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HW #14

Problem 44: X & Y are independent R.V's w/ $Var(X) = Var(Y)$. WTFind $Cov(X+Y, X-Y)$.

$$Cov(X+Y, X-Y) = Cov(1X+1Y, 1X-1Y)$$
$$= Cov(X,X) + \cancel{Cov(X,Y)}^{0} - \cancel{Cov(X,Y)}^{0} - Cov(Y,Y)$$
$$= Var(X) - Var(Y)$$
$$= 0.$$

Problem 46: U and V independent R.V.'s with mean μ and variances σ^2. Let $Z = \alpha U + V\sqrt{1-\alpha^2}$. Find $E[Z]$ & ρ_{UZ}.

$$E[Z] = E[\alpha U + V\sqrt{1-\alpha^2}] = \alpha\mu + \mu\sqrt{1-\alpha^2}$$
$$= \mu(\alpha + \sqrt{1-\alpha^2})$$

$$\rho_{UZ} = \frac{Cov(U,Z)}{\sqrt{Var(U)Var(Z)}} = \frac{E[UZ] - E[U]E[Z]}{\sqrt{Var(U)Var(Z)}} \quad (*)$$

$$UZ = \alpha U^2 + UV\sqrt{1-\alpha^2}$$
$$E[UZ] = \alpha E[U^2] + E[U]E[V]\sqrt{1-\alpha^2}$$

$\Rightarrow$ (*)'s numerator: $\alpha E[U^2] + E[U]E[V]\sqrt{1-\alpha^2} - E[U]\mu(\alpha + \sqrt{1-\alpha^2})$

$$\alpha E[U^2] + \cancel{\mu^2\sqrt{1-\alpha^2}} - \mu^2\alpha - \cancel{\mu^2\sqrt{1-\alpha^2}}$$
$$\alpha(E[U^2] - \mu^2)$$
$$\alpha Var(U)$$
$$\alpha\sigma^2$$

Plug into (*): $\rho_{UZ} = \dfrac{\alpha\sigma^2}{\sigma^2} = \boxed{\alpha}$

Problem 47: X & Y are independent R.V.'s and $Z = Y - X$.
WTFind expressions for the covariance and the correlation of X & Z in terms of the variance of X & Y.

$$\begin{aligned} Cov(X,Z) &= Cov(X, Y-X) = Cov(1X + 0Y, 1Y - 1X) \\ &= Cov(X,Y) + 0 - Cov(X,X) + 0 \\ &= Cov(X,Y) - Var(X) \\ &= -Var(X) = -\sigma_X^2 \end{aligned}$$

$$\rho_{XZ} = \frac{Cov(X,Z)}{\sqrt{Var(X)\,Var(Z)}}$$

$$\begin{aligned} Var(Z) &= Var(Y-X) \\ &= Var(Y) + Var(X) = \sigma_Y^2 + \sigma_X^2 \end{aligned}$$

$$= \frac{-\sigma_X^2}{\sqrt{\sigma_X^2(\sigma_Y^2 + \sigma_X^2)}} = -\frac{\sigma_X}{\sqrt{\sigma_Y^2 + \sigma_X^2}}$$

Problem 48: Given $U = a + bX$ & $V = c + dY$
WTS $|\rho_{UV}| = |\rho_{XY}|$

$$\rho_{UV} = \frac{Cov(U,V)}{\sqrt{Var(U)\,Var(V)}} = \frac{Cov(a+bX, c+dY)}{\sqrt{Var(a+bX)\,Var(c+dY)}}$$

First $Cov(a+bX, c+dY) = Cov(bX, dY) = bd\,Cov(X,Y)$
Then $Var(a+bX) = b^2 Var(X)$ & $Var(c+dY) = d^2 Var(Y)$

So putting it all together $\Rightarrow$

$$= \frac{bd\,Cov(X,Y)}{(b^2 d^2\, Var(X)\,Var(Y))^{1/2}}$$

$$= \frac{Cov(X,Y)}{(Var(X)\,Var(Y))^{1/2}}$$

Add absolute values to complete the proof ∎

Note: $E[Y|X=x] = \mu_Y + \rho(x-\mu_X)\frac{\sigma_Y}{\sigma_X}$

$E[X|Y=y] = \mu_X + \rho(y-\mu_Y)\,\sigma_X/\sigma_Y$

<u>Problem 69</u>: Suppose bivariate normal dist. has $\mu_X = \mu_Y = 0$, and $\sigma_X = \sigma_Y = 1$.

Sketch the contours of the density and the lines $E[Y|X=x]$ and $E[X|Y=y]$ for $\rho = 0$; $\rho = 0.5$; and $\rho = 0.9$.