,644 = 0.999590$$

is their probability of surviving. Happily probabilities of dying are low in developed countries at such early adult ages.

Another way of expressing the pace of death within cohorts is by means of hazard rates, used most widely in advanced work. The hazard rate is the counterpart of the population growth rate. Just as we measure population growth with slopes of logarithms of population size, so we can measure cohort losses from slopes of logarithms of numbers of survivors. We insert a minus sign to

make the hazard rate into a positive number, since cohorts grow smaller, not larger, as they age:

The **hazard rate** for a cohort is minus the slope of the logarithm of the number of cohort survivors as a function of age.

For example, the cohort of boys born in the United States in 1980 started out with 1,853,616 members. As many as 1,836,853 of them survived to their first birthday. We calculate the hazard rate from the logarithms of  $l_0$  and  $l_1$ :

$$\begin{aligned} h &= -\frac{\log(1,836,853) - \log(1,853,616)}{1} \\ &= -\frac{14.42356 - 14.43261}{1} = 0.009085 \end{aligned}$$

If we write the hazard rate in the interval starting at age  $x$  as  $h_x$  (omitting any subscript for  $n$ ), our formulas for cohort survivorship resemble our formulas for exponential population growth:

$$l_{x+n} = l_x e^{-(n)(h_x)}$$

## 3.2 Probabilities of Dying

A hazard rate is a rate like  $R$ , whereas  ${}_nq_x$  is a probability. The word “probability” suggests a random process. Here we introduce randomness into our picture by always referring in principle to a randomly selected member of our cohort. At a deeper level, the occurrence of death appears partly random and partly determined by causes, themselves partly random and partly determined by prior causes. For understanding the mathematics, it may be helpful to imagine the devil playing with a deck of cards. The ace of spades means death. The deck is shuffled. If the devil draws the ace of spades, you die. If the devil draws any other card, you survive. But, it being the devil, the deck may be stacked with more than one ace of spades.

It takes some experience to become accustomed to demography's traditional subscript notation. As mentioned in Section 3.1 in the expression  ${}_nq_x$ , the left subscript gives the width of the age interval and the right subscript gives the starting age. Thus  ${}_{10}q_{20}$  is the probability of dying between 20 and 30, not between 10 and 20. One must also be careful not to confuse  ${}_nq_x$  with  $n$  multiplied by  $q_x$ , which would look superficially like  $n$  multiplied by  $l_x$ . It is wise in the latter case to write  $(n)(l_x)$  with parentheses. Finally note that  ${}_{10}q_{20}$  goes from 20.00000 to 29.99999. We call this interval “the interval from 20 to 30” (including exact age 20, excluding exact age 30), but some other authors call it “the interval from 20 to 29”.

Demographers frequently find themselves with data for one set of age intervals when they need answers for different intervals. They may have data for one-year-wide intervals and need answers for five-year-wide intervals. They may have data for fifteen-year intervals and need answers for five-year intervals. They may have tables for ages 25 and 30 and need to know how many women survive to a mean age of childbearing of, say 27.89 years. Problems that involve working out  ${}_nq_x$  values for different  $x$  and  $n$  are called “ $nq_x$ -conversions”. They provide valuable practice in the basic skills of demography.

From our analogy with population growth comes the key point that we get from  $l_x$  to  $l_{x+n}$  by **multiplication**. We go from  $l_{65}$  to  $l_{85}$  by multiplying by  $1 - {}_{20}q_{65}$ . We go on from  $l_{85}$  to  $l_{100}$  by a further multiplication by  $1 - {}_{15}q_{85}$ . So we go all the way from  $l_{65}$  to  $l_{100}$  by multiplying by the product  $(1 - {}_{20}q_{65})(1 - {}_{15}q_{85})$

Survival Probabilities Multiply

The important thing to remember is that we do not multiply the  ${}_nq_x$  values. We multiply the  $1 - {}_nq_x$  values. To survive ten years you must survive the first year **and** the second year **and** the third year, etc. These “ands” mean multiplication. To die, you can die in the first year **or** in the second year **or** in the third year,



etc., but you only need do it once, and there is no multiplication. So, while we are ultimately interested in  $q$ , we work with  $1 - q$ .

When we do not have direct data for short intervals of interest, like one-year-wide intervals, we need an assumption. The basic assumption we shall always make is this one: *Assume the probability of dying is constant within each interval where we have no further information.*

Thus if we do not know  ${}_1q_{20}$  or  ${}_1q_{21}$  but we do know  ${}_2q_{20}$ , we assume that the probability of dying is constant between ages 20 and 22, that is, that  ${}_1q_{20} = {}_1q_{21}$ , so that they both equal some value  $q$ . Then  $(1 - q)^2$  has to equal  $1 - {}_2q_{20}$ . More generally, for  $y$  between  $x$  and  $x + n - 1$ ,

$$\begin{aligned} (1 - {}_1q_y)^n &= 1 - {}_nq_x \\ 1 - {}_1q_y &= (1 - {}_nq_x)^{1/n} \\ {}_1q_y &= 1 - (1 - {}_nq_x)^{1/n} \end{aligned}$$

For example, for the cohort of U.S. women born in 1980,  ${}_2q_{20} = 0.000837$ . We calculate  ${}_1q_{20}$  to be  $1 - (1 - .000837)^{1/2}$  or 0.000419. For the cohort of women born in 1780 in Sweden, where early data can be found,  ${}_5q_{20} = 0.032545$ . Then

$${}_1q_{20} = 1 - (1 - .032545)^{1/5} = 0.006595.$$

For another example, suppose we know that  ${}_5q_{80} = 0.274248$ , and we want to find the probability of dying each year which would, if constant, account for the observed five-year mortality and survivorship? We calculate

$$1 - (1 - {}_5q_{80})^{1/5} = 1 - (1 - 0.2742479)^{1/5} = 0.062098$$

Sometimes more complicated conversion problems arise. For example, we might have sources which told us three values:

$$l_{65} = 0.915449; \quad l_{75} = 0.799403; \quad {}_{35}q_{65} = 0.930201.$$

We might want the probability of surviving from 70 to 100 for comparison with some other source. (The numbers here actually come from a forecast of survival for the U.S. cohort of women born in 1980.) Our unknown  $1 - {}_{30}q_{70}$  equals  $1 - {}_{35}q_{65}$  divided by  $1 - {}_5q_{65}$ . The numerator is  $1 - 0.930201$ . We calculate the denominator as  $(1 - {}_{10}q_{65})^{5/10}$  since we have no other information between 65 and 75. That denominator is the square root of  $l_{75}/l_{65}$  or 0.934471. Our survival probability from 70 to 100 thus comes out to be .074694.

The best way to solve complicated conversion problems is to begin by drawing a diagonal line on a Lexis Diagram. Then mark off each age for which there is information about survivorship at that age. Next mark off ages which are the endpoints of intervals over which there is information about mortality within the interval. Between each marked age, assume a constant probability of dying, and apply the conversion formulas.

### 3.3 Columns of the Cohort Life Table

#### 3.3.1 King Edward's Children

Since the Seventeenth Century it has been customary to arrange  $l_x$  and  ${}_nq_x$  as columns in a table along with a special set of other measures of mortality. Such a table is called a “life table” or “lifetable”. The rows correspond to age groups. The columns correspond to different measures. From a mathematical point of view, all the main columns of the lifetable contain essentially the same information, in the sense that any one can be computed from any other (given some basic background information). But they present that information from different perspectives and organize it to be directly usable for different applications. Some columns focus on survivors, some on the dying, some on average life remaining. The choice of the columns that go into the lifetable and the notation for them is fixed by tradition. Each of the columns

is a function of age, so the columns of the lifetable are sometimes called “life table functions”.

We introduce the cohort lifetable with an example drawn from the Fourteenth Century. King Edward III of England reigned from 1337 to 1377. He had eleven children, and birth and death dates are known for all but one of them. Many of these children figure as major characters in Shakespeare’s history plays. They may be familiar to you from Richard II and Henry IV Parts I and II and from other stories of the Wars of the Roses. We shall treat these children, born between 1330 and 1346, as a cohort, and calculate a cohort lifetable from our data on their ages at death.

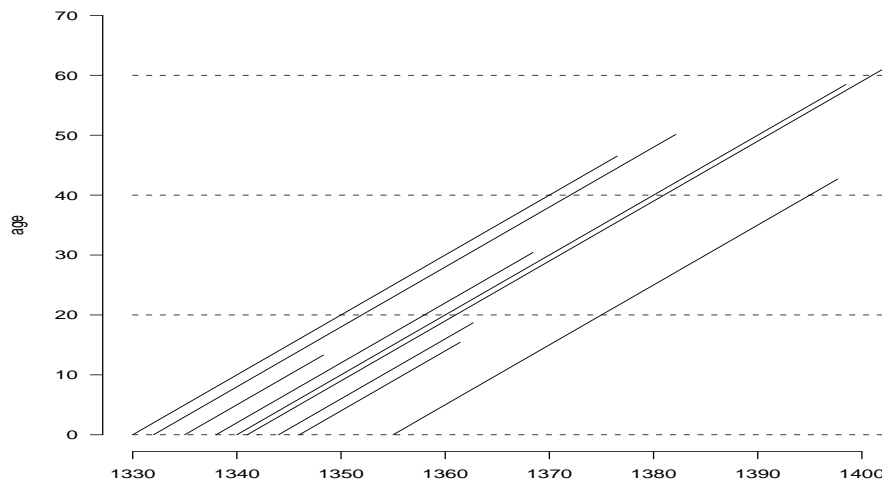
*Children of King Edward III of England (1312-1377)*

1330-1376	Edward, The Black Prince
1332-1382	Isabel
1335-1348	Joan
1336- ?	William of Hatfield (died young)
1338-1368	Lionel of Antwerp, Duke of Clarence
1340-1398	John of Gaunt, Duke of Lancaster
1341-1402	Edmund Langley, Duke of York
1342-1342	Blanche
1344-1362	Mary
1346-1361	Margaret
1355-1397	Thomas of Woodstock, Duke of Gloucester

For our first cohort lifetable, we shall restrict ourselves to the ten children for whom dates of birth and death are known. This will mean that our lifetable will tend to understate mortality for the cohort, since we do know that William of Hatfield, the omitted child, died young. For a better answer, we could fill in William of Hatfield’s death date with a plausible value – a process called “imputation” – and construct a new lifetable applying to all eleven children rather than to our ten.

We begin our lifetable with a column labeled  $x$ . The entry in this column is the **starting** age for the age group. This is followed

Figure 3.1: Lexis Diagram for the Children of King Edward III



by  $l_x$  and then by  $n$ , the length of the age interval, the difference between the current value of  $x$  and the value for the next age group found in the next row. Generally, lifetables are constructed with one-year intervals or with five-year intervals. A table with five-year-wide intervals except for a one-year-wide and a four-year-wide interval at the start is the most common form of lifetable. It is called an “abridged lifetable” in older works to distinguish it from the longer table with one-year-wide intervals. Our example, however, uses two ten-year-wide intervals and two twenty-year-wide intervals. This provides practice with intervals of different sizes. Also, with a cohort of only ten members, splitting up into narrower age groups would serve little purpose.

The survivorship column  $l_x$  leads off the cohort lifetable. The first entry,  $l_0$  is the radix, the initial size of the cohort at birth. In our example it equals ten, the number of children whose lives and deaths we are following in the table. By leaving out William of Hatfield, we conveniently obtain a round number radix which is also an actual starting cohort size. As we have said, in many applications we form a cohort out of a sample drawn from the

whole set of people of interest to us, so that we have a round number like 1,000 or 100,000 for the radix. Another alternative is to take the radix  $l_0$  to be equal to 1, in which case the  $l_x$  values give the proportion rather than the number of cohort members surviving to age  $x$ . We discuss the dependence of life table values on the radix in a later subsection.

We can read the  $l_x$  values off the Lexis Diagram. The value of  $l_{10}$  is the number of princes and princesses surviving at least till age 10, that is to say, the number of lifelines which cross a horizontal line for age 10. All but Blanche survive, so  $l_{10} = 9$ . Only six lifelines cross the mark at age 20, so  $l_{20} = 6$ . Five cross 40, whence  $l_{40} = 5$ , and one crosses 60, whence  $l_{60} = 1$ .

The last listed age is 60. This last row of the lifetable applies to all ages 60 and up. It is called the “open-ended age interval” since it has no maximum age. It requires special treatment for some columns of the lifetable. We write  $\infty$ , the mathematical symbol for infinity, for the length of this interval. But of course no one survives for ever. We write  $\infty$  because we are not setting some upper limit of our own.

The column  $l_x$  is closely related to the percentage distribution of length of life or age at death. That percentage distribution is given by  $100 * (1 - l_x/l_0)$ . In a cohort with the same lifetable as our example, a person who died at 20 would be in the fortieth percentile or age at death. Sixty percent of the cohort would die later and forty percent earlier. As  $l_x$  goes down, the percentile goes up, so  $l_x/l_0$  is sometimes called the “reverse cumulative distribution” of age at death.

Next to the  $l_x$  column it is good practice to write in the width of the age group  $n$  so that it is not forgotten in later calculations. It is easy to make mistakes by using the wrong  $n$ . Not all lifetables have this column.

The next column of the lifetable contains the probability of dying in the interval given that one is alive at the start. This is the  ${}_nq_x$  measure that we have already studied. Its formula is

$${}_nq_x = 1 - (l_{x+n}/l_x)$$

For the first age group we have  $1 - 9/10$  which is 0.100. For the second age group,  $1 - 6/9$  or .333, then .167, then .800, and then what? There is no  $l_{x+n}$  value for this last row. However, what is the probability, if you are alive at age 60, that you will die at some age above 60? The probability is 1. Death is certain. So we can fill in the last entry automatically. It is always 1. We now have a life table with the columns shown in Table 3.2.

Table 3.2: Four columns of King Edward's family life table.

$x$	$l_x$	$n$	${}_nq_x$
0	10	10	0.100
10	9	10	0.333
20	6	20	0.167
40	5	20	0.800
60	1	$\infty$	1.000

We now add a column which gives deaths between ages  $x$  and  $x + n$ . The formula is  ${}_nd_x = l_x - l_{x+n}$ . All this does is to count the lifelines that end in each age interval on the Lexis Diagram. In our table, these values are 1,3,1,4, and 1.

The next column is quite a specialized column in the life table. This column gives information about how early or how late in the interval the  ${}_nd_x$  death in the interval occur on average. It is called  ${}_na_x$ . In words, it is the average number of years lived in the interval from  $x$  to  $x+n$  by those dying within the interval. For King Edward's children, we know the ages at death with precision, so we can calculate this average directly. Often, however, we cannot calculate  ${}_na_x$  directly from data. In that case, we make  ${}_na_x$  equal  $n/2$ . That is, we assume that those who die in the interval die on average half-way through it. Later, we shall have a special convention for age groups of infants and young children, but for now we use  ${}_na_x = n/2$  whenever we cannot calculate it from data.

For the princes and princesses, we can calculate it. For the first interval, Blanche is the only one who dies. She dies in the same year in which she is born. We do not know exactly when, but we

shall assume that she dies half-way through it, and set  $_{10}a_0 = 0.5$  for Blanche.

Joan, Mary, and Margaret die between ages 10 and 20. Joan dies at age 13, that is to say, between 13 and 14. We write 13.5 for her death age. Then we subtract 10, which is the start of the interval, to have the number of years lived in the age interval. Proceeding in the same way with Mary and Margaret, we find  $_{10}a_{10} = (13.5 - 10 + 18.5 - 10 + 15.5 - 10+)/3 = 5.8$ .

Lionel gives us  $_{20}a_{20} = (30.5 - 20)/1 = 10.5$ .

For Edward the Black Prince, famous for his game of tennis, along with Isabel, John, and Thomas, we find  $_{20}a_{40} = (6.5 + 10.5 + 18.5 + 2.5)/4 = 9.50$ . These figures are rather near half-way through the interval, which is something of a coincidence considering how few people we are dealing with. For the open interval we have only Edmund of Langley, Duke of York, with  $_{\infty}a_{60} = 1.5$ .

This calculation has taken some time in this case. But in most cases you can fill in  $n/2$  instantly and go on to the next columns of the table.

### 3.3.2 From ${}_nL_x$ to $e_x$

The remaining columns of the lifetable all depend on a calculation of cohort person-years lived. The person-years lived column is called  ${}_nL_x$ , and it is one of the four most important columns of the lifetable, along with  $l_x$  and  ${}_nq_x$  which we have already studied and with  $e_x$  which we shall soon describe. The formula is

$${}_nL_x = (n)(l_{x+n}) + ({}_na_x)({}_nd_x)$$

This quantity is the number of Cohort Person-Years Lived between ages  $x$  and  $x + n$

The value of  ${}_nL_x$  is made up of two contributions. Those who survive the whole interval,  $l_{x+n}$  in number, each contribute a full  $n$  years to  ${}_nL_x$ . Those who die during the interval,  ${}_nd_x$  in number,

contribute on average  ${}_na_x$  years to  ${}_nL_x$ . For any age group in which we set  ${}_na_x = n/2$ , this formula simplifies. We have  ${}_nd_x = l_x - l_{x+n}$ , so the simplified formula is

$${}_nL_x = (n/2)(l_x + l_{x+n}).$$

Here are the columns which we have filled in so far:

Table 3.3: The first seven columns of a life table

x	$l_x$	$n$	${}_nq_x$	${}_nd_x$	${}_na_x$		${}_nL_x$
0	10	10	0.100	1	0.5	$90 + 0.5 =$	90.5
10	9	10	0.333	3	5.8	$60 + 17.5 =$	77.5
20	6	20	0.167	1	10.5	$100 + 10.5 =$	110.5
40	5	20	0.800	4	9.5	$20 + 38.0 =$	58.0
60	1	$\infty$	1.000	1	1.5	$0 + 1.5 =$	1.5

There are some refinements in the definition of person-years lived that are possible. For those familiar with the calculus, where we have continuous values for  $l_x$  we can form the integral

$${}_nL_x = \int_x^{x+n} l_a da$$

Continuing with the columns of the life table, we have the life table death rate, the ratio of deaths to person-years lived, whose formula is  ${}_nm_x = {}_nd_x / {}_nL_x$ . This is the age-specific counterpart of the crude death rate. It is a rate, measured per units of time. The life table death rate measured over a very short interval starting at  $x$  is very close to the hazard rate. For the first row of our example, we have  ${}_{10}m_0 = 1/90.5 = 0.011050$

If we add up all the person-years to be lived beyond age  $x$ , we obtain  $T_x$ , the person-years of life remaining for cohort members who reach age  $x$ . The formula is  $T_x = {}_nL_x + {}_nL_{x+n} + {}_nL_{x+2n} + \dots$ . This is easiest to compute by first filling in the whole  ${}_nL_x$ , and then cumulating sums from the bottom. In our example,  $T_{60}$  is 1.5,  $T_{40}$  is  $1.5 + 58.0 = 59.5$  and so forth.



The main use of  $T_x$  is for computing the expectation of further life beyond age  $x$ ,  $e_x$ . The  $T_x$  person-years will be lived by the  $l_x$  members of the cohort who reach age  $x$ , so  $e_x$  is given by  $e_x = T_x/l_x$ . The expectation of life at age zero, that is, at birth, is often called the “life expectancy”.

It is important to remember that  $e_x$  is the expectation of future life beyond age  $x$ . It is not an average age at death. To obtain the average age at death for cohort members who all survive to age  $x$ , we add  $x$  and  $e_x$ . I suggest writing this sum  $e_x + x$  as the last column of the lifetable. Not all lifetables include it, but I find that it is useful for checking whether the total ages are plausible. The expectation of future life  $e_x$  does not always go down. It often goes up after the first few years of life, because babies who survive infancy are no longer subject to the high risks of infancy. The column  $x + e_x$ , however, always goes up, and this fact provides a check on the calculations in the lifetable.

Filling in the remaining columns in our example, we have Table 3.6.

Table 3.4: Right-hand columns of a lifetable

$x$	${}_n m_x$	$T_x$	$e_x$	$x + e_x$
0	.011	338.0	33.8	33.8
10	.040	247.5	27.5	37.5
20	.009	170.0	28.3	48.3
40	.070	59.5	11.9	51.9
60	.667	1.5	1.5	61.5

The expectation of life at birth is often taken as an index of overall mortality. But it gives a bad idea of lifespan, since it weights infant mortality which may be heavy.  $e_{10} + 10$  is a better index of lifespan.

### 3.3.3 The radix

We now return to the radix  $l_0$ , which plays the role of the

cohort's initial size. This initial size has a name of its own, the "radix", which is the Latin word meaning "root". The phrase "plays the role" reminds us that the radix need not be the whole size of an actual cohort. The cohort of U.S. males born between 1945 and 1950, for instance, according to the U.S. Census Bureau's Demographic Analysis, had an initial size of 9,025,382. This is not a convenient number on which to build a lifetable. It is much more convenient to have a starting size which is a round power of ten like 1000 or 100,000 or 1. Demographers feel free to choose a radix to suit their tastes. The best way to think about this choice is to think of drawing a random sample of some size like 1,000 or 100,000 from the cohort. We think of the lifetable as following this "sample cohort" through life. Since our sample is a random sample, the survival of the sample cohort mirrors the survival of the whole cohort. If the sample cohort starts with 100,000 members and has 30,000 left alive at age 80, then the real cohort with 9,025,382 members would have  $(30,000/100,000) \times 9,025,382$  members alive at age 80. If we think of taking a sample of size one from the cohort, we can then have a radix  $l_0 = 1$  and we can interpret  $l_x$  as the expected *proportion* (out of 1) of the cohort surviving to age  $x$ .

This freedom to set the radix at any value we choose provides useful flexibility. It is easier to read a lifetable with  $l_0$  equal to 100,000 than with  $l_0$  equal to something like 9,025,382. Conceptually, however, the  $l_x$  column is a column of survivors, and it is worthwhile to picture an actual group of people – a whole cohort or a sample from a cohort – starting with  $l_0$  members and living out their lives, surviving, and dying.

It is important to keep straight what depends on the choice of the radix  $l_0$  and what does not. Doubling the size of the sample cohort at birth changes some quantities and has no effect on other quantities. Which quantities do change with a change of radix? Here is a list:

**YES:**  $l_x, {}_nL_x, {}_nd_x, \dots$

**NO:**  ${}_nq_x, {}_nm_x, e_x, \dots$

### 3.3.4 Combining Single-Sex Life Tables

Because men and women typically die at somewhat different rates, we usually construct separate lifetables for them. Sometimes, however, we want to have a combined-sex lifetable. It is easy to build one. The survivors of a cohort consist of all the women who survive plus all the men who survive. We add the survivors together and construct the other columns of the lifetable as usual from the combined  $l_x$  column.

If our single-sex lifetables have an arbitrary radix like 1 or 100,000, we also need to know what fraction of the cohort at birth were girls. We shall be discussing the “fraction female at birth” in Chapter Four. We use the symbol  $f_{fab}$ , where the “fab” stands for “female at birth”. As we shall see, when we do not know the true  $f_{fab}$  for a cohort, we use a default value  $f_{fab} = 0.4886$ . As long as the radix used in the single-sex tables is the same, the formula for combining them is then

$$l_x^{both} = f_{fab}l_x^{fem} + (1 - f_{fab})l_x^{male}$$

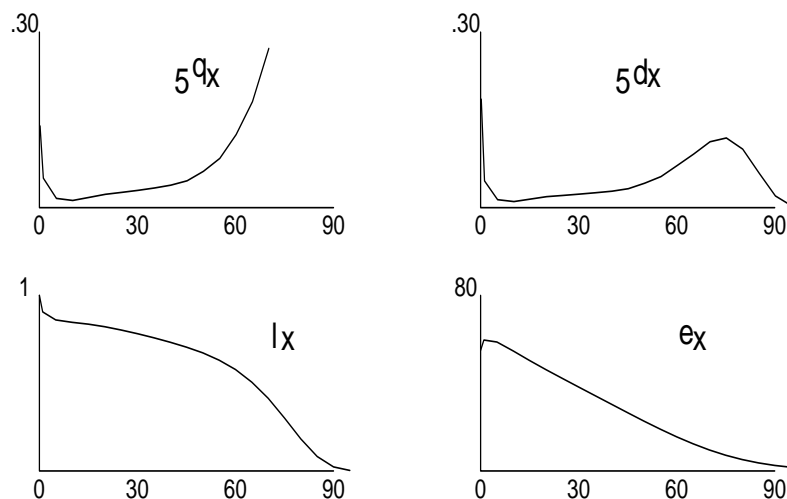
This survivor based approach for combining lifetables will be used for period as well as for cohort lifetables. We do not average  ${}_nq_x$  values or  ${}_nm_x$  values, but instead always use the survivorships  $l_x$ .

## 3.4 Formulas for Cohort Life Tables

We collect together in this section the formulas that we have developed in the preceding sections for the columns of a cohort life table. The different lifetable functions express the same basic information about survivorship from different points of view. Figure 3.2 shows typical shapes of four of these functions of age.

The probability of dying in the upper left of Figure 3.2 drops after infancy and rises steadily in old age. The number of deaths to members of the cohort in the upper right show a bulge in old age

Figure 3.2: Typical Shapes of Lifetable Functions



which decreases as there are fewer and fewer members of the cohort left to die at the ever-increasing rates. The number of cohort members surviving is shown at the lower left, and the expectation of further life at the upper right, with a slight rise after infancy, and a steady decrease into old age.

$l_0$  is the radix, the initial number of people in the sample cohort at birth, typically 100,000 or 1,000 or 1 or another power of 10.

$l_x$  is the number of people in the sample cohort out of the original  $l_0$  who survive to age  $x$ . In a cohort we can observe  $l_x$  directly from the cohort life histories. If the radix  $l_0$  equals 1,  $l_x$  becomes the probability of surviving to age  $x$ . If the radix  $l_0$  equals 100,  $l_x$  becomes the percentile in the distribution of ages at death of a person dying at age  $x$ .

${}_nq_x = 1 - (l_{x+n}/l_x)$  equals the probability of dying within the next  $n$  years for those aged  $x$ . It is a probability conditional on surviving to age  $x$ .

${}_n\mathbf{d}_x = l_x - l_{x+n}$  equals cohort deaths between ages  $x$  and  $x + n$ .

${}_n\mathbf{a}_x$  is the average years per person lived between  $x$  and  $x + n$  by those dying in interval, observed from cohort life histories if possible, or set equal to  $n/2$  in the absence of other information.

${}_n\mathbf{L}_x = (n)(l_{x+n}) + {}_na_x d_x$  equals person years lived by the cohort between ages  $x$  and  $x + n$ . Whenever  ${}_na_x = n/2$ , we then have  ${}_nL_x = \frac{n}{2}(l_x + l_{x+n})$ .

${}_n\mathbf{m}_x = {}_nd_x / {}_nL_x$  equals the life table death rate, deaths to the cohort divided by person-years lived at risk.

$\mathbf{T}_x = {}_nL_x + {}_nL_{x+n} + {}_nL_{x+2n} + \dots$  equals the person years of life remaining for the cohort beyond age  $x$ .

$\mathbf{e}_x = T_x / l_x$  equals the expectation of life remaining for those attaining age  $x$ .

$\mathbf{e}_x + \mathbf{x}$  equals the expectation of total life for those attaining age  $x$ .

$\mathbf{h}(\mathbf{x})$  is the hazard rate at age  $x$ , equal to minus the slope of the logarithm of  $l_x$  at age  $x$ . The hazard rate does not have its own column in the traditional lifetable, but it is one of the most important measures of mortality.

## 3.5 A Typical Life Table Problem

For practice, in this section we take a typical lifetable problem, and work through the solution in detail. The problem is as follows: For Japanese men in 1963 the expectation of life at birth was 67.287 years and the expectation of life at age ten was 59.783 years. A proportion equal to 0.97428 were measured to survive from age zero to age one, and a proportion 0.96728 from age zero to age five. Can we tell what proportion were measured to survive from age zero to age ten ? If so, what is that proportion ?

To solve this problem, we first list the quantities that we are given. These are  $e_0$ ,  $e_{10}$ ,  $l_1$ , and  $l_5$ . We assume a radix of unity. We want to find  $l_{10}$ .

We write down an equation which connects the known value of the expectation of life at birth with the unknown value of the proportion surviving to age 10:

$$e_0 = T_0 = {}_1L_0 + {}_4L_1 + {}_5L_5 + l_{10}e_{10}$$

Since we do not have specific information about the distribution of deaths within intervals, we assume  ${}_na_x = n/2$  in all age groups in this problem. Then we can write

$$e_0 = (1/2)(l_0 + l_1) + (4/2)(l_1 + l_5) + (5/2)(l_5 + l_{10}) + l_{10}e_{10}$$

$$\begin{aligned} 67.287 &= (1/2)(1 + 0.97428) + (4/2)(0.97428 + 0.96728) \\ &\quad + (5/2)(0.96728 + l_{10}) + l_{10}59.783 \end{aligned}$$

We now collect the terms which include  $l_{10}$ :

$$\begin{aligned} 67.287 - (1/2)(1.97428) - 2(0.97428 + 0.96728) - (2.5)(0.96728) \\ = l_{10}(62.283) \end{aligned}$$

Dividing through by 62.283 gives us our answer for  $l_{10}$ :

$$(67.2870 - 0.9871 - 3.8831 - 2.4182)/62.283 = 0.9633 = l_{10}$$

### 3.6 \*Hazards and Survivors

We return in this section to the study of hazard rates, which play a prominent role in the advanced study of mortality. The hazard rate, being a rate, has units of 1/time, that is, it is a rate per unit time. The faster that members of the cohort are dying, the higher is the hazard rate. It can take any value equal to or

greater than zero. We often write the hazard rate as a function  $h_x$  depending on age  $x$ . Then we speak of the “hazard function”. The hazard rate is also often called the “force of mortality”. Some authors prefer the Greek letter  $\mu$ , pronounced “mu” or “myoo” for the hazard rate.

Since it is equal to minus the slope of the logarithm of the survivorship function  $l_x$ , the hazard rate plays a role in formulas very like the role of  $R$ . It enters into an exponential formula for survival from  $x$  to  $x + n$ :

$$l_{x+n} = l_x \exp(-h(x) n)$$

The formula connecting the hazard rate to the probability of dying  ${}_nq_x$  has two minus signs, one inside and one outside the exponential function:

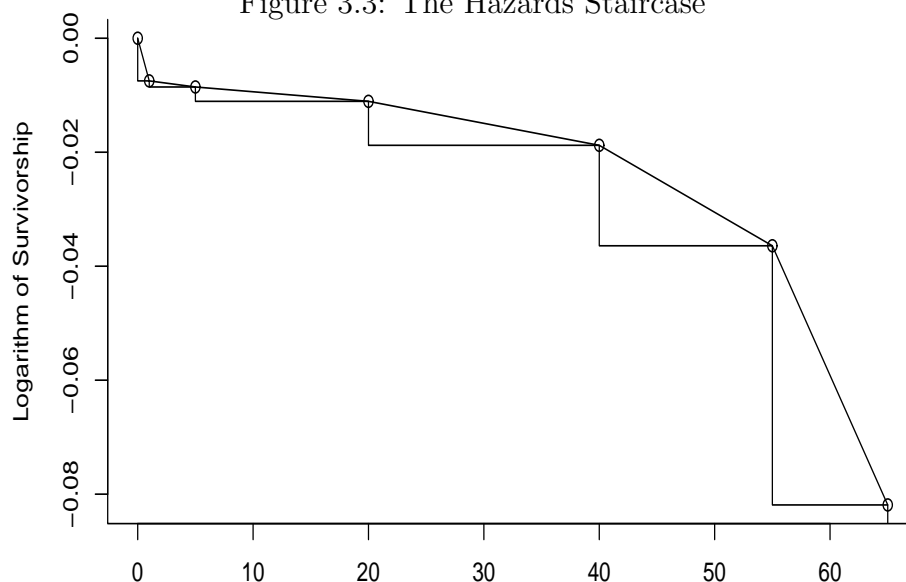
$${}_nq_x = 1 - \exp(-h(x) n)$$

We can use hazard rates to construct survivorship curves in the same way we used growth rates to construct population curves for China in Chapter One. As before, we work on a logarithmic scale. The hazard rate is minus the slope of the curve on this logarithmic scale. We may imagine our cohort descending a staircase shaped like the graph in Figure 3.3. This graph is based on a prediction of the lifetime experience of the cohort of girls born in the United States in 1985, constructed by the methods of Lee and Carter described in Chapter Seven.

Heights along the staircase in Figure 3.3 are values of the logarithm of the number of survivors,  $\log(l_x)$ . Each step corresponds to an age interval. The width of the step is the width of the age interval  $n$ . The drop is the decrease in number of survivors measured on a logarithmic scale. The steepness, the drop at this step divided by the width of the step, is the hazard rate. The picture is like Figure 1.2, where the logarithm of population size increased in steps whose steepness was given by the growth rate  $R$ . There the slope was upward, given by the “rise” over the “run”. Here the slope is downward, given by the “drop” over the “run”. When we

know the slope, the hazard rate, we find the drop by multiplying the slope by the width of the step, the “run”  $n$ .

Figure 3.3: The Hazards Staircase



In Figure 3.3, the first step, extending across the first year of life is a steep one. Hazards for infants are greater than for toddlers or teenagers. After infancy, the steps are gentle. Gradually they steepen again, as the cohort descends the staircase toward later ages with higher hazards. In order to show the early steps clearly, later ages with much steeper steps are excluded. In this graph the decreases in survivors are quite small on a logarithmic scale.

The use of hazard to construct a survivorship curve is illustrated in Table 3.5, which shows data for constructing the curve in Figure 3.3. The information given us is in ordinary type. Entries from our calculations are in *italic type*.

Our cohort starts out with 1000 girl babies drawn from the whole cohort of girls born in the U.S. in 1985. The logarithm of cohort survivors starts out at the value shown in the first row, namely

$$\log(l_0) = \log(1000) = 6.907750$$



Table 3.5: Predicted Hazards and Survivorships for the Cohort of U.S. Women Born in 1985.

x	n	hazard	drop	$\log l_x$	$l_x$
0	1	0.007488	.007488	6.907755	1000
1	4	0.000267	.001068	6.900267	.993
5	15	0.000168	.002520	6.899200	
20	20	0.000386			
40	15	0.001175			
55	10	0.004547			
65	10	0.013059			
75	10	0.030235			
85	10	0.106180			
95	5	0.215550			
100	$\infty$	0.400000			

Each row is a step. At each step we calculate the drop by multiplying the width of the step given in the second column by the hazard rate in the third column. The first drop is 1 times 0.007488. The second drop is  $(4) * (.000267) = 0.001068$ . At each step we subtract the drop just calculated, in the fourth column, from our logarithm of survivors to obtain the logarithm at the next step.

$$\log(l_{x+n}) = \log(l_x) - n h(x)$$

At the first step,  $\log(l_1) = \log(l_0) - 0.007488 = 6.900267$ . We then use the exponential function to recover  $l_x$  itself from  $\log(l_x)$ , finding  $\exp(6.900267) = 0.993$ .

The age intervals in Table 3.5 and Figure 3.3 are mostly fairly wide and are useful when we want to study average hazards over stages of life or summarize information in compact tables. For detailed studies it is preferable to use intervals  $n = 1$  year wide, or even narrower intervals. Figure 3.4 shows a smooth survivorship curve based on one-year-wide intervals for the same predictions.

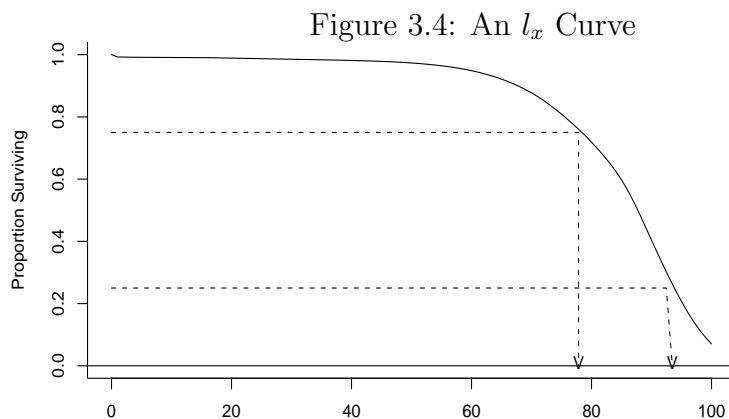
The same reasoning which let us express period person-years

as an area in Chapter Two, adding up little rectangles, helps us add up all the little decrements to  $\log(l_x)$  from all the little steps in our staircase plot. The overall drop in  $\log(l_x)$  is given by the area under the  $h(x)$ -curve:

$$\log(l_x) = \log(l_0) - \int_0^x h(a) da$$

Taking exponentials of both sides, we obtain an elegant formula for survivorships in terms of hazards:

$$l_x = l_0 e^{-\int_0^x h(a) da}$$



The  $l_x$  curve can be used to compare ages at death on a percentage basis. If we want to find the age by which 25% of the cohort die, leaving 75% alive, we draw a horizontal line of height  $0.75l_0$  across the graph till it hits the  $l_x$  curve. Then we draw a vertical line down to the axis to find what age  $x$  corresponds to this point. On this curve,  $x = 78$  years. Similarly, to find the age by which 75% of the cohort die, we draw a horizontal line of height  $0.25l_0$ , and find  $x = 94$  years.

## 3.7 \*Models for Cohort Mortality

### 3.7.1 \*The Gompertz Model for Old Age

Mortality is ordinarily a story about the very young and the fairly old. Evolution driven by natural selection arranges, as best it can, to keep death rates low for creatures of reproductive age. Hazard functions typically are shaped somewhat like the letter “J”, high in infancy, low in early adulthood, and rising into old age. Demographers have put effort into developing special models for infant mortality and for old-age mortality. Here we introduce two such models, the Gompertz model for old-age mortality and the Bourgeois-Pichat model for infant mortality. These are our first examples of model lifetables, which will be studied more extensively in Chapter Seven.

For older ages, mathematical curves for describing cohort mortality are coming back into fashion. The Gompertz model is the most popular of all of these. Over the last decades, models of overall mortality that use conveniently simple mathematical curves have come into disrepute, because we know that the patterns of mortality are not simple. Different causes of death predominate at different ages and there is no reason that their combination should be some familiar curve given by a simple formula. Late in the lifespan, however, the situation changes. At advanced ages, the idea of a definite cause of death is less well-defined. Often older people have many contributing infirmities at the time of death, and the choice of one cause over others is arbitrary. Belief in a simple curve as a summary of the overall outcome may be more plausible.

The Gompertz model, first published by Benjamin Gompertz in 1825, is a model for the hazard function at adult ages. It says that this hazard function takes the form of an exponential curve.

$$h(x) = \alpha e^{\beta x}$$

This formula is equivalent to the provision that the graph of the logarithm of the hazard function as a function of age is a straight

line:

$$\log(h(x)) = \log(\alpha) + \beta x$$

Like most models, the Gompertz model has quantities called *parameters* which are chosen on a case by case basis to make the general model fit the specific case. Here the parameter  $\alpha$  determines the initial level of the hazard and the parameter  $\beta$ , which specifies the slope on a logarithmic scale, determines the rapidity of increase of mortality with age, what is often called “senescent” mortality.

With any model, when we are given values of the parameters, we want to be able to calculate predictions from the model. This is the first goal. We also want, when we are given a set of observations, to find the values of the parameters that make the predictions of the model fit the observations as well as possible. That is to say, we want, when we are given data, to estimate the parameters. This is the second goal.

Pursuing our first goal with the Gompertz model, suppose we are given parameter values  $\alpha = 0.000133$  and  $\beta = 0.080$ . These values are based on unusually good data for the cohort of Swedish women born in 1910 from the Human Mortality Database (2005). What is the predicted hazard rate at age 85, when women enter the category of the “oldest old”?

$$h(85) = (0.000133)e^{(85)(0.080)} = 0.119414$$

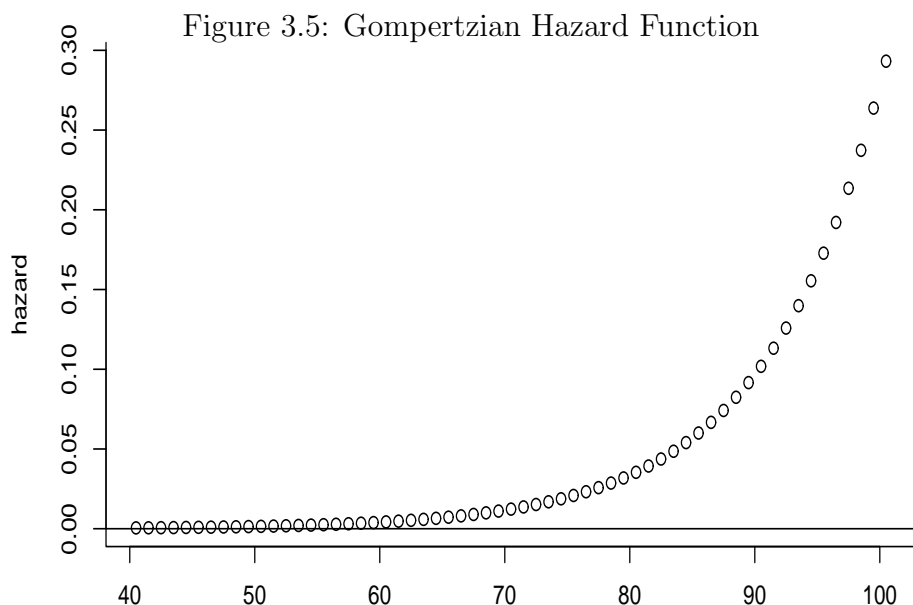
If the Gompertz model continued to hold at extreme ages, how high would the hazard rate rise by age 100?

$$h(100) = (0.000133)e^{(100)(0.080)} = 0.396467.$$

In reality, exponential increases in hazard rates begin to slacken for centenarians and supercentenarians.

When we calculate hazard rates from the Gompertz model, we often also want to calculate survivorships. Happily, our formulas from Chapter Two for the area under an exponential curve match up with our formula for survivorships in terms of areas under the hazard curve to give us a formula for Gompertz survivorships:

$$l_x = l_0 e^{-(\alpha/\beta)(e^{\beta x} - 1)}$$



Here we have an exponential inside an exponential. In practice, Gompertz models are only valid for adult ages over, say 40, and we want to apply the formula to ratios like  $l_x/l_{40}$  rather than to  $l_x/l_0$ .

For example, we can ask what survivorship between 70 and 90 would be implied by a Gompertz model with our Swedish parameter values  $\alpha = 0.000133$  and  $\beta = 0.080$ . It is easy to work with the logarithms:

$$\log(l_{90}/l_{70}) = -(.000133/.080) \left( e^{(.080)(90)} - e^{(.080)(70)} \right) = -1.777220$$

The probability of surviving comes out to be  $\exp(-1.777220) = 0.169$ .

Deaths to members of the cohort at age  $x$  are calculated by multiplying the hazard rate at age  $x$  by the survivors to age  $x$ , that is, by  $h(x)l_x$ . The curve of cohort deaths slopes upward at younger ages, reaches a peak or “mode”, and then drops as fewer members are still alive at risk of dying. The age of the peak is called the “modal age at death”. It is possible to use calculus to

find the age at which the slope changes sign, and the modal age at death under a Gompertz model turns out to be

$$x_{mode} = (1/\beta) \log(\beta/\alpha)$$

For our Swedish fitted model, the modal age at death is 80 years.

The formula for the modal age at death is especially handy, because there is no simple exact formula for life expectancy under a Gompertz model. For low-mortality populations, however, there is a handy approximation for life expectancy beyond age 40 when a Gompertz model applies:

$$e_{40} + 40 = (1/\beta) (\log(\beta/\alpha) - 0.577215)$$

The constant 0.577215 is called Euler's Constant, named after Leonhard Euler who took the first steps toward the creation of stable population theory studied in Chapter Ten.

We have been working forward from parameter values to predictions from the model, pursuing our first goal. Our second goal with any model is to work back from observed data to estimates of parameters. Our Gompertz case illustrates a simple approach that demographers use over and over. By taking logarithms, the Gompertz formula for the hazard rate can be turned into an equation with the same form as equations for straight lines:

$$\log(h(x)) = \log(\alpha) - \beta x$$

We can make a plot with observed values of  $\log(h(x))$  on the vertical axis and corresponding ages  $x$  on the horizontal axis. When we fit a straight line to the points on the plot, the slope is an estimate of  $\beta$  and the exponential function applied to the intercept is an estimate of  $\alpha$ .

Standard formulas for fitting straight lines to plots come from the statistical technique of linear regression. The theory does not matter to us here. What we need are the expressions for the estimated slope and intercept:

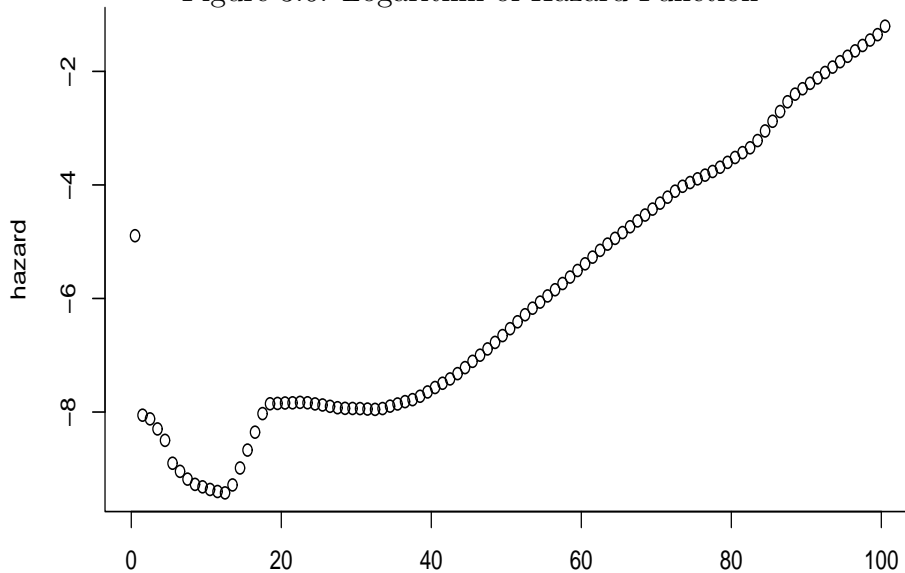
$$slope = \frac{mean(X * Y) - mean(X) * mean(Y)}{mean(X * X) - mean(X) * mean(X)}$$

$$\text{intercept} = \text{mean}(Y) - (\text{slope}) * \text{mean}(X)$$

We write capital  $Y$  for the values on the vertical or “Y”-axis, in this case  $\log(h(x))$ , and capital  $X$  for the values on the horizontal or “X”-axis, in this case ages  $x$ . The product  $X * X$  is the same as  $X^2$ , but writing  $X * X$  emphasizes the symmetry between numerator and denominator.

For illustration, let us fit a Gompertz model to hazard rates for the cohort of U.S. women born in 1985 portrayed in Table 3.5 and Figure 3.3. Logarithms of single-year hazard rates are shown in Figure 3.6. At younger ages, the curve has several bumps, but beyond about age 40 it settles into a straight line pattern, as the Gompertz model requires. This graph derives from forecasts, but recorded cohort data often show the same behavior. Such Gompertzian patterns are found in data from many human societies and from many other species, from species of flies, worms, and birds to horses and chimpanzees. This empirical fact has, as yet, no full theoretical explanation.

Figure 3.6: Logarithm of Hazard Function



For brevity, we work with hazards for wider age groups using the entries above age 40 and up to 100 in Table 3.5. Our X-axis

values are midpoints of the intervals. Our Y-values come from logarithms of the hazards. Bear in mind that logarithms come in twice in this picture. The hazards themselves are slopes of logarithms of survivors, and we are now taking the logarithms of these already logarithmic quantities.

Table 3.6: Fitting Gompertz Parameters.

	X	Y	X*X	X*Y
	47.5	-6.746	2256	-320.435
	60.0	-5.393	3600	-323.580
	70.0	-4.338	4900	-303.660
	80.0	-3.499	6400	-279.920
	90.0	-2.243	8100	-201.870
	97.5	-1.534	9506	-149.565
sum	445	-23.753	34762	-1579.03
mean	74.16	-3.959	5793.75	-263.1717

Our estimate of the slope  $\beta$  is given by

$$\frac{-263.1717 - (74.16)(-23.753)}{5793.75 - (74.166)(74.166)} = \frac{30.44181}{293.0556} = 0.103877$$

The intercept, where the straight line would cross the vertical axis, is given by

$$-3.959 - (0.103877)(74.16667) = -11.66321$$

Our estimate of  $\alpha$  is the exponential of this quantity,  $86 * 10^{-6}$ . We see that our forecasts for U.S. women growing up in the new century imply a  $\beta$  value quite close to the one for Swedish women from the last century, but the level parameter  $\alpha$  decreases a great deal.

### 3.7.2 \* Bourgeois-Pichat's Model for Infants

We turn now to the beginning of life. The timing of infant deaths often has a characteristic pattern, which provides us with



a second example of a mortality model. This model, discussed by Pressat (1972:90-94), was introduced by Jean Bourgeois-Pichat in 1951 in an article in the journal *Population*. Medical advances which now save the lives of many premature infants in well-off societies have reduced the applicability of the model, but it continues to offer good practice in demographic estimation and remains relevant to historical settings and countries with more limited medical technology.

Bourgeois-Pichat's model takes the form

$$1 - l_x/l_0 = a + b[\log(1 + 365x)]^3$$

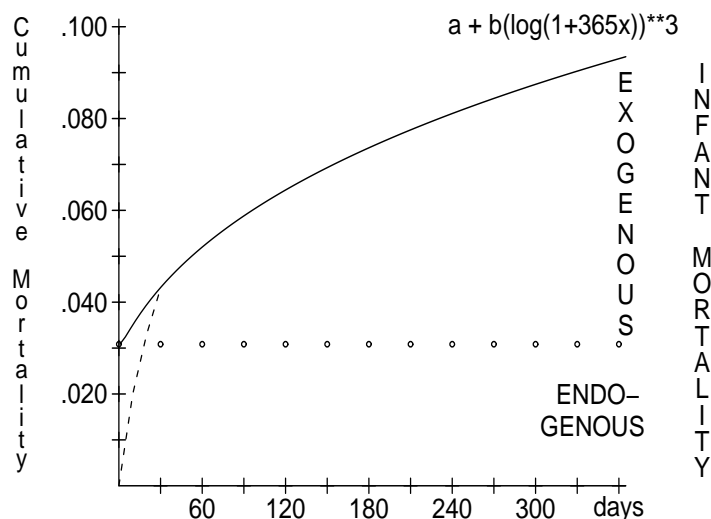
On the left-hand side of this equation is the cumulative probability of dying by age  $x$ , which is 1 minus the proportion surviving to age  $x$ . Usually  $x$  is measured in years. Here we are interested in fractions of the first year of life. We convert to days since birth by multiplying by 365. We transform age with the cube of a logarithmic function, multiply by a constant  $b$  and add a constant  $a$ . The shape of the resulting predicted curve of the cumulative probability of dying as a function of age is shown in Figure 3.7.

The constants  $a$  and  $b$  are the “parameters” of the model. Each population has its own values of  $a$  and  $b$  which tailor the general model to the specific population. Different populations with different values of  $a$  and  $b$  have curves of different height and different steepness, but the curvature is much the same. Bourgeois-Pichat fitted precise data primarily from cumulative mortality after the first month. The curve was then extrapolated back to  $x = 0$ .

It is important to note that the solid line in Figure 3.7 does not start at a height of zero at age zero, even though we know that cumulative mortality has to start at zero. The model only applies after the first month. True cumulative mortality come up from zero, like the dotted line in the picture, joining the solid line given by the model typically about 30 days after birth.

The interpretation is that the model picks up the *exogenous* component of mortality, due to infections, environmental influences, etc. The first month mortality is a mixture. Part is ex-

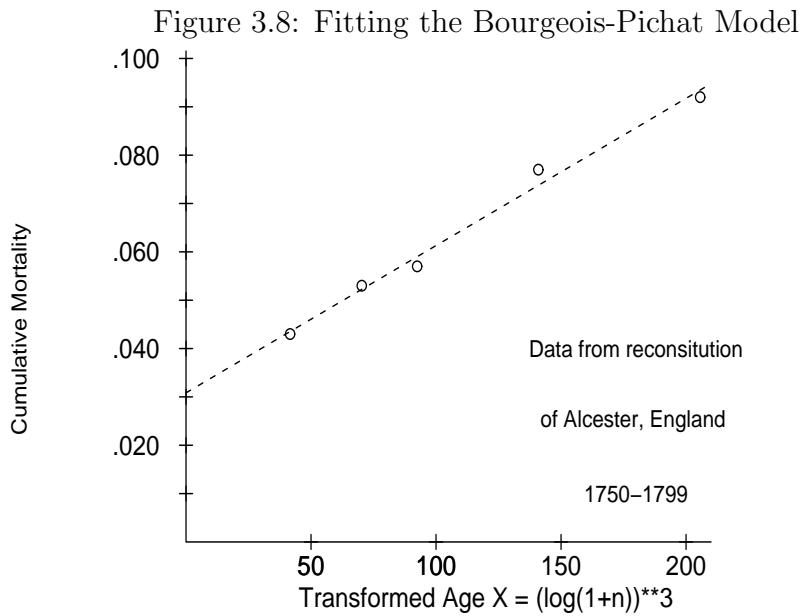
Figure 3.7: Bourgeois-Pichat's Model for Infant Mortality



ogenous, but part is *endogenous*. “Exo” is a Greek root for outside, “endo” for inside. Endogenous mortality is thought to reflect intrinsic biological “error rates” that are a prolongation of spontaneous fetal mortality and compromise viability. Now, with the ability to prolong the life of non-viable newborns, and with the rise in premature births that survive, the one-month distinction is not so valid as it was in the 1950s.

We separate out the endogenous mortality as a residual by extrapolating back the exogenous mortality to the age of zero. All the rest is the endogenous mortality. So to find it we must find the intercept  $a$ . To apply the model to a specific case, we must estimate the parameters  $a$  and  $b$  from data. This model has a specially convenient property which we find over and over again among demographic models. After a transformation, as with the Gompertz, the model takes the form of the equation for a straight line. The transformation takes age  $x$  into a new X-axis value  $X = [\log(1 + 365x)]^3$ .

Then  $a$  and  $b$  are the intercept and slope of a straight line relating  $X$  to  $Y = 1 - l_x/l_0$ .



We proceed to use the same approach based on linear regression as we have used with the Gompertz model. Consider data from the parish reconstitution of Alcester by E. A. Wrigley shown in the table.

Table 3.7: Data from parish reconstitution of Alcester.

days	X	Y	X*X	X*Y
31	41.63	.043	1733	1.79
61	70.30	.053	4942	3.73
91	92.46	.057	8548	5.27
181	140.93	.077	19862	10.85
365	205.65	.092	42294	18.92
mean	110.19	.064	15476	8.11

Plugging in values for Alcester in the expressions for slope and

intercept from the previous section, we have

$$\frac{8.111 - (110.19)(.064)}{(15476 - (110.19)^2)} = \frac{1.014}{3332} = .000308$$

The intercept is the endogenous mortality estimate. The exogenous mortality estimate is the total minus the intercept.

Wrigley and Schofield used this model to adjust for underregistration of infant deaths. For periods of poor registration, such as times of war, they assumed that registers tended to miss deaths that happened shortly after birth. They also assumed that endogenous mortality only changes quite slowly. Then they could use periods of good registration to estimate background levels of endogenous mortality. Rapid apparent changes in endogenous mortality could then be re-interpreted as changes in the underregistration of infant deaths.

### 3.8 Annuities and Insurance

The earliest applications of life-table methods were to annuities. The idea is to assure yourself a steady income, for instance after retirement, by buying now from the Annuity Company for a single sum of money  $P$ , the purchase price, a policy under which the Company agrees to pay you a yearly (annual) benefit amount  $B$  for as long as you live. Annuities and insurance are social institutions you learn about when you reach a certain stage of life, having a job and a family. In an annuity, you pay now and collect benefits as long as you live. If you die soon, the company wins. If you live long, the company loses. They have to set the purchase price to break even or come out ahead. We shall deal here with “actuarially fair rates”, where the profit is zero. In practice, a profit margin is added on.

The purpose of buying an annuity is to share risk. If you are going to die soon, you would want to spend a lot of money while you are alive to enjoy it. But suppose you turned out then to live

a long time. You would have run out of money. You can remove your risk of not having enough to live on if you happen to live a long time by buying an annuity.

The purchase price should depend on how old you are. When the British parliament first tried to raise money by selling annuities with a profit margin independent of age, Dutch financiers had their teen-age daughters buy annuities, and parliament lost money.

For the purpose of purchasing annuities, we shall consider all the members of your cohort buying annuities at the same time. Then some members of the cohort live longer and others live less long. Imagine, then, all  $l_x$  members of a your birth cohort buy annuities when you are aged  $x$ . You live  ${}_nL_x$  person-year in the next  $n$  years, and receive  $B$  dollars each year, so in the next  $n$  years the benefit per cohort member is

$$B \frac{{}_nL_x}{l_x}$$

The total benefits over all periods per cohort member amount to

$$B_nL_x + B_nL_{x+n} + B_nL_{x+2n} + B_nL_{x+3n}\dots = BT_x = BT_x.$$

The total purchase amount is  $Pl_x$ . Equating the two, we find that in this case  $P = Be_x$ .

Thus if you buy an annuity for 10,000 dollars a year now, at age 20, when your expectation of further life is about 53 for men and 60 for women, it looks as if it would cost you more than half a million dollars. But annuities don't cost such huge sums. Why not? The answer is that the company can invest the money and earn interest while it is waiting to pay future benefits. Time elapses between purchase and receipt of benefits.

To take interest into account, it is useful to imagine the insurance company having a separate bank account for each future five-year period. It has to invest the money for early periods in short-term investments, whereas it can invest the money for periods long in the future in long-term investments. We do the calculation by figuring how much money it has to put into each account

now in order to have enough to pay out the benefits from that account when the time comes.

In the following formula, each addend stands for benefit money received during a different period. The average times between purchase and receipt of benefits in the periods are

$$n/2 \dots n + n/2 \dots 2n + n/2 \dots 3n + n/2 \dots$$

Suppose there is an interest rate  $i$  being offered by banks, so that the Annuity Company only need put in the bank  $B/(1+i)$  this year in order to have  $B$  to pay out next year. Then the money that they need to put in the bank to finance benefits for each of the periods is

$$\frac{B}{l_x} \left( \frac{{}_nL_x}{(1+i)^{n/2}} + \frac{{}_nL_{x+n}}{(1+i)^{n+n/2}} + \frac{{}_nL_{x+2n}}{(1+i)^{2n+n/2}} + \dots \right)$$

Annuities are very simple, if you keep clear in your mind the relationship between the amount that you, the annuitant, will receive from the company and the amount the company must put in the bank at the beginning to have money ready later to pay out.

Lifetables with open-ended intervals at the end introduce one subtlety into the computation. What average time should apply for the earning of interest in the open-ended interval? Our rule is to replace  $n/2$  with  $e_{xmax}$  for the open-ended interval, since people alive at the start of the interval will live about  $e_{xmax}$  further years.

As an example of an annuity calculation, suppose King Edward III bought annuities for all five of his surviving children when they were age 40. He made a number of Scottish widows on his expeditions into Scotland. Suppose the Scottish Widows Fund could collect 10 percent per year interest. Suppose the annuities were to pay his children a thousand pounds a years. Then what should a fair purchase price have been?

$$\frac{1000}{5} \left( \frac{58}{1.10^{10}} + \frac{1.5}{1.10^{21.5}} \right) = \frac{22361 + 193}{5} = \frac{22554}{5} = L4511$$

For insurance, you also buy a policy from a company, but the company promises to pay you an amount when you die, not while

you live. There are different kinds of policies. The simplest is term insurance, where you buy a new policy each year, get the benefit if you die that year, and “lose out” if you live.

With chance  ${}_nd_x/l_x = {}_nq_x$  you receive B.

With chance  $l_{x+n}/l_x = 1 - {}_nq_x$  you receive 0, because you don't die.

Here there is no significant lapse of time between buying the policy and receiving the benefit, if you do, so opportunities for the company to earn interest play a smaller role.

For fully funded insurance, you pay the whole price of your policy P now, at the beginning, just as you do with an annuity, but you receive the benefit B at your death. Now the time lapse and opportunity for interest may matter. If all members of your cohort buy policies at age x, the amount of benefits received by you per person from the company amounts to

$$B \frac{{}_nd_x}{l_x} + B \frac{{}_nd_{x+n}}{l_x} + B \frac{{}_nd_{x+2n}}{l_x} + B \frac{{}_nd_{x+3n}}{l_x} \dots$$

With the same average lengths of time between purchase of the policy and receipt of benefits as in the formulas for annuities, the amount per person the company must put in the bank at time of purchase to be able to finance the benefits it promises is

$$\frac{B}{l_x} \left[ \frac{{}_nd_x}{(1+i)^{n/2}} + \frac{{}_nd_{x+n}}{(1+i)^{n+n/2}} + \frac{{}_nd_{x+2n}}{(1+i)^{2n+n/2}} + \dots \right]$$

These formulas can be obtained from the formulas for annuities by replacing  ${}_nL_x$  with  ${}_nd_x$ .

### 3.9 Mortality of the 1400s and 2000s

The lifetable for Edward III's children is quite a good picture of mortality in England in 1400, despite all the reasons that one

might think of for which this small sample of unusual people should be different from the rest of the population. The two anomalies are the low level of infant deaths, underregistered here, as elsewhere, for we have omitted William of Hatfield, and the abbreviated life course after age 60, which would usually be longer. Note the heavy early female mortality, again a frequent feature of medieval life and death. In this family, deaths of mothers in childbearing are not the principal source of early adult women's mortality. They would have been an important but not an overwhelming cause of mortality for most families.

It is instructive to think about the changes between these days and our own. At the one end, infant and child mortality has dropped dramatically in this century, until the death of a baby is an unusual event. Today families who lose a child may feel singled out for tragedy, whereas a century ago, it was part of life for the majority of families and most children experienced a brother or sister's death.

The other great change is the survival of large numbers of people, especially women, into their eighties and beyond. Psalm 90, verse 10, is engrained in all our minds.

The days of our years are three-score years and ten;  
And if by reason of strength, they be fourscore years,  
Yet is their strength labour and sorrow.

From Biblical times down to the times your parents and grandparents remember, active healthy life beyond age 80 – four score – was rare. This is not to say that some people did not live to great age. The expectation of life at birth is no indicator of maximum ages attained by humans, for they are affected by infant mortality. In our lifetable, the value for  $e_0$  is 33, but John of Gaunt lives past 60. The Pharaoh Ramases II of Egypt, the opponent of Moses, is recorded as having lived to an age like 92. His mummy survives as confirming evidence. But these cases were extremely rare. Today almost all of us have family members or neighbors who live active lives into their late 80s and 90s, and active and independent



people who have celebrated their hundredth birthdays are to be found in many communities. The result is a change in attitudes about what it means to be old and what kind of life can be lived at extreme old age.

The members of the cohort in our example provide a striking illustration of these changing attitudes. John of Gaunt, the Duke of Lancaster, dies at age 58. For us, 58 no longer even seems like old age. But Shakespeare presents John of Gaunt as an ancient man. He is described, as he totters onto the stage, as “Old John of Gaunt, Time-Honoured Lancaster”. Then he gives his dying speech:

This royal throne of kings, this sceptred isle,  
This earth of majesty, this seat of Mars,  
This other Eden, demi-paradise,  
This fortress built by Nature for herself  
Against infection and the hand of war,  
This happy breed of men, this little world,  
This precious stone set in the silver sea.  
Which serves it in the office of a wall  
Or as a moat defensive to a house,  
Against the envy of less happier lands,  
This blessed plot, this earth, this realm, this England.

## Highlights of Chapter Three:

Hazard Rates

The Cohort Lifetable

${}_nq_x$  Conversions

Annuities

## Key Formulas

$$\begin{aligned} h_x &= (-1/n) \log(l_{x+n}/l_x) \\ (1 - {}_nq_x)^{1/n} &= 1 - {}_1q_x \\ l_{x+n} &= l_x(1 - {}_nq_x) \\ {}_nL_x &\approx (n/2)(l_x + l_{x+n}) \\ e_x &= ({}_nL_x + {}_nL_{x+n} + \dots)/l_x \end{aligned}$$

## Further Reading

Namboodiri and Suchindran (1987) gives a more detailed introduction to lifetables.

## Exercises for Chapter Three

1. A cohort of 1000 white male babies born in the U.S. in 1976 is forecast to have the numbers of deaths by age shown in the following table. (This is a real forecast.) Adopt a radix of 1,000 and calculate the hazard rate  $h$ , the probability of dying  ${}_nq_x$  and the survivors  $l_x$  for each age interval, stating carefully what  $x$ ,  $n$ , and  $x+n$  are in each case. Do these numbers seem realistic to you in the light of your experiences growing up in your community, and in terms of your expectations for the future? (For an African-American comparison, see the exercises for Chapter Eight.)
2. a) For Chinese boys in 1981,  ${}_1q_0 = .038528$  and  ${}_4q_1 = .015878$ . What is  ${}_5q_0$ ?

Ages :	0-1	1-5	5-18	18-22	22-30	30-65	65+
Deaths :	15	2	4	4	8	132	835

b) For Malaysian girls in 1985,  ${}_5q_0 = .020320$  and  ${}_5q_5 = .002795$ . What is  ${}_{10}q_0$ ?

c) For Swedish women in 1780,  ${}_5q_{40} = .062756$ . What is  ${}_1q_{40}$ ?

3. a) In Guatemala in 1985,  $l_0 = 1.0$ ,  $l_5 = .91301$ , and  $l_{10} = .90394$ . What is  ${}_1q_5$ ?

b) In Israel in 1954,  $l_{40} = .91264$ ;  $l_{41} = .91046$ ;  ${}_{10}q_{30} = .014853$ ; What is  ${}_2q_{39}$ ?

4. Many graduate students today have parents born around 1947. A forecast of  $l_x$  values for the U.S. white female cohort born in 1947 is shown in the following table. The radix is unity, and  $T_{80}$  is 4.530 years. Fill in all the entries in a lifetable (beyond age 50) for these people that you can.

Ages	50	55	60	65	80
Survivors	.8888	.8640	.8248	.7651	.4554

5. Suppose  $T_{85} = .125$ ,  $T_{80} = .525$ ,  $T_{75} = 1.15$ , and  $T_{70} = 2.0$ . Suppose also that  $l_{90} = 0$ . What are  ${}_5L_{75}$ ,  $l_{75}$ , and  $e_{75}$ ?

6. For the children of King Edward III (see the data in the exercises for Chapter Two), how much would the expectation of life at birth change if William of Hatfield were added with a death age of one month?

7. Can it ever happen that  ${}_nq_x$  is smaller than  $({}_nq_{x+2n})/2$ ?

8. Can it ever happen that  ${}_nL_x$  is smaller than  $({}_nL_{x+2n})/2$ ?

9. If  ${}_1q_0 = .0500$ ;  ${}_2q_1 = .0100$ ;  ${}_3q_2 = .0043$ ;  ${}_5q_3 = .0098$  and  ${}_2q_3 = .0040$ , then what is  ${}_5q_0$ ?

10. If  ${}_5d_0 = .2000$ ;  ${}_{10}d_5 = .1000$ ;  ${}_{15}d_{10} = .0500$ , and  $l_0 = 1$ , then what is  $l_{15}$ ?

11. A cohort of American men of size 86,306 at age 60 is depleted to 69,071 at age 70, to 38,701 by age 80, and to 8,061 by age 90, and to 121 by age 100, and to nothing by age 103. Assume prevailing long-term interest rates of 8%. How much should the Teachers' Annuity Association charge a sixty-year-old male for an annuity of 20,000 dollars a year to break even without profit?

12. A population maintains itself in a stationary state with the addition of 1000 births and 200 immigrants aged exactly 20 each year. Write down a formula for the size of this population. One or more life-table quantities may occur in your formula.

\*13. Here are some predicted survivorships for U.S. young people now in their early twenties:  $l_{70} = 0.767636$ ;  $l_{80} = 0.549265$ ;  $l_{90} = 0.253711$ ; and  $l_{100} = 0.037174$ . The Gompertz model predicts that the logarithms of hazard rates at advanced ages should be a linear function of age. By drawing a suitable graph, or otherwise, evaluate how well the predicted survivorships fit a Gompertz model.

\*14. Suppose in Question 4 that instead of knowing  $T_{80}$  you know that  $l_{85} = 0.3260$ ,  $l_{90} = 0.1821$ ,  $l_{100} = 0.0197$  and no one survives beyond 105. Calculate  ${}_nq_x$ ,  ${}_nL_x$  and  $e_x$  for  $x$  beyond 50.

\*15. By experimenting with various cases, find out whether it is always, sometimes, or never true that the hazard rate per year in a one-year age interval is greater than the probability of dying in that one-year interval.

\*16. From the reconstitution of the English village of Colyton for 1750 to 1799 from Wrigley "Births and Baptisms", *Population Studies* 31, 281-312, one learns that the probability of infants dying within 31 days of birth appears to be .046. The cumulative probability of dying within 61 days of birth is .054, within 91 days of birth it is .062, and within 181 days of birth it is .071. Estimate the endogenous mortality rate by the method of Jean Bourgeois-Pichat.

17. For Kuwaiti women in 1980, estimates show  ${}_5q_{65} = 0.154$ ,  ${}_5q_{70} = 0.183$ ,  ${}_5q_{75} = 0.305$ , and  ${}_5q_{80} = 0.409$ . For men, the estimates are  ${}_5q_{65} = 0.188$ ,  ${}_5q_{70} = 0.304$ ,  ${}_5q_{75} = 0.402$ , and  ${}_5q_{80} = 0.458$ . Consider a cohort of 53 percent women and 47 percent men at age 65. Compute values of  ${}_5m_x$  for a combined-sex lifetable from age 65 to 85. Plot  $\log({}_5m_x)$  versus  $x$ . How near do the points come to a straight line?

\*18. Use calculus to find the slope of the curve of cohort deaths as a function of age under the Gompertz model. In this way, verify the formula for the modal age at death and find an expression for the value of the hazard rate at the modal age at death.



# Chapter 4

## Cohort Fertility

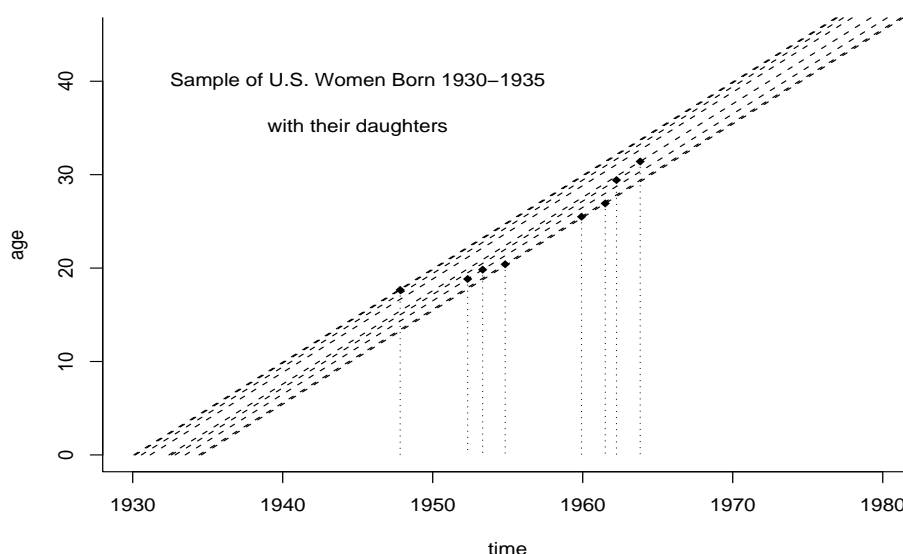
### 4.1 Generational Renewal

In our studies so far we have dwelt mainly on the beginning and ending of lifelines on a Lexis diagram, on the start of life when one is born into a cohort and on the end of life when one takes one's exit by dying. Indeed, on our Lexis diagrams we have been marking nothing at all along the lifeline, as if nothing happened in between. It is reminiscent of a modern poem in which the poet thinks of her father and the inscription on his tombstone: a date of birth, a date of death, and nothing in between but a dash.

In between, among other things, comes childbearing. We focus here on a cohort of women – we shall consider the men's role later – and we focus at the beginning on their daughters. Figure 4.1 is a Lexis diagram for a sample of ten women drawn from the 5,994,000 women born into the five-year birth cohort born between 1930 and 1935 in the U.S.A. One of these women died four months after birth. You can see the tiny lifeline on the middle left. Another woman died at the age of 30. You can see the lifeline end. The remaining eight women survived through to the end of the ages of childbearing. Two of them had two daughters each, four of them had a single daughter, and two of them had no daughters. Each of these births is a droplet along the mother's lifeline. On this

diagram, a dotted line shows the droplet dropping to the axis, to the point where each daughter begins her own lifeline on the Lexis diagram. Visually, the lifelines of the cohort of mothers resemble the left edge of a rainbow, and the cohort's daughters are like a gentle rain falling on the earth beneath.

Figure 4.1: Cohort Fertility on a Lexis Diagram



In our picture we are seeing a process of generational renewal. If we call the cohort of women the “first generation”, then their daughters are the second generation. (The cohort’s daughters’ daughters are the third generation, and so on.) The ratio of the total number of daughters borne by cohort members to the initial number of women in the cohort is a generational replacement ratio, the ratio of the size of the second generation to the first. We call this ratio the “Net Reproduction Ratio” or, less precisely but more commonly, the “Net Reproduction Rate” or “NRR”.

Published data more commonly record numbers of babies rather than numbers of daughters, so we need to convert from babies of both sexes to daughters when we calculate an NRR. The conversion factor, as previously mentioned, is the “fraction female at birth” or  $f_{fab}$ . Take as an example the cohort of U.S. women of



all races born in 1934. These were the mothers who generated the peak of the baby boom. The 1,054,933 women in the 1934 cohort had a total of 3,231,688 babies, of whom approximately 0.4877 or 1,576,094 were daughters. The NRR is therefore given by

$$NRR = \frac{3,231,638 * .4877}{1,054,933} = 1.494$$

In a closed population, if cohort after cohort each has a Net Reproduction Ratio greater than 1, then we expect each generation to be larger than the next, and so we expect a growing population. If cohorts have NRR values equal to one over the course of many generations, then we expect a stationary population, and if cohorts all have NRR values less than 1, then we expect a contracting population.

When aggregate statistics for cohorts and their babies are available, the calculation of the NRR is elementary. Table 4.1 shows some birth cohorts of U.S. women, with the number of babies borne by the cohort, the fraction female at birth, and the number of women in the cohort.<sup>1</sup> shows We multiply the first two columns and divide by the third to obtain the NRR.

Table 4.1: Generation sizes and the NRR

Cohort	Babies	$f_{fab}$	Cohort Size	NRR
1910		.4871		.959
1922	3,579,318	.4866	1,408,000	1.237
1934	3,231,638	.4877	1,054,933	1.494
1947	3,788,342	.4871	1,884,884	.979

Frequently, the fraction female at birth is not given in published data. In such cases it is necessary to have a default value to use. This fraction is generally a little less than one-half, since more boys than girls are conceived and slightly more boys than girls are born in most populations. The default value we adopt is

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<sup>1</sup>To construct this table, several approximations have been employed.

$f_{fab} = 0.4886$ . When the true value is known, we always use it, but when it is unknown, we use 0.4886.

The advantage of using a special number like 0.4886 for our default rather than a common number like 0.5000 is ease of recognition. The number 0.4886 occurs nowhere in formulas except as  $f_{fab}$ , whereas 0.5000 may occur in formulas for many other reasons. This fraction 0.4886 was chosen because it was the U.S. fraction female at birth during the years when many students now studying demography were born. Demographers more commonly quote “sex ratios”, rather than fractions female. The sex ratio at birth implied by our default value is  $.5115/.4886 = 104.67$ .

the ratio of boy

A word about names for the NRR is in order. The name “Net Reproduction Rate” is more common than “Net Reproduction Ratio”, but to call the NRR a “rate” is a misnomer. A rate in demography is a value per unit of time, per month, per year, per decade. The NRR is a pure ratio. It is not expressed in any units of time. Eugene Grebenik, long the editor of the demographic journal *Population Studies* insisted on this point, and rightly so. We follow his practice in calling the NRR the “Net Reproduction Ratio.” Another name in occasional use is “Net Reproductive Ratio”.

The NRR is like an input-output ratio. The potential future mothers starting life in a cohort are the “input”. Their baby daughters in the next generation are the output. The essential feature of an input-output ratio is that input must be measured in the same units as output. We need to compare apples to apples or oranges to oranges, not apples to oranges. Since we are measuring input as a count of females, we need to measure output as a count of females. We have women as input, so we need daughters as output, not babies of both sexes. Furthermore, we have newborn women as input – we count the size of the cohort at birth, not at some later age. Therefore, we need to count newborn daughters as output, not daughters at some later age. We need to be comparing like to like. As an alternative, we could take women at the start of childbearing as input. But then we would need to take daughters surviving to the start of childbearing as output. Mortality comes

into the NRR, but only once, through the mortality of the cohort of potential mothers.

Some members of a cohort die before beginning or completing childbearing. Their deaths reduce the eventual total number of daughters and so affect the NRR. The NRR is a measure of reproduction net of the effects of mortality, that is, remaining after mortality has been taken into account. Mortality diminishes a cohort's production of offspring just as taxes diminish a person's spendable income. The Net Reproduction Ratio is like a person's net income after taxes. There is also a Gross Reproduction Ratio. It is like gross, pre-tax income and excludes losses due to mortality.

The word “net” derives from a Latin root meaning “shining”. The Net Reproduction Ratio is the shining measure of demography, one of the most important quantities demographers study. We spell out all the main aspects of the NRR in the following definition:

The **Net Reproduction Ratio** is the expected number of daughters per newborn prospective mother who may or may not survive to and through childbearing.

## 4.2 Cohort Parity and the NRR

There are a number of alternative ways of calculating an NRR, depending on the kinds of data ready to hand. Often cohort childbearing is reported not in terms of a lifetime total number of babies but in terms of a distribution of children ever born. The number of live births that a woman has is known as her *parity*. This word is spelled the same as the more familiar word “parity” meaning “equality” which occurs in phrases like “parity of pay”. But the word “parity” in demography is derived from a different root, the same root as for “parent”, for “parturition” which means childbirth, or for “post partum” which means “after childbirth”. A woman is “nulliparous” when she has never borne children.

We write  $w(j)$  for the proportion of women in a cohort who have parity  $j$ . Unless otherwise noted, parity distributions are usually computed after all members of the cohort have completed childbearing. Thus a proportion  $w(0)$  of cohort members have borne no children, while  $w(1)$  of them have borne one child,  $w(2)$  have borne two children, and so forth. The sum of all the  $w(j)$  is 1. We count children ever born, not children surviving.

If we are given the distribution of women by parity for a cohort, we can find the NRR by the following formula :

$$NRR = (0 * w(0) + 1 * w(1) + 2 * w(2) + 3 * w(3) \dots)(f_{fab})$$

Table 4.2 shows the distribution of parity for the cohort of U.S. women born in 1934. As we have seen, this cohort began with 1,054,933 women. A fraction 0.4877 of their children were daughters.

Table 4.2: Parity distribution for U.S. women born in 1934

Parity j	0	1	2	3	4
$w(j)$	.079	.097	.233	.239	.166
Parity j	5	6	7	8	9+
$w(j)$	.090	.047	.029	.012	.008

We assume that most of the 9+ category consists of parity 9 women and calculate the NRR to be

$$\begin{aligned} NRR &= (0 * 0.079 + 1 * 0.097 + 2 * 0.233 \dots + 9 * 0.008)(0.4877) \\ &= 1.4860 \end{aligned}$$

In terms of  $w(j)$  we can define “parity progression ratios”  $a(j)$ . The zero-parity progression ratio  $a(0)$  is the proportion of those with at least zero children (all cohort members) who subsequently had at least one child. Let  $T(j)$  be the proportion with at least  $j$  children ever born, that is,  $j$  children or more. Then  $a(j) = \frac{T(j+1)}{T(j)}$ .

## 4.3 Age-Specific Fertility

The approaches to the Net Reproduction Ratio explained in the previous sections, in terms of a generational replacement ratio and in terms of cohort parity, both help to bring out the concept clearly. In practice, however, by far the most usual method for calculating the NRR makes use of age-specific fertility rates..

An age-specific fertility rate is like a crude birth rate insofar as it has babies in the numerator and person-years in the denominator. But it is unlike a crude birth rate, insofar as the babies are only the babies born to women in a particular age range and the person-years are only person-years lived by the women within that age range. Note that there are two restrictions on the person-years. They have to be lived within the particular range of ages, and they have to be lived by women, not (as with the crude birth rate) by men and women.

We take an age interval from  $x$  to  $x + n$ . For a cohort, we write  ${}_nf_x$  for the ratio formed by taking babies of both sexes born to women in the cohort while the women are between ages  $x$  and  $x + n$  and dividing by the cohort person-years lived by women in the cohort between those ages. (As we shall see in Chapter Six, for a period age-specific fertility rate we shall use a capital  $F$  and write  ${}_nF_x$  for the ratio of babies born to women aged  $x$  to  $x + n$  in the period to period-person-years lived by women between those ages.) The abbreviation “ASFR” stands for age-specific fertility rate.

The **cohort age-specific fertility rate** (ASFR)  ${}_nf_x$  is the number of children borne by women in the cohort between ages  $x$  and  $x + n$  per person-year lived by women in the cohort between ages  $x$  and  $x + n$ .

Age-specific fertility rates are rates, not probabilities. They have units of 1/time. Babies are persons, so the babies in the

numerator cancel the persons-part of the person-years in the denominator, leaving 1/years. Doubling the width of the age interval would increase both the numerator and the denominator, and would not drastically change the rate. An age-specific fertility rate  ${}_nf_x$  is the counterpart for fertility of the age-specific mortality rate  ${}_nm_x$  in the life table, which has the same denominator but a numerator with deaths in place of births.

It is usual to concentrate on age-specific fertility rates for women, since women's age is a more obvious determinant of fertility. We can if we wish count births by age of father and divide by person-years-lived by men in the age interval. Such male ASFRs are rarely used, since ages of fatherhood are less narrowly restricted biologically and socially and since data on fathers' ages are rarely tabulated.

We must bear in mind that "female" ASFRs pertain to female parents but to both male and female babies. Sons and daughters enter into the numerator, person years for mothers into the denominator. If the numerator is further restricted to daughters, the resulting rate should be labeled as a "daughters-only" ASFR, for instance by  ${}_nf_x^{daughters}$  or by multiplying  ${}_nf_x$  by the fraction female at birth. Mathematical demographers often work with daughters-only rates and sometimes this restriction is mentioned in the text but omitted from the notation.

## 4.4 ASFRs and the NRR

A Net Reproduction Ratio is most often calculated from a table of age-specific fertility rates. As we shall see in Chapter Six, this way of calculating the NRR is the easiest one to generalize from cohort-based to period-based rates.

Since  ${}_nf_x$  has babies divided by person-years, to recover a count of babies we need to multiply back by person-years. These person-years are the person-years lived by the female members of the cohort, and can borrow them from the  ${}_nL_x$  column of the female

cohort lifetable. After we form the products  ${}_nF_x {}_nL_x$ , we need to add them up over all ages of childbearing. We also need to convert from babies to daughters, and divide by the initial cohort size. These steps give the following age-based formula for the NRR:

$$NRR = \sum {}_nf_x {}_nL_x f_{fab}/l_0$$

The notation  $\sum$  means “add up over all the age intervals with different starting ages  $x$ ”. The symbol  $\sum$  is the capital Greek letter *sigma* and stands for “sum”. Here we do not need to list the ages  $x$  that go into the sum, because ages outside the childbearing span have  ${}_nf_x$  equal to zero and add zero to the sum.

For an example of an age-specific calculation, we take data for a sample of 1000 women from the same cohort of U.S. women born in 1934 treated in previous sections. We break down the 3080 babies they bore, of whom 1502 were daughters, by the age of the mother at birth. Data are shown in Table 4.3.

Table 4.3: A Cohort NRR from Age-Specific Rates

$x$	${}_5f_x$	${}_5L_x$	Babies
0	0.000000	4770	0
5	0.000000	4726	0
10	0.000000	4712	0
15	0.081524	4698	383
20	0.239692	4681	1122
25	0.197984	4662	923
30	0.103947	4637	482
35	0.031494	4604	145
40	0.004604	4561	21
45	0.000888	4503	4
50	0.000000	4421	0

We multiply the  ${}_nf_x$  column by the  ${}_nL_x$  column to obtain the column for babies (rounding for the sake of better agreement with

our earlier calculations). We sum the column for babies, obtaining 3080, multiply by the actual fraction female at birth for this sample, 0.487662, and divide by the radix to obtain an NRR of  $3080 * 0.487662 / 1000$  or 1.502.

Two other summary measures employed throughout demography can be calculated from age-specific fertility rates. They are the Total Fertility Rate or TFR and the Gross Reproduction Ratio or GRR. Both are usually calculated from period rather than cohort data and they will be discussed again in Chapter Six. However, the concepts of the TFR and GRR are cohort concepts, just like the concept of the NRR. Both the TFR and the GRR are measures of fertility rather than generational renewal. Both of them remove the effects of mortality. They tell us how many babies or daughters a cohort would produce in the absence of mortality. In the absence of mortality, each member of a cohort would live  $n$  person-years in the interval from  $x$  to  $x + n$ . We replace  ${}_nL_x/l_0$  by  $n$  in our formulas. When we keep babies of both sexes, we obtain the TFR. When we restrict to daughters by multiplying by the fraction female at birth, we obtain the GRR.

$$TFR = \sum ({}_nf_x)(n)$$

$$GRR = \sum ({}_nf_x)(n)(f_{fab})$$

If all the age intervals in a data table have the same width  $n$ , we can add up the  ${}_nf_x$  column and multiply by  $n$  at the end to obtain the TFR. Then we multiply by  $f_{fab}$  to obtain the GRR. The TFR implied by this table is close to 3.3.

## Highlights of Chapter Four:

The Net Reproduction Ratio

Parity

Age-specific Fertility Rates.



### Key Formulas

$$NRR = (0 * w(0) + 1 * w(1) + 2 * w(2) + 3 * w(3) \dots)(f_{fab})$$

$$NRR = \sum n f_x \quad n L_x f_{fab} / l_0$$

$$TFR = \sum (n f_x)(n)$$

$$GRR = \sum (n f_x)(n)(f_{fab})$$

## Exercises for Chapter Four

1. When we compare two countries at the same point in time, does the country with the higher value of  $e_0$  always have the lower Crude Death Rate? Please explain your reasoning.
2. Find the TFR and GRR implied by the data in Table 4.3 to six decimal places. How close is the GRR to the NRR?
3. Table 4.4 shows estimates of cohort age-specific fertility rates and person-years lived for the cohort of Swedish women born in 1800. The radix is 1000. Find the cohort NRR, TFR, and GRR. How close is the GRR to the NRR?

Table 4.4: Data for the 1800 Cohort of Swedish Women

$x$	${}_5f_x$	${}_5L_x$
15	0.0122	3134
20	0.1038	3036
25	0.2211	2930
30	0.2408	2808
35	0.2131	2663
40	0.1136	2509
45	0.0182	2351

4. Some 34 baby girls were born in a single week in July at the Lucille Packard Wing of Stanford Hospital. Fifty years later, a

survey is sent out asking them, among other questions, how many daughters they have borne. It turns out that one of them died as an infant. All 33 of the others respond to the survey. The numbers of daughters that they list are shown below. Calculate the Net Reproduction Ratio for this July cohort.

Data :

1,0,1,1,2,1,1,0,1,3,1,2,2,1,4,1,0,  
1,0,2,0,1,2,0,0,1, 1,1,0,3,2,3,0.

# Chapter 5

## Population Projection

### 5.1 Transition Matrices

Leslie Matrices are tables used for population *projection*. Official presentations of projections are often filled with disclaimers cautioning the reader that projections are not *predictions*. They do not tell us what the world *will* be like but only what the world *would* be like if a particular set of stated assumptions about future vital rates turned out to be true. The assumptions may or may not bear any relation to what actually happens.

Such disclaimers, however, are disingenuous. Projection is not just a game with computers and pieces of paper. We do projections for a purpose, and that purpose is to foresee, as best we can, the future state of the population. The choice of credible assumptions about vital rates is as much a part of the art and science of projection as are the formulas we use to implement the calculations. Current thinking about plausible assumptions for predicting populations around the world is presented in a recent book called *Beyond Six Billion* edited by Bongaarts and Bulatao (2000).

Over recent history, the record of demographers at guessing future vital rates for use in projection has not been good. The demographic community as a whole failed to predict the baby boom

of the 1950s and 1960s, and then it failed to predict the baby lull of the 1970s and 1980s. It largely failed to predict the continuing trend toward lower mortality in industrialized countries. These failures show that we do not yet understand the mechanisms that drive demographic change. We need deeper theories with better predictive power.

Although demographers have a poor record of prediction, their record is probably better than the records of economists, weather forecasters, and those who bet on football games. Although the choice of assumptions about the future course of fertility, mortality, marriage, divorce, and immigration is fraught with problems, the methods for using those assumptions to calculate future population sizes and age distributions are very well-developed and satisfactory. In this chapter we study such methods of calculation. Much of the mathematics of projection was developed treating time and age as continuous variables, but projections on computers generally are done with populations split into age groups and with discrete steps of time. Here we study the discrete versions of projection, which use matrices and project populations one step at a time.

Sophisticated projections treat a population classified by many characteristics including race, ethnicity, marital status, income, education, and locality as well as sex and age. But the basic ideas are well-illustrated by simpler projections which focus on a single sex and on all races and ethnicities together and which subdivide the population only by age. We shall concentrate here on this simple case. Matrices used to project the size and age distribution of a population forward through time. are called “Leslie matrices”. It was P. H. Leslie who published a description of their use in 1945. The same theory was developed a few years earlier by H. Bernardelli and by E. G. Lewis, but their work did not become widely known until Leslie published his. Leslie matrices are a special case of “transition matrices”. When demographers project a distribution of marital status, parity, educational level, or other variables into the future, they use general transition matrices. When they project age structure, they use the special kind of transition matrices that Leslie defined.

The kind of calculations we shall be doing are closely related to the subject of Markov chains in probability theory. Generally speaking, however, we shall be projecting expected numbers of people rather than probabilities, and that makes the demographic theory slightly different from the theory of Markov chains.

We begin by defining a transition matrix.

A transition matrix is a table with rows and columns showing the expected number of people who  
END up in the state with the label on the ROW  
per person  
at the START in the state with the label on the COLUMN.

A Leslie matrix is a special case of a transition matrix in which the states correspond to age groups. With Leslie matrices, the processes of transition are surviving and giving birth. The Leslie matrix describes a “one-step transition”. We project the population forward one step at a time. The time between START and END, the projection step, should be equal to the width  $n$  of all the age groups.

The fact that the step size has to equal the age group width is an important point. For projection, we generally pick one sex, usually females, and we divide the female population into age groups of width  $n$ . This width may be one year, five years, fifteen years, or some other convenient number. If we are using age groups that are one year wide, then we have to project forward one year at a time, so to project ten years into the future requires ten projection steps. If we are using five-year age groups, then we have to project forward five years at a time, so to project ten years into the future would require only two projection steps.

For our application, we shall assume a closed population. This assumption can be relaxed in more advanced applications. People enter the population only by being born to members already in the population, and people leave it only by dying. Our projection

treats birth as a possible transition along with survival. That means that the people who END up in some state may not be the same people who START in any one of the states. They may, instead, be the babies of the people who start in the various states. We are concerned with the expected numbers in the state for the row per person in the state for the column, however the people are channeled there.

## 5.2 Structural Zeros

The logic of the transition process is built into a transition matrix through the pattern of zeros that occur. Some age groups owe no part of their numbers at the end of the step to certain other age groups. Suppose we have  $n = 5$ . We have five-year-wide age groups and we are projecting forward five years in one step. No teenagers owe their numbers to 40-year-olds five years before. Thus the value of the Leslie matrix element in the row for 15 to 20-year-olds and the column for 40-year-olds has to be zero. This is a “structural zero”. We know it is zero because of the logic of the processes of aging and childbirth. We do not have to consult empirical data to see whether the expected number comes out to be zero. We can tell beforehand.

Most of the elements of a Leslie matrix are structural zeros, and we can fill them in immediately. Continue assuming five-year-wide age groups, and consider the third row of a Leslie matrix, giving expected numbers of 10 to 15-year-olds. We expect no 10 to 15-year olds at the end of five years per person aged 0 to 5 at the start, because no 0 to 5 year old can be over age 10 in only five years. So the entry in the first column in this row is a structural zero. The next entry need not be zero, however, because we do expect some 10 to 15-year-olds at the end of five years per person aged 5 to 10 at the start. Any such person who survives five years will make it into the state labeled by the row.

In this example, the entry in the third row in the third column is another structural zero. No one 10 to 15 at the start can stay

10 to 15 five years later. The next entry is also zero. No one can get younger. So, in the third row, only the entry in the second column is not a structural zero.

The same logic tells us that, below the first row, all elements are structural zeros except the sub-diagonal, the set of entries one below the main diagonal. No one can jump an age group, stay in the same age group, or get younger. They can only move (if they survive) into the next age group. We see how important it is for the projection step to be the same as the age-group width. If it were not, we would not have this clear pattern.

So far, we have excluded the first row. What about the first row, for people who end up aged 0 to 5 at the end of five years? No one can survive into this row. But these elements are not structural zeros, because there can be babies born during the projection step who are found in this age group at the end of the step. The number of babies depends on the number of potential parents (mothers, in a females-only projection) in the various age groups at the start. So the entries in the first row are not structural zeros.

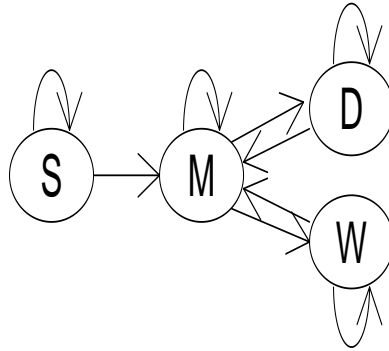
Whether the upper-left element of a Leslie matrix equals zero or not depends on the age-group width. With five-year-wide age groups, we do not expect there to be any babies in five years to people 0 to 5 at the start. But if  $n$  were 15, then we would expect there to be babies in the next fifteen years to people aged 0 to 15 at the start. How wide the age group has to be before we see a non-zero element in the upper-left is an empirical question which depends on the youngest ages of fertility. Therefore the upper left element, although it is often zero in practice, is not regarded as a structural zero.

When we have filled in all the structural zeros of the Leslie matrix, it looks like the following diagram.

	0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
0 to 5	<i>kids</i>	<i>kids</i>	<i>kids</i>	<i>kids</i>	<i>kids</i>
5 to 10	<i>survivors</i>	0	0	0	0
10 to 15	0	<i>survivors</i>	0	0	0
15 to 20	0	0	<i>survivors</i>	0	0
20 to 25	0	0	0	<i>survivors</i>	0

Another way of representing the information about structural zeros is to make a diagram of permitted transitions. We list all the states, and draw an arrow from one state to another one if there is a non-zero element for that column-row pair, that is, if any people can show up in the “receiver” state at the point of the arrow per person in the “sender” state at the shaft of the arrow.

Figure 5.1: Permitted Transitions among Marital Statuses



Such arrow diagrams, like programmer flow charts, are particularly useful for more general projections, in which our transitions are not between age groups but between other states, for example, between marital status states. Consider a model with four states, the states of being single (that is, never married), married, widowed, and divorced. Suppose the projection step is too short for us to expect anyone to get both married and divorced or divorced and remarried within a single projection step. In other words, suppose the projection step is short enough that multiple transitions are not numerically significant. Then the arrow diagram for permitted transitions takes the form shown in Figure 5.1.



The structural zeros in the transition matrix corresponding to Figure 5.1 go into slots marked “0” in the following matrix:

$$\begin{array}{c}
 \begin{array}{c} \textit{Single} \\ \textit{Married} \\ \textit{Widowed} \\ \textit{Divorced} \end{array}
 \begin{pmatrix}
 \begin{array}{c} \textit{Single} \end{array} & \begin{array}{c} \textit{Married} \end{array} & \begin{array}{c} \textit{Widowed} \end{array} & \begin{array}{c} \textit{Divorced} \end{array} \\
 \begin{array}{c} x \\ x \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ x \\ x \\ x \end{array} & \begin{array}{c} 0 \\ x \\ x \\ 0 \end{array} & \begin{array}{c} 0 \\ x \\ 0 \\ x \end{array}
 \end{pmatrix}
 \end{array}$$

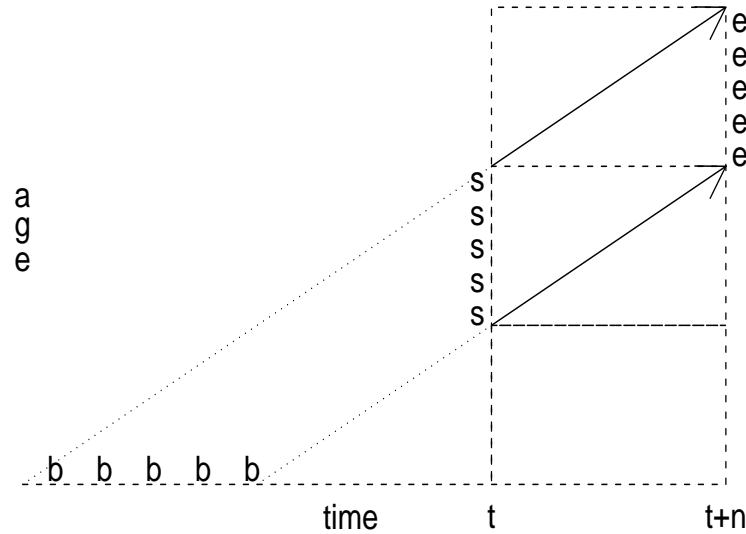
### 5.3 The Leslie Matrix Subdiagonal

We generally denote a matrix by a single capital letter like  $A$ . We pick out an element in a particular row and column with subscripts. The first subscript is for the row and the second subscript is for the column. Thus  $A_{3,2}$  is the element in the third row and second column, for survivors from the second age group to the third age group. By contrast,  $A_{2,3}$  is the element in the second row and third column. This is a structural zero, since no member of the third age group can grow younger and turn up in the second age group at the end of a projection step. This notation for elements of matrices is universal. For our particular matrices it may seem confusing, because the subscript for the destination age group comes first, and the subscript for the origin age group second, in the reverse of chronological order. The notation can be illustrated by writing

$$A_{to,from} \text{ or } A_{row,column}$$

We now want to develop a formula for the elements along the subdiagonal of the Leslie matrix, which represent transitions of survival. We continue to use five-year-wide age groups, and consider  $A_{3,2}$ , the expected number of people aged 10 to 15 at the end, per person aged 5 to 10 at the start. The age group of 5 to 10-year-olds at the start, at time  $t$ , is composed of cohorts born

Figure 5.2: Contributions to the Leslie Matrix Subdiagonal



between times  $t - 10$  and  $t - 5$ . We follow the experience of some of these cohorts on the Lexis diagram in Figure 5.2.

On the diagram, the “b” symbols represent cohorts at birth, the “s” symbols represent them at the start of the projection step, and the “e” symbols represent them at the end. The basic assumption which we make is that we may ignore changes in sizes of cohorts at birth inside each five-year period, pretending that all changes in initial cohort sizes occur in jumps between periods. We also assume that the same lifetable applies to all the one-year cohorts in this five-year group of cohorts.

Let us trace the experience of some of the cohorts on the diagram. The earliest cohort, today’s 10-year-olds, had some size  $l_0$  at birth. Now, at the start of the projection step, it has  $l_{10}$  members left. The latest cohort contributing to the age group, today’s 5-year-olds, had, by assumption, the same size  $l_0$  at birth, and now has  $l_5$  members left. The whole five-year age group is about five times as large as the average size of its youngest and

oldest cohorts, and now amounts to

$$(5/2)(l_5 + l_{10}).$$

Five years from now, at the end of the projection step, as the cohorts reach the points marked with “e” on Figure 5.2, the oldest cohort will have  $l_{15}$  members left and the youngest cohort will have  $l_{10}$  members left. The whole age group will amount to

$$(5/2)(l_{10} + l_{15}).$$

We recognize these sizes as our usual approximations for  ${}_5L_5$  and  ${}_5L_{10}$  from the person-years-lived column of the lifetable. Had we split our age group into arbitrarily many small cohorts and added them all up, we would have obtained the corresponding  ${}_nL_x$  values exactly.

The subdiagonal element of the Leslie Matrix is the ratio of the 10 to 15-year olds at the end to the 5 to 10-year-olds at the start, that is

$$A_{3,2} = \frac{{}_5L_{10}}{{}_5L_5}$$

In general, with the bottom age of the age group  $x$  expressed in terms of the column number  $j$  by  $x = (j - 1)n$ , we have

$$A_{j+1,j} = \frac{{}_nL_{x+n}}{{}_nL_x}$$

Each subdiagonal element of a Leslie Matrix is a ratio of big-L values. The age label on the *numerator* comes from the *row*. The age label on the *denominator* comes from the *column*. The Leslie Matrix with survivorships filled in looks like the following:

	0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
0 to 5	<i>kids</i>	<i>kids</i>	<i>kids</i>	<i>kids</i>	<i>kids</i>
5 to 10	$\frac{{}_5L_5}{{}_5L_0}$	0	0	0	0
10 to 15	0	$\frac{{}_5L_{10}}{{}_5L_5}$	0	0	0
15 to 20	0	0	$\frac{{}_5L_{15}}{{}_5L_{10}}$	0	0
20 to 25	0	0	0	$\frac{{}_5L_{20}}{{}_5L_{15}}$	0

## 5.4 \*The Leslie Matrix First Row

Besides the subdiagonal, the only other elements in a Leslie matrix which are not structural zeros are found in the first row. These are the elements that take account of population renewal, as babies are born to parents and survive to be counted at the end of the projection step. The formulas for the elements in the first row are more complicated than those for the subdiagonal, and we develop them one step at a time.

The formulas we develop are for projecting the female population, and daughters rather than babies of both sexes will be at stake in the first row. Whenever we do projections, we must have the same kinds of people coming out as going in. We cannot have babies of both sexes coming out without having parents of both sexes going in, and that would require a very odd kind of fertility rates, fertility rates in which numbers of potential mothers and fathers are both mixed together in the denominators. For this reason, we do one-sex projections. Since ages of motherhood are more regular than ages of fatherhood, we generally use the female population for projection. The same methods let us do one-sex projections for the male population, with fertility rates for sons and fathers. But putting males and females together in a consistent way is a difficult and still largely unsolved problem in population projection.

When we are doing a projection of the female population, we must remember that we need fertility rates for *daughters only*. Since most age-specific fertility rates are for babies of both sexes, we need to multiply by the fraction female at birth  $f_{fab}$  or by our default fraction 0.4886.

We need a formula for the expected number of kids aged 0 to  $n$  at the end of the projection step per person aged  $x$  to  $x+n$  at the start. Let us write  $j(x)$  for the column of the Leslie Matrix corresponding to people aged  $x$  to  $x+n$  at the start. Then  $j(x)$  satisfies

$$x = (j(x) - 1)(n)$$

Our formula will turn out to be

$$A_{1,j(x)} = \frac{nL_0}{2l_0} \left( {}_nF_x + {}_nF_{x+n} \frac{{}_nL_{x+n}}{{}_nL_x} \right) f_{fab}$$

We build up this formula by introducing the complications one at a time.

First, the entry in column  $j(x)$  gives daughters who have entered the population by being born to mothers aged  $x$  to  $x+n$  at the start during the  $n$  years covered by the projection step. How many daughters per potential mother should there be? A first guess would be to multiply the daughters-only age-specific fertility rate for this age group,  $({}_nF_x)(f_{fab})$  by the years at risk during the step, namely  $n$ . (In the remaining equations in this section, the fertility rates are all daughters-only fertility rates, but we omit the superscripts in some equations for simplicity.) We conclude that the simplest formula we might have for the  $j$ -th element of the first row is

$$(n)({}_nF_x)(f_{fab})$$

Second, however, we need to recognize that not all births produce kids who survive to the end of the projection interval. The first age group counts kids aged zero to  $n$  at the end of the step, not newborns. So we need a formula for the proportion of babies born during the  $n$  years who survive to be counted at the end. The babies born early in the period have to survive to be nearly  $n$  years old. Those born just before the end of the period do not have to survive long at all. We need a formula which averages these survivorships. This problem is very much like the problem of finding a formula for the subdiagonal elements, except that in this case we need survival from birth, at the bottom of a Lexis diagram, up to the age group from age zero to age  $n$  at the interval's end. For the subdiagonal, we were dealing with lifelines in a parallelogram. For the first row, we are dealing with lifelines within the right triangle whose base covers the interval from  $t$  to  $t+n$  on the horizontal axis and whose right-hand side stretches up from age 0 to age  $n$  above time  $t+n$ .

As before, we ignore changes in initial cohort size within our interval, and as before we illustrate the process with an example

for  $n = 5$ . Out of any  $l_0$  girls born at the start of the interval,  $l_5$  survive to age 5 and reach the end of the interval. Out of any  $l_0$  girls born at the end of the interval, all  $l_0$  reach the end since they are already at the end of the interval when they are born. Girls born in between must survive to ages in between. On average, we have  $(5/2)(l_0 + l_5)$  reaching the end for  $(5/2)(l_0 + l_0)$  born during the interval. We recognize the expressions for  ${}_5L_0$  and  $5l_0$ . For general  $n$ , the ratio is  ${}_nL_0/(nl_0)$  survivors per baby. Multiplying by our previous formula for the number of babies per mother, we have an improved candidate for our formula, namely

$$\frac{{}_nL_0}{(n)(l_0)}(n)({}_nF_x^{dau})$$

We see that the two factors of  $n$  cancel each other out. Although there will be no such factors in our final formula, it is valuable to notice that they do come into the reasoning. The formula is much harder to understand if we do not remember that there were factors of  $n$  originally involved.

Third, we introduce the further complication that results from the aging of our population of potential mothers during the projection interval. Women aged 20 to 25 actually spend approximately half of their time over the next 5 years in the 20 to 25 age group and half in the 25 to 30 age group. So they experience two different fertility rates. The fertility component ought to look more like  $(1/2)({}_5F_x + {}_5F_{x+5})$  when we have five-year-wide age groups, and similarly for  $n$ -year-wide groups. Thus our third candidate formula for the  $j$ -th element of the first row is

$$\frac{{}_nL_0}{2nl_0}(n)({}_nF_x + {}_nF_{x+n})f_{fab}$$

Fourth, we must acknowledge that not all women survive from one age group to the next. We need to adjust the fertility rate of the next older age group for the probability of surviving to the next older age group, which we already have from the subdiagonal element formula to be

$$\frac{{}_nL_{x+n}}{{}_nL_x}$$

Now we have our complete official formula for the Leslie Matrix element in the first row and  $j(x)$ -th column :

$$A_{1,j(x)} = \frac{nL_0}{2l_0} \left( {}_nF_x + {}_nF_{x+n} \frac{{}_nL_{x+n}}{{}_nL_x} \right) f_{fab}$$

There is one further subtlety which we mention in passing. We have derived our formula leaving open the choice of the interval width  $n$ . Usually,  $n$  will be one year or five years, or some other small number of years, and age-specific fertility for the youngest age group from zero to  $n$  will be zero. We say that the youngest age group is part of the “pre-procreative span”. All higher mammals have substantial pre-procreative spans. Newborns do not immediately start producing infants. This is not, however, true of non-human populations like populations of neutrons inside atomic bombs. It is a special characteristic of the kind of species to which we belong.

Occasionally we want Leslie Matrices with wider intervals, ten, fifteen, or even twenty years wide. Such intervals are longer than the human pre-procreative span. The fertility for the youngest age group will then not equal zero, and it is possible for children born during the projection step to grow up and have children during the projection step. Our formulas so far make no allowance for such grandchildren. A full correction for the effect of grandchildren is complicated. But a rough-and-ready correction is obtained by taking  $2{}_nF_0$  instead of  ${}_nF_0$  in the formula for the upper-left element of the Leslie Matrix. When fertility in the first age group is zero, this change has no effect, and when it is not zero, it improves the accuracy of the projections.

The formula we have derived is the most commonly used formula for the elements in the first row of a Leslie Matrix. We have, however, made some simplifications and one could imagine still more complicated formulas. For example, we could try to reckon up more exactly the person-years that women starting in the 20 to 25 group spend in that age group and in the succeeding one, taking into account the survivorships throughout the interval. The assumption we have built into our formula would only be

exactly true if anyone who died in the interval died exactly at the boundary between age groups. Our formula is good enough for all practical purposes, but it is an interesting conceptual exercise to figure out what a more precise formula would be.

Returning to the formula we have, we can rewrite it in a new format which suggests an interesting identity. We multiply and divide our  ${}_nF_x$  factor by  ${}_nL_x$  so as to have a single denominator for the whole expression.

$$\frac{{}_nL_0}{{}_nL_0} \frac{{}_nF_x{}_nL_x + {}_nF_{x+n}{}_nL_{x+n}}{{}_nL_x} f_{fab}$$

The denominator depends on the age group  $x$ . But if we multiply the element for age group  $x$  by  ${}_nL_x$  we remove this dependence. Even better, multiply the element for age group  $x$  by  ${}_nL_x/{}_nL_0$  and add together the terms for  $x = 0, x = n, x = 2n$ , etc. The sum is

$$\frac{1}{{}_nL_0} \left( \sum {}_nF_x{}_nL_x + \sum {}_nF_{x+n}{}_nL_{x+n} \right) f_{fab}$$

Notice that the first term in the second sum is exactly the same as the second term in the first sum. The second term in the second sum is the same as the third term in the first sum, and so forth. The only difference is in the first term. The first term in the first sum is for the group from 0 to  $n$ . The first term in the second sum is for the group from  $n$  to  $2n$ . So the second sum has no term for the group from 0 to  $n$ . When  $n$  is small, that makes no difference, since the fertility of that group is zero. When  $n$  is large, if we insert our special correction for grandchildren, we then have twice the term for the youngest age group shown in the last equation, and we can use the “extra” term to make up for the missing term in the second sum. The result is

$$\sum A_{1,j(x)n} L_x / {}_nL_0 = \sum {}_nF_x L_x f_{fab} / {}_nL_0$$

We recognize the sum on the right as the NRR. Whenever  ${}_nF_0$  is zero, or whenever  ${}_nF_0$  is non-zero but we use our upper-left-element correction, we can recover the NRR from the Leslie Matrix in this way.



The factors by which we have to multiply the elements in the first row can themselves be calculated from the subdiagonal elements of the Leslie matrix. They are products of subdiagonal elements. Thus we conclude that if the first age-group is narrow enough to fall within the pre-procreative span, then we can find the NRR from the Leslie Matrix elements, and indeed we have a simple formula for doing so. But we cannot fully disentangle fertility from mortality or recover fertility rates themselves from Leslie Matrix elements.

## 5.5 Projecting Mares and Fillies

We now work out an example of a Leslie Matrix and a population projection using it. It is convenient to have a low-dimensional matrix with which to work. Instead of a human population, we consider a population of horses living at a stable being raised for racing and for pets. In this way we obtain a Leslie Matrix with five-year age groups but with only three rows and columns. The mothers are mares. The daughters are fillies. We are projecting a female population, so we need daughter-only age-specific fertility rates, which, in this case, are filly-only age-specific fertility rates. The basic data on survivorship and fertility are as follows:

$$l_0 = 1.0; l_5 = .9; l_{10} = .6; l_{15} = 0$$

$${}_5F^{dau}_0 = .00; {}_5F^{dau}_5 = .40; {}_5F^{dau}_{10} = .30$$

We have three parts of the Leslie Matrix to fill in. First, we have the structural zeros. Second, we have the subdiagonal elements. Third we have the first row.

The formula for the elements in the first row has been discussed from an advanced point of view in the preceding section. In order to be able to use Leslie Matrices for projection, it is not necessary to understand in detail where each of the terms comes from. The formula is

$$\frac{{}_nL_0}{2l_0} \left( {}_nF_x + {}_nF_{x+n} \frac{{}_nL_{x+n}}{{}_nL_x} \right) (f_{fab})$$

This is the formula for the element in the first row and  $j$ -th column, where  $j$  satisfies

$$x = (j - 1)(n)$$

The first step in writing down the Leslie Matrix is to fill in the structural zeros. In this case, we have a three by three matrix which resembles a tic-tac-toe board. In a three by three Leslie Matrix there is only one chance to win tic-tac-toe, namely in the first row. The structural zeroes block all other runs. After filling in the structural zeros, our matrix looks like the following diagram:

$$\begin{array}{rcc} & \begin{array}{c} 0 \text{ to } 5 \\ 5 \text{ to } 10 \\ 10 \text{ to } 15 \end{array} & \begin{array}{c} 5 \text{ to } 10 \\ 10 \text{ to } 15 \end{array} & \begin{array}{c} 10 \text{ to } 15 \end{array} \\ \begin{array}{c} 0 \text{ to } 5 \\ 5 \text{ to } 10 \\ 10 \text{ to } 15 \end{array} & \left( \begin{array}{ccc} \textit{fillies} & \textit{fillies} & \textit{fillies} \\ \textit{survivors} & 0 & 0 \\ 0 & \textit{survivors} & 0 \end{array} \right) \end{array}$$

The next step is to compute the survivorship ratios for the subdiagonal

$$A_{2,1} = \frac{(5/2)(.9 + .6)}{(5/2)(1.0 + .9)} = 0.7895$$

$$A_{3,2} = \frac{(5/2)(.6 + .0)}{(5/2)(.9 + .6)} = 0.4000$$

The next step is to compute the first row. For this we need

$${}_5L_0 = (5/2)(1.0 + .9) = 4.75$$

Now we compute the upper-left element. Is it zero? The fertility rate for the first age group is zero. But this does not make the upper-left element zero. Some horses who are zero to five at the start of the projection will be between five and ten during some of the projection step over the next five years, and they will have non-zero fertility.

Horses in the wild would have higher early fertility. But we imagine that young fillies in this stable are being taken to races and are not introduced to colts until they have had some years

of racing and other amusements. For our upper-left element, we have

$$A_{1,1} = \frac{4.75}{2} \left( 0 + 0.400 \frac{3.75}{4.75} \right) = 0.7500$$

$$A_{1,2} = \frac{4.75}{2} \left( 0.400 + 0.300 \frac{1.50}{3.75} \right) = 1.2350$$

$$A_{1,3} = \frac{4.75}{2} (0.300 + 0.000) = 0.7125$$

So the final matrix takes the form

$$\begin{pmatrix} .7500 & 1.2350 & .7125 \\ .7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

Now suppose we are given information on the number of horses at the start in the different age groups, suppose 0, 4, and 2. We want to calculate the number in these age groups five years later.

Our notation for the sizes of age groups is organized on the same basis as our lifetable notation. We write  ${}_nK_x(t)$  for the count of persons (or horses) aged  $x$  to  $x+n$  at time  $t$ . We treat these counts as the elements of a vector whose first element is the size of the first age group. The second element is the size of the second age group, etc.. We write  $K(t)$  without subscripts for the whole vector whose elements are  ${}_nK_0$ ,  ${}_nK_n$  and so on.

Let us first calculate the number of horses five years later in the bottom age group zero to 5. We refer to the first ROW of the Leslie Matrix. We have

$$\begin{aligned} & (0.7500 \text{ fillies per young mare at start}) \\ & \quad \text{times } (0 \text{ young mares at start}) \\ & \qquad \qquad \qquad \text{plus} \\ & (1.2350 \text{ fillies per midaged mare at start}) \\ & \quad \text{times } (4 \text{ mid-aged mare at start}) \\ & \qquad \qquad \qquad \text{plus} \\ & (0.7125 \text{ fillies per older mare at start}) \\ & \quad \text{times } (2 \text{ older mares at start}). \end{aligned}$$

That equals

$$(.7500) * (0) + (1.2350) * (4) + (.7125) * (2) = 6.365$$

so we expect 6.365 young fillies at the end of the projection step.

Now for the midaged mares at the end. Referring to the second ROW of the Leslie Matrix, we have

$$\begin{aligned} & ( 0.7895 \text{ midaged mares per young mare } ) \\ & \text{times } (0 \text{ young mares } ) \end{aligned}$$

plus

$$\begin{aligned} & ( 0 \text{ midaged mares per midaged mare } ) \\ & \text{times } (4 \text{ midaged mares } ) \end{aligned}$$

plus

$$\begin{aligned} & ( 0 \text{ midaged mares per older mare } ) \\ & \text{times } (2 \text{ older mares } ) \end{aligned}$$

equals zero midaged mares at the end of the projection step.

For the older mares at the end, referring to the third ROW of the Leslie Matrix, we have

$$(0) * (0) + (.4000) * (4) + (0) * (2) = 1.6000$$

Our answers, however, are coming out not in whole numbers but in fractions. What is 1.60 of a horse? We must remember that the Leslie Matrix elements give "expected numbers" which, in a statistical sense, are averages. A value of 1.60 might occur if you had 2/5 of a chance of having a single horse and 3/5 of a chance of having two horses. Then 1.60 would be your expected number of horses, an average over the possibilities.

Notice that the use of the Leslie Matrix comes right out of its definition. You do not need to know matrix algebra to do projection. But if you do know matrix algebra, you will notice that this is a matrix multiplication. If A is a Leslie Matrix, and

$K(t)$  is a column vector whose elements are the counts at time  $t$  in each age group, then  $K(t)$  for  $t = 5$  is the matrix product of the matrix  $A$  with the vector  $K(0)$  :

$$K(5) = AK(0)$$

The rule for matrix multiplication is like a dance. As your left hand goes across the row, your right hand goes down the column, you multiply and add up.

The matrix notation makes it easy to see what happens in the next period.

$$K(2n) = AK(n) = AAK(0)$$

Here  $AA = A^2$  is not ordinary but matrix multiplication. In general

$$K(tn) = A^t K(0)$$

In our crude model of exponential growth, we had an equation that looked exactly like this one, but it had ordinary numbers rather than vectors and took no account of age. Now we have an equation with matrices and vectors, which take account of in age. The equation has the same form as before but a new and richer interpretation. It is the generalization to populations with structure of the crude rate model with which our study of populations began.

## Highlights of Chapter Five:

Transition matrices.

Leslie matrices.

Structural zeros.

The Leslie Matrix Subdiagonal

The Leslie Matrix First Row

Projecting with Matrices

## Key Formulas

$$\begin{aligned}
 A_{j+1,j} &= \frac{{}_nL_{x+n}}{{}_nL_x} \\
 A_{1,j(x)} &= \frac{{}_nL_0 f_{fab}}{2l_0} \left( {}_nF_x + {}_nF_{x+n} \frac{{}_nL_{x+n}}{{}_nL_x} \right) \\
 K(tn) &= A^t K(0)
 \end{aligned}$$

## Further Reading

Projection matrices are widely used in population biology along with demography. Chapter 3 of Keyfitz and Caswell (2005) *Applied Mathematical Demography* has a valuable introduction.

## Exercises for Chapter Five

1. Let  $A$  be the Leslie Matrix for projecting a population of horses through a five year interval and  $K(0)$  an initial vector of age-group counts at time  $t = 0$  given by

$$A = \begin{pmatrix} .8894 & 1.3650 & .2600 \\ .7058 & 0 & 0 \\ 0 & .4200 & 0 \end{pmatrix}; K(0) = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$$

a) Find the population vector at  $t = 5$  years.

b) Find the population vector at  $t = 10$  years.

2. Consider a set of kinship roles as specified below. At any given time, a person's "state" is defined to be the role that the person has most recently assumed. For instance, a man who has just had a new nephew born enters the state "uncle". If he has a son born shortly afterwards, he then moves to the state "father". Draw an arrow diagram showing the permitted transitions among these states. The set of kinship roles to consider are the following: grandmother, grandfather, mother, father, uncle, aunt, sister, brother, son, daughter, grandson, granddaughter, nephew, niece.

3. Using the lifetable person-years lived entries the cohort of U.S. women born in 1934 given in Chapter Four, calculate the subdiagonal entries of a Leslie matrix with  $n = 5$  year wide age groups.
4. Using the ages-specific fertility rates for the cohort of U.S. women born in 1934 given in Chapter Four, calculate the entries in the first row of a Leslie matrix with  $n = 5$  year wide age groups.
5. The matrix  $A$  shown below is a Leslie Matrix for projecting the female population of Argentina. There are three age groups, each 18 years wide. The starting population for 1992 includes 3.9 million girls aged zero to eighteen, 3.3 million women aged eighteen to thirty-six, and 2.8 million women aged thirty-six to fifty-four.

$$A = \begin{pmatrix} .551 & .556 & .037 \\ .962 & 0 & 0 \\ 0 & .909 & 0 \end{pmatrix}$$

- a) How long is the interval of time covered by a single projection step?
- b) What is the total population of women up to age 54 after one projection step?
- c) What is the total population of women up to age 54 after three projection steps? To what year would this total apply?
- d) At what rate would the population of Argentina be growing according to this projection over three projection steps?

\*6. The first row of a 10 by 10 Leslie Matrix with five-year-wide age groups for projecting the population of Honduras in 1965 forward in time contains elements

.000, .001, .152, .466, .629, .607, .521, .331, .125, .022

The first 13 elements in the  ${}_5L_x$  (person-years lived) column of the women's life table with a radix of 100 are

474, 458, 452, 447, 441, 435, 426, 416, 405, 391, 374, 352, 322

Design a computer program to compute the subdiagonal elements of the Leslie Matrix  $A$ , and to place the elements of the first row, the subdiagonal, and the structural zeros into the positions in a ten by ten matrix where they belong. What is the matrix product of  $A$  with the vector whose elements are the first ten values of  ${}_5L_x$ ?



# Chapter 6

## Period Fertility

### 6.1 Period Measures

It is now time to take an important step forward in the kinds of measures that we calculate. Up to now, we have been studying *cohort* measures like the NRR and the expectation of life at birth, based on observations of the life experience of cohorts. Now, we shall consider how to calculate versions of these measures based on *period* data. Alongside a “cohort NRR” we shall have a “period NRR”. Alongside a “cohort lifetable” we shall have a “period lifetable”.

The concepts of all these measures are fundamentally cohort concepts. They describe features of the life course of individuals. When we move to period versions of our measures, the concepts do not change, but the kinds of data do change.

Our motivation for calculating measures based on period data is timeliness. We cannot, strictly speaking, determine a cohort’s NRR until the last member of the cohort has completed childbearing, and we cannot determine a full cohort lifetable until the last member of a cohort has died. The most recent cohort NRRs now available pertain to cohorts born in the 1940s, before the parents of many readers of this book. The most recent complete cohort

lifetables pertain to cohorts born in the Nineteenth Century, the great grandparents of young people today. Clearly, the fertility of our parents' and grandparents' generation is not very relevant as a description of today's and tomorrow's childbearing. The lifetime mortality experience of the cohorts to which people now in their nineties belong is not a good description of the risks of dying that most of us face. Cohort measures are out of date long before they are complete.

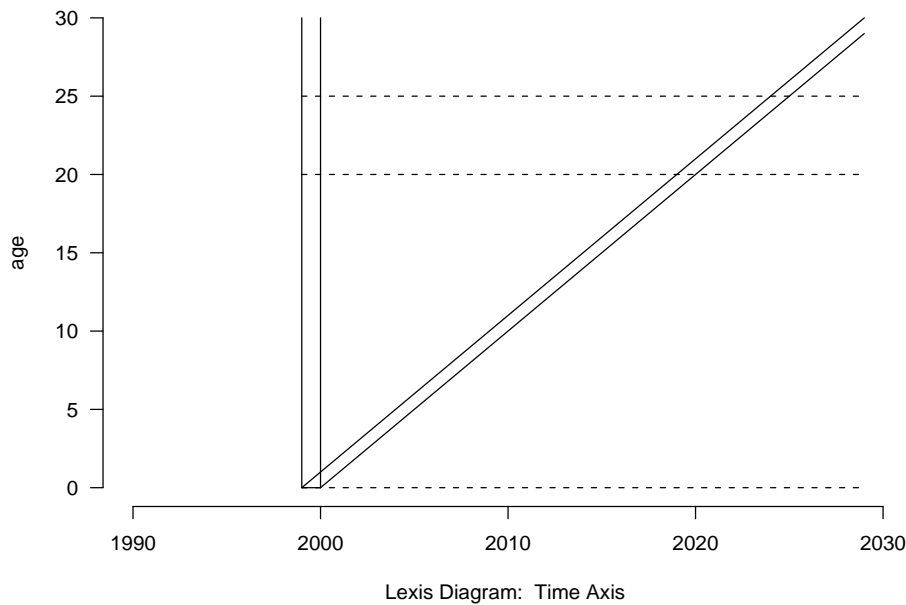
To fill in the picture of fertility or mortality before we have the full story requires making assumptions. One way of doing so is to make forecasts of future fertility and mortality and fill in the future life experience of recent cohorts on the basis of guesses. There is no unique right way of doing so. After all, most of today's young people have not made up their minds about childbearing, and none of us know whether risks of dying will be worse or better when they are old. In the face of this uncertainty, demographers adopt an assumption which is as neutral as possible, the assumption that today's rates will stay the same, rather than going up or down. This assumption provides a baseline. More complicated forecasts can always be calculated and compared with the baseline, but the assumption of unchanging rates gives common ground on which to start.

Our neutral assumption, even though events eventually prove it wrong, allows comparisons from place to place and time to time which are easily understood. If we built into our measures guesses about specific ways that rates are to change in the future, we would have to read the fine print every time we read a number in a demographic publication.

When we calculate a period measure, we are always playing a *Game of Pretend*. We pretend that the age-specific rates we see today for different age groups continue unchanged into the future. The concept is illustrated on the Lexis Diagram in Figure 6.1.

This Lexis Diagram has a narrow vertical strip for the period of the present year. The dotted horizontal lines in the middle single out the age group of 20-25-year-olds. Events of births and deaths that occur in the small rectangle where the strip for the

Figure 6.1: From Period to Cohort on a Lexis Diagram.



present year intersects the band between the dotted lines determine the period age-specific rates of fertility and mortality for this age group in this year. In our game of pretend, these rates carry forward in time (to the right on the diagram) and apply to this age group in every year of time out into the future.

Toward the lower left of the diagram, a new-born cohort starts its way up its diagonal path from the base of the Lexis diagram. When the cohort reaches its early twenties, its lifelines cross into the horizontal dotted band for the 20 to 25-year-olds. We imagine that its fertility and mortality rates are the rates determined by the experience of today's 20 to 25-year-olds, carried forward unchanged in time. Each age interval defines a similar horizontal band on the Lexis diagram. As the cohort ages, it encounters these bands one after another. In childhood, it encounters rates from data for present-day children. In its thirties and forties it encounters rates from data for present-day parents, aunts, and uncles. In its old-age, it encounters rates from data for present-day grandparents. This imaginary cohort is called the *synthetic cohort*. Its life experience stretching over time is put together from the expe-

riences of members of different age groups at one point of time, today.

The rule for calculating a period measure is a simple one: Take the formula for the cohort version of the measure. Replace all the cohort age-specific rates with period-age specific rates. Then evaluate the formula.

In the T.V. quiz show Jeopardy, the emcee gives a phrase that is the answer, and the contestants have to come up with the question to which this phrase is the answer. Suppose the emcee says, “Period NRR”. That’s the answer. What’s the question? The question is, “What would a *cohort* have for an NRR if the present *period’s* age-specific rates persisted forever? “ We can replace the words “period NRR” with any other period measure. The answer is “Period Lifetable”. What is the question? The question is, “What would a *cohort* have for a lifetable if the present *period’s* age-specific rates persisted forever. ” The definition of a period rate has the word cohort in it. This may sound surprising, but it is essential. The concepts are still cohort concepts. What changes is the source of the data. Age-specific rates for periods are substituted for age-specific rates for cohorts.

## 6.2 Period Age-Specific Fertility

As we have seen, cohort age-specific fertility  ${}_nf_x$  is defined to be a quotient whose numerator is the count of babies born to the cohort between ages  $x$  and  $x+n$  and whose denominator is cohort person-years lived. Period age-specific fertility  ${}_nF_x$ , written with a capital F, is defined to be a quotient whose numerator is the count of babies born in the period to population members between ages  $x$  and  $x+n$ . Its denominator is the period person-years lived by people between ages  $x$  and  $x+n$  in the period. The width of the age range,  $n$ , is not generally the same as the duration of the period which, if necessary, we denote by  $T$ . Period rates, in which  $n$  and  $T$  typically differ, are unlike Leslie Matrices, in which the

age group width and the length of the projection step have to be the same.

Most of the time, we calculate period rates for a period one year in length. Consider age-specific rates for 2004. The numerator counts babies born in 2004, and the denominator counts person-years lived in the one year 2004 by people aged  $x$  to  $x + n$ , for instance, aged 20 to 25. We often have one year of time but five years of age. When we have a one-year-long period, the period person-years lived, or PPYL, is estimated to be the mid-year population times the one-year period length, which has the same numerical value as the mid-year population itself. In less common cases with a longer period like a whole decade, we have to multiply the mid-period population by the length of the period, say 10 years, to get the PPYL. But the larger denominator is balanced out by the larger numerator, which includes ten years of babies rather than one.

If we write BIRTHS(x) as shorthand for the babies born in the period to women in the age group and PPYL(x) as shorthand for the corresponding person-years lived in the period in the age group, the period age-specific fertility rate has the following form:

$${}_nF_x = \frac{\text{BIRTHS}(x)}{\text{PPYL}(x)} = \frac{\text{BIRTHS}(x)}{{}_nK_x T}$$

We are dealing with events and lifelines on a Lexis Diagram inside a rectangle whose height is  $n$  and whose base is  $T$ . If we count deaths instead of births, we obtain period age-specific mortality rates, written  ${}_nM_x$ , from which we can calculate lifetable entries and  ${}_nL_x$  values as described in Chapter Seven. Together, these  ${}_nF_x$  and  ${}_nL_x$  values provide the ingredients for period versions of the  $NRR$ ,  $TFR$ , and  $GRR$ .

Like the cohort  ${}_nf_x$ , the period  ${}_nF_x$  is a *rate*, with units of 1/time. In principle, it can be larger than 1.0. Just as increasing  $T$  increases both numerator and denominator, so increasing the width  $n$  of the age group increases both numerator and denominator, and the effects tend to balance out. The rate  ${}_{10}F_{20}$  is expected to be close to  ${}_1F_{25}$ , and certainly not ten times as big.

Panama in 1960 had 41,700 women aged 20 to 25 who gave birth to 13,015 babies. We calculate  ${}_5F_{20}$  to be  $13,015/41,700 = 0.312110$ .

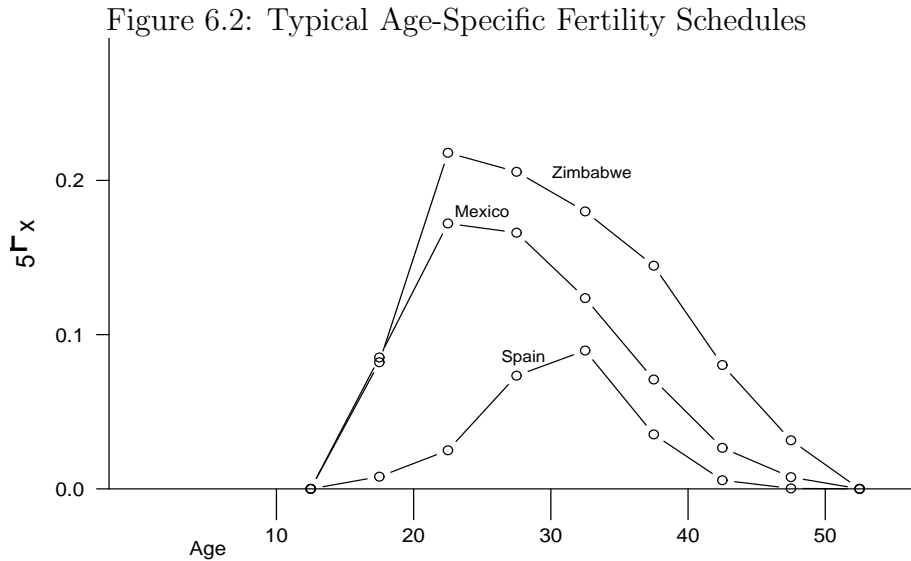


Figure 6.2 shows graphs of age-specific fertility for three contrasting countries drawn from the United Nations *Demographic Yearbook* for 1999. Zimbabwe in 1992, reporting a TFR of 4.71, has the kind of outward-bending curve at older ages typical of countries with only modest family limitation. Mexico in 1995, reporting a TFR of 3.26, had peak fertility at young ages not far below Zimbabwe, but family limitation has squeezed down the curve at older ages. Spain in 1997, reporting a TFR of 1.19, has had the lowest total fertility in the world in number of recent years. The curve for Spain peaks later, reflecting postponements of child-bearing, and drops steeply, bending inward, at older ages, a feature signalling pronounced family limitation which will be discussed in Chapter Eleven.

## 6.3 The Period NRR

The formula for the period NRR is exactly the same as the formula for the cohort NRR presented in Chapter Four, except that period age-specific fertility rates  ${}_nF_x$  replace cohort age-specific fertility rates  ${}_nf_x$  and period life table  ${}_nL_x$  values replace cohort life table  ${}_nL_x$  values. The calculation of the period life table values will be discussed in Chapter Seven. Here we simply assume that they are available. The formula is straightforward:

$$NRR = \sum {}_nF_x {}_nL_x f_{ab}/l_0$$

Table 6.1 gives period age-specific fertility and years lived for the year 2000 for women in India. The radix is 1000. The table is set up to illustrate the calculation of the NRR and other summary measures.

Table 6.1: Calculating Fertility Measures, India, 2000

$x$	${}_5F_x$	${}_5L_x$	babies	$x + n/2$	product
15	0.051	4396		17.5	
20	0.182	4343		22.5	
25	0.168	4282		27.5	
30	0.109	4219		32.5	
35	0.054	4153		37.5	
40	0.024	4074		42.5	
45	0.008	3964		47.5	
			=B		
			/ $l_0$		/ B
	=TFR				= $\mu$
	*.4886		*.4886		
	=GRR		=NRR		

We concentrate first on the  $NRR$ . We compute the babies that would be born to the *synthetic cohort*. The synthetic cohort has

age-specific fertility  ${}_nf_x$  equal to the period rates  ${}_nF_x$  and person-years of life given by the  ${}_nL_x$  values from the period life table. We multiply the second and third columns together for the implied number of babies from each range of ages, add up the values to find total babies  $B$ , divide by the radix (in this case 1000), and multiply by the default fraction female at birth to convert from babies per woman to daughters per woman. The result is the period  $NRR$ .

With a little extra work, we are also able to find out how young or old, on average, the mothers are at the babies' births. Since our data are grouped into five-year-wide intervals, we do not know the exact ages of mothers, but we can use the middle age in each age group as an approximation. We have  $0.051 * 4396$  or about 224 babies whose mothers were about 17.5 years old at their births. The sum of their mothers' ages is 224 times 17.5 or about 3920. Another 790 or so babies had mothers about 22.5 years old, contributing another 1084 times 22.5 to the sum for the last column of Table 6.1. To find the average, we add up all contributions, and divide by the total number of babies  $B$  already obtained by summing the column labeled "babies". The quotient is the average age of mothers from the synthetic cohort at their babies' births.

This average age is called the "synthetic cohort mean age at childbearing". It is often denoted by the Greek letter  $\mu$  (mu), which resembles the letter "u" but is in fact the original form of our letter "m". In most countries this average age is about 26, 27, or 28. Women in countries with early marriage also tend to continue to bear children late in their reproductive span, leading to average ages that are not very different from countries with late marriage but substantial fertility control.

The "babies" that enter into the calculation of the period  $NRR$  are babies that *would be born* if today's period age-specific rates continued unchanged into the future. They are the babies of the synthetic cohort. The synthetic cohort is imaginary, so these babies are imaginary. They come from a game of "Pretend". Chapter Four dealt with real cohorts and real babies, and that meant



that the person-years in the formula for  ${}_nf_x$  were the same as the person-years in the formula for the cohort  $NRR$ . With period measures, the period person-years in the formula for  ${}_nF_x$  are not the same as the synthetic cohort person-years in the formula for the period  $NRR$ . We use mid-period population counts  ${}_nK_x$  to find the period person-years (PPYL) for the denominators of the  ${}_nF_x$ . Then we forget about the  ${}_nK_x$  and use the period life table  ${}_nL_x$  to calculate the synthetic cohort's babies that contribute to the period  $NRR$ .

We must keep in mind that the period  $NRR$  is an abstraction. It is a useful abstraction that tells us what the long-term implications of present-day fertility and mortality would be for population growth. But in the real world rates are always changing, age-specific fertility and mortality do not remain constant, and the long-term implications of short-term rates expressed in the  $NRR$  are not telling us about the real future. The period  $NRR$  for the United States has dropped below 1. If today's age-specific rates continued unchanged forever, eventually the size of the U.S. population (excluding net immigration) would begin to decline. Newspapers often misunderstand this conclusion and publish articles that suggest that the period  $NRR$  less than 1 means that population growth has ceased. That is a serious mistake. In fact, even net of immigration, the United States has one of the fastest growing populations of any developed country, with births exceeding deaths every year. Today's period  $NRR$  tells us what *would* happen in the long term, not what actually happens today.

## 6.4 Period TFR and GRR

The Net Reproduction Ratio, as we have seen in Chapter Four, is a measure of generational growth. It reflects the joint action of fertility and mortality. If we want to measure the overall level of fertility on its own, we need measures that depend only on the age-specific fertility rates. Cohort versions of such measures, the widely-used Total Fertility Rate (TFR) and the somewhat rarely-

used Gross Reproduction Ratio ( $GRR$ ), have been presented in Chapter Four. Period versions of the  $TFR$  and  $GRR$  are just as easy to calculate as the period  $NRR$ .

For the period TFR we add up the products of period age-specific fertility rates  ${}_nF_x$  times the widths  $n$  of the age intervals. For the period  $GRR$  we multiply by the fraction female at birth to convert from all babies to daughters. The formulas assign  $n$  full person-years of risk of childbearing to every  $n$ -year-wide age interval. When the widths of the age intervals are all the same, we can first add the  ${}_nF_x$  and then multiply the sum by the common value of  $n$ . The calculations are indicated in the second column of Table 6.1.

The formulas are the same as the cohort formulas, except that the period  ${}_nF_x$  replaces the cohort  ${}_nf_x$ .

$$\begin{aligned} \text{Period TFR} &= \sum ({}_nF_x)(n) \\ \text{Period GRR} &= \sum ({}_nF_x)(n)f_{fab} \end{aligned}$$

There are some logical relationships among our indices. The  $NRR$  is always less than or equal to the  $GRR$  (since mortality can only decrease the net total of daughters), and the  $GRR$  is always less than or equal to the  $TFR$  (since daughters are a subset of sons and daughters). These relationships are highlighted in Table ta.ngt.

Table 6.2: Relationships for the  $NRR$ ,  $GRR$ , and  $TFR$

$$\frac{\text{NRR} \leq \text{GRR}}{\text{( Daughters Only )}} \leq \text{TFR}$$

$$\frac{\text{GRR}}{\text{( Deaths Excluded )}} \leq \text{TFR}$$

Along with the  $NRR$ , the  $TFR$  is one of the most widely cited measures in demography. Total fertility rates can change quite

rapidly, and changes in the TFR signal major shifts in demographic experience, like the Baby Booms following the Second World War in many developed countries and the “Baby Lulls” that followed the Baby Booms. Total Fertility Rates from the *Statistical Abstract of the United States* are shown in Table 6.3. The 1992 figure is an average of a TFR for whites of 1.994 and a TFR for blacks and others of 2.442.

Table 6.3: Recent U.S Total Fertility Rates

Years	1940-45	1945-50	1950-55	1955-60	1960-65	1965-70
TFR	2.520	2.985	3.337	3.690	3.449	2.622
Years	1970-75	1975-80	1980-85	1985	1986	1987
TFR	2.094	1.774	1.817	1.844	1.836	1.871
Year	1988	1989	1990	1991	1992	1993
TFR	1.934	2.014	2.081	2.073	2.065	2.046
Year	1994	1995	1996	1997	1998	1999
TFR	2.036	2.019	2.027	2.033	2.059	2.075
Year	2000	2001				
TFR	2.056	2.034				

The TFR is not exactly the same as the expected total of children for women who do live through ages at childbearing. Women who survive to fifty-five might not be a typical subset of all women. They might, for instance, have had lower fertility in their twenties than women with poorer prospects for survival. We use the fertility for all women in their twenties, those who will not survive to older ages along with those who will.

## 6.5 \*Log(GRR) Plots

The Net Reproduction Ratio which is a measure of generational growth, as we have said, summarizes the combined effects

of fertility and mortality. In trying to understand the course of population change, it is often valuable to decompose a change in growth rates into the contributions of changes in fertility and changes in mortality. The full story involves different changes at different ages and may be quite complex. But there is one simple decomposition which lets us grasp an overall picture in quite a vivid way.

This decomposition leads to plots called “Log GRR Plots” which plot the logarithm of the  $GRR$  on the vertical axis against an index of survivorship on the horizontal axis. The index is chosen so that the contours of constant growth on these plots are simple straight lines.

To introduce this technique, we first define the “net maternity function”. Set  $n = 1$  for one-year rates, and plot daughters-only age-specific fertility rates  ${}_1F_x * f_{fab}$  as a function of  $x$ . The area under this curve is the GRR.

Now plot  ${}_1F_x * (f_{fab}) {}_1L_x/l_0$  on the same plot against  $x$ . This new curve lies below the old one. The area under the new curve is the NRR. This curve whose area defines the NRR is called the “net maternity function.” Assume a radix of 1.0. Then we get the bottom curve by multiplying the top curve by a factor  ${}_1L_x$  which is roughly equal to  $l_x$ . The average adjustment factor over mothers of all ages is close to the adjustment at the average age  $\mu$  of mothers, which is the proportion  $l_\mu$ . We expect then that the  $NRR$  will be approximately equal to the product  $GRR l_\mu$ . When we take logarithms this becomes a linear relation:

$$\begin{aligned}\log(NRR) &\approx \log(GRR) + \log(l_\mu) \\ \log(GRR) &\approx \log(NRR) + (-1) \log(l_\mu)\end{aligned}$$

$$\text{Fertility Index} = \text{Intercept} + \text{Slope} * \text{Mortality Index}$$

Suppose we have a set of combinations of fertility and mortality that keep the same rate of generational growth, that is to say,

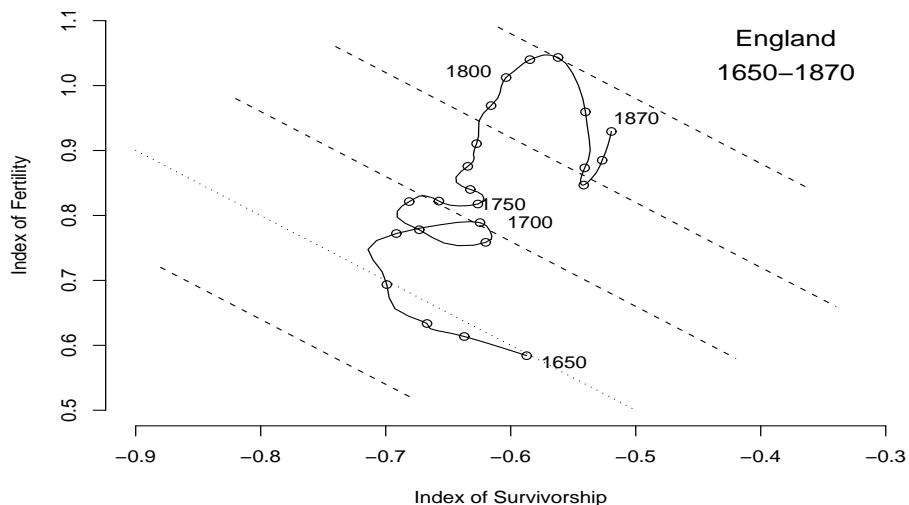
the same NRR. Then, when we plot of  $\log(\text{GRR})$  against  $\log(l_\mu)$ , we obtain a straight line whose slope is  $-1$  and whose intercept is  $\log(\text{NRR})$ . In other words, the “contour” of points representing combinations with this level of growth is a straight line. When we change the NRR, we obtain another line parallel to the first. The contours of growth form a grid of diagonal lines sloping from upper-left to lower-right. Equal separations on this plot correspond to unit changes in the logarithm of the NRR, not in the *NRR* itself.

In practice, we generally have measurements of GRR and NRR, and we plot the log of the GRR on the vertical axis against  $\log(\text{NRR}/\text{GRR})$  on the horizontal axis. This helps us to decompose the changes in growth rates into the relative contributions of changes in fertility and mortality.

The classical picture of demographic transition envisions a reduction in mortality followed by a reduction in fertility. That would be represented by a curve moving to the right and then trending downwards. The movement to the right takes the population up the “ski slope” toward higher growth, and the movement downward brings it back over the diagonal growth contours to lower growth.

A good example of this display is from Wrigley and Schofield (1981) *The Population History of England*, redrawn in Figure 6.3. In the diagram we see how lines of constant generational growth are mainly passed by changes in fertility rather than mortality. These changes in overall fertility are in fact understood to be driven largely by changes in marriage ages and marriage rates. This evidence shook the old belief that England and other European countries had gone through a demographic transition similarly in nature to that which Third World Countries began to take after World War II.

Notice that the diagonal lines here are labeled by  $R$  not by  $\log(\text{NRR})$ . We convert from logarithms of Net Reproduction Ratios to Crude Growth Rates using an approximation that comes straight out of our Crude Rate Model for exponential growth. That model tells us to expect a population of initial size  $K$  to

Figure 6.3: Log *GRR* Plot for England

grow to  $K \exp(RT)$  in time  $T$ . The time required for one generation is the cohort mean age at childbearing  $\mu$ . Plugging in  $\mu$  for  $T$ , we have growth by a factor of  $\exp(R\mu)$  in the time it takes on average for one generation, but in one generation the population grows by a factor equal to the *NRR*. That implies

$$\begin{aligned} e^{R\mu} &\approx NRR \\ R &\approx \log(NRR)/\mu \end{aligned}$$

Here  $R$  is the Growth Rate. In Chapter Ten, when we study stable populations, we shall learn that this equation is quite a good approximation when vital rates have been unchanging for a long time, and then the growth rate  $R$  is equal to a quantity written “ $r$ ” called Lotka’s intrinsic rate of natural increase. When we label our diagonal contours by the value of  $R$ , equal separations on the plot (that is to say, equal separations on the scale of logarithms of *NRR* values) correspond to equal difference in the implied growth rates.

## 6.6 Age-Standardized Rates

The period measures we have been studying, like the period TFR, GRR, and NRR, are preferred to simple period measures like the Crude Birth Rate mainly because they are not influenced by the age distribution of the population. Bringing twenty-year-olds into a population will raise the number of births and raise a crude measure like the CBR but will leave the period TFR, GRR, and NRR unchanged, provided that the age-specific rates remain unchanged. It is sometimes desirable to keep the simplicity of a crude rate while removing the effects of the observed age distribution. We can do so by calculating “age-standardized” rates. Such rates are useful for quick comparisons among widely differing countries or areas.

To calculate age-standardized rates, we pick one special population to define our standard. Here we base our standard on the population of the whole world in 2000 as estimated by the United Nations (2001:38). Often, a standard is based on a conveniently available set of population counts like the population of England in 1950 or Brazil in 1980. We typically compute age-standardized rates when we have a set of countries to compare among each other. For each country, we multiply its age-specific rates by the standard population counts, add up the products, and divide by the total standard population to obtain our age-standardized rate.

Table 6.4 shows the calculation of an age-standardized birth rate for France in 2000. The standard population counts (in millions of person-years) in the third column are multiplied by the French age-specific in the fourth column to give the implied number of babies (in millions) in the last column. Since ordinary age-specific fertility rates credit all the births to women, we multiply the count of men by zero. Adding up the columns, we find that 88.819 million babies would be produced by the 6,057 million people in our standard population, for a standardized crude birth rate of  $88.819/6057 = 0.014539$ . The process of age standardization has raised the observed CBR of 0.013228 by about 10%. When we turn to a new country, for instance, Senegal, we keep the same

Table 6.4: Age-standardized birth rates.

Sex and age $x$	Width $n$	Standard ${}_nK_x$	France ${}_nF_x$	Product =Babies
Women				
0	15	882	0	0
15	5	270	0.008	2.107
20	5	248	0.056	13.864
25	5	245	0.134	32.726
30	5	232	0.118	27.483
35	5	209	0.050	10.531
40	5	182	0.012	2.108
45	5	164	0.000	0
50	$\infty$	574	0	0
Men				
0	$\infty$	3,051	0	0

standard population counts but we replace the rates from France by the rates from Senegal. The results give us simple comparisons of fertility levels freed from direct effects of age structure.

A similar calculation with period age-specific mortality rates  ${}_nM_x$  applied to standard counts leads to an Age-Standardized Death Rate. Age-specific marriage rates applied to standard counts leads to an Age-Standardized Marriage Rate, and so forth. The most widely used of these measures is the Age-Standardized Death Rate, because age structure has so much influence on the CDR. The CDR in the United States is about 9 per thousand, and the CDR for the whole world is about 9 per thousand. Does it follow that the United States has no better mortality than the world average? Hardly. The United States has lower age-specific mortality rates at every age than the world as a whole. But the U.S. also has a much higher percentage of its population in older, high-risk age groups. The favorable mortality rates are disguised by the effect



of the risk-prone age structure. The age-standardized rates clear up the picture.

It is an important fact that the highest Crude Death Rates are not found in countries with the most severe mortality. But with any reasonable choice of standard population, the highest Age-Standardized Death Rates are found in countries with the most severe mortality.

The form of standardization which we have been studying is called “Direct Standardization.” The rates come from the countries under study. The counts of people at risk come from the standard population. This form of standardization is contrasted with a less frequent form called “Indirect Standardization” in which rates come from a standard population and counts of people at risk from the populations under study. We shall meet examples of indirect standardization in Chapter 12.

We can use the idea of standardization to gain insight into the meaning of the Total Fertility Rate. We pick our standard population to be an artificial population with the same constant number  $k$  of women in every one-year-wide age group at every instant of time. The number  $k$  which we can choose at will plays the same role for period measures as the radix  $l_0$  for cohort measures. Every square box on a Lexis diagram which covers one year of age and  $T$  years of time includes  $kT$  woman-years from this standard population. The period TFR times  $kT$  is the number of babies that would result from applying the period’s age-specific fertility rates to this standard population.

We could obtain a similar result by taking a sample of the same fixed size  $k$  in every one-year-wide age group, a sample drawn from the  ${}_1K_x$  women in the observed period population. We give each sampled woman one person-year of exposure, and treat births to these women in the period as a standardized sample of childbirths. We could plot each such childbirth as a point on a Lexis diagram with  $x$  equal to the mother’s age at giving birth and  $t$  equal to the time of giving birth. A period of length  $T$  corresponds to a vertical box on the Lexis diagram rising from a base of length  $T$ . If we count the childbirths from our standardized sample within

the box and divide by  $kT$ , we obtain the period TFR.

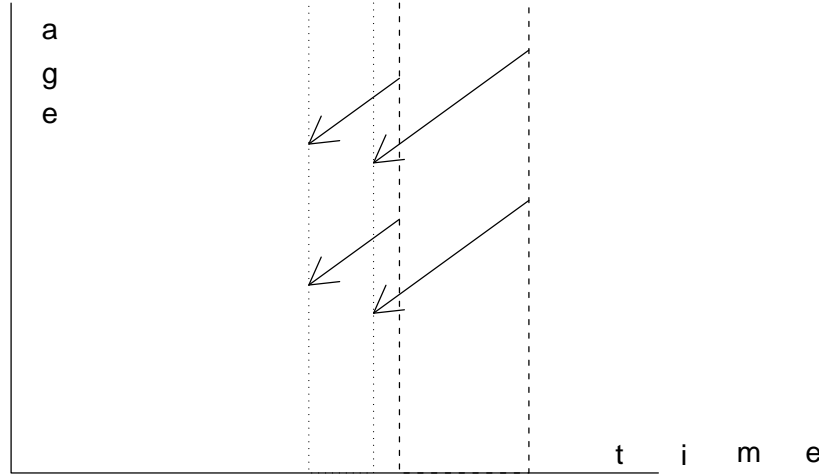
## 6.7 \*Tempo and Quantum

As we have seen in Section 6.6, the period Total Fertility Rate is a measure of fertility standardized for the number of women at risk of childbearing. It is often interesting to take standardization a step further and calculate a measure which is standardized for average ages of childbearing. The procedure is simple. We choose some age  $A_s$  to be our standard age, for instance  $A_s = 25$ . We write  $A(t)$  for the average age of childbirth (as described in Section 6.6), as a function of time. Then we shift every childbirth in our standardized sample forward or backward along the mother's diagonal lifeline on the Lexis diagram by the difference  $A(t) - A_s$ . A birth that takes place at mother's age  $x$  at time  $t$  is now assigned to take place at mother's age  $x - A(t) + A_s$  at time  $t - A(t) + A_s$ . Some such shifts are shown in Figure 6.4.

As long as our choice of  $A_s$  is sensible enough to avoid negative ages, these diagonal shifts leave every cohort TFR unchanged. In every period the new average age of childbirths for our shifted standardized sample is constant at  $A_s$ . We can count the shifted childbirths in a period box of width  $T$ , divide as before by  $kT$ , and obtain a counterpart of the period TFR which is standardized for period average ages at childbirth.

In Figure 6.4, the average age  $A(t)$  is increasing, as it currently is in many countries including the United States. Notice what happens to the box for a 10-year period from  $t$  to  $t + 10$  marked by heavy dashed lines. The left-hand edge is shifted back to time  $t - A(t) + A_s$ , while the right-hand edge is shifted back by a larger amount to time  $t + 10 - A(t + 10) + A_s$ , producing the box marked by dotted lines. The base of the shifted box is only  $10 - (A(t + 10) - A(t))$  wide, instead of 10 units wide. Provided the average ages within the box are between the average ages at the edges, the births in the original box all reappear in the new box. Thus the

Figure 6.4: Birth Age Standardization



numerators for TFRs are the same, but the denominators change from  $10k$  to

$$10k(1 - (1/10)(A(t + 10) - A(t))).$$

The left edge of the new box starts at  $t - A(t) + A_s$  instead of  $t$ . For a period of length  $T$ , our standardized TFR, call it  $S$ , is given by the fraction

$$S(t - A(t) + A_s) = \frac{TFR(t)}{1 - (1/T)(A(t + T) - A(t))}$$

If  $A(t)$  is smooth and  $T$  is tiny, the fraction in the denominator is close to the derivative of  $A(t)$ .

This form of age standardization remove the effects of the thinning out in time that occurs when women postpone births later and later in their lives. The chosen standard age  $A_s$  does not affect the denominator, but it does affect the lag by which the standardized TFR tracks the ordinary TFR. When  $A(t)$  is falling, births are being compressed in time, and the denominator removes

the effect of that compression. This kind of birth-age standardization can be applied separately to groups of births. For instance, births can be separated out by birth order, into first births, second births, and so forth, a version of  $A(t)$  can be calculated for each birth order, a standard age chosen for that order, and births of that order shifted to make that  $A(t)$  equal that standard age. An overall standardized TFR at a time  $\tau$  can be obtained by summing the standardized TFR values for the birth orders, keeping the lags between  $t$  and  $\tau = t - A(t) + A_x$  in mind.

A measure very close to the standardized TFR just described was introduced by Bongaarts and Feeney (1998). They separate out births by birth order and, using one-year-wide age groups, estimate  $A(t+1) - A(t)$  by half the difference between the average age in the year following  $t + 1$  and the average age in the year preceding  $t$ . Their formula has the same denominator but omits the lag. The lag is not of much practical importance, but it is essential to the consistency of the measure and the interpretation in terms of shifted births.

Demographers use the word “tempo” to refer to the timing of births (or other events) within a person’s lifecourse, and “quantum” to refer to the lifetime number of births (or other events). The distinction was brought into prominence by Norman Ryder, who emphasized cohort-based changes in fertility. In recent years, major swings in fertility have tended to respond to chronological period influences, cutting across cohorts. Adjustments for changing ages at birth are meaningful, because there is something we care about that is left invariant by the adjustment, namely the cohort quantum of fertility. Despite some proposals to the contrary, tempo adjustments for mortality do not make sense, because the quantum of mortality does not vary, since every person dies exactly once. Mortality is entirely a matter of “tempo”, that is, of the timing of deaths in the life course. Changing ages at death reflect real changes in mortality, whereas changing ages at child-birth may not reflect real changes in the total numbers of children that individuals have.

## Highlights of Chapter Six:

The strategy for defining period measures.

The period NRR, GRR, and TFR.

Age-standardized rates

Quantum and tempo

## Key Formulas

$$\begin{aligned} \text{Period TFR} &= \sum ({}_nF_x)(n) \\ \text{Period GRR} &= \sum ({}_nF_x)(n)(f_{fab}) \\ \log(NRR) &\approx \log(GRR) + \log(l_\mu) \end{aligned}$$

## Further Reading

Bongaarts and Feeney (1998) presents a discussion of quantum and tempo shifts in fertility.

## Exercises Chapter Six

1. The number of redwood trees in a certain triangular valley along the Jedediah Smith River is believed to have changed very little over the last thousand centuries. The expectation of life of trees that survive to the age of 10 is a further lifespan of 500 years. Trees about age 10 can be recognized because they are about as tall as a human adult. Suppose there are 2000 trees in the valley as tall or taller than human adults. How many ten-year-old trees should you expect to find in the forest?

This is a question that requires some reflective thought. Explain briefly and clearly the reasoning that leads you to your answer.

2. Calculate the period age-specific fertility rates, and the  $NRR$ ,  $GRR$ ,  $TFR$ , and mean age at childbearing for women in Togo in 1961 from data in Table 6.6. Togo in 1961 is an example of a country with high fertility and mortality unaffected by fertility decline. Relatively few data from this period are available for Africa, so Togo in this epoch makes a particularly interesting example. The source of these data is Keyfitz and Flieger (1968), an earlier version of Keyfitz and Flieger (1985). The table includes two columns from the female life table, census counts of women by age, and counts of births by age of mother and deaths of women by age. There were 41315 boy babies and 42855 girl babies total, and the total female population was 813295. The radix is 100,000. Be careful to distinguish between the places in your calculation where you need to use  ${}_nK_x$  and the places where you need to use  ${}_nL_x$ .

3. The following table shows Age-Specific Death Rates ( $ASDR$ ) for women from Togo in 1961 in three broad age groups. Take as your standard the age structure for the Metropolitan Statistical Area of Oakland, California in 1990, and calculate a standardized crude death rate for Togo based on this standard. The roughly 2 million people in the Oakland California Metropolitan Statistical Area in the 1990 Census comprise about 140 thousand (7.0 per cent) under age 5, about 1570 thousand (78.5 per cent ) between 5 and 60, and 290 thousand (14.5 per cent ) above 60.

Age Group	0-5	5-60	60+
Togo $ASDR$	0.063014	0.011403	0.071386

\*4. Values of the  $NRR$  and  $GRR$  for the People's Republic of China from Keyfitz and Flieger (1990:213) are shown in Table 6.5. Plot  $\log(GRR)$  on the vertical axis against  $\log(NRR/GRR)$  on the X-axis. In two sentences state what your plot suggests about the course of China's fertility transition.

5. Calculate the  $TFR$  and  $NRR$  for Nigeria in 1990 from the data in 6.7. The source of the fertility data for 1990 is Foote, Hill,

Table 6.5: Modern Chinese *GRR* and TFR

Date	1950	1955	1960	1965	1970	1975	1980	1985	1990
GRR	3.030	2.620	2.880	2.910	2.310	1.410	1.148	1.148	1.044
NRR	1.890	1.760	2.100	2.460	2.040	1.290	1.054	1.075	.991

and Martin, editors (1993) *Demographic Change in Sub-Saharan Africa*, National Academy Press, Washington, D.C. The population data are for 1990 and the lifetable values are for 1987, both from the U.S. Bureau of the Census, International Data Base, Tables 094 and 014.

Table 6.6: Data from Togo 1961

Starting Age x	Life- table ${}_nq_x$	Life- table ${}_nL_x$	Women in Period Pop. ${}_nK_x$	Births to Women in Period	Deaths to Women in Period
0	.1168	93 403	39,335	0	4,935
1	.1568	318 077	126,628	0	5,523
5	.0414	364 591	138,467	0	1,257
10	.0256	352 481	60,153	0	309
15	.0578	337 775	48,564	7,150	578
20	.0367	321 570	67,096	21,910	502
25	.0621	306 003	80,746	25,305	1,034
30	.0595	287 031	53,670	14,825	659
35	.0596	270 049	51,975	9,935	638
40	.0668	253 276	32,022	3,625	441
45	.0943	232 925	32,307	1,420	638
50	.0999	210 025	20,478	0	431
55	.1001	189 929	19,913	0	416
60	.2305	158 855	12,990	0	674
65	.2052	123 386	13,490	0	618
70+	1.0000	198 549	15,461	0	1,702



Table 6.7: Nigerian Data for 1990

Starting Age x	Life- table ${}_nq_x$	Life- table ${}_nL_x$	Women in Period Pop. ${}_nK_x$	Births to Women in Period	Deaths to Women in Period
0	0.0891	932	1,852,488	0	175745
1	0.1015	3412	5,866,210	0	159972
5	0.0208	4323	6,334,683	0	26669
10	0.0144	4050	5,258,577	0	15197
15	0.0159	3978	4,290,838	617881	13731
20	0.0171	3918	3,591,651	958971	12391
25	0.0177	3854	2,958,508	810631	10562
30	0.0213	3786	2,396,667	532060	10330
35	0.0265	3713	2,152,562	348715	11559
40	0.0387	3624	1,958,264	186035	15451
45	0.0541	3506	1,590,604	106570	17703
50	0.0754	3344	1,276,143	0	20010
55	0.1043	3128	1,013,831	0	22314
60	0.1382	2849	778,240	0	23098
65	0.1971	2506	559,879	0	24484
70	0.2880	2092	363,544	0	24467
75	0.4912	1595	193,504	0	25198
80	1.0000	518	83,143	0	21652



# Chapter 7

## Period Mortality

### 7.1 Period Life Tables

Our approach to building period lifetables is essentially the same as our approach to period fertility measures. We make the assumption that the age-specific rates for the period continue unchanged into the future. We work out the lifetable that an imaginary cohort of newborn babies would experience under this idealized, neutral assumption about the future. This imaginary cohort is our *synthetic cohort*.

With fertility measures, the age-specific rates enter directly into the calculations, and we simply substitute the period rates for the cohort rates in the formulas. The situation with lifetables is a bit more challenging, because we have not been building up our lifetables directly from the age-specific rates. Instead, we have been constructing cohort lifetables by starting with the observed survivorships  $l_x$ . From  $l_x$  we have been computing  ${}_nq_x$ , and from these two columns we have been constructing the other columns, including  ${}_nd_x$  and  ${}_nL_x$ . From  ${}_nd_x$  and  ${}_nL_x$  come the age-specific mortality rates  ${}_nm_x$ , which are the quotients  ${}_nd_x/{}_nL_x$ . These age-specific mortality rates have not been very prominent up to now, and they have been computed late in the process of building the lifetable.

For a period lifetable, however, we want to be able to start with the age-specific rates, because it is the age-specific rates that we are assuming to continue unchanged into the future. The solution is to use algebra to solve for  ${}_nq_x$  in terms of  ${}_nm_x$ . Then we can substitute the period age-specific mortality rates  ${}_nM_x$ , written with capital “M”, into the formula in place of  ${}_nm_x$ , and so obtain our period  ${}_nq_x$  values. We can go on and calculate all the other columns from  ${}_nq_x$ , ending up with the whole period lifetable.

The formula expressing the q’s in terms of the m’s is derived in a later section. Here what matters is the answer. After substituting  ${}_nM_x$  for  ${}_nm_x$  we have the formula:

$${}_nq_x = \frac{({}_nM_x)}{1 + ({}_nM_x)({}_nM_x)}$$

Let us review again why we need to take this different route to the construction of the lifetable. For a period, we do not have information on lifetime cohort survival, as we do in principle for cohorts. What we have are period counts of deaths and counts of people. Contrast the sort of data we had for the cohort life table for the children of King Edward III. For every person, we had a death and an age at death. With period data, there will be many people for whom dates at death are not available, because many people will not die in the period. The data that will be available are, first, deaths in the period, and, second, a mid-period count of people by sex and age. We know that multiplying the mid-period count by the length of the period gives an estimate of period person-years lived (PPYL). This PPYL is not to be confused with cohort person-years lived for any real cohort nor with cohort person-years for the synthetic cohort, which will turn out to quite different, and which will supply our  ${}_nL_x$  column for the period lifetable.

Given deaths  ${}_nD_x$  between ages  $x$  and  $x + n$  in the period and mid-period counts of people  ${}_nK_x$ , we are in a position to compute an age-specific death rate  ${}_nM_x$  for each age group.

$${}_nM_x = \frac{{}_nD_x}{({}_nK_x)(1 \text{ year of time})}$$

With this  ${}_nM_x$  value, we set out to compute  ${}_nq_x$  by our formula:

$${}_nq_x = \frac{({}_n)({}_nM_x)}{1 + (n - {}_na_x)({}_nM_x)}$$

To calculate the denominator, we need to know not only  ${}_nM_x$  but also  ${}_na_x$ , the average years lived per person by people dying in the interval. For most age intervals, we can substitute the approximate value  ${}_na_x = n/2$ . But, for the first few age groups and the last age group in a lifetable, it is better to employ special formulas.

We take our special formulas for  ${}_na_x = n/2$  from empirical work by Nathan Keyfitz and Wilhelm Flieger (1968:138). For the first year of life the  ${}_1a_0$  value depends on  ${}_1M_0$ :

$${}_1a_0 = 0.07 + 1.7 ({}_1M_0)$$

For very low mortality, the average is about one month or 0.07 of a year. For the United States, about half of all babies who die do so in the first month of life. It requires an age-specific rate almost as big as  ${}_1M_0 = 0.200$  to imply an average  ${}_1a_0$  near six months.

For a four-year-wide age group from age 1 to age 5, we use the following value:

$${}_4a_1 = 1.5$$

For the last age group, as we shall shortly see, we use the reciprocal of the last age-specific mortality rate.

We work through period  ${}_nq_x$  calculations with an example based on data from Table 7.1 for U.S. African-American males in 1992. We start with the infant age group. We begin by calculating the observed period age-specific mortality rate for the one-year period.

$${}_1M_0 = \frac{{}_1D_0}{{}_1K_0} = \frac{6298}{321671} = 0.019579$$

The observed death rate in the first year of life is about 20 per thousand per year.

How early in the interval do those who die in it die?

$${}_1a_0 = 0.07 + 1.7 {}_1M_0 = 0.07 + 1.7 * 0.019579 = 0.1032843$$

Table 7.1: U.S. Male Mortality Data from 1992

Age x	White		African-American	
	Deaths ${}_nD_x$	Population ${}_nK_x$	Deaths ${}_nD_x$	Population ${}_nK_x$
0	12,625	1,616,724	6,298	321,671
1	2,690	6,314,554	965	1,243,557
5	1,605	7,535,211	529	1,410,667
10	2,093	7,421,986	633	1,409,800
15	7,440	7,018,868	2,923	1,338,370
20	10,696	7,899,557	4,246	1,322,741
25	12,825	8,365,949	4,695	1,298,037
30	18,210	9,300,306	6,083	1,309,862
35	21,690	8,835,031	7,308	1,198,819
40	24,726	7,919,923	7,949	989,666
45	28,343	6,553,295	7,493	703,106
50	33,681	5,077,027	8,021	565,138
55	47,042	4,390,294	9,824	467,009
60	74,994	4,335,665	12,380	423,349
65	107,427	3,995,797	14,946	370,951
70	132,273	3,296,606	15,580	272,144
75	142,422	2,316,257	13,782	183,711
80	128,484	1,324,509	11,253	102,582
85	147,419	820,992	11,646	69,665
Total	956,685	104,338,551	146,554	15,000,845

The fraction 0.103 of a year is about 1.2 months or 38 days. Those who die in the first year of life die on average after 38 days. From this information we compute the probability of dying in the first year after birth:

$$\begin{aligned} {}_1q_0 &= \frac{(1)({}_1M_0)}{1 + (1 - {}_1a_0)({}_1M_0)} = \frac{0.019579}{1 + (1 - 0.103284)(0.019579)} \\ &= 0.019241 \end{aligned}$$

It is an easy formula, but one common mistake is to use  ${}_na_x$  instead of  $n - {}_na_x$  in the denominator. The denominator itself is always very close to 1.

For our next interval, we have  $x = 1$  and  $n = 4$ . Half of  $n = 4$  would be 2, but we estimate deaths to fall on average a little earlier than half way through the interval:

$${}_4a_1 = 1.5$$

The difference  $n - {}_4a_1$  is  $4 - 1.5$  or 2.5, the same as half of a five year interval, but here the 2.5 arises from a four-year interval. We proceed with our  ${}_nM_x$  and  ${}_nq_x$  calculations.

$${}_4M_1 = \frac{{}_4D_1}{{}_4K_1} = \frac{965}{1,243,557} = 0.000776$$

$${}_4q_1 = \frac{(4)({}_4M_1)}{1 + (4 - {}_4a_1)({}_4M_1)} = \frac{4 * 0.000776}{1 + 2.5 * 0.000776} = 0.003098$$

A common mistake at this stage is to omit the factor of  $n$  in the numerator. Doing so would spoil every later calculation in the lifetable.

From school age onward, deaths are fairly evenly spread and we rely on the usual approximation:

$${}_na_x = n/2$$

The only other age group requiring special treatment is the final, open-ended interval, for instance, “85 and up” or “100 and up”. An open-ended interval is one that goes on forever. Theoretically, its length  $n$  is infinity, written  $\infty$ . We equate  ${}_{\infty}a_x$  to the reciprocal of the open-ended age-specific death rate:

$${}_{\infty}a_x = 1/({}_{\infty}M_x)$$

We do not need this quantity  ${}_{\infty}a_x$  to compute  ${}_{\infty}q_x$ , because we know that everyone alive at the start of an open-ended interval has to die eventually in the interval, meaning that the probability of dying is 1. This outcome is consistent with our formula. Putting any  ${}_na_x$  equal to the reciprocal of  ${}_nM_x$  gives a probability of 1:

$${}_{\infty}q_x = \frac{(n)({}_nM_x)}{1 + (n - 1/{}_nM_x)({}_nM_x)} = 1$$

We do need  ${}_{\infty}a_x$  to compute  ${}_{\infty}L_x$ , which comes out to equal a simple quotient:

$${}_{\infty}L_x = l_x/({}_{\infty}M_x)$$

For the U.S. African-American males in 1992, we have  ${}_{\infty}M_{85} = 11,646/69,665 = 0.167171$ , so we have  ${}_{\infty}a_{85} = 5.982$  years.

For period life tables, we go on to build up all the other columns from the  ${}_nq_x$  column, not from the  $l_x$  column as we did for cohort life tables. We have just discussed the formulas that lead to  ${}_nq_x$ :

$$\begin{aligned} {}_nM_x &= \frac{{}_nD_x}{{}_nK_x} \\ {}_nq_x &= \frac{(n){}_nM_x}{1 + (n - {}_na_x){}_nM_x} \end{aligned}$$

Given the  ${}_nq_x$ , we next calculate the survivorship column

$$l_{x+n} = l_x(1 - {}_nq_x)$$

From this column, all the rest can be computed by the same procedures as for cohort lifetables:

$${}_nd_x = l_x - l_{x+n}$$



$${}_nL_x = (n)(l_{x+n}) + ({}_na_x)({}_nd_x)$$

When  ${}_na_x$  is equal to  $n/2$ , the formula for life table Person-Years Lived reduces to

$${}_nL_x = (n/2)(l_x + l_{x+n})$$

We go on with the standard formulas for the life table death rate, expected years of life remaining, and expectation of further life:

$$\begin{aligned} {}_nm_x &= \frac{{}_nd_x}{{}_nL_x} \\ T_x &= {}_nL_x + {}_nL_{x+n} + {}_nL_{x+2n} + \dots \\ e_x &= T_x/l_x \\ e_x + x &= x + T_x/l_x \end{aligned}$$

## 7.2 \* Deriving the ${}_nq_x$ formula

We construct a life table from period data by making the cohort age-specific death rates in the life table,  ${}_nm_x$ , equal the period age-specific death rates  ${}_nM_x$ . We derive our formula for the period  ${}_nq_x$  by means of algebra. We start with the definition of  ${}_nm_x$ :

$${}_nm_x = \frac{{}_nd_x}{{}_nL_x}$$

First re-express  ${}_nd_x$  and  ${}_nL_x$  in terms of  $l_x$ :

$$\begin{aligned} \frac{{}_nd_x}{{}_nL_x} &= \frac{l_x - l_{x+n}}{(n)l_{x+n} + ({}_na_x)(l_x - l_{x+n})} \\ &= \frac{l_x - l_x(1 - {}_nq_x)}{(n)l_x(1 - {}_nq_x) + ({}_na_x)(l_x - l_x(1 - {}_nq_x))} \\ &= \frac{(l_x)(1 - 1 + {}_nq_x)}{(l_x)(n - (n)({}_nq_x) + ({}_na_x)({}_nq_x))} \end{aligned}$$

Thus

$${}_nm_x = \frac{{}_nq_x}{n - (n - {}_na_x)({}_nq_x)}$$

Now multiply both sides by the denominator on the right.

$$({}_nm_x)(n) - ({}_nm_x)(n - {}_na_x)({}_nq_x) = {}_nq_x$$

Collect terms in  ${}_nq_x$ :

$$({}_nq_x)(1 + (n - {}_na_x)({}_nm_x)) = ({}_nm_x)(n)$$

Now substitute  ${}_nM_x$  for  ${}_nm_x$  to achieve the desired formula:

$${}_nq_x = \frac{(n)({}_nM_x)}{1 + (n - {}_na_x)({}_nM_x)}$$

When we have a period covering one year of time, there is an equivalent formula. Since  ${}_nM_x = {}_nD_x / {}_nK_x$ , we can substitute the quotient of deaths and mid-year population and multiply both numerator and denominator by  ${}_nK_x$  to obtain

$${}_nq_x = \frac{(n)({}_nD_x)}{{}_nK_x + (n - {}_na_x)({}_nD_x)}$$

This formula saves the step of calculating  ${}_nM_x$ , but the formula in terms of  ${}_nM_x$  has the advantage that the denominator is known to be close to 1 and therefore can be inspected easily during the computation.

Our next goal is to come to an intuitive understanding of why the denominator in this formula is what it is. We consider a period one year long, and fix attention on a single  $n$ -year-wide age group. Our deaths  ${}_nD_x$  come from *one* year of time and  $n$  years of age. We make the basic assumption that the changes in birth numbers within  $n$ -year intervals can be ignored. In other words, we approximate the graph of births over time as a step function which is flat within  $n$ -year intervals and changes between them. Then all the one-year birth cohorts contributing to a single  $n$ -year-wide age group can be regarded as having the same initial size, which we will take as our radix  $l_0$ . This gives enough basic information to reconstruct the formula with heuristic arguments.

Now  ${}_nq_x$  is the probability of dying between  $x$  and  $x+n$  conditional on surviving to age  $x$ , that is, conditional on belonging

to the  $l_x$  members of one's cohort who do survive to age  $x$ . So the count of people at risk for  ${}_nq_x$  in the  $n$  cohorts contributing to the age group is  $nl_x$ . But the number of people in the period population at risk of dying from whom the  ${}_nD_x$  deaths come is  ${}_nK_x$ . Let us focus on the age group of 20 to 25-year-olds. Under our assumption of constant initial cohort size within intervals,  ${}_5K_{20}$  includes  $l_{20}$  20-year-olds, but only  $l_{21}$  21-year-olds and only  $l_{24}$  24-year-olds. The cohort from which the 24-year-olds come has been depleted by previous deaths over the age range from 20 to 24. It is therefore smaller than it was at age 20 when it was at risk of the whole rate  ${}_5q_{20}$ . To work back from the period population at risk  ${}_5K_{20}$  to the original cohort at risk  $5l_{20}$  we have to add back an allowance for the depletion of the older, earlier cohorts represented in the period population.

If we draw a graph of population by single years of age, and make it flat in one-year intervals, so that it resembles a step function, then the curve itself follows the  $l_x$  curve. The period population  ${}_5K_{20}$  is the area under this curve between  $x = 20$  and  $x = 25$ . To visualize the process, we can write in names in each square under the curve.

One year later, all the people have aged one year, or moved one step to the right on our diagram. The shape of the picture is exactly the same, since we have assumed unchanging rates and constant initial cohort size. But some of the people who were below the curve a year ago have moved above the curve by moving a year to the right. They are the people who die in the year. We see that the deaths in one year of time from these five years of age equals the size of the drop in the curve from the left-hand side to the right-hand side.

The whole area  $(5)(l_{20})$  is made up of the area under the curve,  ${}_5K_{20}$ , plus the area above it, which is roughly triangular in shape. The vertical dimension of the triangle is  ${}_5D_x$ . The horizontal dimension depends on how early or late in the interval the deaths occur. If deaths are early, the triangular region is fat and big. If they occur late, the region is thin. The average horizontal dimension of the triangular region is the average number of years

not-lived in the interval by those dying in the interval, which is  $n - {}_n a_x$ . To get the full rectangle, we have to add together  ${}_n K_x$  and  $(n - {}_n a_x){}_n D_x$ .

Then

$${}_n q_x = \frac{({}_n d_x)}{({}_n l_x)} = \frac{({}_n D_x)}{{}_n K_x + (n - {}_n a_x){}_n D_x}$$

### 7.3 The 1660s and Laws of Mortality

The lifetables we have been constructing so far have been lifetables based on complete empirical data. We have information for all ages on deaths and populations at risk, so we can base our lifetable entries for each age on data for that age. For many applications, that is not the case, and we only have partial information from which to build our table. We may only have information about certain ages, or the information for certain ages may be much more reliable than that for other ages, or we may only have general ideas about the level of mortality but we may need a full lifetable to make projections or calculations. In this case, we seek to fill in the missing parts of a lifetable from whatever empirical information we do have using mathematical formulas. A lifetable constructed from mathematical formulas in this way is called a model lifetable.

The first of all lifetables, made by the Englishman John Graunt in 1662, was in fact a model lifetable. Graunt used two pieces of empirical information based on his observations of survivorship around him, and he filled in the rest of the lifetable with a formula. Now, you can see that this practice is only going to work if mortality rates at different ages tend to obey a formula. Is the pattern of mortality sufficiently similar in different societies and environments that a few pieces of information are enough to tell you the whole story? Now, model lifetables have become somewhat disreputable among academic demographers in the last few decades, because of overuse and misuse. But they are the single

demographic method most widely used by practical demographers and non-demographers around the world. For many years one could practically define the community of users of demography by the presence of Coale and Demeny's book *New Model Lifetables and Stable Populations* on the bookshelf.

To understand the fascination and the limitations of model lifetables, we should go back to the year of the first model lifetable, set out in the *Natural and Political Observations Mentioned in a following Index and made upon the Bills of Mortality* by John Graunt, Citizen of London. The year of Graunt's book is 1662. Two years earlier, in 1660, the Royal Society had been founded by King Charles II. Four years later, the year 1666 was called the "*Annus Mirabilis*" or "Marvelous Year".

The Great Fire of London occurred in 1666, following an outbreak of plague which was the last of the great periodic outbreaks of plague in London. These periodic outbreaks of plague had in fact been the motive for the collection of John Graunt's data source, the Bills of Mortality. London merchants wanted early warning of outbreaks of plague, and they paid a subscription to have the numbers of burials collected from the parish registers of London and published each week. When they saw a sharp rise in burials, they stopped adding to their inventories, shut their shops, and took their families out into the country. It was these weekly reports, collected (with interruptions) since 1592, that formed John Graunt's data for his study, the first true work of demographic research.

During the plague, the universities closed, and Isaac Newton, who was a student at Trinity College, Cambridge, was sent home to Lincolnshire. As the story goes, it was for this reason that he was sitting in his father's garden when an apple fell on his head, and he said to himself, "F equals GMm over r-squared." This is the picture we want to have in our minds when we think about the origins of ideas of model lifetables. John Graunt was working during the years when Newton and his friends were developing the theory of universal gravitation and the science of mechanics. They were finding laws given by simple mathematical formulas

which would describe phenomena as diverse as apples falling from trees, moons circling Saturn, and tides rising and falling in the ocean. The formulas that Newton discovered about physics were taken to be the rules that God had programmed into the world to make it work when he created it. In those days, there was no usual distinction made between the physical and the human world. The planets and the human body were both seen as part of the same creation. Thus the successes of simple mathematical formulas in physics suggested that there might be simple mathematical formulas with which God had also programmed the processes of life and death in the human body and the social world. A world view of this kind leads one naturally to look for simple formulas for lifetables, laws of life and death which parallel laws of falling bodies. Model lifetables are still called “Laws of Mortality”, dating from these days.

As time has gone on, the unity of creation has become less a part of our intellectual credo, and the hope for simple mathematical formulas for human processes has waned with the discovery of more and more diversity as empirical evidence has accumulated. Nonetheless, the basic motive to hope for simple formulas for lifetables is deeply ingrained, and the possibility that there are partly genetic programs for longevity which can be described by mathematical formulas for extreme old-age mortality has recently had a resurgence. It is partly a matter of philosophical preferences whether one resists the temptation to reduce the variety of empirical variation among mortality patterns to a set of mathematical curves or whether one feels that with mathematical formulas one is discovering something real about the way human beings are put together.

## 7.4 Graunt’s Model Lifetable

Graunt describes his lifetable in his Section 9:

Whereas we have found, that of 100 quick Concep-

tions about 36 of them die before they be six years old,  
and that perhaps but one survivith 76...

What is a “quick Conception”? The “quicken” of the fetus, when it first begins to move in the womb, is taken in medical and religious tradition, as the start of life. “Quick” is derived from “kviker” in old Norse, which is ultimately connected to the Latin “vivus” for “alive”. Here Graunt is making no strong distinction between quick conceptions and live births. The number 100 is the initial size of his cohort, measured either from quickening or birth, in other words, the radix  $l_0$  for his lifetable.

Notice carefully the two pieces of empirical information on the basis of which Graunt filled in the whole set of lifetable entries. With his wide experience, Graunt chose his empirical quantities strategically. One – the number dying before six years old, or equivalently the number surviving to six years – reflected the severity of infant mortality. The other – the number surviving to 76 – reflected late adult lifespan. These are the two aspects of mortality which vary most from time to time and place to place. They were good choices to make, if we want to know two things that will help us know most about what is special about the mortality in a particular case, so that we can fill in the rest of the table with general features common to all cases.

Graunt describes his formula for filling in the missing entries as follows:

... we, having seven Decads between six and 76,  
we sought six mean proportional numbers between 64,  
the remainder, living at six years, and the one, which  
survives 76, and find that the numbers following are  
practically near enough to the truth; for men do not  
die in exact Proportions, nor in Fractions: from whence  
arises this table following, Viz. ...

So we have a form of words to describe the formula and we also have the numbers that come out of the formula in Graunt's

special case. He gives them both as  ${}_{10}d_x$  values and as  $l_x$  values. Let us consider the  $l_x$  values:

x	0	6	16	26	36	46	56	66	76	80
$l_x$	100	64	40	25	15	10	6	3	1	0

What does Graunt mean by “six mean proportional numbers”? Notice that the ratios of successive  $l_x$  values are quite stable, or, what is the same thing, the  ${}_{10}q_x$  values between 6 and 76 are nearly constant. In fact we have

$x$	$l_x$	${}_nq_x$
0	100	0.360
6	64	0.375
16	40	0.375
26	25	0.360
36	16	0.375
46	10	0.400
56	6	0.500
66	3	0.667
76	1	1.000
80	0	

Graunt warns us that he is avoiding fractions and only applying his rule in a way that produces whole numbers. In a fascinating piece of historical detective work, the French demographer Hervé LeBras has worked a rule by which Graunt very likely arrived at his whole numbers. For our purposes, however, the chief point is that Graunt was coming close to assuming that  ${}_{10}q_x$  is constant between the ages of 6 and 76. In other words, he was (nearly) assuming the same rule to build his table as the rule we use to calculate  ${}_nq_x$  in the absence of information.

In Graunt’s table for London in 1662, the table has  ${}_nq_x = .375$  or so. But Graunt’s rule can be used for other cases. With any pair of empirical values for  $l_6$  and  $l_{76}$ , we can fill in the rest of the entries in the table by Graunt’s rule. These two quantities are called *parameters*. Parameters are values which fix the choice of



a particular lifetable from among the tables in the model family defined by the same formula.

Graunt's family has two parameters, which, by tradition we call by the Greek letters  $\alpha$ , "alpha", and  $\beta$ , "beta." Here  $\alpha = l_6$  and  $\beta = l_{76}$ . What is  $_{10}q_x$  in terms of alpha and beta? We must have

$$\frac{\beta}{\alpha} = \frac{l_{76}}{l_6} = (1 - {}_{10}q_x)^7$$

so

$${}_{10}q_x = 1 - (\beta/\alpha)^{1/7}$$

Consider choosing  $\alpha$  and  $\beta$  and finding a model lifetable from Graunt's family to apply to United States women in 1994. We have  $\alpha = l_6 \approx 0.988$  and  $\beta = l_{76} \approx 0.720$ . What would  $l_{26}$  be according to the model lifetable? We have

$$l_{26} = \alpha(\beta/\alpha)^{2/7} = 0.988 * (720/988)^{2/7}$$

There are a few more points to notice about Graunt's table. Note that he insisted on whole numbers, since he was not comfortable with a notion that the entries in a table represented means or statistical expectations. It takes time to become comfortable with such a notion; we often feel ourselves that the counts projected by a Leslie matrix "ought" by rights to be whole numbers and not fractions. Note also that there is more approximation in Graunt's printed table for high ages than for low ones. This is due to the smaller populations at risk. One unit is a smaller fraction of all survivors at higher ages. Even without rounding, this pattern of higher errors with higher ages is typical of lifetable data. As the count of people at risk of dying drops, the proportional errors in the estimates tends to increase.

Note also that Graunt did not end his table with an open-ended interval, but with  $l_{80} = 0$ . This may have been based on observation. But it is more likely derived from the limit of four-score years cited in the 90th Psalm.

Graunt's table is a good example with which to begin the study of model lifetables, because it shows the essential features of a

model lifetable in a straightforward way. In this section, we introduce some terminology for thinking about model lifetables. There are no general terms in use in the demographic literature for most of these ingredients, so some new terms are introduced here.

Whenever we construct a lifetable, we lead off with the calculation of some column, and then express the other columns in terms of this leading column. For cohort lifetables, we led off with  $l_x$  and calculated other columns from it. For period lifetables, we led off with  ${}_nq_x$  and calculated other columns from that one. We call the column with which we choose to lead off the construction of a model lifetable the “leader” or “leading column”. The model formula is a formula for the leading column. The other columns are calculated from it by the usual techniques. For Graunt’s model tables, the leading column is  ${}_nq_x$ . Many model families have this same leading column. The Brass family which we shall study has  $l_x$  for its leading column. The Lee-Carter family has  ${}_nm_x$ .

We have already discussed the parameters of the lifetable. The next important ingredient is the “formula”, the mathematical expression for computing the leading column from the parameters.

Graunt’s formula is all symbols. There are no special constants in the formula which depend on age. But for many models there are sets of age-specific constant which enter into the formula and help determine what the different lifetables in the model family have in common. These constants define a kind of “backbone” or “spine” for the table, while the parameters let the model bend to fit particular circumstances. We call such sets of constants the “spine” of the table. Spines for the families we are studying are found toward the end of the chapter in Tables 7.4 and 7.5.

We shall study several examples of model lifetable families. Before we do, a word is in order about the various uses of model lifetables. Five purposes are particularly worth mentioning:

1. To extrapolate a whole lifetable from data at certain ages.
2. To identify suspicious data which clash with the expectations expressed in the model.

3. To supply a standardized set of mortality assumptions for population projection.
4. To develop formulas for indirect estimators of demographic rates which rely on “usual” or “typical” mortality patterns.
5. To recover rates of births and deaths from counts of births and deaths under model-table assumptions with the method of “inverse projection”.

## 7.5 Coale-Demeny Model Lifetables

The most widely-used model lifetables are those first published by Ansley Coale and Paul Demeny in 1966 and extended from age 85 to age 100 in Coale and Demeny (1983). The volume is called *Regional Model Life Tables and Stable Populations*.

Like John Graunt’s model lifetable family, the Coale-Demeny model is essentially a two-parameter family. There is a parameter for mortality level and a parameter for mortality curve shape. But the parameter for shape takes only four values. It indexes four subfamilies, which we could call shapes #1, #2, #3, and #4.

Coale and Demeny’s inspiration was to call the shapes not #1, #2, #3, and #4 but “North”, “South”, “East”, and “West”. These names conveyed to their readers a comfortable familiarity. We all know south from north and east from west. When readers came to the Coale-Demeny tables with data from a particular country, they came, often, with a prior notion of which shape-family was needed. China is east. Zimbabwe is south. Readers naturally expected to go to the East pages for China or the South pages for Zimbabwe.

Those who read the fine print in Coale and Demeny’s book found out that these expectations are wrong. East here is not like China but like Austria. South is not like Zimbabwe but like Italy. North is like Iceland, well enough, but West is like Israel and White South Africa. Coale and Demeny’s database for

building these models included no Asiatic populations, no black African populations, no pre-Victorian historical populations, and no hunter-gatherer populations.

In more detail, the Coale and Demeny tables are based on 326 empirical lifetables taken almost exclusively from Europe. The “East” family is based on 31 tables including Austria, Germany, Hungary, North Italy, Czechoslovakia, and Poland. Asia is out of the picture entirely. The “North” family is based on tables from Sweden 1851-1890, Norway 1856-80 and 1946-55, and Iceland in 1950. The “South” family is based on tables including Italy, Portugal 1919-1958, and Spain. Latin America and Africa have no role in the “South” family. The “West” family is a grab-bag of tables that did not come into the other families, and includes tables from France, Israel, New Zealand, and South Africa among others. Not only were there no tables from Latin America and most of Africa or Asia, but there were no tables from earlier historical populations, and certainly no prehistoric tables.

Coale and Demeny did not include other tables because high-quality tables for Asia, Africa, Latin America, or earlier history and prehistory were not available at the time. But the limitations of the data base were only obvious to users who read the fine print, and the result has been a lot of foolish demography. There has been a tendency for patterns of mortality that did not fit the Coale-Demeny model to be taken for “bad data”. When one realizes that no African data except data for white South Africans went into the construction of the Coale Demeny models, it is perhaps not surprising to find that the models often do not capture typical African mortality patterns. Certainly, there are flaws in much data from Africa, but the fact that the data do not agree with the Coale-Demeny models is not a good basis for dismissing the African data out of hand.

For each of the four choices for the shape family, that is, north, south, east, or west, there is a one-parameter family of tables with different levels. These levels are labeled from 1 to 25, and these correspond to values for women of expectation of life at birth which go up from 20 in increments of 2.5 years. Thus the female table

for level 3 has a value of  $e_0$  of 25. A higher mortality “Level” in the Coale-Demeny tables means a lower level of mortality. This terminology is extremely unfortunate and has generated confusion over many years. The male table for each level does not have the same value of  $e_0$  as the female table, but rather a value chosen by a complicated formula which will be explained below.

The published Coale-Demeny tables combine two mathematical models called the A-B and the A-prime-B-prime models. The formula for the A-B model is

$${}_nq_x = A_x + B_x\epsilon$$

and the formula for the A-prime-B-prime model is

$${}_nq_x = (1/10000) * \exp(\log(10) * (A'_x + B'_x\epsilon))$$

The constants are the “spine” of the table, the age-specific effects. These constants were chosen by a series of regressions, one for each age-group, regressing the value of  ${}_nq_x$  for each country or period against the corresponding value of  $\epsilon = e_{10}$ . This fact about the construction is confusing, because the values of  $\epsilon$  in the constructed Coale-Demeny tables do not end up corresponding exactly to the values of  $e_{10}$  even though they were constructed on this basis. The spines are given in Tables 7.4 and 7.5.

The final tables are chosen as an amalgamation of the two systems. In the middle range of ages where the A-B system value for  ${}_nq_x$  is larger than the A-prime-B-prime system value, the model uses the average of the two values. At ages for which the A-B values are smaller but the slope of the A-B values as a function of age are higher, the model uses the A-B values. Elsewhere it uses the A-prime-B-prime values. From a mathematical point of view, the result is something of a hodge-podge, but the tables remain useful for practical purposes. Examples of Coale-Demeny lifetables from the West model family are shown for Level 15 in Tables 7.2 and 7.3.

Table 7.2: Females in Coale-Demeny West Lifetables Level 15

Age(x)	1000q(x)	d(x)	1000m(x)	l(x)	L(x)	e(x)	Age(x)
0	93.94	9394	100.23	100000	93723	55.00	0
1	50.44	4570	13.04	90606	350414	59.67	1
5	15.84	1363	3.20	86037	426435	58.76	5
10	12.27	1039	2.47	84673	420874	54.67	10
15	17.52	1465	3.53	83635	414657	50.32	15
20	22.85	1877	4.62	82170	406343	46.17	20
25	26.07	2094	5.28	80292	396437	42.19	25
30	29.59	2314	6.00	78199	385440	38.25	30
35	33.56	2546	6.82	75885	373312	34.34	35
40	38.38	2815	7.82	73338	359935	30.44	40
45	45.93	3239	9.39	70523	344842	26.55	45
50	62.12	4180	12.81	67284	326389	22.70	50
55	84.21	5314	17.55	63104	302768	19.04	55
60	124.95	7221	26.58	57790	271621	15.55	60
65	181.22	9164	39.70	50569	230853	12.40	65
70	272.06	11265	62.59	41405	179991	9.56	70
75	395.30	11915	98.54	30140	120916	7.17	75
80	548.56	9998	154.26	18226	64811	5.22	80
85	719.76	5922	242.45	8228	24427	3.68	85
90	869.29	2004	377.83	2306	5305	2.52	90
95	1000.00	301	607.02	301	496	1.69	95

Table 7.3: Males in Coale-Demeny West Lifetables Level 15

Age(x)	1000q(x)	d(x)	1000m(x)	l(x)	L(x)	e(x)	Age(x)
0	111.36	11136	120.05	100000	92761	51.80	0
1	51.06	4537	13.21	88864	343481	57.24	1
5	15.89	1340	3.21	84327	417949	56.25	5
10	11.68	969	2.35	82987	412608	52.12	10
15	17.65	1447	3.56	82018	406615	47.71	15
20	25.03	2017	5.07	80570	398012	43.52	20
25	27.16	2133	5.50	78554	387648	39.57	25
30	31.01	2370	6.30	76420	376413	35.60	30
35	37.27	2760	7.59	74050	363629	31.66	35
40	47.08	3356	9.63	71291	348399	27.78	40
45	60.39	4102	12.44	67935	329827	24.03	45
50	82.22	5248	17.12	63832	306565	20.40	50
55	111.97	6560	23.67	58584	277177	17.00	55
60	159.48	8297	34.54	52024	240210	13.81	60
65	223.76	9784	50.14	43728	195156	10.94	65
70	316.95	10758	74.76	33943	143897	8.35	70
75	489.54	11350	129.64	23185	87550	6.01	75
80	638.01	7551	195.93	11835	38539	4.38	80
85	794.04	3402	298.93	4284	11380	3.10	85
90	914.31	807	452.62	882	1782	2.14	90
95	1000.00	76	696.12	76	109	1.46	95

## 7.6 \*Lee-Carter Models

In the early 1990s, Ronald Lee and Larry Carter (1992) introduced what is effectively a new system of model lifetables in the course of proposals for forecasting mortality. Lee and Carter were primarily interested in modeling the changes over time in the mortality of a single country or population rather than the changes from country to country. Each country therefore has a separate set of age-specific constants or spines determining the model for that country. There is a single parameter  $k$ . A given value of  $k$  determines the lifetable for a particular time. By forecasting  $k$  as a function of time into the future, we can forecast the whole lifetable.

The leader or leading entry of the Lee-Carter system is the  ${}_n m_x$  column of the lifetable. The formula for the Lee-Carter system is

$${}_n m_x = e^{a_x + k b_x}$$

Thus the logarithm of the lifetable death rate for any age group is a linear function of the single parameter  $k$ , and the intercept and slope of this linear function are the age-specific (spine) constants  $a_x$  and  $b_x$ . The  $a_x$  constants determine the initial level of mortality corresponding to  $k = 0$  and the  $b_x$  constants determine the rate of change of the lifetable death rates on a logarithmic scale.

## 7.7 Brass Relational Logit Models

### 7.7.1 The Brass System

The Brass Relational System of life tables was introduced by William Brass (1974). The leading column of Brass's model tables is given by a formula for  $l_x$ , in contrast to the Coale and Demeny Model, which has a formula for  ${}_n q_x$ . Brass's formula is

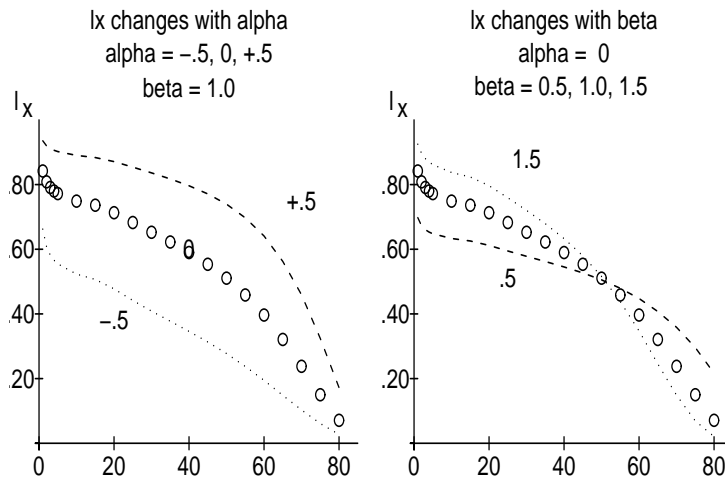
$$l_x = \frac{1}{1 + \exp(-2\alpha - 2\beta Y_x)}$$



Here the “spine” of the system,  $Y_x$ , are constants which determine the age-specific shape of the model, playing the role of the  $A_x$  and  $B_x$  in the Coale-Demeny System. These constants (including modifications under age 15 due to Ewbank *et al.* (1983)) are given in Table 7.4. They are the same for males as for females and they are repeated in Table 7.5.

The parameters, chosen to fit empirical data or expectations, are  $\alpha$  and  $\beta$ , which play the role of  $e_{10}^f$  and N-S-E-W in the Coale-Demeny System. In fact, like  $e_{10}^f$ , the  $\alpha$  parameter is a level parameter. Positive  $\alpha$  means higher survivorship, weaker mortality, as shown on the left-hand side of Figure 7.1.

Figure 7.1: Shapes of Brass Model Lifetables



The  $\beta$  parameter is a shape parameter which determines the balance between childhood and old-age mortality, like the Coale and Demeny NSEW choice.  $\beta > 1$  means better early survivorship compared to later adult survivorship. On the right-hand side of Figure 7.1 we see the effects. A crossover is evident. Higher infant mortality for  $\beta = 0.5$  than for  $\beta = 1$  accompanies lower adult mortality. Crossovers are of great interest in demography. A crossover is evident in data for U.S. black males compared to

white males, and there is an ongoing debate as to whether it is due to poor age reporting or an effect of the selection of less frail individuals into older age groups.

Let us work out two cases of the life table. First consider taking  $\alpha = 0$  and  $\beta = 1$ . Then

$$l_5 = \frac{1}{1 + \exp(-0 - 2(1)(0.6100))} = 0.772064$$

This gives Brass's General Standard Life Table. Now consider taking  $\alpha = +0.40$  and  $\beta = 0.90$ . We have

$$l_5 = \frac{1}{1 + \exp(-(2)(0.40) - (2)(0.90)(0.6100))} = 0.869665$$

### 7.7.2 Estimating Brass's alpha and beta

We could fit the parameters of the Brass system by generating sets of  $l_x$  values for different parameter values and hunting combinations which matched observed tables. That is the usual process for fitting Coale and Demeny tables. But the advantage of the Brass system is that there are direct calculations to find good choices of  $\alpha$  and  $\beta$  to match a set of observed  $l_x$ . The strategy depends on solving for  $\alpha + \beta Y_x$  as a function of  $l_x$  :

$$l_x = \frac{1}{1 + \exp(-2\alpha - 2\beta Y_x)}$$

$$l_x + l_x \exp(-2\alpha - 2\beta Y_x) = 1$$

$$\exp(-2\alpha - 2\beta Y_x) = \frac{1 - l_x}{l_x}$$

$$\alpha + \beta Y_x = -(1/2) \log\left(\frac{1 - l_x}{l_x}\right) = (1/2) \log\left(\frac{l_x}{1 - l_x}\right)$$

The function on the right has a name, the “logit”, like cosines, exponentials, tangents. Not all authors have the  $1/2$  in the definition. The use of this logit function is the reason that Brass's system is called a “logit” system, though it is important to remember that

the model for the  $l_x$  is not given by the logit function itself. The logit function occurs when we solve back in terms of  $l_x$ .

Now we see that if we plot  $(1/2) \log(l_x/(1-l_x))$  on the vertical axis versus  $Y_x$  on the horizontal axis,  $\alpha$  should be the intercept and  $\beta$  the slope of a straight line going through the scatter of points (one point for each age group).

We can use the technique of linear regression to find the intercept and slope if we are familiar with it. Or we can plot the points on a graph and use a ruler to draw in a line chosen by eye. The intercept is where the graph crosses the vertical axis. The slope is the rise over the run. This is particularly simple if we base our fit on only two age groups, for there is a unique straight line going through any two points. Suppose, for instance, that we have partial data on Afghan women for 2005 that suggest that about 70% survive to age 20 and about 40% to age 60. We are interested in guessing the extent of infant mortality.

First we compute the logit values  $Z_x = (1/2) \log(l_x/(1-l_x))$  for the vertical axis. Then we write down the values of  $Y_{20}$  and  $Y_{60}$  from Table 7.4:

$x$	$l_x(obs.)$	$Z_x(l_x)$	$Y_x(table)$
20	.70	+.424	+.455
60	.40	-.203	-.210

Now we plot  $Z_x$  against  $Y_x$  and find the line through the points. The slope  $\beta$  equals  $(.424 - (-.203)) / (.455 - (-.210)) = 0.943$ . The intercept  $\alpha$  is obtained by multiplying the slope times the mean of  $Y_x$  and subtracting this quantity from the mean of  $Z_x$ . We find  $\alpha = 0.1105 - (0.943)(0.1225) = -0.005$ , quite close to zero. Now we can use these parameter estimates to predict the value of  $l_1$ . We find from Table 7.4 that  $Y_1 = 0.83659$ . Thus

$$l_1 = \frac{1}{1 + \exp(-2 * (-0.005) - 2 * 0.943 * 0.83659)} = 0.8275$$

Our rough estimates of adult survival suggest that about 83% of Afghan girls are surviving their first year, and about 17% are dying.

### 7.7.3 Creating A Brass Standard

Up to now, we have considered Brass's family very much like the Coale-Demeny family or any other family specified by formulas with constants like the  $Y_x$ . But there is an important difference. In the Coale-Demeny system, no particular level is special. Level 16 is no more special than level 7. In the Brass system, however,  $\alpha = 0$  and  $\beta = 1$  picks out a special table, Brass's General Standard. If we had started with the  $l_x$  values for this table, we could have found the  $Y_x$  values by solving as before for  $\alpha + \beta Y_x$  which now equals  $Y_x$ .

Now, we could do this starting with some other life table  $l_x$  column, instead of Brass's favorite life table. We could pick our own favorite life table, make this the standard, and solve for the  $Y_x$  values which would make this table correspond to  $\alpha = 0$  and  $\beta = 1$ . In this way, Brass's system provides a way of relating all life tables to any one life table from which we build our standard. For this reason, the system is called a system of "relational model tables".

A particularly important application is to use the present-day life table of a country as the standard, and then find parameter values to match each of the previous life tables. In this way we see the progression of  $\alpha$  and  $\beta$  toward zero and one over time.

Table 7.4: Spines for Model Lifetables: Females

Age	Brass	Lee-Carter		Coale-Demeny		Model West	
x	$Y_x$	$a_x$	$b_x$	$A_x$	$B_x$	$A'_x$	$B'_x$
0	NA	-3.92433	0.078257	0.53774	-0.008044	5.8992	-0.05406
1	0.83659	-6.93213	0.093867	0.39368	-0.006162	7.4576	-0.08834
5	0.61005	-7.77580	0.079217	0.10927	-0.001686	6.2018	-0.07410
10	0.54619	-7.90922	0.073205	0.08548	-0.001320	5.9627	-0.07181
15	0.51316	-7.31074	0.049845	0.10979	-0.001672	5.9335	-0.06812
20	0.45500	-7.13160	0.065845	0.13580	-0.002051	5.9271	-0.06577
25	0.38287	-6.99188	0.071709	0.15134	-0.002276	5.8145	-0.06262
30	0.31502	-6.72443	0.067089	0.17032	-0.002556	5.6578	-0.05875
35	0.24966	-6.36938	0.059199	0.18464	-0.002745	5.3632	-0.05232
40	0.18157	-5.95403	0.050174	0.19390	-0.002828	4.9600	-0.04380
45	0.10741	-5.52067	0.042689	0.20138	-0.002831	4.5275	-0.03436
50	0.02120	-5.09810	0.038014	0.25350	-0.003487	4.4244	-0.03004
55	-0.08319	-4.68188	0.036358	0.31002	-0.004118	4.3131	-0.02554
60	-0.21003	-4.24738	0.035323	0.43445	-0.005646	4.3439	-0.02295
65	-0.37459	-3.80469	0.358670	0.53481	-0.006460	4.2229	-0.01773
70	-0.58184	-3.31377	0.036197	0.69394	-0.007713	4.1838	-0.01376
75	-0.86730	-2.81015	0.036219	0.84589	-0.008239	4.1294	-0.00978
80	-1.28571	-2.31577	0.031089	NA	NA	NA	NA
85	NA	-1.68808	0.019838	NA	NA	NA	NA

Table 7.5: Spines for Model Lifetables : Males

Age	Brass	Lee-Carter		Coale-Demeny		ModelWest	
x	$Y_x$	$a_x$	$b_x$	$A_x$	$B_x$	$A'_x$	$B'_x$
0	9.99999	-3.65769	0.119426	0.63726	-0.009958	5.8061	-0.05338
1	0.83659	-6.73867	0.134723	0.40548	-0.006653	7.1062	-0.08559
5	0.61005	-7.41419	0.109848	0.10393	-0.001662	5.4472	-0.06295
10	0.54619	-7.41771	0.097766	0.07435	-0.001183	5.0654	-0.05817
15	0.51316	-6.52649	0.031743	0.09880	-0.001539	4.8700	-0.05070
20	0.45500	-6.21919	0.036692	0.14009	-0.002183	5.0677	-0.05156
25	0.38287	-6.28896	0.045556	0.15785	-0.002479	5.2660	-0.05471
30	0.31502	-6.17578	0.055107	0.18260	-0.002875	5.3438	-0.05511
35	0.24966	-5.86428	0.057766	0.21175	-0.003312	5.2792	-0.05229
40	0.18157	-5.43656	0.055230	0.25049	-0.003864	5.0415	-0.04573
45	0.10741	-4.96639	0.047765	0.27894	-0.004158	4.6666	-0.03637
50	0.02120	-4.49716	0.038602	0.33729	-0.004856	4.4506	-0.02961
55	-0.08319	-4.05596	0.032368	0.38425	-0.005190	4.2202	-0.02256
60	-0.21003	-3.64113	0.025959	0.48968	-0.006300	4.1851	-0.01891
65	-0.37459	-3.25210	0.022699	0.59565	-0.007101	4.1249	-0.01491
70	-0.58184	-2.85550	0.020672	0.73085	-0.007911	4.1051	-0.01161
75	-0.86730	-2.44076	0.024456	0.98976	-0.008695	4.1133	-0.00895
80	-1.28571	-2.03691	0.025078	NA	NA	NA	NA
85	-9.99999	-1.55980	0.018542	NA	NA	NA	NA

## Highlights of Chapter Seven:

Concept of a Model Lifetable

Parameters, Formula, Spines

John Graunt's Lifetable

Coale-Demeny Model Lifetables

Brass Relational Model Tables

## Key Formulas

$$\begin{aligned}
 {}_nM_x &= \frac{{}_nD_x}{({}_nK_x)(T)} \\
 {}_nq_x &= \frac{({}_n)({}_nM_x)}{1 + (n - {}_na_x)({}_nM_x)} \\
 \text{Brass } l_x &= \frac{1}{1 + \exp(-2\alpha - 2\beta Y_x)} \\
 \text{Brass } \alpha + \beta Y_x &= (1/2) \log(l_x/(1 - l_x))
 \end{aligned}$$

## Exercises for Chapter Seven

1. Consider the counts for African-American male deaths and age-group sizes in the Table 7.1. Calculate the expectation of life at birth from these data.
2. Suppose you believe that deaths under age 1 and over age 85 in the data for Question 1 are underreported, and the counts on the sheet need to be raised by 10 percent of the given values to approximate the true values. By what percentage would the expectation of life at birth, as calculated directly from the given data, be changed if you substituted the corrected “true” death counts for the given death counts? Be sure to express your answer

as a percentage of the original unadjusted expectation of life. (This problem does not lend itself to shortcuts.)

3. In the Austrian census of 1965 there were 65,607 boys under the age of one year and 63,083 girls. Deaths of boy babies less than a year old totaled 2,056 and deaths of girl babies totaled 1,617. In the country then called Ceylon and now called Sri Lanka at mid-year 1962, the infants under the age of one numbered 229,000 boys and 224,000 girls. During the year, 10,561 boys and 8,581 girls under the age of one were reported to have died. What are the ratios of  ${}_1q_0$  for males to  ${}_1q_0$  for females in Austria and in Ceylon ?

4. From the Coale and Demeny Model West Life Tables for Level 15, calculate the first part of the combined-sex life table implied by the separate male and female life tables down to age 10.

5. An estimated lifetable for Chinese women in the early 1990s is a reasonable fit to a Brass model at later ages, with parameters  $\alpha = 1.25$  and  $\beta = 0.93$ . Estimate the proportion of women dying before age 60 and the lifetable person-years lived between 60 and 70.

6. The United States Internal Revenue Service publishes a lifetable called Table 90CM for calculations affecting gift taxes and inheritance taxes. The table currently in effect has  $l_{55} = 0.89658$  and  $l_{80} = 0.47084$ . Fit a Brass model using these two pieces of information. What are your estimates of  $\alpha$  and  $\beta$ ? What value of  ${}_1q_0$  would you predict with the Brass model you have fitted?

7. In the Internal Revenue Service lifetable from Table 90CM, we have  $l_6 = 0.98850$  and  $l_{76} = 0.57955$ . If John Graunt had fitted his model table to this modern data, what would prediction of  $l_{55}$  have been?

8. Accurate full lifetables for contemporary Bangladesh are not



available. However, about 89 percent of all babies are now surviving to the age of 1.0, and about 75 percent of women survive to age 35. Find values of the Brass “alpha” and “beta” parameters to make the Brass model fit these two  $l_x$  values.

9. Use your estimates from Question 6 to predict the proportion of women surviving to age 65 in Bangladesh today.

\*10. Show by your own algebraic manipulations that the formula

$${}_nq_x = \frac{({}_n)({}_nM_x)}{1 + (n - {}_na_x){}_nM_x}$$

implies

$${}_nM_x = {}_nm_x$$

\*11. Consider the 1962 data from Ceylon for boys in Question 3. Calculate two estimates of  ${}_1q_0$ , one by the usual period life table formula shown in Question 8 and the other by assuming that the hazard rate  $h_x$  for ages between zero and one equals the observed value of  ${}_1M_0$ . Is there any value of  ${}_1a_0$  which would make these estimates agree? (For extremely old ages, setting the hazard rate equal to  ${}_nM_x$  often gives better values for  ${}_nq_x$  than the usual formula with the unrealistic value  ${}_na_x = n/2$ .)



# Chapter 8

## Heterogeneous Risks

### 8.1 Heterogeneity

Most of the methods we have studied so far have been applied to one population at a time with interest in one outcome at a time. We take, for example, a cohort of men or of women, making no distinctions among the different cohort members, and we study the outcome “death”, making no distinction among different kinds of deaths. Or we take women of a given age and study the outcome of next childbirth, making no distinction among different women and no distinction among different kinds of births. We assume a homogeneous population and a homogeneous outcome. When data are plentiful, we can adopt a strategy of “divide and conquer”, splitting our population into homogeneous subpopulations, studying each one separately, and studying each kind of outcome separately, one at a time. When data are limited or when we care about finer and finer distinctions, we need methods that take account of differences, that is to say methods addressing heterogeneity.

Initially, in this chapter, we study heterogeneous outcomes; the prime example is the study of deaths from different causes. There are multiple sources for the decreases or “decrements” in numbers of survivors, leading to “multiple decrement” lifetables presented

in Section 8.2. They are based on a theory of “competing risks” in which different risks like cancer and heart disease compete, as it were, to carry a person off from life. The approach described in Section 8.3 was first worked out by the great French mathematician Daniel Bernoulli. Today the mathematics behind competing risks is expressed in terms of hazard functions as in Section 8.4.

Similar mathematics can be applied to study risks of a single outcome when the risks vary from individual to individual within a population. A great deal of research is devoted to studying the effects of measured variables like education, ethnicity, or blood pressure on survival, that is to say, on the duration of life. The same ideas are useful for studying the duration of other states like marriage or residence in a particular city or country. Especially powerful methods are available when it is reasonable to make an assumption of “proportional hazards” described in Section 8.5. The resulting approach, invented by Sir David Cox in 1972, is used throughout demography. We work through an example drawn from the journal *Demography* in Section 8.6.

Risks that vary from individual to individual due to effects of variables that have been measured, as studied in Sections 8.5 and 8.6, constitute “observed heterogeneity”. No data set can hope to capture and record all relevant differences among individuals. Other differences remain as “unobserved heterogeneity.” Although unmeasured, they may leave their mark on the statistical distribution of outcomes over time. We will not go into this advanced topic in this book, but it is an important area of ongoing demographic research.

## 8.2 Multiple Decrements

In developed societies today, something like two-thirds of deaths are blamed on cancer, stroke, and heart disease, and only a few percent on infectious diseases. A century ago, infections dominated this picture; the reduction in their share of deaths represents one of the triumphs of modern human history. Around the

world, however, infections still claim a large proportion of lives, and infections with the HIV virus resulting in deaths from AIDS have become the greatest challenge to public health in our times.

Demographers and epidemiologists track progress against diseases with “Multiple Decrement Lifetables.” Multiple Decrement Lifetables are called by this name because the population is suffering losses, or decrements, from a multiplicity of causes. There is only one way of being dead, but many ways of dying.

When a number of causes of death are carrying off members of a population, the deaths observed from a disease like AIDS do not fully reflect the killing power of the disease. Some people who would have died of AIDS die of something else before AIDS kills them. Similarly, in the presence of AIDS, other causes will seem less lethal than they are, because AIDS kills people who would have died of these other causes. The methods of this section are designed for separating out the effects of causes of death acting on their own from the observed rates of mortality when they all act together.

Zimbabwe in southern Africa has very nearly the highest rate of HIV infections in the world. The United Nations estimates that the overall probability of dying for men aged 30 to 35 in Zimbabwe in 2000 is  ${}_5q_{30} = 0.113390$ . A little over 4/5-ths of the deaths observed in the age group are attributed to AIDS.

The probability of dying of AIDS *in the presence of other causes* is 4/5-ths of the overall probability of dying. We have  $(4/5)(0.113390) = 0.090712$ .

We want a formula for calculating what the probability of dying of AIDS would be *in the absence* of other causes. We write this special probability with a star on the label for the cause:

The formula we need is

$$\begin{aligned} {}_5q_{30}^{AIDS*} &= 1 - (1 - {}_5q_{30})^{4/5} \\ &= 1 - (1 - 0.113390)^{4/5} = 0.091790 \end{aligned}$$

Similarly, the probability of dying of other causes *in the absence*

of AIDS would be

$$\begin{aligned} {}_5q_{30}^{OTHER*} &= 1 - (1 - {}_5q_{30})^{1/5} \\ &= 1 - (1 - 0.113930)^{1/5} = 0.023782 \end{aligned}$$

These formulas will be derived in Section 8.4. Notice their similarity to our familiar formulas for  ${}_nq_x$ -conversions from Chapter Three. To convert from an 5-year probability to a 4-year probability and a 1-year probability we would calculate

$$\begin{aligned} {}_4q_x &= 1 - (1 - {}_5q_x)^{4/5} \\ {}_1q_x &= 1 - (1 - {}_5q_x)^{1/5} \end{aligned}$$

To survive the whole 5-year interval, a person must survive for 4 years AND for 1 more year. Similarly, to survive the whole battery of causes of death, a person must survive the 4/5ths share of AIDS deaths AND survive the 1/5th share of deaths from other causes. In both sets of formulas, the *survivorships* multiply. For causes of death, the reason for multiplication is an idealized assumption that the causes of death act independently, as discussed in Section 8.3.

To make our formulas fully general, we need some notation. We use letters  $A, B, C \dots$  for causes of death, and write  ${}_nD_x^A$ ,  ${}_nD_x^B$ , etc. for deaths observed from these causes. The essential quantities for our calculations are the *shares* of deaths by cause:

$$s^A = {}_nD_x^A / {}_nD_x \quad \text{and} \quad s^B = {}_nD_x^B / {}_nD_x \quad \text{etc.}$$

In our example from Zimbabwe,  $A$  is AIDS,  $B$  is all other causes,  $s^A = 4/5$  and  $s^B = 1/5$ .

Just as we have deaths  ${}_nD_x^A$  from cause  $A$ , we have death rates  ${}_nM_x^A$  for cause  $A$ , obtained by dividing the deaths from cause  $A$  by the period person-years lived by everyone in the age group. These are “cause-specific death rates” (strictly speaking, cause, age, and sex-specific death rates). Since the denominator, the person-years lived, is the same regardless of cause, we can calculate the shares from the death rates or from the deaths and

get the same answers. The deaths for all the causes add up to total deaths. The death rates for all the causes add up to the total death rate. the shares add up to 1. We can also reverse the formulas:

$$\begin{aligned} {}_nD_x^A &= (s^A)({}_nD_x) \\ {}_nM_x^A &= (s^A)({}_nM_x) \end{aligned}$$

There is a parallel equation with  $q$  in place of  $D$  or  $M$ .

$${}_nq_x^A = (s^A)({}_nq_x)$$

As we have seen in our example, this is a kind of probability of dying, but not the kind that is most interesting to us. It is probability of dying of cause  $A$  in the *presence* of other causes. It mixes together the fatal effects of  $A$  with intervening fatal effects of other causes.

This probability is most easily computed directly from the share  $s^A$  and the overall  ${}_nq_x$ , but it can also be computed from the period life table formula, as long as we have the death rate from cause  $A$  in the numerator but the death rate from all causes (which affect the person-years lived) in the denominator:

$${}_nq_x^A = \frac{({}_nM_x^A)}{1 + (n - {}_na_x)({}_nM_x)}$$

The kind of probability of more interest to us is the probability of dying of cause  $A$  in the *absence* of other causes. There is no very standard notation for the different kinds of probabilities. Here we write a star beside the cause for any quantity computed in the absence of other causes. We calculate this probability using  $s^A$  as an exponent:

$${}_nq_x^{A*} = 1 - (1 - {}_nq_x)^{s^A}$$

The differences between the two kinds of probabilities may not be large numerically, but they are important for studying trends. In our example from Zimbabwe,  ${}_5q_{30}^A$  is 0.099394, whereas  ${}_5q_{30}^{A*}$  is 0.100121.

Worldwide standards for classifying deaths by cause, with thousands of detailed categories, are published in the International Classification of Disease. Doctors or hospital assistants generally record one primary cause, sometimes along with some secondary causes, on each death certificate. This is a routine process. Deep investigation might often reveal that the true causes of death were other than the apparent causes given on the death certificate. Especially among the elderly, the choice of one of several contributing causes to list as the primary cause may be quite arbitrary. Despite efforts at standardization, diagnostic practices differ from time to time and country to country, and aggregation into broad categories is subject to wide variation. Cause of death data should always be regarded with caution.

Table 8.1 shows cause-specific death rates for ages 80 to 85 for 1981-1990 for Japan, the current world leader in low mortality. Popular rather than medical names for categories of disease are used to avoid lengthy specifications. The row for “Diseases of other organs” includes pneumonia and diseases of the stomach, liver, and kidneys.

Table 8.1: Cause-Specific Death Rates for Japan

Ages 80 to 85 for 1981 to 1990

Cause of Death	Rates	
	Females	Males
Accidents and Violence	.000515	.001070
Infectious Diseases	.000221	.000504
Cancers	.003876	.008498
Heart Disease	.002021	.003685
Stroke	.001920	.003187
Diseases of Other Organs	.000971	.002127
Remaining Causes	.001370	.002434

Let us calculate what the male probability of dying of other



causes would be in the absence of cancer. First we find the overall value of  ${}_5M_{80}$  by adding up the cause-specific rates in the third column. The sum is .021505. The corresponding overall  ${}_5q_{80}$  from the period life table formula is  $(5)(.021505)/(1 + (2.5)(.021505)) = 0.102039$ . The share of cancers  $s^C$  is  $.008498/.021505 = 0.395166$ , so the share of other causes is 0.604836. The probability of dying of these other causes between ages 80 and 85 in the absence of cancer would amount to  $1 - (1 - 0.102039)^{0.604836} = 0.063024$ .

A life table built up out of such probabilities for every age group is called a “cause-elimination life table”. It tells us how much progress would be made if a cure for cancer could be found. The progress is a little less than we would expect, because some of the people who would have died of cancer would be dying of other causes in the age interval.

We may also concentrate on causes  $A, B, C \dots$  by themselves, each in the absence of all the others. We build a life table from the probabilities  ${}_nq_x^{A*}$  for all the age groups, and another from the  ${}_nq_x^{B*}$ , and so forth. We write  $l_x^{A*}, l_x^{B*} \dots$  for the survivorship columns of these tables, and use similar notation for other life table columns. These tables are called “associated single-decrement life tables” while the table built from the overall  ${}_nq_x$  values is the multiple-decrement life table. The single decrement tables describe the effects of causes in isolation from each other, which combine when taken together into the multiple decrement life table.

### 8.3 \*Competing Risks

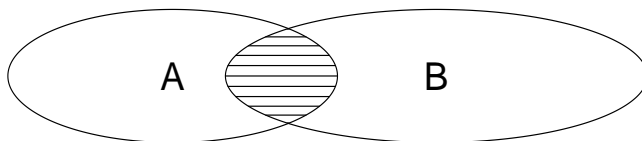
Understanding the formulas for associated single and multiple decrement life tables involves the idea of “competing risks”. We think of causes of death  $A, B, C \dots$  competing to carry off each member of the population. The deaths that would occur from each cause acting on its own differ from those that do occur when all the causes compete together.

Probabilities of dying in the presence of other causes add together whereas probabilities of dying in the absence of other causes, written with stars in the formulas, do not: (We temporarily omit n's and x's for shorter formulas.)

$$\begin{aligned} q &= q^A + q^B \\ q &= q^{A*} + q^{B*} - (q^{A*})(q^{B*}) = 1 - (1 - q^{A*})(1 - q^{B*}) \end{aligned}$$

Why must we subtract the product  $(q^{A*})(q^{B*})$  from the sum of the starred q's? The answer is “double jeopardy”. Figure 8.1 is a Venn diagram. The oval on the left denotes people who would be killed by  $A$  acting on its own, the one on the right people killed by  $B$  acting on its own. Acting on its own, in the absence of other causes,  $A$  would kill some of the same people as  $B$ , namely, the people in the shaded intersection, who are counted twice if we simply add the starred q's together, and have to be subtracted to obtain the correct overall probability of dying.

Figure 8.1: Double Jeopardy



In principle we cannot observe double jeopardy. We see people die of one thing or another, and we do not know whether, had they been spared, they would have died before the end of the age interval of another cause. Sometimes we do observe something rather like double jeopardy, when medicine succeeds in bringing people back from the brink of death with heroic interventions. Some patients rescued by heart transplants die shortly afterward of lung cancer. But most of the time we are dealing with a mathematical abstraction.

Two separate assumptions are involved. One is the multiplication law for *independent* competing risks. The other is an assumption of *steady shares* of deaths by cause inside each interval over which we observe the totals.

The assumption of independent competing risks is a strong assumption. It is violated whenever people susceptible to dying from some particular cause are at heightened risk of dying of another. Smokers are particularly likely to die of lung cancer and at higher risk of heart disease. In practice, demographers try to make the independence assumption more nearly valid by restricting calculations to subpopulations which are as much alike as possible with respect to known risk factors. For instance, the formulas may be applied separately to smokers and non-smokers. For advanced work, complex models for dependent competing risks are also available, including the stochastic risk factor models of Kenneth Manton and his collaborators.

The assumption of steady shares is a sensible expedient. When we have no information about changes in shares inside an interval, we assume shares are constant. It is the same kind of assumption we have made for converting  ${}_nq_x$  values for different  $n$ . When we have no extra information, we assume constant rates within intervals.

We gain insight into the working of competing risks from an analogy inspired by the trials of medieval knights who had to “run the gauntlet” down a row of adversaries striking at them from both sides. Imagine the adversaries as devils, A-devils and B-devils and C-devils competing to claim victims for their causes. Thousands of them line up along the gauntlet, scattered so that the shares of each kind of devil are much the same in every part of the line. Each devil strikes out once on its own as the knight runs by, each with the same chance of hitting him. The knight has to run the whole gauntlet, getting up and moving on whenever he is hit. He may be hit more than once, by an A-devil, by a B-devil, by another A-devil, by a C-devil. The first devil who succeeds in hitting him has the right to claim him in the end for that devil’s cause. But he may not be hit. The knight, fortified with grace, may dodge all the devils and come safely through the gauntlet, a survivor.

The way the devils strike out separately on their own is the assumption of independent competing risks. The constant mix of devils down the line is the assumption of steady shares. With

thousands of devils, each step is too short to worry about double jeopardy at any single step. Double jeopardy occurs when the knight is hit by more than one devil in the line. Take away the A-devils, we have a cause-elimination lifetable. Take away all devils but the A-devils, we have the single-decrement lifetable for A. The chance that at least one A-devil hits the knight somewhere along the line is the probability of dying of *A* in the absence of other causes. The probability of dying of *A* in the presence of other causes is the chance of an A-devil winning the competition, being first to hit the knight. The chance that one or more A-devils hit the knight is not proportional to the number of A-devils; it is more complicated. But if the knight is hit, the chance that the first to hit him is an A-devil *is* proportional to the number of A-devils. That is the important feature of the picture. The shares of devils are the same as the shares of deaths.

## 8.4 \*Risks and Hazard Functions

Formulas for competing risks are neatly proved using hazard functions. The hazard function is a name for the hazard rate or “force of mortality” when it is regarded as a function of age. It was introduced at the start of Chapter 3 by analogy with the growth rate in the exponential model for population size. The hazard function equals minus the slope of the logarithm of surviving cohort size as a function of age.

To express competing risks in terms of hazard functions, we start with separate hazard functions  $h_x^A$ ,  $h_x^B$ ,  $h_x^C \dots$  for causes *A*, *B*, *C*.... We put no stars on the hazard function labels for reasons discussed at the end of this section. Throughout this section, we assume a radix  $l_0$  of 1. When survivorships multiply, their logarithms add. The assumption of independent competing risks is an addition rule for hazard functions:

$$h_x = h_x^A + h_x^B + h_x^C \dots$$

The assumption of steady shares is a requirement for hazards in constant proportion within each age interval:

$$\frac{h_x^A}{h_x^A + h_x^B + h_x^C \dots} = s^A \quad \text{for } x \text{ within the interval}$$

To derive our formulas for competing risks from these two assumptions, we need two facts from calculus. First, since the logarithm of survivorship  $l_x$  has  $-h_x$  for its slope, we recover  $l_x$  itself by taking the exponential function of minus the area under the hazard function:

$$l_x = \exp \left( - \int_0^x h_a da \right)$$

Second, the survivorship  $l_x$  itself has  $-h_x l_x$  for its slope ( since the derivative of  $l_x$  is the same as  $l_x$  times the derivative of  $\log(l_x)$  ) so the area under  $h_a l_a$  between  $x$  and  $x+n$  is the life table deaths:

$${}_n d_x = l_x - l_{x+n} = \int_x^{x+n} h_a l_a da$$

We say  $h_a$  “acts on”  $l_a$  to produce the deaths.

Using the first fact, we see that the addition rule for hazards

$$l_x = \exp \left( - \int_0^x (h_a^A + h_a^B + h_a^C \dots) da \right)$$

along with

$$l_x^{A*} = \exp \left( - \int_0^x h_a^A da \right)$$

implies the multiplication rule for survivorships:

$$l_x = l_x^{A*} l_x^{B*} l_x^{C*} \dots$$

$$1 - {}_n q_x = l_{x+n}/l_x = (1 - {}_n q_x^{A*})(1 - {}_n q_x^{B*})(1 - {}_n q_x^{C*}) \dots$$

Using the second fact, when the hazard for cause  $A$  acts on the survivors from all causes, we obtain deaths from  $A$  in the presence of all causes:

$${}_n d_x^A = \int_x^{x+n} h_a^A l_a da = \int_x^{x+n} s^A h_a l_a da = s^A ({}_n d_x)$$

These are deaths in a cohort lifetable. When we divide by person-years lived, we obtain a corresponding formula for the lifetable death rates  ${}_n m_x$  and when we equate  ${}_n m_x$  to the period rate  ${}_n M_x$ , we see that the shares for the hazard rates have to be the same as the observed shares of deaths in the period population.

When we let  $h_a^A$  act on the survivors from  $A$  acting alone, we obtain our formula for the probability of dying of  $A$  in the absence of other causes:

$$\begin{aligned} (1 - {}_n q_x^{A*}) &= \exp \left( - \int_x^{x+n} h_a^A da \right) = \exp \left( - \int s^A h_a da \right) \\ &= \left( \exp \left( - \int h_a da \right) \right)^{s^A} = (1 - {}_n q_x)^{s^A} \end{aligned}$$

We now see why there is no need for a star on the labels on the hazards. The hazard rate for  $A$  is the same in the presence of other causes as in the absence of other causes. What makes the difference is the survivors that it acts on. When  $h^A$  acts on survivors from all causes, we have deaths in the presence of other causes. When  $h^A$  acts on survivors from  $A$  acting alone, we have deaths in the absence of other causes.

Table 8.2: Survivorship to Age 80 in Single Decrement Tables for Japan, 1981-1990

Cause of Death	Female $l_{80}$	Male $l_{80}$
Accidents and Violence	.976	.944
Infectious Diseases	.991	.982
Cancers	.880	.773
Heart Disease	.913	.862
Stroke	.917	.875
Diseases of Other Organs	.958	.908
Other Causes	.938	.899
All Causes ( $l_{80}$ )	.643	.442

Our derivations using hazard functions not only explain why we compute probabilities in the way we do, but they also give handy relationships for further calculations. For example, the multiplication law for  $l_x$  values shortens calculations of cause elimination. Table 8.2 shows the  $l_{80}^A$ ,  $l_{80}^B \dots$  values for Japan for the causes listed in Table 8.1. Each all-cause  $l_{80}$  entry in the final row is the product of the numbers above it in the table. To eliminate causes, divide by their single-decrement survivorships. For instance, the effect of eliminating both heart disease and stroke would be to increase survival to age 80 for females from 0.643 to  $0.643/((0.914)(0.917))$  which is 0.838.

## 8.5 \*Proportional Hazards

With multiple decrement lifetables we have been studying differences among risks from different causes of death that all impinge on the same person. We now turn our attention to differences among risks from person to person. The mathematics is similar, but the purpose is new.

Every lifetable groups everyone in some cohort together. The lifetable itself does not separate out people by their characteristics or the dangers that they run. When we want to distinguish among groups of people, we can do separate lifetables. We can have separate lifetables for women and men, for villagers and cityfolks, for drinkers and teetotalers. But if we do separate tables for small groups, we may have unstable answers. We would also be failing to take advantage of similarities in age patterns of mortality from group to group.

Proportional hazard models help us study hazard rates with individual-level data when there are measurements of background variables (called covariates or cofactors) associated with lower and higher risk but when the shape of the age pattern of mortality is likely to be common to everyone. We consider different individuals with identification numbers  $i = 1, i = 2 \dots$ , and write  $h(x, i)$  for

the hazard function at age  $x$  that applies to the  $i$ -th individual. We choose a notation with parentheses instead of subscripts and superscripts in order to distinguish easily between hazards from person to person and hazards from cause to cause. We let  $i = 0$  stand for an imaginary individual whose hazard function  $h(x, 0)$ , called the “baseline hazard”, is the standard age pattern to which other hazard functions are compared. In this section we introduce the model with a special case with one covariate. In Section 8.6 we present the general model.

For our special case, we let sex be our covariate. We take the hazard function for females as our baseline, and assume that the hazard function for any male is the same constant multiple  $e^\beta$  of the hazard function for a female at every age. In other words, we assume that the hazard functions for females and males are *proportional* to each other.

$$\begin{aligned} h(x, i) &= h(x, 0) && \text{for females} \\ h(x, i) &= e^\beta h(x, 0) && \text{for males} \end{aligned}$$

We express the effect in exponential notation as  $e^\beta$  instead of  $\beta$  for the sake of an analogy with linear regression explained in Section 8.6. For younger ages, our proportionality assumption for males and females would be foolish, but for extreme ages it is more plausible. For our example, we apply our model to the most extreme ages to which human beings have been known to survive.

Table 8.3 show the five longest lifespans from a particular collection of carefully validated data published by Robine and Vaupel (2002). We want to see how we would use this information along with the assumption of proportional hazards to estimate  $\beta$ , the effect of being male. This example is highly simplified for the sake of illustrating the method. For a realistic application, a larger data set and computer-based calculations would be required.

The key idea is due to Sir David Cox (1972). Imagine a competition in which we start at age 115 and see which of our five individuals is the first to die. Earlier in this chapter we had a competition among causes of death, which we could describe loosely as a competition among organs of the body as to which is to be the



Table 8.3: Record Lifespans

Name	Sex	Date of Birth	Date of Death	Lifespan
J. Calment	F	21 Feb. 1875	4 Aug. 1997	122
M. Meilleur	F	29 Aug. 1880	16 Apr. 1998	117
C. Mortensen	M	16 Aug. 1882	25 Apr. 1998	115
C. Hughes	F	1 Aug. 1877	17 Mar. 1993	115
A. Jennings	F	12 Nov. 1884	20 Nov. 1999	115

first to fail. Here, instead, we have a competition among individuals of a population as to which is to be first to die. The picture is different, but the mathematics is the same. With causes of death, the probability that  $A$  comes first is given by the ratio of  $h_x^A$  to the sum  $h_x^A + h_x^B + \dots$ . With individuals, the probability that  $i$  dies first is given by the ratio of  $h(x, i)$  to the sum  $h(x, 1) + h(x, 2) + \dots$ . Under the assumption of proportional hazards, this ratio does not depend on age  $x$  but only on the effects of the covariates, in this case on  $\beta$ .

Just as in Sections 8.3 and 8.4, two assumptions are involved. The first is an assumption of independent competing risks. This time that assumption is fairly innocent. The deaths of different individuals are to be independent of each other. The assumption would be violated if the population included members of the same families or the same mountain-climbing teams, but for large populations or randomly selected samples it is easy to accept. This time the second assumption, of steady shares or of proportional hazards, is the strong assumption. We discuss it further in Section 8.6.

When we run our competition on the data in Table 8.3, the model says that the chance of Mrs. Jennings dying first is

$$\frac{h(x, 0)}{h(x, 0) + h(x, 0) + e^\beta h(x, 0) + h(x, 0) + h(x, 0)} = \frac{1}{4 + e^\beta}$$

The model gives each of the other three women the same chance,

but it gives Mr. Mortensen a different chance,

$$\frac{e^\beta h(x, 0)}{h(x, 0) + h(x, 0) + e^\beta h(x, 0) + h(x, 0) + h(x, 0)} = \frac{e^\beta}{4 + e^\beta}$$

In fact, Mrs. Jennings dies first. The chance the model assigns to the event that actually occurs is called the “likelihood”. This likelihood is a function of the unknown parameter  $\beta$ , so we have a “likelihood function” of  $\beta$ .

We can tell nothing from a single outcome. If we only know that the first to die was a woman, the probability of a man dying might be anything. It might be zero. But we can repeat our competition. At the next round, after Mrs. Jennings’ death, four individuals remain, three women with chances  $1/(3 + e^\beta)$  of being the next to die, and one man with a chance  $e^\beta/(3 + e^\beta)$ . A woman dies. The likelihood is  $1/(3 + e^\beta)$ . At the next round, two women and one man remain. The man, Mr. Mortensen, dies. The likelihood is  $e^\beta/(2 + e^\beta)$ . At the next round, the likelihood is  $1/2$ . At the final round, one woman remains, Madame Jeanne Calment. She dies, with likelihood 1. The process is like a game of musical chairs, with one fewer place for a survivor at each round. In reality, it is no game. But it is happy to dwell on the remarkable survival of these five individuals. Madame Calment may be the oldest human being who has ever lived. Her story is told by Allard, Lebre, and Robine (1994).

The different rounds are independent of each other. The death of each of these individuals has no effect on the survival of the others. Since probabilities of independent events multiply, we take the product of the likelihoods from each round. This product is called a “partial likelihood” because it only takes into account the order of deaths and not the ages at death. Sir David Cox realized that the partial likelihood is all we need for estimating the effects of covariates. We look for the value of the parameter  $\beta$  which maximizes the partial likelihood, that is, which makes the observed order of deaths as likely as possible under the model. Hunting maxima usually requires powerful computer programs, but in our simple example we can plug in trial values for  $\beta$  and

choose the one that gives the highest value for the partial likelihood

$$\lambda = \frac{1}{4 + e^\beta} \frac{1}{3 + e^\beta} \frac{e^\beta}{2 + e^\beta} \frac{1}{2} \frac{1}{1}$$

Figure 8.2: Partial Likelihood

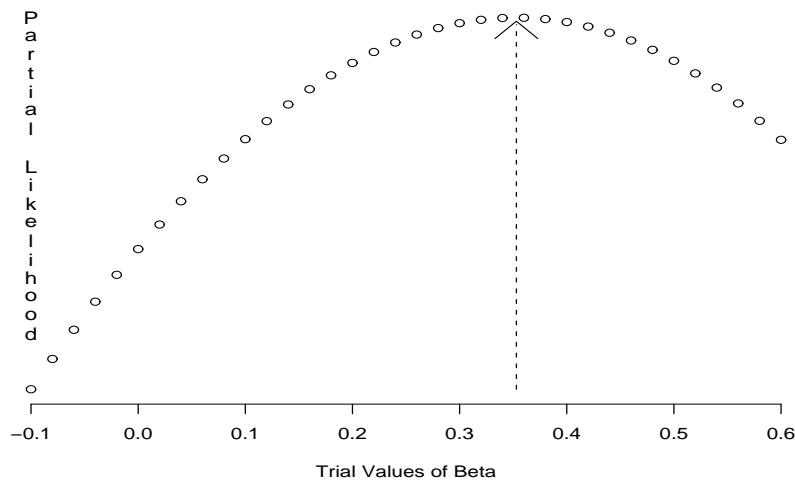


Figure 8.2 shows the partial likelihood for 36 trial values of  $\beta$ . At first the numerator increases more rapidly and later the denominator, so that the partial likelihood attains a maximum, in this case close to  $\beta = 0.353$ . The outcome means that the hazard function for men at these extreme ages is estimated to be about  $e^\beta = 1.423$  times the hazard for women. With only five data points, we would not expect this estimate to be reliable, but in fact ratios like 1.4 or 1.5 are often seen at more moderate older ages. Estimates of the male disadvantage at ages around 100 from larger data sets given by Hill, Preston, and Rosenwaike (2002) range between 1.1 and 1.3.

Computers can find a maximum for the partial likelihood and thus find estimates of effects even when many covariates and many cases are involved. Often ages at death are “censored”, that is,

unseen because a person is still alive when the period of observation ends. Cox's method handles these cases easily. The person counts in the denominators of the partial likelihood for all deaths that occur while the person is still in observation and therefore could have been next to die. The person contributes no factor of his or her own to the likelihood, because he or she is never seen to be the next to die. For instance, suppose we had only known that Mrs. Hughes was still alive in March 1993 and did not know her age at death. Then we would remove the factor  $1/(3 + e^\beta)$  from the likelihood, and search for a value of  $\beta$  to maximize the product of the four remaining factors corresponding to observed ages at death.

## 8.6 Cox Regression

Sir David Cox's approach makes the estimation in proportional hazard models very similar to estimation in the most widely used of all statistical techniques, linear regression. In the general setting the covariates are variables  $Z(1, i)$ ,  $Z(2, i) \dots$  that take numerical values for each person  $i$ . For each covariate, we have a coefficient  $\beta_1$ ,  $\beta_2 \dots$  which has to be estimated from data and which measures the strength of the effect of that characteristic on the hazard function. The proportional hazard model is given by the equation.

$$h(x, i) = h(x, 0)e^{\beta_1 Z(1, i) + \beta_2 Z(2, i) \dots}$$

The baseline hazard  $h(x, 0)$  which depends only on age  $x$  is multiplied by a factor which depends on who the individual is but not on his or her age. The factor for the  $i$ -th individual is an exponential function of a sum formed by multiplying each coefficient  $\beta$  by its variable  $Z(1, i)$  and adding them together. The sum or "linear combination" is very like the linear combinations in linear regression. We make the effects multiplicative, that is to say, proportional, by taking the exponential of the sum, which is the product of exponentials.

$$e^{\beta_1 Z(1, i) + \beta_2 Z(2, i) \dots} = e^{\beta_1 Z(1, i)} e^{\beta_2 Z(2, i)} \dots$$

The effects of the different covariates multiply.

This multiplicative relationship means that the effects are assumed to be independent of age and of each other. The assumption that they are independent of age is a serious restriction, but the assumption that they are independent of each other is not. If the effects of two variables depend on the levels of each other, for example, if the effect of smoking depends on whether the individual is a man or a woman, we can define a combined variable or “interaction”, for instance a variable that takes the value 1 for male smokers and zero for everyone else. The art of defining interactions, adding or deleting variables, and testing for statistical significance is a subject for books on statistics or statistical demography. Cox regression is very like ordinary linear regression in this regard. Here we concentrate on drawing demographic implications out of proportional hazard models. We take an example from an unusual and thought-provoking application of proportional hazard modeling published in the journal *Demography* by Robert Hummer and colleagues (1999). The researchers measured effects of religious attendance on U.S. adult mortality. Their data came from 21,204 respondents to a survey taken as part of the U.S. National Health Interview Survey whose survival was followed from 1987 until 1995. One question on the survey asked “How often do you go to church, temple, or other religious services?”. The first variable  $Z(1, i)$  takes the value 1 if individual  $i$  answered “Never”, and otherwise equals zero. The second variable equals 1 for “Less than once a week”, and zero otherwise. The third variable equals 1 for “Weekly”, and zero otherwise. The omitted category, for individuals for whom  $Z(1, i) = Z(2, i) = Z(3, i) = 0$ , is composed of people answering “More than once a week.” Such variables taking values of 1 and 0 and picking out a category of respondents are called “indicator variables”.

Hummer’s basic model also included indicators for male sex, black race, and residence outside the south. The individual’s starting age at the time of the survey is a variable measured in single years of age backwards or forwards from age 65. The variable which plays the role of age or duration  $x$  in the proportional hazard model itself is the years of additional age from the time of the

survey until the individual died or went out of observation in 1995. The researchers calculated a partial likelihood by the methods of Section 8.5, and found estimates of the coefficients  $\beta_1, \beta_2 \dots$  shown in Table 8.4.

Table 8.4: Proportional Hazard Coefficients

Betas for the Basic Model of Hummer et al. (1999)

Religious Attendance	$\beta$	Demographic Variables	$\beta$
1. Never	.63	4. Male sex	.42
2. Less than weekly	.27	5. Black race	.44
3. Weekly	.14	6. Outside south	-.13
(More than weekly)	.00	7. Starting age minus 65	.09

The journal article by Hummer *et al.* (1999) does not report values of the baseline hazard. For our example, we assume that the baseline hazard for survival from 0 to 3 years from the survey is 0.016, from 3 to 6 years is 0.021, and from 6 to 9 years is 0.028. The baseline is the hazard when all the variables equal zero, that is for white women in the south, age 65 at the time of the survey, who attend religious services more than once a week.

We want to use the estimates in Table 8.4 to calculate survival probabilities as they depend on religious attendance. Under the baseline hazard, the probability of surviving 9 years beyond the survey from age 65 to age 74 is given by

$$\begin{aligned} l_{74}/l_{65} &= \exp\left(-\int_0^9 h(x, 0)dx\right) \\ &= \exp(-3 * 0.016 - 3 * 0.021 - 3 * 0.028) = 0.823 \end{aligned}$$

For white women in the south who never attend church, the hazard is multiplied at all ages by  $e^{\beta_1} = e^{0.63}$ . The probability of surviving is reduced to

$$\begin{aligned} l_{74}/l_{65} &= \exp\left(-\int_0^9 e^{0.63} h(x, 0)dx\right) \\ &= \exp(-e^{0.63} 0.195) = 0.693 \end{aligned}$$

We have exponentials inside exponentials, because survivorship depends exponentially on the hazards, and the hazards depend exponentially on the coefficients.

For a more complicated case, we ask what the chance of surviving 4 years from the time of the survey would be for a black male in California who was 70 at the time of the survey and attended church less than once a week. The proportionality factor for his hazard is given by the exponential of a sum:

$$\begin{aligned} \exp(\beta_1 Z(1, i) + \dots \beta_7 Z(7, i)) \\ &= \exp(.63 * 0 + .27 * 1 + .14 * 0 + .42 * 1 \dots \\ &\quad + .44 * 1 - .13 * 1 + .09 * (70 - 65)) \\ &= \exp(1.45) = 4.263 \end{aligned}$$

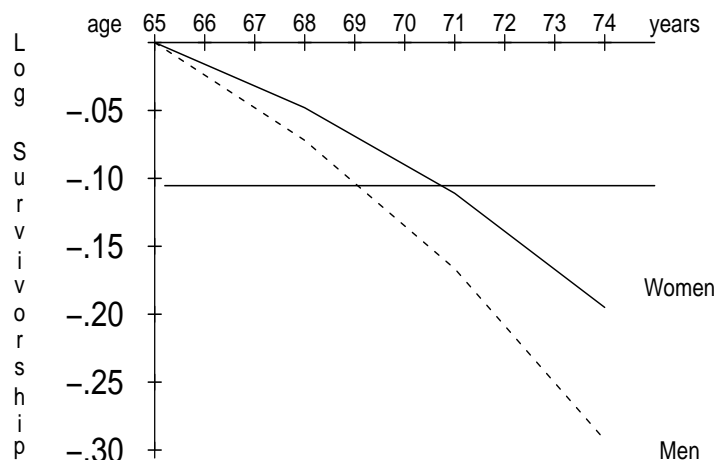
His probability of survival from 70 to 74 would be

$$\begin{aligned} l_{74}/l_{70} &= \exp\left(-\int_0^4 e^{1.45} h(x, 0) dx\right) \\ &= \exp(-4.263 * 0.069) = 0.745 \end{aligned}$$

In any proportional hazard model like this one in which a person's initial age is one of the covariates, treated as a numerical variable rather than broken down into categories, an assumption about the form of period lifetables is being built into the model. For simplicity, suppose initial age is the only covariate, call it  $Z$ , and call its coefficient  $\beta$ . After a duration of  $x$  years, people age  $a$  are those who were age  $Z = a - x$  in the survey, and the model sets their hazard rate equal to  $e^{\beta(a-x)}h(x, 0)$  which equals  $e^{\beta a}(e^{-\beta x}h(x, 0))$ . Here  $x$  is the duration of time since the survey and  $a$  is current age, so the period lifetable  $x$  years after the survey for every  $x$  is based on a hazard rate that is an exponential function of age  $a$ . In other words, the period lifetables are being assumed to fit the Gompertz Model studied in Chapter 3. For adult ages, this assumption is often reasonable, but it needs to be kept in mind.

Calculation using a proportional hazard model can also tell us about percentiles of the survival distribution. Figure 8.3 graphs

Figure 8.3: Log Survivorship with Proportional Hazards



the logarithm of  $l_{65+x}/l_{65}$  as a function of duration  $x$  since the time of the survey. The solid line is computed from the baseline hazard. The dotted line is computed for individuals with all the same characteristics as the baseline group except that it applies to men instead of women. The hazard for men is  $e^{\beta_4} = e^{.42}$  or about 1.5 times the hazard for women at every duration  $x$ . The hazards for women and men are minus the slopes of the curves graphed in Figure 8.3. The hazards were assumed constant over three-year intervals, so the curves are straight within the intervals, and form broken lines. The curves both start at zero, and the men's curve has 1.5 times the slope of the women's curve at every point, so the men's curve itself is 1.5 times the women's curve at every point.

Suppose we ask, how long would it take, according to the model, for 10% of the women and for 10% of the men to die? With 90% survivors, we need  $l_{x+65}/l_{65} = 0.90$  so we need  $\log(Y) = \log(0.90) = -0.1053$ . We draw a dashed line across the figure, and measure the values of  $x$  for the places at which the line crosses our two curves. By inspection, starting at age 65, it takes about 4 years till age 69 for 10% of men to die and about 5.5 years till



age 70.5 for 10% of women to die. For a precise value, we use linear interpolation. The crossing point for men lies between  $x = 3$  and  $x = 6$ . The curve reaches  $-.072$  at  $x = 3$  and has slope  $1.5 * (-.021) = -0.0315$ , so to drop  $-.1053 - (-.072)$  farther, we need to go  $(-.1053 + .0720)/(-.0315) = 1.0571$  farther, taking us to age  $65 + 3 + 1.0571 = 69.0571$  years.

Computations with percentiles of survivorships are of broader importance, because they allow us to simulate the predictions of models. Percentiles always have a uniform statistical distribution. Even though surveys asking university professors “Are you in the top ten percent of your field?” often find 80% answering “Yes”, in fact, in any ranking, the proportion of people in the top ten percent is ten percent. We can ask the computer to give us a thousand independent uniform random numbers between zero and one. For each one, we take its logarithm, imitate the calculation with Figure 8.3, and find the value of  $x$  that corresponds to this percentile. We end up with a sample of a thousand survival times predicted by the model. With our proportional hazard model, for each kind of person we compute the proportionality factor for the person, multiply all values of the baseline curve in Figure 8.3 by that factor, and use that curve in our percentile simulation. We then have a simulated sample of survival times for different kinds of people drawn from different groups in the population.

Computer programs for maximizing the partial likelihood and estimating the coefficients in proportional hazard models often go on to estimate the baseline hazard function and survival curve using methods that will not be described here. Estimates of baseline survival curves can be used to check on the validity of the proportional hazards assumption. Suppose, for instance, that we fit separate proportional hazard models with the same covariates to separate subsamples of men and women. We compute baseline survival curves, take their logarithms (as we would to draw Figure 8.3), then multiply by  $-1$  to have positive numbers, and take logarithms a second time, plotting the resulting curves against age or duration. The second step with logarithms turns proportional effects into additive effects, so if the proportional hazard assumption is valid, the resulting curves for men and women should be

parallel to each other, with a constant difference between them. This technique of “log-minus-log” plots is one of several handy tools treated in advanced courses in statistical demography.

We have only presented the basic model from Hummer *et al.* (1999). The authors go on to fit six other models which include variables for health, socio-economic status, social ties, and habits like smoking and drinking. By comparing the changes in coefficients when variables are added to the model, they find that some of the effect of religious attendance on enhanced survival appears to operate through promotion of stronger social ties and healthier habits. Healthier people also attend more often. However, the mediating variables measured in their data set do not wholly account, in a statistical sense, for the association between religious attendance and survival. The authors suggest that other mediating factors, harder to measure, like lower stress or better coping resources may play a role. Observational studies of this kind can never establish causal relationships in the way that experiments can, but they do help point attention to features of experience that turn out to be aligned, sometimes in unsuspected ways, with demographic outcomes. The study by Hummer *et al.* is a good example of the use demographers make of the methods of Cox regression and proportional hazard models.

### Highlights of Chapter Eight:

Probabilities of dying of a case  $A$  in the presence of other causes.

Probabilities of dying of  $A$  in the absence of other causes.

Double jeopardy.

Two assumptions behind formulas for competing risks.

The baseline hazard and the proportional hazard model.

Two assumptions behind formulas for proportional hazards.

## Key Formulas

$$\begin{aligned} {}_nq_x^A &= (s^A)({}_nq_x) \\ {}_nq_x^{A*} &= 1 - (1 - {}_nq_x)^{s^A} \\ h(x, i) &= h(x, 0) \exp(\beta_1 Z(1, i) + \beta_1 Z(1, i) \dots) \end{aligned}$$

## Further Reading

The book on lifetables by Namboodiri and Suchindran (1987) gives a full account of most of the topics in this chapter. Our formulas for competing risks were developed in detail by Chin-Long Chiang (1968) at U.C. Berkeley.

## Exercises for Chapter Eight

\*1. Table 8.5 shows cause-specific mortality for Colombia. Treat the column for “Above Age 60” as if these were rates for ages 65 to 75. For men in this age group, what would the probability of dying in the *absence* of both heart disease and stroke?.

\*2. Express the slope of the logarithm of  $l_x$  at the point  $x$  as the limit for  $n$  going to zero of the slope between  $x$  and  $x + n$ . Use this expression to show that the hazard rate at  $x$  is also the limit as  $n$  goes to zero of  ${}_nm_x$  and of  ${}_nq_x/n$ . What is the relationship to the slope of  $l_x$  itself?

\*3. By analogy with the devils’ gauntlet of Section 8.3, here is a lineup for two competing risks  $A$  and  $B$ : B A B A B A A A B B A A B A A B B A A A. The order has been generated at random under the assumption of steady shares. Throw a pair of dice once for each letter in the lineup. The victim is hit if the dice come up with two aces (that is, 1 and 1, or “ambsace”). In your random experiment, which risk turns out to win the victim by being the first to land a hit? Either by repeating the experiment or by combining appropriate formulas under the assumption of

Table 8.5: Cause-Specific Death Rates for Colombian Men in 1990

MALES	all ages	0 to 14	14 to 60	60 and up
Cancer	0.00059	0.00005	0.00033	0.00637
Diabetes	0.00006	0.00000	0.00003	0.00068
Heart disease	0.00096	0.00005	0.00043	0.01157
Stroke	0.00036	0.00001	0.00016	0.00439
Cirrhosis	0.00004	0.00000	0.00004	0.00034
Other diseases	0.00190	0.00230	0.00063	0.01135
Accidents	0.00148	0.00040	0.00204	0.00216
Unknown	0.00029	0.00015	0.00011	0.00292
All Causes	0.00567	0.00296	0.00377	0.03979

steady shares, find the probability of a win for  $A$  in the presence of  $B$ , the probability of a hit for  $A$  in the absence of  $B$ , and the probability of double jeopardy, that is, at least one hit for both  $A$  and  $B$ .

4. In Table 8.3, suppose Mrs. Hughes had only been known to survive until March 1993 rather than to die at that time. Re-estimate the coefficient  $\beta$  for the ratio of male to female hazards in a proportional hazard model.

\*5. A proportional hazards model has been fitted to Malaysian mortality data for 1985 by the methods of Sir David Cox. The baseline hazard  $h(0, t)$  (for babies whose mothers have no formal education) is estimated to be 0.014387 per month in the first month of life and 0.001544 per month in the next eleven months of life.  $Z(i)$  is mother's education in years, and  $\beta = -.111$  is the estimated strength of the (loglinear) effect of mother's education. Find the probability of a girl baby surviving to her first birthday if her mother has completed sixth grade.

\*6. Among girl babies whose mothers have completed sixth grade and who do die before their first birthdays, by what age have the first 5% died?

\*7 Among girl babies whose mothers have completed sixth grade and who do die before their first birthdays, what is the median age at death?



## Chapter 9

# Marriage and Family

### 9.1 The Complexity of Marriage

Marriage is full of complexities, not only for human beings who enter into it, but also for demographers who study it. Not all models of population growth and change include marriage as a component. There is, for instance, no explicit mention of marriage in the Balancing Equation. It is the first topic we have treated that does not appear explicitly in the balancing equation. But marriage enters implicitly into all models through its impacts on other processes. Marriage obviously affects fertility, initiating exposure to significant risks of childbearing for many. Marriage also affects mortality. Married people live longer than non-married and never-married people. It remains a puzzle whether this effect occurs primarily because marriage enhances chances of survival by providing nurturance and support or, alternatively, because healthier people with better life chances are more likely to find spouses, marry, and remain married.

Not only does marriage affect other demographic processes, but it is affected by them. Becoming widowed, for example, is a change in marital status determined not by marriage rates but by mortality rates applying to a person's spouse. Whenever we study a process, it is easier to begin by studying the process as far as we

can on its own in isolation from other processes. We shall do that with marriage too, but we shall not be able to get as far with our models before we have to bring other demographic processes into the picture.

The rates of transitions into and out of marriage are important not just for their own sake, but for what they mean in terms of the nature of the family. It is in the study of marriage that demography has its closest ties to sociology and psychology and some of its most newsworthy relevance in debates over family values.

Fundamentally, of course, we are interested in marriage because it is a lifecycle event that happens to many or most of us. It is a process over which we have some control and about which we can make informed choices. We had no choice about being born and we have limited control over dying. But entering or leaving marriage is always partly up to us.

The subject of marriage is called “nuptiality”, from a Latin word for the wedding ceremony. Nuptiality is a complicated subject, not only because of its interactions with fertility, mortality, and the sociology of the family, but also because of problems of definition. What is a “marriage”? Should consensual unions be treated as marriages for demographic purposes? Should any or all cohabiting couples be treated among the married or among the single? All demographic processes are subject to some fuzziness of definition. Are brain-dead patients with beating hearts dead or alive? If a foetus is moved from its mother’s womb to artificial life-support systems, has it been born? But with marriage, the fuzziness in definitions is more extensive than with death and birth.

Should couples who are separated but not divorced be counted as married? Should same-sex unions be counted as marriages? Should couples in historical data who were betrothed be counted as married prior to the wedding ceremonies? Legal definitions of marriage are poorly suited in many respects to the needs of demographers. On the other hand, having a different definition for every different purpose is an invitation to confusion, and there is no consensus on how far the legal definitions should be broadened



in demographic usage.

Nuptiality is also complicated because marriage and divorce rates are so volatile. We often pretend demographic rates are unchanging when we devise methods for calculating demographic measures from data. Rates of marriage formation and dissolution change rapidly, and in the past their changes were often the most rapid and important regulators of population growth.

All these problems are especially difficult for demographers today, because changes in the nature of marriage are happening especially rapidly today. There have always been cohabitations and consensual unions. But in a few decades in countries like Sweden and Austria they have gone from being rarities to being something like a new norm. Within two generations in the United States rates of divorce and remarriage have grown so much that anthropologists talk of “serial polygamy” – having several spouses, one after the other rather than all at the same time. Similarly rapid changes in the timing and role of marriage are occurring throughout the newly economically developed countries of east and southeast Asia. It is hard to define measures that are not rapidly obsolete. Demographers must continually be coming to grips with social change.

There are also technical reasons that nuptiality in general is harder to quantify than, say, mortality. A person can have several events of marriage formation and dissolution along his or her lifeline, and these events divide the lifeline into successive episodes of being married and being not-married whose duration generally matters to the rates of transition. No person has successive episodes of being dead and being alive. Duration of life is just the same thing as age, whereas duration of marriage is different from age for either partner. Finally, there are multiple ways of exiting from the state of marriage, through being divorced, through being widowed, or through dying.

In the introduction to marriage here, we shall concentrate on the simpler aspects of the subject and largely avoid these complexities. We shall devote most of our attention to first marriages, and we shall also assume that mortality rates during ages of first marriage are low enough to be neglected.

## 9.2 An Analogy

Among all the complexities of nuptiality in general, the one process that is comparatively simple is first marriage. First marriage is the transition from being never-married to being ever-married, and it can happen at most once. We shall use the word “single” to mean “never-married”. In our terminology, those who have been previously married are either widowed, divorced, or married, but not “single” again. Being single in this sense is a state that you can never regain once you have left it. In this respect, first marriage is like death. We draw an analogy between exiting out of singlehood and exiting out of life itself. This analogy gives us demographic measures for first marriage that are close counterparts of the measures we have already defined for the lifetable.

In medieval and Renaissance Florence, it is said, marriages were actually recorded in the death registers, the “Books of the Dead”, and if my sources are correct, the registers were kept, ironically enough, in the Baptistry, the glorious building with the “Doors of Paradise” sculpted by Ghiberti next to the cathedral, the Duomo, in Florence.

Let us develop our analogy between first-marriage and dying, and spell out what about marriage corresponds to what columns of the lifetable. All members of a cohort start out single. We restrict attention for this analysis to the members of the cohort who remain alive through all the ages of interest to us, for example, to cohort members who are still alive and able to respond to a survey taken when they are in late middle age.

The following table shows numbers still single by age for those members of the cohort of U.S. women born in 1930 who were alive in 1990 and responded to the Marriage and Family Supplement of the Current Population Survey (CPS). There were 661 such respondents. These women grew up in the Depression and got married in the late 1940s and 1950s, creating a marriage boom along with the baby boom.

Table 9.1: First marriages for the U.S. cohort of 1930.

age $x$	number single in cohort	cohort first marriages
0	661	0
10	661	280
20	381	315
30	66	29
40	37	3
50	34	0
60	34	0

From the number of cohort members single at age  $x$  we can find the proportion single at age  $x$  by dividing by the initial cohort size 661. This step is like choosing a radix of 1 for a cohort lifetable. The proportion still single at age  $x$  in our marriage table then corresponds to the proportion still alive at age  $x$  in our lifetable,  $l_x$ . The proportion marrying between ages  $x$  and  $x + n$ , that is, the proportion exiting the single state, corresponds to  ${}_nd_x$ , the proportion exiting life between ages  $x$  and  $x + n$ . The probability of marrying in the next  $n$  years for those still single at age  $x$  is like  ${}_nq_x$ .

Although the proportion still single at age  $x$  is analogous to  $l_x$  in its definition, the typical shape of such a curve is different from familiar  $l_x$  curves. The proportion single as a function of  $x$  is entirely flat during ages of infancy and childhood. It drops rapidly during prime ages of marriage in the late teens and twenties, and then flattens out at older ages. The  $l_x$  curve typically has its drops during infancy, is flat during the teens and twenties, and drops again at older ages. Similarly, a graph of cohort first marriages by age has a single hump centered at some age in the twenties, whereas the  ${}_nd_x$  column of the life table has two humps, one in infancy and one at older ages.

There is one other notable difference between first-marriage tables and lifetables, between singleness and survivorship – not

everyone eventually marries whereas everyone does eventually die. When we calculate a measure of timing, like a cohort's mean age at first marriage, the only members who contribute are those who do marry. How early or late in life marriages occur among those who do marry is only part of the story. How many never marry at all is the other part. We distinguish the timing of marriage from its level of universality. For cohorts, the mean age at first marriage among those ever-marrying is the most common measure of timing, and the proportion ever-marrying is the most common measure of universality.

Just as there are model lifetables, there are model schedules for first marriage. Most models take the proportion ever-marrying as one of the parameters, and then have one or more parameters for marriage timing which jointly determine the mean age at first marriage. For instance, the model of Ansley Coale and Donald McNeil has three parameters, proportion ever-marrying, minimum age at marriage, and a slope parameter governing the pace of marriage in its prime years. Along with the Coale-McNeil model, the most widely used model for first marriage is by Hernes.

Mean age at first marriage and proportions ever-marrying are mathematically separate measures. But around the world and over history they have been closely associated with each other. In 1965 the English demographer John Hajnal brought a striking demographic contrast to the attention of social scientists. His paper was called "The European Marriage Pattern in Perspective". It is reprinted in Glass and Eversley (1965). Hajnal pointed out that late ages at marriage and high proportions never marrying were found together throughout most of northwestern Europe before the Twentieth Century. This pattern, has come to be called the "European Marriage Pattern", although it only applies to part of Europe and only to part of European history. It contrasts dramatically with early marriage and nearly negligible proportions never-marrying throughout most of Asia and Africa and in the southern and eastern portions of Europe up to recent times. Hajnal contrasted 17% never-marrying among Belgian women of 1900 to 1% never-marrying among Bulgarian women in 1900, and he contrasted mean ages at marriage for women of 28.8 in Venice

around 1700 to 21.3 years in Serbia around 1900. (Hajnal's mean ages were for all marriages, but similar contrasts in mean ages of first marriages are not hard to find.) Many exceptions to Hajnal's generalization have been pointed out since 1965, Nonetheless, his broad-brush picture – based on the two measures we are studying – remains an important element of our understanding of what was special about the part of the world in which the industrial and scientific revolutions first took place.

### 9.3 The SMAFM

The letters SMAFM are the demographer's acronym for the "Singulate Mean Age at First Marriage". This important measure is also, less properly, often called the SMAM, for "Singulate Mean Age at Marriage", but it is always a measure of the mean age at first marriage.

A cohort's mean age at first marriage is a straightforward concept. When we do not have cohort data, and attempt to calculate a period version of a cohort mean age at first marriage, certain complexities arise. Some years before his paper on European marriage patterns, John Hajnal introduced a method for computing a measure of cohort mean age at first marriage from the kind of period data on marriage that is frequently available from censuses. The citation is John Hajnal (1953) "Age at Marriage and Proportions Marrying" in the journal *Population Studies*.

Hajnal's SMAFM is a very commonly used demographic measure. The only pieces of information required to calculate it are the proportions never-married in each of the age groups in the period population. The word "singulate" in the name has nothing to do with "single" people. "Singulate" is a technical term from statistics. Hajnal's measure has a special form which makes it usable in principle with "singular" statistical distributions. "Singular" distributions are those which are infinitely jagged on small scales, like the fractal curves popularized by Benoit Mandelbrot, found

in books on chaos theory and studied by characters in “Jurassic Park”. The distributions we use in demography are not singular, but the name from statistics has stuck.

The Singulate Mean is a way of inferring mean age at first marriage when we do not have data on ages *at* marriage but only data on proportions married *by* certain ages. Censuses often do not give us marital histories or ages at marriages but do give a breakdown of the population by marital status. That is the ideal kind of data for calculating the SMAFM.

Like any calculation of a cohort measure from period data, the SMAFM requires that we pretend that today’s age-specific values in the period population persist unchanged into the future. In this case, the values in question are the proportions never-married by age group. Three things about first marriage ages are prerequisites for using a method like singular means:

1. Ages are never negative numbers.
2. Being ever-married is a state which, once attained, can never be lost.
3. Only the subset of people who do all eventually attain the state – that is, only the ever-marrying subset of the population – figure in the calculation.

The singulate mean is a form of calculation that can be used with processes other than first marriage if these conditions are met. We can calculate a singulate mean age of college graduation – once a graduate, always a graduate – so long as we restrict ourselves to the subset of those who do eventually graduate. But we cannot calculate a singulate mean bank balance. Bank balances can be negative as well as positive. Nor can we calculate a singulate mean age over all marriages. Being married, as opposed to being ever-married, is a state which once attained can all too easily be lost.

In practice, Hajnal’s SMAFM makes three further assumptions.

4. that proportions married by age have not been changing markedly in recent decades;
5. that those listed as married, widowed, or divorced are in fact the ever-married, even though we know that some do report themselves as single, particularly in the past;
6. that death is independent of marital status at young ages.

None of these assumptions is ever strictly true. Proportions married do tend to change fairly rapidly, as we have said. Especially in the past, some of the divorced and widowed tended to report themselves as single. Some of the never-married with children born out of wedlock may have reported themselves as widowed. Furthermore, mortality does vary by marital status, depleting the population of never-married persons more than the population of the ever-married. Because its assumptions are not strictly met, the SMAFM is always no more than an imperfect and approximate measure. Nevertheless it is a very widely usable and effective one.

We illustrate the calculation of the SMAFM with an example from Egypt. The table gives the most recent data for Egypt from the U.N. Demographic Yearbook of 1990, pages 894-895:

The first columns two columns come from the Egyptian Census of September 1986. We divide ever-married women by total women to get the proportion ever-married in the age group from  $x$  to  $x+n$ ,  $F(x)$ . We then look down the  $F(x)$  column till we reach the largest value before the first decrease. We call this value  $F_{ult}$ . It is our estimate of the proportion ultimately marrying, that is, the proportion ever-married in the population. In this table  $F_{ult} = 0.963$ . It occurs in the age-group from 35 to 40. It is marked with a star. The first decrease occurs with the next value, 0.958. Even though there is an increase afterwards, to 0.972, it has no bearing on  $F_{ult}$ , which is determined by the position of the first decrease. If there is no decrease in a table, then the last value of  $F(x)$  supplies  $F_{ult}$ .

Table 9.2: Data from Egypt, 1990.

Age x	Total Egyptian Women	Ever- Married Women	F(x)	G(x)	nG(x)
0	4,533,002	0	0.000	1.000	5.000
5	3,469,468	0	0.000	1.000	5.000
10	2,153,358	0	0.000	1.000	5.000
15	1,764,959	366,014	0.207	0.785	3.923
20	1,889,617	1,145,262	0.606	0.371	1.853
25	1,824,119	1,573,377	0.863	0.104	0.521
30	1,484,115	1,376,066	0.927	0.037	0.185
35	1,486,419	1,431,268	*0.963	0.000	0.000
40	1,070,386	1,025,612	0.958		
45	1,011,132	983,191	0.972		
SMAFM					21.482

We next calculate  $G(x) = 1 - F(x)/F_{ult}$  for all rows up to the row for  $F_{ult}$ . This is our estimate of the proportion of those who do ultimately marry who are still single in the age group from  $x$  to  $x + n$ . We omit all the rows beyond  $F_{ult}$ . For the next column, we multiply  $G(x)$  by the width of the age group  $n$ . The sum of the values in this final column is the Singulate Mean Age at First Marriage, the SMAFM.

The only complicated part of this calculation is the selection of  $F_{ult}$ . (The rule given here is not in Hajnal's original article.) If the  $F(x)$  had come from cohort rather than period data, then  $F(x)$  would have to increase at all ages, and  $F_{ult}$  would be the last value in the table. However, because we are using period data, the observed proportion ever-married can drop as we look at older age groups, and in many cases like this one it bounces around. Although we have designed our measure assuming that marriage rates have not been changing, in fact they do change. The older the ages of people, the more time has generally gone by between their first marriages and the date of the census, and



the more opportunity there has been for marriage rates to change. Differential death rates by marital status may also operate. The effects are generally not large in absolute terms, but they may be large enough compared to the proportions of people having their first marriages at older age groups to make the observed proportions bounce up and down as they do here.

In calculating the SMAFM, it is essential to start at age zero. Even though there are no marriages before age 15 in these data, the three age groups below age 15 contribute fully 15 years out of the total sum of around 21.5. If they are omitted, absurd singulate mean ages at first marriage like 6 years or 7 years result.

The concept behind the SMAFM is a cohort concept, even though the estimate is generally an estimate based on period data. The Singulate Mean Ages at First Marriage for men and women indicate when in the lives of men and women their first marriages occur. These means can be very different from the average ages of grooms and brides marrying (for the first time) in a population in a given year. The latter statistic depends heavily on the age structure of the population. Even if only a small proportion of women marry at age 15, if the population consists mostly of fifteen-year-olds, the average age of brides will be close to 15.

## 9.4 \*The Singulate Mean Formula

In symbols, the formula for the Singulate Mean Age at First Marriage is

$$SMAFM = \sum(n) \left( 1 - \frac{F(x)}{F_{ult}} \right)$$

This formula is easiest to understand in terms of our analogy with lifetable quantities. The proportion still single out of those ever-marrying in a cohort is analogous to  $l_x/l_0$ . We ask, what correspond to the proportion still single out of those ever-marrying in the age group  $x$  to  $x + n$  in a cross-sectional, period population?

We have already answered this question in our study of the formulas for Leslie matrices. Suppose we have  $n$  successive one-year birth cohorts, each of the same size  $l_0$ , born between  $x + n$  and  $x$  years ago. The first-born of these cohorts has  $l_{x+n}$  surviving members today. The last-born of the cohorts has  $l_x$ . Altogether, the number of surviving members of all  $n$  cohorts is the area under the  $l_x$  curve from  $x$  to  $x + n$ , which is  ${}_nL_x$ . The proportion surviving is  ${}_nL_x/(nl_0)$ .

Now the estimated proportion still single out of those ever-marrying observed in the period population in the age group from  $x$  to  $x + n$  is  $1 - F(x)/F_{ult}$ . Therefore,

$$n(1 - \frac{F(x)}{F_{ult}}) \text{ corresponds to } \frac{{}_nL_x}{l_0}$$

$$SMAFM = \sum n(1 - \frac{F(x)}{F_{ult}}) \text{ corresponds to } \sum \frac{{}_nL_x}{l_0} = \frac{T_0}{l_0} = e_0$$

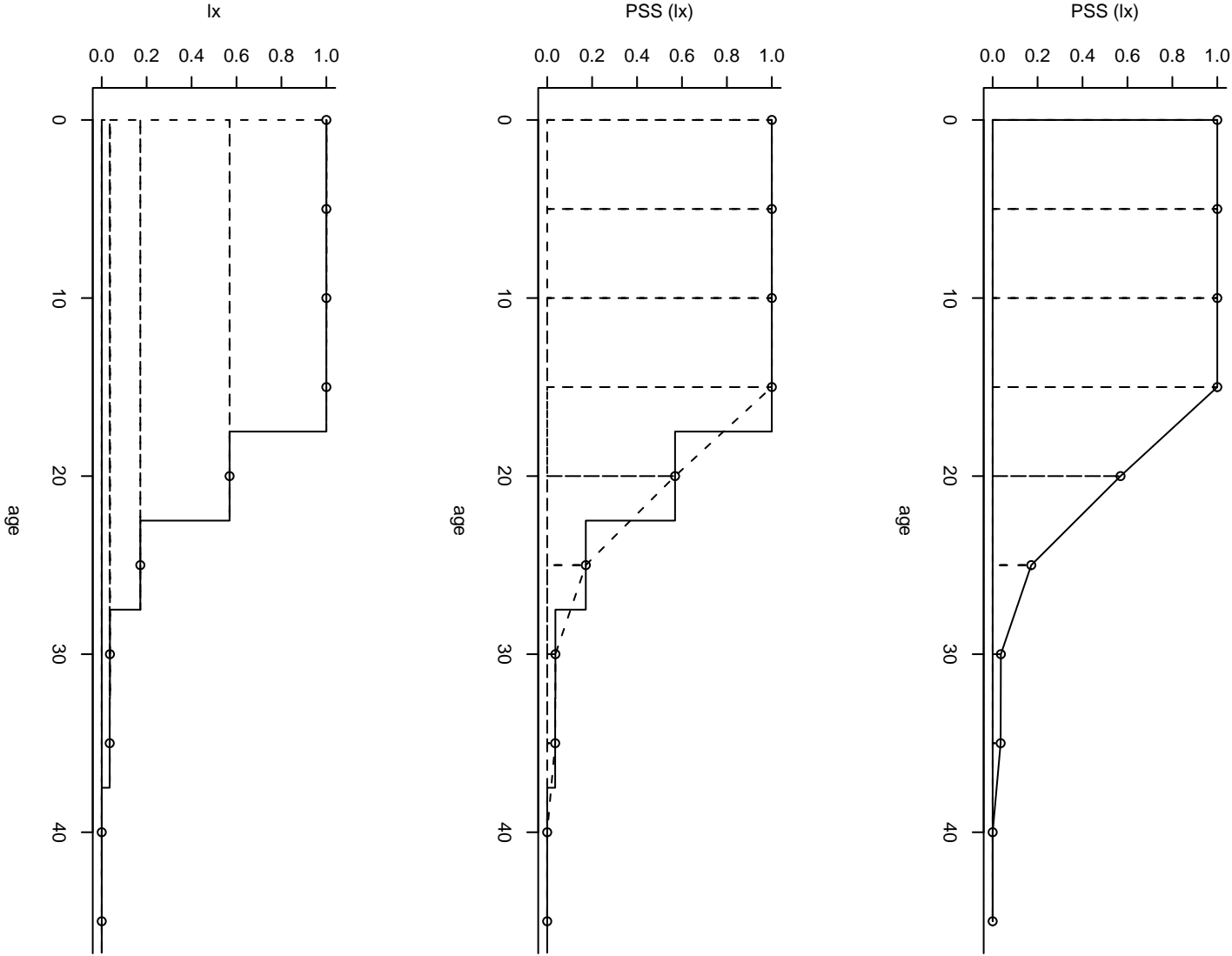
Thus the Singulate Mean Age at First Marriage is just the “expectation of single life” at age zero, and the singulate mean formula corresponds exactly to the standard lifetable formula for expectation of life.

We can give this formula a revealing geometric interpretation. The top panel of Figure 8.1 shows a cohort’s proportion still single (PSS) as a function of age  $x$ . Let us assume a radix  $l_0 = 1$ , so the PSS corresponds to  $l_x$ . The area under this curve between  $x$  and  $x + n$  is, as usual,  ${}_nL_x$ . The total area under the curve is  $\sum_n L_x = T_0 = e_0$  since the radix is unity.

Each of the segments under the curve is approximately formed by a rectangle with a triangle on top of it. The area of the triangle is the same as the area of a rectangle of the same height with half the base. So the area under the curve is approximately the same as the area under a step function which goes down in steps in the middle of each interval, as shown in the middle panel of the figure. The total area is made up of the sums of the areas of the vertical blocks.

Now notice that we can carve up the same area in another way. Instead of vertical blocks, we carve the area into horizontal

Figure 9.1: Singulate Means as Areas



blocks. The process is shown in the bottom panel of the figure. The length of the top block is  $15 + 5/2 = 17.5$  years. The height of the curve at the top of the block is  $l_{15}$  and the height of the curve at the bottom of the block is  $l_{20}$ , so the height of the block is  $l_{15} - l_{20} = {}_5d_{15}$ . The area of the block is therefore  $17.5{}_5d_{15}$ . The length of the next block is  $20 + 5/2$  and its height is  ${}_5d_{20}$ , for an area of  $22.5{}_5d_{20}$ . The total area under the curve is

$$\sum (x + n/2)_n d_x$$

We start the sum at age zero, even though the first few terms do not contribute in our particular example because in our picture  ${}_n d_x = 0$  before the age of 15 when people begin to exit the single state. What we have done is to give a general geometrical proof that

$$\sum_n L_x = \sum (x + n/2)_n d_x$$

In words, that the expectation of life at birth is the average cohort age of death.

To make the singulate mean calculation work we need several things to be true about our picture:

1. We need a left-hand edge (the vertical axis) to bound the area. We can only apply the singulate mean calculation to quantities like age which are never negative (or at least which are all bounded below by the same value which can define a left-hand edge).
2. We need a bottom edge (the horizontal axis) to bound the area. We have to restrict ourselves to the subgroup of people who actually do attain the state of interest, in this case the subgroup of ever-marrying people. The curve has to go down to zero. Otherwise the area is not defined.
3. We need a non-increasing curve. If our curve went down and then back up, we would not have whole horizontal blocks, but bits and pieces of blocks. The sums based on the horizontal approach would become uninterpretable.

These three conditions are the conditions we have already mentioned for using the singulate mean approach with quantities other than ages at death or marriage.

Let us go back to an approximation we made which was not in fact necessary. In our picture, we approximated the three-sided top bit of each bar with a triangle whose base is half the interval length,  $n/2$ . We could have reproduced the area of the bit exactly by taking a triangle whose base was  ${}_n a_x$ , the lifetable quantity which is the average years lived in the interval per person for those exiting (life or singlehood) during the interval. Then we would have had

$$\sum (x + {}_n a_x) {}_n d_x$$

Let us examine the  $\sum x {}_n d_x$  part of this sum. Taking  $n = 5$ , we arrange terms in the following table:

Table 9.3: A Telescoping Sum

$$\begin{array}{rcccc} 0l_0 & -0l_5 & & & \\ & +5l_5 & -5l_{10} & & \\ & & +10l_{10} & -10l_{15} & \\ & & & +15l_{15} & -15l_{20} \end{array}$$

Notice that each negative term partly cancels the next positive term. We end up with only half as many terms. This is an example of a “telescoping” sum, which folds up in the way older refracting telescopes used to do. The result is the following:

$$5l_5 + 5l_{10} + 5l_{15} + 5l_{20} + \dots$$

If we write the sum as a sum over values of  $x$  starting at zero, the first term in this sum is  $0 + 5$  or  $x + n$  for  $x = 0$ . So the sum can be written  $\sum (n)l_{x+n}$ . The other part of our original sum was  $\sum {}_n a_x d_x$ . Putting them together, we have

$$\sum (n)l_{x+n} + {}_n a_x d_x$$

which is exactly  $\sum L_x$  as we claimed.

We have now proved the equivalence of  $\sum_n L_x$  with  $\sum (x +_n a_x)_n d_x$  in two ways, geometrically and algebraically. We can also write down the same proof in terms of calculus. The switch from areas of horizontal blocks to areas of vertical blocks is the method known as “integration by parts”: We start with

$$\sum_0^\infty (x +_n a_x) \frac{l_x - l_{x+n}}{n} n$$

As we let our interval width  $n$  go to zero, we have more and more smaller and smaller terms. The quotient  $\frac{l_x - l_{x+n}}{n}$  becomes the derivative  $\frac{-dl_x}{dx}$ . The quantity  $_n a_x$  becomes negligible, and  $n$ , which is the little change in  $x$  in each interval, is written as  $dx$ . The sum becomes an integral, and we have

$$\int_0^\infty x \frac{-dl_x}{dx} dx$$

The formula for integration by parts for two smooth functions  $u(x)$  and  $v(x)$  is

$$\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b v \frac{du}{dx} dx$$

In our application,  $u(x) = x$ ,  $v(x) = -l_x$ ,  $a = 0$ , and  $b = \infty$ . So we have

$$\int_0^\infty x \frac{-dl_x}{dx} dx = [-xl_x]_0^\infty - \int_0^\infty -l_x dx$$

The terms involving  $-xl_x$  at the endpoints zero and infinity vanish, because  $x$  is zero at zero (ages are all non-negative), and  $l_x$  goes strictly to zero (everyone eventually attains the state like death or like being ever-married in the group with which we are concerned). Therefore, the right-hand side is just

$$\int_0^\infty l_x dx = e_0 l_0$$

In our formula,  $l_x$  is a non-increasing function. We do not use this fact when we do our integration by parts. The formulas would still be true if  $l_x$  were some other function that could increase. But in that case we could no longer interpret

$$\int_0^\infty x \frac{-dl_x}{dx} dx$$

as a mean.

## Highlights of Chapter Nine:

Complexities of marriage.

The European Marriage Pattern

The Singulate Mean Age at First Marriage

## Key Formulas

$$SMAFM = \sum(n) \left( 1 - \frac{F(x)}{F_{ult}} \right)$$

## Exercises for Chapter Nine

1. Table 9.4 shows percents ever-married by age for cross-sectional data on U.S. women at two periods, with one tabulation by race. Compute the difference between the singulate mean age at first marriage for white women in 1990 and the singulate mean age at first marriage for black women in 1990. Assume, in the absence of data, that 11.2 percent of white women and 7.5 percent of black women were ever-married in the 15 to 20 year age group.

2. Tables 9.5 9.6 shows percents of U.S. women divorced by age and percents remarried by age. Calculate the nearest quantities you can to a singulate mean age at first divorce for all races in 1990 and a singulate mean age at remarriage for all races in 1990. Do these quantities make demographic sense. Comment briefly on the conceptual or practical limitations involved in applying the singulate mean approach to each of these data sets.

Table 9.4: Percent Ever-married Among U.S. Women 1975 and 1990

Category	All races 1975	All races 1990	White 1990	Black 1990
20 to 25	62.5	38.5	41.3	23.5
25 to 30	87.2	69.0	73.2	45.0
30 to 35	93.1	82.2	85.6	61.1
35 to 40	95.5	89.4	91.4	74.9
40 to 45	95.8	92.0	93.4	82.1
45 to 50	95.9	94.4	95.1	89.7
50 to 55	95.8	95.5	96.1	91.9

Table 9.5: Percent of Ever-Married Women Divorced After First Marriage among U.S. Women 1975 and 1990

Category	All races 1975	All races 1990	White 1990	Black 1990
20 to 25	11	13	13	10
25 to 30	17	19	20	18
30 to 35	20	28	29	27
35 to 40	22	34	35	36
40 to 45	21	36	35	45
45 to 50	21	35	36	40
50 to 55	18	30	29	39



Table 9.6: Percent of Divorced Women Remarried After Divorce  
U.S. Women 1975 and 1990

Category	All races 1975	All races 1990	White 1990	Black 1990
20 to 25	48	38	39	-
25 to 30	60	52	53	44
30 to 35	64	60	61	42
35 to 40	70	65	67	54
40 to 45	70	67	70	50
45 to 50	70	66	67	55
50 to 55	74	63	65	50

Source: U.S. Census Bureau (1992) "Marriage, Divorce, and Remarriage in the 1990s", Report P23-180.



# Chapter 10

## Stable Age Structures

### 10.1 Age Pyramids

We turn now to the study of age structure. There is a very full and satisfactory theory which accounts for the relative numbers of young and old men and women in a population. The basic idea is to obtain formulas for how a population will be distributed by age if the population has been closed to migration and if its birth and death rates have been unchanging for a long time. The actual age distribution of the population naturally differs from this theoretical age distribution. Each of the deviations is then explained by reference to particular events of migration and particular changes in rates in the prior history of the population. Thus the age distribution of each population has *general* features which it shares with populations with the same vital rates and *special* features which are derived from its own particular history.

Two pairs of age pyramids illustrate the contrast. Figures 10.1 and 10.2 show two examples of the sorts of age structures that would occur in theory in closed populations with unchanging vital rates. Figure 10.4 and Figure 10.3 show two examples of age pyramids of actual countries in the last few decades, each of which has features that reflect the country's own recent history.

Figure 10.1: Stable Age Pyramid with High Growth

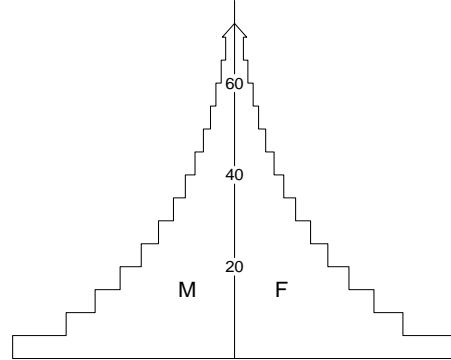
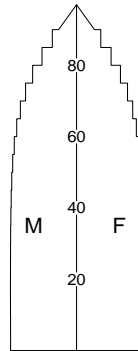


Figure 10.2: Stable Age Pyramid with Zero Growth



The graphical diagrams of age structure shown in these figures are called “Age Pyramids” because they are shaped more or less like Egyptian or Mayan pyramids. An age-pyramid is made up of a pair of bar graphs, one for men and one for women, turned on their sides and joined. The vertical axis corresponds to age. For each age group, the bar coming off the axis to the right represents the number of women in that age group, the bar to the left the number of men. The young are at the bottom, the old at the top. Demographers often use the words “age pyramid”, “age distribution”, and “age structure” interchangeably. The age pyramid represents the distribution of the population by both age and sex. On the pyramids graphed here, the open-ended age group at the

top is shown with triangles instead of bars.

Figure 10.3: Actual Age Pyramid with High Growth

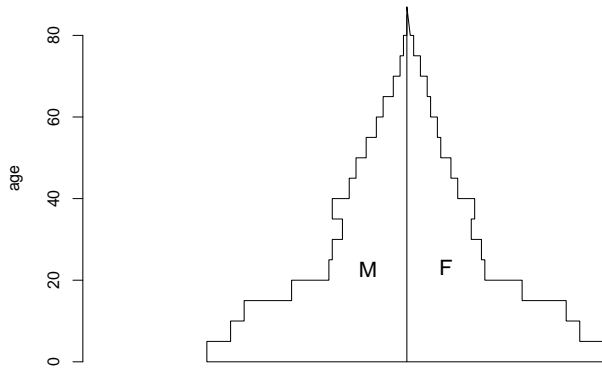
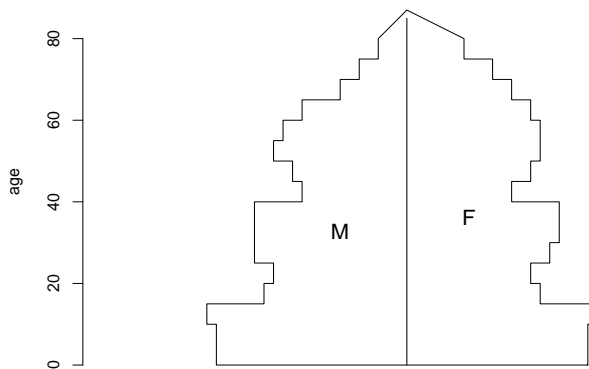


Figure 10.4: Actual Age Pyramid with Moderate Growth



A population which has been produced by keeping age-specific rates of fertility and mortality constant over a long period of time is called a *stable population*. As we shall see, such a population has an age pyramid which is determined uniquely when we know the population's life table and its long-term growth rate. The proportions in each age group in a stable population do not change over time. The numbers in each age group may, however, change over time, for the population may be growing or declining in size, depending on what the growth rate happens to be.

Demographers use the word “stable” in a technical sense, distinguishing it from the word “stationary”. In a stable population,

the rates stay the same, but the population size may change. In a *stationary* population, both the age-specific rates and the population size remain the same. The growth rate is zero. A stationary population is a special case of a stable population which satisfies the extra condition of having “ZPG”, zero population growth.

The mathematical study of formulas for the stable age pyramid and uses of these formulas is called “Stable Population Theory”. It is an elegant theory which goes back more than 200 years to work of Leonhard Euler in 1760, developed by Laplace and Alfred Lotka and in our own time by Nathan Keyfitz and Ansley Coale. Keyfitz and Caswell (2005) is an excellent source.

We first find a formula for the numbers in the age groups of a stationary population, worked out in Section 10.2. We go on, in Section 10.5 to see how our formula can be modified to take into account effects of non-zero rates of growth.

## 10.2 Stationary Populations

We study age distributions for stationary populations by constructing an example of a stationary population using the concept of a synthetic cohort developed in Chapter Six. When we wanted to define period measures like the period NRR, we introduced a “Game of Pretend”, imagining a current set of age-specific rates continuing unchanged into the future. A cohort of babies born today are to have the rates of today’s 20-year-olds when they reach 20, and the rates of today’s 80-year-olds when they reach 80. As we have emphasized, the lifecourse experience of such an imaginary “synthetic cohort” is pieced together or “synthesized” from the experience of different age groups in today’s period population. A synthetic cohort differs from real cohorts whose members’ lives reflect the accumulating trends and changes in demographic rates that actually occur.

We can carry this idea further and think of launching a new copy of a synthetic cohort on its lifecourse every year. Because

we are interested in stationary populations, we make the initial sizes of these cohorts constant at some number  $B$  of births per year. The cohorts are all imaginary, invented by us. Later cohorts are not meant to be the children of earlier cohort members or of anyone else. What we care about is their survival. Because age-specific mortality rates are assumed to be unchanging, the number of survivors to age  $x$  in each cohort is the same quantity  $Bl_x/l_0$ .

We gather up all the members from different cohorts alive at some instant of time far in the future. The number age  $x$ , belonging to the cohort born  $x$  years in the past, is  $Bl_x/l_0$ . The count for a whole age group between age  $x$  and  $x + n$  can be found by summing up the  $l_x$  values (or taking the area under the  $l_x$  curve), just as we did to find formulas for Leslie Matrices in Chapter Four. The sum of little- $l$  values is a big- $L$  value,  ${}_nL_x$ . Our population is stationary; because  $B$  is not changing over time, age-group sizes are not changing over time. We see the same age pyramid, whatever instant in the future we pick. With our construction, we gain a formula for counts by age in a stationary population:

$${}_nK_x = B \frac{{}_nL_x}{l_0}$$

Our formula applies to any stationary population which is a special case of a stable population with  $R = 0$ , that is to say, to any stationary population produced by unchanging age-specific rates. Some demographers use the word “stationary” more broadly for any population whose total size is not changing, whatever the rates. But, from this point on, we reserve the word stationary for populations in which the size of every age group is unchanging, thanks to unchanging age-specific rates. Our formula for  ${}_nK_x$  is a formula for the counts of individuals in age groups. When we are interested in proportions instead of counts, we have to divide by total population. Births per year divided by total population equals the Crude Birth Rate  $b$ . To avoid having an extra symbol for totals, we write total population as the population from age 0 up to age infinity,  ${}_{\infty}K_0$ . Births per year  $B$  divided by total population  ${}_{\infty}K_0$  equals the Crude Birth Rate  $b$ . Our formula for

*proportions* in age groups in a stationary population has  $b$  in place of  $B$ :

$$\frac{{}_nK_x}{{}_\infty K_0} = b \frac{{}_nL_x}{l_0}$$

Proportions have to sum to 1. The values  ${}_nL_x/l_0$  on the right add up to  $T_0/l_0 = e_0$ . Thus we recover our Basic Identity for Stationary Populations. When  $R = 0$ ,  $1 = b e_0$ .

A country's observed age distribution often differs from the age distribution that would prevail under zero population growth. For example, in Pakistan in 2000, 42.0% of the population was less than 15 years old, and 3.6% were over age 65 according to the United Nations (2003:362). With a radix  $l_0 = 1$ , lifetables show  $T_0 = 60.00$ ,  $T_{15} = 46.53$  and  $T_{65} = 7.55$ . Using our formulas, we see that with zero population growth there would be a Crude Birth Rate  $b = 1/e_0 = 1/60$ , with  ${}_{15}L_0 = T_0 - T_{15} = 13.47$ , and  $(1/60)(13.47) = 0.224$  or only 22.4% less than 15 years old.

### 10.3 The Synthetic Cohort

Our construction in Section 10.2 establishes a correspondence between lifetables and stationary populations. The population corresponding to the synthetic cohort for a given lifetable is sometimes called the "Stationary Equivalent Population". The lifetable quantity  ${}_nL_x$  in the formula is in units of person-years, and if we take  $B$  to have units of people born per year and  $l_0$  to have units of people, we see that  $B {}_nL_x / l_0$  is in units of people, as it should be. There is a subtlety here. In taking  $B$  to have units of people per year, we are implicitly establishing a correspondence between person years in the lives of the synthetic cohort and persons in the stationary equivalent population.

One way to understand this correspondence is by an extended metaphor. Among poets of the Sixteenth Century like John Donne, mental pictures were often communicated with extended metaphors called "Metaphysical Conceits".



In our metaphor, we imagine a film director making a monumental epic film which portrays the experiences of people from various age-groups during a single year, for instance the Millennial Year A.D. 2000. The director picks a number  $B$  and hires  $B$  newborn babies to be his actors. When they are infants, they play the infant roles in his film. They play the part of one character for a whole year, and then change parts for the next year. When they are one-year-olds, the director films the roles depicting one-year-olds, and splices the sequences into his movie. Each year his team of actors plays the roles suited to their age. The characters are all people living in 2000, but the filming of the events as played by the actors occurs long afterwards, when the actors are old enough to play the roles.

In our metaphor, the actors survive or die according to the period lifetable for 2000. Their experiences on the movie set may be more like the world of 2000 than like the world going on outside around them. So the survivors of the cohort of actors when they reach a particular age, as given by the lifetable, is the same as the number of characters of that age that they can play. If all these characters had been brought together at some point in 2000, their numbers at each age would have matched the numbers in the stationary equivalent population. Each character would be paired with an actor surviving to that age. Some age groups in the actual population for 2000 may be over-represented in the film, and some age groups may be under-represented. The availability of actors dictates the group of characters that can be portrayed.

In our metaphor, a person-year in the life of an actor corresponds to a person depicted in the film. Each actor plays each role for one year. This provision establishes a correspondence between person-years and people which mirrors the different units that can be assigned to the demographic quantity  ${}_nL_x$ . The number of roles is much greater than the number of actors. Perhaps a thousand characters could be brought to the screen with an original group of a dozen actors. Each actor, in his time, plays many parts.

Our metaphor owes much to William Shakespeare, who made

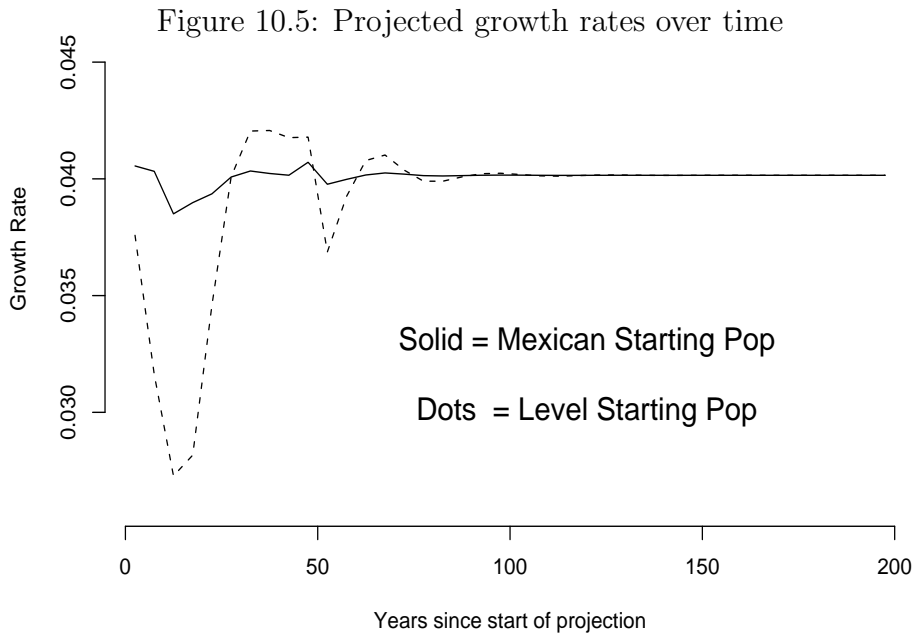
his character the melancholy Jacques into an impromptu demographer in *As You Like It*:

All the world's a stage,  
And all the men and women merely players:  
They have their exits and their entrances;  
And one man in his time plays many parts,  
His acts being seven ages. At first the infant,  
Mewling and puking in the nurse's arms.  
And then the whining school-boy, with his satchel,  
And shining morning face, creeping like snail  
Unwillingly to school. And then the lover,  
Sighing like furnace, with a woeful ballad  
Made to his mistress' eyebrow. Then a soldier,  
Full of strange oaths and bearded like the pard,  
Jealous in honor, sudden and quick in quarrel,  
Seeking the bubble reputation  
Even in the cannon's mouth. And then the justice,  
In fair round belly with good capon lined,  
With eyes severe and beard of formal cut,  
Full of wise saws and modern instances;  
And so he plays his part. The sixth age shifts  
Into the lean and slipper'd pantaloon,  
With spectacles on nose and pouch on side,  
His youthful hose, well saved, a world too wide  
For his shrunk shank; and his big manly voice  
Turning again toward childish treble, pipes  
And whistles in his sound. Last scene of all,  
That ends this strange eventful history,  
Is second childishness and mere oblivion,  
Sans teeth, sans eyes, sans taste, sans everything.

## 10.4 Consequences of Unchanging Rates

Stable populations are more general than stationary populations, because they may have any growth rate, whereas stationary populations have zero growth. To generalize our formulas, we need to study the effects of keeping age-specific fertility and mortality rates constant at levels for which births do not necessarily balance deaths. We continue to restrict our attention to closed populations either of men or of women. The consequences of unchanging rates can be seen by projecting a population forward over many steps using the Leslie Matrices of Chapter Five.

The behavior of the Growth Rate  $R$  in the face of unchanging age-specific rates is illustrated in Figure 10.5. The lines come from projections using a Leslie Matrix based on rates for Pakistani women in 2000. For the solid line, the projection starts with the female population of Pakistan in 2000. For the dashed line, the projection starts instead with the female population of Russia in 2000.



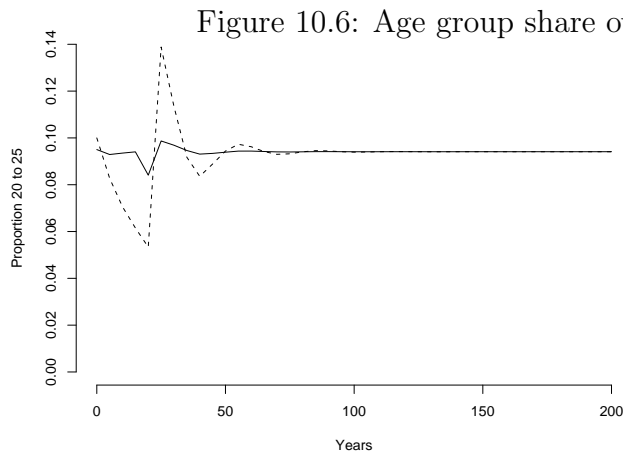
The outstanding feature of Figure 10.5 is the flattening out of both lines as time goes by. The growth rates bounce around for a couple of generations, and then they settle down to nearly constant values. What is more, they differ at the start, but they settle down to the same constant value. Under the pressure of constant age-specific rates, the growth rates converge to fixed values, and those values are independent of the choice of the initial population to project.

In general, populations projected forward with the same constant age-specific rates may differ in their short-term growth rates, but they share the same constant long-term growth rate. This rate has its own symbol,  $r$ . Capital  $R$  stands for any growth rate. Little  $r$  stands for the special long-term growth rate produced by a long stretch of unchanging demographic rates.

Little  $r$  is often called “Lotka’s  $r$ ” after Alfred Lotka, an mathematical biologist working in the United States in the early Twentieth Century. Its official name is the “intrinsic rate of natural increase”. It is “intrinsic” because it is built in to the fertility and mortality schedules, without regard to population numbers, and it is a rate of “natural increase” because it is defined for closed populations without regard to migration. One way to calculate  $r$  is to construct a Leslie Matrix and project any starting population forward until the growth rate settles down to its long-term value,  $r$ . There will be an extensive discussion of  $r$  later in the chapter, but its first definition is worth stating here:

Lotka’s intrinsic rate of natural increase  $r$  is the long-term growth rate of any population subject to unchanging age-specific rates of fertility and mortality.

The growth rate is not the only population characteristic which settles down to a fixed value when the population is subject for a long time to unchanging demographic rates. Figure 10.6 shows the proportion of the population under age 15 for the same projections as Figure 10.5. Like the growth rates, these proportions bounce around for a couple of generations and then settle down.

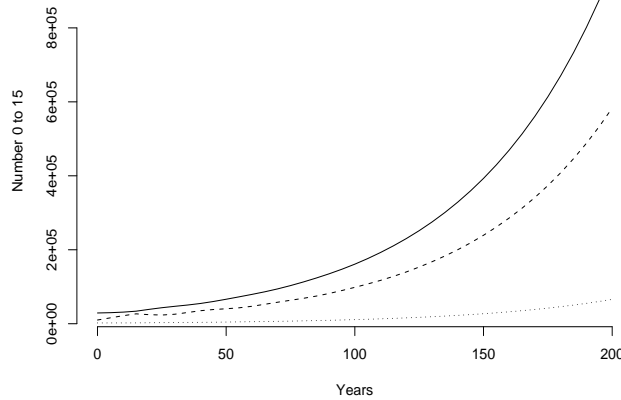


Like the growth rates, the curves derived from different starting populations differ in the short term and come to coincide over the long-term. The same would be true for the proportion in any age group. Under unchanging demographic rates, proportions in age groups stop changing and become independent of the starting population. The population becomes a stable population.

A Stable Population is a population in which the proportions in age groups remain exactly constant over time.

We must always remember that it is the *proportions* in age groups that are constant over time in a stable population, not the *sizes* of the age groups. Figure 10.7 show the sizes of the group of 0 to 15-year-olds over time from the same projections. The curves do not flatten out. They keep rising. Eventually, they settle into exponential growth. The curves based on the two different starting populations both take on the same shape, but they do not come to coincide. The sizes of the age groups at any time in the future do continue to depend on the starting population. They depend both on the size of the starting population and on its age distribution. Starting with a thousand young people about to embark on childbearing is bound to produce bigger future populations than starting with a thousand people over 35 with less childbearing in prospect.

Figure 10.7: Age group size over time



The growth rate is the slope of the logarithm of population size over time. When it settles down to  $r$ , total population is growing exponentially like  $e^{rt}$ . When the proportions in age groups have also settled down to fixed values, each age group, being a fixed fraction of the whole, is also growing exponentially like  $e^{rt}$ . The flow of births, reflected in the size of the youngest age-group of new-borns, is growing in the same way.

In a Stable Population, the size of every age group obeys the formula for exponential growth.

Growth rates and age-group proportions are independent of starting populations, but they are not independent of other inputs to projections. They do depend on the choice of rates that go into a Leslie Matrix. It is easy to experiment with projections using Leslie Matrices based on different lifetables and fertility schedules and obtain different values for  $r$  and for the age-group proportions.

Underlying these properties are the four main features illustrated by Figures 10.5, 10.6, and 10.7:

- A) Growth rates become fixed over time.
- B) Growth rates become independent of starting populations.

- C) Age pyramid proportions become fixed over time.
- D) Age pyramid proportions become independent starting populations.

These features are the consequences of the assumptions which go into the projections:

1. Age-specific rates of giving birth and dying are unchanging over time.
2. The population is closed to migration.
3. The female population can be projected independent of the males.
4. Randomness can be ignored, letting population estimates represent mean or expected values.
5. Children are reckoned in continuous fractions of child.

The first four of these assumptions are familiar from all our work with Leslie matrices. The last assumption, although implicit in what we have been doing, has not been spelled out before. Because we are dealing with expected values, the projected numbers of children from any one mother in the population are typically fractions rather than whole numbers. When we are interested in rates of growth or proportions in age groups, this fact is no handicap. But some kinds of statistics only make sense in terms of whole numbers. For example, if we want to project the mean number of sisters for girls under 20, we cannot do so with Leslie matrices in any straightforward way, because numbers of sisters depend on the distribution of family sizes among the possible whole-number family sizes that there can be, and not simply on the means. Stable population theory can be applied to many aspects of family and kinship, but not to all of them.

## 10.5 Stable Age Pyramids

A stable population grows exactly exponentially, and its growth rate is Lotka's intrinsic rate of natural increase  $r$ . These properties have been inferred from observations of the consequences of unchanging rates in projections shown in the figures in Section 10.4. They have not been proved here, although they can be proved. Our present goal is to use the property of exponential increase to generalize our formulas for the sizes and proportions in age groups. We derived formulas for stationary populations in Section 10.2. Now we want formulas for stable populations with any growth rate  $r$ .

In a stable population, the flow of births grows exponentially:

$$B(t) = B(0)e^{rt}.$$

In Section 10.2, we imagined launching synthetic cohorts year after year with a constant flow of births  $B$ . Suppose instead we launch synthetic cohorts with exponentially changing flows of births, amounting to  $B(t)$  in year  $t$ . Cohort members age  $x$  at time  $t$  were born  $x$  years earlier at time  $t - x$ . The starting size for their cohort depends on  $x$ :

$$B(t - x) = B(0)e^{r(t-x)} = B(0)e^{rt-rx} = B(0)e^{rt}e^{-rx} = B(t)e^{-rx}$$

This expression suggests substituting  $Be^{-rx}$  in place of  $B$  in our formulas. We might worry that the factor  $e^{-rx}$  would get in the way of adding up ages between  $x$  and  $x + n$ , but it turns out, as shown in Section 10.8, that simple substitution gives correct equations consistent with our Leslie Matrix formulas:

$$\text{Stable Counts: } {}_nK_x = B \frac{{}_nL_x}{l_0} e^{-rx}$$

$$\text{Stable Proportions: } \frac{{}_nK_x}{\infty K_0} = b \frac{{}_nL_x}{l_0} e^{-rx}$$

The critical new component is the exponential factor  $e^{-rx}$ . The minus sign is essential. The formula for exponential growth as a



function of time has a plus sign  $e^{+rt}$ . Our new formula, as a function of age, has a minus sign  $e^{-rx}$ . This sign makes sense. “Up in age” means “back in time” in the sense of “back in time of birth”. In a growing population, with  $r > 0$ , the age pyramid has to shrink as we go up in age, because birth cohorts shrink as we go back in time. In declining populations, with  $r < 0$ , the effect is reversed. Older people come from larger cohorts from the past. The exponential factor makes for larger numbers of older people, but declining survival with age makes for smaller numbers and eventually wins out.

The three possibilities  $r > 0$ ,  $r = 0$ , and  $r < 0$  lead to three kinds of characteristic shapes for stable age pyramids. With positive growth the shape is broad at the bottom, like the pyramids at Gizeh or a well-shaped Christmas tree. There are lots of young people and fewer old people. Those born later are more numerous than those born earlier. With zero growth, in stationary populations, the shape is steep, rather like a beehive. The life table  ${}_nL_x$  values determine the slope. With negative growth, the base is narrower and the pyramid typically bulges out, rather like a lightbulb. When an age-pyramid is studied, the first step is to assign it to one of these three classes on the overall basis of its shape.

The neat expression for the total size of a stationary population  $Be_0$  does not have a simple counterpart for stable populations. To find the total size, one has to add up all the age group sizes group by group. We can, however, see that the constant  $B$  is the right constant for all stable populations. For the youngest age group, with  $x = 0$ ,  $e^{-rx}$  is 1. As we make  $n$  very small, like a day or an hour or some other small fraction of a year, we make  ${}_nL_0/l_0$  close to  $n$  and  ${}_nK_0$  close to  $Bn$ . Our age group of newborns comes out, as it should, to  $B$  births per year times the corresponding fraction  $n$  of a year.

## 10.6 Dependency Ratios

The formula for the Stable Age Pyramid has many applications throughout demography. A good example is the study of dependency ratios and the effects of population growth. In an economy, consumption by children and retired people depends on production by people in prime working ages. A dependency ratio measures the balance between these groups. A young-age dependency ratio is often defined with the number of children up to age 15 in the numerator and the number in the economically active ages between 15 and 60 in the denominator. A similar old-age dependency ratio has the number over 60 in the numerator along with the same denominator. Adding the two ratios gives an overall dependency ratio. Along with observed dependency ratios based on current counts, we can calculate ratios that would be found in a stable population with the intrinsic growth rate and life table for that population. Observed ratios are influenced by migration, prior history, and other special features. The systematic effect of growth is easier to see when we calculate ratios from the stable population formula.

Table 10.1 illustrates a stable population calculation for India. The intrinsic growth rate based on information for 2000 is  $r = 0.006$ , less than half the current observed growth rate. The first column shows the starting age  $x$  for each age group in the numerator of the dependency ratio. The second column shows the observed count for men and women in millions and the third column the combined-sex life table  ${}_nL_x$  values inferred from projection data from the United Nations (2003:493). We multiply each  ${}_nL_x$  value by the factor  $e^{-rx}$  in the next column to obtain counts. We do not know the value of the factor  $B$  in the stable formula, but it will not matter. The same value of  $B$  occurs in the numerator and in the denominator and cancels out when we take ratios. Adding up the counts in the fifth column, we obtain a numerator for a young-age dependency ratio of 13.298.

A similar calculation with values from Table 10.2 produces a denominator of 30.717. The young-age dependency ratio for the

Table 10.1: Stable Population for India, Dependent Ages

x	Observed Count (millions)	Lifetable ${}_5L_x$	Factor $e^{-rx}$	Product
0	120.878	4.649	1.000	4.649
5	116.296	4.549	0.970	4.414
10	109.984	4.497	0.942	4.235
	<hr/> 347.158			<hr/> 13.298

stable population equals  $13.298/30.717 = 0.433$ . It is quite a bit below the young-age dependency ratio in the observed population, which equals  $347.158/593.111 = 0.585$ .

Dependency ratios for stable populations are especially interesting, because we can compare what we would see with different growth rates. Had India had an intrinsic growth rate equal to its current Crude Growth Rate of 0.015, with  $r = 0.015$  in our formula we should find a young-age dependency ratio of  $12.740/22.887 = 0.556$ , close to the observed ratio. Had India reached sustained zero population growth, with  $r = 0$  we should have  $13.695/37.573 = 0.364$ .

Old-age dependency ratios for stable populations with different choices of  $r$  can also be calculated from Table 10.2. They are higher for  $r = 0$  and  $r = 0.006$  than for  $r = 0.015$ . Under the current Indian lifetable, it turns out that the reduction in young-age dependents that accompanies lower growth is nearly balanced by an increase in older-age dependents. The overall dependency ratio is 0.687 with  $r$  equal to zero, 0.696 with  $r = 0.006$ , and 0.743 with  $r = 0.015$ . By accident, the overall dependency ratio with the current intrinsic rate of  $r = 0.006$  is very close to the observed overall dependency ratio of 0.714, although the stable age pyramid is not close to the observed age pyramid, since India's fertility rates have been changing and declining over recent decades.

Table 10.2: Data for Older Age Groups in India, 2000

x	Observed Count (millions)	Lifetable ${}_5L_x$	x	Observed Count (millions)	Lifetable ${}_5L_x$
15	100.852	4.466	60	26.743	3.386
20	88.504	4.426	65	20.861	2.948
25	83.604	4.371	70	14.426	2.381
30	75.671	4.301	75	8.617	1.706
35	66.900	4.228	80	4.250	1.023
40	58.114	4.145	85	1.468	0.477
45	48.176	4.042	90	.340	0.158
50	39.033	3.900	95	.047	0.034
55	32.257	3.694	100	.005	0.005

Economists call these kinds of calculations with stable age pyramids “comparative statics”. The word “statics” means the study of things that stay fixed, in contrast to “dynamics”, the study of things in the process of movement or change. With our stable population calculations, we are comparing what we would see in a country whose growth rate stayed fixed at one value, compared to what we would see in a country whose growth rate stayed fixed at some other value. If the growth rate of India fell toward zero, the population would not be stable and we would see fluctuations in the dependency ratio for a while. The dynamic story would be more complicated than the static story, and it would depend on the details of changing rates. Comparative statics gives us information that is easier to interpret and which tells us about the underlying impact of alternative rates of growth.

Whenever we apply the stable population formula, it is essential to use age groups that are as narrow as the data permit. If we have data for five-year-wide age groups, it would be wrong to take  ${}_{15}K_0$  to equal  ${}_{15}L_0e^{0r}$ . Instead we split  ${}_{15}K_0$  into  ${}_5K_0 + {}_5K_5 + {}_5K_{10}$  and add up  ${}_5L_0e^{0r}$  plus  ${}_5L_5e^{-5r}$  plus  ${}_5L_{10}e^{-10r}$ . If we have sepa-

rate data for ages 0 to 1 and 1 to 5, we should work out  ${}_5K_0$  with  ${}_1L_0e^{0r}$  plus  ${}_4L_1e^{-1r}$ .

Often we want to find ratios, and, when we do, the multiplier  $B$  in the stable formula does not matter. For counts rather than ratios we would need to be told the value of  $B$ . For proportions, we have two options. One option is to add up stable population counts for all age groups and use this total population as a denominator for calculating proportions. Another option is available if we have outside information on the crude birth rate  $b$  in the stable population. Then we know that the stable proportion aged  $x$  to  $x + n$  is given directly by  $b {}_nL_x e^{-rx}$ . For the Indian stable population with  $r = 0.006$ , we have  $b = 0.019$ . Referring back to Table 10.1, we calculate a proportion of children under age 15 equal to  $(.019)(13.298) = .253$ .

Economists often make more refined calculations applying age-specific participation rates to counts in stable age groups. Instead of counting everyone above age 60 as a dependent, we can multiply each age-group count by the proportion of the age group who are not in the labor force. To study consumption and production, we can multiply each age-group count by an age-specific consumption rate, do the same with an age-specific production rate, and compare total consumption to total production and their relationship to the underlying rate of population growth as it affects the economy.

For India, as we seen, reductions in counts at young ages are nearly balanced by increases at older ages, when we look at lower growth rates. The net economic effect depends on the costs and productive output of each age group. Costs include education for the young and medical care for the old, and a number of other forms of transfers between age groups and generations. Ronald Lee (1994) has developed detailed frameworks for measuring inter-generational transfers. Contributors to a collection by Peter Laslett and James Fishkin (1992) grapple with issues of justice and fairness that come into play. Calculations with our stable population formula provide the starting point for these studies and for much of economic demography.

## 10.7 The Many Faces of Lotka's $r$

We need a way of finding  $r$  without having to rely on computer projections. . There is a handy approximation which is simple and surprisingly accurate:

$$r \approx \frac{\log(NRR)}{\mu}$$

This approximation is easy to understand. Lotka's  $r$  is the growth rate of the stable population. We start with our usual formula for a growth rate  $R = (1/T) \log(K(T)/K(0))$ . The  $NRR$  is the factor by which the population grows in one generation, so we can substitute it for  $K(T)/K(0)$ . The mean age  $\mu$  is, roughly speaking, the length of time it takes for the daughter generation to come along, so we can replace  $T$  by  $\mu$ . We end up with our approximation for Lotka's  $r$ . Like  $r$ , both the  $NRR$  and the mean age  $\mu$  depend only on  ${}_nL_x$  and  ${}_nF_x$  and the fraction female at birth.

This formula is easy to apply. For instance, at the height of the baby boom, around 1960 the  $NRR$  for the United States was about 1.7. The mean age  $\mu$  was around 27, so we find

$$r \approx \frac{\log(NRR)}{\mu} = \frac{\log(1.7)}{27} = 0.019653$$

For 1987, we have  $r \approx (1/28) \log(.9) = -0.003763$ .

Does the negative value for  $r$  mean that the population of the United States was declining? By no means! Lotka's  $r$  is the long-term growth rate that would occur if the rates for 1987 remained in effect for a very long time. In fact, the rates of fertility went up shortly after 1987. The United States is a good example of a population in which the value of  $r$  has been negative while the current growth rate has been strongly positive. This has occurred because the United States does not have a stable population.

Although there is no necessary relationship between  $r$  and the current growth rate, there is a relationship between  $r$  and the Net

Reproduction Ratio:

$$\begin{aligned} r &> 0 \text{ if and only if } NRR > 1 \\ r &= 0 \text{ if and only if } NRR = 1 \\ r &< 0 \text{ if and only if } NRR < 1 \end{aligned}$$

The approximation for  $r$  in terms of  $\log(NRR)$  suggests that this relationship ought to be true. A mathematical proof will be sketched later in this chapter. We shall see that there is an exact equation with a unique solution which  $r$  satisfies. But there is no direct way to solve this equation, so our approximation is the usual approach.

We sum up our discussion by listing the names and the roles of  $r$ . Little  $r$  goes by several names:

- *The intrinsic rate of natural increase:* It is intrinsic because it is built into the rates of age specific fertility and mortality without reference to the accidents of age structure. It refers only to “natural” increase from an excess of births over deaths, taking no account of migration.
- *Lotka's parameter:* Although it goes back to Euler, little  $r$  became familiar through the work of Alfred Lotka.
- *The Malthusian parameter:* This name, popular in France, can be confusing. Malthus emphasized limits to population growth, whereas  $r$  is the rate at which a population (with unchanging birth and death rates) would grow exponentially if there were no Malthusian limits. The name comes from passages in which Malthus did discuss exponential growth. In evolutionary biology,  $r$  is important because Darwinian natural selection can often be described as favoring genes which promote vital rates implying higher values of  $r$ .

In this section we have emphasized three roles for little  $r$ :

- $r$  is the exact growth rate of the stable population
- $r$  is the long-term growth rate when any initial population is projected forward in time with permanently unchanging rates
- $r$  is the parameter that appears in the  $e^{-rx}$  factor in the formula for the stable age pyramid.

Mathematical demographers also keep in mind two other roles that will be defined and discussed in the next section:

- $r$  is the unique real solution to the Euler-Lotka Equation
- $r$  is the ratio of the leading eigenvalue of the Leslie Matrix to the projection step  $n$ .

## 10.8 \*The Euler-Lotka Equation

### 10.8.1 \*Verifying the Stable Formula

Our approach to Stable Population Theory in Sections 10.4 and 10.5 has been informal. We have observed from the outputs of computer projections that the proportions in age groups appear to settle down to fixed values when rates of fertility and survival remain constant, but we have not proved that they do. Most proofs require more advanced mathematics. But one fact can be established with elementary arguments. We can show that our formula for a stable age structure does indeed give a stable age structure. We do so in this section, because the demonstration gives extra insight into the nature of stable populations.

Consider, first, how we check whether a given vector of counts in age groups is or is not a stable age distribution for a particular



Leslie Matrix. For example, let  $A$  and  $K$  be the following stylized matrix and vector:

$$A = \begin{pmatrix} .9 & .8 & .7 \\ .6 & 0 & 0 \\ 0 & .5 & 0 \end{pmatrix} ; K = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

Is  $K$  a stable population vector for  $A$ ?

For  $K$  to be stable, the proportions in the age groups have to stay the same when we project  $K$  forward using  $A$ , which means that every age group in  $K$  has to change by the same factor. Performing matrix multiplication, we find  $AK$  is the vector with elements 7.4, 2.4, and 1.5. The first age group increases by a factor of  $7.4/4 = 1.85$ . The second decreases by a factor of  $2.4/3 = 0.8$ , and the third by a factor of  $1.5/2 = .75$ . The factors are not the same, so  $K$  is not a stable vector.

The largest factor 1.85 is more than twice as big as the smallest factor 0.75. On a log scale, the range from smallest to largest amounts to  $\log(1.85/0.75) = 0.90$ . If  $K$  were a stable vector, the range on the log scale would be zero, since the factors would all come out to be the same.

We now perform the same test on our stable formula, working with symbols instead of with numbers. We apply our expression for the Leslie Matrix  $A$  to our expression for the stable age vector  $K$ . We obtain an expression for the elements of the new vector, which we temporarily call  $N$  and which is given by the matrix product  $AK$ . We divide each element of the new vector  $N$  by the corresponding element of the starting vector  $K$ , and ask whether the answers come out to be the same.

We are interested in proportions, so constants like  $B$  and  $l_0$  do not matter, and we take  $B = l_0 = 1$ . For legibility, we take the age group width  $n$  equal to 5, but the argument would be the same with any choice of  $n$ . The  $j$ -th age group starts at age  $x = (j-1)n$ , or, in our case, at  $x = 5(j-1)$ .

Our stable formula for the  $j$ -th element of  $K$  is a simple one:

$$K_j = {}_nL_x e^{-rx}$$

Our formula for the Leslie Matrix  $A$  is cumbersome, and we only spell out one typical first-row element in full when we write down the product  $AK$  which gives our new vector  $N$ :

$$\begin{pmatrix} \dots & \dots & \dots & (1/2)({}_5L_0)({}_5F_{15} + {}_5F_{20} \frac{{}_5L_{20}}{{}_5L_{15}})f_{fab} \\ \frac{{}_5L_5}{{}_5L_0} & 0 & 0 & 0 \\ 0 & \frac{{}_5L_{10}}{{}_5L_5} & 0 & 0 \\ 0 & 0 & \frac{{}_5L_{15}}{{}_{10}L_5} & 0 \end{pmatrix} \begin{pmatrix} {}_5L_0 \\ {}_5L_5 e^{-5r} \\ {}_5L_{10} e^{-10r} \\ {}_5L_{15} e^{-15r} \end{pmatrix}$$

The first row, which gives the first age group of our projected population, is the most complicated one, so we postpone dealing with it, and begin with the second row. The second row generates the second element of our new vector  $N$ , the projected count of 5 to 10-year-olds after one projection step:

$$N_2 = A_{2,1}K_1 = \frac{{}_5L_5}{{}_5L_0} {}_5L_0 = {}_5L_5$$

The formula for  $K_2$ , the starting number of 5 to 10-year-olds, is  ${}_5L_5 e^{-5r}$ . The age group increases by a factor of  $N_2/K_2$  or  ${}_5L_5/({}_5L_5 e^{-5r}) = e^{+5r}$ .

For the third age group, we have

$$N_3 = A_{3,2}K_2 = \frac{{}_5L_{10}}{{}_5L_5} {}_5L_5 e^{-5r} = {}_5L_{10} e^{-5r}$$

The  ${}_5L_5$  in  $K_2$  cancels the  ${}_5L_5$  in the subdiagonal of  $A$ , leaving  ${}_5L_{10}$  along with  $e^{-5r}$ . It is  $K_2$  that comes into the product, but it is  $K_3$  with which we want to compare  $N_3$ . The ratio  $N_3/K_3$  for the 10 to 15-year olds is  ${}_5L_{10} e^{-5r}/({}_5L_{10} e^{-10r}) = e^{+5r}$ . This is the same age-group ratio we found before.

We see from these cases that similar cancellations occur for every age group from the second on up, and we obtain the same factor  $N_j/K_j = e^{5r}$  for all  $j > 1$ . Because the factors by which each age group is increasing are coming out to be the same, our formula for  $K$  is passing our test and looking very like a correct formula for a stable age distribution. But the outcome for the first age group is still open.

### 10.8.2 \*Lotka's Equation

We now return to the first age group. For the first age group to increase by the same factor as we have found for the older age groups, we have to have  $N_1 = e^{5r}K_1 = e^{5r}{}_5L_0$ . But the expression we obtain by multiplying out the first row looks quite different at first sight. It is a sum over age groups:

$$N_1 = \sum (1/2)({}_5L_0) \left( {}_5F_x + {}_5F_{x+5} \frac{{}_5L_{x+5}}{{}_5L_x} \right) (f_{fab})({}_5L_x e^{-rx})$$

We have no hope of turning this sum of terms into the single term  $e^{5r}{}_5L_0$  by symbolic cancellation. Cancellation worked for older age groups but it does not work for this youngest group. However, all is not lost. When we divide the sum by  $e^{5r}{}_5L_0$  we obtain the following equation:

$$1 = \sum (1/2) ({}_5F_x {}_5L_x + {}_5F_{x+5} {}_5L_{x+5}) (f_{fab}) e^{-r(x+5)}$$

The right-hand side of this equation is a function of  $r$ . If  $r$  is a number that satisfies this equation, then the first age group will increase by the same factor  $e^{5r}$  as the other age groups, and the vector  $K$  will be a stable vector. We have arrived at an equation for  $r$ .

This equation is the Euler-Lotka Equation or Lotka's Equation for short. The form given here is for discrete age groups, with  $n = 5$ . For a general version, we replace 5 by  $n$ . As we shall see, this equation uniquely determines  $r$ . For any given set of values for  ${}_nF_x$ ,  ${}_nL_x$ , and  $f_{fab}$ , there turns out to be one, and only one, real number  $r$  which can be substituted for  $r$  in the equation to make the sum on the right-hand side add up to 1. .

The condition on the first age group is deeper than the conditions on the older age groups which we studied first. With the older age groups,  $r$  could be any number. We could plug any value for  $r$  into  $e^{-rx}$  in the expression for  $K$  and symbolic cancellation would make the age-group growth factors all come out the same.

It is only the condition on the new children, the first age group, that requires  $r$  to be a particular number, the solution to an equation. This makes sense. The projection of the higher age groups is just a matter of survival, and the fertility rates that help determine the growth rate do not affect it. The formulas for these rows satisfy the required condition for any choice of growth rate  $r$ . It is only when we require a particular ratio of new children to older people that we arrive at a condition pinning down  $r$ .

We have shown that our formula does give a stable age structure, one in which the proportions remain constant when projected forward in time. We have also shown that the factor by which each age group grows in one projection step spanning  $n$  years is  $e^{rn}$ . Thus the solution of the Euler-Lotka Equation is indeed the growth rate of the stable population.

In our approach we have taken our stable population formula and used Leslie Matrix multiplication to show it to be correct. We could, instead, have derived our formula by solving successively for each element of  $K$  in terms of the preceding one using the second, third, and higher rows of the Leslie Matrix. Since the stable age pyramid is only defined up to the constant factor  $B$  which determines the total population size, with ten age groups we would have ten equations, nine for determining  $K_2, K_3$  up to  $K_{10}$  in terms of  $K_1$  and the tenth for determining  $r$ .

In matrix theory, when the matrix product  $AK$  produces a vector, each of whose elements is the same multiple of the corresponding element of  $K$ , the vector  $K$  is called an “eigenvector” of  $A$  and the multiple is called the “eigenvalue”. Eigenvectors and eigenvalues may be complex numbers instead of real numbers. But the stable age distribution is a real-valued eigenvector of the Leslie Matrix, and  $e^{rn}$  is a real-valued eigenvalue.

In the limit as  $n$  to 0, we can write  $f_x = \lim(nF_x)(f_{fab})$  and  $l_x = \lim_n L_x/n$ . We obtain the continuous version of the Euler-Lotka Equation:

$$1 = \int f_x(l_x/l_0)e^{-rx}dx$$

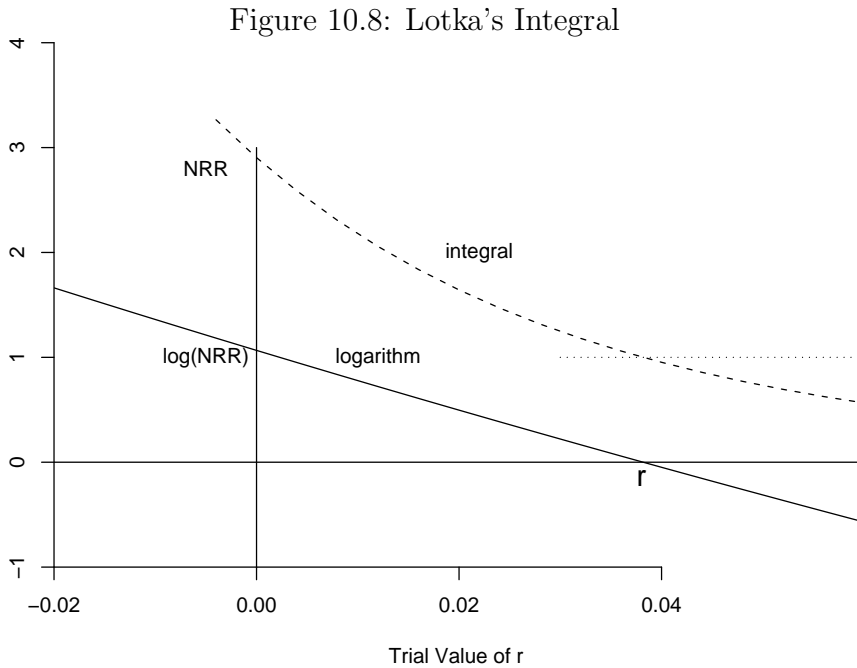
### 10.8.3 \*Existence and Uniqueness of Solutions

We can hunt for a solution to Lotka's Equation by trying out various values of  $r$ . It is convenient to take logarithms, making the left-hand side zero rather than one. In logarithmic form, we are looking for a "root" or "zero" of the equation.

Let us define  $Y(r)$  to be the logarithm of the right-hand side of Lotka's Equation when we substitute a trial value of  $r$ . The true value of  $r$  is the value which makes  $Y(r) = 0$ . For the continuous version of Lotka's Equation,  $Y$  has the form

$$Y(r) = \log \left( \int f_x(l_x/l_0)e^{-rx}dx \right)$$

Figure 10.8 shows a graph of  $Y(r)$  versus  $r$  for one of the combinations from the projection experiment.



The first point to notice from the graph is that  $Y(r)$  is a strictly decreasing function of  $r$ . We can prove this fact to be true.

$$r_1 \leq r_2 \text{ implies } f_x l_x e^{-r_1 x} \geq f_x l_x e^{-r_2 x}$$

We have weak inequality for all  $x$ , since  $f_x$  is zero at some ages. We have strong inequality whenever  $f_x l_x$  is strictly positive. If the product is a continuous function of  $x$ , there is an open set on which they are positive, so the integral must be strictly decreasing in  $r$ . Then

$$\begin{aligned} Y(-\infty) &= +\infty \\ Y(+\infty) &= \log(0) = -\infty \end{aligned}$$

Since the function decreases monotonically from positive to negative values, and is a continuous function of  $r$ , it must cross the axis. The point at which it crosses is the solution to Lotka's Equation, the true value of  $r$ . We notice an important relationship between  $Y(0)$  and the NRR:

$$\begin{aligned} Y(0) &= \log(NRR) > 0 \text{ if } NRR > 1 \\ Y(0) &= \log(NRR) = 0 \text{ if } NRR = 1 \\ Y(0) &= \log(NRR) < 0 \text{ if } NRR < 1 \end{aligned}$$

So the crossing point on the horizontal axis is to the right of zero for growing populations, at zero for stationary ones, and to the left of zero for declining populations, as ought to be the case.

If we know the slope of  $Y(r)$  at the point where  $Y(r)$  crosses zero, we can use it to construct an approximation for  $r$ . Differentiating gives the following expression:

$$\frac{d}{dr} \log\left(\int l_x f_x e^{-rx} / l_0 dx\right) = \frac{\int l_x f_x (-x) e^{-rx} / l_0 dx}{\int l_x f_x e^{-rx} / l_0 dx}$$

At  $r = 0$ , the denominator equals the NRR and the ratio of numerator over denominator is minus the cohort mean age at child-bearing  $\mu$ . In words, minus the slope of the logarithm of Lotka's Integral (the right-hand side of Lotka's equation) is the cohort

mean age at childbearing. Samuel Preston and others have written at length about the consequences of this derivative formula. As Hervé Lebras (1969) pointed out, Carrying out the Taylor Series further would show that

$$0 = \log(NRR) - \mu r + \sigma^2 r^2 / 2 \dots$$

The first two terms lead to our original approximation for  $r$

$$r \approx \frac{\log(NRR)}{\mu}$$

Substituting this approximation for  $r$  into the error term in  $r^2$  gives a better approximation

$$r \approx \frac{\log(NRR)}{\mu} + \frac{\sigma^2}{2\mu} \left( \frac{\log(NRR)}{\mu} \right)^2$$

## 10.9 Population Renewal

Lotka's Equation is the equation which determines little  $r$ . As we have shown in the preceding section, when we have discrete age groups of width  $n$ , Lotka's Equation has the form

$$1 = \sum \frac{{}_n F_{xn} L_x + {}_n F_{x+nn} L_{x+n}}{2l_0} f_{fab} e^{-r(x+n)} = 1$$

This equation cannot be solved analytically in closed form, the way, for example, a quadratic equation can be solved. But for any trial value of  $r$ , we can find out if it is close to a solution by plugging it into the equation and seeing how close the answer is to 1. If the answer is below 1, the trial value is too high. If the answer is above 1, the trial value is too low.

For example, consider our Leslie Matrix for mares and fillies. The life table values (for a radix of 1) and the fillies-only age-specific fertility rates are given below. Is the value of little  $r$  for these horses bigger or less than 80 per thousand per year?

$x$	${}_5L_x$	${}_5F_x^d$	<i>Product</i>
0	4.75	.000	.00
5	3.75	.400	1.50
10	1.50	.300	.45

The first term in our sum for Lotka's Equation is equal to

$$(1/2)(.00 + 1.50)e^{-5r} = 0.750e^{-5r}$$

The next term is

$$(1/2)(1.50 + 0.45)e^{-10r} = 0.975e^{-10r}$$

Similarly, the third term is  $0.225e^{-15r}$  So Lotka's Equation is

$$1 = ? = 0.750e^{-5r} + 0.975e^{-10r} + 0.225e^{-15r}$$

We substitute our trial value  $r = 0.080$  and add up terms:

$$1 = ? = 0.750 * 0.670 + 0.975 * 0.449 + 0.225 * 0.301 = 1.008$$

Our answer is larger than 1.000, indicating that our trial is too small. Let us try a slightly higher value, 0.082. Then we have

$$1 = ? = 0.750 * 0.663 + 0.975 * 0.440 + 0.225 * 0.292 = 0.993$$

Now we are less than 1. Thus the true value of  $r$  must lie between 80 per thousand and 82 per thousand per year, close to 81 per thousand. Our approximation from  $\log(NRR)/\mu$  would have given 0.077, which is fairly close but not perfect. At the rate of 81 per thousand growth, the population of horses would double every 8.55 years.

When the width of our age interval  $n$  is tiny,  ${}_nL_x$  is close to  $nl_x$ . We write  $f_x$  for  ${}_nF_x$  when  $n$  is tiny, and Lotka's Equation takes the simpler form

$$1 = \sum (f_x l_x / l_0) e^{-rx}$$



In a stable population, births are increasing exponentially over time with growth rate  $r$ , just as the size of every age group is increasing. In other words,

$$B(t) = B(0)e^{rt}$$

Women aged  $x$  at time  $t$  were born at time  $t - x$  when there were  $B(t - x) = B(0)e^{r(t-x)} = B(0)e^{rt}e^{-rx}$  births. Of the newborns,  $l_x/l_0$  of them survive to age  $x$ . (We are assuming that  $n$  is tiny, so we do not have to worry about differences between  $x$  and  $x+n$ .) It follows that the number of women aged  $x$  at time  $t$  is

$$B(t - x)l_x/l_0$$

which for a stable population is

$$B(0)e^{rt}e^{-rx}l_x/l_0$$

These women are giving birth to daughters at time  $t$  according to the fertility rates  $f_x$ . Thus the births at time  $t$  must be given by

$$B(t) = \sum B(t - x)f_xl_x/l_0$$

This equation is called the Renewal Equation. In a stable population the Renewal Equation, takes the form

$$B(0)e^{rt} = \sum B(0)f_xe^{rt}e^{-rx}l_x/l_0$$

Now divide both sides by the left-hand side. We are left with

$$1 = \sum e^{-rx}f_xl_x/l_0$$

Here we have obtained Lotka's Equation (in the form for tiny age groups) from the Renewal Equation by using the fact that stable populations grow exponentially at the rate  $r$ .

The Renewal Equation is just a statement of the logic of survival and childbearing. Births at time  $t$  are the sum of births at time  $t$  from parents of age  $x$ , and parents of age  $x$  are the babies of time  $t - x$  who survive to age  $x$ . The equation is true, in fact, even if rates are changing, as long as we put the survival

and fertility rates for the correct cohort alongside each cohort's births. The Renewal Equation describes the process of population renewal: babies are born, age, survive, and give birth to new babies, and the population is renewed. The Renewal Equation combined with exponential growth for stable populations gives us Lotka's Equation and little  $r$ .

We have said that any starting age structure combined with unchanging age-specific rates of childbirth and survival will lead to exponential growth and a stable population. We have not, however, given a mathematical proof of this convergence to stability. The fact that solutions of the Renewal Equation with unchanging rates do converge to exponential functions of time is a deep result called the Renewal Theorem, first proved for a significant class of cases by David Blackwell of the Berkeley Department of Statistics. There is a whole branch of statistics called "Renewal Theory" which treats many kinds of populations which can age, survive, reproduce or be replaced, and in due course die, among them stars, lightbulbs, lions, automobiles, empires, and morningglory flowers.

## 10.10 Population Momentum

In Stable Population Theory we have a very complete account of the age structure and pattern of growth implied by a long period of unchanging vital rates. The next challenge is to understand the effects of changing vital rates. Vital rates can change in a great variety of ways. The simplest sort of change we can study, however, would be a one-time change, from one set of previously unchanging vital rates to another set of previously unchanging vital rates. The situation of greatest practical interest is in the change from a high-growth demographic regime to a lower-growth or even stationary demographic regime. China over the last two decades is the outstanding example of a sudden shift of this kind. In its period of high growth, China acquired an age structure heavily weighted toward younger people. Over the 1970s and 1980s, birth rates fell rapidly. But the bulge of young people, as they pass

through childbearing age, even with the lower birth rates are producing a new bulge of babies. The Crude Rates of Growth do not catch up with the intrinsic rates of growth for a considerable time.

The tendency for populations which have been growing to keep growing for a time, even when birth rates and intrinsic growth rates drop, is known as “Population Momentum”. A population is a bit like an aircraft carrier. Even when the captain on the bridge sees the sandbar ahead, it is not possible to stop the aircraft carrier immediately, and its momentum may carry it forward into the sandbar.

The mathematics of Stable Population Theory leads to useful formulas for the effect of population momentum. The most-studied situation is one introduced by Nathan Keyfitz. It is called the “Keyfitz scenario”. In the Keyfitz scenario, the population has been steadily increasing for a long time and become a stable population. Suddenly at time  $t = 0$  age-specific fertility rates at all ages drop by the same factor, just enough to make the new  $NRR$  equal to 1. The lifetable remains unchanged. The new rates persist for a long time, eventually creating a stationary population age distribution.

The factor by which all the age-specific fertility rates are multiplied in order to achieve a new  $NRR$  of 1 is just 1 over the old  $NRR$ . If  ${}_nF_x$  are the pre-drop rates, the pre-drop  $NRR$  is given by the usual formula:

$$NRR = \sum {}_nF_x {}_nL_x f_{fab}/l_0$$

We divide both sides by this pre-drop  $NRR$ :

$$NRR^{new} = 1 = \sum \frac{{}_nF_x}{NRR} {}_nL_x f_{fab}/l_0$$

Unless otherwise marked,  $NRR$  stands for the pre-drop  $NRR$ .

We establish some notation:

$t = 0^-$  is the time just before the sudden drop in rates at the end of the long run of stable rates:

$t = 0+$  is the time just after the drop;

$t = U$  is a time well into the future when the ultimate stationary age distribution has been established.

During the stable, pre-drop years, births  $B(t)$  are increasing exponentially, so  $\log B(t)$  follows an upward-sloping straight line path. At the drop,  $\log B(t)$  falls by  $\log NRR$ . Afterward,  $\log B(t)$  goes back to increasing, as ever-larger cohorts from the stable period of population growth pass through childbearing. But the increase tapers off after half a generation or so, as smaller post-drop cohorts enter childbearing. So we expect  $\log B(t)$  after the drop to be bigger than  $\log B(0+)$  but not to rise back as high as  $\log B(0-)$ . Keyfitz proposes that  $\log B(t)$  eventually settles out about half-way between  $\log B(0+)$  and  $\log B(0-)$ , at a value  $\log B(U)$  approximately equal to  $\log B(0-) - (1/2) \log NRR$ , implying a value for  $B(U)$  of  $B(0)/\sqrt{NRR}$ .

We now convert from births to population sizes. We divide births before the drop by the Crude Birth Rate before the drop  $b(0-)$  for  $K(0-)$ . We multiply births in the ultimate stationary population by  $e_0$  for  $K(U)$ , making use of the Basic Identity for Stationary Populations which requires  $1/b(U) = e_0$ :

$$K(U) \approx K(0-) \frac{b(0-) e_0}{\sqrt{NRR}}$$

This is Keyfitz's approximation for the effect of population momentum.

Consider applying the Keyfitz formula to China. In 1980, the population of China was estimated at about 985 million, with a Crude Birth Rate of about  $b(0-) = 0.024$  and an  $NRR$  close to 1.5. Had fertility rates dropped instantly to replacement levels by the same factor at all ages, the ultimate stationary population would have been on the order of  $985 * 0.024 * e_0 / \sqrt{1.5}$ . In fact, contrary to the assumptions of the Keyfitz scenario, survival has been increasing substantially. The value of  $e_0$  in the formula is used to convert from births to population in the ultimate stationary population, so a recent value like 70 years is sensible. The

prediction for the ultimate stationary population driven by population momentum would be around 1.350 billion. Because fertility declines have been gradual and have not yet fully reached replacement levels, long-term populations in China are likely to exceed even these high levels.

## Highlights of Chapter Ten:

Stable Populations

Lotka's  $r$

Population Momentum

## Key Formulas

$$\begin{aligned} \text{Stable } {}_nK_x &= B({}_nL_x/l_0)e^{-rx} \\ r &\approx \frac{\log(NRR)}{\mu} \\ 1 &= \sum (1/2) ({}_nF_x {}_nL_x + {}_nF_{x+n} {}_nL_{x+n}) (f_{fab}/l_0)e^{-r(x+n)} \\ 1 &= \int f_x(l_x/l_0)e^{-rx}dx \\ B(t) &= \int B(t-x)f_xl_x/l_0dx \\ K(U) &\approx K(0-)\frac{b(0-)e_0}{\sqrt{NRR(0-)}} \end{aligned}$$

## Further Reading

Nathan Keyfitz (1985) gives a wealth of applications of stable population theory.

## Exercises for Chapter Ten

1. Assume that the cohort lifetables for Swedish men and women for the cohort of 1805 based on a radix of 1000 both have values of  $T_x$  given by the following table. Draw an age pyramid for

the Stationary Equivalent Population corresponding to this cohort lifetable.

x	0	1	5	10	20
$T_x$	41216	40307	37211	33694	27103

x	35	50	65	75
$T_x$	18024	10168	4052	1403

2. Lotka's  $r$  can be approximated in terms of the Net Reproduction Ratio ( $NRR$ ) and the cohort mean age at childbearing  $\mu$  with the formula

$$r \approx \frac{\log(NRR)}{\mu}$$

Using this approximation, estimate  $r$  for the following countries:

Country	Date	NRR	$\mu$
Puerto Rico	1985	1.151	26.296
Mexico	1983	2.141	28.953
Malaysian	1985	1.822	29.766
Grenada	1960	2.874	27.870
Fiji	1964	2.375	28.759

3. Consult the table of data on Malaysian women in 1985. The data are sufficient to calculate a life table and age-specific fertility rates and so to calculate age groups in the stable population. Note that there were 208422 boy babies and 198384 girl babies born in Malaysia in 1985.

a) Using these data, along with your estimate of  $r$  from Question 2, find the ratio of the number of girls of elementary school age, aged 5 to 10, to the number of young adult women aged 20 to 30 in the stable population.

b) In one sentence comment on the similarities and differences between the observed age distribution of Malaysian women and the stable age distribution.

4. The following table gives vital rates for four countries for years in the early 1970s. Suppose the fertility rates in each of these countries had dropped suddenly by the same factor for each age to give a new intrinsic rate of natural increase of zero, and suppose zero growth rates then persisted for many years. Using Keyfitz's formula for population momentum in this scenario calculate what the eventual population size in each of these countries would be after the waves following the drop in fertility died away.

Country	CBR	$e_0$ female	$NRR$	1978 Population
Nigeria	50.3	36.7	2.070	72,220,000
Mexico	42.0	66.6	2.760	66,940,000
China	26.9	64.4	1.610	933,000,000
Philippines	41.0	60.0	2.600	46,350,000

\*5. In 1995, in Sri Lanka, the fertility rates per thousand per year for children of both sexes for mothers in five-year age groups from 0 to 5 up to 45 to 50 were given by

0,0,0, 31.2, 113.9, 146.1, 81, 33.7, 8.9, 1.3.

Suppose the Sri Lankan lifetable came to resemble the lifetable for Japanese women in 1963, whose first eleven  ${}_5L_x$  values for a radix of 100 are

488, 486, 485, 482, 479, 476, 472, 467, 460, 449 and 433.

Use your population projection program to calculate what the intrinsic rate of natural increase of the Sri Lankan population would then be.

\*6. Consider the approximate value of Lotka's  $r$  calculated for Malaysian women from 1985 rates. Using life-table values and age-specific fertility rates from the data for Malaysia, plug this value of  $r$  into Lotka's Equation and find out whether it gives a value less than, equal to, or greater than 1. Does your answer imply that your approximation is too low, just right, or too high?

\*7. The cohort mean age at childbearing implied by the 1985 fertility and mortality rates of Malaysian women for 1985 is  $\mu = 29.766$ . The cohort variance in age at childbearing implied by these rates is  $\sigma^2 = 38.562$ . Find the value of  $r$  which satisfies

$$0 = \log(NRR) - \mu r + (1/2)\sigma^2 r^2.$$

Is this a better approximation of Lotka's intrinsic rate than the approximation used in Question 6?

\*8. Suppose a strict government, determined to maintain a stationary population, issued licenses for childbirth and prevented those without licenses from giving birth. A license is issued to some couple each time there is some death in the population, so the number of births equals the number of deaths. Write down an equation for the number of births in year  $t$  as a function of the numbers of births in prior years and of the life table entries.

The source of this table of data for women in Peninsular Malaysia in 1985 is Keyfitz and Flieger (1990), page 384. The total estimated population of both sexes is 12 980 968, with a crude birth rate of .03134 and a crude death rate of 0.00527. The female expectation of life at birth is 72.863. Note that the births in the table are for babies of both sexes.



Table 10.3: Age-specific Data on Malaysian Women in 1985

x	Census count	Deaths in year	Childbirths
0	191,935	2,937	0
1	694,671	925	0
5	748,606	419	0
10	735,407	355	130
15	687,089	437	17,669
20	659,034	507	103,486
25	575,833	554	137,470
30	472,378	578	91,283
35	379,068	616	44,009
40	280,371	735	11,465
45	265,362	982	1,219
50	200,618	1,336	75
55	175,121	1,785	0
60	132,110	2,458	0
65	109,605	2,934	0
70	69,457	3,493	0
75	51,593	3,548	0
80	22,739	2,279	0
85	17,457	2,566	0



# Chapter 11

## Fertility Limitation

### 11.1 “Natural” Fertility

The decline of fertility in most Western and many Eastern societies and the persistence of high fertility and high rates of population growth in sectors of the Third World are the demographic facts which shape, probably more than any others, the contemporary scene. Demographers have therefore devoted a great deal of attention to devising measures to track processes of fertility decline and particularly processes of conscious fertility limitation. This chapter presents the principal measures now in use both for historical and contemporary data.

The sources of data available on populations before and during fertility decline, though rich in many respects, almost always have certain critical limitations. Each of the measures we shall be studying has been designed to get around the particular limitations of some data source and make the most of some set of partial information.

The modern study of fertility limitation began with Louis Henry, who worked at the Institut National d'Études Démographiques, better known as INED, in Paris in the years following the Second World War. His ideas were taken up and extended by the

Cambridge Group for the History of Population and Social Structure in England under the leadership of Peter Laslett, E. Anthony Wrigley, and Roger Schofield, and then at the Office for Population Research at Princeton University under Ansley Coale. Louis Henry saw that, to understand the onset of fertility limitation practices in a society, it was important to be able to describe the pattern of fertility before these practices came into use. He was interested in conscious, intentional fertility limitation, what we now call “family planning”, and he was particularly interested in being able to analyze data which would distinguish between the absence and the presence of practices by which couples attempt to stop childbearing after desired family size targets have been achieved. Data on couples’ intentions with regard to family formation do not, of course, exist to any extent for previous centuries, and even today the gap between reported intentions and observed behavior is often great. Henry therefore asked what kind of tell-tale signs would appear in the kind of fertility data that do exist from earlier centuries, that would unambiguously indicate family limitation of a modern, conscious kind.

What Henry focussed on is called “parity-specific control”. Parity, as we have already said, is the demographer’s word for the number of children a person has had. When we look at women of the same age, say 30, and look at their rates of having another birth in the next year, then women who have already had several children may have much lower rates of having another birth than women who have had, say, only one child. In other words, it may be that the age-specific fertility rates *depend on parity*. This is dramatically true in the contemporary United States, where many couples, when they reach parity two, have reached their desired family size and use contraception from that point on. The fertility rates for those who have not reached parity two or more are much higher than for those who have. In earlier periods, the targets might be greater than two, but if people did have targets in mind and if they were able to take steps to avoid further births after targets were reached, then we would expect to see some parities at which the fertility rates would depend on parity. When that is so, we call it evidence of “parity-specific control”.

In an article in the *Eugenics Quarterly* in 1961, Louis Henry proposed using the term “natural fertility” in a technical demographic sense to mean “fertility in the absence of parity-specific control”. In some ways, this choice of words was unfortunate. It suggests that human populations were in some sort of “state of nature” before they began to practice modern versions of family limitation. It also suggests that parity-specific control is the only important form of fertility control, which is not true. Nonetheless, Henry’s definition of “natural fertility” has one great virtue. It is an operational definition. Whether there is parity-specific control can be determined from data on fertility rates by parity and age. We can measure whether fertility is “natural fertility” in Henry’s sense by measuring whether women of the same age but different parities have different fertility rates.

When sufficient data are available, we pursue Henry’s approach by calculating parity-specific, age-specific fertility rates. When feasible, it is often useful to make our rates marital-status-specific as well. A word is in order about how we calculate an age-specific, parity-specific, marital-status-specific (or anything-else-specific) fertility rate. The adjectives specify a group of women, for instance a group of women with a particular range of ages, a particular parity, and a particular marital status (e.g. married). Then the numerator for the group includes all births in the period to women in this group. The denominator includes all person-years-lived *in this group* by women in the population. The important point is that the denominator only includes person-years during which the women remain members of the group, still being within the age-range, still at the given parity, and still married. If a woman is divorced and later remarries within the period, the person-months or person-years unmarried do not contribute to the denominator. If a woman has another birth, she is then at the next parity, and her person-months or person-years after the birth do not count in the denominator for the original parity group.

The table shows numerators and denominators for parity-specific, age-specific fertility rates for Nigerian women in 1990 from the Nigerian Demographic and Health Survey (calculated from the data tapes by Tracy Brunette). In this example, the rates are not

marital-status-specific. The age group is women 30 to 35 at the time of the survey. The different columns show rates for women of different parities. The period consists of the five years before the survey date. Numerators are shown in the first row. These are the counts of births to women in the group. Denominators are shown in the second row. These are person-years-lived in the group, added up woman by woman by checking the dates of entry and exit into the group. The desired parity-specific, age-specific rates are in the third row.

Table 11.1: Recent Nigerian Parity-Specific Data

Parity	0	1	2	3	4	5	6+
Numerator	60	101	150	202	223	225	146
Denominator	220.4	448.2	582.6	826.6	799.2	791.9	627.7
Rate	0.272	0.225	0.257	0.244	0.279	0.284	0.233

Notice that the age-specific rates at different parities are practically identical. This indicates that we are dealing with a population that has “natural fertility” in Louis Henry’s sense of the word.

Contrast the Nigerian case with the following set of data for U.S. white women 1985 to 1990.

In this table, there is a dramatic fall-off in rates for this age

Table 11.2: Recent U.S. White Parity-Specific Data

Parity	0	1	2	3	4	5+
Rate	0.276	0.246	0.116	0.110	0.124	0.148

group for women who are already at parity 2 and above. Note, however, that those who are at extremely high parities have slightly higher rates. Only women who desire many children are likely to arrive at such high parities, so that the women at the end of this table are a specially selected group. The drop-off in rates at the lower parities makes this a good example of a population practicing parity-specific control and therefore not a natural fertility population in Louis Henry’s sense.

Biologists observe that the differences in age-specific fertility rates from species to species are often much greater than differences among members of the same species. Flies have fertility rates hundreds or thousands of times as high as falcons. In this sense, a species has its own characteristic level of fertility. This perspective tempts one into thinking that levels of fertility for humans subject to “natural fertility” might be pretty much the same from person to person and society to society, and differences would only appear when fertility patterns diverged from natural fertility.

Louis Henry discovered that this expectation is completely wrong. If one takes “natural fertility” to mean the absence of parity-specific control, then there turn out to be huge variations in the overall level of fertility in human societies from time to time and place to place. This is partly because biological and environmental factors affect fertility levels (without introducing parity-specific patterns), partly because biological and environmental factors interact with cultural practices that differ, and partly because not all forms of family limitation are parity-specific.

In thinking about variations in “natural fertility”, it is worthwhile to introduce some technical distinctions in vocabulary. We distinguish three related concepts:

**Fecundity** is the the biological capacity for childbearing. Strictly speaking, fecundity is a characteristic of couples, not of individuals.

**Fecundability** is the probability of conceiving for a woman subject to a continuous exposure to the risk of pregnancy.

**Fertility** is the outcome level of childbearing, which depends not only on fecundity but on the decisions and behaviors of couples within the social, cultural, and environmental context.

These are the distinctions in English. The French call fertility *fécondité* and they call fecundity *fertilité*, creating some potential for confusing the unwary.

There is also a technical vocabulary for infertility.

**Primary sterility** is the lack of capacity ever to have children. Primary sterility may be a condition of individuals or of couples.

**Secondary sterility** is a loss of capacity to have children, after some children have been born.

**Post-partum amenorrhoea** is temporary infecundity for women following childbirth.

**Lactational amenorrhoea** is temporary infecundity accompanying breastfeeding, which generally persists only during intense, largely unsupplemented breastfeeding.

These definitions call attention to some of the reasons that levels of fertility vary widely among societies with natural fertility in Louis Henry's sense. The chief reason is *lactation*. Different cultures have very different norms about nursing and weaning, about ages at which breast-milk is supplemented with other foods, and about the frequency and intensity of nursing. All these variations interact with the human endocrine system to produce different lengths of lactational amenorrhoea.

A second reason for variations in levels of natural fertility is more purely cultural. That is *abstinence*. In most cultures, couples refrain from sexual intercourse for some period following birth, and in some cultures they refrain from intercourse until weaning or even longer. This practice is called "post-partum abstinence",



that is, after-childbirth abstinence. It has a strong affect on birth intervals, and it may have very beneficial effects on the survival chances of infants and the health of the mother. The practice does not generally depend on parity, and so it is not a form of parity-specific control. But in many cases its effect on childspacing is well-understood and to some degree intentional. Whether to regard it as a form of family limitation is a definitional question. Post-partum abstinence is particularly important in Africa. Some Moslems, Indian Brahmins, and other groups also observe ritual abstinence during certain sacred periods. In some cultures, couples adopt abstinence after their first mature child has her or his first birth. In other cultures, total abstinence is rare, but frequency of intercourse declines with duration of marriage in ways that produce similar fertility effects.

A third reason for levels of fertility to vary in "natural fertility" populations is nutrition. It turns out that nutrition affects fecundity only at the extremes. What is at stake is the net balance of nutritional intake compared to the demands placed on the body. In famines, women stop ovulating. The Dutch hunger-winter of 1944, when the Nazis starved the population of Holland during World War II, is a classic example. Women athletes are often amenorrheic, a condition with possibly serious medical implications. Anorexia nervosa is a comparatively familiar cause of amenorrhoea, especially among teenagers. But work in Bengla Desh indicates that effects of nutrition on fecundity are modest until extremes of malnutrition are reached. Even poor and poorly fed Bengla Deshi women have very high fertility.

It was once believed that there was a direct relationship between the attainment of a level of critical body fatness and age at menarche, but that claim turned out to rest on a statistical error. It is possible that much of the effect of nutrition is on the viability of the foetus after conception. But late in pregnancy the body will support the foetus at the expense of the mother, even under famine conditions.

Had there been a direct relationship between nutrition and fecundity at moderate levels of nutritional deprivation, it would

have been of great theoretical interest. One can easily imagine evolutionary processes of natural selection favoring physiological responses that reduced fecundity when food was scarce and increased it when food was plentiful. This would be an example of a “homeostatic” mechanism. Homeo means “same” and static refers to states. Homeostatic means maintaining the same state. A thermostat keeps a house at the same temperature. As the house cools, it turns on the heat and warms it back up. As the house warms, it turns off the heat, till the house cools down. Homeostatic mechanisms in demography are mechanisms which regulate population growth in relation to resources. When resources are plentiful, growth rates are to rise. When resources are scarce, growth rates are to drop. Most species in nature appear to be subject to homeostatic controls of one sort or another. A strong nutrition-fecundity link would have been a biological homeostatic mechanism. It does not appear to exist, except at the extremes. The principal real homeostatic mechanisms for human populations appear to be largely the result of cultural and social institutions.

## 11.2 Proximate Determinants

In studying fertility limitation, demographers have come to distinguish between background social and economic causes of reduced fertility and mechanisms through which these causes impinge on the biological processes of having children. The mechanisms are called proximate determinants. Proxima is the Latin word for “nearest”. Proximate determinants are the nearest causal factors to the actual outcomes that can be measured from ordinary demographic sources like surveys without special medical examinations. Examples of proximate determinants are contraception, induced abortion, post-partum infecundity, and marriage or its counterparts. Methods for estimating the contributions of proximate determinants to fertility levels have been developed. John Bongaarts’ approach is the most widely used. It is presented in Bongaarts and Potter (1983). This topic is sufficiently advanced

that we shall not treat it in detail, but it is of great practical importance for careful work.

### 11.3 Parity Progression Ratios

The most direct way of detecting fertility limitation from data on parity alone is through the calculation of ratios called “parity progression ratios.” The data necessary for this calculation are counts of women by parity. “Tally” is a word which means “count up the numbers in various categories”. We shall speak of “tallying” women by parity, and use the symbol  $t(j)$  to denote the tally of women at parity  $j$ . Often we express  $t(j)$  in numbers per thousand, dividing each count by the total number of women and multiplying by 1000. Tallies of women by completed parity are particularly useful. We choose some age group for ages over the usual ages of reproduction and count up those women by parity. Such a tally of completed parity in numbers per thousand for Italian women around 1980 is shown in the following table, derived from Lutz (1989: Table A.1):

There are two steps in calculating a Parity Progression Ratio or “PPR”. First, we find out how many women (per thousand) there are who have parities of  $j$  or more. In the example there are 3 women per thousand at parities 10 or more, and 3 women at parity 9. Therefore there are  $3 + 3 = 6$  women per thousand at parities 9 or more. There are  $6 + 4 = 10$  at parities 8 or more. We start at the bottom and form a cumulative sum, at each stage adding  $t(j)$  to the sum we have previously obtained for  $(j+1)$  or more. This is the same process we use to obtain the lifetable  $T_x$  column of person-years-lived at ages  $x$  or more. The resulting column is sometimes called the “reverse cumulative distribution”, since we are cumulating in reverse, up from the bottom instead of down from the top.

To compute the Parity Progression Ratio for parity  $j$  we look at the women who reached at least parity  $j$  and see what fraction

Table 11.3: Parity Distribution for Italian Women 1980

parity j	tally t(j)	j or more	PPR
0	58	1000	0.942
1	170	942	0.820
2	387	772	0.499
3	228	385	0.408
4	94	157	0.401
5	36	63	0.429
6	10	27	0.630
7	7	17	0.588
8	4	10	0.600
9	3	6	0.500
10+	3	3	

of them went on to reach at least parity  $j+1$ . Everyone (1000 women per thousand) reaches at least parity zero; that is where everyone starts. In the table, of these 1000 women, 942 reach at least parity 1. Their completed parities are 1 or more. Thus the parity progression ratio for progressing *from parity 0* is  $942/1000$  or 0.942. Of the 942 who reach at least parity 1, some 772 reached at least parity 2, ending up at a parity of 2 or more. The the parity progression ratio  $PPR_1$  for progressing onward *from parity 1* is  $772/942$  or 0.820. The next ratio is  $PPR_2$ , which equals  $385/772$  or 0.492. Notice how the parity progression ratios for Italian women drop off dramatically at parities 2 and above. Modern Italian women have low fertility targets and exercise strong parity-specific control. Traditionally a high-fertility society, in recent years Italy has become one of the lowest-fertility societies in the world. The dropoff in parity progression ratios for those with 2-child families is a clear indication of this pattern.

After the sharp drop at 2, we observe that the parity progres-

sion ratios are more or less constant, at fairly low values, and then eventually rise. With these low parity progression ratios, fewer and fewer women are to be found at high parities. But the women who are found there are a special subpopulation. Most of them are found at high parities because, unlike the majority, they and their spouses want large families or want “as many children as the Lord sends them”. The high-parity categories are largely composed of members of such couples, and we expect their progression rates to be higher than rates in the general population. Such an effect is called a “selection effect” and this table is a good illustration of one.

It is essential to remember that parity progression ratios are labeled by the *starting* parity.  $PPR_0$  is for progression from zero to one.  $PPR_1$  is for progression from 1 to 2. This terminology is the usual one, but it is not absolutely universal, and one should check the fine print when reading tables from around the world.

In our table, we worked forward from a tally of women by parity to the parity progression ratios. We can also work backward. Table 10.4 shows the parity progression ratios for Moroccan women around 1980 from the same source.

We start with 1000 women per thousand, that is to say, everyone. A fraction 0.930 go on from parity 0 to parity 1, leaving 930 per thousand with at least parity 1. Of these, 0.960 go on from parity 1 to at least parity 2, giving  $930 * 0.960 = 893$  women per thousand with at least parity 2. The product of the second column times the third column in each row gives the value for the third column in the following row. In this way, we generate the whole column for the number with at least parity  $j$  for each  $j$ . The number at exactly parity  $j$  is then the number with at least parity  $j$  minus the number with at least parity  $j+1$ . For parity zero, we subtract 930 from 1000 to obtain 70 women per thousand tallied at exactly parity zero. Then we subtract 893 from 930 to obtain 37 women tallied at exactly parity one. In this way, by successive subtraction, we obtain the distribution of women by parity from the parity progression ratios.

In the Moroccan case, we see that the parity progression ratios

Table 11.4: Parity Progression for Moroccan Women 1980

parity j	PPR	j or more	tally t(j)
0	0.930	1000	70
1	0.960	930	37
2	0.956	893	39
3	0.953	854	40
4	0.942	814	47
5	0.926	767	57
6	0.865	710	96
7	0.840	614	98
8	0.756	516	126
9	0.692	390	120
10	0.596	270	109
11	0.602	161	64
12	0.412	97	57
13	0.500	40	20
14	0.450	20	11
15	0.000	9	9

remain high for a long stretch of parities, only falling below 0.500 at parity 12. This is characteristic of a society still largely subject to natural fertility. Parity progression ratios always eventually fall off, both because the background level of secondary sterility in the population grows with increasing parity and age and because women eventually reach ages of lower fecundity and menopause. There is always some degree of random variation in the lengths of birth intervals, so some women reach menopause at lower parities than others, even in the presence of natural fertility. It is also true that there is heterogeneity in fecundity in populations. Some women have a physiological tendency to conceive more rapidly than others. This heterogeneity introduces further variability into the ages at which women reach each parity and therefore at the parities at which women reach menopause. For this reason, even

under natural fertility, parity progression ratios do not remain close to 1.000 at high parities.

To obtain the mean parity for these women, we multiply the  $t(j)$  women by the number  $j$  of children they have had, add up these products, and divide by the total number of women. In our Moroccan case, we have

$$\frac{70 * 0 + 37 * 1 + 39 * 2 + 40 * 3 + \dots + 9 * 15}{1000} = 7.085$$

It is also possible to adapt the arguments we have used for the singulate mean ages at first marriage to show that we can obtain the mean parity directly by adding up the entries in the column for the reverse cumulative distribution. When this is done, it is necessary to ignore the first entry, which is equal to 1000 in our tables. When the sum has been obtained, we then divide by the total number of women to obtain the mean.

Examination of parity progression ratios give strong indications of the desired family sizes in populations exercising parity-specific control. However, we cannot infer the distribution of desired family sizes directly from data on parity. Even if a couple know beforehand how many children they want, not every woman ends up at the parity she and her partner intend. She may practice methods for parity-specific fertility control with complete effectiveness. Every contraceptive method except abstinence has some rate of contraceptive failure. Furthermore, couples who want more children may not succeed in having them, because of the onset of secondary sterility or reduced couple-specific fertility with age. Thus intentions cannot be easily disentangled from biology, and inferences about conscious choice from parity data are necessarily imperfect.

## 11.4 Princeton Indices

The best parity data available today come from sample surveys. The best parity data for the past come from information

recorded in church registers of baptisms, marriages, and burials, analyzed with the aid of methods of “family reconstitution” discussed in Section 9.7. Neither of these kinds of sources supply detailed geographical coverage. In the 1970s, under the leadership of Ansley Coale, a large project was organized to study variations in the onset of fertility limitation province by province across Europe. Known as the European Fertility Project, this set of studies originally aimed at identifying relationships that held up across place and time between fertility limitation and aggregate economic indices of modernization and industrialization.

These studies required a set of indices of fertility limitation which could be calculated with a minimum of data, specifically, to statistical information widely and uniformly available at a provincial or local level across Europe from the mid-1800s on. Such demands on data availability rule out any indices that require relating births to ages or parities of mothers or tabulations of person-years. Two forms of data are, however, widely available. First are counts of births at local levels broken down by the marital status of mothers. Second are counts of women by age and marital status. The former are available from birth registration systems. The latter are available from national Censuses.

At Princeton Ansley Coale developed four indices that can be computed from this minimal sort of data. They are called  $I_f$ ,  $I_g$ ,  $I_h$ , and  $I_m$ , and they are widely known as the Princeton indices. Their chief virtue is that they can be computed for so many different local areas, and thus allow comparisons at a low level of geographical aggregation. They are also designed to measure how favorable the patterns of age at marriage are to high fertility, and to separate out the effects of changing ages of marriage from changes in fertility within marriage, as they both affect total fertility.

The Princeton indices are a form of “indirect standardization”. For direct standardization, we would need age-specific fertility rates for all the places of interest, and that is what we do not have. Instead, we take a standard schedule of age-specific fertility rates. Ansley Coale chose to use the rates for the Hutterites, mem-



bers of a Protestant religious group founded by Jacob Hutter in the 1500s a small number of whom immigrated to North America and founded communities in the general area of North Dakota and adjoining areas of Canada. The Hutterites put a very high value on large families, and they make communal provisions for childrearing which spread the burdens of looking after the young. They have some of the highest sustained fertility rates ever measured in a substantial population, and their numbers have increased at a phenomenal rate over the last century. They have been studied intensively by anthropologists.

The Princeton Indices  $I_f$ ,  $I_g$  and  $I_h$  all compare the number of births that a population actually had in a period with the number that the population would have had if their fertility rates had been equal to the Hutterite rates. Since the Hutterite rates are about as high as any population is observed to have, this procedure amounts to comparing the number of births a population does have to the maximal number it could have had.

The index  $I_f$  is an index of overall fertility, regardless of marital status. In the numerator is the total number of births to all women that the population actually had, derived from birth registration data. In the denominator, we multiply the count of women in the actual population in each age group by the set of Hutterite age-specific fertility rates  ${}_5F_x$  and add up over age to have the total number of births these women would have had if they reproduced at the rate of Hutterites. This index of overall fertility is given by

$$I_f = \frac{\text{births to all women}}{\sum (\text{women } x \text{ to } x+5) {}_5F_x^{Hutt}}$$

For this calculation, one needs the values of the Hutterite age-specific fertility rates, which are fixed constants that never change when one is calculating Princeton indices. These numbers are found in Table 11.5.

Let us calculate the Princeton  $I_f$  index of overall fertility for U.S. women in 1970. The numerator for our calculation is the total number of live births in that year, which was about 3,731,000. For the denominator, we use the numbers in the first two columns in

Table 11.5: Hutterite Fertility Rates

Age x	15	20	25	30	35	40	45
Rate	.300	.550	.502	.407	.406	.222	.061

the following table. In the first column are the ages of age groups. In the next column are the total number of women counted in the 1970 Census. In the third column we copy down the Hutterite fertility rates. We then multiply column 2 by column 3 to obtain the number of babies these women would have had if they had reproduced at Hutterite rates. We add these numbers up, and the answer is our denominator.

Table 11.6: Calculating  $I_f$  for U.S. Women in 1970

Age x	Total Women	Hutterite Rates	Babies
0	8,408,838	.000	0
5	9,787,751	.000	0
10	10,198,731	.000	0
15	9,463,501	.300	2,839,050
20	8,453,752	.550	4,649,564
25	6,855,426	.502	3,441,424
30	5,834,646	.407	2,374,701
35	5,694,428	.406	2,311,938
40	6,162,141	.222	1,367,995
45	6,264,605	.061	382,141
total			17,366,813

Had they been reproducing at Hutterite rates, the 49 million U.S. women between the ages of 15 and 50 in 1970 would have had

17 million babies. In fact, they had only 3,731,000 babies. The value of  $I_f$  is the ratio of the number they had to the number they would have had:

$$I_f = \frac{3,731,000}{17,366,813} = 0.214835$$

This value indicates that U.S. women were only having about 20% of the births they would have had under maximal fertility levels. It is therefore evidence of strong fertility limitation. Values of  $I_f$  are usually in the range from 0 to 1, although nothing mathematical prevents indices greater than 1 from occurring in populations with even higher fertility than the Hutterites.

The Princeton index of marital fertility,  $I_g$  is calculated in the same way as  $I_f$ , except that the births in the numerator are restricted to legitimate births, that is, births within marriage, and the women in the denominator are married women only.

$$I_g = \frac{\text{births to married women}}{\sum (\text{married women } x \text{ to } x+5) {}_5F_x^{Hutt}}$$

U.S. women in 1970 had 3,332,000 of their births within wedlock. That gives the numerator for  $I_g$ . For the denominator, we have the following table:

We use the same Hutterite rates for our overall and our marital fertility calculations. The Hutterites had high proportions marrying at young ages, so there is little difference between overall and marital fertility rates for them. We see that had all married women had Hutterite rates, they would have had 11.5 million babies rather than the 3.3 million that they actually had. The Princeton Index of Marital Fertility is therefore given by

$$I_g = \frac{3332000}{11498901} = 0.289767$$

Marital fertility was therefore about 30% of its presumed maximum value.

How can it be that marital fertility is about 30% of its maximum and overall fertility is only about 20% of its maximum?

Table 11.7: Calculating  $I_g$  for U.S. Women in 1970

Age x	Married Women	Hutterite rates	Babies
0	0	.000	0
5	0	.000	0
10	0	.000	0
15	1,073,147	.300	321,944
20	5,054,321	.550	2,779,877
25	5,616,300	.502	2,819,383
30	5,055,678	.407	2,057,661
35	5,211,111	.406	2,115,711
40	4,976,644	.222	1,104,815
45	4,910,001	.061	299,510

The answer, of course, is that the age pattern of marriage for U.S. women in 1970 was not conducive to high fertility. Many women married after a significant proportion of their fertile span had passed, therefore reducing the overall number of children they would have had with Hutterite rates.

The Princeton Index of Marriage,  $I_m$  measures how conducive the marriage pattern is to high fertility. To calculate it, we take the denominator from the  $I_g$  measure and divide it by the denominator from the  $I_f$  measure. The quotient shows how many births the married women could have had compared to the number of births married plus unmarried women could have had, under Hutterite rates.  $I_m$  is never greater than 1. For U.S. women in 1970 it comes out to be

$$I_m = \frac{11498901}{17366813} = 0.6621192$$

There is one other Princeton Index which is not particularly important in its own right but which is useful for deriving an equation among the indices. This is the index of illegitimate or

non-marital fertility,  $I_h$ . The numerator is births out of wedlock and the denominator is the number of births that the unmarried population of women would have had in the extreme case in which they had had Hutterite fertility rates. We see that the numerator is the difference between the numerators of  $I_g$  and  $I_f$  and the denominator is the difference between the denominators of  $I_g$  and  $I_f$ . It follows that the four Princeton Indices satisfy the equation

$$I_f = I_g * (I_m) + I_h * (1 - I_m)$$

When illegitimate fertility is a small part of all fertility, as it often is, the second term on the right-hand side of this equation can be neglected, and we have a simple multiplicative approximate relationship:

$$I_f \approx I_g * I_m$$

This equation is one of the most useful features of the Princeton Indices. We multiply our measure of how high fertility within marriage is by our measure of how favorable to overall fertility the marriage pattern is, and we obtain, to good approximation, our measure of overall fertility. This equation also tells us that the for a fixed level of overall fertility as measured by  $I_f$ , the level of marriage  $I_m$  required to sustain it varies like one over the level of marital fertility  $I_g$ . We can plot values of  $I_m$  versus  $I_g$  for different places or periods, and lines of constant  $I_f$  are then hyperbolas. A number of informative diagrams of this form can be found in the book edited by Ansley Coale and Susan Watkins (1986).

## 11.5 Coale and Trussell's $M$ and $m$

In the early 1970s Ansley Coale and James Trussell at Princeton University introduced two indices of marital fertility called  $M$  and  $m$ , or “big  $M$ ” and “little  $m$ ”. These indices depend only on the age-schedule of marital fertility, and they do not make any use of parity-specific information. They are therefore, at best, rather indirect tools for detecting the presence of parity-specific fertility

control or of deviations from “natural fertility” in Louis Henry’s sense of the word. However, they have the virtue of simplicity, and they are very widely used, especially among historical demographers.

The Coale-Trussell  $M$  and  $m$  are parameters in a family of model marital fertility schedules. These models are very much like model lifetables. Instead of giving the values of  ${}_5m_x$  or  $l_x$  by a formula as a function of parameters, the Coale-Trussell model gives the values of the age-specific marital fertility rates  ${}_5F_x^{marital}$  as a function of the parameters  $M$  and  $m$ . Just as with model lifetables, there are a set of constants, playing the same role as the spines of the lifetable. Coale and Trussell’s constants are called  $n(x)$  and  $\nu(x)$ . Here  $\nu$  is the Greek letter “nu”, which looks like “v” but corresponds to our letter “n”. The model is presented in Coale and Trussell (1974).

The formula for the Coale-Trussell model for the age-specific marital fertility rate is

$${}_5F_x^{marital} = Mn(x)e^{-m\nu(x)}$$

**M** is a parameter denoting the background level of natural fertility;

**m** is a parameter denoting the extent of fertility limitation.

$n(x)$  is an assumed schedule for age-specific marital fertility under conditions of natural fertility.

$\nu(x)$  is an assumed set of weights for the impact of fertility limitation on the rates for age  $x$ .

The values of  $n(x)$  and  $\nu(x)$  are given in the following table. They are constants which do not change from application to application. The values shown here are those in recent use, as given in Coale and Watkins (1986:12) (They differ slightly from values originally in use.)

Table 11.8: Coale-Trussell Constants

Age $x$	15	20	25	30
$n(x)$	.360	.460	.431	.396
old $\nu(x)$	.000	.000	.316	.814
new $\nu(x)$	.000	.000	.279	.667

Age $x$	35	40	45
$n(x)$	.321	.167	.024
old $\nu(x)$	1.048	1.404	1.667
new $\nu(x)$	1.042	1.414	1.670

The index  $M$  is typically less than 1.0, although mathematically speaking it does not have to be so. Values close to 1.0 indicate natural fertility near maximum levels. In the absence of parity-specific control, with  $m$  values near zero, marital fertility rates may still be well below maximum, reflecting breast-feeding practices and other influences. For example, in Bangladesh between 1971 and 1975,  $M$  was estimated to be about 0.70 with  $m$  at zero. The model predicts a marital fertility rate for women 30 to 35 of

$${}_5F_{30}^{marital} = (0.70)n(30) = (0.70)(0.396) = 0.277$$

This prediction is closed to the observed value. For industrialized countries  $M$  values as low as about 0.40 are common.

The index  $m$  is usually between 0 and 2, although again it does not have to be so. Values less than about 0.20 are interpreted as indicating the absence of fertility control. Values on the order of 1.00 indicate active control. For example, among Taiwanese women in 1978, the  $m$  index had grown to 1.03, with a  $M$  value of 0.79. Age-specific marital fertility according for women 30 to 35 is predicted to be

$${}_5F_{30}^{marital} = (0.79) * n(30) * \exp(-1.03 \nu(30))$$

$$= 0.79 * 0.396 * \exp(-1.03 * 0.814) = 0.312 * 0.432 = 0.135$$

This prediction is an overestimate. The observed value was 0.080. These numbers are based on graphs in Knodel (1983:67).

It is important to bear in mind that this model is a model for *marital* fertility. Age-specific marital fertility for age  $x$  to  $x + n$  is a ratio whose numerator is the number of births to married women in the age-group and whose denominator is the number of person-years-lived in the married state in the age group. Both the numerator and denominator are restricted to married women. The more familiar ordinary age-specific fertility rates have denominators that count person-years for all women, married and unmarried, and they are typically much smaller in the younger age groups than the marital fertility rates. In younger age groups, many women are not married, and they bring down the overall age-specific fertility rates, but not the marital age-specific fertility rates that go with the Coale-Trussell model.

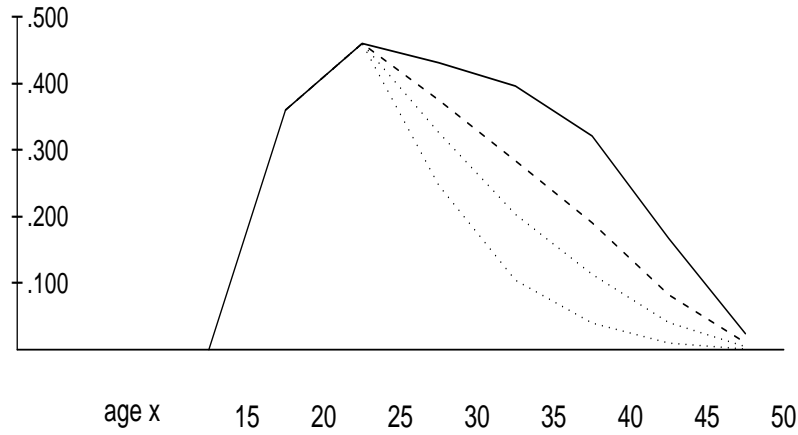
The  $n(x)$  schedule is an average of age-specific marital fertility schedules for ten populations believed by Louis Henry to be natural-fertility populations. The  $\nu(x)$  schedule is estimated from 48 populations with parity-specific control. The  $n(x)$  schedule is very like the Hutterite fertility schedule used for the  $I_f$  and  $I_g$  indices, and either could have been used in place of the other. The  $\nu(x)$  schedule is estimated from 48 populations with parity-specific control.

The figure shows a graph of the shape of age-specific marital fertility predicted by the Coale-Trussell model for three values of the family limitation parameter  $m$ , namely,  $m = 0$ ,  $m = .5$ , and  $m = 1$ . Stronger family limitation goes with an inward-bending, increasingly convex shape to the curve.

The main use of the model is to take a set of observed age-specific marital fertility rates and find the values of the parameters  $M$  and  $m$  which give the best fit between model predictions and observations. We put the model in a form which makes it easy to estimate  $M$  and  $m$  by dividing both sides by  $n(x)$  and taking



Figure 11.1: Coale-Trussell Model for Marital Fertility



logarithms:

$$\log \left( \frac{{}_5F_x^{marital}}{n(x)} \right) = \log(M) - m\nu(x)$$

Let us write  $X$  for the impact constants  $\nu(x)$  and  $Y$  for the left-hand-side values  $\log({}_5F_x^{marital}/n(x))$ . Then this equation has the form of a straight line:

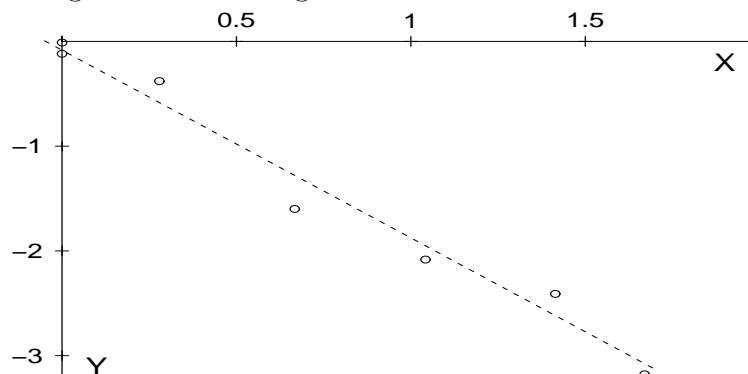
$$Y = (\text{intercept}) + (\text{slope})X$$

The intercept of the line is the logarithm of the M-parameter, and the slope of the line is the negative of the m-parameter. So if we plot the values of  $X$  for the various age groups against the corresponding values of  $Y$  and fit a straight line, finding its intercept and slope, we can estimate M and m by

$$M = \exp(\text{intercept})$$

$$m = -(\text{slope})$$

Figure 11.2: Fitting the Coale-Trussell Model



We have used a similar line-fitting approach to estimate the parameters of Bourgeois-Pichat's exogenous mortality model and Brass's relational logit lifetable model..

The conventional way to estimate the slope and intercept of a straight line fitted to a set of points is by "linear regression". We shall not discuss linear regression here, but simply present the formulas:

$$\text{slope} = \frac{\text{Av}(X * Y) - \text{Av}(X) * \text{Av}(Y)}{\text{Av}(X * X) - \text{Av}(X) * \text{Av}(X)}$$

$$\text{intercept} = \text{Av}(Y) - \text{slope} \text{Av}(X)$$

In these formulas,  $\text{Av}(X * X)$  is just the average of  $X^2$ , and  $\text{Av}(X) * \text{Av}(X)$  is just the square of the average of  $X$ . Writing both these terms out as in the formula, however, helps to avoid confusing the one with the other. To find the values, it is useful to write down four columns of numbers, with  $X$ ,  $Y$ ,  $X * X$ , and  $X * Y$ . We then find the average of each column, and plug the values into the formulas for intercept and slope.

Table 11.9 shows the calculations for Taiwanese women in 1979 from data in Knodel(1983).

Plugging into the formulas, we have

$$\text{slope} = \frac{-1.755 - (0.750) * (-1.378)}{0.944 - (0.750) * (0.750)}$$

Table 11.9:  $M$  and  $m$  for Taiwanese women in 1979

age	${}_5F_x$	$n(x)$	$X$	$Y$	$X*X$	$X*Y$
15	0.360	0.360	0.000	0.000	0.000	0.000
20	0.460	0.460	0.000	0.000	0.000	0.000
25	0.295	0.431	0.316	-0.379	0.100	-0.120
30	0.080	0.396	0.814	-1.599	0.663	-1.302
35	0.040	0.321	1.048	-2.083	1.098	-2.183
40	0.015	0.167	1.404	-2.410	1.971	-3.384
45	0.001	0.024	1.667	-3.178	2.779	-5.298
Av	0.179	0.308	0.750	-1.378	0.944	-1.755

$$= \frac{-0.722}{0.381} = -1.891$$

and

$$\text{intercept} = -1.378 - (-1.891) * (0.750) = 0.04025$$

So we have  $M = e^{0.040} = 1.041$  and  $m = +1.891$ . The value of little  $m$  near 2 indicates very strong family limitation among these women. The value of big  $M$ , greater than 1, if taken at face value, would seem to indicate a background level of natural fertility even higher than the Hutterites. In the presence of such strong family limitation, however, the value of  $M$  is not very meaningful. The low fertility levels at high age groups tend to be measured with comparatively wide margins of error, and higher estimates of  $M$  tend to go along automatically with higher estimates of  $m$ ; that would be one explanation of the suspiciously high estimate of  $M$ . The important part of the story is the high value of  $m$ , and the pronounced pattern of family limitation that it reflects.

In practice, values of  $M$  and  $m$  are often calculated only using the age groups from 20-25 up to 40-45. The 15-20 year age group, and even the 20-25 year age group, tend to be distorted by pre-nuptial pregnancies, especially in historical data. The 45-50 year

age group, due to its position on the far right of the graph, tends to have too much influence over the fitted straight line. Some researchers use weighted fitting methods, but the simplicity of the direct fitting discussed here is also an advantage that tells in its favor.

The Coale-Trussell model relies on women at later ages being mainly at higher parities, since it uses the shape of a curve by age to give clues to the pattern of fertility by parity. The assumptions built into the model are often but not always reasonable ones. When marriage is concentrated at early ages and childbearing is concentrated early in marriage, the model works well. When significant numbers of couples marry in their thirties or postpone childbearing into their thirties and forties, the age groups over 35 will have women at low parity still in the process of family formation. Everyone may be using contraception and everyone may have low total desired family sizes, but the high marital fertility for the older low-parity women will tend to fool the Coale-Trussell model. The present-day United States is an example in which the model does not function particularly well. Historical series in which modest family limitation comes into prominence during periods of falling age at marriage also tend to strain the model, and changes in  $m$  may be confused with changes in  $M$ . In such settings,  $M$  and  $m$  should always be interpreted with caution. The  $M$  and  $m$  measures are often used with data from family reconstitution of parish registers, but family reconstitution typically provides direct measures of parity-specific age-specific fertility rates. In that case, exclusive use of  $M$  and  $m$  is hard to justify, since  $M$  and  $m$  make no use of the parity-specific information.

At the time when  $I_f$  and  $I_g$  and  $M$  and  $m$  were coming into use, Louis Henry's focus on parity-specific control was shared by most demographers. Over recent decades, it has become more common to emphasize child-spacing behavior as a form of family limitation along with parity-dependent, target-driven stopping behavior. The most fully developed method for measuring the combined effects of spacing and stopping behavior on overall fertility is called Cohort Parity Analysis. It was developed by Paul David, then of Stanford University, and his colleagues, and is pre-

sented in David *et al.* (1988).

## Highlights of Chapter Eleven:

Natural fertility.

Parity-specific control

Parity progression ratios

Princeton  $I_f$ ,  $I_g$ , and  $I_m$

Coale and Trussell's M and m

## Key Formulas

$$\begin{aligned}
 I_f &= \frac{\text{births to all women}}{\sum (\text{women } x \text{ to } x+5) {}_5F_x^{Hutt}} \\
 I_g &= \frac{\text{births to married women}}{\sum (\text{married women } x \text{ to } x+5) {}_5F_x^{Hutt}} \\
 I_f &= I_g * (I_m) + I_h * (1 - I_m) \\
 {}_5F_x^{marital} &= Mn(x)e^{-m\nu(x)}
 \end{aligned}$$

## Further Reading

Coale and Watkins (1986) is a good account of the historical study of fertility limitation.

## Exercises for Chapter Eleven

1. Table 11.10 shows age-specific marital fertility rates for Portugal in 1930 and in 1960 taken from Massimo Livi Bacci (1971) *A Century of Portuguese Fertility*, along with rates for the Hutterites of North Dakota. For these three populations, estimate

Table 11.10: Portuguese Fertility 1930 and 1960

age	15	20	25	30	35	40	45
1930 Portugal	.452	.398	.291	.222	.176	.084	.015
1960 Portugal	.415	.350	.245	.160	.111	.051	.005
Hutterite	.300	.550	.502	.407	.406	.222	.061

the Coale-Trussell indices of marital fertility level  $M$  and family limitation  $m$  using data from all age groups. Are your answers evidence for the presence of family limitation ?

2. Table 11.11 shows the number of married women and the number of total women by age in Berlin in 1900. The data were supplied by researcher Patrick Galloway. The number of yearly legitimate births in Berlin in 1900 is estimated to have been 42,186 out of a total of 49,638 births in Berlin in that year. Find the Princeton index  $I_g$  at this time of transition.

Table 11.11: Married and Total Women, Berlin, 1900

age x	married women	total women
15	1538	91358
20	28710	114464
25	55417	99644
30	62076	88886
35	55293	75729
40	47197	66448
45	36906	54485

3. The Princeton Marriage Index  $I_m$  for Berlin in 1900 is about 0.464. In the same year, Frankfurt had  $I_g = 0.497$  and  $I_m = 0.454$ . Muenster had  $I_g = 0.764$  and  $I_m = 0.396$ . Trier had  $I_g = 0.766$  and  $I_m = 0.342$ . From this information, what is your best guess as

to the ordering of Berlin, Frankfurt, Trier, and Muenster in terms of total fertility as measured by the Princeton Index  $I_f$ ?

4. Table 11.12 shows counts of all women and single women from the 1920 U.S. census, before the period when divorce was common. Table 11.13 shows total live births in the United States during the Twentieth Century and illegitimate births starting in 1940. Extrapolate the proportion of illegitimate births back to 1920, and calculate as many of the Princeton indices for 1920 as you can.

Table 11.12: Marital Status of U.S. Women in 1920

Ages	Total Women	Single Women
14 to 15	1,012,968	1,007,088
15 to 18	2,861,030	2,711,081
18 to 20	1,895,743	1,426,569
20 to 25	4,749,976	2,164,051
25 to 30	4,548,258	1,048,285
30 to 35	3,940,410	588,119
35 to 45	6,760,934	767,882
45 to 55	4,845,398	464,838
55 to 65	3,069,807	257,029
65 to up	2,450,144	173,442

Source: Historical Statistics of the United States, 1975.

Table 11.13: U.S. Births (Thousands) 1910 to 1970

Date	Total Births	:	Date	Total Births	Illegitimate Births
1910	2777		1940	2360	89
1920	2950		1950	5534	142
1930	2618		1960	4258	224
			1970	3731	399





# Chapter 12

## Conclusion

In this book we have only been able to focus on the most essential demographic methods. What we have covered provides a foundation for studying many further fascinating areas, which we have hardly been able to touch on so far. Stable population theory has many further ramifications. An excellent book for pursuing them is the Third Edition of *Applied Mathematical Demography* by Nathan Keyfitz and Hal Caswell (2005). Newer work in mathematical demography, not yet comprehensively treated in textbook form, includes random population models, population feedback and homeostatic control, and models with the population, the economy, and the environment intertwined.

In the last few years, active collaboration between demographers, biologists, and geneticists has led to the new field of “Biodemography”. Careful measurement and modeling of old-age mortality for non-human as well as human species are leading to new demographic methods as well as new comparative understanding. The rich mathematical theories of population genetics are beginning to be appreciated and employed in demographic research. Genetic mapping may in the near future begin to give demographers new tools for characterizing biological heterogeneity in populations that has previously been treated as if it were unobservable.

Following on pathbreaking work of William Brass, demogra-

phers have developed a repertory of special-purpose methods for dealing with faulty and partial data. This is a subject in itself, which we have only been able to mention briefly.

Much of the day-by-day research that goes on in demography makes use of methods and models which come out of the field of statistics. These include many refinements of proportional hazard models, log-linear contingency table models, structural equation models with what are called “identifying restrictions”, logistic regression, splines, autoregressive and moving average models for demographic time series, and non-parametric likelihood-based models for unobserved heterogeneity. Even a summary would require a whole separate book.

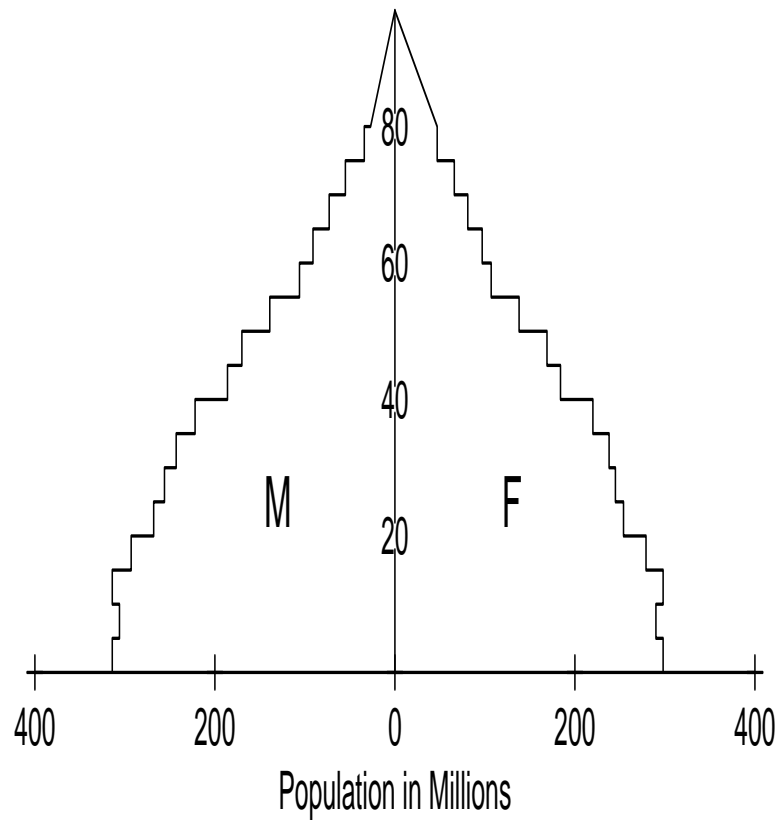
The lifetables in this book are based on age and age alone. An extensive field of “multi-state” lifetables has developed, building on the basic ideas of transition matrices that we have studied in the context of Leslie Matrix projection.

Computer simulation has become a central tool for studying the demography of kinship. Most of our models have ignored the kinship ties that connect individuals in real populations. Computers can generate artificial populations with specified sets of vital rates, and track the relationships of kinship among them.

The excitement of demography comes not only out of the elegance and ingenuity of its methods but also out of the relevance of its subject matter. The future of our planet depends on bringing the growth of our population under control, and many of the methods presented in this book have direct application in monitoring progress toward that goal. Demographic forecasts are a key component of social planning and personal planning, and again many of the methods have direct application to the difficult problems of prediction. Demographic prediction is so difficult because it depends on the values and choices of human actors – specifically, for example, on the readers of this book. The decisions you (and the rest of humans like you) make about marriage, parenthood, and survival-promoting behaviors will create the elements of the Leslie Matrices of the future, write the script for the march of cohorts up the world’s Lexis diagram, and empower the Renewal Equation to

do its work.

\* \* \*



# Appendix A

## Useful Formulas

Growth Rate:  $R = (1/T) \log(K(T)/K(0))$

Exponential Growth:  $K(t+n) = K(t)e^{Rt}$

Survival from hazards:  $l_{x+n} = l(x)e^{-h_x \cdot n}$

Interval Conversions:  $(1 - {}_1q_x)^n = 1 - {}_nq_x$

Hazard Rate:  $h_x = -(1/n) \log(l_{x+n}/l_x)$

Exponential Slope:  $\frac{d}{dx} \alpha e^{\beta x} = \alpha \beta e^{\beta x}$

Exponential Area:  $\int_a^b \alpha e^{\beta x} dx = (\alpha/\beta)(e^{\beta b} - e^{\beta a})$

Gompertz Model:  $h(x) = \alpha e^{\beta x}$ ;  $l_x = \exp\left((-\alpha/\beta)(e^{\beta x} - 1)\right)$

Period Lifetable:  ${}_nq_x = \frac{({}_nM_x)}{1 + ({}_na_x)({}_nM_x)}$

Age Specific Death Rate:  ${}_nM_x = {}_nD_x / {}_nK_x$

First Age Factor:  ${}_1a_0 = 0.07 + 1.7({}_1M_0)$ .

Second Age Factor:  ${}_4a_1 = 1.5$

Last Age Factor:  ${}_{\infty}a_x = 1/({}_{\infty}M_x$

Survivorship:  $l_{x+n} = l_x(1 - {}_nq_x) = l_x - {}_nd_x$

Person-Years Lived:  ${}_nL_x = ({}_nl_{x+n}) + ({}_na_x)({}_nd_x)$

Lifetable death rate:  ${}_nm_x = {}_nd_x / {}_nL_x$

Remaining Person-Years:  $T_x = {}_nL_x + {}_nL_{x+n} + {}_nL_{x+2n} \dots$

Expectation of Life:  $e_x = T_x / l_x$

In the Presence of Other Causes:  ${}_nq_x^A = ({}_nq_x)s_A$

In the Absence of Other Causes:  ${}_nq_x^{A*} = 1 - (1 - {}_nq_x)^{s_A}$

$$\text{Annuity Price: } \frac{B}{l_x} \left( \frac{{}_nL_x}{(1+i)^{n/2}} + \frac{{}_nL_{x+n}}{(1+i)^{n+n/2}} \cdots + \frac{{}_\infty L_{x\max}}{(1+i)^{x\max-x+e_{x\max}}} \right)$$

$$\text{Brass's Logit System: } l_x = \frac{1}{1 + \exp(-2\alpha - 2\beta Y_x)}$$

$$\text{Brass Estimation: } \alpha + \beta Y_x = (1/2) \log(l_x/(1 - l_x))$$

$$\text{Leslie Matrix Top Row: } \frac{{}_nL_0}{2l_0} \left( {}_nF_x + {}_nF_{x+n} \frac{{}_nL_{x+n}}{{}_nL_x} \right) f_{fab}$$

$$\text{Leslie Matrix Subdiagonal: } \frac{{}_nL_{x+n}}{{}_nL_x}$$

$$\text{Lotka's Equation: } 1 = \sum (1/2) ({}_nF_{xn}L_x + {}_nF_{x+nn}L_{x+n}) (f_{fab}/l_0) e^{-r(x+n)}$$

$$\text{Stable Age Pyramid : } {}_nK_x^{stable} = B({}_nL_x) e^{-rx}$$

$$\text{Lotka's Parameter: } r \approx \log(NRR)/\mu$$

$$\text{Momentum: } \frac{K(+\infty)}{K(-0)} = \frac{b(-0)e_0}{\sqrt{NRR(-0)}}$$

$$\text{Coale-Trussell Model: } {}_nF_x^{marital} = Mn(x) \exp(-m\nu(x)).$$

Coale-Trussell Estimation:  $\log({}_nF_x^{marital}/n(x)) = \log(M) - m\nu(x)$ .

Princeton Indices:  $I_f \approx I_g I_m$

Singulate Mean Age =  $\sum(n)(1 - F(x)/F_{ult})$

Bourgeois-Pichat's Formula  ${}_d q_0 = a + b(\log(1 + d))^3$

$$I_f = \frac{\text{Actual Births}}{\sum(\text{women } x \text{ to } x+n)_n F_x^{Hutt}}$$

$$I_g = \frac{\text{Legit. Births}}{\sum(\text{married women } x \text{ to } x+n)_n F_x^{Hutt}}$$

$$I_h = \frac{\text{Illegit. Births}}{\sum(\text{unmarried women } x \text{ to } x+n)_n F_x^{Hutt}}$$

$$I_m = \frac{\sum(\text{married women } x \text{ to } x+n)_n F_x^{Hutt}}{\sum(\text{women } x \text{ to } x+n)_n F_x^{Hutt}}$$

$$\text{Linear Regression slope} = \frac{\text{mean}(X * Y) - \text{mean}(X) * \text{mean}(Y)}{\text{mean}(X * X) - \text{mean}(X) * \text{mean}(X)}$$

$$\text{Linear Regression intercept} = \text{mean}(Y) - (\text{slope}) * \text{mean}(X)$$



# References

Allard, Michel, Victor Lebre and Jean-Marie Robine (1994) *Les 120 Ans De Jean Calment: Doyenne De L'Humanité*, Le cherche midi, Paris.

Bongaarts, John, and Rodolfo Bulatao, editors (2000) *Beyond Six Billion* National Academy Press, Washington, D.C.

Bongaarts, John, and Griff Feeney “On the quantum and tempo of fertility” (1998) *Population and Development Review* 24, 271-291.

Bongaarts, John, and J. Potter (1983) *Fertility, Biology, and Behavior*, Academic Press, New York.

Brass, William (1974) “Perspectives in Population Projection” *Journal of the Royal Statistical Society, Series A* 137, 532-583.

Chiang, Chin Long (1968) *Introduction to Stochastic Processes in Biostatistics* John Wiley and Sons, Inc., New York.

Coale, Ansley, and Paul Demeny (1983) *Regional Model Life Tables and Stable Populations*, Second Edition, Academic Press, New York.

Coale, Ansley, and James Trussell (1974) “Model Fertility Schedules”, *Population Index* 185-258.

Coale, Ansley, and Susan Watkins, editors (1986) *The Decline of Fertility in Europe*, Princeton University Press, Princeton, New Jersey.

Cohen, Joel (1995) *How Many People Can the Earth Support?*, W.W. Norton and Company, New York.

Cox, David (1972) "Regression models and life tables", *Journal of the Royal Statistical Society, Series B*, 34: 187-202.

David, Paul, T. Mroz, W. Sanderson, K. Wachter, and D. Weir (1988) "Cohort Parity Analysis : Statistical Estimates of the Extent of Fertility Control" *Demography* 25: 163-188.

Demeny, Paul, and Geoffrey McNicoll, editors (2003) *The Macmillan Encyclopedia of Population*, Thompson-Gale Publishing, New York.

Ewbank, Douglas C., J. Gomez de Leon, and M. Stoto (1983), "A Reducible Four-Parameter System of Model Life Tables" *Population Studies*, 37, 91-104.

Foote, Karen, K. Hill, and L. Martin, editors (1993) *Demographic Change in Sub-Saharan Africa*, National Academy Press, Washington, D.C.

Glass, David, and D. E. C. Eversley (1965) *Population in History* Aldine Publishing Company, Chicago.

Hajnal, John (1953) "Age at Marriage and Proportions Marrying" *Population Studies*. 7: 111-136.

Hill, Mark, Samuel Preston, and Ira Rosenwaike (2000) "Age Reporting Among White Americans Aged 85+ " *Demography* 37:175-186.

Human Mortality Database (2005) available at [www.mortality.org](http://www.mortality.org), University of California, Berkeley, USA, and Max Planck Institut für Demografische Forschung, Rostock, Germany.

Hummer, Robert A., R. G. Rogers, C. B. Nam, and C. G. Ellison (1999) "Religious Involvement and U.S. Adult Mortality" *Demography* 36: 273-285.

Keyfitz, Nathan (1985) *Applied Mathematical Demography* Second Edition, Springer Verlag, Berlin. (A new edition is expected soon.)

Keyfitz, Nathan, and Hal Caswell (2005) *Applied Mathematical Demography* Third Edition, Springer, New York.

Keyfitz, Nathan, and Wilhelm Flieger (1990) *World Population Growth and Aging* University of Chicago Press, Chicago, Illinois.

Knodel, John (1983) "Natural Fertility: Age Patterns, Levels, and Trends" in *Determinants of Fertility in Developing Countries, Volume 1* edited by R. Bulatao and R.D. Lee, page 67.

Laslett, Peter (1971) *The World We Have Lost*, Second Edition, Methuen and Company, London.

Laslett, Peter, and James Fishkin, Editors (1992) *Justice between Age Groups and Generations* Yale University Press, New Haven, CT.

Lee, Ronald (1994) "The Formal Demography of Population Aging, Transfers, and the Economic Life Cycle", *Demography of Aging* Committee on Population, National Research Council, edited by Linda Martin and Samuel Preston, National Academy Press, 1994.

Livi-Bacci, Massimo (1971) *A Century of Portuguese Fertility*, Princeton University Press, Princeton, New Jersey.

Livi-Bacci, Massimo (1997) *A Concise History of World Population* Second Edition, Blackwell Publishing, Oxford, England.

Lee, Ronald D., and Larry Carter (1992) "Modeling and Forecasting the Time Series of U.S. Mortality" *Journal of the American Statistical Association* 87, 659-671.

Lutz, Wolfgang (1989) *Distributional Aspects of Human Fertility* Academic Press, New York.

Namoodiri, Krishnan, and C.M. Suchindran (1987) *Life Table Techniques and Their Applications* Academic Press Inc., New York, NY.

Pollard, John H. (1973) *Mathematical Models for the Growth of Human Populations* Cambridge University Press, Cambridge, England.

Pressat, Roland (1972) *Demographic Analysis*. Aldine Publishing Company, Chciago, Illinois. Jean Bourgeois-Pichat (1951) *Population* proposed the following model:

Preston, Samuel, P. Heuveline, and M. Guillot (2001) *Demography: Measuring and Modeling Population Processes* Blackwell Publishing, Oxford, England.

Robine, Jean-Marie, and J. W. Vaupel (2001) “Supercentenarians, slower ageing individuals or senile elderly?” *Experimental Gerontology* 36: 915-930.

Rohde, Douglas L. T., S. Olson, S., and Joseph T. Chang (2004) “Recent Common Ancestors in Structured Populations” *Nature* 431: 562–566.

Strauss, William, and Neil Howe (1991) *Generations* Quill Press, William Morrow, New York.

United Nations (2001) *World Population Prospects: The 2000 Revision*, Volume 1, United Nations, New York.

Wachter, Kenneth W. (1980) “Ancestors at the Norman Conquest” in *Genealogical Demography* edited by Bennett Dyke and Warren Morrill, Academic Press, New York.

Wrigley, E. A., and Roger Schofield (1981) *The Population History of England* Oxford University Press, Oxford, England.

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