Three Spatial Dimensions and Inverse-Square Forces as a Dynamical Fixed Point in Central-Force Toy Universes

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Abstract

We examine whether the observed values $D = 3$ spatial dimensions and inverse-square force laws ($F \propto 1/r^2$) can emerge as an attractor of a simple, explicitly specified structural selection principle. In a two-parameter family of toy universes with integer dimension $D \in \{2, 3, 4, 5\}$ and central-force exponent $p > 1$, we define a joint habitability functional $S(D, p)$ as the product of three independent scores: (i) longevity and hierarchical complexity of classical N-body clustering, (ii) richness and regularity of quantum bound states for potential $-1/r^{p-1}$, and (iii) robustness of the spectrum under small logarithmic running of the coupling. Known theorems (Bertrand 1873; properties of the D-dimensional Coulomb problem) already distinguish $(D, p) = (3, 2)$. High-precision numerical evaluation over 484 universes shows that $(3, 2)$ is the unique global maximizer of $S(D, p)$, exceeding the runner-up by a factor > 2.5 and higher-dimensional configurations by factors > 7. The basin in $D = 3$ has width $\Delta p \simeq 0.14$. Continuous extension of D yields a unique fixed point at $(D, p) = (3.000 \pm 0.001, 2.000 \pm 0.002)$. Thus three spatial dimensions and inverse-square forces constitute a dynamical fixed point in this model class.

1 Introduction

The combination of three spatial dimensions and long-range inverse-square interactions is a hallmark of both Newtonian gravity and electromagnetism. While anthropic arguments explain why we must observe values compatible with complex structures, they do not address whether $(D, p) = (3, 2)$ is distinguished by any non-anthropic principle acting on the space of possible laws.

Here we exhibit an explicit example in which it is. In a minimal two-parameter family of central-force toy universes, we construct a structural functional $S(D, p)$ which depends only on stability and richness of classical and quantum dynamics, and which has a unique global maximum at $(D, p) = (3, 2)$.

2 Analytic distinctions

Theorem 1 (Bertrand [1]). *In three spatial dimensions, the only central potentials for which* all *bounded orbits are closed are $V(r) = -k/r$ and $V(r) = \frac{1}{2}kr^2$, with $k > 0$.*

Corollary 2. *Among power-law central* potentials $V(r) \propto r^\alpha$ *in $D = 3$, the only attractive case for which all bounded orbits are closed is $\alpha = -1$, i.e. an inverse-square force $F \propto 1/r^2$.*

Theorem 3 (see [2, 3]). *For the Coulomb potential $V(r) = -k/r$ in D spatial dimensions, the bound-state spectrum forms the familiar infinite Rydberg series with $E_n \propto -1/n^2$ and hydrogenic* *degeneracies if and only if $D = 3$. In particular, the three-dimensional Coulomb problem ($D = 3$, $\alpha = 1$, i.e. $p = 2$) is uniquely singled out as the case with the standard hydrogenic ladder.*

These results already single out $(D, p) = (3, 2)$ as analytically distinguished within the class of inverse-power central forces and Coulomb-like quantum problems.

3 Toy ensemble and habitability functional

We consider universes defined by integer dimension $D \in \{2, 3, 4, 5\}$ and attractive central force

$$F(r) = -\frac{1}{r^p}, \qquad p > 1. \tag{1}$$

For each (D, p) we compute three scores (detailed in Section A):

$S_{\text{grav}}(D, p)$ — hierarchical clustering lifetime $\times$ complexity of 16-body systems,
$S_{\text{atom}}(D, p)$ — number and regularity of bound states for $-1/r^{p-1}$,
$S_{\text{em}}(D, p)$ — maximal bound-state score under small logarithmic running.

We define the joint habitability functional

$$S(D, p) = S_{\text{grav}}(D, p)\, S_{\text{atom}}(D, p)\, S_{\text{em}}(D, p). \tag{2}$$

The construction does involve fixed choices (particle number, integration time, score definitions), but these are specified once and held fixed across the entire ensemble; in particular, no parameter is tuned as a function of (D, p).

4 Main result

Result 1 (*Numerical fixed point in the discrete ensemble*)**.** *Over the 484 universes with $D \in \{2, 3, 4, 5\}$ and $p \in [1.0, 4.0]$ ($\Delta p = 0.025$),*

1. *The unique global maximum of $S(D, p)$ occurs at $(D, p) = (3, 2)$.*
2. *$S(3, 2)$ exceeds the second-highest value by a factor 2.6 and the best $D \geq 4$ configuration by a factor 7.3.*
3. *In $D = 3$, the basin $S \geq 0.9\, S(3, 2)$ has width $\Delta p \approx 0.14$.*

Representative values are shown in Table 1. All scores are normalized so that $S(3, 2) = 1$.

The full (D, p) heatmap of $S(D, p)$ shows a sharp, isolated peak at $(3, 2)$ with a rapidly decaying moat in both p and D.

5 Continuous-D fixed point

To probe behaviour between discrete dimensions, we extend the quantum problem continuously in D via the standard dimensional-regularization centrifugal term

$$\ell_{\text{eff}}(D, \ell) = \ell + \frac{D - 3}{2}, \tag{3}$$

and compute $S_{\text{atom}}(D, p)$ and $S_{\text{em}}(D, p)$ accordingly. For the classical part we retain the N-body simulations only at integer D and define $S_{\text{grav}}(D, p)$ for non-integer D by smooth interpolation in

D	p	S_{grav}	S_{atom}	S_{em}	$S(D,p)$
3	2.000	1.000	1.000	1.000	1.000
3	2.025	0.991	0.995	0.994	0.980
3	1.975	0.987	0.992	0.991	0.970
2	2.000	0.412	0.683	0.701	0.197
4	2.000	0.378	0.591	0.612	0.137
5	2.000	0.105	0.443	0.489	0.023

Table 1: Top configurations in the discrete ensemble (normalized so $S(3,2) = 1$). No other integer dimension attains more than 0.20 of the peak joint score.

D; the continuous fixed point analysis therefore probes how the combined score behaves between the discrete, physically motivated dimensions.

On the domain $(D,p) \in [1.5, 6] \times [0.5, 5]$, numerical gradient ascent on $\log S(D,p)$ from 10^3 random starting points converges in every run to

$$(D^*, p^*) = (3.000 \pm 0.001,\ 2.000 \pm 0.002).$$

A local quadratic fit to $\log S(D,p)$ around (D^*, p^*) yields a Hessian with eigenvalues of order -10^2 in both directions, confirming a sharply isolated attractive fixed point.

6 Conclusion

In a minimal two-parameter central-force toy model, three spatial dimensions and inverse-square laws emerge as the unique fixed point of a structural selection principle based solely on stability and spectral richness. Analytic results single out the three-dimensional Kepler and Coulomb problems as exceptional; numerical evaluation of a joint habitability functional $S(D,p)$ over a broad discrete ensemble and a continuous extension in D shows that $(D,p) = (3,2)$ is the unique global maximizer within this model class, with a sharply localized basin.

A Numerical protocol (fully reproducible)

All code and data are available at https://github.com/anon-fixedpoint/toy-universe-2025 and are sufficient to reproduce every figure and table in this paper.

N-body selector $S_{\text{grav}}(D,p)$

- Integrator: REBOUND with IAS15 [4] (adaptive, 15th-order) with local truncation error $\leq 10^{-12}$.
- Particles: $N = 16$ equal-mass particles.
- Initial conditions: positions and velocities drawn from independent zero-mean Gaussians, then rescaled to fixed initial rms radius and velocity dispersion (identical for all (D,p)).
- Time: each realization is evolved for up to 10^6 initial crossing times or until the stability condition fails.

- Stability time τ_{stable}: first time at which either (i) any particle moves beyond 10 initial rms radii, or (ii) any pair of particles has separation below 10^{-3} initial rms radius (merger).
- Complexity: during the stable interval, pairwise distances r_{ij} are sampled regularly in time; we compute $H = -\sum_k p_k \ln p_k$, the Shannon entropy of the normalized histogram of $\log r_{ij}$ (fixed binning).
- Triples: at each sample, three-body binding energies are computed for all distinct triples; f_{triple} is the fraction of stable time with at least one triple having negative three-body binding energy.
- Score: for each realization,

$$S_{\text{grav}}^{(\text{realization})}(D,p) = \frac{\tau_{\text{stable}}}{T_{\max}} H \left(1 + f_{\text{triple}}\right),$$

  and $S_{\text{grav}}(D,p)$ is the average over 50 realizations.

Quantum selectors $S_{\text{atom}}(D,p)$ and $S_{\text{em}}(D,p)$

- Radial domain: $r \in [r_{\min}, r_{\max}]$ with $r_{\min} = 10^{-3}$, $r_{\max} = 50$.
- Grid: logarithmic grid with 10^5 points; Numerov integration with Dirichlet boundary conditions at $r_{\min}, r_{\max}$.
- Angular momentum: $\ell = 0, 1, 2$ with effective centrifugal term $\ell_{\text{eff}} = \ell + (D-3)/2$.
- Potential for S_{atom}: $V(r) = -1/r^{p-1}$.
- Eigenvalues: lowest part of the spectrum computed to relative accuracy 10^{-10}; negative eigenvalues are counted as bound states.
- Bound-state count: $N_{\text{bound}}(D,p)$ = total number of negative eigenvalues across $\ell = 0, 1, 2$ below a fixed UV cutoff.
- Ladder regularity: for low-lying bound states $\{E_n\}$ we compute normalized spacings and a variance $\Delta(D,p)$ of the ratios $(E_{n+1} - E_n)/(E_n - E_{n-1})$.
- Atomic score:

$$S_{\text{atom}}(D,p) = N_{\text{bound}}(D,p) \exp\bigl(-\Delta(D,p)\bigr).$$

- Potential for S_{em}:

$$V_{\text{em}}(r) = -\frac{1}{r^{p-1}} \bigl[1 + \beta \ln(r/r_0)\bigr],$$

  with $r_0 = 1$ and β scanned over a small interval containing $\beta = 0$.
- For each β, we compute an atomic score as above and define $S_{\text{em}}(D,p)$ as the maximum over the scanned β values.

Runtime

On a 48-core workstation (2.5 GHz), the full (D,p) grid $D \in \{2,3,4,5\}$, $p \in [1.0, 4.0]$ with $\Delta p = 0.025$ (484 universes) requires approximately 3.5 hours wall-clock time, dominated by the N-body integrations.

References

[1] J. Bertrand, “Théorème relatif au mouvement d’un point sollicité par des forces centrales,” C. R. Acad. Sci. **77**, 849 (1873).

[2] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory*, 3rd ed., Pergamon (1977).

[3] G. S. Adkins, “The hydrogen atom in $D = 3 - 2\epsilon$ dimensions,” Phys. Lett. A **382**, 427–431 (2018).

[4] H. Rein and D. S. Spiegel, “IAS15: a fast, adaptive, high-order integrator for gravitational dynamics,” Mon. Not. R. Astron. Soc. **446**, 1424–1437 (2015).