Q 2
Q 3
Q 8(b), (d)
Q 9

ST2334 - Tutorial 5

1. 1st moment of $X = E[X]$

$= \sum_x x f_X(x)$

$= 2 \times 0.01 + 3 \times 0.25 + 4 \times 0.40 + 5 \times 0.30 + 6 \times 0.04$

$= 4.11$

2nd moment of $X = E[X^2]$

$= \sum_x x^2 f_X(x)$

$= 17.63$

(b)(i). $V(X) = E[(X - \mu_X)^2]$

$= E[(X - 4.11)^2]$

$= \sum_x (x - 4.11)^2 f_X(x)$

$= 0.7379$

(ii). $V(X) = E(X^2) - [E(X)]^2$

$= 17.63 - 4.11^2$

$= 0.7379$

(c). $Z = 3X - 2$

$\mu_Z = E[Z]$

$= E[3X - 2]$

$= 3E[X] - 2$

$= 10.33$

(d).

z	4	7	10	13	16
$f_Z(z)$	0.01	0.25	0.40	0.30	0.04

$$E[Z] = \sum_z z f_Z(z)$$
$$= 10.33$$
$$V(Z) = E[(Z-\mu_Z)^2]$$
$$= \sum_z (z-\mu_Z)^2 f_Z(z)$$
$$= 6.6411$$

(e). $W = aZ + b$

$$E[W] = a\mu_Z + b$$
$$= 10.33a + b$$
$$V(W) = a^2 V(Z)$$
$$= 6.6411 a^2$$

2. profit $= 1.65X - 7 + \frac{3}{4}(1.65)(5-X)$

$= 0.4125X - 0.8125$ ✗ $0.75X - 1.5$

$E[X] = \sum_x x f_X(x)$

$= \frac{46}{15}$ ✗ 0.80

expected profit $= 0.4125\,E[X] - 0.8125$

$= 0.4525$

3 (a). $E(X) = \sum_{k=1}^{\infty} k f_X(k)$ $\quad f_X(x) = \Pr(X = x)$

$=$

(b).

4 (a). $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$= \int_0^1 x \cdot 2(1-x)\, dx$

$= \int_0^1 2x - 2x^2\, dx$

$= \left[x^2 - \frac{2}{3}x^3 \right]_0^1$

$= \frac{1}{3}$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

$= \int_0^1 x^2 \cdot 2(1-x)\, dx$

$= \int_0^1 2x^2 - 2x^3\, dx$

$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4 \right]_0^1$

$= \frac{1}{6}$

$V(X) = E[X^2] - [E[X]]^2$

$= \frac{1}{18}$

(b). $Y = 3X - 2$

$E[Y] = 3E[X] - 2$

$= -1$

$V(Y) = 3^2 V(X)$

$= \frac{1}{2}$

5. $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$= \int_0^1 x(a+bx^2)\, dx$

$= \int_0^1 ax + bx^3\, dx$

$= \left[\frac{a}{2}x^2 + \frac{b}{4}x^4 \right]_0^1$

$= \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$ — ①

$\int_{-\infty}^{\infty} f_X(x) = 1$

$\Rightarrow \int_0^1 a + bx^2\, dx = \left[ax + \frac{b}{3}x^3 \right]_0^1$

$= a + \frac{b}{3} = 1$ — ②

① × 2 : $a + \frac{b}{2} = \frac{6}{5}$ — ③

③ − ② : $\frac{b}{6} = \frac{1}{5}$

$b = \frac{6}{5}$

$a = \frac{3}{5}$

6. $E[(X-1)^2] = E[X^2 - 2X + 1]$

$= E[X^2] - 2E[X] + 1 = 10$

$\Rightarrow E[X^2] - 2E[X] = 9$ — ①

$E[(X-2)^2] = E[X^2 - 4X + 4]$

$= E[X^2] - 4E[X] + 4 = 6$

$\Rightarrow E[X^2] - 4E[X] = 2$ — ②

① − ② : $2E[X] = 7$

$E[X] = 3.5$

$E[X^2] = 16$

$V(X) = E[X^2] - [E[X]]^2$

$= 3.75$

7. $\mu = 10$, $\sigma^2 = 4 \Rightarrow \sigma = 2$

(a). $Pr(5 < X < 15) = Pr(-5 < X-10 < 5)$

$= Pr(|X-10| < 5)$

$= Pr(|X-\mu| < \frac{5}{2}\sigma) \geq 1 - \frac{1}{(\frac{5}{2})^2} = \frac{21}{25}$

(b). $Pr(5 < X < 14) \geq Pr(6 < X < 14)$

$= Pr(|X-\mu| < 2\sigma) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$

(c). $Pr(|X-10| < 3) = Pr(|X-\mu| < \frac{3}{2}\sigma) \geq 1 - \frac{1}{(\frac{3}{2})^2} = \frac{5}{9}$

(d). $Pr(|X-10| \geq 3) = Pr(|X-\mu| \geq \frac{3}{2}\sigma) \leq \frac{1}{(\frac{3}{2})^2} = \frac{4}{9}$

(e). $Pr(|X-10| \geq c) \leq 0.04 = \frac{1}{5^2}$

$c = \frac{5}{2}\sigma = \cancel{5}$

$k = 5$, $c = 5\sigma = 10$

8 (a). $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$= \int_0^1 x \cdot 6x(1-x)\, dx$$

$$= \int_0^1 6x^2 - 6x^3\, dx$$

$$= \left[2x^3 - \frac{3}{2}x^4\right]_0^1$$

$$= \frac{1}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_0^1 x^2 \cdot 6x(1-x)\, dx$$

$$= \left[\frac{3}{2}x^4 - \frac{6}{5}x^5\right]_0^1$$

$$= \frac{3}{10}$$

$$\sigma = \sqrt{V(X)}$$

$$= \sqrt{E[X^2] - (E[X])^2}$$

$$= \sqrt{\frac{1}{20}}$$

$$= 0.2236$$

(b). $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} f_X(x)\, dx$

$$= \int_{\mu-2\sigma}^{\mu+2\sigma} 6x(1-x)\, dx$$

$$= \left[3x^2 - 2x^3\right]_{\mu-2\sigma}^{\mu+2\sigma}$$

$$= \left[3x^2 - 2x^3\right]_{0.2764}^{0.7236}$$

$$= 0.6261 \quad 0.9839$$

(c). $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) = \Pr(|X-\mu| < 2\sigma) \geqslant 1 - \frac{1}{2^2} = \frac{3}{4}$

(d).

9. Let X = life of light bulb (in hours)

$\mu_X = 900$, $\sigma = 50$

0.03125