

Problem 44: X & Y are independent R.V's w/ $\text{Var}(X) = \text{Var}(Y)$.

Find $\text{Cov}(X+Y, X-Y)$.

$$\begin{aligned}\text{Cov}(X+Y, X-Y) &= \text{Cov}(1X+1Y, 1X-1Y) \\ &= \text{Cov}(X, X) + \text{Cov}(X, Y) - \text{Cov}(Y, X) - \text{Cov}(Y, Y) \\ &= \text{Var}(X) - \text{Var}(Y) \\ &= 0\end{aligned}$$

Problem 46: U and V independent R.V's with mean μ and variances σ^2 . Let $Z = \alpha U + V\sqrt{1-\alpha^2}$. Find $E[Z]$ & ρ_{UZ} .

$$\begin{aligned}E[Z] &= E[\alpha U + V\sqrt{1-\alpha^2}] = \alpha\mu + \mu\sqrt{1-\alpha^2} \\ &= \mu(\alpha + \sqrt{1-\alpha^2})\end{aligned}$$

$$\rho_{UZ} = \frac{\text{Cov}(U, Z)}{\sqrt{\text{Var}(U)\text{Var}(Z)}} = \frac{E[UZ] - E[U]E[Z]}{\sqrt{\text{Var}(U)\text{Var}(Z)}} \quad (*)$$

$$E[UZ] = \alpha E[U^2] + E[U]E[V]\sqrt{1-\alpha^2}$$

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$$\begin{aligned}\Rightarrow (*) \text{'s numerator: } & \alpha E[U^2] + E[U]E[V]\sqrt{1-\alpha^2} - E[U]\mu(\alpha + \sqrt{1-\alpha^2}) \\ &= \alpha E[U^2] + \mu^2\sqrt{1-\alpha^2} - \mu^2\alpha - \mu^2\sqrt{1-\alpha^2} \\ &= \alpha(E[U^2] - \mu^2) \\ &= \alpha \text{Var}(U) \\ &= \alpha \sigma^2\end{aligned}$$

$$\text{Plug into } (*) \quad \rho_{UZ} = \frac{\alpha \sigma^2}{\sigma^2} = \boxed{\alpha}$$

Problem 47: X & Y are independent R.V.'s and $Z = Y - X$.

WTFind expressions for the covariance and the correlation of X & Z in terms of the variance of X & Y .

$$\begin{aligned} \text{Cov}(X, Z) &= \text{Cov}(X, Y - X) = \text{Cov}(1X + 0Y, 1Y - 1X) \\ &= \text{Cov}(X, Y) + 0 - \text{Cov}(X, X) + 0 \\ &= \text{Cov}(X, Y) - \text{Var}(X) \\ &= -\text{Var}(X) = -\sigma_X^2 \end{aligned}$$

$$\rho_{XZ} = \frac{\text{Cov}(X, Z)}{\sqrt{\text{Var}(X) \text{Var}(Z)}}$$

$$\begin{aligned} \text{Var}(Z) &= \text{Var}(Y - X) \\ &= \text{Var}(Y) + \text{Var}(X) = \sigma_Y^2 + \sigma_X^2 \end{aligned}$$

$$= \frac{-\sigma_X^2}{\sqrt{\sigma_X^2(\sigma_Y^2 + \sigma_X^2)}} = \frac{-\sigma_X}{\sqrt{\sigma_Y^2 + \sigma_X^2}}$$

Problem 48: Given $U = a + bX$ & $V = c + dY$

WTS $|\rho_{UV}| = |\rho_{XY}|$

$$\rho_{UV} = \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U) \text{Var}(V)}} = \frac{\text{Cov}(a + bX, c + dY)}{\sqrt{\text{Var}(a + bX) \text{Var}(c + dY)}}$$

$$\text{First } \text{Cov}(a + bX, c + dY) = \text{Cov}(bX, dY) = bd \text{Cov}(X, Y)$$

$$\text{Then } \text{Var}(a + bX) = b^2 \text{Var}(X) \quad \& \quad \text{Var}(c + dY) = d^2 \text{Var}(Y)$$

$$\begin{aligned} \text{So, putting it all together } \Rightarrow &= \frac{bd \text{Cov}(X, Y)}{(b^2 d^2 \text{Var}(X) \text{Var}(Y))^{1/2}} \\ &= \frac{\text{Cov}(X, Y)}{(\text{Var}(X) \text{Var}(Y))^{1/2}} \end{aligned}$$

Add absolute values to complete the proof \square

Note: $E[Y|X=x] = \mu_y + \rho(x - \mu_x) \frac{\sigma_y}{\sigma_x}$
 $E[X|Y=y] = \mu_x + \rho(y - \mu_y) \frac{\sigma_x}{\sigma_y}$

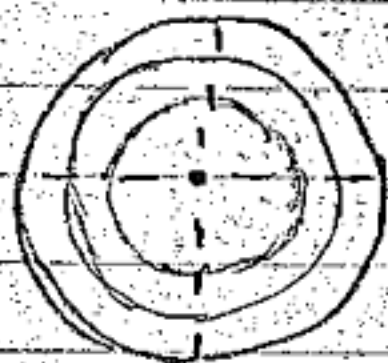
Problem 69: Suppose bivariate normal dist. has $\mu_x = \mu_y = 0$,
 and $\sigma_x = \sigma_y = 1$.

Sketch the contours of the density and the lines
 $E[Y|X=x]$ and $E[X|Y=y]$ for $\rho = 0$, $\rho = 0.5$, and $\rho = 0.9$.

$E[Y|X=x] = 0$

$E[X|Y=y] = 0$

$\rho = 0$



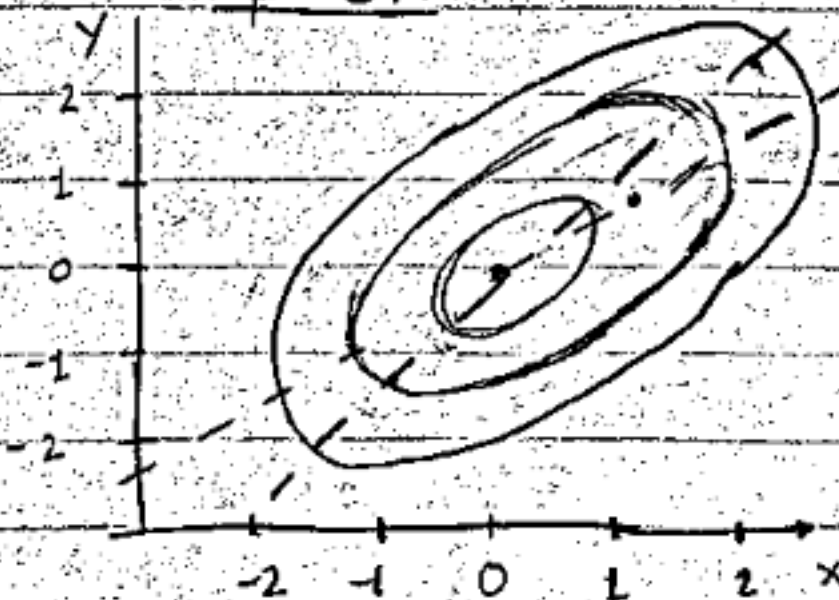
$\rho = 0.5$



$E[X|Y=y] = 0.5y$

$E[Y|X=x] = 0.5x$

$\rho = 0.9$



$E[X|Y=y] = 0.9y$

$E[Y|X=x] = 0.9x$