1 Uniform Continuity

Definition 1. *Let $S \subseteq dom(f)$. The function f is* ***continuous on*** *S if for every $x_0 \in S$ and every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in S$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$.*

Here, δ may depend on both ϵ and x_0.

Definition 2. *A function f is* ***uniformly continuous on*** *S if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, x' \in S$ with $|x - x'| < \delta$, we have $|f(x) - f(x')| < \epsilon$.*

Here, δ depends only on ϵ, not on the particular points in S.

Example. *The function $f(x) = \frac{1}{x}$ on $(0, 1)$ is continuous but not uniformly continuous.*

Theorem 1. *If f is continuous on $[a, b]$ (where $a, b \in \mathbb{R}$), then f is uniformly continuous on $[a, b]$.*

Proof. Suppose for contradiction that f is not uniformly continuous on $[a, b]$. Then there exists $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$, there exist $x_n, y_n \in [a, b]$ with:

$$|x_n - y_n| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

By Bolzano-Weierstrass, there exists a convergent subsequence $(x_{n_k}) \to x_0 \in [a, b]$. Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, we also have $y_{n_k} \to x_0$.

By continuity:

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0) = \lim_{k\to\infty} f(y_{n_k}).$$

But this contradicts $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$ for all k. □

2 Operations on Continuous Functions

Theorem 2 (Composition)**.** *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof. Let (x_n) be a sequence with $x_n \to x_0$. By continuity of f:

$$f(x_n) \to f(x_0).$$

By continuity of g:

$$g(f(x_n)) \to g(f(x_0)).$$

Thus $(g \circ f)(x_n) \to (g \circ f)(x_0)$, so $g \circ f$ is continuous at x_0. □

Theorem 3. *If f and g are continuous at x_0, then $\max(f, g)$ is continuous at x_0.*

Proof. We have the identity:

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|.$$

Since f and g are continuous at x_0, so are $f + g$ and $f - g$. The absolute value function is continuous, so $|f - g|$ is continuous. Therefore, $\max(f, g)$ is continuous at x_0. $\square$