ST2334 - Special Probability Distributions

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4.1 — Discrete Uniform Distribution

- **Definition 4.1**: Random variable X assumes the values $x_1, x_2, \dots, x_k$,

  with equal probability

  → X has a discrete uniform distribution and p.f. is given by:

$$f_X(x) = \begin{cases} \frac{1}{k} & x = x_1, x_2, \dots, x_k; \\ 0 & \text{otherwise} \end{cases}$$

- **Theorem 4.1**: Mean and variance of the discrete uniform distribution:

$$\mu = E(X) = \sum_{\text{all } x} x f_X(x) = \sum_{i=1} x_i \frac{1}{k} = \frac{1}{k}\sum_{i=1}^{k} x_i$$

$$\sigma^2 = V(X) = \sum_{\text{all } x} (x-\mu)^2 f_X(x) = \frac{1}{k}\sum_{i=1}^{k} (x_i-\mu)^2$$

or

$$\sigma^2 = E(X^2) - \mu^2 = \frac{1}{k}\left(\sum_{i=1}^{k} x_i^2\right) - \mu^2$$

4.2 - Bernoulli and Binomial Distributions

4.2.1 - Bernoulli Distributions

special case of binomial distribution $(n = 1)$

coded 1 and 0

- Bernoulli experiment : a random experiment with only two possible outcomes
- Definition 4.2 : A random variable X has a Bernoulli distribution if p.f. of X is :

$$f_X(x) = p^x(1-p)^{1-x}, \quad x = 0, 1;$$

where parameter p satisfies $0 < p < 1$.

$f_X(x) = 0$ for other X values

- $(1-p)$ is often denoted as q
- $Pr(X=1) = p$ and $Pr(X=0) = 1-p = q$
- Theorem 4.2 : Mean and variance of a Bernoulli distribution :

$$\mu = E(X) = p$$

$$\sigma^2 = V(X) = p(1-p) = pq$$

4.2.2 - Binomial Distributions

- Definition 4.3 : A random variable X has a binomial distribution (i.e. $X \sim B(n,p)$) with two parameters n and p, if p.f. of X is :

$$Pr(X=x) = f_X(x) = \binom{n}{x}p^x(1-p)^{n-x} = \binom{n}{x}p^x q^{n-x},$$

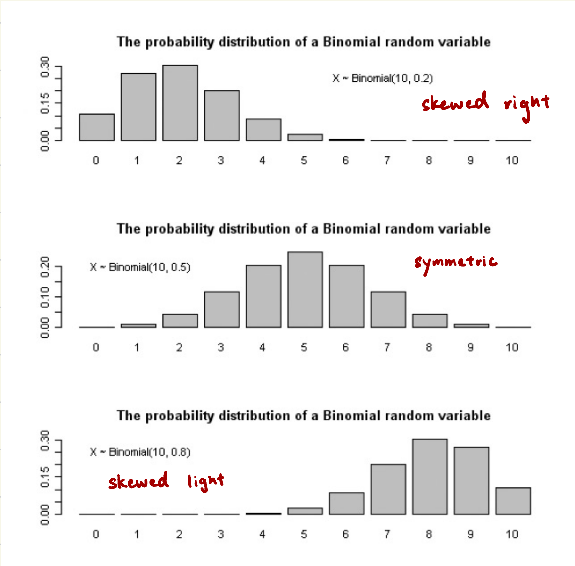
for $x = 0, 1, \ldots, n$ where p satisfies $0 < p < 1$, $q = 1-p$,

and n ranges over the positive integers

- X is the number of successes that occur in n independent Bernoulli trials
- Theorem 4.3 : Mean and variance of a binomial distribution for $X \sim B(n,p)$:

$$\mu = E(X) = np$$

$$\sigma^2 = V(X) = np(1-p) = npq$$



- Conditions for a binomial experiment :

① Only consists of n repeated Bernoulli trials

② Only two possible outcomes : success and failure in each trial

③ $Pr(\text{success}) = p$ is the same constant in each trial

④ Trials are independent

- The RV X is the number of successes among the n trials in a binomial experiment

$\longrightarrow X \sim B(n, p)$

4.3 - Negative Binomial Distribution

x

- Let X be a random variable represents the number of trials to produce the k successes in a sequence of independent Bernoulli trials

$X \sim NB(k, p)$

- X follows a negative binomial distribution with parameters k and p
- p.f. of X: $Pr(X = x) = f_X(x) = \binom{x-1}{k-1} p^k q^{x-k}$

  for $x = k, k+1, k+2, \dots$

- Mean and variance of a negative binomial distribution:

$$E(X) = \frac{k}{p}$$

$$Var(X) = \frac{(1-p)k}{p^2}$$

- $k = 1$: geometric distribution i.e. $X \sim NB(1, p) \rightarrow X \sim Geo(p)$

t, may be of any length e.g. minute, day, week, month, year

4.4 – Poisson Distribution

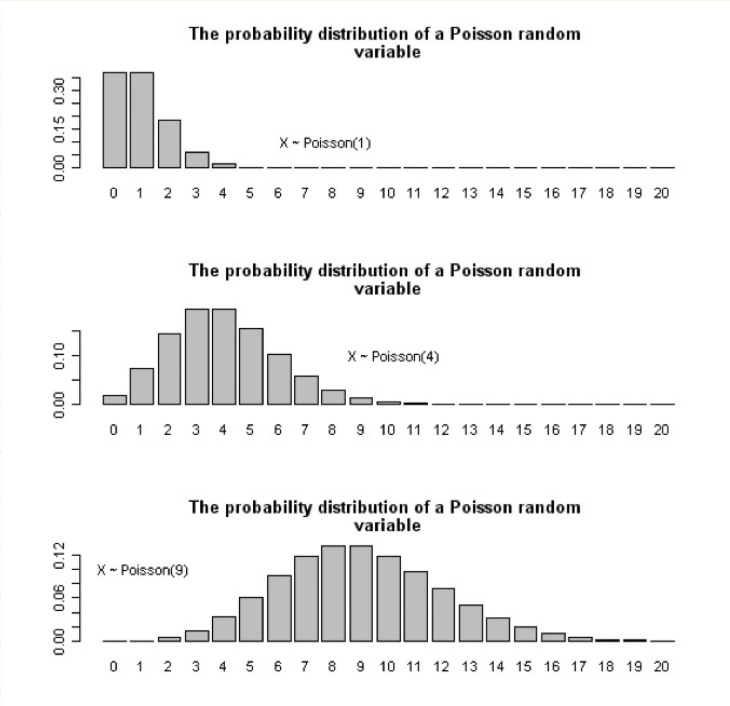
- **Poisson experiments**: experiments yielding numerical values of a random variable X, the number of successes occurring during a given time interval or in a specified region
- Properties of Poisson experiments:
  1. Number of successes occurring in one time interval or specified region are independent of those occurring in any other disjoint time interval or region of space
  2. Probability of a single success occurring during a very short time interval or in a small region is proportional to the length of the time interval or the size of the region and does not depend on the number of successes occurring outside this time interval or region
  3. Probability of more than one success occurring in such a short time interval or falling in such a small region is negligible
- Definition 4.4: **Poisson random variable**: number of successes X in a Poisson experiment

  Poisson distribution: probability distribution of the Poisson random variable X, p.f. given by:

$$f_X(x) = Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots$$

where λ : average number of successes occurring in the given time interval or specified region

and $e \approx 2.71828\ldots$



X with parameter λ

- **Theorem 4.4** : Mean and variance of a Poisson random variable :

$$E(X) = \lambda$$

$$V(X) = \lambda$$

4.5 - Poisson Approximation to the Binomial Distribution

- **Theorem 4.5**: Let X be a binomial random variable with parameters n and p, i.e. $Pr(X=x) = f_X(x) = \binom{n}{x} p^x q^{n-x}$, where $q = 1-p$.

  Suppose $n \to \infty$ and $p \to 0$ in such a way that $\lambda = np$ remains a constant as $n \to \infty$.

  Then X will approximately have Poisson distribution with parameter np, i.e. $\lim_{\substack{p \to 0 \\ n \to \infty}} Pr(X=x) = \frac{e^{-np}(np)^x}{x!}$

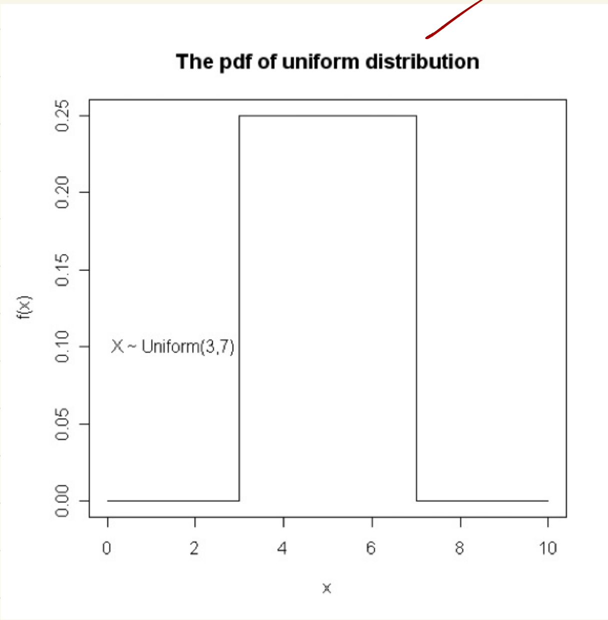
- If p is close to 1, we can still use Poisson distribution to approximate binomial probabilities by interchanging what we have defined to be a success and a failure so that changing p to a value close to zero

4.6 – Continuous Uniform Distribution

- **Definition 4.5** : A random variable has a uniform distribution over the interval $[a,b]$, $-\infty < a < b < \infty$, denoted by $U(a,b)$, if its p.d.f. is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b; \\ 0 & \text{otherwise.} \end{cases}$$

a.k.a. rectangular distribution



- **Theorem 4.6** : Mean and variance of a continuous uniform random variable:

If X is uniformly distributed over $[a,b]$, then:

$$E(X) = \frac{a+b}{2}$$

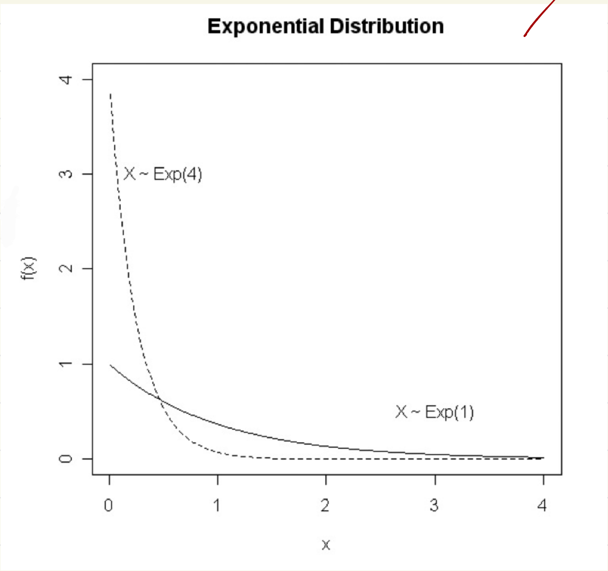
$$V(X) = \frac{1}{12}(b-a)^2$$

4.7 - Exponential Distribution

- **Definition 4.6**: A continuous random variable X assuming all non-negative values has an **exponential distribution** with parameter $\alpha > 0$ if its p.f. is given by:

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note: $\int_{-\infty}^{\infty} f(x)\, dx = 1$



- **Theorem 4.7**: **Mean and variance** of an exponential random variable:

If X has an exponential distribution with parameter $\alpha > 0$, then:

$$E(X) = \frac{1}{\alpha}$$

$$V(X) = \frac{1}{\alpha^2}$$

- p.d.f. can be written in the form: $f_X(x) = \begin{cases} \frac{1}{\mu} e^{-\frac{x}{\mu}} & x > 0; \\ 0 & \text{otherwise.} \end{cases}$

Then $E(X) = \mu$ and $V(X) = \mu^2$.

- **Theorem 4.8** : Suppose X has an exponential distribution with parameter $\alpha > 0$. Then for any two positive numbers s and t, we have :

"No memory" property of exponential distribution

$$Pr(X > s+t \mid X > s) = Pr(X > t)$$

- Application of exponential distribution : frequently used as a model for the distribution of times between the occurrence of successive events e.g. customers arriving at a service facility or calls coming into a switchboard

4.8 - Normal Distribution

$-\infty < x < \infty$

- **Definition 4.7**: The random variable X assuming all real values has a normal (Gaussian) distribution if its p.d.f. is given by:

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right), \quad -\infty < x < \infty,$$

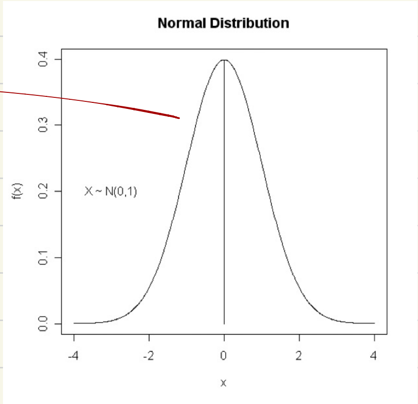
where $-\infty < \mu < \infty$ and $\sigma > 0$.

parameters of the normal distribution

- Properties of the normal distribution:

① Graph is bell-shaped, symmetrical about vertical line $x = \mu$

normal curve



② Maximum point occurs at $x = \mu$, value: $\frac{1}{\sqrt{2\pi}\,\sigma}$

③ Normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean

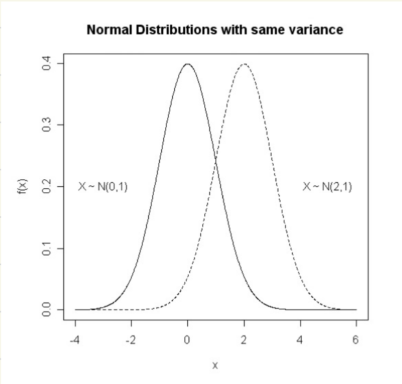
④ Total area under the curve and above the horizontal axis is equal to 1

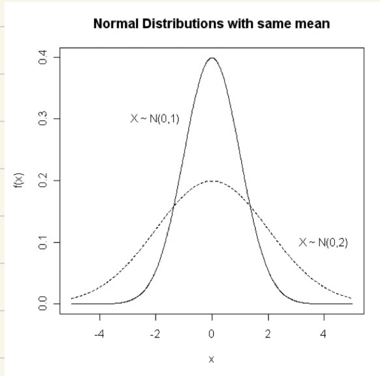
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1$$

⑤ $E(X) = \mu$ and $V(X) = \sigma^2$

⑥ Two normal curves are identical in shape if they have the same σ^2, but they are centered at different positions when their means are different

(6)





(7) As σ increases, the curve flattens; and as σ decreases, the curve sharpens

(8) If X has distribution $N(\mu, \sigma^2)$, and if $Z = \frac{(X-\mu)}{\sigma}$, then

Z has the $N(0, 1)$ distribution. i.e. $E(Z) = 0$ and $V(Z) = 1$

$\therefore$ Z has a standardized normal distribution and p.d.f. of Z is:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

- Let $z_1 = \frac{x_1 - \mu}{\sigma}$ and $z_2 = \frac{x_2 - \mu}{\sigma}$. Then

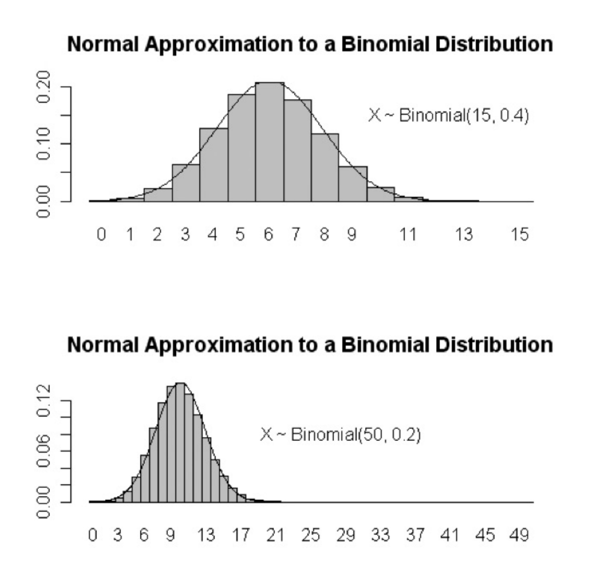
$$Pr(x_1 < X < x_2) = Pr(z_1 < Z < z_2)$$

- Standardized normal distribution is tabulated $\Rightarrow$ useful
- $\Phi(z) = Pr(Z \leq z)$

4.9 – Normal Approximation to the Binomial Distribution

- When $n \to \infty$ and $p \to \frac{1}{2}$, we can use normal distribution to approximate the binomial distribution. In fact, even when n is small and p is not extremely close to 0 or 1, the approximation is fairly good.
- A good rule of thumb is to use the normal approximation only when

  $np > 5$ and $n(1-p) > 5$



- **Theorem** : If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$, then as $n \to \infty$,

  $Z = \dfrac{X - np}{\sqrt{npq}}$ is approximately $\sim N(0,1)$

- Continuity correction :

① $Pr(X = k) \approx Pr(k - \frac{1}{2} < X < k + \frac{1}{2})$

② $Pr(a \leq X \leq b) \approx Pr(a - \frac{1}{2} < X < b + \frac{1}{2})$

$Pr(a < X \leq b) \approx Pr(a + \frac{1}{2} < X < b + \frac{1}{2})$

$Pr(a \leq X < b) \approx Pr(a - \frac{1}{2} < X < b - \frac{1}{2})$

$Pr(a < X < b) \approx Pr(a + \frac{1}{2} < X < b - \frac{1}{2})$

③ $Pr(X \leq c) = Pr(0 \leq X \leq c) \approx Pr(-\frac{1}{2} < X < c + \frac{1}{2})$

④ $Pr(X > c) = Pr(c < X \leq n) \approx Pr(c + \frac{1}{2} < X < n + \frac{1}{2})$