WRITTEN PORTION

$$\begin{bmatrix} \omega_1 = 3 \\ \omega_2 = 3 \\ \omega_1 - \omega_2 = 1 \\ \omega_1 + \omega_2 = 7 \end{bmatrix}$$

Problem 6: $\vec{Y} = X\vec{\beta} + e_i$

A. $$\begin{bmatrix} 3 \\ 3 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

B. LS estimate: $\hat{\beta} = (X'X)^{-1}X'\vec{Y}$

$$\hat{\beta} = \left(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 7 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 3 \end{bmatrix}$$

C. Find the estimate for σ^2

$$\Rightarrow \hat{Y} = X\vec{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 11/3 \\ 3 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 3 \\ 2/3 \\ 20/3 \end{bmatrix}$$

$$\Rightarrow Y - \hat{Y} = \begin{bmatrix} -2/3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix}; \quad \|Y - \hat{Y}\| = \sqrt{(-2/3)^2 + (1/3)^2 + (1/3)^2}$$

$$= \sqrt{(6/9)}$$

$$\Rightarrow \|Y - \hat{Y}\|^2 = 6/9 = \frac{2}{3}$$

$$\hat{\sigma}^2_{MLE} = \frac{\|Y - \hat{Y}\|^2}{n-p} = \frac{\frac{2}{3}}{2} = \boxed{\frac{1}{3}}$$

Written Portion

D. Find the estimated std errors of the LS estimates.

Look back at $(X'X)^{-1}$

$$S_{\hat{\beta}_i} = \sqrt{c_{ii}} \cdot s = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$

$$S_{\hat{\omega}_1} = S_{\hat{\omega}_2} = \frac{1}{3}$$

E. Estimate $\omega_1 - \omega_2$ and its standard error.

Estimate of $\omega_1 - \omega_2$ is $\hat{\omega}_1 - \hat{\omega}_2 = \frac{11}{3} - 3 = \frac{2}{3}$

Recall $s_{\hat{\beta}_i} = s\sqrt{c_{ii}}$.

So $$s_{\hat{\omega}_1 - \hat{\omega}_2} = \sqrt{\frac{1}{3}} \cdot \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{3}$$

F. Test the null hypothesis $H_0: \omega_1 = \omega_2$.

$$t_{n-p} \sim \frac{\hat{\beta}_i - \beta_i}{s_{\hat{\beta}_i}}$$

$$t_2 \sim \frac{\frac{2}{3}}{\frac{\sqrt{2}}{3}} = \frac{2}{\sqrt{2}}$$ So, $2 * \left(1 - pt\left(\frac{2}{\sqrt{2}}, df = 2\right)\right) \approx 0.30$

$\Rightarrow$ This means that there is very weak evidence to suggest ω_1 and ω_2 are different from one another.

9

Problem 15:

Find the LS estimate for $\beta x = y$ to points (x_i, y_i), $i = \{1, \dots, n\}$.
Using the Matrix approach.

$$\Rightarrow \vec{Y} = X\vec{\beta} \Leftrightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \vec{\beta}$$

$\vec{\beta} = [\beta]$. So, WTFind $\vec{\beta}$. Recall $\vec{\beta} = (X'X)^{-1} X' \vec{Y}$

$$(X'X)^{-1} = \left([x_1 \dots x_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)^{-1} = \left(\sum_{i=1}^{n} x_i^2 \right)^{-1} = 1/\sum x_i^2$$

$$\vec{\beta} = \left(1/\sum x_i^2 \right) \left([x_1 \dots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right)$$

$$= \left(1/\sum x_i^2 \right) \left(\sum_{i=1}^{n} x_i y_i \right) = \boxed{\frac{\sum_{i=1}^{n} x_i y_i}{\sum x_i^2}} \checkmark$$

Problem 16:

A) Use the matrix form to find LS estimate for β_0 & β_1.
Where $y_i = \beta_0 x_i + \beta_1 x_i^2$ where $i = \{1, \dots, n\}$.

$$\vec{Y} = X\vec{\beta} \Leftrightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & x_1^2 \\ \vdots & \vdots \\ x_n & x_n^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

Again, $\vec{B} = (X'X)^{-1} X' \vec{Y}$

$$(X'X)^{-1} = \left(\begin{bmatrix} x_1 & \dots & x_n \\ x_1^2 & \dots & x_n^2 \end{bmatrix} \begin{bmatrix} x_1 & x_1^2 \\ \vdots & \vdots \\ x_n & x_n^2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} \sum x_i^2 & \sum x_i^3 \\ \sum x_i^3 & \sum x_i^4 \end{bmatrix}^{-1}$$

$$= \frac{1}{\sum x_i^2 \sum x_i^4 - (\sum x_i^3)^2} \begin{bmatrix} \sum x_i^4 & -\sum x_i^3 \\ -\sum x_i^3 & \sum x_i^2 \end{bmatrix}$$

HW #15

$$X'Y = \begin{bmatrix} x_1 & \cdots & x_n \\ x_1^2 & \cdots & x_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

$$\vec{\beta} = \frac{1}{(\sum x_i^2 \sum x_i^4) - (\sum x_i^3)^2} \begin{bmatrix} \sum x_i^4 & -\sum x_i^3 \\ -\sum x_i^3 & \sum x_i^2 \end{bmatrix} \cdot \begin{bmatrix} \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

$$\vec{\beta} = \frac{1}{\sum (x_i^2) \sum x_i^4 - (\sum x_i^3)^2} \begin{bmatrix} \sum x_i^4 \sum x_i y_i - \sum x_i^3 \sum x_i^2 y_i \\ -\sum x_i^3 \sum x_i y_i + \sum x_i^2 \sum x_i^2 y_i \end{bmatrix}$$

$$\vec{\beta} = \begin{bmatrix} \dfrac{\sum x_i^4 \sum x_i y_i - \sum x_i^3 \sum x_i^2 y_i}{\sum x_i^2 \sum x_i^4 - (\sum x_i^3)^2} \\ \dfrac{\sum x_i^2 \sum x_i^2 y_i - \sum x_i^3 \sum x_i y_i}{\sum x_i^2 \sum x_i^4 - (\sum x_i^3)^2} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

B. Expression for the covariance matrix.

Recall $\Sigma_{\hat{\beta}_0 \hat{\beta}_1} = \sigma^2 (X'X)^{-1}$.

So $$\Sigma_{\hat{\beta}_0 \hat{\beta}_1} = \frac{\sigma^2}{\sum x_i^2 \sum x_i^4 - (\sum x_i^3)^2} \begin{bmatrix} \sum x_i^4 & -\sum x_i^3 \\ -\sum x_i^3 & \sum x_i^2 \end{bmatrix}$$

Problem 29: Assume X_1 & X_2 are uncorrelated r.v.'s with variance σ^2. Use matrix methods to show $Y = X_1 + X_2$ and $Z = X_1 - X_2$ are uncorrelated.

Givens. $\Rightarrow$ $Var(X_1) = Var(X_2) = \sigma^2$

Recall, $Y = X_1 + X_2$

$Z = X_1 - X_2$

$$\Sigma_{YZ} = \begin{bmatrix} Var(Y) & Var(Z,Y) \\ Var(Y,Z) & Var(Z) \end{bmatrix} = \begin{bmatrix} Var(X_1 + X_2) & Cov(X_1 - X_2, X_1 + X_2) \\ Cov(X_1 + X_2, X_1 - X_2) & Var(X_1 - X_2) \end{bmatrix}$$

10

$Var(X_1 + X_2) = Var(X_1) + Var(X_2) = 2\sigma^2$

$Var(X_1 - X_2) = Var(X_1) + Var(X_2) = 2\sigma^2$.

$Cov(X_1 - X_2, X_1 + X_2) = Cov(X_1 + X_2, X_1 - X_2) =$

$Cov(X_1, X_1) - Cov(X_1, X_2) + Cov(X_1, X_2) - Cov(X_2, X_2)$.

$= 1 - 0 + 0 - 1 = 0.$

$$\Sigma_{YZ} = \begin{bmatrix} 2\sigma^2 & 0 \\ 0 & 2\sigma^2 \end{bmatrix}$$

The off-diagonals indicate it is not correlated. It being Z & Y.

Problem 30: $X_1, \ldots, X_n$ w/ $Var(X_1) = \sigma^2$ and $Cov(X_i, X_j) = \rho\sigma^2$ for $i \neq j$. Use matrix methods to find $Var(\bar{X})$.

$Var(\bar{X}) = 1/n^2 [Var(X_1) + Var(X_2) + \ldots + Var(X_n)]$

$= n\sigma^2 / n^2 = \dfrac{\sigma^2}{n}$

0

MATRIX METHODS.

30. Recall $\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$

$$\bar{X} = \begin{bmatrix} \frac{1}{n} & \dots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

$\boxed{\text{Recall from Thm A \& Thm B p. 568.}}$

$$Var(\bar{X}) = \begin{bmatrix} \frac{1}{n} \dots \frac{1}{n} \end{bmatrix} \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \dots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & & \rho\sigma^2 \\ \vdots & & \ddots & \\ \rho\sigma^2 & & \rho\sigma^2 & \sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma^2}{n} + \frac{\rho\sigma^2(n-1)}{n} & \dots & \frac{\sigma^2}{n} + \frac{\rho\sigma^2(n-1)}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$$

$$= \frac{\sigma^2 + \rho\sigma^2(n-1)}{n^2} + \dots + \frac{\sigma^2 + \rho\sigma^2(n-1)}{n^2}$$

$$= \frac{n(\sigma^2 + \rho\sigma^2(n-1))}{n^2} = \frac{\sigma^2}{n} + \frac{\rho\sigma^2}{n} \cdot \frac{n-1}{n}.$$

$$= \frac{\sigma^2}{n} + \frac{n-1}{n}(\rho\sigma^2)$$