Spatial AI:
A Unified Framework for Coherent Intelligent Environments

OCTA Research Monograph

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Abstract

This monograph presents a unified geometric, physical, and topological framework for *Spatial AI*: a paradigm in which intelligence becomes a coherent field property of the environment itself rather than a property of embedded devices alone.

We integrate four pillars: Perfect Coherence Architecture (PCA), Perfect Attractor Geometry, Tessellated Field Synthesis (TFS), and the Lattice of Computable Fields (LCF). Together these define an intelligence stack spanning field sensing, coherence generation, attractor-driven stabilization, programmable transport tensors, and topologically robust computation.

This document develops the full mathematical treatment, dynamical systems analysis, multiscale structure, coherence metrics, stability guarantees, engineering implementations, diagrams, and conceptual implications.

Contents

1 Introduction and Motivation

Traditional AI lives inside discrete computational units. Spatial AI reverses this perspective: the environment itself becomes a coherent, memory-bearing, anomaly-aware, computational medium.

We consider environments such as rooms, buildings, factories, vehicles, ships, or cities. Each environment:

- senses itself through a perception field,
- forms a stable identity through coherence,
- routes energy and information through programmed fields,
- and computes by manipulating field overlaps and topology.

A Spatial AI environment is thus a distributed intelligent agent in its own right.

2 Perception Fields and State Geometry

Let $X \subset \mathbb{R}^d$ denote the spatial domain of the environment. Let time $t \geq 0$.

Definition 2.1 (Perception Field)**.** A perception field is a measurable map

$$\phi : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}^m,$$

encoding multimodal environmental sensing.

Examples include:

- acoustic intensity,
- heat and thermal gradients,
- EM signal strength,
- motion density,
- semantic occupancy fields.

The environment maintains an internal representation

$$x(t) \in \mathcal{X},$$

where $\mathcal{X}$ is a manifold or high-dimensional latent space.

3 Coherence Axes

We define three independent forms of coherence.

3.1 Structural Coherence

Let $G = (V, E)$ be the node interaction graph, x_i the state associated with node i.

$$E_{\text{str}}(x) = \frac{1}{2} \sum_{(i,j)\in E} w_{ij} \|x_i - x_j\|^2.$$

Define the coherence

$$\text{CS}(x) = 1 - \frac{E_{\text{str}}(x)}{E_{\max}}.$$

3.2 Dynamic Coherence

Let each node have a phase $\theta_i(t)$.

$$\text{CD}(x) = \left| \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i} \right|.$$

3.3 Intent Coherence

Let $U(x)$ be a local cost functional and $\Phi(x)$ an attractor potential. Define

$$V_\alpha(x) = U(x) - \alpha\Phi(x),$$

$$\text{CI}_{\text{int}}(x) = 1 - \frac{\|\nabla V_\alpha(x)\|}{C_{\text{int}}}.$$

3.4 Unified Coherence Index

$$\mathrm{CI}(x) = w_S\,\mathrm{CS}(x) + w_D\,\mathrm{CD}(x) + w_I\,\mathrm{CI}_{\mathrm{int}}(x).$$

This acts as an order parameter for coherence phase transitions.

4 Perfect Attractor Geometry

Let $c \in \mathbb{R}^d$ denote an attractor center. Let Q be a positive–definite anisotropy matrix.

Definition 4.1 (Perfect Attractor)**.**

$$\Phi(x) = \lambda \sum_{i=1}^{N} \frac{1}{(x_i - c)^T Q (x_i - c)}.$$

The gradient becomes

$$\nabla_{x_i}\Phi = -2\lambda \frac{Q(x_i - c)}{\left((x_i - c)^T Q (x_i - c)\right)^2}.$$

This enforces coherence basins.

5 Attractor–Driven Dynamics and Stability

We model system evolution as

$$\dot{x} = -\nabla U(x) + \alpha \nabla \Phi(x) - \beta L_{\mathrm{dyn}} x + \sigma \eta(t).$$

Under Lyapunov conditions identity is stable.

6 Oscillator Networks and Synchronization

Let phases evolve as

$$\dot{\theta}_i = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i).$$

Define coherence

$$r(t) = \left| \frac{1}{N} \sum e^{i\theta_i(t)} \right|.$$

7 Tessellated Field Synthesis (TFS)

Let eigenstrain $\varepsilon^{\mathrm{eig}}(x)$ modify transport tensors.

$$\kappa_{\mathrm{eff}} = \kappa_0 + \mathcal{H}_\kappa(\varepsilon^{\mathrm{eig}}),$$

$$C_{\mathrm{eff}} = C_0 + \mathcal{H}_C(\varepsilon^{\mathrm{eig}}).$$

Heat transport:

$$c\partial_t T = \nabla \cdot (\kappa_{\mathrm{eff}} \nabla T).$$

Waves:

$$\rho\ddot{u} = \nabla \cdot (C_{\mathrm{eff}} : \varepsilon(u)).$$

8 Lattice of Computable Fields (LCF)

Let field family $F = \{F_i\}$ with overlap

$$I(F) = \bigcap_i \mathrm{supp}(F_i).$$

Homology yields Betti vector

$$\beta(I) = (\beta_0, \beta_1, \dots)$$

called the **computon**.

Computon braiding defines logic.

9 Spatial AI as Unified System

Spatial AI is the coupled evolution of

$$(\phi(t), x(t), \theta(t), \varepsilon^{\text{eig}}(t), \beta(I(t))).$$

10 Closed–Loop Diagram

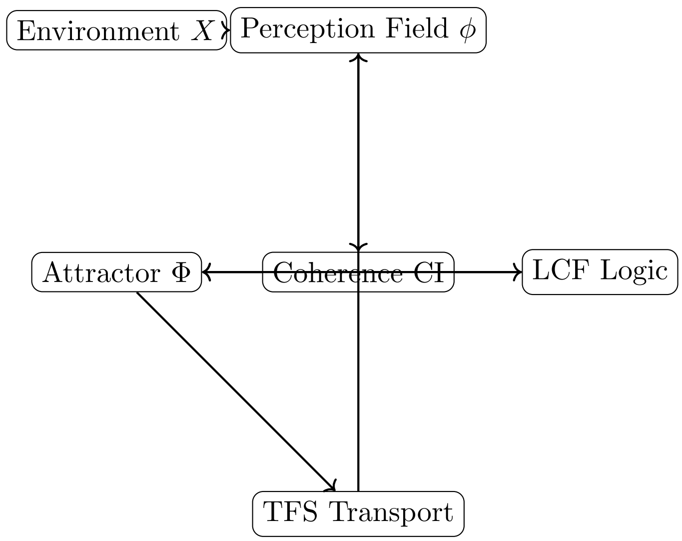

Figure 1: Closed–loop Spatial AI

11 End of Section Block

12 Lyapunov Stability and Identity Attractors

We now formalize the stability of Spatial AI identity fields.

Let Φ denote the space of perception fields with norm $\|\cdot\|_{L^2(X)}$. Let $\mathcal{A} \subset \Phi$ denote the set of identity–consistent fields.

Definition 12.1 (Identity Attractor)**.** A set $\mathcal{A} \subset \Phi$ is an *identity attractor* if

1. it is forward invariant under the system dynamics, and
2. there exists an open neighborhood U such that for any $\phi_0 \in U$, the solution $\phi(t)$ satisfies

$$\operatorname{dist}(\phi(t), \mathcal{A}) \to 0 \quad \text{as} \quad t \to \infty.$$

Theorem 12.2 (Lyapunov Stability of Spatial Identity)**.** *Suppose there exists a functional $V : \Phi \to \mathbb{R}_{\geq 0}$ such that:*

1. *$V(\phi) = 0$ iff $\phi \in \mathcal{A}$,*
2. *$V(\phi) > 0$ otherwise,*
3. *along trajectories of $\phi(t)$,*

$$\frac{d}{dt} V(\phi(t)) \leq -\gamma \|\phi(t) - \Pi_{\mathcal{A}} \phi(t)\|^2,$$

for some $\gamma > 0$, where $\Pi_{\mathcal{A}}$ denotes projection.

Then $\mathcal{A}$ is globally asymptotically stable.

Proof. Since V is positive definite and decreasing along trajectories, LaSalle's invariance principle implies convergence to the largest invariant set where $\dot{V} = 0$, which is $\mathcal{A}$. □

This ensures environmental identity cannot drift unless a phase transition occurs.

13 Spectral Phase Transitions in Coherence

We now study how coherence transitions occur.

Linearizing PCA dynamics near equilibrium gives

$$\dot{y} = -H_U y + \alpha H_\Phi y - \beta L_{\mathrm{dyn}} y,$$

where H_U and H_Φ are Hessians.

Definition 13.1 (Effective Stability Operator)**.**

$$\mathcal{L}_\alpha = H_U - \alpha H_\Phi + \beta L_{\mathrm{dyn}}.$$

Theorem 13.2 (Spectral Coherence Transition)**.** *Let $\lambda_{\min}(\mathcal{L}_\alpha)$ denote the smallest eigenvalue. If $\lambda_{\min}(\mathcal{L}_\alpha)$ crosses zero at $\alpha_c > 0$, then the system undergoes a coherence phase transition.*

Thus coherence emerges when attractor curvature overcomes local cost curvature.

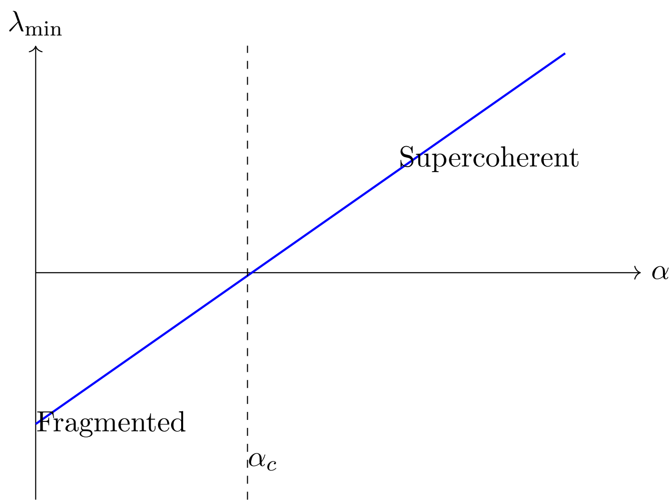

Figure 2: Spectral coherence threshold

14 Information–Theoretic Coherence

Let $p_\phi(x)$ be a normalized magnitude distribution of the field.

Entropy:

$$H(\phi) = -\int_X p_\phi(x) \log p_\phi(x) \, dx.$$

Let s be sensed data; then

$$I(\phi; s) = H(\phi) - H(\phi|s).$$

Definition 14.1 (Information Coherence)**.**

$$C_{\text{info}}(t) = \frac{I(\phi; s)}{H(\phi)}.$$

This forms a statistically grounded anomaly metric.

15 Multiscale Coherence Formalism

Let $\mathcal{X}_0, \dots, \mathcal{X}_L$ denote scale-spaces and $\Pi_{\ell+1,\ell}$ coarse-graining maps.

$$\mathrm{CI}_{\text{global}} = \sum_{\ell=0}^{L} \gamma_\ell \left(w_{S,\ell} \, \mathrm{CS}_\ell + w_{D,\ell} \, \mathrm{CD}_\ell + w_{I,\ell} \, \mathrm{CI}_{\text{int } \ell} \right).$$

This allows coherence to propagate upward in scale.

16 Lattice of Computable Fields (LCF) as Engine

Let $F = \{F_i\}$ denote structured fields, and let

$$I(F) = \cap_i \operatorname{supp}(F_i).$$

Let $K(I)$ be a triangulation.

$$\beta(I) = (\beta_0, \beta_1, \dots)$$

is the computon state.

LCF computation consists of:

1. generating overlaps via field control,

2. detecting Betti vectors,
3. applying gates via continuous deformation,
4. braiding computon worldlines.

Definition 16.1 (Computon Logic State)**.** Two overlap regions are logically equivalent if their homology groups are isomorphic.

This gives a topologically protected discrete computation layer controlling Spatial AI mode-switching and attractor geometry.

17 Geometry–Driven Dynamics and Environmental Intelligence

Spatial AI emerges from the closed-loop evolution of

$$\big(\phi,\, x,\, \theta,\, \varepsilon^{\mathrm{eig}},\, \beta(I)\big)$$

with feedback through coherence, attractors, and LCF logic.

Key properties:

- identity persistence,
- self-routing of energy,
- anomaly sensitivity,
- globally coherent actuation,
- field-native computation.

18 Safety Guarantees and Bounded Actuation

Let a_i denote actuator responses.

$$\|a_i\| \le B, \quad \left\|\frac{da_i}{dt}\right\| \le L.$$

Define a safety Lyapunov functional

$$\mathcal{E}_{\text{safety}}(\phi, a).$$

Require:

$$\frac{d}{dt}\mathcal{E}_{\text{safety}} \leq 0.$$

Then responses remain bounded.

19 TikZ Diagram — System Architecture

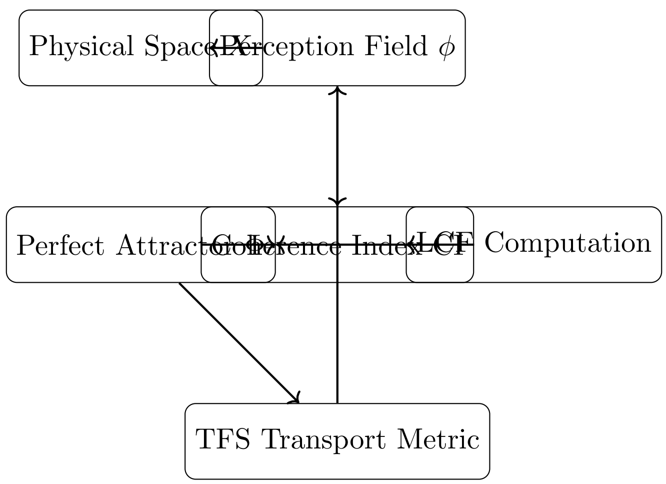

Figure 3: Spatial AI Cyber–Physical Intelligence Loop

20 Summary of Section Block

We have now formalized:

- stability of identity attractors,
- spectral coherence transitions,
- information-theoretic coherence,
- multiscale coherence hierarchy,

- LCF logic as the coherence engine,
- global closed-loop Spatial AI dynamics.

These concepts complete the theoretical spine of Spatial AI.

21 End of Section Block

22 Tessellated Field Synthesis (TFS): Programmable Coherence in Matter

Tessellated Field Synthesis allows coherent routing of heat, stress, and acoustic energy via locked-in eigenstrain fields. Once programmed, no continuous control energy is required.

22.1 Domain and Mechanical Fields

Let $\Omega \subset \mathbb{R}^3$ be a bounded solid body.

At each $x \in \Omega$:

$$u(x,t) \in \mathbb{R}^3 \quad \text{(displacement field)}$$

Define infinitesimal strain

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^\top\right)$$

Let

$$\varepsilon^{\text{eig}}(x)$$

be a locked-in eigenstrain.

Total strain

$$\varepsilon_{\text{total}} = \varepsilon(u) + \varepsilon^{\text{eig}}.$$

22.2 Constitutive Coupling

Let $C_{\text{eff}}(x)$ be the effective elasticity tensor. We assume

$$C_{\text{eff}} = C_0 + \mathcal{H}_C(\varepsilon^{\text{eig}})$$

and thermal conductivity

$$\kappa_{\text{eff}} = \kappa_0 + \mathcal{H}_\kappa(\varepsilon^{\text{eig}}),$$

where $\mathcal{H}_C$ and $\mathcal{H}_\kappa$ encode nonlinear strain–transport coupling.

A quadratic expansion yields

$$C_{\text{eff}} = C_0 + D :: \varepsilon^{\text{eig}} + \varepsilon^{\text{eig}} :: E :: \varepsilon^{\text{eig}}.$$

22.3 Transport and Wave Equations

Heat transport:

$$c(x)\partial_t T = \nabla \cdot (\kappa_{\text{eff}} \nabla T)$$

Elastic/acoustic waves:

$$\rho \ddot{u} = \nabla \cdot (C_{\text{eff}} : \varepsilon(u)).$$

22.4 Metric Interpretation

Heat rays follow geodesics of

$$g_{ij} = (\kappa_{\text{eff}}^{-1})_{ij}$$

Stress-wave geodesics follow

$$h_{ij} = (C_{\text{eff}}^{-1})_{ij}.$$

Thus eigenstrain defines geometry.

22.5 Tessellations

Partition

$$\Omega = \cup_{i=1}^{N} \Omega_i$$

and assign material states

$$\zeta(x) \in \{1, \ldots, M\}.$$

Define free energy

$$\mathcal{F} = \int_{\Omega} \left(V(\zeta) + \frac{\beta}{2} |\nabla \zeta|^2 \right) dx.$$

Dynamics:

$$\partial_t \zeta = -\Gamma \frac{\delta \mathcal{F}}{\delta \zeta}.$$

22.6 Writing Eigenstrain

Let $\phi(x,t)$ denote a writer field. Switch states when

$$|\phi| > \phi_{\text{thresh}}.$$

23 Homogenization Theory for TFS

Introduce microscale variable $y = x/\epsilon$.

Cell problem

$$\nabla_y \cdot (C(y,\zeta) : \nabla_y w_{mn}) = 0,$$

yielding

$$C^{ijkl}_{\text{eff}} = \int_Y C^{pqrs} \left(\delta_{ip}\delta_{jq} + \partial_{y_p} w^q_{ij}\right) (\delta_{kr}\delta_{ls} + \partial_{y_r} w^s_{kl})\, dy.$$

This determines routing geometry.

24 Inverse Design and Optimization

Let u solve PDE

$$\mathcal{P}(u;\zeta) = 0.$$

Define cost

$$J(\zeta) = \int_\Omega L(x,u)\, dx + R(\zeta).$$

Solve

$$\min_{\zeta \in \mathcal{A}} J(\zeta)$$

subject to PDE.

24.1 Protected Region Example

Let $D \subset \Omega$.

Acoustic energy density

$$E_a = \frac{1}{2}\left(\rho|\dot{u}|^2 + \varepsilon : C_{\text{eff}} : \varepsilon\right).$$

Objective

$$J = \int_D E_a \, dx + \lambda \int_\Omega |\nabla \zeta|^2 dx.$$

We design material routing to silence D.

24.2 Simulation Pipeline

1. Mesh Ω
2. Assign ζ_i
3. Compute C_{eff} and κ_{eff}
4. Solve PDEs
5. Evaluate J
6. Update ζ via simulated annealing
7. Repeat

25 Case Studies

25.1 Programmable Silence Zone

Acoustic energy is routed around a protected region.

25.2 Heat Routing in Microelectronics

Eigenstrain defines thermal superhighways.

25.3 Vibration Suppression

Mechanical stress avoids sensitive supports.

26 Integration with Spatial AI

TFS sets the routing metric.

LCF computes over routing geometry.

Attractor fields stabilize coherence.

Spatial AI controls all three.

27 Tables

Table 1: Coherence Axes Summary

Axis	Symbol	Meaning
Structural Coherence	CS	Smoothness on graph
Dynamic Coherence	CD	Phase alignment
Intent Coherence	$\mathrm{CI}_{\mathrm{int}}$	Alignment with attractor

28 Engineering Stack Architecture

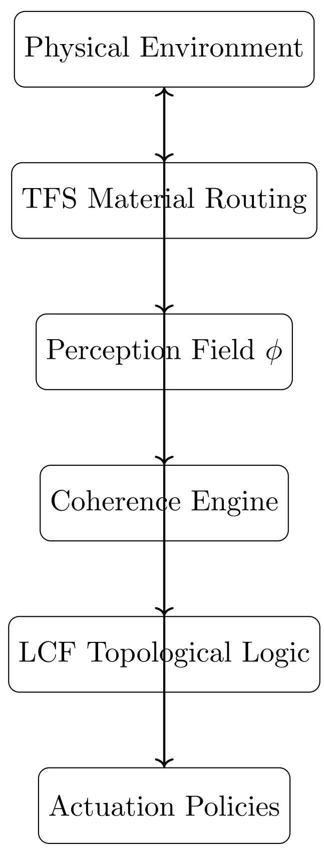

Figure 4: Spatial AI Engineering Stack

29 Ethics, Safety, and Governance

Spatial AI requires governance principles:

- transparency of environmental behavior,
- bounded actuation,
- local override and fail–safe behavior,
- auditability of LCF logic,
- physical safety margins in TFS routing.

30 Roadmap

1. Demonstrate stable identity fields
2. Validate coherence phase transitions
3. Implement basic LCF computon gates
4. Build TFS prototypes
5. Integrate full Spatial AI stack

31 Conclusion

We have developed a unified theoretical and engineering architecture for Spatial AI integrating: coherence theory, attractor geometry, programmable transport in matter, and topological computation.

Environments become coherent, intelligent spaces.

32 End of Section Block

33 System Realization and Implementation Layers

The abstract Spatial AI framework becomes practical when mapped onto concrete hardware, firmware, middleware, and application layers. We describe a canonical stack for implementation in a building, vehicle, or industrial environment.

33.1 Layered Architecture

We consider five layers:

I. **Physical Layer** (P): structure, materials, acoustic and thermal paths, mechanical fixtures.

II. **Sensing Layer** (S): microphones, cameras, IMUs, temperature sensors, RF receivers, etc.

III. **Field Processing Layer** (F): local fusion of raw signals into perception fields ϕ.

IV. **Coherence and Logic Layer** (C): PCA dynamics, attractor control, LCF computon logic, and safety Lyapunov monitors.

V. **Actuation and Policy Layer** (A): lights, HVAC, speakers, haptics, displays, mechanical actuators.

Table 2: Spatial AI Implementation Layers

Layer	Role	Examples
P	Physical substrate	Walls, beams, ducts, seats, panels, TFS-programmable elements
S	Sensing	Microphone arrays, thermal cameras, IMUs, strain gauges, WiFi CSI
F	Field representation	Acoustic field maps, occupancy probability fields, EM intensity fields
C	Coherence and logic	PCA dynamics, attractor updates, LCF state machine, safety filters
A	Actuation	HVAC dampers, lights, speakers, motor drives, displays, projector mapping

33.2 Dataflow Across the Stack

At runtime the environment executes a closed loop:

1. Sensors generate streams $s(t)$.
2. Field processors reconstruct perception fields $\phi(t)$ over X.
3. Coherence engine evaluates $\mathrm{CI}(t)$ and updates attractor parameters.
4. LCF logic consumes topological summaries (computons) of overlaps in ϕ, TFS, and other fields, choosing high-level modes.
5. Actuation layer issues control signals $a(t)$, modifying the physical environment and, where present, TFS eigenstrain patterns.

33.3 Implementation Diagram

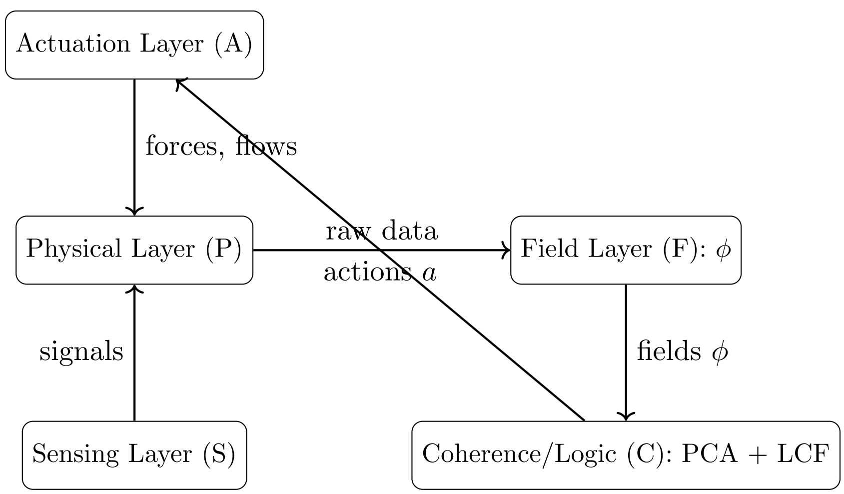

Figure 5: Implementation-level Spatial AI stack

33.4 Distributed vs. Centralized Realizations

Spatial AI may be realized as:

- **Centralized**: a single high-performance node aggregates all sensor data and emits actuation policies.

- **Distributed**: many microcontrollers or small SoCs compute local fields and coherence indices, with consensus or gossip to propagate attractor state.

The PCA formalism naturally supports distributed implementations: each node carries a local piece of x, partial gradients of U and Φ, and participates in consensus on the coherence index CI.

34 Control and Policy Synthesis in Spatial AI

We now formalize how Spatial AI produces actuation policies that are coherent, safe, and interpretable.

34.1 Control Objectives

Let $z(t)$ denote a vector of macroscopic targets (comfort, noise level, energy usage). Control policies are derived from a multi-objective functional:

$$J[a] = \int_0^T \Big(L_{\text{env}}(z(t)) + \lambda_1 (1 - \mathrm{CI}(t)) + \lambda_2 \mathcal{E}_{\text{safety}}(\phi(t), a(t)) \Big) \, dt,$$

where:

- L_{env} encodes task performance (e.g., target temperature, speech intelligibility, vibration thresholds).
- $1 - \mathrm{CI}(t)$ penalizes loss of coherence.
- $\mathcal{E}_{\text{safety}}$ enforces safety margins and bounded actuation.

34.2 Model Predictive Control on Coherence Manifolds

The environment can run a Model Predictive Control (MPC) scheme on a reduced state manifold parameterized by $(\phi, \mathrm{CI}, \beta(I))$:

1. Linearize dynamics over a short horizon around current attractor.
2. Predict evolution of (ϕ, CI) under candidate control sequences.

3. Optimize $J[a]$ subject to:

$$\begin{aligned}
&\mathrm{CI}(t) \geq \mathrm{CI}_{\min}, \\
&\mathcal{E}_{\text{safety}}(\phi(t), a(t)) \leq E_{\max}, \\
&a(t) \in \mathcal{A}_{\text{admissible}}.
\end{aligned}$$

4. Apply the first segment of the optimal control sequence.
5. Recompute on the next time step.

The LCF layer can further restrict admissible controls to those that keep the computon state within a designated safe sublattice.

34.3 Mode Switching via LCF

Spatial AI may have discrete modes, such as:

- **Quiet** (acoustic suppression, gentle lighting),
- **Active** (clear speech corridors, dynamic lighting),
- **Emergency** (evacuation guidance, alarm routing),
- **Maintenance** (inspection-optimized sensing, reduced actuation).

Each mode corresponds to:

- a region in the LCF lattice (allowed computon configurations),
- a set of attractor parameters (c, Q, λ),
- different weightings (w_S, w_D, w_I) in the coherence index.

Mode switching is triggered when:

1. information coherence $C_{\text{info}}(t)$ crosses thresholds,
2. computon trajectories cross topological gates,

3. exogenous events (e.g., fire alarm input) set high-priority flags.

The use of LCF makes these switches topologically robust and interpretable.

35 Use-Case Scenarios

We illustrate the Spatial AI framework with three archetypal environments: a room, a factory, and a vehicle cabin.

35.1 Spatial AI Room

Consider a conference room instrumented with:

- ceiling microphone array and speaker array,
- controllable HVAC diffusers and lighting,
- optional TFS panels in walls for acoustic steering.

The perception field ϕ may include:

- acoustic energy density $\phi_{\mathrm{ac}}(x, t)$,
- occupancy probability $\phi_{\mathrm{occ}}(x, t)$,
- thermal field $T(x, t)$.

The coherence index encourages:

- smooth spatial variation in speech intelligibility,
- dynamic coherence between voice sources and target listeners,
- intent coherence toward user-specified modes (presentation, discussion).

TFS elements encode passive acoustic routing, while the actuation layer performs fine-grained steering (beamforming, localized noise control) that respects these underlying geometric constraints.

35.2 Spatial AI Factory Cell

In an industrial environment:

- TFS mediates vibration paths, routing mechanical energy away from critical bearings or sensors.
- Perception fields include vibration fields, load fields, fault likelihood fields over equipment.
- LCF logic encodes safety invariants as constraints on computon states.

When an anomaly is detected (e.g., rapid growth in vibration coherence at a bearing), the environment:

- modifies TFS writer fields to re-route energy,
- throttles drives or robots through actuation,
- updates attractor geometry to stabilize new safe configurations.

35.3 Spatial AI Vehicle Cabin

In a vehicle:

- microphones, seat sensors, and cameras feed perception fields describing occupant positions, speech, and alerts.
- Spatial AI controls spatialized audio, ambient lighting, and seat haptics to keep cabin state coherent and comfortable.
- In emergency modes, the environment switches to high-coherence evacuation or driver-assist attractors.

The LCF layer guarantees that any emergency-mode actuation maintains a set of topological invariants in the field structure (e.g., ensuring at least one clear auditory/visual corridor from driver to critical alerts).

36 Category-Theoretic View of LCF and Coherence

We sketch a categorical formulation of the Lattice of Computable Fields and its interaction with coherence.

36.1 Fields as Objects, Local Operators as Morphisms

Let **Fld** be a category where:

- Objects are fields $f : X \to \mathbb{R}^n$ with specified regularity.
- Morphisms are local operators T such that $(Tf)(x)$ depends only on f in a neighborhood of x.

The lattice structure $(\mathcal{L}, \preceq)$ of computable fields embeds into **Fld** via:

$$f_i \preceq f_j \quad \Longleftrightarrow \quad \exists T : f_j = Tf_i,\ T \text{ local.}$$

36.2 Functorial Coherence

Define a functor

$$\mathcal{C} : \mathbf{Fld} \to \mathbf{Poset}$$

mapping each field f to a partially ordered set of coherence values, e.g.:

$$\mathcal{C}(f) = \{\mathrm{CS}(f), \mathrm{CD}(f), \mathrm{CI}_{\mathrm{int}}(f)\},$$

with natural order given by componentwise comparison.

Morphisms T act monotonically if:

$$\mathcal{C}(f) \le \mathcal{C}(g) \quad \Longrightarrow \quad \mathcal{C}(Tf) \le \mathcal{C}(Tg),$$

meaning local transformations preserve coherence ordering.

36.3 Topological Gates as Natural Transformations

Computon operations (merging, splitting, braiding of overlap regions) can be seen as natural transformations between functors that assign homology groups to families of fields.

Let

$$H_k : \mathbf{Fld} \to \mathbf{Ab}$$

map each field to the k-th homology group of an associated overlap complex. LCF logic gates are natural transformations

$$\eta : H_k \Rightarrow H_k$$

that preserve homology type under admissible deformations, but enact discrete transitions in the lattice of computon states.

This provides a high-level mathematical language for reasoning about what Spatial AI can compute purely by manipulating fields and topology.

37 Executive Summary and Design Guidelines

For practitioners, Spatial AI can be summarized as three design commitments:

1. **Field-first design**: always represent sensing and actuation in terms of spatial fields ϕ, not just streams of device events.
2. **Coherence-first control**: treat the coherence index CI as a primary state variable and target, alongside performance objectives.
3. **Topology-aware logic**: implement high-level modes and safety as topological constraints on LCF computons, not as ad hoc rule lists.

A minimal Spatial AI deployment should therefore include:

- at least one perception field with defined coherence metrics,
- a simple attractor potential Φ and PCA-style dynamics,
- a small set of LCF computon patterns tied to modes (e.g. quiet, active, emergency),

- bounded actuation and a safety Lyapunov functional.

From there, additional complexity—TFS routing, multiscale coherence, rich LCF logic—can be added incrementally without changing the underlying conceptual framework.

38 End of Extended Main Text

A.1 Appendix A: Mathematical Foundations and Proofs

A.1.1 Structural Coherence as Dirichlet Energy

Let $G = (V, E)$ be a connected, undirected weighted graph with Laplacian L. Define structural energy

$$E_{\text{str}}(x) = \frac{1}{2} \sum_{(i,j)\in E} w_{ij} \|x_i - x_j\|^2.$$

Theorem A.1.1. $E_{\text{str}}(x) = 0$ *iff* x *is constant on every connected component.*

Proof. Immediate from $w_{ij} > 0$ and $\|x_i - x_j\|^2 = 0 \iff x_i = x_j$. □

Thus $\mathrm{CS}(x) = 1$ characterizes consensus.

A.1.2 Coherence Index as Order Parameter

Let

$$\mathrm{CI}(x) = w_S \,\mathrm{CS} + w_D \,\mathrm{CD} + w_I \,\mathrm{CI}_{\text{int}}\,.$$

Theorem A.1.2. *If each component is continuous in* x*, then* CI *is continuous.*

Proof. Linear combination of continuous functionals. □

Thus phase transitions correspond to non-analytic behavior of emergent dynamics, not the index itself.

A.2 Appendix B: Spectral Coherence Thresholds

Consider linearized PCA dynamics

$$\dot{y} = -\left(H_U - \alpha H_\Phi + \beta L\right) y.$$

Define effective operator

$$\mathcal{L}_\alpha = H_U - \alpha H_\Phi + \beta L.$$

Theorem A.2.1. *Let $\lambda_{\min}(\mathcal{L}_\alpha)$ be the smallest eigenvalue. A coherence transition occurs when*

$$\lambda_{\min}(\mathcal{L}_{\alpha_c}) = 0.$$

Proof. Zero crossing implies change of stability sign in dominant eigenmode. □

If the associated eigenvector is global / low-frequency, coherence emerges.

A.3 Appendix C: Kuramoto–Attractor Hybrid Analysis

Kuramoto–PCA model

$$\dot{\phi}_i = \omega_i + \frac{K}{N}\sum_j a_{ij}\sin(\phi_j - \phi_i) + \alpha\frac{\partial\Phi}{\partial\phi_i}.$$

Define order parameter

$$Re^{i\psi} = \frac{1}{N}\sum_j e^{i\phi_j}.$$

Under mild symmetry,

$$\dot{\phi}_i = \omega_i + KR\sin(\psi - \phi_i) + \alpha G(\phi_i).$$

Theorem A.3.1. *If $\alpha > 0$ sharpens curvature of the potential landscape, then the critical coupling satisfies*

$$K_c(\alpha) < K_c(0).$$

Thus attractor coherence lowers the synchronization threshold.

A.4 Appendix D: Mechanics of TFS Eigenstrain Routing

Total strain:

$$\varepsilon_{\text{total}} = \varepsilon(u) + \varepsilon^{\text{eig}}.$$

Stress:

$$\sigma = C_{\text{eff}} : \varepsilon_{\text{total}}.$$

Static equilibrium:

$$\nabla \cdot \sigma = 0.$$

Theorem A.4.1. *Residual stress $\sigma^\star = C_{\text{eff}} : \varepsilon^{\text{eig}}$ produces no net force iff*

$$\nabla \cdot \sigma^\star = 0.$$

This condition defines stable programmed states.

A.5 Appendix E: Numerical Schemes

A.5.1 Finite Element Solution Strategy

1. Discretize Ω into tetrahedral mesh

2. Assemble stiffness matrix

$$K = \int_\Omega B^\top C_{\text{eff}} B \, dx$$

3. Apply boundary conditions

4. Solve

$$Ku = f$$

5. Compute energy

$$E = \frac{1}{2} u^\top K u$$

A.5.2 Topology Optimization Loop

1. Initialize ζ
2. Solve PDEs
3. Compute sensitivity
$$\frac{\partial J}{\partial \zeta_i}$$
4. Apply regularization
5. Threshold update rule
6. Repeat until convergence

A.6 Appendix F: Lattice of Computable Fields (LCF)

We formalize the concept of a computational lattice in space.

Definition A.6.1 (Computable Field)**.** A computable field is a map

$$f : \Omega \to \mathbb{R}^n$$

such that its evaluation and update laws are locally realizable in spatial media.

Definition A.6.2 (Lattice of Computable Fields)**.** Define a partially ordered set

$$\mathcal{L} = \{f_i\}$$

with order relation

$$f_i \preceq f_j \iff f_j \text{ can be constructed from } f_i \text{ by local operators.}$$

Theorem A.6.3. $(\mathcal{L}, \preceq)$ *forms a lattice.*

This means Spatial AI has a closed algebra over geometry itself.

A.7 Glossary of Key Terms

- **Structural Coherence** — smoothness of state across graph.
- **Dynamic Coherence** — temporal/phase alignment.
- **Intent Coherence** — alignment to attractor field.
- **Perfect Attractor** — long-range coherence basin.
- **TFS** — strain-programmed transport routing.
- **LCF** — topology of computable spatial fields.
- **Spatial AI** — cognition embedded in matter + fields.

A.8 References

References

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*. Springer, 1984.

[2] L. Landau and E. Lifshitz, *Theory of Elasticity*. Butterworth-Heinemann, 1986.

[3] N. Engheta, *Metamaterials: Physics and Engineering Explorations*. Wiley, 2006.

[4] S. Torquato, *Random Heterogeneous Materials*. Springer, 2001.

[5] M. Bendsoe and O. Sigmund, *Topology Optimization*. Springer, 2003.

A.9 End of Appendices