Limits of Functions

Definition 1. *Let $S \subseteq \mathbb{R}$, $a \in \mathbb{R} \cup \{\pm\infty\}$, and suppose a is the limit of some sequence in S. Let $L \in \mathbb{R} \cup \{\pm\infty\}$. We write*

$$\lim_{x \to a^S} f(x) = L$$

if f is defined on S and for every sequence (x_n) in S with $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

This is read as: "the limit, as x tends to a along S, of $f(x)$."

Remark 1. *If $\mathrm{dom}(f) = S$, then f is continuous at a if and only if*

$$\lim_{x \to a^S} f(x) = f(a).$$

Remark 2. *If $\lim_{x\to a^S} f(x) = L$ exists, then it is unique.*

Definition 2. *Let $a \in \mathbb{R}$. We write*

$$\lim_{x \to a} f(x) = L$$

if there exists $\delta > 0$ such that for $S = (a-\delta, a) \cup (a, a+\delta)$, we have

$$\lim_{x \to a^S} f(x) = L.$$

(Note: f need not be defined at a.)

Definition 3 (One-sided limits)**.** *Let $a \in \mathbb{R}$.*

- *We write $\lim_{x\to a^-} f(x) = L$ if for some $\delta > 0$ and $S = (a-\delta, a)$, we have*
$$\lim_{x \to a^S} f(x) = L.$$
- *We write $\lim_{x\to a^+} f(x) = L$ if for some $\delta > 0$ and $S = (a, a+\delta)$, we have*
$$\lim_{x \to a^S} f(x) = L.$$

Definition 4 (Limits at infinity)**.** *We write* $\lim_{x\to\infty} f(x) = L$ *if for some* $c \in \mathbb{R}$ *and* $S = (c, +\infty)$*, we have*

$$\lim_{x\to\infty^S} f(x) = L.$$

Similarly, we define $\lim_{x\to-\infty} f(x) = L$*.*

Example 1.

$$\lim_{x\to 4} x^3 = 64,$$

since $x \mapsto x^3$ *is continuous. In fact, all polynomials are continuous.*

Example 2.

$$\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}.$$

Example 3.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4, \quad \textit{for } x \neq 2.$$

Example 4.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \frac{1}{2}.$$

Theorem 1 (Limit Laws)**.** *Suppose*

$$\lim_{x\to a^S} f_1(x) = L_1 \quad \textit{and} \quad \lim_{x\to a^S} f_2(x) = L_2.$$

Then:

1. $\lim_{x\to a^S}(f_1 + f_2)(x) = L_1 + L_2$
2. $\lim_{x\to a^S}(f_1 \cdot f_2)(x) = L_1 L_2$
3. *If* $L_2 \neq 0$ *and* $f_2(x) \neq 0$ *for* $x \in S$*, then*

$$\lim_{x\to a^S} \frac{f_1}{f_2}(x) = \frac{L_1}{L_2}.$$

Theorem 2 (Composition of Limits)**.** *Let f be defined on $S \subseteq \mathbb{R}$, and suppose*

$$L = \lim_{x \to a^S} f(x).$$

Let g be defined on $\{f(x) : x \in S\} \cup \{L\}$, and suppose g is continuous at L. Then

$$\lim_{x \to a^S} (g \circ f)(x) = g(L).$$

Proof. Let (x_n) be a sequence in S with $\lim_{n\to\infty} x_n = a$. Then

$$L = \lim_{n \to \infty} f(x_n).$$

Since g is continuous at L, we have

$$g(L) = \lim_{n\to\infty} g(f(x_n)) = \lim_{n\to\infty} (g \circ f)(x_n).$$

□

Example 5. *If $\lim_{x\to a} f(x) = L$ exists and is finite, then*

$$\lim_{x\to a} |f(x)| = |L|,$$

since $g(x) = |x|$ is continuous.

Theorem 3 (ε-δ Definition of Limit)**.** *Let f be defined on $S \subseteq \mathbb{R}$, and let $a \in \mathbb{R}$ be a limit point of S. Let $L \in \mathbb{R}$. Then*

$$\lim_{x \to a^S} f(x) = L$$

if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in S$,

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Corollary 1. *Let f be defined on (a, b). Then*

$$\lim_{x \to a^+} f(x) = L$$

if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (a, b)$,

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Corollary 2. *Let f be defined on $(a - \delta_1, a + \delta_1)$ for some $\delta_1 > 0$. Then*

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \quad \textit{and} \quad \lim_{x \to a^+} f(x) = L.$$