Q 2
Q 3
Q 8(b), (d)
Q 9

ST2334 – Tutorial 5

1. 1st moment of $X = E[X]$

$$= \sum_x x f_X(x)$$

$$= 2 \times 0.01 + 3 \times 0.25 + 4 \times 0.40 + 5 \times 0.30 + 6 \times 0.04$$

$$= 4.11$$

2nd moment of $X = E[X^2]$

$$= \sum_x x^2 f_X(x)$$

$$= 17.63$$

nth moment of a distribution about zero: $E[X^n]$

nth moment of a distribution about mean: $E[(X-\mu)^n]$

$E(X)$: 1st moment about zero

$V(X)$: 2nd moment about mean

(b)(i). $V(X) = E[(X-\mu_X)^2]$

$$= E[(X-4.11)^2]$$

$$= \sum_x (x-4.11)^2 f_X(x)$$

$$= 0.7379$$

(ii). $V(X) = E(X^2) - [E(X)]^2$

$$= 17.63 - 4.11^2$$

$$= 0.7379$$

(c). $Z = 3X - 2$

$\mu_Z = E[Z]$

$$= E[3X-2]$$

$$= 3E[X] - 2$$

$$= 10.33$$

$E(aX+b) = aE(X) + b$

$V(aX+b) = a^2 V(X)$

(d).

z	4	7	10	13	16
$f_Z(z)$	0.01	0.25	0.40	0.30	0.04

$$E[Z] = \sum_z z f_Z(z)$$

$$= 10.33$$

$$V(Z) = E[(Z-\mu_Z)^2]$$

$$= \sum_z (z-\mu_Z)^2 f_Z(z)$$

$$= 6.6411$$

(e). $W = aZ + b$

$$E[W] = a\mu_Z + b$$

$$= 10.33a + b$$

$$V(W) = a^2 V(Z)$$

$$= 6.6411\, a^2$$

Let profit = Y.

$Y = 1.65X + \frac{3}{4}(1.2)(5-X) - 5(1.2)$ $\Rightarrow$ Expected profit = $E(Y)$

2. $\text{profit} = 1.65X - 7 + \frac{3}{4}(1.65)(5-x)$

$= \cancel{0.4125X - 0.8125}$ $0.75X - 1.5$

$E[X] = \sum_x x f_X(x)$

$= \cancel{\frac{46}{15}}$ 0.80

expected profit $= 0.4125\, E[X] - 0.8125$

$= 0.4525$

$f_X(x) = Pr(X=x)$

3 (a). $E(X) = \sum_{k=1}^{\infty} k f_X(k)$

$Pr(X \geq 1) = Pr(X=1) + Pr(X=2) + Pr(X=3) + Pr(X=4) + \ldots$

$Pr(X \geq 2) = Pr(X=2) + Pr(X=3) + Pr(X=4) + \ldots$

$Pr(X \geq 3) = Pr(X=3) + Pr(X=4) + \ldots$

$\vdots \quad = \quad \vdots$

$\therefore \sum_{k=1}^{\infty} Pr(X \geq k) = 1\,Pr(X=1) + 2\,Pr(X=2) + 3\,Pr(X=3) + \ldots$

$= \sum_{x=1}^{\infty} x\,Pr(X=x) = E(X)$

(b). Let X_1, X_2, X_3 denote respectively the number obtained in the first, second, third die.

Then $M = \min\{X_1, X_2, X_3\}$ for $k = 1, 2, \ldots, 6$.

For $k = 1, 2, \ldots, 6$:

$Pr(M \geq k) = Pr(X_1 \geq k, X_2 \geq k, X_3 \geq k)$ Independent

$= Pr(X_1 \geq k)\,Pr(X_2 \geq k)\,Pr(X_3 \geq k)$

$= \left(\frac{6-(k-1)}{6}\right)\left(\frac{6-(k-1)}{6}\right)\left(\frac{6-(k-1)}{6}\right)$

$= \left(\frac{7-k}{6}\right)^3$

$\therefore E(M) = \sum_{k=1}^{\infty} Pr(M \geq k)$

$= \sum_{k=1}^{6} Pr(M \geq k)$

$= \sum_{k=1}^{6} \left(\frac{7-k}{6}\right)^3$

$= \ldots$

4 (a). $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$= \int_0^1 x \cdot 2(1-x)\, dx$

$= \int_0^1 2x - 2x^2\, dx$

$= \left[x^2 - \frac{2}{3}x^3 \right]_0^1$

$= \frac{1}{3}$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

$= \int_0^1 x^2 \cdot 2(1-x)\, dx$

$= \int_0^1 2x^2 - 2x^3\, dx$

$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4 \right]_0^1$

$= \frac{1}{6}$

$V(X) = E[X^2] - \left[E[X]\right]^2$

$= \frac{1}{18}$

(b). $Y = 3X - 2$

$E[Y] = 3E[X] - 2$

$= -1$

$V(Y) = 3^2 V(X)$

$= \frac{1}{2}$

5. $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$= \int_0^1 x(a+bx^2)\, dx$

$= \int_0^1 ax + bx^3\, dx$

$= \left[\frac{a}{2}x^2 + \frac{b}{4}x^4 \right]_0^1$

$= \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$ — ①

$\int_{-\infty}^{\infty} f_X(x) = 1$

$\Rightarrow \int_0^1 a + bx^2\, dx = \left[ax + \frac{b}{3}x^3 \right]_0^1$

$= a + \frac{b}{3} = 1$ — ②

① × 2 : $a + \frac{b}{2} = \frac{6}{5}$ — ③

③ − ② : $\frac{b}{6} = \frac{1}{5}$

$b = \frac{6}{5}$

$a = \frac{3}{5}$

6. $E[(X-1)^2] = E[X^2 - 2X + 1]$

$= E[X^2] - 2E[X] + 1 = 10$

$\Rightarrow E[X^2] - 2E[X] = 9$ — ①

$E[(X-2)^2] = E[X^2 - 4X + 4]$

$= E[X^2] - 4E[X] + 4 = 6$

$\Rightarrow E[X^2] - 4E[X] = 2$ — ②

① − ② : $2E[X] = 7$

$E[X] = 3.5$

$E[X^2] = 16$

$V(X) = E[X^2] - [E[X]]^2$

$= 3.75$

7. $\mu = 10$, $\sigma^2 = 4 \Rightarrow \sigma = 2$

(a). $Pr(5 < X < 15) = Pr(-5 < X - 10 < 5)$

$= Pr(|X-10| < 5)$

$= Pr(|X-\mu| < \frac{5}{2}\sigma) \geq 1 - \frac{1}{(\frac{5}{2})^2} = \frac{21}{25}$

(b). $Pr(5 < X < 14) \geq Pr(6 < X < 14)$

$= Pr(|X-\mu| < 2\sigma) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$

(c). $Pr(|X-10| < 3) = Pr(|X-\mu| < \frac{3}{2}\sigma) \geq 1 - \frac{1}{(\frac{3}{2})^2} = \frac{5}{9}$

(d). $Pr(|X-10| \geq 3) = Pr(|X-\mu| \geq \frac{3}{2}\sigma) \leq \frac{1}{(\frac{3}{2})^2} = \frac{4}{9}$

(e). $Pr(|X-10| \geq c) \leq 0.04 = \frac{1}{5^2}$

$c = \frac{5}{2}\sigma = \cancel{5}$

$k = 5$, $c = 5\sigma = 10$

8 (a). $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)\ dx$

$= \int_0^1 x \cdot 6x(1-x)\ dx$

$= \int_0^1 6x^2 - 6x^3\ dx$

$= \left[2x^3 - \frac{3}{2}x^4 \right]_0^1$

$= \frac{1}{2}$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\ dx$

$= \int_0^1 x^2 \cdot 6x(1-x)\ dx$

$= \left[\frac{3}{2}x^4 - \frac{6}{5}x^5 \right]_0^1$

$= \frac{3}{10}$

$\sigma = \sqrt{V(X)}$

$= \sqrt{E[X^2] - (E[X])^2}$

$= \sqrt{\frac{1}{20}}$

$= 0.2236$

(b). $Pr(\mu - 2\sigma < X < \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} f_X(x)\ dx$

$= \int_{\mu-2\sigma}^{\mu+2\sigma} 6x(1-x)\ dx$

$= \left[3x^2 - 2x^3 \right]_{\mu-2\sigma}^{\mu+2\sigma}$

$= \left[3x^2 - 2x^3 \right]_{0.2764}^{0.7236}$

$= 0.6261$ X 0.9839

Probably careless somewhere

(c). $Pr(\mu - 2\sigma < X < \mu + 2\sigma) = Pr(|X-\mu| < 2\sigma) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$

(d). Consistent

9. Let X = life of light bulb (in hours)

$\mu_X = 900$, $\sigma = 50$

Symmetric: $Pr(X < 700) = Pr(X > 1100)$

$= \dfrac{Pr(X < 700 \text{ or } X > 1100)}{2}$ → $k = 4$; apply Chebyshev's

$= \dfrac{Pr(|X - 900| > 4(50))}{2}$