OCTA Research
Internal Technical Note

Perfect Attractor Phase Engine

Unified v4: Lyapunov Geometry, Quasicrystal Crust, and Robust Phases under Noise and Heterogeneity

OCTA Research Phase Dynamics Program

Version: Unified v4 (Birth Token 17)

Abstract. This document formalizes the Unified v4 implementation of the *Perfect Attractor Phase Engine* developed within OCTA Research. The engine is a Kuramoto-type phase system driven by a Lyapunov functional with Euclidean, hyperbolic, and quasicrystal geometries, plus a discrete quasicrystal "crust" and a birth token. We define the functional, its gradient flow, and a set of diagnostics (Hessian spectra, shell order parameters, phase diagrams, noise and frequency robustness labs). We then summarize the empirical evidence that, for suitable parameter choices—especially in the quasicrystal regime with birth token 17—the engine realizes a robust "PERFECT" attractor phase with a large basin of attraction and stability under noise and frequency heterogeneity. We conclude with an OCTA Research view of how this engine can be used as a generic alignment and phase-locking module in larger AGI and networked systems, and we provide a methods section that mirrors the Unified v4 Python implementation.

Contents

1 Non-technical summary (OCTA Research view)

This section explains, in plain language, what the Unified v4 engine is doing and what it shows.

1.1 What the engine actually is

- We have a network of N oscillators (phases) living on a graph. Each node carries an angle θ_i.
- The graph can be embedded in different geometries: Euclidean, hyperbolic, or a quasicrystal embedding.
- There is a single scalar function $T(\theta)$—the *Perfect-T functional*—which assigns an "energy" to any configuration of phases. It has three parts:
  1. Pairwise mismatch between connected nodes.
  2. Mismatch to a coarse "sector" pattern.
  3. Mismatch to a quasicrystal phase pattern defined by a $[1, -1, 1, 0]$ crust in 4D.
- The dynamics are simply gradient descent on T (plus optional intrinsic frequencies and noise).

In short: the engine is a carefully constructed Kuramoto-like system where the evolution of phases is completely driven by a single energy function T that prefers (1) neighbours to align, (2) sectors to be consistent, and (3) phases to line up with a specific quasicrystal pattern.

1.2 What is guaranteed (mathematically)

Because the dynamics are gradient descent on T, we have a clean Lyapunov property:

- The energy $T(\theta(t))$ never increases over time in the deterministic case.
- The rate of decrease is exactly minus the squared norm of the gradient.

This is a strong structural statement: without any approximations, the deterministic dynamics cannot run away or oscillate; they always move downhill in T.

1.3 What we see empirically

In Unified v4, we run a large set of experiments:

- **Canonical runs** in each geometry, with fixed parameters, tracking global phase-lock, quasicrystal alignment, and energy.
- **Shell diagnostics**: inner/mid/outer shells show that coherence and alignment spread from the center to the boundary.
- **Hessian spectra**: we compute eigenvalues of the Hessian of T at the final state to see how "stiff" or fragile the attractor is.
- **Parameter scans**: we vary (α, β, γ) and classify the resulting final states into phases (PERFECT, QC-LOCKED, FRUSTRATED, etc.).
- **N-scaling**: we repeat for $N = 50, 100, 150, 300$ to see how behaviour scales with system size.

- **Crust and birth-token labs**: we test different 4D crust vectors and different birth tokens.
- **Noise and frequency robustness labs**: we add Langevin noise and heterogeneous intrinsic frequencies and measure whether the attractor structure survives.

The key empirical result for OCTA Research is:

- In the quasicrystal geometry, with parameters $(\alpha, \beta, \gamma) = (0.5, 0, 1)$ and birth token $b = 17$, the system almost always flows into a *PERFECT* phase: high global phase-lock, strong quasicrystal alignment, low energy, stiff Hessian, large basin (about 98% of random initial conditions) and strong robustness to both noise and intrinsic frequency heterogeneity.

1.4 What we can do with it

Within OCTA, this engine can be treated as:

- A **generic alignment module**: if you feed some subsystem into this geometry + crust + birth-token pattern, it will tend to collapse into a highly aligned quasicrystal phase.
- A **phase-locking primitive**: used to coordinate large ensembles (e.g., neurons, agents, or distributed nodes) into a coherent global rhythm that is robust against noise and small heterogeneities.
- A **geometry probe**: by switching between Euclidean, hyperbolic, and quasicrystal embeddings, we can see which geometry supports which type of attractor (PERFECT vs coherent glass vs frustrated).

From an OCTA AGI perspective, this gives you a concrete, measurable way to talk about global coherence, stiffness, and robustness in a high-dimensional system—not as metaphors, but as explicit numerical invariants.

2 Model definition

2.1 Graph, geometry, and shells

Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, N\}$ and weighted adjacency matrix $W = (W_{ij})_{i,j=1}^N$, $W_{ij} \geq 0$, $W_{ii} = 0$. The empirical studies focus on an average degree of approximately 14.

We consider three geometries:

- **Euclidean geometry:** vertices embedded as $x_i \in \mathbb{R}^2$ with distances $d_{ij} = \|x_i - x_j\|$.
- **Hyperbolic geometry:** vertices embedded in a hyperbolic model (e.g. Poincaré disk) with metric $d_{\mathcal{H}}(x_i, x_j)$.
- **Quasicrystal geometry:** vertices embedded in a four-dimensional quasicrystal space $q_i \in \mathbb{R}^4$, with a projection to the physical plane.

In each case, W_{ij} is constructed from distances via a decaying kernel and normalized (details may vary between implementations, but the only property used here is $W_{ij} \geq 0$ and sparsity).

Definition 2.1 (Shell decomposition)**.** *Let $r_i \geq 0$ denote a radial coordinate of vertex i in its embedding (e.g., Euclidean radius, hyperbolic radius, or projected radius). A* shell decomposition *is a partition*

$$V = S_0 \sqcup S_1 \sqcup S_2$$

into inner, middle, and outer shells, typically obtained by quantiles of $\{r_i\}_{i=1}^N$. We refer to S_0 as the inner shell, S_1 as mid-shell, and S_2 as outer shell.

2.2 Phase variables and fields

Each vertex i carries a phase $\theta_i \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Write $\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{T}^N$.

In addition, we define two fixed fields over the vertices:

- A **sector field** $\varphi_{\mathrm{sec}} : V \to \mathbb{T}$, encoding a coarse partition of space into angular sectors. A simple form is

  $$\varphi_{\mathrm{sec}}(i) = \arg(x_i) + 2\pi \frac{b \bmod p}{p},$$

  where p is the number of sectors and $b \in \mathbb{N}$ is a *birth token.* The shift $2\pi(b \bmod p)/p$ represents a discrete "phase offset" keyed by b.

- A **quasicrystal phase field** $\varphi_{\mathrm{qc}} : V \to \mathbb{T}$, derived from a quasicrystal embedding $q_i \in \mathbb{R}^4$ and a crust vector $c = (c_1, \ldots, c_4) \in \mathbb{R}^4$.

  For Unified v4, the canonical choice is

  $$c = (1, -1, 1, 0).$$

  One convenient construction is

  $$\varphi_{\mathrm{qc}}(i) = \arg\left(\sum_{k=1}^{4} c_k q_{ik}\right),$$

  where arg is a mapping to phase in $[0, 2\pi)$.

The pair (c, b) plays the role of a discrete code or "birth crust" that selects a specific quasicrystal phase pattern.

2.3 Order parameters

Definition 2.2 (Global Kuramoto order parameter)**.** *The global Kuramoto order parameter is*

$$R(\theta)\mathrm{e}^{\mathrm{i}\psi(\theta)} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j},$$

where $R(\theta) \in [0,1]$ and $\psi(\theta)$ is the mean phase.

Definition 2.3 (Shell-resolved order parameters)**.** *For a shell $S \subset V$ with $|S| > 0$, define*

$$R_S(\theta)\mathrm{e}^{\mathrm{i}\psi_S(\theta)} = \frac{1}{|S|}\sum_{j\in S} \mathrm{e}^{\mathrm{i}\theta_j}.$$

We denote the inner, middle, and outer shells by S_0, S_1, S_2 and write $R_{\mathrm{shell},k}$ for S_k.

Definition 2.4 (Alignment order parameters)**.** *Define the alignment with the quasicrystal and sector fields by*

$$A_{\mathrm{qc}}(\theta) = \frac{1}{N}\sum_{i=1}^{N} \cos\big(\theta_i - \varphi_{\mathrm{qc}}(i)\big), \qquad A_{\mathrm{sec}}(\theta) = \frac{1}{N}\sum_{i=1}^{N} \cos\big(\theta_i - \varphi_{\mathrm{sec}}(i)\big).$$

For a shell S,

$$A_{\mathrm{qc},S}(\theta) = \frac{1}{|S|}\sum_{i\in S} \cos\big(\theta_i - \varphi_{\mathrm{qc}}(i)\big).$$

When $A_{\mathrm{qc}}(\theta) \approx 1$, the configuration is strongly aligned with the quasicrystal phase field; $A_{\mathrm{sec}}(\theta) \approx 1$ indicates strong sector alignment.

3 Perfect-T functional and dynamics

3.1 Perfect-T functional

Definition 3.1 (Perfect-T functional)**.** *Given $\alpha, \beta, \gamma \geq 0$, define*

$$\begin{aligned} T_{\mathrm{pair}}(\theta; W) &= \sum_{i,j=1}^{N} W_{ij}\big(1 - \cos(\theta_i - \theta_j)\big), \\ T_{\mathrm{sector}}(\theta; \varphi_{\mathrm{sec}}) &= \sum_{i=1}^{N}\big(1 - \cos(\theta_i - \varphi_{\mathrm{sec}}(i))\big), \\ T_{\mathrm{qc}}(\theta; \varphi_{\mathrm{qc}}) &= \sum_{i=1}^{N}\big(1 - \cos(\theta_i - \varphi_{\mathrm{qc}}(i))\big). \end{aligned}$$

The Perfect-T functional is

$$T(\theta) = \alpha T_{\mathrm{pair}}(\theta; W) + \beta T_{\mathrm{sector}}(\theta; \varphi_{\mathrm{sec}}) + \gamma T_{\mathrm{qc}}(\theta; \varphi_{\mathrm{qc}}).$$

Each term is nonnegative. Up to an additive constant, $T(\theta)$ measures total misalignment: neighbour mismatch, sector mismatch, and quasicrystal mismatch.

3.2 Deterministic gradient flow

The deterministic dynamics are defined as gradient flow of T:

$$\frac{d\theta}{dt} = -\nabla_\theta T(\theta). \tag{1}$$

Theorem 3.2 (Lyapunov monotonicity)**.** *For any solution $\theta(t)$ of (1),*

$$\frac{d}{dt}T(\theta(t)) = -\|\nabla_\theta T(\theta(t))\|_2^2 \leq 0.$$

Proof. By the chain rule,

$$\frac{d}{dt}T(\theta(t)) = \nabla_\theta T(\theta(t)) \cdot \frac{d\theta}{dt} = \nabla_\theta T(\theta(t)) \cdot (-\nabla_\theta T(\theta(t))) = -\|\nabla_\theta T(\theta(t))\|_2^2.$$

□

This is the theoretical basis for the numerically observed "Max positive dT/dt violation ≈ 0" in all canonical runs.

3.3 Stochastic dynamics with frequencies and noise

Unified v4 augments (1) with intrinsic frequencies ω_i and Langevin noise:

$$d\theta_i(t) = -\frac{\partial T}{\partial \theta_i}(\theta(t))\,dt + \omega_i\,dt + \sqrt{2D}\,dB_i(t), \tag{2}$$

where:

- $\omega_i \sim \mathcal{N}(0, \omega_{\text{std}}^2)$ are i.i.d. intrinsic frequencies,
- $D \geq 0$ is the noise intensity,
- $B_i(t)$ are independent standard Brownian motions.

In this setting, T is no longer a strict Lyapunov functional, but it still heavily shapes the effective energy landscape. Unified v4 measures how robust the attractors of the deterministic system are under non-zero D and ω_{std}.

4 Phase classification and composite scores

4.1 Phase types

Let θ^* be the final phase vector at time T_{total}. Write

$$R^* = R(\theta^*), \quad A_{\text{qc}}^* = A_{\text{qc}}(\theta^*), \quad A_{\text{sec}}^* = A_{\text{sec}}(\theta^*), \quad T^* = T(\theta^*).$$

Definition 4.1 (Phase types (heuristic decision rule))**.** *Based on $(R^*, A_{\text{qc}}^*, A_{\text{sec}}^*, T^*)$, Unified v4 classifies final states as:*

- ***PERFECT:*** *R^* near 1, A_{qc}^* near 1, T^* small. Global phase-lock and strong quasicrystal alignment at low energy.*

- ***QC-LOCKED:*** *high R^* and high A^*_{qc}, but with residual energy or sector tension.*
- ***SECTOR-LOCKED:*** *strong sector alignment, weaker or ambiguous quasicrystal alignment.*
- ***FRUSTRATED:*** *intermediate R^* with high T^*; the system is coherent but trapped at high energy.*
- ***MIXED:*** *intermediate or ambiguous; none of the above criteria clearly dominate.*

Threshold values are fixed in the code to ensure consistent classification across experiments.

4.2 Composite scores J and P^*

Two scalar composite scores are used:

Definition 4.2 (Score J for parameter scans)**.** *For some experiments, a scalar score*

$$J = J(R^*, T^*, A^*_{\text{qc}})$$

is used to rank parameter sets (α, β, γ). A typical form is

$$J = w_R R^* + w_A A^*_{\text{qc}} - w_T T^*,$$

with positive weights (w_R, w_A, w_T). Larger J indicates better global phase-lock, better quasicrystal alignment, and lower energy.

Definition 4.3 (Score P^* for v3/v4 parameter rankings)**.** *In Unified v3/v4, a related composite score P^* is used, again of the form*

$$P^* = \tilde{w}_R R^* + \tilde{w}_A A^*_{\text{qc}} - \tilde{w}_T T^*,$$

with weights tuned so that P^* lies in a convenient numerical range for ranking. Larger P^* indicates a more "perfect" configuration.

The exact numerical weights are implementation details; from a theory standpoint, J and P^* are monotone in (R^*, A^*_{qc}) and anti-monotone in T^*.

5 Canonical regimes and empirical behaviour

We summarize the key empirical results for the canonical runs in each geometry. Unless stated otherwise:

$$N = 150, \quad \text{avg degree} \approx 14, \quad T_{\text{total}} = 80, \quad \Delta t = 0.02, \quad \text{birth token } b = 17.$$

5.1 Euclidean canonical regime

Parameters: $(\alpha, \beta, \gamma) = (1.0, 0.0, 1.0)$.

Time slices and shells

t	$R(t)$	$T(t)$	$A_{\mathrm{qc}}(t)$
0	0.078	597.640	−0.043
40	0.992	4.798	0.875
80	0.988	2.769	0.945

Shell-resolved order parameters at $t = 80$:

shell	R_{shell}	$A_{\mathrm{qc,shell}}$
0	0.996	0.987
1	0.984	0.951
2	0.981	0.915

Hessian and phase type

Final-state Hessian:

$$\lambda_{\min}(H) \approx 2.07 \times 10^{-1}, \quad \lambda_{\max}(H) \approx 1.97 \times 10^{1},$$

with no negative eigenvalues at tolerance 10^{-8}, cluster count 1.

Empirically, this regime realizes a stiff single-cluster attractor with strong quasicrystal alignment across all shells. Phase type: PERFECT.

5.2 Hyperbolic canonical regime

Parameters: $(\alpha, \beta, \gamma) = (1.5, 0.5, 1.0)$.

Time slices and shells

t	$R(t)$	$T(t)$	$A_{\mathrm{qc}}(t)$
0	0.078	805.877	0.109
40	0.846	118.276	0.451
80	0.825	118.149	0.513

Shell-resolved:

shell	R_{shell}	$A_{\mathrm{qc,shell}}$
0	0.924	0.789
1	0.751	0.585
2	0.697	0.340

Final-state Hessian:

$$\lambda_{\min}(H) \approx 2.23 \times 10^{-1}, \quad \lambda_{\max}(H) \approx 2.44 \times 10^{1},$$

no negative eigenvalues, cluster count 1.

This regime behaves as a *coherent glass*: moderate-to-high coherence, but high energy $T(t)$ that remains large, especially in the outer shells.

5.3 Quasicrystal canonical regime

Parameters: $(\alpha, \beta, \gamma) = (0.5, 0.0, 1.0)$.

Time slices and shells

t	$R(t)$	$T(t)$	$A_{\mathrm{qc}}(t)$
0	0.078	269.606	-0.043
40	0.600	50.043	-0.070
80	0.980	3.241	0.925

Shell-resolved:

shell	R_{shell}	$A_{\mathrm{qc,shell}}$
0	0.992	0.939
1	0.973	0.922
2	0.965	0.920

Final-state Hessian:

$$\lambda_{\min}(H) \approx 1.95 \times 10^{-1}, \quad \lambda_{\max}(H) \approx 9.15 \times 10^{0},$$

no negative eigenvalues, cluster count 1.

Again we obtain a stiff, single-cluster attractor with strong quasicrystal alignment that is nearly uniform across shells. Phase type: PERFECT.

6 Scaling, spectra, and basins

6.1 N-scaling

For each canonical geometry and parameter set, N-scaling experiments were run for $N \in \{50, 100, 150, 300\}$.

Euclidean, $(\alpha, \beta, \gamma) = (1, 0, 1)$

N	R_{final}	T_{final}	$A_{\mathrm{qc,final}}$	$\lambda_{\min}(H)$	clusters
50	0.999	1.105	0.908	1.95×10^{-1}	1
100	0.996	1.837	0.930	1.95×10^{-1}	1
150	0.990	2.670	0.944	2.05×10^{-1}	1
300	0.974	4.054	0.969	2.10×10^{-1}	1

Hyperbolic, $(\alpha, \beta, \gamma) = (1.5, 0.5, 1)$

N	R_{final}	T_{final}	$A_{\mathrm{qc,final}}$	$\lambda_{\min}(H)$	clusters
50	0.993	29.930	0.282	3.47×10^{-1}	1
100	0.962	79.845	0.159	1.09×10^{-1}	1
150	0.867	99.517	0.518	2.92×10^{-1}	1
300	0.812	271.720	0.342	1.86×10^{-1}	1

Quasicrystal, $(\alpha, \beta, \gamma) = (0.5, 0, 1)$

N	R_{final}	T_{final}	$A_{\mathrm{qc,final}}$	$\lambda_{\min}(H)$	clusters
50	0.994	0.745	0.947	1.89×10^{-1}	1
100	0.986	1.483	0.957	2.00×10^{-1}	1
150	0.976	1.751	0.973	2.03×10^{-1}	1
300	0.948	2.471	0.989	2.08×10^{-1}	1

Remark 6.1. *For Euclidean and quasicrystal geometries, the minimal Hessian eigenvalue stays bounded away from 0 as N grows, indicating a stiff attractor with no soft directions appearing at larger scales. Hyperbolic remains locally stable but at significantly higher energy.*

6.2 Spectral gap and locking times

Let $L = D - W$ be the graph Laplacian with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Unified v4 reports:

geometry	λ_2	$\lambda_{\max}$	$t_{R\geq 0.95}$	$t_{A_{\mathrm{qc}}\geq 0.90}$
Euclidean	0.557	19.067	40.0	40.0
Hyperbolic	0.426	15.296	∞	∞
Quasicrystal	0.447	16.673	40.0	40.0

Here $t_{R\geq 0.95}$ is the first time when $R(t) \geq 0.95$ (if such time exists before T_{total}), and similarly for A_{qc}.

6.3 Contraction and basin statistics

For Euclidean geometry (canonical parameters), two independent trajectories from random initial conditions are compared via a mean circular distance $D(t)$:

t	$D(t)$
0	1.537
20	0.248
40	0.005
60	0.000
80	0.000

This suggests a unique global attractor (modulo global phase) for this regime.

For hyperbolic geometry, basin statistics (over many random initial conditions, canonical parameters) are:

metric	mean	std
R_{final}	0.715	0.281
T_{final}	121	35
$A_{\mathrm{qc,final}}$	0.491	0.0563
$A_{\mathrm{sec,final}}$	0.344	0.104
clusters	1.1	0.3
$\lambda_{\min}(H)$	0.318	0.0696

7 Energy landscape probes

For each canonical geometry, Unified v4 perturbs a converged state θ^* by small random displacements $\delta\theta$ and measures

$$\Delta T = T(\theta^* + \delta\theta) - T(\theta^*), \quad \|\nabla T(\theta^* + \delta\theta)\|.$$

7.1 Euclidean

$$T_* \approx 65.303, \quad \Delta T_{\mathrm{mean}} \approx 9.904, \quad \Delta T_{\mathrm{std}} \approx 1.260,$$

$$\|\nabla T\|_{\mathrm{mean}} \approx 1.343, \quad \|\nabla T\|_{\mathrm{std}} \approx 0.084,$$

with fraction of samples having $\Delta T > 0$ equal to 1.00.

7.2 Hyperbolic

$$T_* \approx 103.043, \quad \Delta T_{\text{mean}} \approx 12.640, \quad \|\nabla T\|_{\text{mean}} \approx 1.713,$$

again all sampled $\Delta T > 0$.

7.3 Quasicrystal

$$T_* \approx 1.933, \quad \Delta T_{\text{mean}} \approx 4.637, \quad \Delta T_{\text{std}} \approx 0.585,$$

$$\|\nabla T\|_{\text{mean}} \approx 0.628, \quad \|\nabla T\|_{\text{std}} \approx 0.039,$$

with fraction of $\Delta T > 0$ equal to 1.00.

Proposition 7.1 (Empirical local minima)**.** *For each canonical geometry, the combination of*

- *positive-definite Hessian at θ^*,*
- *$\Delta T > 0$ for all tested perturbations,*

indicates that θ^* is a strict local minimum of T.

8 Crusts, birth tokens, and phase codes

8.1 Crust scan (quasicrystal)

For quasicrystal geometry with $(\alpha, \beta, \gamma) = (0.5, 0, 1)$ and birth token $b = 17$:

crust	R_{final}	T_{final}	$A_{\text{qc,final}}$	P^*
canonical $[1, -1, 1, 0]$	0.944	30.772	0.867	0.481
all_pos $[1, 1, 1, 1]$	0.934	29.298	0.866	0.479
alt $[1, -1, -1, 1]$	0.797	13.489	0.917	0.486
shifted $[0, 1, -1, 1]$	0.935	32.851	0.841	0.461

No crust is uniquely optimal in P^*, but the family of crusts including $(1, -1, 1, 0)$ and $(1, -1, -1, 1)$ all support strong quasicrystal alignment.

8.2 Birth token scan (quasicrystal)

For quasicrystal geometry with $(\alpha, \beta, \gamma) = (1.0, 0.2, 1.0)$:

b	R_{final}	T_{final}	A_{qc}	A_{sec}	phase
1	0.963	69.069	0.861	−0.234	QC-LOCKED
5	0.967	76.983	0.793	−0.198	FRUSTRATED
17	0.995	23.508	0.911	0.327	QC-LOCKED
23	0.893	80.747	0.711	−0.153	FRUSTRATED
29	0.955	68.294	0.866	−0.265	QC-LOCKED
31	0.949	65.217	0.880	−0.206	QC-LOCKED

The birth token b acts as a discrete switch between QC-LOCKED and FRUSTRATED phases at fixed continuous parameters.

9 Phase frequencies and robustness

9.1 Phase frequencies (canonical quasicrystal)

For quasicrystal geometry with $(\alpha, \beta, \gamma) = (0.5, 0, 1)$ and birth token $b = 17$, over 50 random initial conditions:

phase	count	frequency
PERFECT	49	0.98
QC-LOCKED	0	0.00
SECTOR-LOCKED	0	0.00
FRUSTRATED	0	0.00
MIXED	1	0.02

Proposition 9.1 (Dominant basin of the PERFECT phase)**.** *In this canonical regime, the PERFECT phase occurs in 49 out of 50 runs from random initial conditions, suggesting that the PERFECT attractor occupies a dominant fraction of the basin of attraction at* $N = 150$*.*

9.2 Noise robustness

For the same canonical quasicrystal regime, the noise robustness lab varies D in (2):

D	R_{mean}	T_{mean}	$A_{\text{qc,mean}}$	phase frequencies
0	0.963	2.946	0.966	PERFECT: 0.97, MIXED: 0.03
10^{-4}	0.963	2.954	0.966	PERFECT: 0.97, MIXED: 0.03
$5 \cdot 10^{-4}$	0.924	4.842	0.933	PERFECT: 0.97, MIXED: 0.03
10^{-3}	0.972	1.988	0.972	PERFECT: 0.93, FRUSTRATED: 0.07
$5 \cdot 10^{-3}$	0.964	3.263	0.966	PERFECT: 1.00
10^{-2}	0.929	6.044	0.942	PERFECT: 1.00

Proposition 9.2 (Noise stability window (empirical))**.** *For D in the range* $[0, 10^{-2}]$*, the PERFECT phase remains dominant. Coherence and quasicrystal alignment stay high, and the PERFECT frequency is between* ≈ 0.93 *and* 1.0*.*

9.3 Frequency robustness

Frequency robustness lab: same canonical regime, varying ω_{std}:

ω_{std}	R_{mean}	T_{mean}	$A_{\text{qc,mean}}$	phase frequencies
0.000	0.954	3.197	0.962	PERFECT: 1.00
0.050	0.982	1.656	0.971	PERFECT: 1.00
0.100	0.966	2.763	0.957	PERFECT: 1.00
0.200	0.920	6.496	0.928	PERFECT: 0.97, MIXED: 0.03
0.300	0.947	4.675	0.946	PERFECT: 0.97, QC-LOCKED: 0.03

Proposition 9.3 (Frequency heterogeneity tolerance (empirical))**.** *The PERFECT phase remains dominant for ω_{std} up to at least* 0.2–0.3*, with high coherence and quasicrystal alignment and PERFECT frequency* ≈ 0.97–1.0*.*

10 Conjectures and OCTA Research interpretation

10.1 Conjectures

Conjecture 10.1 (Geometry-dependent perfect attractor)**.** *For graphs with Euclidean or quasicrystal geometry and suitable parameters* $(\alpha, \beta, \gamma, c, b)$*, the gradient flow* (1) *converges, from almost all initial conditions (up to global phase), to a unique attractor* θ^* *with*

$$R(\theta^*) \approx 1, \quad A_{\mathrm{qc}}(\theta^*) \approx 1, \quad T(\theta^*) \textit{ small},$$

and this attractor persists as the dominant state under small D *and moderate* ω_{std}*.*

Conjecture 10.2 (Crust and birth token as phase codes)**.** *For fixed geometry and continuous parameters* (α, β, γ)*, the crust vector* c *and birth token* b *act as a discrete phase code that selects between PERFECT, QC-LOCKED, FRUSTRATED, and other phases. Certain pairs, such as* $c = (1, -1, 1, 0)$ *and* $b = 17$ *in the quasicrystal geometry, maximize the basin and robustness of the PERFECT attractor within a family of comparable codes.*

Conjecture 10.3 (Hyperbolic coherent glass)**.** *On hyperbolic geometries with reasonable parameters, the Perfect-*T *functional produces coherent states that are locally stable minima of* T*, but with significantly higher energy compared to Euclidean and quasicrystal regimes. These states behave as coherent glasses: partially ordered, high-energy, and slow to relax, providing metastable diversity rather than full lock.*

10.2 OCTA Research applications

Within the broader OCTA program, this engine can be treated as a reusable building block:

- **Alignment module:** A subsystem that, when fed a population of agents/neurons/nodes, pushes them into a globally coherent, quasicrystal-aligned phase with high robustness to noise and heterogeneity.
- **Phase-locking core:** A core component for synchronizing large distributed systems, potentially coupled into higher-level OCTA controllers or Hivemind layers.
- **Geometry control knob:** A controlled way to switch between low-energy perfect attractors (Euclidean/quasicrystal) and higher-energy coherent glass states (hyperbolic), by choosing the embedding.
- **Token-coded regimes:** By changing the birth token b and crust c, OCTA can switch between different macroscopic phase behaviours using only discrete codes, supplying a compact interface layer between symbolic configuration and emergent dynamics.

10.3 Next steps

Natural next steps for OCTA Research include:

- Analytical study of the continuum limit ($N \to \infty$) in quasicrystal geometry with the $[1, -1, 1, 0]$ crust.
- Rigorous characterization of the PERFECT basin measure under random initial conditions.

- Embedding the Perfect Attractor Phase Engine into larger OCTA architectures (e.g., as a "coherence spine" for spiking networks or as a consensus layer in P3P / NoBlok-style meshes).
- Exploration of alternative crusts and higher-dimensional embeddings (e.g., extending c to longer quasicrystal codes) to see whether richer phase taxonomies emerge.

11 Methods and implementation details (Unified v4)

This section mirrors the Unified v4 Python implementation. The goal is to make the code and the theory one-to-one, so that all numerical results in this note can be reproduced by running the engine with the same flags.

11.1 Global simulation parameters

Unless otherwise specified, the following defaults are used:

- Number of oscillators: $N = 150$.
- Average graph degree: ≈ 14.
- Total simulation time: $T_{\text{total}} = 80$.
- Time step: $\Delta t = 0.02$ (Euler or Euler–Maruyama).
- Birth token: $b = 17$ (unless explicitly scanned).
- Crust vector (canonical): $c = (1, -1, 1, 0)$.
- Noise intensity (default deterministic runs): $D = 0$.
- Frequency standard deviation (default deterministic): $\omega_{\text{std}} = 0$.

11.2 Core data structures

The engine keeps the following core objects in memory:

- `geom` (string): one of `"euclidean"`, `"hyperbolic"`, `"quasicrystal"`.
- `W`: $N \times N$ sparse adjacency/weight matrix.
- `x`: Euclidean or hyperbolic coordinates (if applicable).
- `q`: Quasicrystal coordinates in $\mathbb{R}^4$ (for quasicrystal geometry).
- `theta`: current phase vector, shape $(N,)$.
- `phi_sec`: sector phase field, shape $(N,)$.
- `phi_qc`: quasicrystal phase field, shape $(N,)$.
- `shell_ids`: integer array $(N,)$ with values in $\{0, 1, 2\}$ indicating the shell index.

11.3 Building the geometry and graph

The function `build_geometry_and_graph(geom, N, avg_degree, birth_token)` performs:

1. Sample base coordinates:
   - Euclidean: sample x_i on a disk or bounded region in $\mathbb{R}^2$.
   - Hyperbolic: sample x_i in a hyperbolic disk model.
   - Quasicrystal: sample $q_i \in \mathbb{R}^4$ (e.g. from a cut-and-project or random 4D lattice) and optionally project to 2D for visualization.
2. Construct pairwise distances d_{ij} (geometry dependent).
3. Build a sparse weight matrix W_{ij} using a decaying kernel, e.g.

$$W_{ij} \propto \mathrm{e}^{-d_{ij}/\ell},$$

   and then prune to achieve average degree ≈ 14.
4. Normalize weights row-wise.
5. Compute radial radii r_i from the embedding and assign shell indices by quantiles to produce S_0, S_1, S_2.

The function returns $(W,\ x,\ q,\ \text{shell\_ids})$ as appropriate.

11.4 Phase fields and quasicrystal crust

Given birth token b and crust vector c, the engine constructs:

- Sector field `phi_sec` via:

$$\varphi_{\text{sec}}(i) = \arg(x_i) + 2\pi \frac{b \bmod p}{p},$$

  with a fixed number of sectors p (e.g. $p = 4$ or $p = 8$).
- Quasicrystal phase field `phi_qc` via:

$$\varphi_{\text{qc}}(i) = \arg\left(\sum_{k=1}^{4} c_k q_{ik}\right).$$

The code supports alternate crusts; the crust scan lab simply reuses this construction with different c.

11.5 Integration of the dynamics

Gradient computation

The gradient of T is computed as:

$$\frac{\partial T_{\text{pair}}}{\partial \theta_i} = \alpha \sum_j W_{ij} \sin(\theta_i - \theta_j),$$
$$\frac{\partial T_{\text{sector}}}{\partial \theta_i} = \beta \sin(\theta_i - \varphi_{\text{sec}}(i)),$$
$$\frac{\partial T_{\text{qc}}}{\partial \theta_i} = \gamma \sin(\theta_i - \varphi_{\text{qc}}(i)).$$

So,

$$\nabla_\theta T(\theta)_i = \alpha \sum_j W_{ij} \sin(\theta_i - \theta_j) + \beta \sin(\theta_i - \varphi_{\text{sec}}(i)) + \gamma \sin(\theta_i - \varphi_{\text{qc}}(i)).$$

Deterministic step

For the deterministic case ($\omega_{\text{std}} = D = 0$), one time step is:

```
grad = grad_T(theta, W, phi_sec, phi_qc, alpha, beta, gamma)
theta = theta - dt * grad
theta = wrap_to_2pi(theta)
```

where `wrap_to_2pi` maps phases into $[0, 2\pi)$.

Stochastic step (Unified v4)

With intrinsic frequencies and Langevin noise:

```
grad = grad_T(theta, W, phi_sec, phi_qc, alpha, beta, gamma)
dtheta = -grad * dt + omega * dt + sqrt(2*D*dt) * normal(0,1,size=N)
theta = theta + dtheta
theta = wrap_to_2pi(theta)
```

Here:

- `omega` is drawn once per run as `omega[i]   N(0, omega`$_s td^2$`)`.
- `normal(0,1,size=N) is a vector of i.i.d. standard normals.`

11.6 Canonical comparison lab

Function: `run_canonical_comparison_lab()`

This lab:

1. Loops over geometries `["euclidean","hyperbolic","quasicrystal"]`.
2. For each geometry, selects canonical (α, β, γ):

$$(1, 0, 1) \text{ (euclidean)}, \quad (1.5, 0.5, 1) \text{ (hyperbolic)}, \quad (0.5, 0, 1) \text{ (quasicrystal)}.$$

3. Initializes `theta` uniformly at random on $[0, 2\pi)$.

4. Simulates from $t = 0$ to T_{total} with the deterministic flow, recording $(R(t), T(t), A_{\text{qc}}(t))$ at $t = 0, 40, 80$.
5. Computes shell-resolved order parameters at $t = 80$.
6. Computes the Hessian of T at the final state via finite differences or analytic second derivatives, then estimates eigenvalues and cluster count.

Outputs are formatted exactly as in the canonical comparison section.

11.7 Parameter scan labs

Function: `run_parameter_scan(geom, alpha_grid, beta_grid, gamma_grid)`

For each geometry:

1. For each triple (α, β, γ) in the Cartesian product of the provided grids:
   (a) Initialize phases randomly.
   (b) Integrate to T_{total}.
   (c) Compute $R^*, T^*, A^*_{\text{qc}}$.
   (d) Compute phase type.
   (e) Compute composite score P^*.
2. Sort all configurations by P^* and report the top few.

The grids used for v4 scans were:

$$\alpha \in \{0.5, 1.0, 1.5\}, \quad \beta \in \{0.0, 0.2, 0.5\}, \quad \gamma \in \{0.5, 1.0, 1.5\}.$$

11.8 Contraction experiment

Function: `run_contraction_experiment(geom, alpha,beta,gamma)`

1. Sample two independent random initial conditions $\theta^{(1)}(0)$ and $\theta^{(2)}(0)$.
2. Using the same graph and fields, run two deterministic simulations in parallel.
3. At times $t \in \{0, 20, 40, 60, 80\}$, compute the circular distance

$$D(t) = \frac{1}{N} \sum_i \left|\text{angle\_diff}(\theta_i^{(1)}(t), \theta_i^{(2)}(t))\right|.$$

4. Report $D(t)$ and $D(0), D(T_{\text{total}})$.

11.9 Basin statistics lab

Function: `run_basin_stats(geom, alpha,beta,gamma, num_runs)`

1. For $k = 1, \ldots,$ `num_runs`:
   - Sample random initial $\theta^{(k)}(0)$.
   - Run deterministic dynamics to T_{total}.
   - Compute $R^{*(k)}, T^{*(k)}, A_{\text{qc}}^{*(k)}, A_{\text{sec}}^{*(k)}, \text{clusters}^{(k)}, \lambda_{\min}^{(k)}$ at final state.
2. Aggregate means and standard deviations over k.

This lab is used in v4 primarily for hyperbolic geometry.

11.10 N-scaling lab

Function: `run_N_scaling(geom, Ns, alpha,beta,gamma)`

1. For each $N \in$ `Ns`:
   - Build geometry and graph with given `geom` and N.
   - Run a deterministic simulation from random initial condition.
   - Compute $R^*, T^*, A_{\text{qc}}^*, \lambda_{\min}(H)$, cluster count.
2. Report as a table across N.

11.11 Spectral + time-to-lock lab

Function: `run_spectral_time_lab(geom, alpha,beta,gamma)`

1. Compute Laplacian L from W and obtain eigenvalues $0 = \lambda_1 \leq \cdots \leq \lambda_N$; record λ_2 and λ_N.
2. Run deterministic dynamics; at each time step, check whether:
$$R(t) \geq 0.95, \quad A_{\text{qc}}(t) \geq 0.90, \quad \|\nabla T(\theta(t))\| \leq 0.10.$$
3. Record the first time each threshold is crossed; if not crossed by T_{total}, record ∞.

11.12 Energy landscape lab

Function: `run_energy_landscape_lab(geom, alpha,beta,gamma, eps, samples)`

1. From a converged θ^*, approximate $T_* = T(\theta^*)$.
2. For each sample $s = 1, \ldots,$ `samples`:
   - Draw $\delta\theta$ with each component uniform in $[-\varepsilon, \varepsilon]$.
   - Compute $T(\theta^* + \delta\theta)$ and $\Delta T = T(\theta^* + \delta\theta) - T_*$.
   - Compute $\|\nabla T(\theta^* + \delta\theta)\|$.
3. Aggregate mean and standard deviation of ΔT and gradient norm; compute fraction of samples with $\Delta T > 0$.

11.13 Crust scan lab

Function: `run_crust_scan(geom="quasicrystal", alpha,beta,gamma, crust_list)`

For each crust vector c in `crust_list`:

1. Build φ_{qc} from q and c.
2. Run deterministic dynamics to T_{total}.
3. Compute $R^*, T^*, A^*_{\text{qc}}, P^*$.

11.14 Birth token lab

Function: `run_birth_token_lab(geom="quasicrystal", alpha,beta,gamma, b_list)`

For each birth token b in `b_list`:

1. Rebuild sector field φ_{sec} using b.
2. Run deterministic dynamics to T_{total}.
3. Compute $R^*, T^*, A^*_{\text{qc}}, A^*_{\text{sec}}$ and phase type.

11.15 Phase frequency lab

Function: `run_phase_frequency_lab(geom, alpha,beta,gamma, b, num_runs)`

1. Fix `geom`, (α, β, γ), birth token b.
2. For each run $k = 1, \ldots,$ `num_runs`:
   - Sample random initial $\theta^{(k)}(0)$.
   - Run deterministic dynamics to T_{total}.
   - Classify final phase type.
3. Count the frequency of each phase type (PERFECT, QC-LOCKED, SECTOR-LOCKED, FRUSTRATED, MIXED).

11.16 Noise robustness lab

Function: `run_noise_robustness_lab(geom, alpha,beta,gamma, b, D_values)`

1. Fix (α, β, γ), birth token b.
2. For each noise level D in `D_values`:
   - For several runs (e.g. 30 or 50):
     - Draw random initial $\theta(0)$.
     - Draw frequencies ω_i with ω_{std} fixed (often 0 here, focusing on noise).
     - Run stochastic dynamics (2).
     - Compute $R^*, T^*, A^*_{\text{qc}}$ and phase type.
   - Aggregate mean values and phase frequencies for this D.

11.17 Frequency robustness lab

Function: `run_frequency_robustness_lab(geom, alpha,beta,gamma, b, omega_std_values)`

1. Fix (α, β, γ), birth token b, and noise D (often a small constant or zero).
2. For each ω_{std} in `omega_std_values`:
   - For several runs:
     - Draw random initial $\theta(0)$.
     - Draw $\omega_i \sim \mathcal{N}(0, \omega_{\mathrm{std}}^2)$.
     - Run dynamics with these frequencies.
     - Compute $R^*, T^*, A^*_{\mathrm{qc}}$ and phase type.
   - Aggregate mean values and phase frequencies.

11.18 Reproducibility from the code to this note

The Unified v4 code can be mapped to this document as follows:

- The function that builds the graph and geometry corresponds to Section 2.
- The function that computes `T`, its gradient, and optional stochastic terms corresponds to Sections 3.1–3.3.
- The canonical comparison, parameter scans, contraction, basin, N-scaling, spectral, energy, crust, birth-token, phase frequency, noise, and frequency labs correspond line-by-line to Sections 4–7.
- All numerical values printed in this note are direct outputs of those labs with fixed seeds and parameter grids.

From an OCTA Research standpoint, this means that the Perfect Attractor Phase Engine is not just a conceptual model: it is a fully specified, reproducible computational object whose behaviour is captured both in code and in this internal technical note.

Prepared for internal use by OCTA Research.