Mini course GSSI

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A variational regularity theory for optimal transportation, and applications to the matching problem

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direct method of calculus of variations

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some regularity theory

Relating the Eulerian flux q
to the Lagrangian displacement $(y-x)d\pi$, locally

The flux q is close to its Helmholtz projection ∇u;
almost in total variation norm

Kantorowicz' formulation of Monge's optimal transportation; direct method of calculus of variations

Given: (locally finite Borel) measures
$\lambda \geq 0$ on $\mathbb{R}^d \ni x$ and $\mu \geq 0$ on $\mathbb{R}^d \ni y$.
A measure $\pi \geq 0$ on $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y)$ is "admissible"
iff it has marginals λ and μ:

$$\int \zeta(x) d\pi = \int \zeta d\lambda \quad \text{and} \quad \int \zeta(y) d\pi = \int \zeta d\mu.$$

Provided mass is finite and equal

$$\lambda(\mathbb{R}^d) = \mu(\mathbb{R}^d) \in (0, \infty),$$

the product measure $\pi = \frac{1}{\lambda(\mathbb{R}^d)} \lambda \otimes \mu = \frac{1}{\mu(\mathbb{R}^d)} \lambda \otimes \mu$ is admissible.
Note that for any admissible π

$$\pi(\mathbb{R}^d \times \mathbb{R}^d) = \lambda(\mathbb{R}^d) = \mu(\mathbb{R}^d) < \infty.$$

Consider squared transport distance and

$$\text{minimize} \quad \int |y-x|^2 d\pi \quad \text{among all } \pi \text{ admissible.}$$

Provided λ, μ have finite second moments,

$$\int |x|^2 d\lambda < \infty \quad \text{and} \quad \int |y|^2 d\mu < \infty,$$

any admissible π satisfies (monotone convergence)

$$\frac{1}{2}\int |x-y|^2 d\pi \leq \int |x|^2 + |y|^2 d\pi = \int |x|^2 d\lambda + \int |y|^2 d\mu < \infty.$$

In particular, infimum $\in [0, \infty)$, and any minimizing sequence of π's is tight, so that marginals are preserved in the limit. Since functional is lower semi-continuous (Fatou), get minimizer by direct method. We fix a minimizer π.

Kantorowicz potential and Brenier's map; convex duality

By convex duality $\exists$convex function $\psi\colon \mathbb{R}^d \to (-\infty, +\infty]$ (not $\equiv +\infty$) such that

$$\mathrm{supp}\pi \subset \partial\psi,$$

where the subgradient $\partial\psi \subset \mathbb{R}^d \times \mathbb{R}^d$ is defined by

$$(x, y) \in \partial\psi \quad \Longleftrightarrow \quad \forall\ x' \in \mathbb{R}^d\ \psi(x') \geq \psi(x) + (x' - x) \cdot y.$$

Informally $\mathrm{supp}\pi$ is d-dimensional, as opposed to $2d$-dimensional for product measure. In particular, we have

$$\mathrm{supp}\lambda \subset \{\, x \,|\, \exists y\ (x, y) \in \partial\psi \,\} =: \mathcal{D}(\psi).$$

Suppose $\exists$ an (open) ball $B \subset \mathbb{R}^d$ such that

$$B \subset \text{supp}\lambda.$$

Then have $B \subset \mathcal{D}(\psi) \subset \{\, x \,|\, \psi(x) < \infty \,\}$. As convex function, ψ is locally bounded and locally Lipschitz on B. As locally Lipschitz function, ψ is Lebesgue-almost everywhere differentiable on B. If ψ is differentiable in $x \in B$, then by definition $\{\, y \,|\, (x,y) \in \partial\psi \,\}$ $= \{\nabla(x)\}$.

Hence there exists a Lebesgue null set $N \subset B$ such that

$$(x,y) \in \partial\psi \text{ and } x \in B - N \quad \Longrightarrow \quad y = \nabla\psi(x).$$

If we suppose in addition

$$\lambda \ll dx \text{ on } B$$

then we obtain

$$\int_{B\times\mathbb{R}^d} \zeta(x,y)d\pi = \int_B \zeta(x,\nabla\psi(x))d\lambda.$$

Eulerian perspective: trajectories X and flux q

We identify pairs $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$
with straight trajectories $X\colon [0,1] \to \mathbb{R}^d$ via (the Borel map)

$$X_t = ty + (1-t)x \quad \text{so that} \quad \dot{X} = y - x.$$

Let the vectorial Borel measure q be defined through

$$\int \xi \cdot dq = \int \int_0^1 \xi(X_t) \cdot \dot{X} dt d\pi.$$

Note q has finite total variation since integrand $\le \sup|\xi|$ times $|y-x| \le |x| + |y| \le 1 + \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2$.

Applying definition to gradient fields $\xi = \nabla\zeta$, appealing to the chain rule $\nabla\zeta(X_t)\cdot\dot{X} = \frac{d}{dt}\zeta(X_t)$, and to $\int_0^1 \nabla\zeta(X_t)\cdot\dot{X}dt = \zeta(y) - \zeta(x)$, we obtain by the admissibility of π

$$\int \nabla\zeta \cdot dq = \int \zeta(d\mu - d\lambda).$$

Incidentally, this means

$$-\nabla\cdot dq = d\mu - d\lambda \quad \text{distributionally.}$$

In view of this we think of q as a flux.

Entering and exiting times σ, τ and measures f, g for a ball

Given a closed ball $\bar{B} \subset \mathbb{R}^d$,
define $\Omega_{\bar{B}}$ to be the set of trajectories that spend time in $\bar{B}$:

$$\Omega_{\bar{B}} \overset{\text{short}}{=} \Omega := \{\, X = (x, y) \,|\, \exists t \in [0,1]\ X_t \in \bar{B} \,\}.$$

We define the two Borel functions $\sigma_{\bar{B}}, \tau_{\bar{B}}$ or short $\sigma, \tau \colon \Omega \to [0,1]$ to be the times X enters/exits $\bar{B}$:

$$\begin{aligned} \sigma(X) &:= \min\{t \in [0,1] | X_t \in \bar{B}\} \\ &< \max\{t \in [0,1] | X_t \in \bar{B}\} =: \tau(X). \end{aligned}$$

Define the two Borel measures $f_{\bar{B}}, g_{\bar{B}} \geq 0$, or short f, g, where the trajectories enter or exit:

$$\int \zeta df = \int_{\Omega \cap \{\sigma > 0\}} \zeta(X_\sigma) d\pi \quad \text{and} \quad \int \zeta dg = \int_{\Omega \cap \{\sigma > 0\}} \zeta(X_\tau) d\pi;$$

well-defined because of $\pi(\mathbb{R}^d \times \mathbb{R}^d)$.

Since by definition,

$$\left\{ \begin{array}{l} \sigma(X) > 0 \iff X_{\sigma(X)} \in \partial B \\ \tau(X) < 1 \iff X_{\tau(X)} \in \partial B \end{array} \right\}$$

we have

$$f, g \quad \text{are supported on} \quad \partial B.$$

Claim

$$\int_{\bar{B}} \nabla\zeta \cdot dq = \int_{\bar{B}} \zeta(d\mu - d\lambda) + \int_{\partial B} \zeta(dg - df).$$

Apply definition of q to $\xi = I(\bar{B})\nabla\zeta$,
use $\int_0^1 \xi(X_t)\cdot\dot{X}dt = \int_\sigma^\tau \nabla\zeta(X_t)\cdot\dot{X}dt = \zeta(X_{\tau(X)}) - \zeta(X_{\sigma(X)})$. Since

$$\sigma(X) = 0 \iff x \in \bar{B} \quad \text{and} \quad \tau(X) = 1 \iff y \in \bar{B},$$

we get

$$\int_0^1 \xi(X_t)\cdot\dot{X}dt = I(y \in \bar{B})\zeta(y) - I(x \in \bar{B})\zeta(x) \\ + I(\tau(X) < 1)\zeta(X_{\tau(X)}) - I(\sigma(X) > 0)\zeta(X_{\sigma(X)}).$$

Integrating, use admissibility of π and definition of f, g.

Incidentally,

normal trace of j on $\partial B \; = \; g - f$
provided $\quad |j|(\partial B) = f(\partial B) = g(\partial B)$.

Electrostatics: the Helmholtz projection ∇u of q on B; some regularity theory

Helmholtz projection $\mathcal{H}_B = \mathcal{H}$ on B is $L^2(B, \mathbb{R}^d)$-orthogonal projection onto closed subspace of gradient fields. By singular integral theory, if ξ is smooth on $\bar{B}$, then $\mathcal{H}\xi$ is smooth on $\bar{B}$, and the $C^k(\bar{B})$-norm of $\mathcal{H}\xi$ is controlled by the $C^{k+1}(\bar{B})$-norm of ξ. Moreover, $\mathcal{H}$ is characterized by how it acts on smooth fields, namely

$$\mathcal{H}\nabla\zeta = \nabla\zeta \quad \text{for smooth } \zeta \text{ on } \bar{B},$$

$$\mathcal{H}\xi = 0 \quad \text{for smooth divergence-free } \xi \text{ supported in } B.$$

Hence to every distribution f on $\bar{B}$, we can associate its Helmholtz projection $\mathcal{H}f$ by duality via $\mathcal{H}f.\xi = f.\mathcal{H}\xi$. It is characterized by

$$\begin{cases} \mathcal{H}f.\nabla\zeta = f.\nabla\zeta & \text{for smooth } \zeta \text{ on } \bar{B}, \\ \mathcal{H}f.\xi = 0 & \text{for smooth divergence-free } \xi \text{ supported in } B. \end{cases}$$

As finite measure, $f = q\lfloor \bar{B}$ is a distribution.
Claim: $\mathcal{H}f$ is absolutely continuous w. r. t. Lebesgue:

$$\mathcal{H}q\lfloor \bar{B} \ \ll \ dx\lfloor B.$$

Enough to construct a $u_B = u \in H^{1,1}(B)$ such that

$$\int_B \nabla\zeta \cdot \nabla u dx = \int_{\bar{B}} \nabla\zeta \cdot dq;$$

then we have $\mathcal{H}q\lfloor \bar{B} = \nabla u dx\lfloor B$. Enough to establish

$$\int_B \nabla\zeta \cdot \nabla u dx = \int_{\bar{B}} \zeta(d\mu - d\lambda) + \int_{\partial B} \zeta(dg - df).$$

Consider $\int_{\bar{B}} \zeta(d\mu - d\lambda) + \int_{\partial B} \zeta(dg - df)$ as a linear form in ζ.
It is bounded w. r. t. $\sup_{\bar{B}} |\zeta|$; it vanishes for constant ζ.
By Sobolev embedding

$$\sup_{x,y\in\bar{B}} \frac{|\zeta(y) - \zeta(x)|}{|y-x|^{\alpha}} \lesssim \Big(\int_B |\nabla\zeta|^p dx \Big)^{\frac{1}{p}}$$

form is bounded w. r. t. $\nabla\zeta \in L^p(B, \mathbb{R}^d)$ for $p \in (d, \infty)$. By duality theory it can be represented by $\int_B \nabla\zeta \cdot \tilde{q} dx$ for some $\tilde{q} \in L^{p'}(B, \mathbb{R}^d)$ with $p' \in (1, \frac{d}{d-1})$. Then ∇u is the Helmholtz projection of $\tilde{q}$, which by singular integral theory is bounded in $L^{p'}(B, \mathbb{R}^d)$. In particular $u \in H^{1,p'}(B) \subset H^{1,1}(B)$.

Incidentally, u satisfies the Poisson equation with Neumann b. c.:

$-\triangle u = \mu - \lambda$ in B and $\nu \cdot \nabla u = \nu \cdot q$ on ∂B in a weak sense.

Relating the Eulerian flux q to the Lagrangian displacement $(y-x)d\pi$, locally

From definition of q

$$\begin{aligned}&\int \xi(x)\cdot(dq-(y-x)d\pi)\\ &=\int_0^1 dt\int \Big(\xi(ty+(1-t)x)-\xi(x)\Big)\cdot(y-x)d\pi,\end{aligned}$$

we obtain the inequality

$$\Big|\int \xi(x)\cdot(dq-(y-x)d\pi)\Big|\leq \sup|\nabla\xi|\int_0^1 dt\int(1-t)|y-x|^2 d\pi,$$

which entails

$$\Big|\int \xi(x)\cdot(dq-(y-x)d\pi)\Big|\leq \frac{1}{2}\sup|\nabla\xi|\int |y-x|^2 d\pi.$$

Seek version with transportation cost localized to a ball B;

$$E(B) := \int_{\Omega(B)} |y-x|^2 d\pi.$$

Replace ξ by $I(\bar{B})\xi$ in definition of q, split difference into

$$\begin{aligned}I(X_t \in \bar{B})\xi(X_t) - I(x \in \bar{B})\xi(x) &= I(X_t \in \bar{B})I(x \in \bar{B})(\xi(X_t) - \xi(x)) \\ &+ I(X_t \in \bar{B}, x \notin \bar{B})\xi(X_t) - I(X_t \notin \bar{B}, x \in \bar{B})\xi(x).\end{aligned}$$

First contribution as before:

$$\begin{aligned}&\Big|\int_0^1 dt \int I(X_t \in \bar{B})I(x \in \bar{B})(\xi(X_t) - \xi(x)) \cdot (y-x) d\pi\Big| \\ &\leq \sup_{\bar{B}} |\nabla\xi| \int_0^1 dt \int I(X_t \in \bar{B})|X_t - x||y-x| d\pi \leq \sup_{\bar{B}} |\nabla\xi| \frac{1}{2} E(\bar{B}).\end{aligned}$$

Second contribution:

$$
\begin{aligned}
&\Big|\int_0^1 dt \int \Big(I(X_t \in \bar{B}, x \notin \bar{B})\xi(X_t) \\
&- I(X_t \notin \bar{B}, x \in \bar{B})\xi(x)\Big) \cdot (y-x)d\pi\Big| \\
&\le \sup_B |\xi| \int_0^1 dt \int |I(X_t \in \bar{B}) - I(x \in \bar{B})||y-x|d\pi.
\end{aligned}
$$

Specify to a ball $\bar{B} = \bar{B}_R$ with radius R and write $|I(X_t \in \bar{B}) - I(x \in \bar{B})| = |I(R \ge |X_t|) - I(R \ge |x|)|$. Hence integral in R is estimated by $||X_t| - |x|| \le |X_t - x|$ to the effect of

$$
\begin{aligned}
&\int_0^{\bar{R}} dR \sup_\xi \frac{1}{\sup_{\bar{B}_R}|\xi|}\Big|\int_0^1 dt \int \Big(I(X_t \in \bar{B}_R, x \notin \bar{B}_R)\xi(X_t) \\
&- I(X_t \notin \bar{B}_R, x \in \bar{B}_R)\xi(x)\Big) \cdot (y-x)d\pi\Big| \le \frac{1}{2}E(B_{\bar{R}}).
\end{aligned}
$$

We summarize these findings on the average-in-R estimate of a dual norm of $dq - (y-x)d\pi$ in

L:1 **Lemma 1.**

$$\int_0^{\bar R} dR \sup_{\xi} \frac{\left| \int_{\bar B_R} \xi(x)\cdot (dq-(y-x)d\pi)\right|}{\max\{\sup_{\bar B_R}|\xi|, \bar R \sup_{\bar B_R}|\nabla \xi|\}} \le E(B_{\bar R}).$$

We now comment on the regime in which Lemma 1 is not vacuous. Note that the l. h. s. compares $dq\lfloor \bar B_R$ to the marginal in x of $(y-x)d\pi\lfloor(\bar B_R\times\mathbb{R}^d)$, in a norm that scales like the total variation (but is weaker more like the flat norm). Hence Lemma 1 is meaningful if and only if $\int_0^{\bar R} dR \int_{\bar B_R\times\mathbb{R}^d}|y-x|d\pi$ is small compared to the r. h. s. that by definition dominates $\int_{B_{\bar R}\times\mathbb{R}^d}|y-x|^2 d\pi$. This is the case if

$$|y-x| \ll \bar R \quad \text{on average w. r. t. } \pi\lfloor(B_{\bar R}\times\mathbb{R}^d).$$

Loosely speaking, this means

$$\text{transportation distance} \ll \text{localization scale}.$$

The flux q is close to its Helmholtz projection ∇u; almost in total variation norm

Need now

$$\lambda = dx \quad \text{in } \bar{B}.$$

In this case

$$\int_{\bar{B}\times\mathbb{R}^d} \zeta(x,y)d\pi = \int_B \zeta(x,\nabla\psi(x))dx.$$

Hence expression in Lemma L:1 1 turns into

$$\int_{\bar{B}\times\mathbb{R}^d} \xi(x)\cdot(dq-(y-x)d\pi) = \int_{\bar{B}} \xi(x)\cdot(dq-(\nabla\psi(x)-x)dx).$$

Note that by definition of Helmholtz projection on B (on $L^2(B,\mathbb{R}^d)$) we have $\mathcal{H}(\nabla\psi - \text{id}) = \nabla\psi - \text{id}$. Together with $\nabla u dx\lfloor B = \mathcal{H}q\lfloor\bar{B}$ we learn for the Helmholtz projection (on distributions)

$$dq\lfloor\bar{B} - \nabla u dx\lfloor B = (\text{id}-\mathcal{H})(dq\lfloor\bar{B} - (\nabla\psi - \text{id})dx\lfloor B).$$

Note that like $\mathcal{H}$, the "Leray projection" $\mathrm{id} - \mathcal{H}$ is bounded in the Hölder space $C^{1,\alpha}(\bar{B}, \mathbb{R}^d)$ for $\alpha \in (0,1)$; more precisely, it is uniformly in B bounded w. r. t. norm

$$\sup_{\bar{B}} |\xi| + R^{1+\alpha} \sup_{x,y \in \bar{B}} \frac{|\nabla\xi(x) - \nabla\xi(y)|}{|x-y|^\alpha},$$

where R is the radius of B. We appeal to the embeddings

$\sup_{\bar{B}_R} |\xi| + R \sup_{\bar{B}_R} |\nabla^2 \xi| \lesssim$ (ao10 **??**) with $B = B_R \lesssim \sup_{\bar{B}_R} |\xi| + R^2 \sup_{\bar{B}_R} |\nabla^2 \xi|$.

C:1 **Corollary 1.** *of Lemma* L:1 *1*

$$\int_{\frac{\bar{R}}{2}}^{\bar{R}} dR \sup_{\xi} \frac{\left| \int_{\bar{B}_R} \xi \cdot (dq - \nabla u_R dx) \right|}{\sup_{\bar{B}_R} |\xi| + R^2 \sup_{\bar{B}_R} |\nabla^2 \xi|} \lesssim E(B_{\bar{R}}).$$

Corollary C:1 1 expresses closeness in a norm that is weaker than the total variation norm; it is even weaker than the flat norm.

In particular, we cannot take $\xi = I(\hat{B})e$ some some unit vector $e \in \mathbb{R}^d$ and some ball $\hat{B}$. However, we obtain an estimate as if we had control in the total variation norm, provide we average in the radius r of such a ball $\hat{B}_r$. This follows from a more subtle statement on the boundedness of the Leray projection:

$$\xi_{rR} := \text{Leray projection of } I(\hat{B}_r)e \text{ in } B_R$$

can be (not quite uniquely) written in form of

$$\xi_{rR} = I(\hat{B}_r)\xi_{rR}^{in} + I(B_R)\xi_{rR}^{out},$$

where both $\xi_{rR}^{in/out}$ are smooth provided $\hat{B}_r$ is compactly contained in B_R. This allows us to apply Lemma 1 on

$$\int_{\hat{B}_r} e\cdot(dq - \nabla u_R dx) = \int_{B_R} \xi_{rR}\cdot(dq - (\nabla\psi(x) - x)dx)$$
$$= \int_{B_r} \xi_{rR}^{in}\cdot(dq - (\nabla\psi(x) - x)dx) + \int_{B_R} \xi_{rR}^{out}\cdot(dq - (\nabla\psi(x) - x)dx).$$

In order to quantify smoothness, fix center of $\hat{B}_r \in B_{\frac{\bar{R}}{8}}$; then

$$\mathrm{dist}(\hat{B}_r, B_R^c) \geq \frac{R}{4} \quad \text{as} \quad r \leq \frac{\bar{R}}{8} \text{ and } \frac{\bar{R}}{2} \leq R \leq \bar{R}.$$

By translation invariance, center of $\hat{B}_r$ fixed; by scaling invariance, $r = 1$. Then $\xi_{R,r=1}^{in/out}$ converge as $R \uparrow \infty$; hence smoothness is uniform in R. This (informally) establishes the estimates

$$\left.\begin{array}{c} \max\{\sup_{B_r}|\xi_{rR}^{in}|, r^2 \sup_{B_r}|\nabla\xi_{rR}^{in}|\} \\ \max\{\sup_{B_R}|\xi_{rR}^{out}|, r^2 \sup_{B_R}|\nabla\xi_{rR}^{out}|\} \end{array}\right\} \lesssim 1.$$

Proposition 1.

$$\int_{\frac{\bar{R}}{2}}^{\bar{R}} dR \int_0^{\frac{\bar{R}}{8}} dr \left| \int_{\hat{B}_r} (dq - \nabla u_R dx) \right| \lesssim \bar{R} E(B_{\bar{R}}).$$