On the Chua Circuit

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1 Introduction

The Chua Circuit is an electronic circuit named after Leon Chua, who suggested it in 1983. It was reported by Matsumoto in 1984 [3] that (a simplified version of) this system exhibited chaotic behaviour in the form of a chaotic attractor, although it is a simple autonomous circuit.

The circuit in question is shown in figure 1. It contains four passive elements – two capacitors C_1, C_2, one inductor L and a resistor G – and one non-linear resistor R. In the original circuit the non-linearity of R was given by a three-component piecewise linear function, dependent on the voltage over the resistor [3, 4, 1], giving the current through it. We will, however, instead study a resistor with a non-linearity that is given by the function

$$\phi(x) = \rho x^3 - \sigma x \tag{1}$$

as suggested in [2, p. 379]. The study will also follow the "exploration" given in this publication and expand on it.

The dynamics of the original circuit can be described by the following equations [4, eq. (1.1)]:

$$\begin{cases} C_1 \dfrac{dv_{C_1}}{dt} = G(v_{C_2} - v_{C_1}) - f(v_{C_1}) \\ C_2 \dfrac{dv_{C_2}}{dt} = G(v_{C_1} - v_{C_2}) + i_L \\ L \dfrac{di_L}{dt} = -v_{C_2} \end{cases} \tag{2}$$

This set of equations can be rescaled to transform it into [1, eq. (1.1)]:

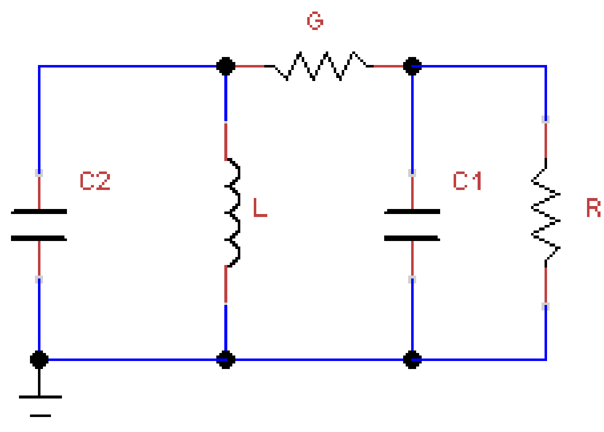


Figure 1: The Chua Circuit. The resistor R has a non-linear dependence on the voltage. After [3, 4].

$$\begin{cases} \dfrac{dx}{d\tau} = a(y - \phi(x)) \\ \dfrac{dy}{d\tau} = x - y + z \\ \dfrac{dz}{d\tau} = -by \end{cases} \tag{3}$$

We will now, in accordance with [2], modify this circuit by taking $\phi(x)$ as in eq. 1 and looking at the following system:

$$\begin{cases} x' = a(y - \phi(x)) \\ y' = x - y + z \\ z' = -by \end{cases} \tag{4}$$

2 Investigating the equilibria

First, we will locate the equilibria of this system. Note that we will take the parameters $a, b, \rho, \sigma > 0$. This is because the first two correspond to physical parameters that are positive and the other two are positive by assumption.

Equating the right part of equation 4 with zero, we get from the z' part, noting that $b \neq 0$, that $y = 0$. Furthermore, the second equation tells us then that $z = -x$. And lastly, the first equation tells us to look for the roots of $\phi(x)$. We will factor this into:

$$\phi(x) = x(\rho x^2 - \sigma) = 0, \tag{5}$$

from which we deduce that $x = 0$ or $x = \pm\sqrt{\frac{\sigma}{\rho}}$.

This leaves us with three equilibrium points for the system:

$$X_0 = (x, y, z)_0 = (0, 0, 0) \tag{6}$$

$$X_\pm = (x, y, z)_\pm = (\pm\sqrt{\frac{\sigma}{\rho}}, 0, \mp\sqrt{\frac{\sigma}{\rho}}), \tag{7}$$

one equilibrium in the origin and two in the xz-plane, symmetric in the origin. Note that the location of the equilibria $X_\pm$ is completely determined by the parameters in ϕ, and that these fail to exist if ρ and *sigma* do not have the same sign.

2.1 Classification of the origin

We will now linearize the system about the equilibria and classify these points based on this. For X_0 (the origin), we have the following linearization:

$$X' = A_0 X, \tag{8}$$

$$A_0 := \begin{pmatrix} -a\phi'(0) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix} = \begin{pmatrix} a\sigma & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix}. \tag{9}$$

Now, to study the eigenvalues of A_0, we note that $tr(A_0) = a\sigma - 1$ and $det(A_0) = ab\sigma > 0$. For any real $n \times n$ matrix, we know that the trace is equal to the sum and the determinant to the product of the eigenvalues, counting multiplicities. For the eigenvalues of A_0 we have three cases:

1. All eigenvalues are zero
2. All eigenvalues are real and non-zero
3. One eigenvalue is real and two are complex and non-real, and all are non-zero.

We can distinguish these cases by studying the characteristic polynomial, which is given by:

$$p_{A_0}(\lambda) = -\lambda^3 + (a\sigma - 1)\lambda + (a\sigma + a - b)\lambda + ab\sigma. \tag{10}$$

Zero eigenvalues This case is fundamentally impossible, since in this case, we would have $0 = det(A_0) = ab\sigma > 0$, which is a contradiction. Therefore, all eigenvalues are non-zero.

Real and non-zero eigenvalues We will study this case with the help of the computer algebra program Wolfram Mathematica. Using the script in appendix A, we can compare the coefficients of the characteristic polynomial with a factorization that would arize when all roots are real and non-zero. We hereby note that either all eigenvalues must be positive, or two must be negative and one positive, since the determinant is positive. The automated algebraic analysis shows that the former case is impossible, so in this case we must have two negative eigenvalues and one positive.

We can prove this more rigorously for the subcase $0 < a\sigma \leq 1$. Let $\lambda_i \in \mathbb{R} (i = 1, 2, 3)$ be the eigenvalues of A_0. Then we have $ab\sigma = det(A_0) = \lambda_1\lambda_2\lambda_3 > 0$. If without generality we assume that $\lambda_1 \leq \lambda_2 \leq \lambda_3$, we can again distinguish two cases. If $\lambda_1 < 0$, we must have $\lambda_2 < 0$ and $\lambda_3 > 0$. This corresponds to a saddle with a stable plane, spanned by the (generalized) eigenvectors corresponding to λ_1 and λ_2.

If $\lambda_1 > 0$, we automatically have $\lambda_i > 0 (i = 1, 2, 3)$. This corresponds to a source.

Now we investigate the influence of the parameters a, b, σ on the eigenvalues. If $0 < a\sigma \leq 1$, we see that:

$$0 \geq a\sigma - 1 = tr(A_0) = \sum_{i=1}^{3} \lambda_i.$$

Therefore:

$$\lambda_3 \leq -(\lambda_1 + \lambda_2)$$

So, if $\lambda_1 > 0$, we get $0 < \lambda_3 \leq 0$, which can't be true. Therefore we must have in this case (i.e. if there are only real eigenvalues and $0 < a\sigma \leq 1$) that $\lambda_1, \lambda_2 < 0$ and $\lambda_3 \leq |\lambda_1 + \lambda_2|$. So, then the origin is a saddle with a stable plane.

Now to study the subcase of $a\sigma > 1$, we observe that if $\lambda_1, \lambda_2 < 0$, we get $\lambda_3 > -(\lambda_1 + \lambda_2) = |\lambda_1 + \lambda_2| > 0$. And if $\lambda_1, \lambda_2 > 0$, we get $\lambda_3 > -(\lambda_1 + \lambda_2)$, which is perfectly possible, since we know that $\lambda_3 > 0$. Therefore, on these criteria, both cases (source and saddle) can occur.

Note that if we include the third requirement $-(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = a - b + a\sigma$, which arises from equating $p_{A_0}(\lambda)$ with its factorization, we can proof it also for this subcase. However, we feel that such a proof is mere very precise bookkeeping, which can best be done by a computer algebra system.

One real, two non-real eigenvalues Now, let $\lambda_1 \in \mathbb{R}, \lambda_2, \lambda_3 \in \mathbb{C}$ be the eigenvalues of A_0. Then we immediately know that $\lambda_3 = \bar{\lambda_2}$, the complex conjugate of λ_2. From the trace and determinant we deduce:

$$\begin{gathered}0 < ab\sigma = \lambda_1|\lambda_2|^2 \implies \lambda_1 > 0 \\ a\sigma - 1 = \lambda_1 + \lambda_2 + \bar{\lambda_2} = \lambda_1 + 2Re(\lambda_2).\end{gathered}$$

So we see that the real eigenvalue must be strictly positive (since $|\lambda_2|^2 > 0$. Furthermore, if $0 < a\sigma \leq 1$, the real part of λ_2 and λ_3 is strictly negative.

When $a\sigma > 1$, we can again factor the characteristic polynomial:

$$\begin{gathered}p_{A_0}(\lambda) = k(\lambda - (\mu + \nu i))(\lambda - (\mu - \nu i))(\lambda - z) \\ (k, \mu, \nu, z \in \mathbb{R}).\end{gathered}$$

Again we can let Mathematica do the algebraic work, from which we get $Re(\lambda_1) = Re(\lambda_2) < 0$ for all (positive) values of the parameters.

Type of equilibrium So, we can deduce that in every case we will always have two eigenvalues with negative real part and one real positive eigenvalue. The question now arises whether the eigenvalues are real, or two are non-real. We can determine this by looking at the cubic discriminant of the characteristic polynomial. The algebraic expression for this is too large to include in this report, but we can tell you that it is a continuous function of the parameters. We can therefore fix some parameter values and then check the behaviour of the eigenvalues near these points.

With this in mind, we will now fix $b = 14, \rho = \frac{1}{16}$ and $\sigma = \frac{1}{6}$. We can now determine the roots of the discriminant with respect to a. A numerical calculation shows that two roots are real and negative and the other two are non-real. Now, we can calculate the discriminant for one positive value of a, e.g. $a = 1$. This gives a negative value of the discriminant, so we can conclude that for these parameters and all positive values of a, we get a negative discriminant, which corresponds to two non-real and one real eigenvalue.

2.2 Classification of $X_\pm$

We follow the same road as with the origin. The linearized system is:

$$Y' = A_\pm Y \tag{11}$$

$$A_\pm := \begin{pmatrix} -2a\sigma & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix}. \tag{12}$$

We now have $det(A_\pm) = -2a\sigma b < 0$ and $tr(A_\pm) = -(2a\sigma + 1) < 0$ and the same three possibilities as above. Again, we immediately note that an eigenvalue zero is not possible.

Real eigenvalues We first note that because $-2ab\sigma < 0$, we must have either one or three negative eigenvalues. A numerical search shows that there are instances of the parameters available for each of these cases. We therefore conclude that both cases are possible: a one-dimensional stable manifold with a two-dimensional unstable manifold (saddle) and a three-dimensional stable manifold (a sink).

Two non-real, one real eigenvalues This case turns out to be troubling. It is possible to find values of the parameters such that the real part of the non-real eigenvalues is zero. This indicates that for these parameter values this method of classification is undetermined.

We can however conclude that the real eigenvalue is always negative (because of the negative determinant). But apart from that, the real part of the non-real eigenvalues can be negative, positive and zero.

Type of eigenvalues We can of course again use the discriminant of the characteristic polynomial to determine whether the eigenvalues will be real or non-real. Again we fix the parameters as in the case of the origin and using the same methods we find a negative discriminant for all positive values of a, in a neighbourhood of $b = 14, \sigma = \frac{1}{6}$ and $\rho = \frac{1}{16}$.

Therefore, in this neighbourhood in the parameter space, we have one real and two non-real eigenvalues, which gives the troubling behaviour of above. I.e. for some parameter values, there may not exist a conjugation between the original and the linearized system. One example is $a = \frac{3}{2}(-1 + \sqrt{29})$. It is however to be noted that the eigenvalues depend continuously on the coefficients in the characteristic polynomial, which depend continously on the parameters. Therefore all bifurcations with respect to the real part of the non-real eigenvalues of the linearized system occur at the points in the parameter space for which the real part is equal to zero.

3 Bifurcations

We already showed that the origin is strictly a saddle point. To see what types of bifurcations might occur in the system when a is varied, we ran a simulation with an initial point close to one of the equilibrium points away from the origin and had a vary from 4 to 14.

The equilibrium point we started near acts as a spiral sink for low values of; $4 \leq a \leq 6$. When $6 \leq a \leq 6.64$, the solution orbit approaches a circular limit set around the equilibrium point. This limit set seems to disappear for $6.64 \leq a \leq 8.3$, but is probably just too small to see, for these values of a, the orbit approaches the equilibrium point, circles it and moves away to a larger limit set surrounding it, which implies that the equilibrium point is no longer a spiral sink, but rather a saddle point of some kind. The outer orbit limit is drawn to the origin, eventually splitting it into two orbit limits, one of which contin-ues to move toward the origin as a increases. When $8.68 \leq a \leq 9.37$ the outer orbit limit is close enough to the origin to be affected by it and for $a \geq 8.8$ the orbit is pulled through the origin and around the equilibrium point on the other side.

The circular limit set around the first equilibrium point is more clearly present around the second equilibrium point, and fluctuates in size as a increases from 9.37 to 10.7, steadily increasing its size from $a = 10.1$ onwards.

When a increases beyond 10.7, the orbit instantly moves away from its previous orbit and approaches a new limit set away from all 3 equilibrium points.

For an illustration of these changes, see figure 2, which shows numerous numerical approximations for $b = 14, \rho = \frac{1}{16}$ and $\sigma = \frac{1}{6}$ and various values of a. The calculations all started at the same initial point, namely $(1.63, 0, -1.63)$, near one of the equilibria.

References

[1] L. O. Chua, M. Komuro, and T. Matsumoto. The double scroll family. *IEEE Transactions on Circuits and Systems*, 33(11):1072–1118, November 1986.

[2] M. W. Hirsh, S. Smale, and R. L. Devaney. *Differential Equations, Dynamical Systems, and an Introduction to Chaos*. Academic Press, 3rd edition, 2012. ISBN: 9780123820112.

[3] T. Matsumoto. A chaotic attractor from chua's circuit. *IEEE Transactions on Circuits and Systems*, 31(12):1055–1058, December 1984.

[4] T. Matsumoto, L. O. Chua, and M. Komuro. The double scroll. *IEEE Transactions on Circuits and Systems*, 32(8):797–818, August 1985.

A Mathematica script

We used the following script with Wolfram Mathematica 7.0 to determine which eigenvalues are possible for the linearized system. This is the version used for the origin. Note that it can simply be modified to determine the behaviour for the other equilibria.

It works by computing the characteristic polynomial of the linearized system and making an assumption about the nature of its roots. Then, by comparing the coefficients of the characteristic polynomial and the expanded factorization, it deduces a set of requirements on the parameters for the system to be consistent. It then attempts to find a set of parameters for which certain restrictions on the eigenvalues hold. If no such set exists, it returns an empty list.

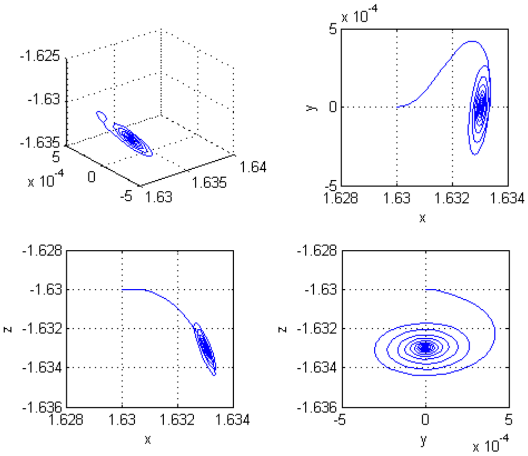
```
Clear[a, b, n]
m = {{a s, a, 0}, {1, -1, 1}, {0, -b, 0}};
m // MatrixForm
match = And @@ (Equal @@@
    Transpose[{CoefficientList[Expand[k (l - x) (l - y) (l - z)], l],
      CoefficientList[CharacteristicPolynomial[m, l], l]}])
FindInstance[
 match && a > 0 && b > 0 && s > 0 && x > 0 && y > 0 && z > 0, {a, b,
  s, x, y, z, k}]

match = And @@ (Equal @@@
    Transpose[{CoefficientList[
       Expand[k (l - (x + I y)) (l - (x - I y)) (l - z)], l],
      CoefficientList[CharacteristicPolynomial[m, l], l]}])
FindInstance[
 match && a > 0 && b > 0 && s > 0 && x < 0 && y >= 0 && z > 0, {a, b,
  s, x, y, z, k}]
FindInstance[
 match && a > 0 && b > 0 && s > 0 && x > 0 && y >= 0 && z > 0, {a, b,
  s, x, y, z, k}]
```

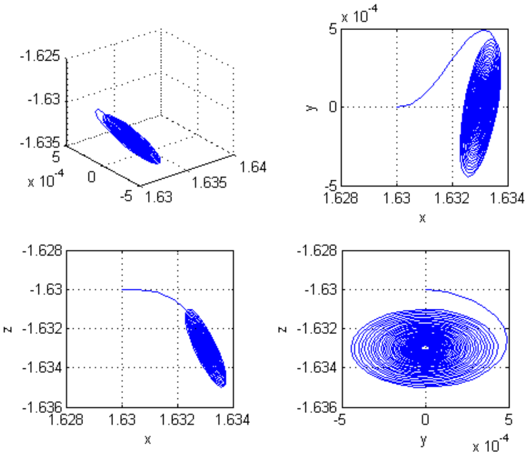
B Numerical approximations

Figure 2: The graphs of numerical approximations of the solution curve for $b = 14$, $\sigma = \frac{1}{6}$ and $\rho = \frac{1}{16}$.

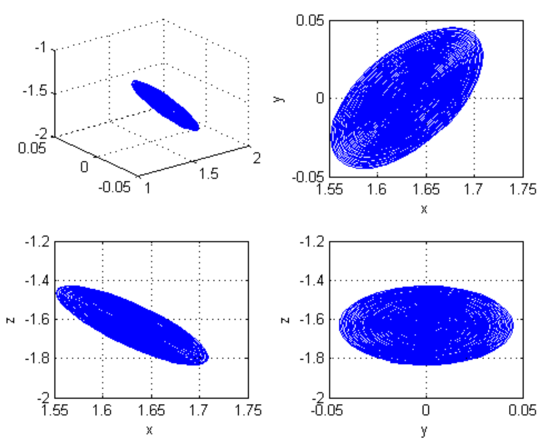
(a) $a = 5$



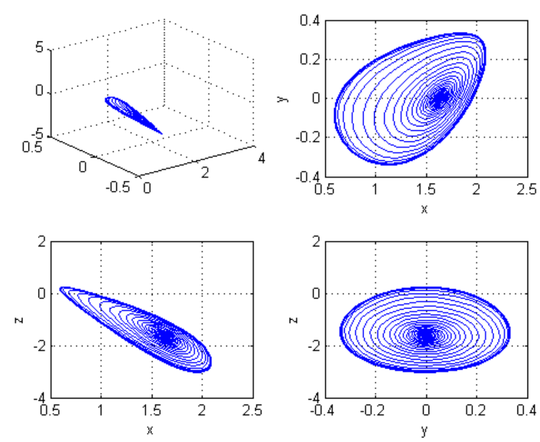
(b) $a = 6.3$

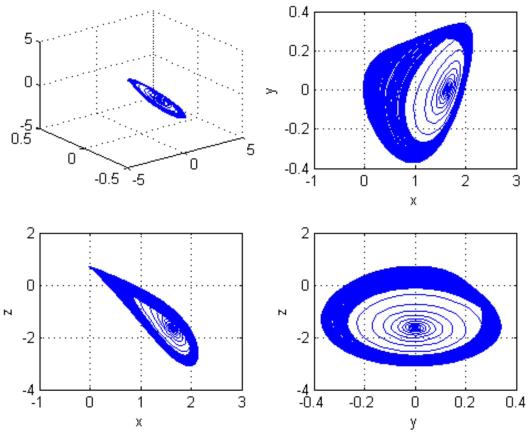


(c) $a = 6.68$

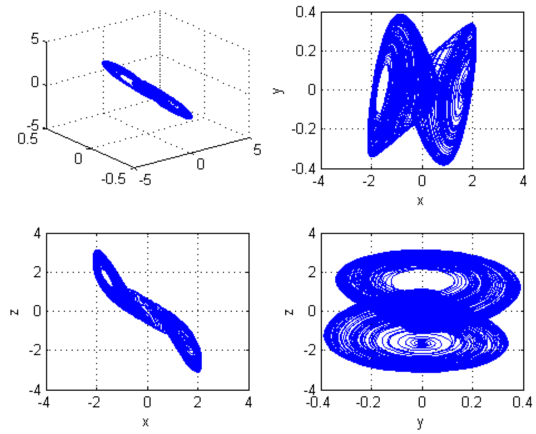


(d) $a = 7.5$

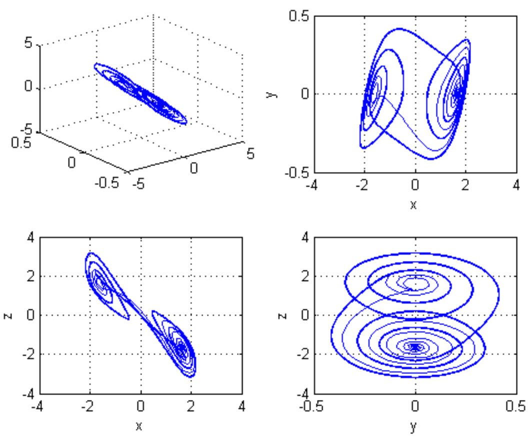




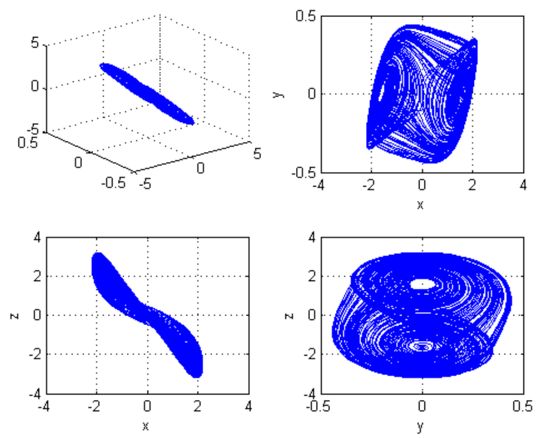
(e) $a = 8.5$



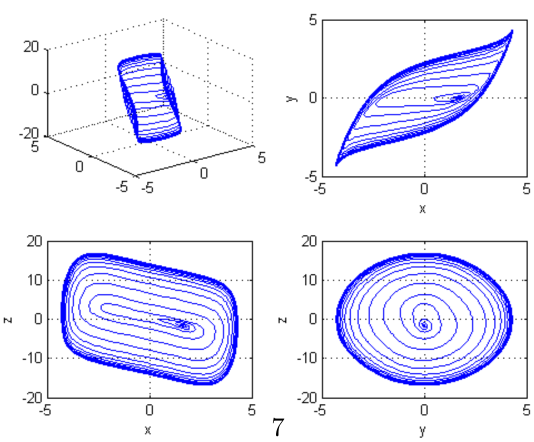
(f) $a = 9$



(g) $a = 9.8$



(h) $a = 10.5$



(i) $a = 12$