On the Chua Circuit

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April 2, 2013

1 Introduction

The Chua Circuit is an electronic circuit named after Leon Chua, who suggested it in 1983. It was reported by Matsumoto in 1984 [3] that (a simplified version of) this system exhibited chaotic behaviour in the form of a chaotic attractor, although it is a simple autonomous circuit.

The circuit in question is shown in figure 1. It contains four passive elements – two capacitors C_1, C_2, one inductor L and a resistor G – and one non-linear resistor R. In the original circuit the non-linearity of R was given by a three-component piecewise linear function, dependent on the voltage over the resistor [3, 4, 1], giving the current through it. We will, however, instead study a resistor with a non-linearity that is given by the function

$$\phi(x) = \rho x^3 - \sigma x \tag{1}$$

as suggested in [2, p. 379]. The study will also follow the "exploration" given in this publication and expand on it.

The dynamics of the original circuit can be described by the following equations [4, eq. (1.1)]:

$$\begin{cases} C_1 \dfrac{dv_{C_1}}{dt} = G(v_{C_2} - v_{C_1}) - f(v_{C_1}) \\ C_2 \dfrac{dv_{C_2}}{dt} = G(v_{C_1} - v_{C_2}) + i_L \\ L \dfrac{di_L}{dt} = -v_{C_2} \end{cases} \tag{2}$$

This set of equations can be rescaled to transform it

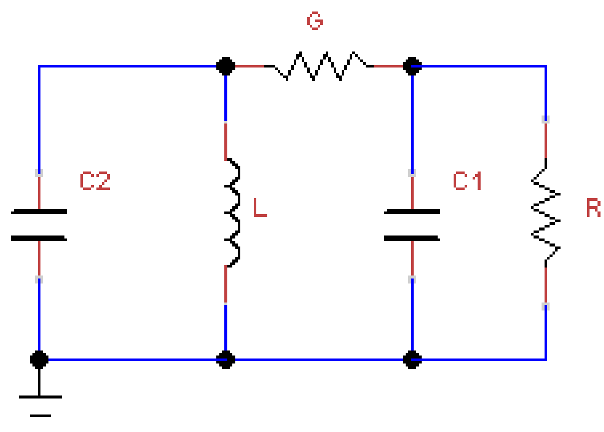

Figure 1: The Chua Circuit. The resistor R has a non-linear dependence on the voltage. After [3, 4].

into [1, eq. (1.1)]:

$$\begin{cases} \dfrac{dx}{d\tau} = a(y - \phi(x)) \\ \dfrac{dy}{d\tau} = x - y + z \\ \dfrac{dz}{d\tau} = -by \end{cases} \tag{3}$$

We will now, in accordance with [2], modify this circuit by taking $\phi(x)$ as in eq. 1 and looking at the following system:

$$\begin{cases} x' = a(y - \phi(x)) \\ y' = x - y + z \\ z' = -by \end{cases} \tag{4}$$

2 Investigating the equilibria

First, we will locate the equilibria of this system. Note that we will take the parameters $a, b, \rho, \sigma > 0$. This is because the first two correspond to physical parameters that are positive and the other two are positive by assumption.

Equating the right part of equation 4 with zero, we get from the z' part, noting that $b \neq 0$, that $y = 0$. Furthermore, the second equation tells us then that $z = -x$. And lastly, the first equation tells us to look for the roots of $\phi(x)$. We will factor this into:

$$\phi(x) = x(\rho x^2 - \sigma) = 0, \tag{5}$$

from which we deduce that $x = 0$ or $x = \pm\sqrt{\frac{\sigma}{\rho}}$.

This leaves us with three equilibrium points for the system:

$$X_0 = (x, y, z)_0 = (0, 0, 0) \tag{6}$$

$$X_\pm = (x, y, z)_\pm = (\pm\sqrt{\frac{\sigma}{\rho}}, 0, \mp\sqrt{\frac{\sigma}{\rho}}), \tag{7}$$

one equilibrium in the origin and one in the xz-plane, symmetric in the origin. Note that the location of the equilibria $X_\pm$ is completely determined by the parameters in ϕ, and that these fail to exist if ρ and $sigma$ do not have the same sign.

2.1 Classification of the origin

We will now linearize the system about the equilibria and classify these points on this. For X_0 (the origin), we have the following linearization:

$$X' = A_0 X, \tag{8}$$

$$A_0 := \begin{pmatrix} -a\phi'(0) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix} = \begin{pmatrix} a\sigma & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix}. \tag{9}$$

Now, to study the eigenvalues of A_0, we note that $tr(A_0) = a\sigma - 1$ and $det(A_0) = ab\sigma > 0$. For any real $n \times n$ matrix, we know that the trace is equal to the sum and the determinant to the product of the eigenvalues, counting multiplicities. For the eigenvalues of A_0 we have three cases:

1. All eigenvalues are zero
2. All eigenvalues are real and non-zero
3. One eigenvalue is real and two are complex and non-real, and all are non-zero.

Bare with us, because this will get confusing.

Zero eigenvalues This case is fundamentally impossible, since in this case, we would have $0 = det(A_0) = ab\sigma > 0$, which is a contradiction. Therefore, all eigenvalues are non-zero.

Real and non-zero eigenvalues Let $\lambda_i \in \mathbb{R}(i = 1, 2, 3)$ be the eigenvalues of A_0. Then we have $ab\sigma = det(A_0) = \lambda_1\lambda_2\lambda_3 > 0$. If without generality we assume that $\lambda_1 \leq \lambda_2 \leq \lambda_3$, we can again distinguish two cases. If $\lambda_1 < 0$, we must have $\lambda_2 < 0$ and $\lambda_3 > 0$. This corresponds to a saddle with a stable plane, spanned by the (generalized) eigenvectors corresponding to λ_1 and λ_2.

If $\lambda_1 > 0$, we automatically have $\lambda_i > 0 (i = 1, 2, 3)$. This corresponds to a source.

Now we investigate the influence of the parameters a, b, σ on the eigenvalues. If $0 < a\sigma \leq 1$, we see that:

$$0 \geq a\sigma - 1 = tr(A_0) = \sum_{i=1}^{3} \lambda_i.$$

Therefore:

$$\lambda_3 \leq -(\lambda_1 + \lambda_2)$$

So, if $\lambda_1 > 0$, we get $0 < \lambda_3 \leq 0$, which can't be true. Therefore we must have in this case (i.e. if there are only real eigenvalues and $0 < a\sigma \leq 1$) that $\lambda_1, \lambda_2 < 0$ and $\lambda_3 \leq |\lambda_1 + \lambda_2|$. So, then the origin is a saddle with a stable plane.

Now to study the subcase of $a\sigma > 1$, we observe that if $\lambda_1, \lambda_2 < 0$, we get $\lambda_3 > -(\lambda_1 + \lambda_2) = |\lambda_1 + \lambda_2| > 0$. And if $\lambda_1, \lambda_2 > 0$, we get $\lambda_3 > -(\lambda_1 + \lambda_2)$, which is perfectly possible, since we know that $\lambda_3 > 0$. Therefore, on these criteria, both cases (source and saddle) can occur.

One real, two non-real eigenvalues Again let $\lambda_1 \in \mathbb{R}, \lambda_2, \lambda_3 \in \mathbb{C}$ be the eigenvalues of A_0. Then we immediately know that $\lambda_3 = \bar{\lambda_2}$, the complex conjugate of λ_2. From the trace and determinant we deduce:

$$0 < ab\sigma = \lambda_1 |\lambda_2|^2 \implies \lambda_1 > 0$$
$$a\sigma - 1 = \lambda_1 + \lambda_2 + \bar{\lambda_2} = \lambda_1 + 2Re(\lambda_2).$$

So we see that the real eigenvalue must be strictly positive (since $|\lambda_2|^2 > 0$. Furthermore, if $0 < a\sigma \leq 1$, the real part of λ_2 and λ_3 is strictly negative.

When $a\sigma > 1$, the only restriction we can impose is that the real part of λ_2 and λ_3 is strictly greater than $-\frac{\lambda_1}{2}$.

Type of equilibrium From this we can deduce that as long as $a \leq \frac{1}{\sigma}$, the origin will be a saddle point, with a stable plane. When a becomes greater than $\frac{1}{\sigma}$, there is a possible bifurcation, changing the origin into a source.

2.2 Classification of $X_{\pm}$

References

[1] L. O. Chua, M. Komuro, and T. Matsumoto. The double scroll family. *IEEE Transactions on Circuits and Systems*, 33(11):1072–1118, November 1986.

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[4] T. Matsumoto, L. O. Chua, and M. Komuro. The double scroll. *IEEE Transactions on Circuits and Systems*, 32(8):797–818, August 1985.