2. Rational Numbers

1 From Natural Numbers to Rational Numbers

Definition 1.1. The set of integers $\mathbb{Z}$ is obtained from $\mathbb{N}$ by including additive inverses and zero:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

This ensures that subtraction is always defined within $\mathbb{Z}$.

Definition 1.2. The set of rational numbers $\mathbb{Q}$ is defined as:

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z},\ n \neq 0 \right\} / \sim$$

where $\frac{m_1}{n_1} \sim \frac{m_2}{n_2}$ if and only if $m_1 n_2 = m_2 n_1$.

2 Algebraic Numbers and Irrationality

Definition 2.1. A number α is called *algebraic* if it satisfies a polynomial equation:

$$c_n \alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1 \alpha + c_0 = 0$$

where $c_0, c_1, \dots, c_n \in \mathbb{Z}$, $c_n \neq 0$, and $n \geq 1$.

Theorem 2.2 (Rational Root Theorem)**.** *Let $c_0, c_1, \dots, c_n \in \mathbb{Z}$ with $c_n \neq 0$ and $c_0 \neq 0$. Suppose $\gamma \in \mathbb{Q}$ satisfies:*

$$c_n \gamma^n + c_{n-1}\gamma^{n-1} + \cdots + c_1 \gamma + c_0 = 0$$

Write $\gamma = \frac{c}{d}$ in lowest terms (i.e., $\gcd(c, d) = 1$). Then c divides c_0 and d divides c_n.

Proof. Substituting $\gamma = \frac{c}{d}$ into the equation and multiplying through by d^n gives:

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0$$

Since d divides the left-hand side and $\gcd(d, c) = 1$, we conclude $d \mid c_n$. Similarly, since c divides the left-hand side and $\gcd(c, d) = 1$, we have $c \mid c_0$. □

Corollary 2.3. *Any rational solution to the monic polynomial equation:*

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0 \quad \textit{with } c_0, \dots, c_{n-1} \in \mathbb{Z},\ c_0 \neq 0$$

must be an integer that divides c_0.

Example 2.4 (Irrationality of $\sqrt{2}$)**.** The number $\sqrt{2}$ is irrational.

Proof. Suppose $\sqrt{2} = \frac{c}{d}$ in lowest terms. Then $c^2 = 2d^2$, so c is even. Write $c = 2k$, then $4k^2 = 2d^2$, so $d^2 = 2k^2$, making d even as well. This contradicts $\gcd(c, d) = 1$. □

Example 2.5 (Irrationality of $6^{1/3}$)**.** The cube root of 6 is irrational.

Proof. $\sqrt[3]{6}$ satisfies $x^3 - 6 = 0$. By the Rational Root Theorem, any rational root must be an integer dividing 6. Checking $\pm 1, \pm 2, \pm 3, \pm 6$ shows none satisfy the equation. □

Example 2.6 (Irrationality of Nested Radical)**.** The number $\sqrt{2 + \sqrt[3]{5}}$ is irrational.

Proof. Let $\gamma = \sqrt{2 + \sqrt[3]{5}}$. Then $\gamma^2 = 2 + \sqrt[3]{5}$, so $(\gamma^2 - 2)^3 = 5$. Expanding gives:

$$\gamma^6 - 6\gamma^4 + 12\gamma^2 - 13 = 0$$

By the Rational Root Theorem, any rational root must be an integer dividing 13. Checking $\pm 1, \pm 13$ shows none satisfy the equation. □

Homework Problems