Basic Concepts of Probability

Basic Properties : Events

1. $(A \cap B)' = A' \cup B'$
2. $(A \cup B)' = A' \cap B'$
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. $A \cup B = A \cup (B \cap A')$
6. $A = (A \cap B) \cup (A \cap B')$

De Morgan's Law

1. $(A_1 \cup A_2 \cup \dots \cup A_n)' = A_1' \cap A_2' \cap \dots \cap A_n'$
2. $(A_1 \cap A_2 \cap \dots \cap A_n)' = A_1' \cup A_2' \cup \dots \cup A_n'$

Permutation and Combination

$${}_nP_r = n(n-1)(n-2)\dots(n-(r-1)) = \frac{n!}{(n-r)!}$$

$${}_nC_r = C^n_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$${}_nC_r \, r! = {}_nP_r$$

Binomial Coefficient → $\binom{n}{r}$

1. $\binom{n}{r} = \binom{n}{n-r}$ for $r = 0, 1, \dots, n$
2. $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for $1 \le r \le n$
3. $\binom{n}{r} = 0$ for $r < 0$ or $r > n$

Basic Properties : Probability

1. $Pr(A) = Pr(A \cap B) + Pr(A \cap B')$
2. $Pr(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^{n} Pr(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr(A_i \cap A_j) + \dots + (-1)^{n+1} Pr(A_1 \cap A_2 \cap \dots \cap A_n)$

Inclusion-Exclusion Principle

Conditional Probability

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

1. $Pr(S|A) = 1$
2. $B_1, B_2, \dots$ are mutually exclusive (disjoint) events $\longrightarrow Pr\left(\bigcup_{i=1}^{\infty} B_i \middle| A\right) = \sum_{i=1}^{\infty} Pr(B_i|A)$

Multiplication Rule of Probability

$$Pr(A \cap B) = Pr(A)\,Pr(B|A)$$
$$= Pr(B)\,Pr(A|B), \text{ providing } Pr(A) > 0, Pr(B) > 0$$

In general, $Pr(A_1 \cap A_2 \cap \dots \cap A_n)$

$$= Pr(A_1)\,Pr(A_2|A_1)\,Pr(A_3|A_1 \cap A_2) \dots Pr(A_n|A_1 \cap A_2 \cap \dots \cap A_n),$$

providing $Pr(A_1 \cap A_2 \cap \dots \cap A_{n-1}) > 0$.

The Law of Total Probability

Let $A_1, A_2, \dots, A_n$ be a partition of S. Then for any event B,

$$Pr(B) = \sum_{i=1}^{n} Pr(B \cap A_i) = \sum_{i=1}^{n} Pr(A_i)\,Pr(B|A_i)$$

Bayes' Theorem

Let $A_1, A_2, \dots, A_n$ be a partition of S. Then

$$Pr(A_k|B) = \frac{Pr(A_k)\,Pr(B|A_k)}{\sum_{i=1}^{n} Pr(A_i)\,Pr(B|A_i)}$$

$$= \frac{Pr(A_k)\,Pr(B|A_k)}{Pr(B)}$$

Random Variables

Discrete Random Variables		Continuous Random Variables
probability (mass) function - Must satisfy: ① $f(x_i) \geq 0 \quad \forall x_i$ ② $\sum_{i=1}^{\infty} f(x_i) = 1$	$f_X(x)$	probability density function (p.d.f.) - Must satisfy: ① $f(x) \geq 0 \quad \forall x \in R_X$ ② $\int_{R_X} f(x)\,dx = 1$ or $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ③ $\forall c, d$ s.t. $c < d$, $(c,d) \subset R_X$, $\Pr(c \leq X \leq d) = \int_c^d f(x)\,dx$
$F(x) = \sum_{t \leq x} f(t) = \sum_{t \leq x} \Pr(X = t)$ ① $\Pr(a \leq X \leq b) = \Pr(X \leq b) - \Pr(X < a)$ $= F(b) - F(a-1)$ ② $a = b \longrightarrow \Pr(X = a) = F(a) - F(a-1)$	Cumulative Distribution Function (CDF) $F_X(x)$	$F(x) = \int_{-\infty}^{x} f(t)\,dt$ ① $\Pr(a \leq X \leq b) = \Pr(a < X \leq b)$ $= F(b) - F(a)$ ② $f(x) = \frac{d\,F(x)}{dx} \longrightarrow$ if the derivative exists
$\mu_X = E(X) = \sum_i x_i f_X(x_i) = \sum_x x f_X(x)$ $E[g(x)] = \sum_x g(x) f_X(x)$ $\sigma_X^2 = V(x) = E[(X-\mu_X)^2] = \sum_x (x-\mu_x)^2 f_X(x)$	Mean (Expectation) and Variance μ_X σ_X^2	$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$ $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$ $\sigma_X^2 = V(x) = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$

$g(x) = x^k \longrightarrow$ k-th moment of X

① $E(aX + b) = aE(X) + b \longrightarrow$ a, b are constants

② $V(X) = E(X^2) - [E(X)]^2$

③ $V(aX + b) = a^2 V(X)$

Chebyshev's Inequality

$\Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$

$\Pr(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

2D RVs , Conditional Probability Distributions