ST2334 - Tutorial 7   Week 9

1(a).

$f_{X,Y}(x,y)$		$x=2$	$x=4$	$f_Y(y)$
y	1	0.10	0.15	0.25
	3	0.20	0.30	0.50
	5	0.10	0.15	0.25
$f_X(x)$		0.40	0.60	

$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall\, x,y \in R_{X,Y} \longrightarrow$ X and Y are independent

(b). $f_{Y|X}(y|2) = \begin{cases} 0.25 & y=1; \\ 0.50 & y=3; \\ 0.25 & y=5; \\ 0 & \text{0 otherwise} \end{cases}$

$E(Y|X=2) = \sum_y y\, f_{Y|X}(y|2)$

$= 3$

(c). $f_{X|Y}(x|y=3) = \begin{cases} 0.40 & x=2; \\ 0.60 & x=4; \\ 0 & \text{otherwise} \end{cases}$

$E(X|Y=3) = \sum_x x f_{X|Y}(x|2)$

$= 3.2$

(d). $E(2X-3Y) = 2E(X) - 3E(Y)$

$= 2\sum_x x f_X(x) - 3\sum_y y f_Y(y)$

$= 2(3.2) - 3(3)$

$= -2.6$

(e). $E(XY) = \sum_x \sum_y xy\, f_{X,Y}(x,y)$

$= 9.6$

(g). $V(X) = E[(X-\mu_X)^2]$

$= \sum_x (x-\mu_X)^2 f_X(x)$

$= 0.96$

$V(Y) = E[(Y-\mu_Y)^2]$

$= \sum_y (y-\mu_Y)^2 f_Y(y)$

$= 2$

(g). $\sigma_{X,Y}^2 = E(XY) - \mu_X\mu_Y$

$= 9.6 - 3.2(3)$

$= 0$

$\sigma_{X,Y} = 0$

$\rho_{X,Y} = \dfrac{\sigma_{X,Y}^2}{\sqrt{V(X)}\sqrt{V(Y)}}$

$= 0$

2.

$f_{X,Y}(x,y)$		$x=0$	$x=1$	$x=2$	$f_Y(y)$
y	0	0.01	0.01	0.03	0.05
	1	0.03	0.08	0.07	0.18
	2	0.03	0.06	0.06	0.15
	3	0.07	0.07	0.13	0.27
	4	0.12	0.04	0.03	0.19
	5	0.08	0.06	0.02	0.16
$f_X(x)$		0.34	0.32	0.34	

Z :

	$x=0$	$x=1$	$x=2$
0	-10	-2	6
1	-7	1	9
2	-4	4	12
3	-1	7	15
4	2	10	18
5	5	13	21

$f_Z(z)$:

Y	$x=0$	$x=1$	$x=2$
0	-7.13		
1	-6.74		
2			
3			
4			
5			

?

Let profit = Z.

$$Z = 8X + 3Y - 10$$

$$\begin{aligned} E(Z) &= E(8X + 3Y - 10) \\ &= 8E(X) + 3E(Y) - 10 \\ &= 8(1) + 3(2.85) - 10 \\ &= 6.55 \end{aligned}$$

$$\begin{aligned} V(Z) &= V(8X + 3Y - 10) \\ &= 8^2 V(X) + 3^2 V(Y) \\ &= 8^2(0.68) + 3^2(2.1275) \\ &= 62.6675 \end{aligned}$$

3.

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0 \le x \le 1, 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

(a).

$$f_X(x) = \int_0^1 \frac{2}{3}(x+2y)\,dy$$
$$= \frac{2}{3}\left[xy+y^2\right]_0^1$$
$$= \begin{cases} \frac{2}{3}(x+1), & 0 \le x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)\,dx$$
$$= \frac{2}{3}\left[\frac{x^2}{2}+2xy\right]_0^1$$
$$= \frac{2}{3}\left[\frac{1}{2}+2y\right]$$
$$= \begin{cases} \frac{1}{3}(1+4y), & 0 \le y \le 1; \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x)f_Y(y) = \frac{2}{3}(x+1)\left[\frac{1}{3}(1+4y)\right]$$
$$= \frac{2}{9}(x+4xy+4y+1) \neq f_{X,Y}(x,y)$$

$\therefore$ X are dependent.

(b).

$$E(X) = \int_0^1 x \cdot \frac{2}{3}(x+1)\,dx$$
$$= \frac{2}{3}\int_0^1 x^2+x\,dx$$
$$= \frac{2}{3}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1$$
$$= \frac{5}{9}$$

$$E(X^2) = \int_0^1 x^2 \cdot \frac{2}{3}(x+1)\,dx$$
$$= \frac{2}{3}\int_0^1 x^3+x^2\,dx$$
$$= \frac{2}{3}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1$$
$$= \frac{7}{18}$$

$$\sigma_X^2 = E(X^2) - [E(X)]^2$$
$$= \frac{13}{162}$$

(c).

$$E(Y) = \int_0^1 y \cdot \frac{1}{3}(1+4y)\,dy$$
$$= \frac{1}{3}\int_0^1 y+4y^2\,dy$$
$$= \frac{1}{3}\left[\frac{y^2}{2}+\frac{4y^3}{3}\right]_0^1$$
$$= \frac{11}{18}$$

$$E(Y^2) = \int_0^1 y^2 \cdot \frac{1}{3}(1+4y)\,dy$$
$$= \frac{1}{3}\int_0^1 y^2+4y^3\,dy$$
$$= \frac{1}{3}\left[\frac{y^3}{3}+y^4\right]_0^1$$
$$= \frac{4}{9}$$

$$\sigma_Y^2 = E(Y^2) - [E(Y)]^2$$
$$= \frac{23}{324}$$

(d).

4.

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0 \le x \le 1, 0 \le y \le 1; \\ 0, & \text{otherwise} \end{cases}$$

(a).

$$f_X(x) = \int_0^1 \tfrac{3}{2}(x^2+y^2)\,dy = \tfrac{3}{2}\left[x^2y + \tfrac{y^3}{3}\right]_0^1 = \tfrac{3}{2}\left(x^2+\tfrac{1}{3}\right) = \begin{cases} \tfrac{1}{2}(3x^2+1), & 0 \le y \le 1; \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^1 \tfrac{3}{2}(x^2+y^2)\,dx = \tfrac{3}{2}\left[\tfrac{x^3}{3} + xy^2\right]_0^1 = \tfrac{3}{2}\left(\tfrac{1}{3}+y^2\right) = \begin{cases} \tfrac{1}{2}(1+3y^2), & 0 \le x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

$f_X(x)\,f_Y(y) = \frac{1}{4}(3x^2+1)(3y^2+1) \neq f_{X,Y}(x,y) \longrightarrow$ X and Y are dependent.

(b).

$$E(X) = \int_0^1 x \cdot \tfrac{1}{2}(3x^2+1)\,dx = \tfrac{1}{2}\int_0^1 3x^3 + x\,dx = \tfrac{1}{2}\left[\tfrac{3x^4}{4} + \tfrac{x^2}{2}\right]_0^1 = \tfrac{5}{8}$$

$$E(X^2) = \int_0^1 x^2 \cdot \tfrac{1}{2}(3x^2+1)\,dx = \tfrac{1}{2}\int_0^1 3x^4 + x^2\,dx = \tfrac{1}{2}\left[\tfrac{3x^5}{5} + \tfrac{x^3}{3}\right]_0^1 = \tfrac{7}{15}$$

$$V(X) = E(X^2) - [E(X)]^2 = \tfrac{73}{960}$$

(c).

$$E(Y) = \int_0^1 y \cdot \tfrac{1}{2}(1+3y^2)\,dy = \tfrac{5}{8}$$

$$E(Y^2) = \dots = \tfrac{7}{15}$$

$$V(Y) = \tfrac{73}{960}$$

(d).

$$Cov(X,Y) = \int_0^1\int_0^1 \overbrace{(x-\mu_x)(y-\mu_y)}^{xy - \mu_x y - \mu_y x + \mu_x\mu_y} \cdot \tfrac{3}{2}(x^2+y^2)\,dx\,dy$$

$$= \tfrac{3}{2}\int_0^1\int_0^1 x^2$$

(e).

$$E(X+Y) = E(X) + E(Y) = \tfrac{5}{4}$$

$$V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$$

5. $$f_{X,Y}(x,y) = \begin{cases} x+y, & 0 \le x \le 1, 0 \le y \le 1; \\ 0, & \text{otherwise}. \end{cases}$$

(a). $$f_X(x) = \int_0^1 x+y \; dy = \left[xy + \frac{y^2}{2}\right]_0^1 = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^1 x+y \; dx = \left[\frac{x^2}{2} + xy\right]_0^1 = \begin{cases} \frac{1}{2} + y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^1 x\left(x + \frac{1}{2}\right) \; dx = \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \frac{7}{12}$$

$$E(Y) = \ldots = \frac{7}{12}$$

$x^2y + xy^2$

$$E(XY) = \int_0^1\int_0^1 xy \cdot (x+y) \; dx \, dy = \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_{x=0}^{x=1} dy = \int_0^1 \frac{y}{3} + y^2$$

6. $Var(X) = 5$, $Var(Y) = 3$, $Z = -2X + 4Y - 3$

(a). $$\begin{aligned} Var(Z) &= Var(-2X + 4Y - 3) \\ &= (-2)^2 Var(X) + 4^2 Var(Y) + \cancelto{0}{2(-2)(4)\,Cov(X,Y)} \\ &= (-2)^2(5) + 4^2(3) \\ &= 68 \end{aligned}$$

(b). $$\begin{aligned} Var(Z) &= Var(-2X + 4Y - 3) \\ &= (-2)^2 Var(X) + 4^2 Var(Y) + 2(-2)(4)\,Cov(X,Y) \\ &= (-2)^2(5) + 4^2(3) - 16(1) \\ &= 52 \end{aligned}$$

(c). $$\begin{aligned} \sigma_{X,Y} &= \frac{Cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} \\ &= \frac{1}{\sqrt{5}\,\sqrt{3}} \\ &= 0.2582 \end{aligned}$$

7(a). $f_X(x) = \begin{cases} \frac{1}{10} & x = 1, 2, 3, \dots, 10; \\ 0 & \text{otherwise} \end{cases}$

(b). $Pr(X < 4) = Pr(X=1) + Pr(X=2) + Pr(X=3)$

$= \frac{3}{10}$

(c). $E(X) = \sum_x x f_X(x)$

$= 5.5$

$E(X^2) = \sum_x x^2 f_X(x)$

$= 38.5$

$V(X) = E(X^2) - [E(X)]^2$

$= 8.25$