# ST2334 – Chapter 2 – Concepts of Random Variables

2.1 – Introduction

2.2 – Discrete Probability Distributions

2.3 – Continuous Probability Distributions

2.4 – Cumulative Distribution Function

2.5 – Mean and Variance of a Random Variable

2.6 – Chebyshev's Inequality

## 2.1 – Introduction

- Some function of the outcome > actual outcome

### 2.1.1 – Random Variable

- Random variable : a function $X$ which assigns a number to every element $s \in S$.

event in $S$

- $R_X = \{ x \mid x = X(s), s \in S \}$

### 2.1.2 – Equivalent Events

A consists of all sample points, s, in S for which $X(S) \in B$.

- Suppose $A = \{ s = S \mid X(s) \in B \}$
  Then A and B are equivalent events and $Pr(B) = Pr(A)$.

## 2.2 – Discrete Probability Distributions

### 2.2.1 – Discrete Random Variable

- Definition 2.3 : Let X be a random variable

$R_X$, the range space

Number of possible values of X is finite / countable infinite

$\longrightarrow$ X is a discrete random variable

### 2.2.2 – Probability Function

a.k.a probability mass function, p.m.f.

- Each value of X has a probability function $f(x)$
- Probability distribution of X : collection of pairs $(x_i, f(x_i))$
  - Must satisfy :

① $f(x_i) \geq 0 \quad \forall x_i$

② $\sum_{i=1}^{\infty} f(x_i) = 1$

### 2.2.3 – Another View of Probability Function

- Specifying a mathematical model for a finite population

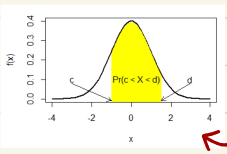
# 2.3 – Continuous Probability Distributions

## 2.3.1 – Continuous Random Variables

- **Definition 2.4** : Suppose $R_X$, the range space of a random variable $X$, is an interval or a collection of intervals.
  Then $X$ is a continuous random variable.

## 2.3.2 – Probability Density Function

- **Definition 2.5** : Let $X$ be a continuous random variable.
  The probability density function $f(x)$ satisfies : p.d.f.
  ① $f(x) \geq 0 \quad \forall\ x \in R_X$
  ② $\int_{R_X} f(x)\,dx = 1$ or $\int_{-\infty}^{\infty} f(x)\,dx = 1$ since $f(x) = 0$ for $x \notin R_X$.
  $(c, d) \subset R_X$
  ③ $\forall\ c, d$ s.t. $c < d$, $Pr(c \leq X \leq d) = \int_c^d f(x)\,dx$




- In a probability statement, $\leq$ and $<$ can be used interchangeably
- $Pr(A) = 0 \not\Rightarrow A = \phi$
- $X$ assumes values only in some interval $[a, b]$
  $\longrightarrow f(x) = 0 \quad \forall\ X$ outside $[a, b]$

## 2.4 - Cumulative Distribution Function

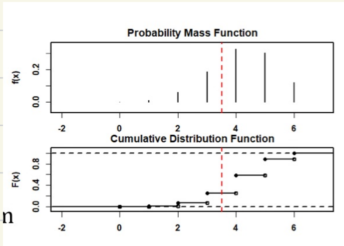
- Definition 2.6 : Let $X$ be a random variable, discrete or continuous.

Cumulative distribution function of $X$ : $F(x) = Pr(X \leq x)$

### 2.4.1 - CDF for Discrete Random Variables

- $X$ is a discrete random variable step function

$\longrightarrow F(x) = \sum_{t \leq x} f(t) = \sum_{t \leq x} Pr(X = t)$




- For any $a, b$ with $a \leq b$,

$Pr(a \leq X \leq b) = Pr(X \leq b) - Pr(X < a)$

$= F(b) - F(a^-)$ largest possible $X < a$

- The only possible values are integers and $a, b$ are integers

$\longrightarrow Pr(a \leq X \leq b) = Pr(X = a$ or $a+1$ or ... or $b)$

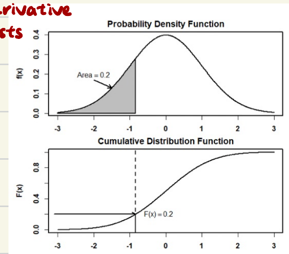
$Pr(a \leq X \leq b) = F(b) - F(a-1)$

- $a = b \longrightarrow Pr(X = a) = F(a) - F(a-1)$

### 2.4.2 - CDF for Continuous Random Variables

- $X$ is a continuous random variable $\longrightarrow F(x) = \int_{-\infty}^{x} f(t)\, dt$
- For a continuous random variable $X$, $f(x) = \frac{d\, F(x)}{dx}$ if the derivative exists
- $Pr(a \leq X \leq b) = Pr(a < X \leq b)$

$= F(b) - F(a)$




- $F(x)$ is a non-decreasing function

i.e. $x_1 < x_2 \longrightarrow F(x_1) \leq F(x_2)$

- $0 \leq F(x) \leq 1$

# 2.5 - Mean and Variance of a Random Variable

## 2.5.1 - Expected Values

- **Definition 2.7a**: $X$ is a discrete random variable, taking on values $x_1, x_2, \dots$ with probability function $f_X(x)$

  → mean / expected value of $X$: $\mu_X = E(X) = \sum_i x_i f_X(x_i) = \sum_x x f_X(x)$

  Not necessarily a possible value of $X$

- **Definition 2.7b**: $X$ is a continuous random variable with probability density function $f_X(x)$

  → mean of $X$: $\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$

  (weighted) average

- The expected value exists provided the sum / integral in the above definitions exists
- Discrete case: $f_X(x) = \frac{1}{N}$ for each $N$ of $x$

  → $E(X) = \sum_i x_i f(x_i) = \frac{1}{N} \sum_i x_i$

## 2.5.2 - Expectation of a Function of a RV

- **Definition 2.8**: For any function $g(x)$ of a random variable $X$ with p.f. / p.d.f. $f_X(x)$,
  1. If $X$ is a discrete r.v. providing sum exists: $E[g(X)] = \sum_x g(x) f_X(x)$
  2. If $X$ is a continuous r.v. providing integral exists: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

- Special case 1:

  $g(x) = (x - \mu_X)^2$ → definition of variance of a given r.v. $X$

- **Definition 2.9**: Let $X$ be a r.v. with p.f. / p.d.f. $f(x)$.

  Then variance of $X$: $\sigma_X^2 = V(X) = E[(X - \mu_X)^2]$

$$= \begin{cases} \sum_x (x-\mu_X)^2 f_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

- $V(X) \geq 0$
- $V(X) = E(X^2) - [E(X)]^2$
- Standard deviation of $X$: $\sigma_X = \sqrt{V(X)}$ positive
- Special case 2:

  $g(x) = x^k$ → $k$-th moment of $X$

## 2.5.3 - Properties of Expectation

① $E(aX + b) = aE(X) + b$ ⟶ a, b are constants

② $V(X) = E(X^2) - [E(X)]^2$

③ $V(aX + b) = a^2 V(X)$

## 2.6 - Chebyshev's Inequality

- We know probability distribution of a r.v. $X \longrightarrow$ we can compute $E(X)$ and $V(X)$, but
  We know probability distribution of a r.v. $X \not\longleftarrow$ we can compute $E(X)$ and $V(X)$
  $\Rightarrow$ we cannot compute quantities e.g. $Pr(|X - E(X)| \leq c)$

  positive constant

  $Pr(|Y| \leq a) = Pr(-a \leq Y \leq a)$

- **Chebyshev's inequality**: Let $X$ be a r.v. with $E(X) = \mu$ and $V(X) = \sigma^2$.

  Probability that value of $X$ lies $\geq k$ s.d. from its mean is $\leq \frac{1}{k^2}$

  Then for any $k > 0$, we have $Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$

  or $Pr(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

- $k > 0$
- Inequality is true $\forall$ distributions with finite mean and variance
- Theorem gives a lower probability on the probability that $|X - \mu| < k\sigma$