Math 440B HW 2

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It is important to remember the following proportionality:

$$Posterior \propto Likelihood \times Prior$$

Problem 5d
The problem opens with the following information:

$$pmf : P(X = 1) = \theta,\ P(X = 2) = 1 - \theta$$

With a set of three observations: $(1,\ 2,\ 2)$.

From this information, the likelihood can be calculated to be $L(\theta) = \theta(1-\theta)^2$.

The posterior density can then be calculated as follows:

$$f_{\Theta|X} = \frac{\theta(1-\theta)^2 \times 1}{\int_0^1 \theta(1-\theta)^2 \times 1\ d\theta}$$

Notice that the denominator can be rewritten:

$$\int_0^1 \theta(1-\theta)^2\ d\theta = \int_0^1 \theta^{2-1}(1-\theta)^{3-1}\ d\theta = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)}$$

Thus allowing $f_{\Theta|X}$ to simplify to

$$\frac{\Gamma(5)}{\Gamma(2)\Gamma(3)}\theta(1-\theta)^2 \sim \beta(2,3)$$

Problem 7b
The probability mass function is given to be $P(X = x) = p(1-p)^{x-1}$ for $x = 1, 2, 3, ...$

Its corresponding likelihood function is given to be $L(\theta) = p^n(1-p)^{\sum_{i=1}^n x_i - n}$.

The log-likelihood function can then be calculated to be $l(\theta) = nln(p) + (\sum_{i=1}^n x_i - n)ln(1-p)$.

Taking its derivative, setting it equal to zero, then solving for theta produces

$$\hat{\theta}_{MLE} = \frac{n}{\sum x} = \frac{1}{\bar{x}}$$

Problem 7d
Now the problem is to find the posterior distribution of p, given a prior that is uniform on the closed interval [0,1].

To solve this problem, consider

$$f_{P|X}(p|x) = \frac{p^n(1-p)^{\sum_{i=1}^n x_i - n}}{\int_0^1 p^n(1-p)^{\sum_{i=1}^n x_i - n} dp}$$

Luckily, integration isn't necessary, and as the question is similar to Problem 5, some steps can be skipped. Looking specifically at the powers in the numerator, I can conclude that the distribution is,

$$\beta(n+1,\ \sum_{i=1}^{n} x_i - n + 1)$$

Letting $n = 1$ allows the simplification of the distribution to be more in line with the answer provided in the back of the book letting the distribution become

$$\beta(2, k_1)$$

With posterior mean

$$\frac{2}{2+k}$$

Problem 62
Part 1:
To show that the Gamma distribution is a conjugate prior of the exponential distribution, I begin by getting the Likelihood equation of the exponential distribution.

$$L(\lambda) = v^n e^{-v \sum \lambda_i}$$

The prior distribution is provided on page 288 of the textbook and is given to be

$$f_\Lambda(\lambda) = \frac{v^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-v\lambda}$$

Multiplying the above will result in a posterior that should be gamma, that is,

$$v^{n+\alpha} e^{-v(\sum \lambda_i + \lambda)} \times \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}$$

where the stuff to the right of the multiplication sign is considered a constant and can be disregarded.

Thus the above distribution is similar to

$$** \, \Gamma(n+\alpha,\ \sum \lambda_i + \lambda)$$

Part 2:
Here there are two cases to consider:

Case 1: Γ with $\mu_1 = 0.5$ and $\sigma_1 = 1$.
Case 2: Γ with $\mu_2 = 10$ and $\sigma_2 = 20$.

I begin with Case 1. The following system of equations can be constructed:

$$\frac{\alpha}{\lambda} = 0.5 \ and \ \frac{\alpha}{\lambda^2} = 1$$

Solving the above system grants the values $\alpha = 0.25$ and $\lambda = 0.5$. Implementing the above values into $**$ grants the distribution $\Gamma(20.25,\ 102.5)$ with corresponding posterior mean ~ 0.197, whose reciprocal grants the time ~ 5.0617 minutes.

Next is Case 2. Solving a system similar to the one earlier, but with μ_2 and σ_2 grants values of $\lambda = \frac{1}{40}$ and $\alpha = \frac{1}{4}$. Once again, plugging these values into $**$ grants the distribution $\Gamma(20.25,\ 102.025)$. The posterior mean in this case is ~ 0.1985 whose reciprocal grants the time 5.038 minutes.

Problem 63
The book provides two cases to consider:
Case 1: $a = b = 1$
Case 2: $a = 0.5,\ b = 5$

Where the above cases are values corresponding to a beta prior distribution. Notice that their situation of finding 3 defective parts in a collection of 100 parts is akin to a $bin(100, p)$ with a total of 3 "successes".

Recall the following about the binomial distribution:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \ \propto \ p^x (1-p)^{n-x}$$

Then recall the following about the beta distribution:

$$f(p) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1} \ \propto \ p^{a-1}(1-p)^{b-1}$$

Multiplying both these distributions together will grant a posterior distribution which can be written as

$$= \ p^{(x+a)-1}(1-p)^{(n+b-x)-1} \sim \beta(x+a,\ n+b-x)$$

Thus the posterior distribution for Case 1 is $\beta(4,\ 98)$ while the posterior distribution for Case 2 is $\beta(3.5,\ 102)$. Each with posterior means 0.0392 and 0.03301 respectively.