

# boris

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## Abstract

boris is a python module which calculates the dispersion characteristics of spin waves (SW) in ferromagnetic films based on the perturbation theory of Kalinikos and Slavin (K&S). Additionally, by calculating contours of the dispersion surface and via an inverse Fourier transform, boris calculates the emission pattern of a SW point source contacting a ferromagnetic medium.

## 1 Introduction

These notes address the problem of spin-wave emission in tangentially-magnetized magnetic films. We aim to prov

## 2 Preliminaries

K&S [3, 4] have solved Maxwell's equation in the magnetostatic limit for a medium described by the linearised Landau-Lifshitz equation, subject to electromagnetic and "exchange" boundary conditions. Using perturbation theory, K&S have obtained an explicit dispersion relation for "spin-waves" in the medium. To wit,

$$\omega_n = \sqrt{(\omega_H + \alpha\omega_M k_n^2)(\omega_H + \alpha\omega_M k_n^2 + \omega_M F_{nn})} \quad (1)$$

where

$$F_{nn} = P_{nn} + \sin^2 \theta \left( 1 - P_{nn} (1 + \cos^2 \phi) + \omega_M \frac{P_{nn}(1 - P_{nn}) \sin^2 \phi}{(\omega_H + \alpha\omega_M k_n^2)} \right), \quad (2)$$

$k_n^2 = k_\zeta^2 + \kappa_n^2$ , and  $\kappa_n = \frac{n\pi}{L}$  where  $L$  is the film thickness. K&S defined  $\omega_H = \mu_0 |g| H_i$  and  $\omega_M = \mu_0 |g| M_0$ , where  $\mu_0$  is the permeability of vacuum,  $|g|$  is the gyromagnetic ratio,  $H_i$  is the magnitude of the internal field, and  $M_0$  is the magnitude of the saturation magnetization. The constant  $\alpha$  is the exchange constant.

We rely on the approximation of totally unpinned surface spins, for which  $P_{nn}$  has the explicit expression

$$P_{nn} = \frac{k_\zeta^2}{k_n^2} - \frac{k_\zeta^4}{k_n^4} F_n \frac{1}{(1 + \delta_{0n})} \quad (3)$$

whereby

$$F_n = \frac{2}{k_\zeta L} [1 - (-1)^n e^{-k_\zeta L}]. \quad (4)$$

We write  $P_{nn}$  in the diagonal approximation, having taken  $n = n'$ . Lastly, we exclusively consider modes with a uniform profile across the film thickness, i.e. everywhere  $n = 0$ .

K&S utilize two coordinate systems. The first  $(\xi, \eta, \zeta)$  system is oriented such that the  $\xi$  direction lies parallel to the film normal vector, if the film is considered as a plane with  $L = 0$ . The upper and lower surfaces of the film lie at  $\xi = \frac{L}{2}$  and  $\xi = -\frac{L}{2}$ , respectively. Furthermore, the direction of spin-wave propagation is oriented along the  $\zeta$  direction, i.e.  $\vec{k} \parallel \hat{\zeta}$ . The second  $(x, y, z)$  system is oriented such that the  $z$  axis lies parallel to the saturation magnetization  $\vec{M}_0$  and the internal static magnetic field  $\vec{H}_i$ . The angle  $\theta$  measures the rotation of the  $z$  axis relative to the  $\xi$  axis and takes values in the range  $[0, \pi]$ . The angle  $\phi$  measures the rotation of the  $z$  axis relative to the  $\zeta$  axis and takes values in the range  $[0, 2\pi]$ . If  $\theta = \frac{\pi}{2}$ ,  $\vec{M}_0$  lies “in-plane”. If then  $\phi = 0$ , then  $\hat{z} \parallel \hat{\zeta}$ , i.e.  $\vec{M}_0 \parallel \vec{k}$ . If instead  $\phi = \frac{\pi}{2}$ , then  $\hat{z} \parallel \hat{\eta}$ , i.e.  $\vec{M}_0 \perp \vec{k}$ . Note that, by choosing the orientation of the axes,  $\vec{k} = (0, 0, k_\zeta)$  in the  $(\xi, \eta, \zeta)$  coordinate system.

We consider the physical quantities  $\omega_H$ ,  $\omega_M$ ,  $\alpha$ , and  $L$  to be fixed. Then, for “in-plane” oriented magnetization, i.e. for fixed  $\theta = \frac{\pi}{2}$ , equation (1) defines a spin-wave dispersion surface  $\omega_n(k_\parallel, k_\perp) = \omega_n(\vec{k})$ . We have written  $(k_\parallel, k_\perp)$  in place of  $(k_z, k_y)$  to denote the components of  $\vec{k}$  relative to the internal static magnetic field.

A dispersion relation connects time oscillations  $e^{i\omega t}$  to spatial oscillations  $e^{i\vec{k} \cdot \vec{r}}$  of wave number  $\vec{k} = (\frac{2\pi}{\lambda_\parallel}, \frac{2\pi}{\lambda_\perp})$ . The dispersion relation is the function  $\omega(\vec{k})$  for which the plane waves  $e^{i\vec{k} \cdot \vec{r}} e^{i\omega(\vec{k})t}$  satisfy the systems of equations.

Suppose we excite magnetostatic waves with a harmonically varying magnetic field of circular frequency  $\omega_{ex}$ . The spatial profile of the excited magnetostatic waves then arises from waves satisfying

$$\omega_n(\vec{k}) = \omega_{ex}. \quad (5)$$

Equation (5) defines a constant frequency contour of the dispersion surface. For harmonic excitation of a non-dissipative system, we may calculate the spatial profile of excited magnetostatic waves by investigating contours of the dispersion surface.

Even with ideally harmonic excitation, dissipation introduces coupling between the eigenmodes of an oscillating system. An external harmonically varying magnetic field excites oscillations not only at circular frequency  $\omega_{ex}$ , but also

within a narrow range  $[\omega_{ex} - \frac{\Delta\omega}{2}, \omega_{ex} + \frac{\Delta\omega}{2}]$  whose characteristic width  $\Delta\omega$  is proportional to the dissipation of the system. For magnetostatic waves, dissipation is phenomenologically modelled via the Gilbert damping parameter  $\alpha$ , i.e.  $\Delta\omega \propto \alpha$ .

With dissipation present, the spatial profile of magnetostatic waves excited by a harmonic magnetic field of frequency  $\omega_{ex}$  is not synthesized from a contour of the dispersion surface, but rather from a family of contours on the dispersion surface with frequencies in the range  $[\omega_{ex} - \frac{\Delta\omega}{2}, \omega_{ex} + \frac{\Delta\omega}{2}]$ . The weights, or the amplitudes,  $F(\vec{k})$  in the synthesis are inversely proportional to  $|\omega_{ex} - \omega_n(\vec{k})|$ .

## 2.1 Group velocity

We derive the group velocity,

$$\vec{v}_g = \nabla_{\vec{k}} \omega_n(\vec{k}) = \frac{\partial \omega_n}{\partial k_\zeta} \hat{k}_\zeta + \frac{1}{k_\zeta} \frac{\partial \omega_n}{\partial \phi} \hat{\phi} \quad (6)$$

first in polar coordinates. Direction of energy propagation parallel to group velocity. Non-collinearity of phase and group velocity. Semi-caustics and caustics.

## 2.2 Magnon effective mass

Corpuscular picture, magnons. The magnon inverse effective mass tensor has elements,

$$\left( \frac{1}{m} \right)_{\parallel, \perp} = \frac{1}{\hbar} \frac{\partial^2 \omega_n}{\partial k_\parallel \partial k_\perp}. \quad (7)$$

Note that, besides the factor  $\frac{1}{\hbar}$ , the magnon inverse effective mass tensor is the Hessian matrix of  $\omega_n(k_\parallel, k_\perp)$ .

## 2.3 Density of modes/states

Caustic direction corresponds to minimum magnon effective mass ( $\rightarrow 0$ ?).

# 3 Steering magnetostatic waves

Validity of convolution as system operator—is magnetic medium under linearized L. L. equation an LTI system? Can we show that system of equations must admit a dispersion equation? An explicit dispersion equation?

Time evolution can be calculated in the linear regime by multiplication of a phase factor  $e^{i\omega t}$  to the Fourier synthesis of the magnetization.

Wave sources. Control of real-space wave source distribution. The excitation of waves in an unbounded medium supporting spin waves contacted by a wave source is modelled by convolution considering waves excited by point sources at the wavesource/medium boundary. The size and shape of the contact control the efficiency of excitation of spin waves and the type of convolution. Consider

the lateral dimensions of the source parallel  $a_{\parallel}$  and perpendicular  $a_{\perp}$  to the static field. The source is assumed to efficiently excite waves with wavevector components  $k_{\parallel} < \frac{\pi}{a_{\parallel}}$  and  $k_{\perp} < \frac{\pi}{a_{\perp}}$ , i.e. waves with wavelengths greater than the source size are not efficiently excited.

Phase dispersion. Anisotropic dispersion (origin: (bi-)gyrotropy).

Possibilities via control of spectrum of excitation. Directional emission for harmonic excitation. Convolution between spectrum of excitation field (e.g. harmonic  $\rightarrow$  delta function) and slowness surface. This convolution is valid, as long as the coupling of the microwave field to the magnetization is constant over the support of the spectrum of microwave field.

## 4 Full rigor

The above approach is based on the explicit dispersion relation derived by K&S in a perturbation theory. In reality, one should obtain the magnetostatic (magnetostatic-exchange) Green's function which solves the equations of motion: Maxwell's equations, electrodynamic boundary conditions, and linearized LLG. Showing that the magnetostatic operator is linear guarantees that an analysis on the basis of Fourier analysis and synthesis is applicable. Furthermore, (?) this shows the system operator is convolution. There is the possibility that one can show that MSW are an LTI system.

Then, using the Green's function, arbitrary sources maybe be modeled via convolution of the spectrum of the source structure and the spectrum of the driving field.

In this approach, non-zero damping introduces a coupling between the eigenmodes, i.e. destroys the orthogonality of the eigenmodes, hindering an expansion of the field into the eigenmodes. Therefore, damping must be accounted for phenomenologically.

#### 4.1 Point source emission pattern

#### 4.2 Smeared point source

#### 4.3 Finite wave source

#### 4.4 Almost arbitrary driving field

#### 4.5 Almost arbitrary driving field and finite wave source

#### 4.6 Diffraction grating

### 5 Measuring magnetostatic waves, practical considerations

#### 5.1 Sampling, mechanical restrictions

#### 5.2 Scattering, optical restrictions

## References

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## A Appendix A: Group Velocity

We derive the group velocity,

$$\vec{v}_g = \nabla_{\vec{k}} \omega_n(\vec{k}) = \frac{\partial \omega_n}{\partial k_\zeta} \hat{k}_\zeta + \frac{1}{k_\zeta} \frac{\partial \omega_n}{\partial \phi} \hat{\phi} \quad (8)$$

first in polar coordinates.

We calculate the derivative  $\frac{\partial \omega_n}{\partial k_\zeta} = \frac{1}{2\omega_n} \frac{\partial \omega_n^2}{\partial k_\zeta}$  via  $\frac{\partial \omega_n^2}{\partial k_\zeta}$ . For notational convenience, we define

$$R = \omega_H + \alpha \omega_M k_n^2 \quad (9)$$

$$E = \omega_M \sin^2 \theta \sin^2 \phi. \quad (10)$$

Then, we have

$$\omega_n = \sqrt{R(R + \omega_M F_{nn})} \quad (11)$$

$$\frac{\partial \omega_n^2}{\partial k_\zeta} = 2\alpha\omega_M k_\zeta [2R + \omega_M F_{nn}] + \omega_M R \frac{\partial F_{nn}}{\partial k_\zeta}. \quad (12)$$

Now, we calculate

$$\frac{\partial P_{nn}}{\partial k_\zeta} = 2\frac{k_\zeta}{k_n^2} - 2\frac{k_\zeta^3}{k_n^4} - 4\frac{k_\zeta^3}{k_n^4} F_n B + 4\frac{k_\zeta^5}{k_n^6} F_n B - \frac{k_\zeta^4}{k_n^4} B \frac{\partial F_n}{\partial k_\zeta} \quad (13)$$

$$\frac{\partial F_n}{\partial k_\zeta} = \frac{-2}{k_\zeta^2 L} + \frac{2(-1)^n e^{-k_\zeta L}}{k_\zeta^2 L} + \frac{2(-1)^n e^{-k_\zeta L}}{k_\zeta} \quad (14)$$

where we have defined  $B = \frac{1}{2}$  if  $n = 0$  and  $B = 1$  if  $n \neq 0$ . Now we have

$$\begin{aligned} \frac{\partial F_{nn}}{\partial k_\zeta} &= \frac{\partial P_{nn}}{\partial k_\zeta} - \frac{\partial P_{nn}}{\partial k_\zeta} \sin^2 \theta (1 + \cos^2 \phi) - \frac{EP_{nn}}{R^2} \frac{\partial R}{\partial k_\zeta} + \\ &+ \frac{E}{R} \frac{\partial P_{nn}}{\partial k_\zeta} + \frac{EP_{nn}^2}{R^2} \frac{\partial R}{\partial k_\zeta} - \frac{2EP_{nn}}{R} \frac{\partial P_{nn}}{\partial k_\zeta} \end{aligned} \quad (15)$$

which yields  $\frac{\partial \omega_n}{\partial k_\zeta}$ . For  $\frac{\partial \omega_n}{\partial \phi}$ , we find

$$\frac{\partial \omega_n}{\partial \phi} = \frac{1}{2\omega_n} \frac{\partial \omega_n^2}{\partial \phi} \quad (16)$$

$$\frac{\partial \omega_n^2}{\partial \phi} = R\omega_M \frac{\partial F_{nn}}{\partial \phi} \quad (17)$$

$$\frac{\partial F_{nn}}{\partial \phi} = P_{nn} \sin^2 \theta \sin 2\phi \left[ 1 + \frac{\omega_M(1 - P_{nn})}{\omega_H + \alpha\omega_M k_n^2} \right]. \quad (18)$$

We may translate these results to rectangular coordinates. In that case,  $\vec{v}_g = \nabla_{\vec{k}} \omega(\vec{k}) = \frac{\partial \omega_n}{\partial k_z} \hat{z} + \frac{\partial \omega_n}{\partial k_y} \hat{y}$ . We have as relations

$$k_\zeta = \sqrt{k_z^2 + k_y^2} \quad (19)$$

$$k_z = k_\zeta \cos \phi \quad (20)$$

$$k_y = k_\zeta \sin \phi. \quad (21)$$

From the chain rule,

$$\frac{\partial \omega_n}{\partial k_z} = \frac{\partial \omega_n}{\partial k_\zeta} \cos \phi + \frac{\partial \omega_n}{\partial \phi} \left( \frac{-\sin \phi}{k_\zeta} \right) \quad (22)$$

$$\frac{\partial \omega_n}{\partial k_y} = \frac{\partial \omega_n}{\partial k_\zeta} \sin \phi + \frac{\partial \omega_n}{\partial \phi} \left( \frac{\cos \phi}{k_\zeta} \right). \quad (23)$$

## B Appendix B: Magnon Effective Mass

Corpuscular picture, magnons. The magnon effective mass tensor has elements,

$$\left(\frac{1}{m}\right)_{\parallel,\perp} = \frac{1}{\hbar} \frac{\partial^2 \omega_n}{\partial k_{\parallel} \partial k_{\perp}}. \quad (24)$$

Note that, besides the factor  $\frac{1}{\hbar}$ , the magnon effective mass tensor is the Hessian matrix of  $\omega_n(k_{\parallel}, k_{\perp})$ .

Once again, we proceed via  $\frac{\partial^2 \omega_n}{\partial k_{\zeta}^2}$ :

$$\frac{\partial^2 \omega_n}{\partial k_{\zeta}^2} = \frac{1}{2\omega_n} \frac{\partial^2 \omega_n^2}{\partial k_{\zeta}^2} - \frac{1}{\omega_n} \left( \frac{\partial \omega_n}{\partial k_{\zeta}} \right)^2 \quad (25)$$

$$\frac{\partial^2 \omega_n^2}{\partial k_{\zeta}^2} = [2R + \omega_M F_{nn}] \frac{\partial^2 R}{\partial k_{\zeta}^2} + R\omega_M \frac{\partial^2 F_{nn}}{\partial k_{\zeta}^2} + \left[ 2 \frac{\partial R}{\partial k_{\zeta}} + 2\omega_M \frac{\partial F_{nn}}{\partial k_{\zeta}} \right] \frac{\partial R}{\partial k_{\zeta}} \quad (26)$$

$$\frac{\partial^2 R}{\partial k_{\zeta}^2} = 2\alpha\omega_M. \quad (27)$$

We lack only  $\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}$ . Having  $\frac{\partial F_{nn}}{\partial k_\zeta}$ , we proceed term-by-term,

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} = \frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_1 + \frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_2 + \frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_3 + \frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_4 + \frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_5 + \frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_6 \quad (28)$$

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_1 = \frac{\partial}{\partial k_\zeta} \frac{\partial P_{nn}}{\partial k_\zeta} = \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \quad (29)$$

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_2 = \frac{\partial}{\partial k_\zeta} \left[ -\frac{\partial P_{nn}}{\partial k_\zeta} \sin^2 \theta (1 + \cos^2 \phi) \right] \quad (30)$$

$$= -\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \sin^2 \theta (1 + \cos^2 \phi) \quad (31)$$

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_3 = \frac{\partial}{\partial k_\zeta} \left[ -\frac{1}{R^2} E P_{nn} \frac{\partial R}{\partial k_\zeta} \right] \quad (32)$$

$$= 2 E P_{nn} \left( \frac{\partial R}{\partial k_\zeta} \right)^2 \frac{1}{R^3} - \frac{1}{R^2} E \frac{\partial R}{\partial k_\zeta} \frac{\partial P_{nn}}{\partial k_\zeta} - \frac{1}{R^2} E P_{nn} \frac{\partial^2 R}{\partial k_\zeta^2} \quad (33)$$

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_4 = \frac{\partial}{\partial k_\zeta} \left[ \frac{1}{R} E \frac{\partial P_{nn}}{\partial k_\zeta} \right] = -\frac{1}{R^2} E \frac{\partial P_{nn}}{\partial k_\zeta} \frac{\partial R}{\partial k_\zeta} + \frac{1}{R} E \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \quad (34)$$

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_5 = \frac{\partial}{\partial k_\zeta} \left[ \frac{1}{R^2} E P_{nn}^2 \frac{\partial R}{\partial k_\zeta} \right] \quad (35)$$

$$= -2 \frac{1}{R^3} E P_{nn}^2 \left( \frac{\partial R}{\partial k_\zeta} \right)^2 + 2 \frac{1}{R^2} E P_{nn} \frac{\partial R}{\partial k_\zeta} \frac{\partial P_{nn}}{\partial k_\zeta} + \frac{1}{R^2} E P_{nn}^2 \frac{\partial^2 R}{\partial k_\zeta^2} \quad (36)$$

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} \Big|_6 = \frac{\partial}{\partial k_\zeta} \left[ -2 \frac{1}{R} E P_{nn} \frac{\partial P_{nn}}{\partial k_\zeta} \right] \quad (37)$$

$$= 2 \frac{1}{R^2} E P_{nn} \frac{\partial R}{\partial k_\zeta} \frac{\partial P_{nn}}{\partial k_\zeta} - 2 \frac{1}{R} E \left( \frac{\partial P_{nn}}{\partial k_\zeta} \right)^2 - 2 \frac{1}{R} E P_{nn} \frac{\partial^2 P_{nn}}{\partial k_\zeta^2}. \quad (38)$$



Hence we need  $\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}$ . Having  $\frac{\partial P_{nn}}{\partial k_\zeta}$ , we proceed term-by-term,

$$\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} = \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_1 + \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_2 + \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_3 + \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_4 + \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_5 \quad (39)$$

$$\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_1 = \frac{\partial}{\partial k_\zeta} \left[ 2 \frac{k_\zeta}{k_n^2} \right] = \frac{2}{k_n^2} - 4 \frac{k_\zeta}{k_n^3} \frac{\partial k_n}{\partial k_\zeta} \quad (40)$$

$$\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_2 = \frac{\partial}{\partial k_\zeta} \left[ -2 \frac{k_\zeta^2}{k_n^3} \frac{\partial k_n}{\partial k_\zeta} \right] \quad (41)$$

$$= -4 \frac{k_\zeta}{k_n^3} \frac{\partial k_n}{\partial k_\zeta} + 6 \frac{k_\zeta^2}{k_n^4} \left( \frac{\partial k_n}{\partial k_\zeta} \right)^2 - 2 \frac{k_\zeta^2}{k_n^3} \frac{\partial^2 k_n}{\partial k_\zeta^2} \quad (42)$$

$$\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_3 = \frac{\partial}{\partial k_\zeta} \left[ -4 \frac{k_\zeta^3}{k_n^4} F_n B \right] \quad (43)$$

$$= -12 \frac{k_\zeta^2}{k_n^4} F_n B + 16 \frac{k_\zeta^3}{k_n^5} F_n B \frac{\partial k_n}{\partial k_\zeta} - 4 \frac{k_\zeta^3}{k_n^4} B \frac{\partial F_n}{\partial k_\zeta} \quad (44)$$

$$\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_4 = \frac{\partial}{\partial k_\zeta} \left[ 4 \frac{k_\zeta^4}{k_n^5} F_n B \frac{\partial k_n}{\partial k_\zeta} \right] \quad (45)$$

$$= 16 \frac{k_\zeta^3}{k_n^5} F_n B \frac{\partial k_n}{\partial k_\zeta} - 20 \frac{k_\zeta^4}{k_n^6} F_n B \left( \frac{\partial k_n}{\partial k_\zeta} \right)^2 + 4 \frac{k_\zeta^4}{k_n^5} B \frac{\partial k_n}{\partial k_\zeta} \frac{\partial F_n}{\partial k_\zeta} + 4 \frac{k_\zeta^4}{k_n^5} F_n B \frac{\partial^2 k_n}{\partial k_\zeta^2} \quad (46)$$

$$\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \Big|_5 = \frac{\partial}{\partial k_\zeta} \left[ -\frac{k_\zeta^4}{k_n^4} B \frac{\partial F_n}{\partial k_\zeta} \right] \quad (47)$$

$$= -4 \frac{k_\zeta^3}{k_n^4} B \frac{\partial F_n}{\partial k_\zeta} + 4 \frac{k_\zeta^4}{k_n^5} B \frac{\partial F_n}{\partial k_\zeta} \frac{\partial k_n}{\partial k_\zeta} - \frac{k_\zeta^4}{k_n^4} B \frac{\partial^2 F_n}{\partial k_\zeta^2}. \quad (48)$$

To conclude, we calculate  $\frac{\partial^2 F_n}{\partial k_\zeta^2}$ ,

$$\frac{\partial^2 F_n}{\partial k_\zeta^2} = \frac{\partial^2 F_n}{\partial k_\zeta^2} \Big|_1 + \frac{\partial^2 F_n}{\partial k_\zeta^2} \Big|_2 + \frac{\partial^2 F_n}{\partial k_\zeta^2} \Big|_3 \quad (49)$$

$$\frac{\partial^2 F_n}{\partial k_\zeta^2} \Big|_1 = \frac{\partial}{\partial k_\zeta} \left[ -2 \frac{1}{k_\zeta^2 L} \right] = 4 \frac{1}{k_\zeta^3 L} \quad (50)$$

$$\frac{\partial^2 F_n}{\partial k_\zeta^2} \Big|_2 = \frac{\partial}{\partial k_\zeta} \left[ 2(-1)^n \frac{1}{k_\zeta^2 L} e^{-k_\zeta L} \right] \quad (51)$$

$$= -4(-1)^n \frac{1}{k_\zeta^3 L} e^{-k_\zeta L} - 2(-1)^n \frac{1}{k_\zeta^2} e^{-k_\zeta L} \quad (52)$$

$$\frac{\partial^2 F_n}{\partial k_\zeta^2} \Big|_3 = \frac{\partial}{\partial k_\zeta} \left[ 2(-1)^n \frac{1}{k_\zeta} e^{-k_\zeta L} \right] \quad (53)$$

$$= -2(-1)^n \frac{1}{k_\zeta^2} e^{-k_\zeta L} - 2L(-1)^n \frac{1}{k_\zeta} e^{-k_\zeta L}. \quad (54)$$

Note the derivative of  $k_n$  with respect to  $k_\zeta$ ,

$$\frac{\partial k_n}{\partial k_\zeta} = \frac{k_\zeta}{k_n} \quad (55)$$

$$\frac{\partial^2 k_n}{\partial k_\zeta^2} = \frac{1}{k_n} - \frac{k_\zeta^2}{k_n^3}. \quad (56)$$

Now, we calculate  $\frac{\partial^2 \omega_n}{\partial \phi^2}$ ,

$$\frac{\partial^2 \omega_n}{\partial \phi^2} = \frac{1}{2\omega_n} \frac{\partial^2 \omega_n^2}{\partial \phi^2} - \frac{1}{\omega_n} \left( \frac{\partial \omega_n}{\partial \phi} \right)^2 \quad (57)$$

$$\frac{\partial^2 \omega_n^2}{\partial \phi^2} = R\omega_M \frac{\partial^2 F_{nn}}{\partial \phi^2} \quad (58)$$

$$\frac{\partial^2 F_{nn}}{\partial \phi^2} = 2 \cos 2\phi \left( P_{nn} \sin^2 \theta + \frac{\omega_M P_{nn} (1 - P_{nn}) \sin^2 \theta}{R} \right). \quad (59)$$

Finally the mixed-derivatives,

$$\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} = \frac{\partial^2 \omega_n}{\partial \phi \partial k_\zeta} = \frac{\partial}{\partial k_\zeta} \frac{\partial \omega_n}{\partial \phi} = \frac{\partial}{\partial k_\zeta} \left[ \frac{1}{2\omega_n} \frac{\partial \omega_n^2}{\partial \phi} \right] \quad (60)$$

$$= \frac{\partial}{\partial k_\zeta} \left[ \frac{1}{2\omega_n} R P_{nn} \omega_M \sin 2\phi \sin^2 \theta + \frac{1}{2\omega_n} R \omega_M^2 \sin 2\phi \frac{P_{nn}(1 - P_{nn}) \sin^2 \theta}{R} \right] \quad (61)$$

$$\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} = \frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} \Big|_1 + \frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} \Big|_2 \quad (62)$$

$$\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} \Big|_1 = \frac{-T}{2\omega_n^2} R P_{nn} \frac{\partial \omega_n}{\partial k_\zeta} + \frac{T}{2\omega_n} P_{nn} \frac{\partial R}{\partial k_\zeta} + \frac{T}{2\omega_n} R \frac{\partial P_{nn}}{\partial k_\zeta} \quad (63)$$

$$\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} \Big|_2 = \frac{-U}{2\omega_n^2} [P_{nn} - P_{nn}^2] \frac{\partial \omega_n}{\partial k_\zeta} + \frac{U}{2\omega_n} \frac{\partial P_{nn}}{\partial k_\zeta} [1 - 2P_{nn}]. \quad (64)$$

The variables  $T$  and  $U$  have been defined for notational brevity:

$$T = \omega_M \sin 2\phi \sin^2 \theta \quad (65)$$

$$U = \omega_M^2 \sin 2\phi \sin^2 \theta. \quad (66)$$

To write down the Hessian matrix, we convert these results to Cartesian coordinates via

$$\frac{\partial^2 \omega_n}{\partial k_z^2} = \frac{\partial^2 \omega_n}{\partial k_z \partial k_\zeta} \frac{\partial k_\zeta}{\partial k_z} + \frac{\partial^2 \omega_n}{\partial k_z \partial \phi} \frac{\partial \phi}{\partial k_z} + \frac{\partial \omega_n}{\partial k_\zeta} \frac{\partial^2 k_\zeta}{\partial k_z^2} + \frac{\partial \omega_n}{\partial \phi} \frac{\partial^2 \phi}{\partial k_z^2} \quad (67)$$

$$\frac{\partial^2 \omega_n}{\partial k_y^2} = \frac{\partial^2 \omega_n}{\partial k_y \partial k_\zeta} \frac{\partial k_\zeta}{\partial k_y} + \frac{\partial^2 \omega_n}{\partial k_y \partial \phi} \frac{\partial \phi}{\partial k_y} + \frac{\partial \omega_n}{\partial k_\zeta} \frac{\partial^2 k_\zeta}{\partial k_y^2} + \frac{\partial \omega_n}{\partial \phi} \frac{\partial^2 \phi}{\partial k_y^2} \quad (68)$$

$$\frac{\partial^2 \omega_n}{\partial k_z \partial k_y} = \frac{\partial^2 \omega_n}{\partial k_z \partial k_\zeta} \frac{\partial k_\zeta}{\partial k_y} + \frac{\partial^2 \omega_n}{\partial k_z \partial \phi} \frac{\partial \phi}{\partial k_y} + \frac{\partial \omega_n}{\partial k_\zeta} \frac{\partial^2 k_\zeta}{\partial k_z \partial k_y} + \frac{\partial \omega_n}{\partial \phi} \frac{\partial^2 \phi}{\partial k_z \partial k_y} \quad (69)$$

$$\frac{\partial^2 \omega_n}{\partial k_z \partial k_y} = \frac{\partial^2 \omega_n}{\partial k_y \partial k_z}. \quad (70)$$

Hence we need

$$\frac{\partial^2 \omega_n}{\partial k_z \partial k_\zeta} = \frac{\partial k_\zeta}{\partial k_z} \frac{\partial^2 \omega_n}{\partial k_\zeta^2} + \frac{\partial \phi}{\partial k_z} \frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} \quad (71)$$

$$\frac{\partial^2 \omega_n}{\partial k_z \partial \phi} = \frac{\partial k_\zeta}{\partial k_z} \frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} + \frac{\partial \phi}{\partial k_z} \frac{\partial^2 \omega_n}{\partial \phi^2} \quad (72)$$

$$\frac{\partial k_\zeta}{\partial k_z} = \cos \phi \quad (73)$$

$$\frac{\partial k_\zeta}{\partial k_y} = \sin \phi \quad (74)$$

$$\frac{\partial^2 k_\zeta}{\partial k_z^2} = \frac{\sin^2 \phi}{k_\zeta} \quad (75)$$

$$\frac{\partial^2 k_\zeta}{\partial k_y^2} = \frac{\cos^2 \phi}{k_\zeta} \quad (76)$$

$$\frac{\partial^2 k_\zeta}{\partial k_y \partial k_z} = \frac{\partial^2 k_\zeta}{\partial k_z \partial k_y} = \frac{-\cos \phi \sin \phi}{k_\zeta} \quad (77)$$

$$\frac{\partial \phi}{\partial k_z} = \frac{-\sin \phi}{k_\zeta} \quad (78)$$

$$\frac{\partial \phi}{\partial k_y} = \frac{\cos \phi}{k_\zeta} \quad (79)$$

$$\frac{\partial^2 \phi}{\partial k_z^2} = \frac{\sin 2\phi}{k_\zeta^2} \quad (80)$$

$$\frac{\partial^2 \phi}{\partial k_y^2} = \frac{-\sin 2\phi}{k_\zeta^2} \quad (81)$$

$$\frac{\partial^2 \phi}{\partial k_y \partial k_z} = \frac{\partial^2 \phi}{\partial k_z \partial k_y} = \frac{1 - 2 \cos^2 \phi}{k_\zeta^2} \quad (82)$$

## C Appendix C: Gyrotropy and Angular Dependence of $\omega_n(\vec{k})$

## D Appendix D: Linearity of MSW operator

## E Appendix E: Aside, the SHM