Lecture Notes: Mean Value Theorem and Taylor's Theorem

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1 Extrema and Derivatives

Theorem 1 (Fermat's Theorem on Local Extrema)**.** *Let f be defined on an open interval containing x_0. If f assumes its maximum (or minimum) at x_0 and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Assume f achieves a maximum at x_0. We proceed by contradiction.

Suppose $f'(x_0) > 0$. Then there exists $\delta > 0$ such that for $0 < |x - x_0| < \delta$:

$$\frac{f(x) - f(x_0)}{x - x_0} > 0$$

If $x > x_0$, then $f(x) > f(x_0)$, contradicting that $f(x_0)$ is a maximum.

Similarly, if $f'(x_0) < 0$, there exists $\delta > 0$ such that for $0 < |x - x_0| < \delta$:

$$\frac{f(x) - f(x_0)}{x - x_0} < 0$$

If $x < x_0$, then $f(x) > f(x_0)$, again contradicting the maximality of $f(x_0)$.

Therefore, we must have $f'(x_0) = 0$. □

2 Rolle's Theorem

Theorem 2 (Rolle's Theorem)**.** *Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

Proof. Since f is continuous on the closed interval $[a, b]$, by the Extreme Value Theorem, there exist points $x_0, x_1 \in [a, b]$ such that:

$$f(x_0) \leq f(x) \leq f(x_1) \quad \text{for all } x \in [a, b]$$

If both x_0 and x_1 are endpoints, then $f(x_0) = f(x_1) = f(a) = f(b)$, so f is constant and $f'(x) = 0$ for all $x \in (a, b)$.

Otherwise, at least one extremum occurs in the interior (a, b). By Fermat's Theorem, the derivative at this point is zero. □

3 Mean Value Theorem

Theorem 3 (Mean Value Theorem)**.** *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x_0 \in (a, b)$ such that:*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define the linear function $L(x)$ representing the secant line connecting $(a, f(a))$ and $(b, f(b))$:

$$L(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Now define $g(x) = f(x) - L(x)$. Then:

$$g(a) = f(a) - L(a) = 0, \quad g(b) = f(b) - L(b) = 0$$

Since $g(a) = g(b) = 0$ and g is continuous on $[a, b]$ and differentiable on (a, b), by Rolle's Theorem there exists $x_0 \in (a, b)$ such that $g'(x_0) = 0$.

But:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

So:

$$g'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{b - a} = 0$$

which gives the desired result. □

Corollary 1. *If f is differentiable on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).*

4 Taylor's Theorem

Theorem 4 (Taylor's Theorem with Lagrange Remainder)**.** *Let f be a real function on (a, b) such that $f^{(k)}(t)$ exists for $k = 1, 2, \ldots, n$ for all $t \in (a, b)$. Let α, β be distinct points in $[a, b]$ and define the Taylor polynomial:*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

Then there exists ξ between α and β such that:

$$f(\beta) = P(\beta) + \frac{f^{(n)}(\xi)}{n!}(\beta-\alpha)^n$$

Proof. Let M be the number defined by:

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n$$

Define the function:

$$g(t) = f(t) - P(t) - M(t-\alpha)^n$$

We need to show that $n!M = f^{(n)}(\xi)$ for some ξ between α and β. Since:

$$g^{(n)}(t) = f^{(n)}(t) - n!M$$

it suffices to show that there exists ξ such that $g^{(n)}(\xi) = 0$.

Note that:

$$g^{(k)}(\alpha) = 0 \quad \text{for } k = 0, 1, \ldots, n-1, \quad \text{and} \quad g(\beta) = 0$$

By repeated application of Rolle's Theorem:

- Since $g(\alpha) = g(\beta) = 0$, there exists ξ_1 between α and β such that $g'(\xi_1) = 0$
- Since $g'(\alpha) = g'(\xi_1) = 0$, there exists ξ_2 between α and ξ_1 such that $g''(\xi_2) = 0$
- Continuing this process, after n steps we find $\xi = \xi_n$ between α and ξ_{n-1} such that $g^{(n)}(\xi) = 0$

This completes the proof. □