Pattern Classification (EET 3035)

Lecture 04

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Parametric Estimation Techniques

Maximum Likelihood Estimation

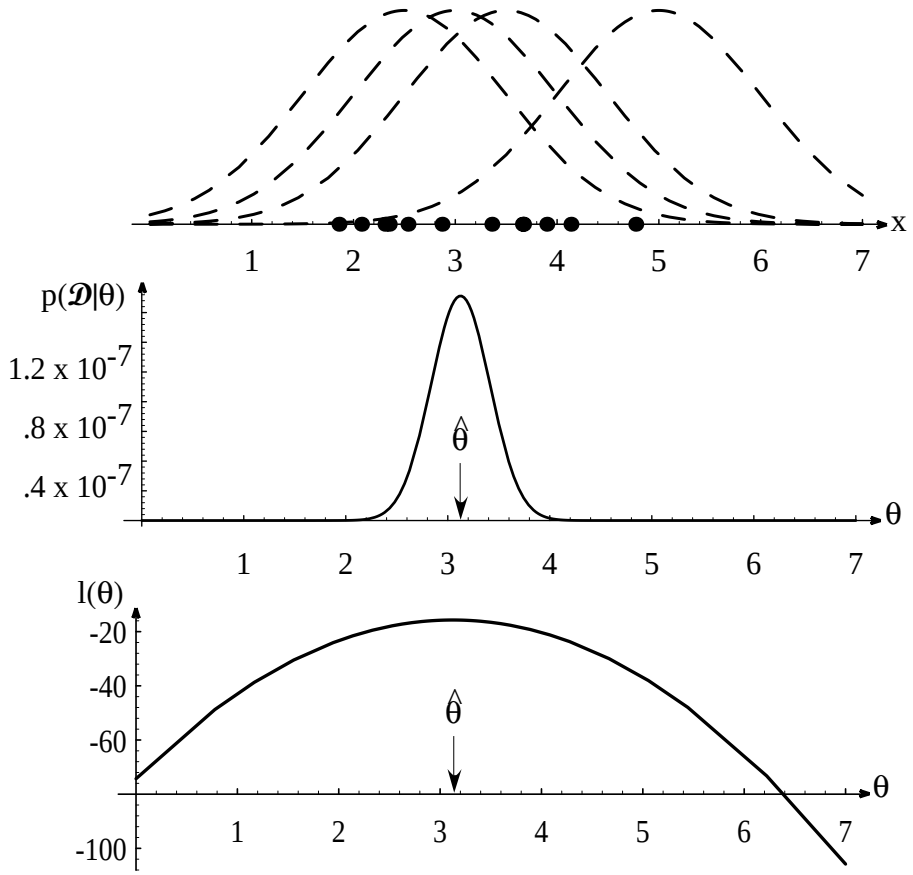
- Let us assume the set $\mathcal{D}_j = \{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n\}$ of independent and identically distributed (i.i.d.) samples drawn from the density $p(\mathrm{x}|\theta_j)$
- That means $\mathcal{D}_i$ does not provide any information about the parameter vector θ_j for $i \neq j$, *i.e.*, samples from one class do not provide any information of the parameter vector of the *probability density function* of another class.
- Thus, we can work with each class separately and omit the class labels (j), so that we write the probability density as $p(\mathrm{x}|\theta)$.
- Thus, the probability of observing $\mathcal{D} = \{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n\}$ is

$$p(\mathcal{D}|\theta) = p(\mathrm{x}_1|\theta) * p(\mathrm{x}_2|\theta) * \cdots * p(\mathrm{x}_n|\theta) = \prod_{k=1}^{n} p(\mathrm{x}_k|\theta)$$

  where n is the number of data samples in set $\mathcal{D}$.
- $p(\mathcal{D}|\theta)$ is also called the likelihood of θ with respect to the set of samples $\mathcal{D}$.

Maximum Likelihood Estimation

- The *maximum-likelihood estimation* of θ is, by definition, the value $\hat{\theta}$ that maximizes $p(\mathcal{D}|\theta)$.



$$p(\mathcal{D}|\theta) = \prod_{k=1}^{n} p(\mathrm{x}_k|\theta)$$

$$l(\theta) = \ln p(\mathcal{D}|\theta)$$

$$= \sum_{k=1}^{n} \ln\, p(\mathrm{x}_k|\theta)$$

Solution

$$\hat{\theta} = \arg\max_{\theta} l(\theta)$$

Maximum Likelihood Estimation-Optimal estimation

- Let $\theta = (\theta_1, \theta_2, \ldots, \theta_p)^T$, and ∇_θ be the gradient operator

$$\nabla_\theta = \begin{bmatrix} \frac{\partial}{\partial \theta_1} \\ \vdots \\ \frac{\partial}{\partial \theta_p} \end{bmatrix}$$

- $\Delta_\theta l(\theta) = 0$
- Example of a specific case: Gaussian distribution
- Multivariate normal population with (μ, Σ)

Gaussian case: Unknown μ

- σ^2 is known, only μ is unknown.

$$\ln p(\mathbf{x}_k|\boldsymbol{\mu}) = -\frac{1}{2}\ln\left[(2\pi)^d|\boldsymbol{\Sigma}|\right] - \frac{1}{2}(\mathbf{x}_k - \boldsymbol{\mu})^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}_k - \boldsymbol{\mu})$$

$$\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{x}_k|\boldsymbol{\mu}) = \boldsymbol{\Sigma}^{-1}(\mathbf{x}_k - \boldsymbol{\mu}).$$

- The maximum likelihood estimate for μ must satisfy

$$\sum_{k=1}^{n} \boldsymbol{\Sigma}^{-1}(\mathbf{x}_k - \hat{\boldsymbol{\mu}}) = \mathbf{0}$$

- Each of the d component of $\hat{\mu}$ must vanish.

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_k$$

Gaussian case: Unknown μ and Σ

- $\theta_1 = \mu$ and $\theta_2 = \sigma^2$ are unknown.
- The log-likelihood of a single point is

$$\ln\, p(x_k|\boldsymbol{\theta}) = -\frac{1}{2}\,\ln\, 2\pi\theta_2 - \frac{1}{2\theta_2}(x_k - \theta_1)^2$$

- Derivative is

$$\nabla_{\boldsymbol{\theta}} l = \nabla_{\boldsymbol{\theta}}\,\ln\, p(x_k|\boldsymbol{\theta}) = \begin{bmatrix} \frac{1}{\theta_2}(x_k - \theta_1) \\ -\frac{1}{2\theta_2} + \frac{(x_k-\theta_1)^2}{2\theta_2^2} \end{bmatrix}$$

- After simplification

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{n} x_k \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n}(x_k - \hat{\mu})^2$$

Example

- Estimate optimal parameter $\hat{\theta}$

$$p(x|\theta) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

using log-maximum likelihood estimation approach.

Non-parametric parameter estimation

Introduction

- We have already seen that for statistical pattern classification, density function are to be known for each class.
- The type of density function, such as the normal or poison, are to be known to estimate the parameters of the densities called *parametric estimation*.
- In most real problems, even the types of the density functions of interest are unknown.

Introduction

- Looking at histograms, scatter plots or tables of the data may suggest that a particular type of class density may be used or some arbitrary density can be used.
- Arbitrary density function can be estimated from the data samples using *nonparametric methods*.
- In addition, most of the classical parametric densities are unimodal, whereas many practical problems involve multimodal densities.
- Non-parametric methods can be used with arbitrary distributions and without the assumption that the forms of the underlying densities are known.

Histogram Method

- A very simple method is to partition the space into a number of equally-sized cells (bins) and compute a histogram.

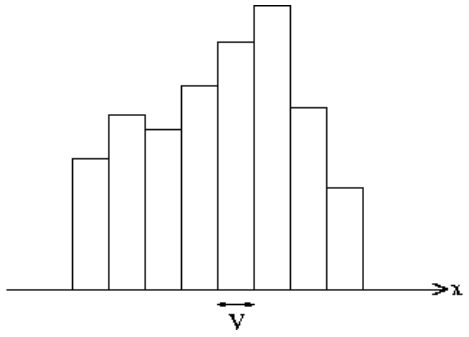


Figure: Histogram in one dimension

- The estimate of the density at a point x becomes

$$p(\mathrm{x}) = \frac{k}{nV}$$

where n is the total number of samples, k is the number of samples in the bin that includes x, and V is the volume of that cell.

- For 1-D feature, V is width of bin. Similarly for 2-D feature, V is the area of the bin.
- Thumb rule to choose the number of intervals to be equal to the square root of the number of samples

Histogram method

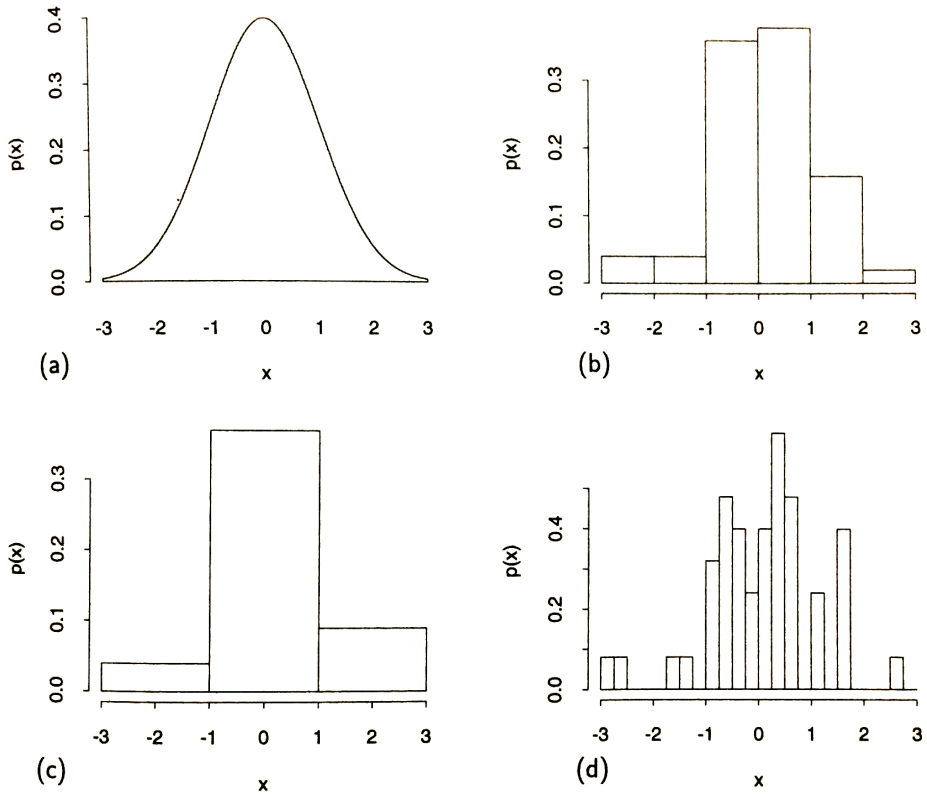


Figure: (a) The true normal density from which 50 random numbers were chosen. (b) A histogram of 50 normally distributed random numbers with six intervals. (c) A histogram of 50 normally distributed random numbers with three intervals. (d) A histogram of 50 normally distributed random numbers with 24 intervals.

Histogram method: Example

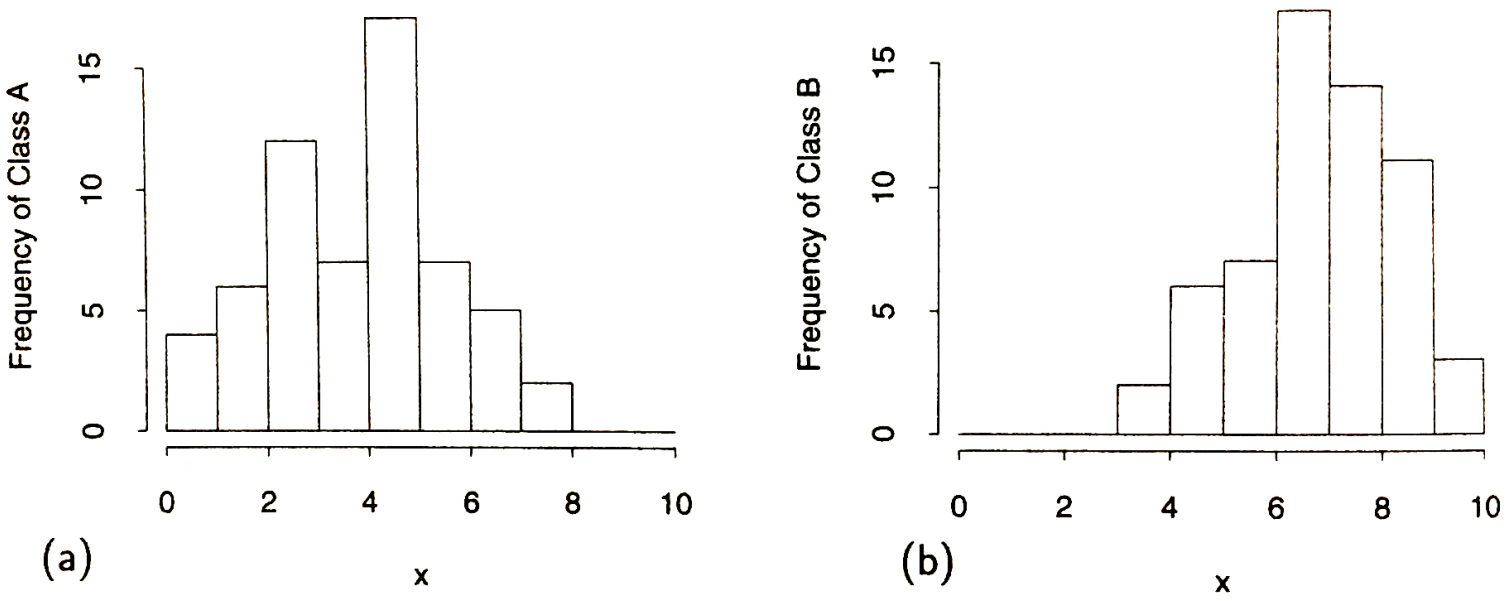
Question: Classification of samples using histograms and Bayes' Theorem
Use the following data to classify a sample with $x = 7.5$, given that $P(A) = P(B) = 0.5$. The following data are the values of feature x for 60 randomly chosen samples from
Class A:

0.80	0.91	0.93	0.95	1.32	1.53	1.57	1.63	1.67	1.74
2.01	2.18	2.27	2.31	2.40	2.61	2.64	2.64	2.67	2.85
2.96	2.97	3.17	3.17	3.38	3.67	3.73	3.83	3.99	4.06
4.10	4.12	4.18	4.20	4.23	4.27	4.27	4.39	4.40	4.46
4.47	4.61	4.64	4.89	4.96	5.12	5.15	5.33	5.33	5.47
5.64	5.85	5.99	6.29	6.42	6.53	6.70	6.78	7.18	7.22

Class B:

3.54	3.88	4.24	4.30	4.30	4.70	4.75	4.97	5.21	5.42
5.60	5.77	5.87	5.94	5.95	6.04	6.05	6.15	6.19	6.21
6.33	6.41	6.43	6.49	6.52	6.58	6.60	6.63	6.65	6.75
6.90	6.92	7.03	7.08	7.18	7.29	7.33	7.41	7.41	7.46
7.61	7.67	7.68	7.68	7.78	7.96	8.03	8.12	8.20	8.22
8.33	8.36	8.44	8.45	8.49	8.75	8.76	9.14	9.20	9.86

Histogram method: Solution



$P(A|7.5) = 0.125$ and $P(B|7.5) = 0.875$, so the sample should be classified into class B.

2-D Histogram method

- Histograms are not restricted to one-dimensional densities, but can be used in any number of dimensions.
- $p(x, y)$ can be approximated by dividing both x and y into intervals, and determining the number of samples that fall within each rectangular histogram bin with dimensions Δx and Δy.
- The volume under the surface of this two-dimensional histogram is to be normalized to equal one, to yield an estimate of the density function $p(x, y)$
- The histogram technique becomes impractical for spaces of high dimension.
- The square root rule of thumb can be generalized to produce an *equal precision rule*. When there are n features, the $(n+1)st$ root is used.

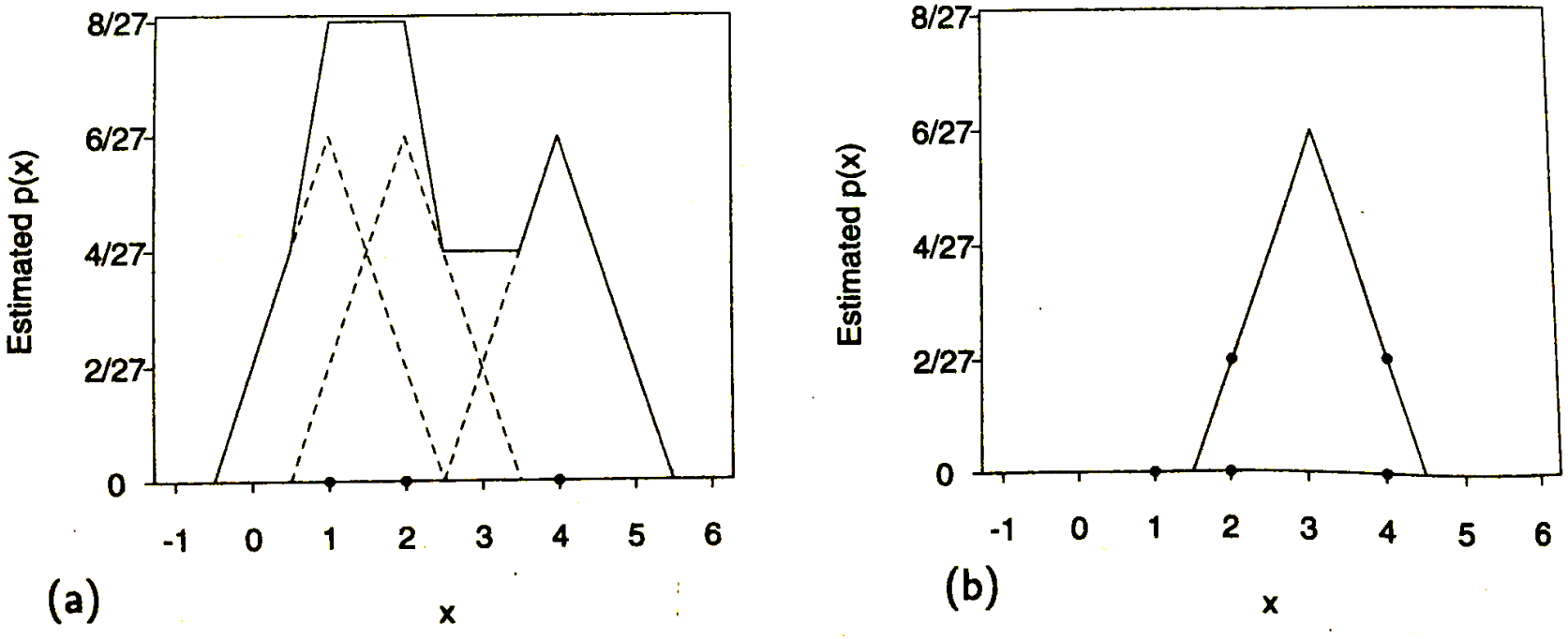
Kernel and Window Estimators

- The samples gives a very rough approximation to the true density function, namely a set of spikes or delta functions, one at each sample value, each with a very small width and a very large height.
- The combined area of all the spikes is one.
- Histogram based density approximation to a continuous density function is not useful in decision making.
- If the delta functions at each sample point are replaced by other function called *Kernels* – such as *rectangles*, *triangles*, or *normal density functions*, which have been scaled so that their combined area equals one-their sum produces a smoother, more satisfactory estimate.

Example

Question: Using a triangle kernel.
Consider the data set with one feature x and three samples at $x = 1, 2$, and 4. We have decided to use a triangular kernel with a base of three units. Plot the estimated density function $p(x)$.

Solution


8/27
6/27
4/27
2/27
0
Estimated p(x)
-1 0 1 2 3 4 5 6
(a)
x
8/27
6/27
4/27
2/27
0
Estimated p(x)
-1 0 1 2 3 4 5 6
(b)
x

Non-parametric Density Estimation

- Suppose that n samples $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are drawn i.i.d. according to the distribution $p(\mathbf{x})$.
- The probability P that a vector $\mathbf{x}$ will fall in a region $\mathcal{R}$ is given by

$$P = \int_{\mathcal{R}} p(\mathbf{x}')d\mathbf{x}'.$$

- The probability that k of the n will fall in $\mathcal{R}$ is given by the binomial law

$$P_k = \binom{n}{k} P^k (1-P)^{n-k}.$$

- The expected value of k is $E[k] = nP$ and the MLE for P is $\hat{P} = \frac{k}{n}$.

Non-parametric Density Estimation

- If we assume that $p(\mathbf{x})$ is continuous and $\mathcal{R}$ is small enough so that $p(\mathbf{x})$ does not vary significantly in it, we can get the approximation
$$\int_{\mathcal{R}} p(\mathbf{x}')d\mathbf{x}' \simeq p(\mathbf{x})V$$
where $\mathbf{x}$ is a point in $\mathcal{R}$ and V is the volume of $\mathcal{R}$.
- Then, the density estimate becomes
$$p(\mathbf{x}) \simeq \frac{k/n}{V}.$$

Non-parametric Density Estimation

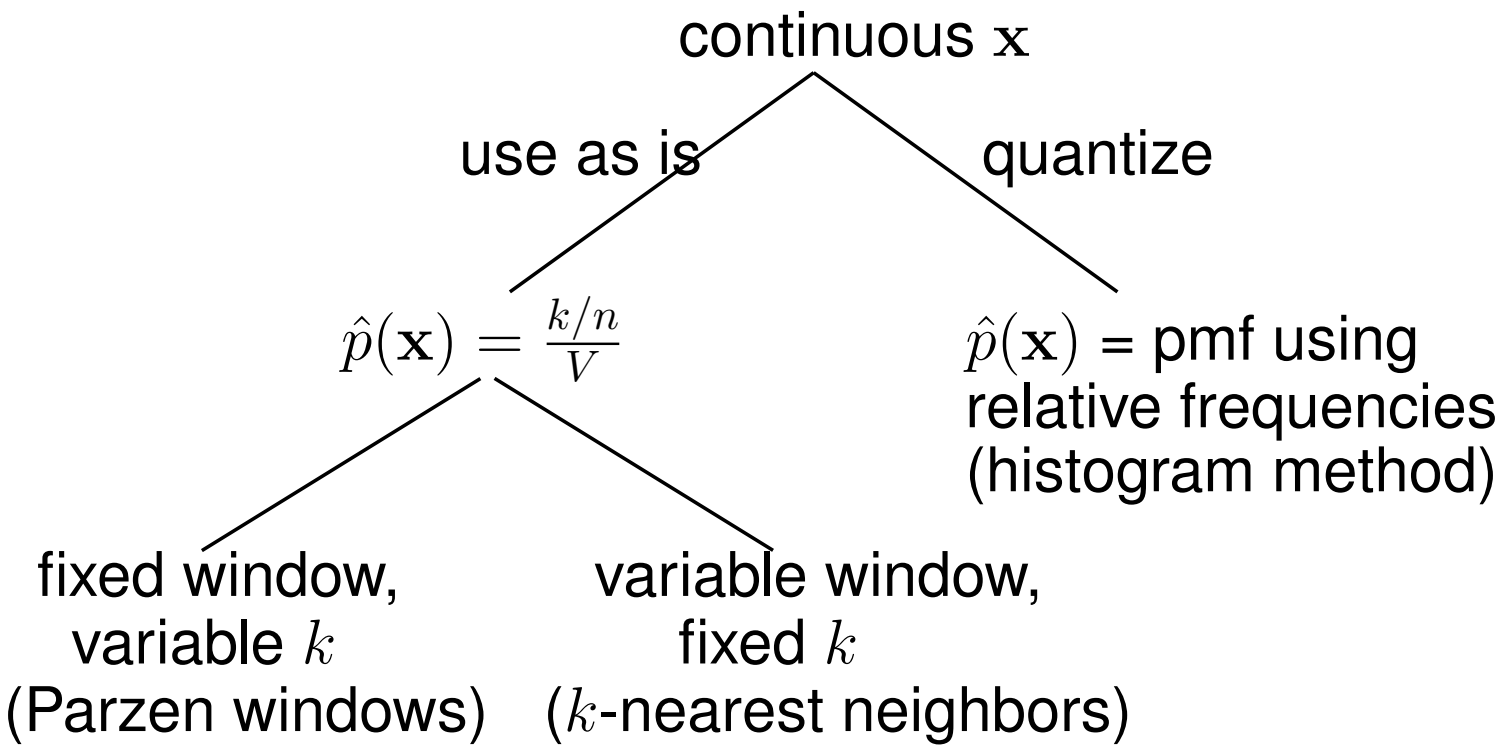
- Let n be the number of samples used, $\mathcal{R}_n$ be the region used with n samples, V_n be the volume of $\mathcal{R}_n$, k_n be the number of samples falling in $\mathcal{R}_n$, and $p_n(\mathbf{x}) = \frac{k_n/n}{V_n}$ be the estimate for $p(\mathbf{x})$.
- If $p_n(\mathbf{x})$ is to converge to $p(\mathbf{x})$, three conditions are required:

$$\lim_{n\to\infty} V_n = 0$$

$$\lim_{n\to\infty} k_n = \infty$$

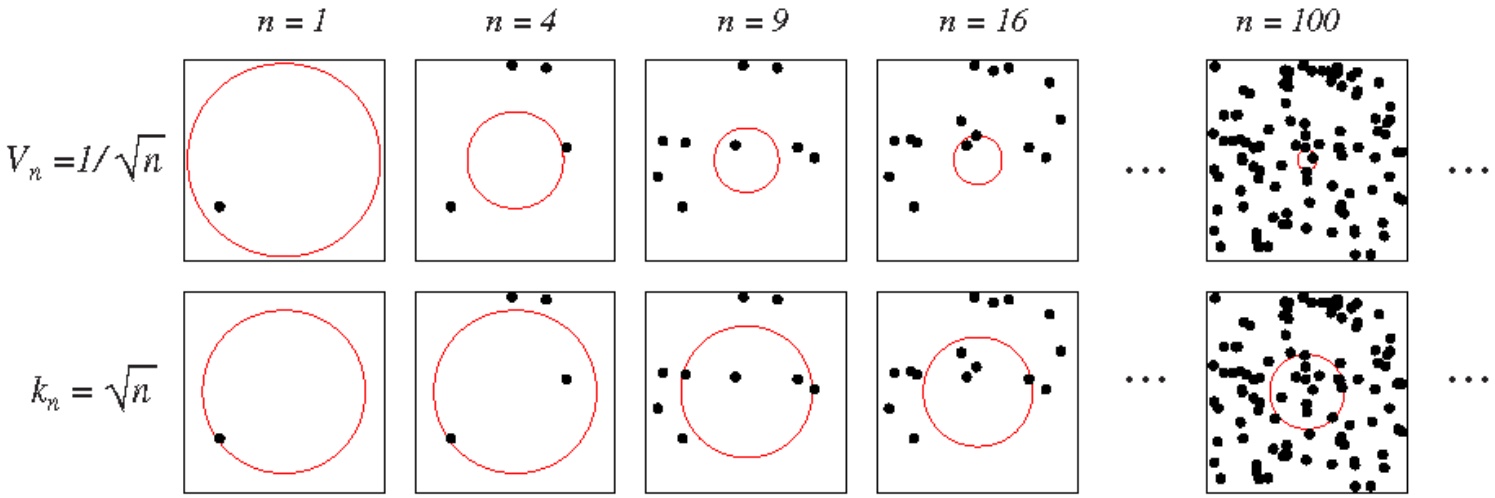
$$\lim_{n\to\infty} \frac{k_n}{n} = 0.$$

Non-parametric Methods



Non-parametric Density Estimation

- Other methods for obtaining the regions for estimation:
  - Shrink regions as some function of n, such as $V_n = 1|\sqrt{n}$. This is the *Parzen window* estimation
  - Specify k_n as some function of n, such as $k_n = \sqrt{n}$. This is the *k-nearest neighbor* estimation



Parzen Window

- Suppose that φ is a d-dimensional window function that satisfies the properties of a density function, i.e.,

$$\varphi(\mathbf{u}) \geq 0 \quad \text{and} \quad \int \varphi(\mathbf{u}) d\mathbf{u} = 1.$$

- A density estimate can be obtained as

$$p_n(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{V_n} \varphi\left(\frac{\mathbf{x} - \mathbf{x}_{\boldsymbol{i}}}{h_n}\right)$$

where h_n is the window width and $V_n = h_n^d$.

Parzen Window

- The density estimate can also be written as

$$p_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\delta_n(\mathbf{x}-\mathbf{x}_{\boldsymbol{i}}) \quad \text{where} \quad \delta_n(\mathbf{x}) = \frac{1}{V_n}\varphi\left(\frac{\mathbf{x}}{h_n}\right).$$

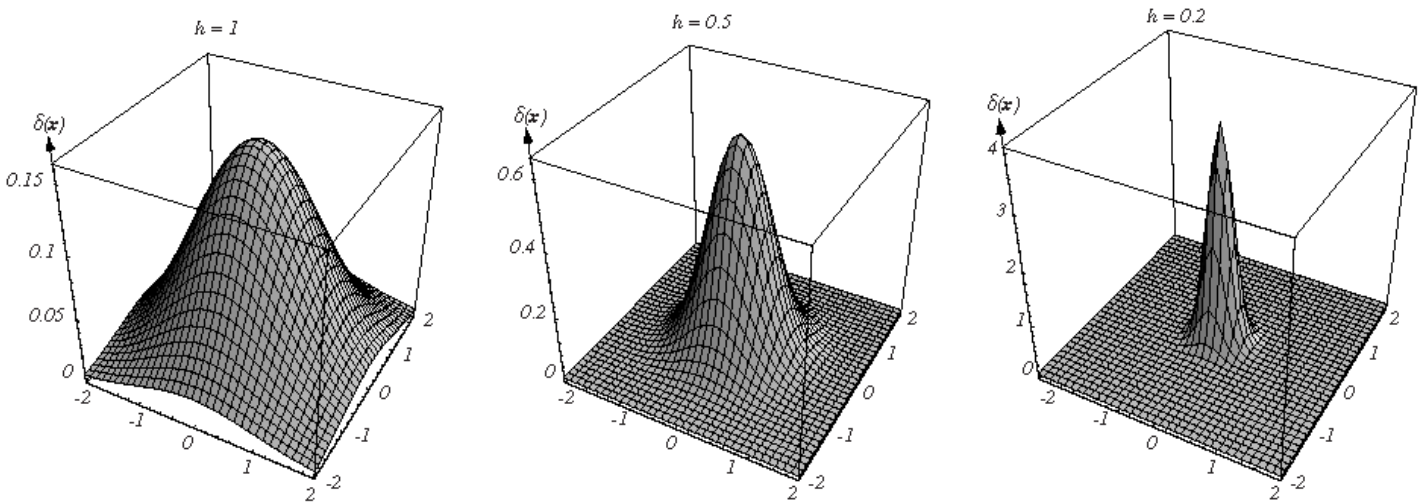


Figure : Examples of two-dimensional circularly symmetric Parzen windows functions for three different values of h_n. The value of h_n affects both the amplitude and the width of $\delta_n(\mathbf{x})$.

Parzen Window

- If h_n is very large, $p_n(\mathbf{x})$ is the superposition of n broad functions, and is a smooth "out-of-focus" estimate of $p(\mathbf{x})$.
- If h_n is very small, $p_n(\mathbf{x})$ is the superposition of n sharp pulses centered at the samples, and is a "noisy" estimate of $p(\mathbf{x})$.
- As h_n approaches zero, $\delta_n(\mathbf{x} - \mathbf{x}_i)$ approaches a Dirac delta function centered at $\mathbf{x}_i$, and $p_n(\mathbf{x})$ is a superposition of delta functions.

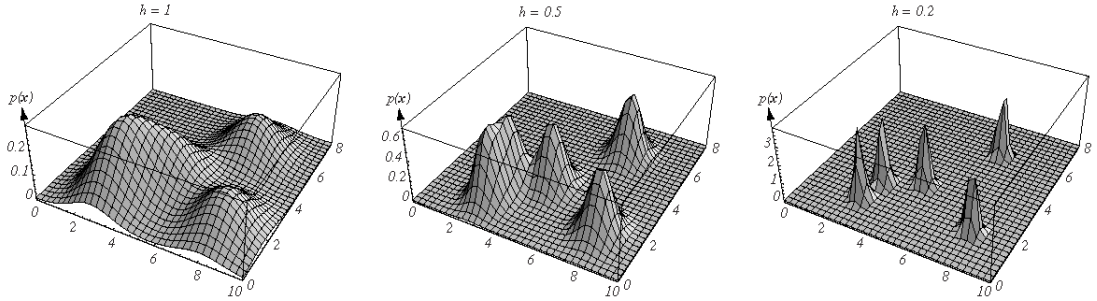


Figure : Parzen window density estimates based on the same set of five samples using the window functions in the previous figure.

Parzen Window

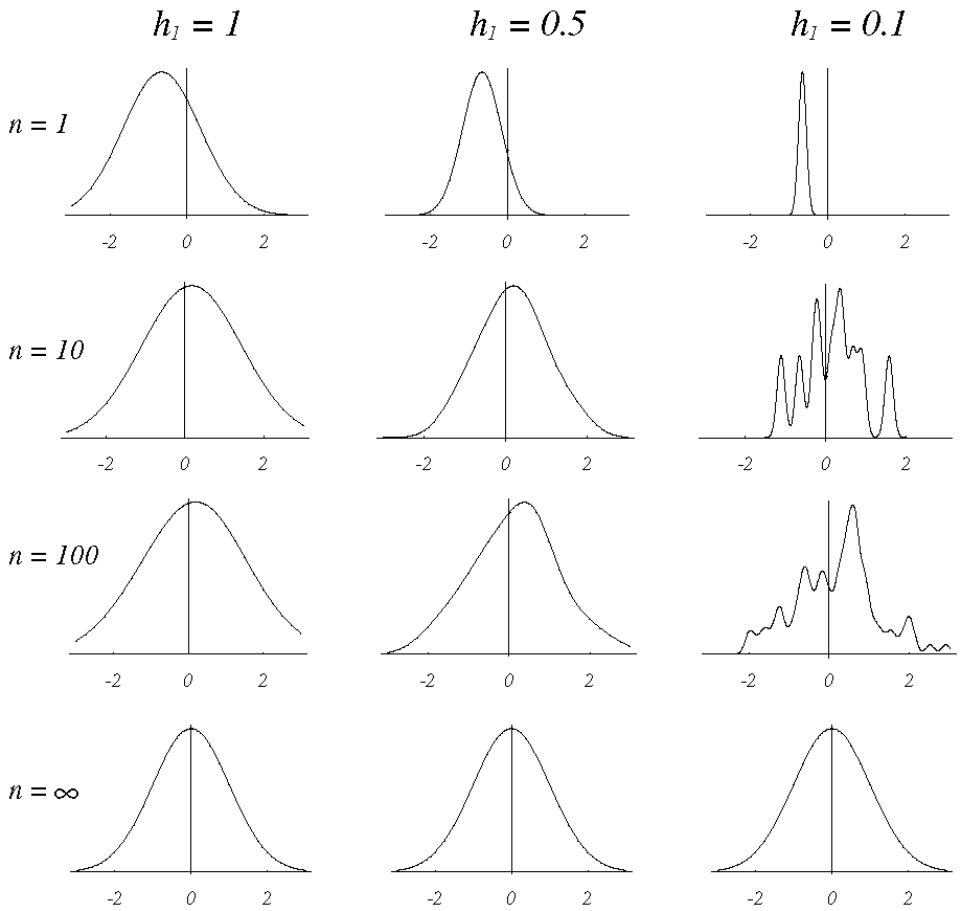


Figure 5: Parzen window estimates of a univariate Gaussian density using different window widths and numbers of samples where $\varphi(u) = N(0,1)$ and $h_n = h_1/\sqrt{n}$.

Parzen Window

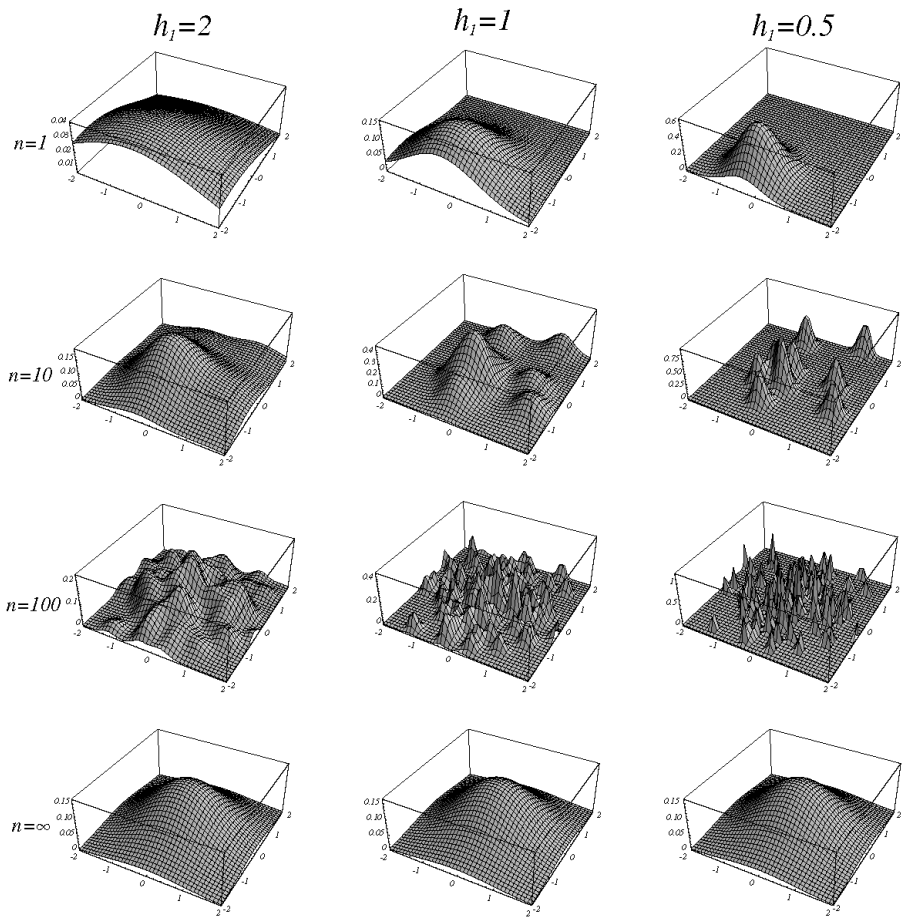


Figure 6: Parzen window estimates of a bivariate Gaussian density using different window widths and numbers of samples where $\varphi(\mathbf{u}) = N(\mathbf{0}, \mathbf{I})$ and $h_n = h_1/\sqrt{n}$.

Parzen Window

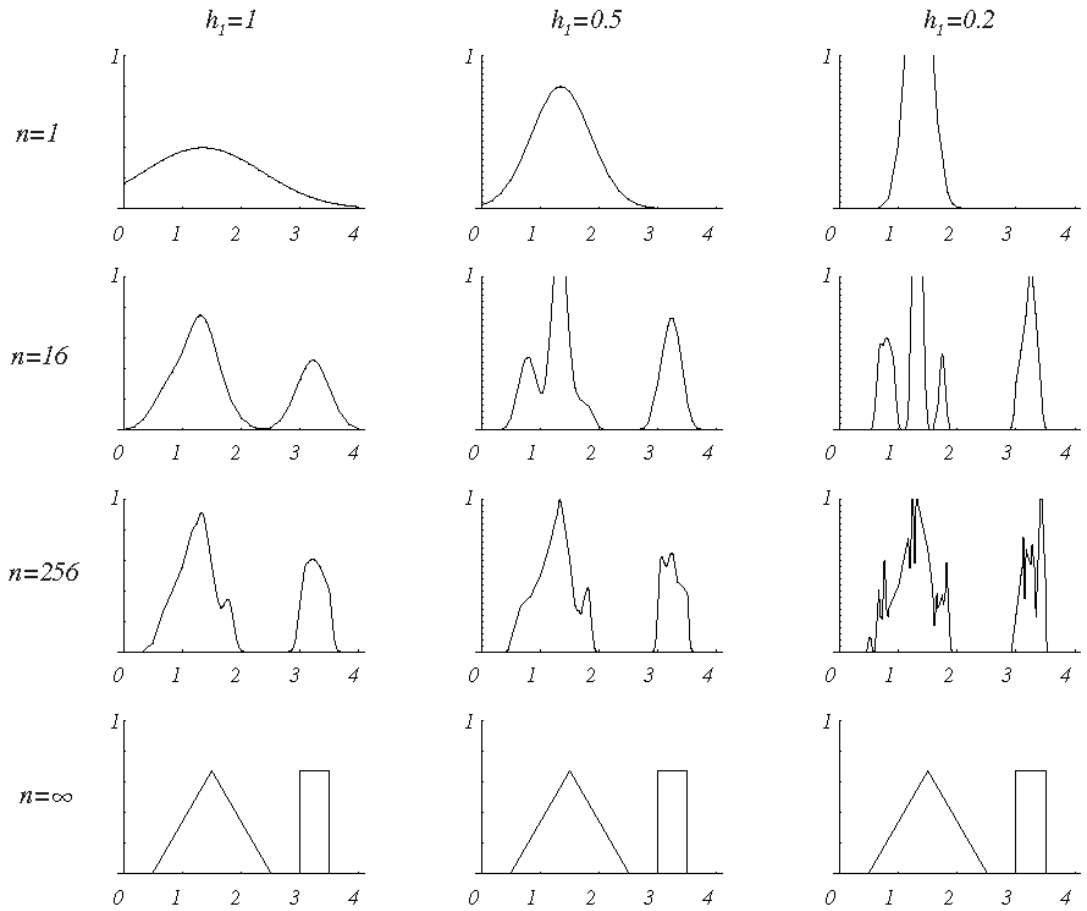


Figure 7: Estimates of a mixture of a uniform and a triangle density using different window widths and numbers of samples where $\varphi(u) = N(0,1)$ and $h_n = h_1/\sqrt{n}$.

Parzen Window

- Densities estimated using Parzen windows can be used with the Bayesian decision rule for classification.
- The training error can be made arbitrarily low by making the window width sufficiently small.
- However, the goal is to classify novel patterns so the window width cannot be made too small.

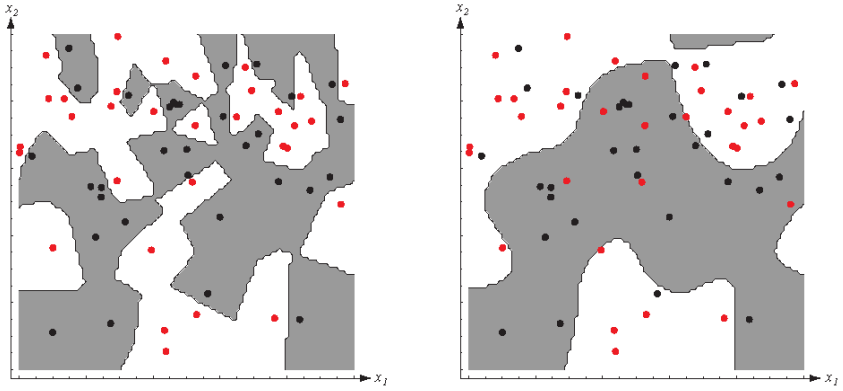


Figure 8: Decision boundaries in 2-D. The left figure uses a small window width and the right figure uses a larger window width.

$k-$Nearest Neighbors

- A potential remedy for the problem of the unknown "best" window function is to let the estimation volume be a function of the training data, rather than some arbitrary function of the overall number of samples.
- To estimate $p(\mathbf{x})$ from n samples, we can center a volume about $\mathbf{x}$ and let it grow until it captures k_n samples, where k_n is some function of n.
- These samples are called the k-nearest neighbors of $\mathbf{x}$.
- If the density is high near $\mathbf{x}$, the volume will be relatively small. If the density is low, the volume will grow large.

$k-$Nearest Neighbors

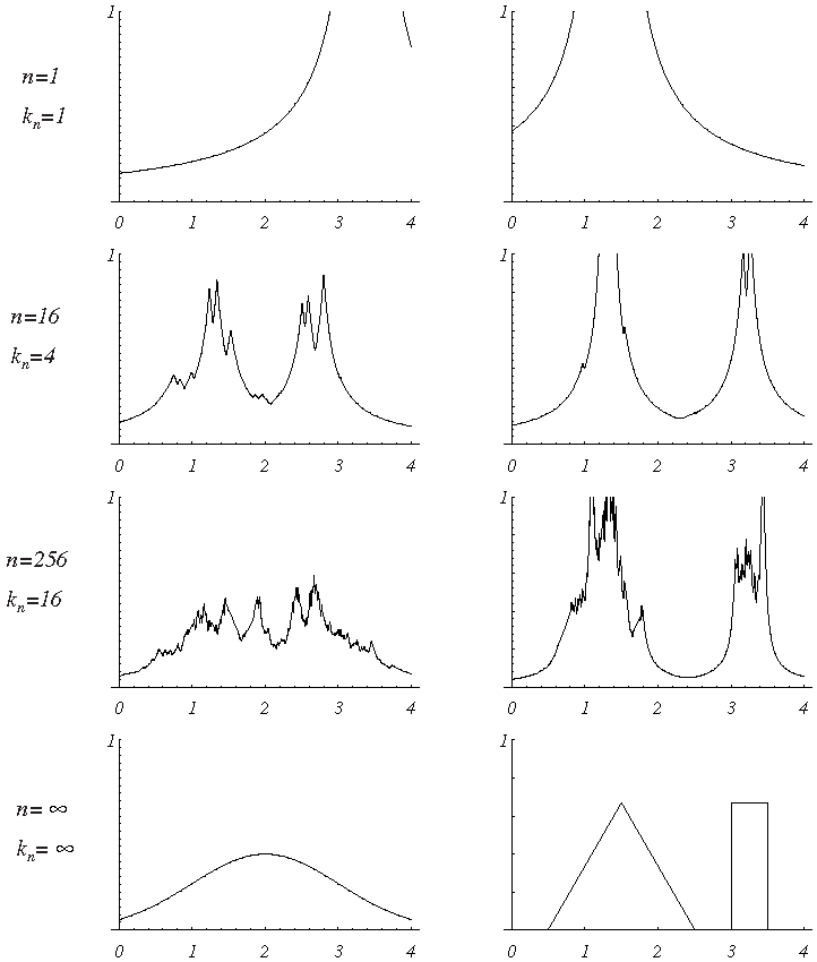


Figure : k-nearest neighbor estimates of two 1-D densities: a Gaussian and a bimodal distribution.

$k-$Nearest Neighbors

- Posterior probabilities can be estimated from a set of n labeled samples and can be used with the Bayesian decision rule for classification.
- Suppose that a volume V around $\mathbf{x}$ includes k samples, k_i of which are labeled as belonging to class w_i.
- As estimate for the joint probability $p(\mathbf{x}, w_i)$ becomes

$$p_n(\mathbf{x}, w_i) = \frac{k_i/n}{V}$$

and gives an estimate for the posterior probability

$$P_n(w_i|\mathbf{x}) = \frac{p_n(\mathbf{x}, w_i)}{\sum_{j=1}^{c} p_n(\mathbf{x}, w_j)} = \frac{k_i}{k}.$$

Non-parametric Methods

- Advantages:
  - No assumptions are needed about the distributions ahead of time (generality).
  - With enough samples, convergence to an arbitrarily complicated target density can be obtained.
- Disadvantages:
  - The number of samples needed may be very large (number grows exponentially with the dimensionality of the feature space).
  - There may be severe requirements for computation time and storage.

Nearest Neighbor Classification

The Nearest Neighbor Classifier

- We have been using Bayesian classifiers that make decisions according to the posterior probabilities.
- We have discussed parametric and non-parametric methods for learning classifiers by estimating the probabilities using training data.
- We will study new techniques that use training data to learn the classifiers directly without estimating any probabilistic structure.
- In particular, we will study the k-nearest neighbor classifier, linear discriminant functions, and support vector machines.

The Nearest Neighbor Classifier

- Given the training data $\mathcal{D} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ as a set of n labeled examples, the *nearest neighbor classifier* assigns a test point $\mathbf{x}$ the label associated with its closest neighbor in $\mathcal{D}$.
- Closeness is defined using a distance function.
- Given the distance function, the nearest neighbor classifier partitions the feature space into cells consisting of all points closer to a given training point than to any other training points.

The Nearest Neighbor Classifier

- All points in such a cell are labeled by the class of the training point, forming a *Voronoi tesselation* of the feature space.

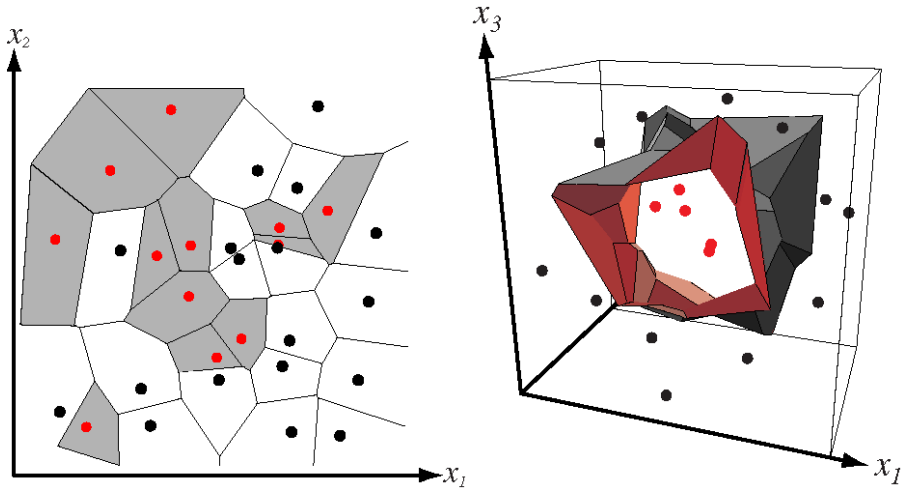


Figure 1: In two dimensions, the nearest neighbor algorithm leads to a partitioning of the input space into Voronoi cells, each labeled by the class of the training point it contains. In three dimensions, the cells are three-dimensional, and the decision boundary resembles the surface of a crystal.

The k-Nearest Neighbor Classifier

- The *k-nearest neighbor classifier* classifies $\mathbf{x}$ by assigning it the label most frequently represented among the k nearest samples.
- In other words, a decision is made by examining the labels on the k-nearest neighbors and taking a vote.

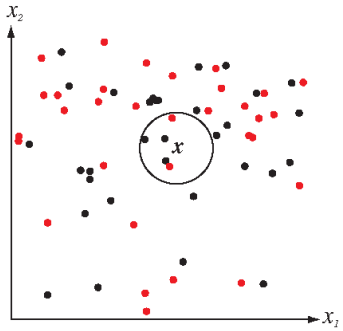


Figure 2: The k-nearest neighbor query forms a spherical region around the test point $\mathbf{x}$ until it encloses k training samples, and it labels the test point by a majority vote of these samples. In the case for $k = 5$, the test point will be labeled as black.

The k-Nearest Neighbor Classifier

- The computational complexity of the nearest neighbor algorithm — both in space (storage) and time (search) — has received a great deal of analysis.
- In the most straightforward approach, we inspect each stored training point one by one, calculate its distance to $\mathbf{x}$, and keep a list of the k closest ones.
- There are some parallel implementations and algorithmic techniques for reducing the computational load in nearest neighbor searches.

Nearest Neighbor Algorithm

Learning Algorithm:

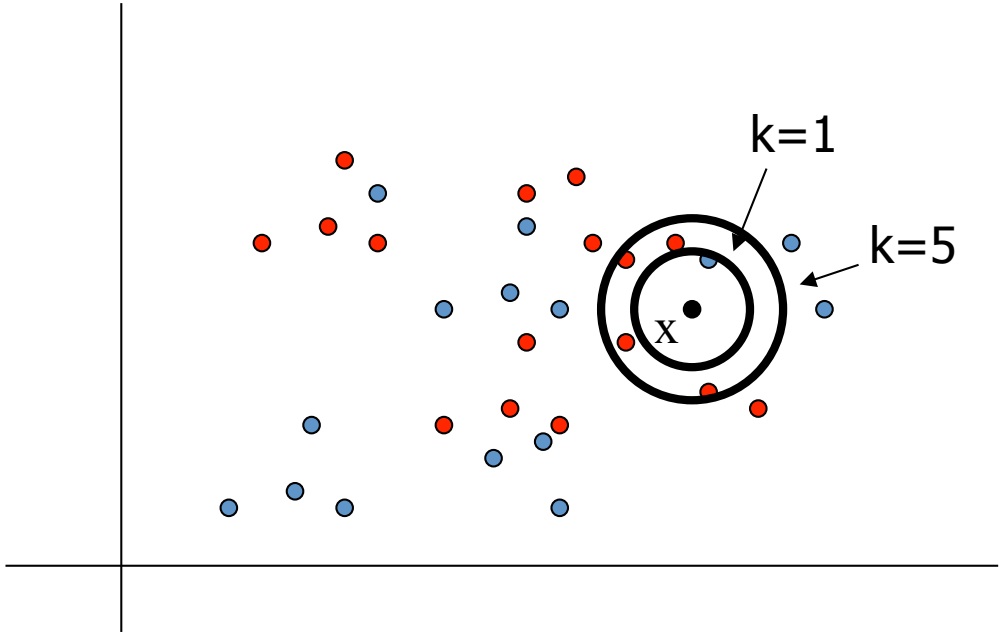
- Store training examples

Prediction Algorithm:

- To classify a new example x by finding the training example (x_i, y_i) that is nearest to x
- Guess the class $y = y_i$

Nearest Neighbor Algorithm

- To classify a new input vector x, examine the k-closest training data points to x and assign the object to the most frequently occurring class



- common values for k: 3, 5

The k-Nearest Neighbor Classifier

- Examples of algorithmic techniques include
  - computing partial distances using a subset of dimensions, and eliminating the points with partial distances greater than the full distance of the current closest points,
  - using search trees that are hierarchically structured so that only a subset of the training points are considered during search,
  - editing the training set by eliminating the points that are surrounded by other training points with the same class label.

Distance Functions

Distance Functions

- The nearest neighbor classifier relies on a *metric* or a *distance function* between points.
- For all points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, a metric $D(\cdot,\cdot)$ must satisfy the following properties:
  - Nonnegativity: $D(\mathbf{x},\mathbf{y}) \geq 0$.
  - Reflexivity: $D(\mathbf{x},\mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
  - Symmetry: $D(\mathbf{x},\mathbf{y}) = D(\mathbf{y},\mathbf{x})$.
  - Triangle inequality: $D(\mathbf{x},\mathbf{y}) + D(\mathbf{y},\mathbf{z}) \geq D(\mathbf{x},\mathbf{z})$.
- If the second property is not satisfied, $D(\cdot,\cdot)$ is called a pseudometric.

Distance Functions

- A general class of metrics for d-dimensional patterns is the *Minkowski metric*

$$L_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{d} |\mathbf{x}_i - \mathbf{y}_i|^p \right)^{1/p}$$

  also referred to as the L_p *norm*.

- The *Euclidean distance* is the L_2 norm

$$L_2(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{d} |\mathbf{x}_i - \mathbf{y}_i|^2 \right)^{1/2}.$$

- The *Manhattan* or *city block distance* is the L_1 norm

$$L_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{d} |\mathbf{x}_i - \mathbf{y}_i|.$$

Distance Functions

- The L_∞ norm is the maximum of the distances along individual coordinate axes

$$L_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1}^{d} |\mathbf{x}_i - \mathbf{y}_i|.$$

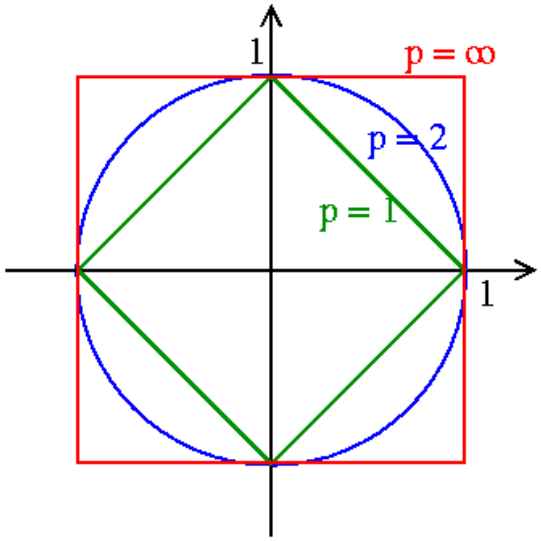


Figure 3: Each colored shape consists of points at a distance 1.0 from the origin, measured using different values of p in the Minkowski L_p metric.

Feature Normalization

- We should be careful about scaling of the coordinate axes when we compute these metrics.
- When there is great difference in the range of the data along different axes in a multidimensional space, these metrics implicitly assign more weighting to features with large ranges than those with small ranges.
- *Feature normalization* can be used to approximately equalize ranges of the features and make them have approximately the same effect in the distance computation.
- The following methods can be used to independently normalize each feature.

Feature Normalization

- *Linear scaling to unit range:*
  Given a lower bound l and an upper bound u for a feature $x \in \mathbb{R}$,
  $$\tilde{x} = \frac{x - l}{u - l}$$
  results in $\tilde{x}$ being in the $[0, 1]$ range.
- *Linear scaling to unit variance:*
  A feature $x \in \mathbb{R}$ can be transformed to a random variable with zero mean and unit variance as
  $$\tilde{x} = \frac{x - \mu}{\sigma}$$
  where μ and σ are the sample mean and the sample standard deviation of that feature, respectively.

Problem on Probability Density Estimation

Question: Following sets of 2-D feature vectors from classes A and B are given

$$\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}2.5\\2\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}4\\1\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right\}\in A$$

$$\left\{\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}4.5\\3\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix},\begin{pmatrix}6\\3\end{pmatrix},\begin{pmatrix}4\\6\end{pmatrix},\begin{pmatrix}7\\3\end{pmatrix}\right\}\in B$$

Using rectangular window of size 3×3, compute $p((3.5,3)^t|A)$ and $p((3.5,3)^t|B)$. Classify $(3.5,3)^t$ if $P(A)=1/3$ and $P(B)=2/3$.

Problem on nearest neighbor rule

Consider the following set of seven 2-dimensional feature vectors:

$$X_1 = (1,0)^t,\ X_2 = (0,1)^t,\ X_3 = (0,-1)^t,$$

$$X_4 = (0,0)^t,\ X_5 = (0,2)^t,\ X_6 = (0,-2)^t,\ X_7 = (-2,0)^t$$

If $X_1, X_2, X_3 \in \omega_1$ and $X_4, X_5, X_6, X_7 \in \omega_2$, sketch the decision boundary resulting from the nearest neighbor rule.

Feature Normalization

- *Normalization using the cumulative distribution function:*
  Given a random variable $x \in \mathbb{R}$ with cumulative distribution function $F_x(x)$, the random variable $\tilde{x}$ resulting from the transformation $\tilde{x} = F_x(x)$ will be uniformly distributed in $[0, 1]$.

- *Rank normalization:*
  Given the sample for a feature as $x_1, \ldots, x_n \in \mathbb{R}$, first we find the order statistics $x^{(1)}, \ldots, x^{(n)}$ and then replace each pattern's feature value by its corresponding normalized rank as

$$\tilde{x}_i = \frac{\underset{x_1,\ldots,x_n}{\text{rank}}\ (x_i) - 1}{n - 1}$$

  where x_i is the feature value for the i'th pattern. This procedure uniformly maps all feature values to the $[0, 1]$ range.

Distance Metric

- *Euclidean distance:* $\mathrm{a} = (a_1, a_2, \ldots, a_d)$ and $\mathrm{b} = (b_1, b_2, \ldots, b_d)$

$$d_e(\mathrm{a}, \mathrm{b}) = \sqrt{\sum_{i=1}^{d} (b_i - a_i)^2}$$

  where d is the dimension of feature vector.

- Absolute differences:

$$d_{cb}(a, b) = \sum_{i=1}^{d} |b_i - a_i|$$

  also called *city block distance*, the *Manhattan distance* or the *taxi-cab distance*

- *Maximum distance:*

$$d_m(a, b) = \max_{i=1}^{d} |b_i - a_i|$$

Distance Metric

- *Minkowski distance:* $\mathrm{a} = (a_1, a_2, \ldots, a_d)$ and $\mathrm{b} = (b_1, b_2, \ldots, b_d)$

$$d_r(a, b) = \left[\sum_{i=1}^{d} |b_i - a_i|^r \right]^{1/r}$$

where r is an adjustable parameter.

References

[1] Richard O Duda, Peter E Hart, and David G Stork. *Pattern classification*. John Wiley & Sons, 2012.

Thank you!