1 Investigating the equilibria

$a, b, \rho, \sigma > 0$

z'

$b \neq 0$

$$y = 0$$

$$z = -x$$

$\phi(x)$

$$\phi(x) = x(\rho x^2 - \sigma) = 0, \tag{1}$$

$$x = 0$$

$$x = \pm\sqrt{\frac{\sigma}{\rho}}$$

$$X_0 = (x, y, z)_0 = (0, 0, 0) \tag{2}$$

$$X_\pm = (x, y, z)_\pm = (\pm\sqrt{\frac{\sigma}{\rho}}, 0, \mp\sqrt{\frac{\sigma}{\rho}}), \tag{3}$$

xz

$X_{\pm}$

ϕ

ρ

$sigma$

1.1 Classification of the origin

X_0

$$X' = A_0 X, \tag{4}$$

$$A_0 := \begin{pmatrix} -a\phi'(0) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix} = \begin{pmatrix} a\sigma & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix}. \tag{5}$$

$$A_0$$

$$tr(A_0) = a\sigma - 1$$

$$det(A_0) = ab\sigma > 0$$

$n \times n$

A_0

Zero eigenvalues

$$0 = det(A_0) = ab\sigma > 0$$

Real and distinct eigenvalues

$\lambda_i \in \mathbb{R}(i = 1, 2, 3)$

A_0

$$0 < ab\sigma = det(A_0) = \lambda_1\lambda_2\lambda_3$$

$$\lambda_1 < \lambda_2 < \lambda_3$$

$\lambda_1 < 0$

$\lambda_2 < 0$

$\lambda_3 > 0$

$\lambda_1 > 0$

$\lambda_i > 0 (i = 1, 2, 3)$

a, b, σ

$$0 < a\sigma \leq 1$$

$$0 \geq a\sigma - 1 = tr(A_0) = \sum_{i=1}^{3} \lambda_i.$$

$$\lambda_3 \leq -(\lambda_1 + \lambda_2)$$

$\lambda_1 > 0$

$$0 < \lambda_3 \leq 0$$

$$0 < a\sigma \leq 1$$

$\lambda_1, \lambda_2 < 0$

$$\lambda_3 \leq |\lambda_1 + \lambda_2|$$

$a\sigma > 1$

$\lambda_1, \lambda_2 < 0$

$$\lambda_3 > -(\lambda_1 + \lambda_2) = |\lambda_1 + \lambda_2| > 0$$

$\lambda_1, \lambda_2 > 0$

$$\lambda_3 > -(\lambda_1 + \lambda_2)$$

$\lambda_3 > 0$

One real, two non-real eigenvalues

$\lambda_1 \in \mathbb{R}, \lambda_2, \lambda_3 \in \mathbb{C}$

A_0

$$\lambda_3 = \bar{\lambda}_2$$

λ_2

$$0 < ab\sigma = \lambda_1 |\lambda_2|^2 \implies \lambda_1 > 0$$
$$a\sigma - 1 = \lambda_1 + \lambda_2 + \bar{\lambda_2} = \lambda_1 + 2Re(\lambda_2).$$

$$|\lambda_2|^2 > 0$$

$$0 < a\sigma \leq 1$$

λ_2

$$\lambda_3$$

$a\sigma > 1$

λ_2

λ_3

$$-\frac{\lambda_1}{2}$$

Type of equilibrium

$$a \leq \frac{1}{\sigma}$$

a

$$\frac{1}{\sigma}$$

1.2 Classification of $X_\pm$