MyToken Homotopic Automated Market Maker

Exchange Rate and Liquidity

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January 26, 2023

Abstract

Automated market makers (AMMs) are used in the DeFi space to enforce desired liquidity characteristics of a token. For instance, a DeFi project may want its token always to be liquid, exponentially appreciating in value as its supply diminishes and therefore being impossible to buy out completely. A constant product marker maker is a straightforward solution for this. In the standard setting, this token would be placed in a liquidity pool along with another suitable token and an equation such as

$$X \times Y = 0$$

used to govern the buying of one token in amounts of the other. X and Y represent quantities of these tokens in the pool, while k is a constant.

If, instead, a limited supply is desired, a constant sum market maker with the equation

$$X + Y = 0$$

at its core would be used. This enforces a constant exchange rate and allows for the complete depletion of a token from the pool i.e. $X = 0$ or $Y = 0$.

Here I describe and characterize an automated market maker arrived at by taking the geometric mean of a Constant Sum Market Maker (CSMM) and a Constant Product Market Maker (CPMM). Such a configuration gives an AMM that displays characteristics of both a CSMM and a CPMM (the CPMM is dominant). This is desirable since apart from liquidity, which a CPMM will always ensure, a CSMM will perform better by other crypto-economic metrics.

Given the present quantities of the currencies in the liquidity pool, the illustrations herein arrive at the immediate exchange rate between the currencies and the amount of one currency to exchange for some amount of the other (taking into consideration that the exchange rate changes marginally for each marginal amount of currency traded).

The characteristics considered are those that pertain to liquidity, particularly the configuration of the automated market maker (AMM) so that the pool is always liquid.

1 Preamble

For two currencies in a market maker with quantities x and y, the equations defining a CSMM and CPMM are

$$a_1 x + a_2 y = k_1$$

Figure 1: Graph of a CSMM

$$b_1 x \cdot b_2 y = k_2$$

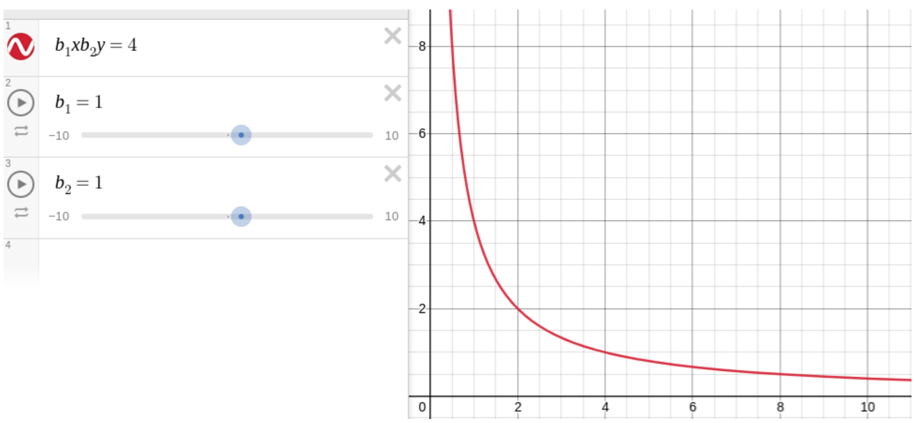

Figure 2: Graph of a CPMM. It is symmetrical about $y = x$.

where their coefficients are positive, real numbers. k_1 and k_2 are the constants for the sum and product, respectively. For convenience, we denote these equations as [1] $A_0(x, y) = k_1$ and $A_1(x, y) = k_2$, respectively.

2 Quantity Mechanism: Construction of the Geometric Mean Market Maker

Within this section, a_1, a_2, b_1 and b_2 are all assumed to be 1 for simplicity. The optimal values for these constants will be established through [2] modeling and simulation or actual trial. We define the

[1] Notation borrowed from [1].
[2] The model and its simulation are covered in a subsequent paper.

geometric weighted mean as

$$A_t(x,y) = A_0(x,y)^{1-t}A_1(x,y)^t$$
$$A_t(x,y) = (x+y)^{1-t}(x\cdot y)^t$$

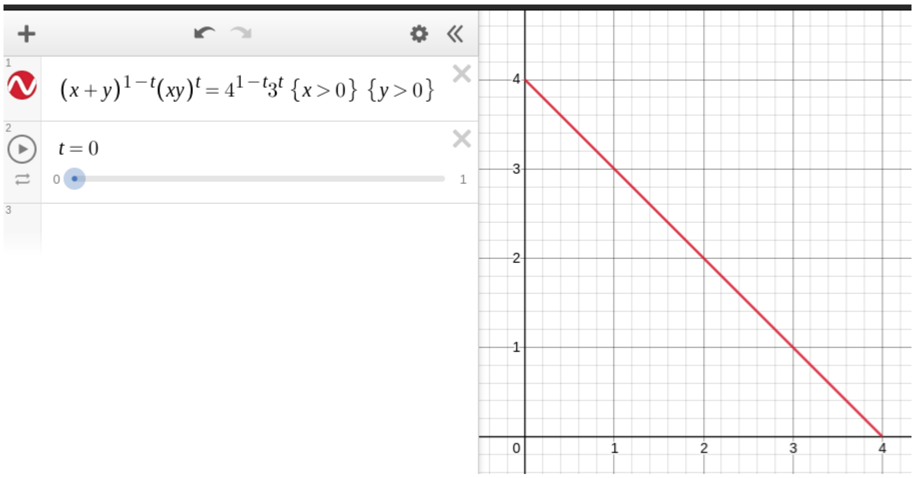

Figure 3: Geometric weighted mean with t=0. This is $A_0(x,y)$.

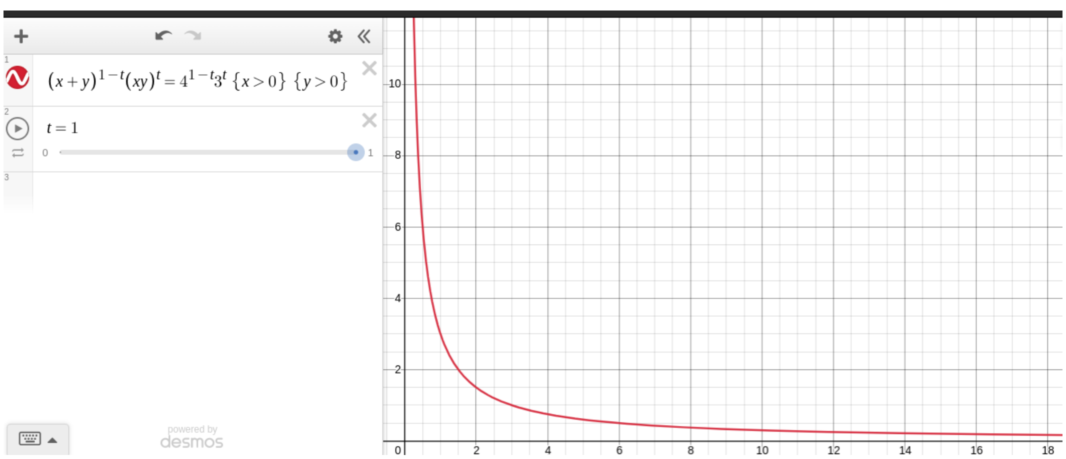

Figure 4: Geometric weighted mean with t=1. This is $A_1(x,y)$.

Notice that this definition is such that when $t=0$, $A_t(x,y) = A_0(x,y)$, the CSMM; and when $t=1$, $A_t(x,y) = A_1(x,y)$, the CPMM.

This construction, for selected values of the weighting parameter t, covers the quantity mechanism in the automated market maker. This is because for whatever quantity of currency x in the market maker, the equation will give the corresponding quantity of the currency y. It is noteworthy, however, that the market maker equation $A_t(x,y) = c$ cannot easily be written explicitly as one variable in terms of the other (contrasted with purely constant sum or constant product market makers). Consequently, we will devise a mechanism to determine the quantity of one currency for a desired quantity of the other.

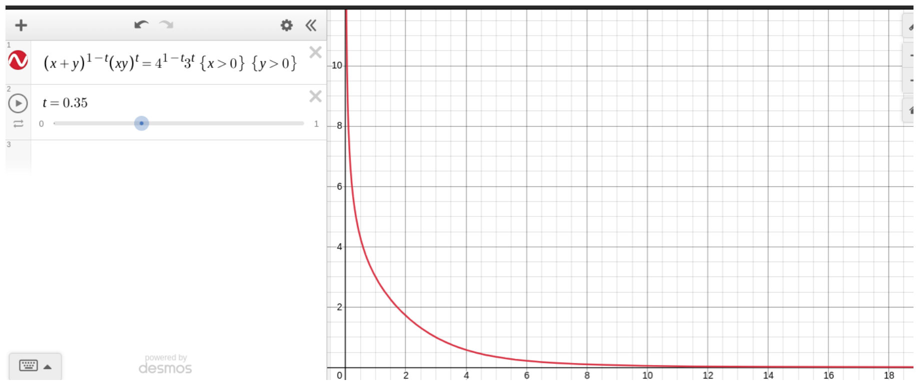

Figure 5: Graph of the geometric weighted mean of a CSMM and CPMM. $t = 0.35$. This is $A_{0.35}(x, y)$.

3 Exchange Rate Mechanism

The exchange rate of currency Y is proportional to the rate of change of the quantity of Y with respect to the quantity of X in the liquidity pool.

$$\text{Exchange rate of Y} \propto \frac{dy}{dx}$$

$$\text{Exchange rate of Y} = P \cdot \frac{dy}{dx}$$

The proportionality factor, P, potentially a function $P = P(x, y, t)$, is a parameter that is chosen to give desired characteristics to the market maker. For simplicity in the following derivations, this parameter is assigned the value 1. Therefore, for given x, y, and t, the expression $\frac{dy}{dx}$ evaluates to a real number, the exchange rate of currency Y in the value of currency X when y and x are the present quantities of the currencies, respectively.

We obtain $\frac{dy}{dx}$ as follows

$$
\begin{aligned}
&\frac{d}{dx}A_t(x,y) = \frac{d}{dx}k \\
&\frac{d}{dx}(x+y)^{1-t}(xy)^t = \frac{d}{dx}k \\
&\frac{d}{dx}(x+y)^{1-t}\cdot(xy)^t + (x+y)^{1-t}\cdot\frac{d}{dx}(xy)^t = 0 \\
&(1-t)(x+y)^{-t}\frac{d}{dx}(x+y)\cdot(xy)^t + (x+y)^{1-t}\cdot t(xy)^{t-1}\frac{d}{dx}(xy) = 0 \\
&(1-t)(x+y)^{-t}\left(1+\frac{dy}{dx}\right)\cdot(xy)^t + (x+y)^{1-t}\cdot t(xy)^{t-1}\left(y+x\frac{dy}{dx}\right) = 0 \\
&\frac{(1-t)(xy)^t(1+\frac{dy}{dx})}{(x+y)^t} + \frac{t(xy)^t(x+y)^{1-t}(y+x\frac{dy}{dx})}{xy} = 0 \\
&\text{multiplying by } (x+y)^t(xy) \text{ and rearranging} \\
&t(xy)^t(x+y)\left(y+x\frac{dy}{dx}\right) + (1-t)(xy)^{1+t}\left(1+\frac{dy}{dx}\right) = 0 \\
&(xy)^t\left[t(x+y)\left(y+x\frac{dy}{dx}\right) + (1-t)(xy)\left(1+\frac{dy}{dx}\right)\right] = 0 \\
&t(x+y)\left(y+x\frac{dy}{dx}\right) + (1-t)(xy)\left(1+\frac{dy}{dx}\right) = 0 \\
&xyt + x^2\frac{dy}{dx}t + y^2t + xy\frac{dy}{dx}t + xy + xy\frac{dy}{dx} - xyt - xy\frac{dy}{dx}t = 0 \\
&\frac{dy}{dx}\left(x^2t + xy\right) + xy + y^2t = 0 \\
&\frac{dy}{dx} = -\frac{xy+ty^2}{xy+tx^2}
\end{aligned}
$$

This is the exchange rate for currency Y for quantities x, y, and weighting factor t. Since $A_t(x,y) = c$ is symmetric about $y = x$, that is $A_t(x,y) = (x+y)^{1-t}(xy)^t = (y+x)^{1-t}(yx)^t = A_t(y,x) = c$ (x and y can be interchanged with no effect), then it follows that

$$\frac{dx}{dy} = -\frac{xy+tx^2}{xy+ty^2}$$

4 Currency Exchange

The section above arrives at the exchange rate given the current quantities of the currencies in the liquidity pool. However, for each unit of one currency exchanged, the exchange rate changes and therefore the rate used before is no longer valid. That is, since $y' = \frac{dy}{dx} = f(x,y,t)$, if we desire to exchange a certain quantity of currency, the present exchange rate, $y'_0 = f(x_0,y_0,t)$, will not apply for this entire quantity since once a unit is exchanged then the quantities have changed to x_1 and y_1 and the new exchange rate is $y'_1 = f(x_1,y_1,t)$. This section derives an expression for how much of one currency to give in exchange for a given quantity of the other, factoring in the changing exchange rate.

Say the present quantity of currency X in the liquidity pool is x and an actor in the MyToken ecosystem desires to exchange Δx amount of this currency for Y, whose current quantity in the pool is y. The quantity of Y to give, Δy, is determined as follows

$$\begin{aligned}
\Delta y &= \int_x^{x+\Delta x} \frac{dy}{dx} dx \\
&= \int_x^{x+\Delta x} -\frac{xy+ty^2}{xy+tx^2} dx \\
&= -\int_x^{x+\Delta x} \frac{1+t\frac{y}{x}}{1+t\frac{x}{y}} dx \\
&= -\int_x^{x+\Delta x} \left[\frac{1}{1+t\frac{x}{y}} + \frac{ty}{x+\frac{t}{y}x^2}\right] dx \\
&= -\int_x^{x+\Delta x} \left[\frac{1}{1+t\frac{x}{y}} + \frac{ty}{x} - \frac{t^2}{1+\frac{t}{y}x}\right] dx \\
&= -\left[\frac{y}{t}\ln\left(1+\frac{t}{y}x\right) + ty\ln(x) - ty\ln\left(1+\frac{t}{y}x\right) + C\right]\Bigg|_x^{x+\Delta x} \\
&= \left[ty\left(\ln\left(1+\frac{t}{y}x\right) - \ln(x)\right) - \frac{y}{t}\ln\left(1+\frac{t}{y}x\right) + C\right]\Bigg|_x^{x+\Delta x} \\
&= \left[ty\ln\left(\frac{1}{x}+\frac{t}{y}\right) - \frac{y}{t}\ln\left(1+\frac{t}{y}x\right) + C\right]\Bigg|_x^{x+\Delta x} \\
&= ty\left[\ln\left(\frac{1}{x+\Delta x}+\frac{t}{y}\right) - \ln\left(\frac{1}{x}+\frac{t}{y}\right)\right] - \frac{y}{t}\left[\ln\left(1+\frac{t}{y}(x+\Delta x)\right) - \ln\left(1+\frac{t}{y}x\right)\right]
\end{aligned}$$

For real x, y, and t, [3]Δy above evaluates to a real number, the quantity of currency Y to exchange for Δx given the initial exchange rate.

5 Liquidity

The market is liquid as long as neither currency has quantity 0 in the market maker. In that case, there would be no quantity of that currency to exchange for the other.
The market maker equation is made to provide liquidity over a wide range of currency quantity values by weighting the constant product term more than the constant sum term. Due to the asymptotes on the extremes of the CPMM curve, a currency becomes asymptotically more valuable as its quantity goes down. This results in the quantity never being 0 in a purely CPMM.

Given that the graph of the weighted market maker is symmetric about the origin, we have two situations:

1. when $t = 0$: purely CPMM curve with no axis intercepts (no currency gets to quantity 0)
2. when $0 < t < 1$: $A_t(x, y) = (a_1x + a_2y)^{1-t}(b_1x \cdot b_2y)^t$ has no intercepts (always liquid). The exchange rate of x relative to y is steeper for lesser t
3. when $t = 1$: purely CSMM curve with relatively short range of liquidity (currencies can get to quantity 0 rather fast)

[3]Δy is not simplified into a quotient of the terms in the logarithms because this is not computationally simpler: generally, a difference is computationally less intensive than a quotient on a computer.

In conclusion, a weighting factor t, $0 < t \leq 1$, ensures liquidity always.

6 Code implementation and example

In python3, the value Δy can be obtained by the following

```python
import sys
import math

x = int(sys.argv[1])
y = int(sys.argv[2])
t = float(sys.argv[3])
delta_x = int(sys.argv[4])

# we represent the natural log function as ln(x) for ease
def ln(x):
   return math.log(x)

print("\nMake sure that your arguments are in the order quantity of X, quantity of Y, t,
    delta x (or delta y)")

delta_y = t*y * ( ln( 1/(x + delta_x) + t/y) - ln( 1/x + t/y ) ) \
         -(y/t)*( ln(1 + (t/y) * (x + delta_x) ) - ln( 1 + (t/y) * x ) )

print(delta_y)
```

this script is saved as `delta_y.py` for the following illustrations.

Suppose the liquidity pool is initiated with a million units of each currency and a weighting factor of $t = 0.35$. Someone intends to buy 17290 units of currency X i.e. $\Delta x = -17290$, what quantity of Y i.e. Δy should they give to the pool in exchange?

First of all, notice that due to the equality of quantities in the pool, the exchange rate is -1

$$\begin{aligned}\frac{dy}{dx} &= -\frac{xy + ty^2}{xy + tx^2} \\ &= -\frac{(10^6 \times 10^6) + (0.35)(10^6)^2}{(10^6 \times 10^6) + (0.35)(10^6)^2} \\ &= -1\end{aligned}$$

This means that when there is a marginal increase in the quantity of one currency, there is an equal (in the limit) decrease in the quantity of the other. Since the constant that we have chosen to relate price and quantity is 1, the exchange rate is -1 at the beginning.
Now,

$$\begin{aligned}A_{0.35}(10^6, 10^6) &= A_0(10^6, 10^6)^{1-0.35} A_1(10^6, 10^6)^{0.35} \\ &= (10^6 + 10^6)^{0.65}(10^6 \cdot 10^6)^{0.35} \\ &= 197546571.70636436 \end{aligned} \tag{1}$$

This is the constant that should be maintained given the above initial conditions of the pool.

Running the following

```
python3 delta_y.py 1000000 1000000 0.35 -17290
```

which is equivalent to

$$\Delta y = 0.35(10^6)\left[\ln\left(\frac{1}{10^6-17290}+\frac{0.35}{10^6}\right)-\ln\left(\frac{1}{10^6}+\frac{0.35}{10^6}\right)\right]-\frac{10^6}{0.35}\left[\ln\left(1+\frac{0.35}{10^6}(10^6-17290)\right)\right.$$
$$\left.-\ln\left(1+\frac{0.35}{10^6}10^6\right)\right]$$

gives $\Delta y = 17368.190503495436$.

Now,

$$A_{0.35}(10^6-17290, 10^6+17368.190503495436) = A_{0.35}(982710, 1017368.190503495436) = 197536233.54578418$$

is about equal to the constant (1). The error arises from the fact that the [4]natural log function evaluates to values with more decimal places than our python script can handle. In actual implementation, more advanced techniques will be employed to enhance precision.

Crucial to note is that the actor in this example receives a lesser quantity than that which they give ($17290 < 17368$). This is because as they buy up currency X there is less of it and it becomes more valuable. Each marginal quantity purchased in this transaction makes X marginally more valuable than it was before. This eventually ensures that a currency cannot be bought out in the pool i.e. the pool is always liquid.

[4]The fact that e is irrational (hence its decimal places do not terminate) means that $\ln(x), x \neq e$ or 0 is also irrational (The Gelfond–Schneider theorem).

References

[1] Alexander Port and Neelesh Tiruviluamala. Mixing constant sum and constant product market makers. *arXiv preprint arXiv:2203.12123*, 2022.