ST2334 - Chapter 1 - Basic Concepts of Probability
Part 2

1.5 - Basic Properties of Probability

1.6 - Conditional Probability

1.7 - Independent Events

1.5 - Basic Properties of Probability

1.5.1 - Some basic properties of probability

① $Pr(\phi) = 0$

② $A_1, A_2, \dots, A_n$ are mutually exclusive $\longrightarrow Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} Pr(A_i)$

③ $Pr(A') = 1 - Pr(A)$

④ $Pr(A) = Pr(A \cap B) + Pr(A \cap B')$

⑤ $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

⑥ $Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C)$
$\qquad - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C)$

$\Rightarrow$ Extension (Inclusion - Exclusion Principle):

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \Pr(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr(A_i \cap A_j)$$
$$+ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \Pr(A_i \cap A_j \cap A_k) - \cdots\cdots$$
$$+ (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n)$$

⑦ $A \subset B \longrightarrow Pr(A) < Pr(B)$

1.5.2 - Sample Spaces Having Finite Outcomes

- Consider $S = \{a_1, a_2, \dots, a_k\}$ and $Pr(a_i) = p_i$. Then:
  1. $0 \le p_i \le 1$ for $i = 1, 2, \dots, k$
  2. $p_1 + p_2 + \dots + p_k = 1$

1.5.3 - Sample Spaces Having Equally Likely Outcomes

- Consider $S = \{a_1, a_2, \dots, a_k\}$ and $Pr(a_1) = Pr(a_2) = \dots = Pr(a_k)$.
  1. $Pr(a_i) = \frac{1}{k}$, $i = 1, 2, \dots, k$
  2. For any event A, $Pr(A) = \frac{\text{Number of sample points in } A}{\text{Number of sample points in } S}$

1.6 – Conditional Probability

1.6.1 – Introduction

- $Pr(A|B)$: conditional probability of A given B has occured
- $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$

1.6.2 – Conditional Probability

- Various postulates of probability for $Pr(B|A)$ (fixed A):

① $0 \leq Pr(B|A) \leq 1$

② $Pr(S|A) = 1$

③ $B_1, B_2, \ldots$ are mutually exclusive (disjoint) events

$\longrightarrow Pr\left(\bigcup_{i=1}^{\infty} B_i \middle| A\right) = \sum_{i=1}^{\infty} Pr(B_i|A)$

$\Rightarrow$ In particular, B_1 and B_2 are disjoint

$\longrightarrow Pr(B_1 \cup B_2 | A) = Pr(B_1|A) + Pr(B_2|A)$

1.6.3 – Multiplication Rule of Probability

- $Pr(A \cap B) = Pr(A)\, Pr(B|A)$ or

$Pr(A \cap B) = Pr(B)\, Pr(A|B)$,

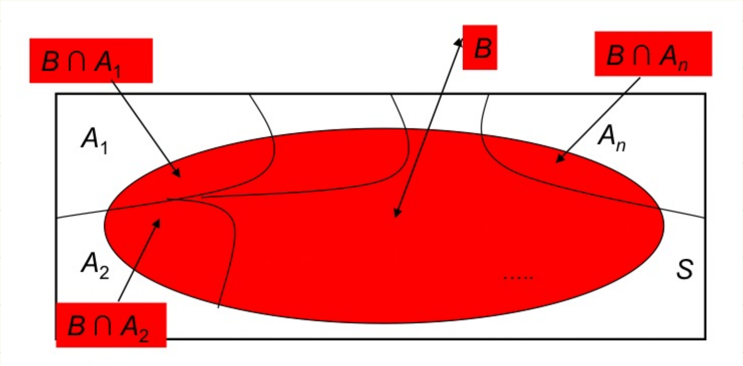
providing $Pr(A) > 0$, $Pr(B) > 0$.

$\Rightarrow$ In general, $Pr(A_1 \cap A_2 \cap \ldots \cap A_n) = Pr(A_1)\, Pr(A_2|A_1)\, Pr(A_3|A_1 \cap A_2) \ldots Pr(A_n|A_1 \cap \ldots \cap A_{n-1})$,

providing $Pr(A_1 \cap \ldots \cap A_{n-1}) > 0$.

1.6.4 – The Law of Total Probability

- Partition: $A_1, A_2, \ldots, A_n$ are partitions of S
  $\rightarrow A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive events
  s.t. $A_i \cap A_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{n} A_i = S$.
- Let $A_1, A_2, \ldots, A_n$ be a partition of S. Then for any event B,
  $Pr(B) = \sum_{i=1}^{n} Pr(B \cap A_i) = \sum_{i=1}^{n} Pr(A_i) Pr(B | A_i)$



1.6.5 – Bayes' Theorem

- Let $A_1, A_2, \ldots, A_n$ be a partition of S. Then

$$Pr(A_k \mid B) = \frac{Pr(A_k)\,Pr(B \mid A_k)}{\sum_{i=1}^{n} Pr(A_i)\,Pr(B \mid A_i)}$$

$$= \frac{Pr(A_k)\,Pr(B \mid A_k)}{Pr(B)}$$

1.7 – Independent Events

1.7.1 – Introduction

1.7.2 – Independent Events

- Two events A and B are independent

$\longleftrightarrow \Pr(A \cap B) = \Pr(A)\Pr(B)$

1.7.3 – Properties of Independent Events

- Suppose $\Pr(A) > 0$, $\Pr(B) > 0$.

① A and B are independent

$\longrightarrow \Pr(B|A) = \Pr(B)$ and $\Pr(A|B) = \Pr(A)$.

② A and B are independent

$\longrightarrow$ A and B cannot be mutually exclusive.

③ A and B are mutually exclusive

$\longrightarrow$ A and B cannot be independent

- For any event:

④ The sample space S and the empty set ϕ are independent of any event

⑤ $A \subset B \longrightarrow$ A and B are dependent unless $B = S$.

- Theorem : A and B are independent

$\longrightarrow$ so are A and B', A' and B, A' and B'.

1.7.4 - n Independent Events

- **Pairwise independent** events: $A_1, A_2, \dots, A_n$ are pairwise independent
  $\Longleftrightarrow Pr(A_i \cap A_j) = Pr(A_i)Pr(A_j)$
  for $i \neq j$ and $i, j = 1, \dots, n$
- **Mutually independent** events: $A_1, A_2, \dots, A_n$ are mutually independent
  (or simply **independent**) $\Longleftrightarrow$ for any subset $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ of $A_1, \dots, A_n$,
  $Pr(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = Pr(A_{i_1})Pr(A_{i_2}) \dots Pr(A_{i_k})$
- Mutual independence $\longrightarrow$ pairwise independence, but
  the converse is not always true.
- Suppose $A_1, A_2, \dots, A_n$ are mutually independent events.
  let $B_i = A_i$ or A_i', $i = 1, 2, \dots, n$.
  Then $B_1, B_2, \dots, B_n$ are also mutually independent events.