boris

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Abstract

boris is a python module which calculates the dispersion characteristics of spin waves (SW) in ferromagnetic films based on the perturbation theory of Kalinikos and Slavin. Additionally, by calculating contours of the dispersion surface and via an inverse Fourier transform, boris calculates the emission pattern of a SW point source contacting a ferromagnetic medium.

1 Preliminaries

Kalinikos and Slavin (K & S) have solved Maxwell's equation in the magnetostatic limit for a medium described by the linearised Landau-Lifshitz equation, subject to electromagnetic and "exchange" boundary conditions. Using perturbation theory, K & S have obtained an explicit expression for "spin-waves" in the medium. To wit,

$$\omega_n = \sqrt{(\omega_H + \alpha\omega_M k_n^2)(\omega_H + \alpha\omega_M k_n^2 + \omega_M F_{nn})} \tag{1}$$

where

$$F_{nn} = P_{nn} + \sin^2\theta \left(1 - P_{nn}\left(1 + \cos^2\phi\right) + \omega_M \frac{P_{nn}(1 - P_{nn})\sin^2\phi}{(\omega_H + \alpha\omega_M k_n^2)}\right) \tag{2}$$

and $k_n^2 = k_\zeta^2 + \kappa_n^2$. We rely on the approximation of totally unpinnned surface spins, for which P_{nn} has the explicit expression

$$P_{nn} = \frac{k_\zeta^2}{k_n^2} - \frac{k_\zeta^4}{k_n^4} F_n \frac{1}{(1 + \delta_{0n})} \tag{3}$$

whereby

$$F_n = \frac{2}{k_\zeta L}[1 - (-1)^n e^{-k_\zeta L}]. \tag{4}$$

In this approximation, $\kappa_n = \frac{n\pi}{L}$ where L is the film thickness. Moreover, we write P_{nn} in the diagonal approximation, having taken $n = n'$. Lastly, we exclusively consider modes with a uniform profile across the film thickness, i.e. everywhere $n = 0$.

K & S defined $\omega_H = \mu_0 |g| H_i$ and $\omega_M = \mu_0 |g| M_0$, where μ_0 is the permeability of vacuum, $|g|$ is the gyromagnetic ratio, H_i is the magnitude of the internal field, and M_0 is the magnitude of the saturation magnetization. The constant α is the exchange constant.

K & S utilize two coordinate systems. The first (ξ, η, ζ) system is oriented such that the ξ direction lies parallel to the film normal vector, if the film is considered as a plane with $L = 0$. The upper and lower surfaces of the film lie at $\xi = \frac{L}{2}$ and $\xi = \frac{-L}{2}$, respectively. Furthermore, the direction of spin-wave propagation is oriented along the ζ direction, i.e. $\vec{k} \parallel \hat{\zeta}$. The second (x, y, z) system is oriented such that the z axis lies parallel to the saturation magnetization $\vec{M}_0$ and the internal static magnetic field $\vec{H}_i$. The angle θ measures the rotation of the z axis relative to the ξ axis and takes values in the range $[0, \pi]$. The angle ϕ measures the rotation of the z axis relative to the ζ axis and takes values in the range $[0, 2\pi]$. If $\theta = \frac{\pi}{2}$, $\vec{M}_0$ lies "in-plane". If then $\phi = 0$, then $\hat{z} \parallel \hat{\zeta}$, i.e. $\vec{M}_0 \parallel \vec{k}$. If instead $\phi = \frac{\pi}{2}$, then $\hat{z} \parallel \hat{\eta}$, i.e. $\vec{M}_0 \perp \vec{k}$. Note that, by choosing the orientation of the axes, $\vec{k} = (0, 0, k_\zeta)$ in the (ξ, η, ζ) coordinate system.

We consider the physical quantities ω_H, ω_M, α, and L to be fixed. Then, for "in-plane" oriented magnetization, i.e. for fixed $\theta = \frac{\pi}{2}$, equation (1) defines a spin-wave dispersion surface $\omega = \omega\left(k_\parallel, k_\perp\right)$. We have written $\left(k_\parallel, k_\perp\right)$ in place of (k_z, k_y).

We derive the group velocity $\vec{v}_g = \nabla_{\vec{k}} \omega(\vec{k}) = \frac{\partial \omega_n}{\partial k_\zeta} \hat{k}_\zeta + \frac{1}{k_\zeta} \frac{\partial \omega_n}{\partial \phi} \hat{\phi}$ (stationary phase approx?).

We calculate the derivative of $\frac{\partial \omega_n}{\partial k_\zeta} = \frac{1}{2\omega_n} \frac{\partial \omega_n^2}{\partial k_\zeta}$ via $\frac{\partial \omega_n^2}{\partial k_\zeta}$. For notational convenience, we define

$$R = \omega_H + \alpha \omega_M k_n^2 \tag{5}$$

$$S = \frac{\omega_M P_{nn}(1 - P_{nn}) \sin^2\theta}{R} \tag{6}$$

$$E = \omega_M \sin^2\theta \sin^2\phi. \tag{7}$$

Then, we have

$$\omega_n = \sqrt{R(R + \omega_M F_{nn})} \tag{8}$$

$$\frac{\partial \omega_n^2}{\partial k_\zeta} = 2\alpha \omega_M k_\zeta \left[2R + \omega_M F_{nn}\right] + \omega_M R \frac{\partial F_{nn}}{\partial k_\zeta}. \tag{9}$$

Now, we calculate

$$\frac{\partial P_{nn}}{\partial k_\zeta} = 2\frac{k_\zeta}{k_n^2} - 2\frac{k_\zeta^3}{k_n^4} - 4\frac{k_\zeta^3}{k_n^4} F_n B + 4\frac{k_\zeta^5}{k_n^6} F_n B - \frac{k_\zeta^4}{k_n^4} B \frac{\partial F_n}{\partial k_\zeta} \tag{10}$$

$$\frac{\partial F_n}{\partial k_\zeta} = \frac{-2}{k_\zeta^2 L} + \frac{2(-1)^n e^{-k_\zeta L}}{k_\zeta^2 L} + \frac{2(-1)^n e^{-k_\zeta L}}{k_\zeta} \tag{11}$$

where we have defined $B = \frac{1}{2}$ if $n = 0$ and $B = 1$ if $n \neq 0$. Now we have

$$\frac{\partial F_{nn}}{\partial k_\zeta} = \frac{\partial P_{nn}}{\partial k_\zeta} - \frac{\partial P_{nn}}{\partial k_\zeta} \sin^2\theta(1+\cos^2\phi) - \frac{EP_{nn}}{R^2}\frac{\partial R}{\partial k_\zeta} + \\ + \frac{E}{R}\frac{\partial P_{nn}}{\partial k_\zeta} + \frac{EP_{nn}^2}{R^2}\frac{\partial R}{\partial k_\zeta} - \frac{2EP_{nn}}{R}\frac{\partial P_{nn}}{\partial k_\zeta} \tag{12}$$

which yields $\frac{\partial \omega_n}{\partial k_\zeta}$. For $\frac{\partial \omega_n}{\partial \phi}$, we find

$$\frac{\partial \omega_n}{\partial \phi} = \frac{1}{2\omega_n}\frac{\partial \omega_n^2}{\partial \phi} \tag{13}$$

$$\frac{\partial \omega_n^2}{\partial \phi} = R\omega_M \frac{\partial F_{nn}}{\partial \phi} \tag{14}$$

$$\frac{\partial F_{nn}}{\partial \phi} = P_{nn}\sin^2\theta \sin 2\phi \left[1 + \frac{\omega_M(1-P_{nn})}{\omega_H + \alpha\omega_M k_n^2}\right]. \tag{15}$$

We may translate these results to rectangular coordinates. In that case, $\vec{v}_g = \nabla_{\vec{k}}\omega(\vec{k}) = \frac{\partial \omega_n}{\partial k_z}\hat{z} + \frac{\partial \omega_n}{\partial k_y}\hat{y}$. We have as relations

$$k_\zeta = \sqrt{k_z^2 + k_y^2} \tag{16}$$

$$k_z = k_\zeta \cos\phi \tag{17}$$

$$k_y = k_\zeta \sin\phi. \tag{18}$$

From the chain rule,

$$\frac{\partial \omega_n}{\partial k_z} = \frac{\partial \omega_n}{\partial k_\zeta}\cos\phi + \frac{\partial \omega_n}{\partial \phi}\left(\frac{-\sin\phi}{k_\zeta}\right) \tag{19}$$

$$\frac{\partial \omega_n}{\partial k_y} = \frac{\partial \omega_n}{\partial k_\zeta}\sin\phi + \frac{\partial \omega_n}{\partial \phi}\left(\frac{\cos\phi}{k_\zeta}\right). \tag{20}$$

1.1 Second derivatives

In order to calculate the magnon effective mass, we need the second derivatives of ω_n with respect to k_ζ and ϕ. We proceed via $\frac{\partial^2 \omega_n}{\partial k_\zeta^2}$:

$$\frac{\partial^2 \omega_n}{\partial k_\zeta^2} = \frac{1}{2\omega_n}\frac{\partial^2 \omega_n^2}{\partial k_\zeta^2} - \frac{1}{\omega_n}\left(\frac{\partial \omega_n}{\partial k_\zeta}\right)^2 \tag{21}$$

$$\frac{\partial^2 \omega_n^2}{\partial k_\zeta^2} = [2R + \omega_M F_{nn}]\frac{\partial^2 R}{\partial k_\zeta^2} + R\omega_M\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} + \left[2\frac{\partial R}{\partial k_\zeta} + 2\omega_M\frac{\partial F_{nn}}{\partial k_\zeta}\right]\frac{\partial R}{\partial k_\zeta} \tag{22}$$

$$\frac{\partial^2 R}{\partial k_\zeta^2} = 2\alpha\omega_M. \tag{23}$$

We lack only $\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}$. Having $\frac{\partial F_{nn}}{\partial k_\zeta}$, we proceed term-by-term,

$$\frac{\partial^2 F_{nn}}{\partial k_\zeta^2} = \left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_1 + \left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_2 + \left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_3 + \left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_4 + \left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_5 + \left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_6 \tag{24}$$

$$\left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_1 = \frac{\partial}{\partial k_\zeta}\frac{\partial P_{nn}}{\partial k_\zeta} = \frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \tag{25}$$

$$\left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_2 = \frac{\partial}{\partial k_\zeta}\left[-\frac{\partial P_{nn}}{\partial k_\zeta}\sin^2\theta(1+\cos^2\phi)\right] \tag{26}$$

$$= -\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\sin^2\theta(1+\cos^2\phi) \tag{27}$$

$$\left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_3 = \frac{\partial}{\partial k_\zeta}\left[-\frac{1}{R^2}EP_{nn}\frac{\partial R}{\partial k_\zeta}\right] \tag{28}$$

$$= 2EP_{nn}\left(\frac{\partial R}{\partial k_\zeta}\right)^2\frac{1}{R^3} - \frac{1}{R^2}E\frac{\partial R}{\partial k_\zeta}\frac{\partial P_{nn}}{\partial k_\zeta} - \frac{1}{R^2}EP_{nn}\frac{\partial^2 R}{\partial k_\zeta^2} \tag{29}$$

$$\left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_4 = \frac{\partial}{\partial k_\zeta}\left[\frac{1}{R}E\frac{\partial P_{nn}}{\partial k_\zeta}\right] = -\frac{1}{R^2}E\frac{\partial P_{nn}}{\partial k_\zeta}\frac{\partial R}{\partial k_\zeta} + \frac{1}{R}E\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} \tag{30}$$

$$\left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_5 = \frac{\partial}{\partial k_\zeta}\left[\frac{1}{R^2}EP_{nn}^2\frac{\partial R}{\partial k_\zeta}\right] \tag{31}$$

$$= -2\frac{1}{R^3}EP_{nn}^2\left(\frac{\partial R}{\partial k_\zeta}\right)^2 + 2\frac{1}{R^2}EP_{nn}\frac{\partial R}{\partial k_\zeta}\frac{\partial P_{nn}}{\partial k_\zeta} + \frac{1}{R^2}EP_{nn}^2\frac{\partial^2 R}{\partial k_\zeta^2} \tag{32}$$

$$\left.\frac{\partial^2 F_{nn}}{\partial k_\zeta^2}\right|_6 = \frac{\partial}{\partial k_\zeta}\left[-2\frac{1}{R}EP_{nn}\frac{\partial P_{nn}}{\partial k_\zeta}\right] \tag{33}$$

$$= 2\frac{1}{R^2}EP_{nn}\frac{\partial R}{\partial k_\zeta}\frac{\partial P_{nn}}{\partial k_\zeta} - 2\frac{1}{R}E\left(\frac{\partial P_{nn}}{\partial k_\zeta}\right)^2 - 2\frac{1}{R}EP_{nn}\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}. \tag{34}$$

Hence we need $\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}$. Having $\frac{\partial P_{nn}}{\partial k_\zeta}$, we proceed term-by-term,

$$\frac{\partial^2 P_{nn}}{\partial k_\zeta^2} = \left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_1 + \left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_2 + \left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_3 + \left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_4 + \left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_5 \tag{35}$$

$$\left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_1 = \frac{\partial}{\partial k_\zeta}\left[2\frac{k_\zeta}{k_n^2}\right] = \frac{2}{k_n^2} - 4\frac{k_\zeta}{k_n^3}\frac{\partial k_n}{\partial k_\zeta} \tag{36}$$

$$\left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_2 = \frac{\partial}{\partial k_\zeta}\left[-2\frac{k_\zeta^2}{k_n^3}\frac{\partial k_n}{\partial k_\zeta}\right] \tag{37}$$

$$= -4\frac{k_\zeta}{k_n^3}\frac{\partial k_n}{\partial k_\zeta} + 6\frac{k_\zeta^2}{k_n^4}\left(\frac{\partial k_n}{\partial k_\zeta}\right)^2 - 2\frac{k_\zeta^2}{k_n^3}\frac{\partial^2 k_n}{\partial k_\zeta^2} \tag{38}$$

$$\left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_3 = \frac{\partial}{\partial k_\zeta}\left[-4\frac{k_\zeta^3}{k_n^4}F_n B\right] \tag{39}$$

$$= -12\frac{k_\zeta^2}{k_n^4}F_n B + 16\frac{k_\zeta^3}{k_n^5}F_n B\frac{\partial k_n}{\partial k_\zeta} - 4\frac{k_\zeta^3}{k_n^4}B\frac{\partial F_n}{\partial k_\zeta} \tag{40}$$

$$\left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_4 = \frac{\partial}{\partial k_\zeta}\left[4\frac{k_\zeta^4}{k_n^5}F_n B\frac{\partial k_n}{\partial k_\zeta}\right] \tag{41}$$

$$= 16\frac{k_\zeta^3}{k_n^5}F_n B\frac{\partial k_n}{\partial k_\zeta} - 20\frac{k_\zeta^4}{k_n^6}F_n B\left(\frac{\partial k_n}{\partial k_\zeta}\right)^2 + 4\frac{k_\zeta^4}{k_n^5}B\frac{\partial k_n}{\partial k_\zeta}\frac{\partial F_n}{\partial k_\zeta} + 4\frac{k_\zeta^4}{k_n^5}F_n B\frac{\partial^2 k_n}{\partial k_\zeta^2} \tag{42}$$

$$\left.\frac{\partial^2 P_{nn}}{\partial k_\zeta^2}\right|_5 = \frac{\partial}{\partial k_\zeta}\left[-\frac{k_\zeta^4}{k_n^4}B\frac{\partial F_n}{\partial k_\zeta}\right] \tag{43}$$

$$= -4\frac{k_\zeta^3}{k_n^4}B\frac{\partial F_n}{\partial k_\zeta} + 4\frac{k_\zeta^4}{k_n^5}B\frac{\partial F_n}{\partial k_\zeta}\frac{\partial k_n}{\partial k_\zeta} - \frac{k_\zeta^4}{k_n^4}B\frac{\partial^2 F_n}{\partial k_\zeta^2}. \tag{44}$$

To conclude, we calculate $\frac{\partial^2 F_n}{\partial k_\zeta^2}$,

$$\frac{\partial^2 F_n}{\partial k_\zeta^2} = \left.\frac{\partial^2 F_n}{\partial k_\zeta^2}\right|_1 + \left.\frac{\partial^2 F_n}{\partial k_\zeta^2}\right|_2 + \left.\frac{\partial^2 F_n}{\partial k_\zeta^2}\right|_3 \tag{45}$$

$$\left.\frac{\partial^2 F_n}{\partial k_\zeta^2}\right|_1 = \frac{\partial}{\partial k_\zeta}\left[-2\frac{1}{k_\zeta^2 L}\right] = 4\frac{1}{k_\zeta^3 L} \tag{46}$$

$$\left.\frac{\partial^2 F_n}{\partial k_\zeta^2}\right|_2 = \frac{\partial}{\partial k_\zeta}\left[2(-1)^n \frac{1}{k_\zeta^2 L} e^{-k_\zeta L}\right] \tag{47}$$

$$= -4(-1)^n \frac{1}{k_\zeta^3 L} e^{-k_\zeta L} - 2(-1)^n \frac{1}{k_\zeta^2} e^{-k_\zeta L} \tag{48}$$

$$\left.\frac{\partial^2 F_n}{\partial k_\zeta^2}\right|_3 = \frac{\partial}{\partial k_\zeta}\left[2(-1)^n \frac{1}{k_\zeta} e^{-k_\zeta L}\right] \tag{49}$$

$$= -2(-1)^n \frac{1}{k_\zeta^2} e^{-k_\zeta L} - 2L(-1)^n \frac{1}{k_\zeta} e^{-k_\zeta L}. \tag{50}$$

Note the derivative of k_n with respect to k_ζ,

$$\frac{\partial k_n}{\partial k_\zeta} = \frac{k_\zeta}{k_n} \tag{51}$$

$$\frac{\partial^2 k_n}{\partial k_\zeta^2} = \frac{1}{k_n} - \frac{k_\zeta^2}{k_n^3}. \tag{52}$$

Now, we calculate $\frac{\partial^2 \omega_n}{\partial \phi^2}$,

$$\frac{\partial^2 \omega_n}{\partial \phi^2} = \frac{1}{2\omega_n}\frac{\partial^2 \omega_n^2}{\partial \phi^2} - \frac{1}{\omega_n}\left(\frac{\partial \omega_n}{\partial \phi}\right)^2 \tag{53}$$

$$\frac{\partial^2 \omega_n}{\partial \phi^2} = R\omega_M \frac{\partial^2 F_{nn}}{\partial \phi^2} \tag{54}$$

$$\frac{\partial^2 F_{nn}}{\partial \phi^2} = 2\cos 2\phi \left(P_{nn}\sin^2\theta + \frac{\omega_M P_{nn}(1-P_{nn})\sin^2\theta}{R}\right). \tag{55}$$

Finally the mixed-derivatives,

$$\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} = \frac{\partial^2 \omega_n}{\partial \phi \partial k_\zeta} = \frac{\partial}{\partial k_\zeta}\frac{\partial \omega_n}{\partial \phi} = \frac{\partial}{\partial k_\zeta}\left[\frac{1}{2\omega_n}\frac{\partial \omega_n^2}{\partial \phi}\right] \tag{56}$$

$$= \frac{\partial}{\partial k_\zeta}\left[\frac{1}{2\omega_n} R P_{nn}\omega_M \sin 2\phi \sin^2\theta + \frac{1}{2\omega_n} R\omega_M^2 \sin 2\phi \frac{P_{nn}(1-P_{nn})\sin^2\theta}{R}\right] \tag{57}$$

$$\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi} = \left.\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi}\right|_1 + \left.\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi}\right|_2 \tag{58}$$

$$\left.\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi}\right|_1 = \frac{-T}{2\omega_n^2} R P_{nn}\frac{\partial \omega_n}{\partial k_\zeta} + \frac{T}{2\omega_n} P_{nn}\frac{\partial R}{\partial k_\zeta} + \frac{T}{2\omega_n} R\frac{\partial P_{nn}}{\partial k_\zeta} \tag{59}$$

$$\left.\frac{\partial^2 \omega_n}{\partial k_\zeta \partial \phi}\right|_2 = \frac{-U}{2\omega_n^2}\left[P_{nn} - P_{nn}^2\right]\frac{\partial \omega_n}{\partial k_\zeta} + \frac{U}{2\omega_n}\frac{\partial P_{nn}}{\partial k_\zeta}\left[1 - 2P_{nn}\right]. \tag{60}$$

The variables T and U have been defined for notational brevity:

$$T = \omega_M \sin 2\phi \sin^2\theta \tag{61}$$

$$U = \omega_M^2 \sin 2\phi \sin^2\theta. \tag{62}$$

Now we may calculate the magnon effective mass.