Digital Filter Design

(Subject Code - EET 3134)

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Linear-Phase FIR filters using Windows

FIR filter design using windowing technique

- Suppose desired impulse response of the filter is $h_d(n)$, which is related to $H_d(\omega)$ by the Fourier transform relation as

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

- Let us design a lowpass filter

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$$h_d(n) = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n} & |n| > 0 \end{cases}$$

FIR filter design using windowing technique

- Continuing with design a lowpass filter, we choose a finite series by truncating

$$H_M(\omega) = \sum_{n=-M}^{M} h_d(n) e^{-jwn}$$

which contains $(2M+1)$ coefficients from $-M$ to M

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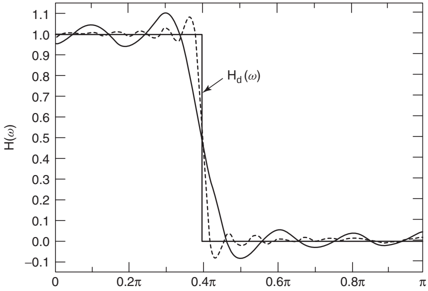
- As M increases, the finite series of $H_M(\omega)$ approximates to ideal response (i.e., desired) $H_d(\omega)$ in the least mean-square sense. That is

$$J(h,\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_M(\omega) - H_d(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{n=-M}^{M}\left(\frac{\sin\omega_c n}{\pi n}\right)e^{-j\omega n} - H_d(\omega)\right|^2 d\omega$$

attains a minimum at all frequencies, except at points of discontinuity.

FIR filter design using windowing technique



- As $M \uparrow$ the number of ripple in the passband and stopband will $\uparrow$.
- At the same time, width between the frequencies at which maximum error occurs in the passband $(0 \leq |\omega| \leq \omega_c)$ and in the stopband $(\omega_c \leq |\omega| \leq \pi)$ will $\downarrow$.

FIR filter design using Rectangular Window

- Truncating of $h_d(n)$ to a length $2M+1$ is equivalent to multiplying $h_d(n)$ by a "rectangular window", defined as

$$w_R(n) = \begin{cases} 1 & -M \le n \le M \\ 0 & otherwise \end{cases} \tag{1}$$

- Thus the impulse response of the FIR filter will be

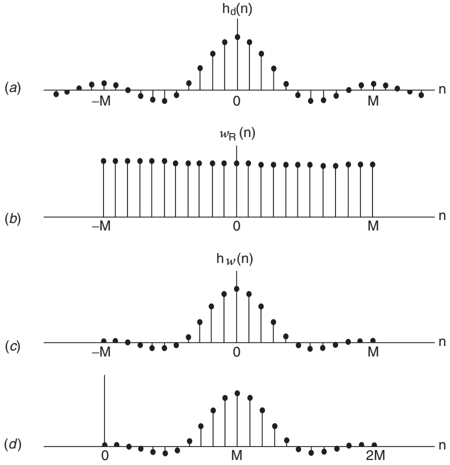
$$h_w(n) = h_d(n) w_R(n) \tag{2}$$

$$= \begin{cases} h_d(n) & -M \le n \le M \\ 0 & otherwise \end{cases} \tag{3}$$

- It is useful to consider the effect of the window function on the desired frequency response $H_d(w)$
- Recall

$$h_d(n) w(n) \leftrightarrow H_d(\omega) * W(\omega) \tag{4}$$

FIR filter design using Rectangular Window



FIR filter design using Window

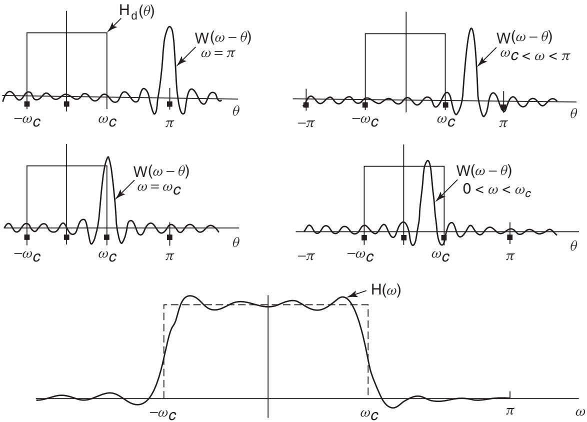
- $W(\omega)$ is the frequency domain representation of window function

$$W(\omega) = \sum_{n=-M}^{M} w(n)e^{-j\omega n} = \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)} \quad (5)$$

- Thus, the convolution will give frequency response characteristic of FIR filter

$$H_M(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\theta)W(\omega-\theta)d\theta \quad (6)$$

Convolution with desired frequency response

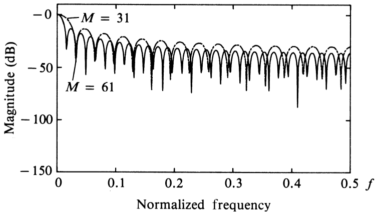


Window function main lobe and side lobe

- Main lobe of $W(\omega)$, centered at $\omega = 0$ has a width define by the first zero crossings on either sides of $\omega = 0$ which occur when

$$[(2M+1)\omega/2] = \pm\pi \tag{7}$$

- The width of the main lobe is $4\pi/(2M+1)$.
- As M increases, the width of the mainlobe and the sidelobe decrease, giving rise to more sidelobes or ripples in the same frequency band.



Gibbs Phenomenon

- It is obvious that if the width of the mainlobe is extremely narrow, the resulting $H_M(\omega)$ will have a sharp drop at $\omega = \omega_c$. If the number of sidelobes or their peak values in $W(\omega)$ increases, so also will the number of ripples and the maximum error in $H_M(\omega)$.
- The other hand, the sidelobes of $W(\omega)$ result in some undesirable ringing effects in the FIR filter.
- This is called Gibbs Phenomenon.
- These undesirable effects are best alleviated by the use of windows that do not contain abrupt discontinuities in their time-domain characteristic and have corresponding low sidelobes in their frequency-domain characteristics.

Different types of Window

Many tapered window have been proposed but few are popular

Bartlett window:[2]

$$w(n) = 1 - \frac{|n|}{M+1}; \quad -M \leq n \leq M$$

Hann window:

$$w(n) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi n}{2M+1}\right)\right]; \quad -M \leq n \leq M$$

Hamming window:

$$w(n) = 0.54 + 0.46\cos\left(\frac{2\pi n}{2M+1}\right); \quad -M \leq n \leq M$$

Blackman window:

$$w(n) = 0.42 + 0.5\cos\left(\frac{2\pi n}{2M+1}\right) + 0.08\cos\left(\frac{4\pi n}{2M+1}\right); \quad -M \leq n \leq M$$

Final Design

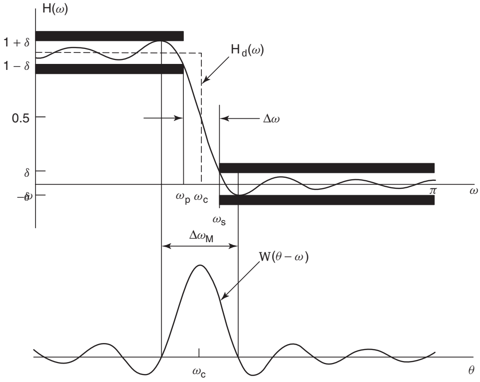


Figure: Frequency response of an ideal filter and final design.

- Maximum attenuation in the stopband of the filter:

$$A_s = -20\log_{10}(\delta_s)$$

- Transition bandwidth $= \Delta\omega = \omega_s - \omega_p$
- Cutoff frequency

$$\omega_c = \frac{\omega_p + \omega_s}{2}$$

Important characteristics of different window

- Relative sidelobe lebel A_{sl} is defined as the difference in dB between magnitude of the mainlobe of the window function chosen and the largest sidelobe.

Type of Window	$\Delta\omega_M$	$\Delta\omega$	A_{sl} (dB)	A_s (dB)
Rectangular	$4\pi/(2M+1)$	$0.92\pi/M$	13	20.9
Bartlett	$4\pi/(M+1)$	—[a]	26.5	—[a]
Hann	$8\pi/(2M+1)$	$3.11\pi/M$	31.5	43.9
Hamming	$8\pi/(2M+1)$	$3.32\pi/M$	42.7	54.5
Blackman	$12\pi/(2M+1)$	$5.56\pi/M$	58.1	75.3

[a] The frequency response of the Bartlett window decreases monotonically and therefore does not have sidelobes. So the transition bandwidth and sidelobe attenuation cannot be found for this window.

- A_s is the minimum attenuation in the stopband.
- $\omega_c = 0.4\pi$ and $M = 128$, Type I FIR filter

Trade-off

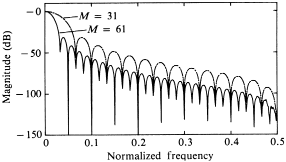
- It can be observed from the table that A_s increases, with fixed value for M, the transition bandwidth $\Delta\omega$ also increases.
- Since we like to have a large value for A_s and a small value of $\Delta\omega$. So we have to make a trade-off between them.
- Window function and $M \rightarrow$ control $\Delta\omega$.
- Minimum attenuation in stopband A_s depends only on the window function we chose, and not on the value of M.
- Two window functions that provide control over both δ_s (hence A_s) and $\Delta\omega$ are
    - Dolph-Chebyshev window (r)
    - Kaiser Window function (β)

Some other standard formulas for different window

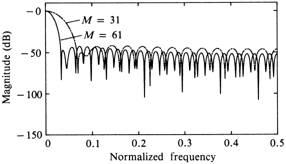
Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2} \right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{M-1}{2}\right)^2 - \left(n - \dfrac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\dfrac{M-1}{2}\right)\right]}$

- All the above function have significantly lower sidelobes compared with to rectangular window.
- For the same value of M, the width of the main lobe is also wider for these windows compared to the rectangular window.

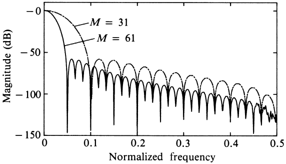
Frequency response



(a) Hanning window



(b) Hamming window



(c) Blackman

- The given table summarize these important frequency domain features of various window function.

Type of window	Approximate transition width of main lobe	Peak sidelobe (dB)
Rectangular	$4\pi/M$	-13
Bartlett	$8\pi/M$	-25
Hanning	$8\pi/M$	-31
Hamming	$8\pi/M$	-41
Blackman	$12\pi/M$	-57

Kaiser Window Function

- Kaiser window is defined as

$$w(n) = \frac{I_0\left\{\beta\sqrt{1-(n/M)^2}\right\}}{I_0\left\{\beta\right\}} \qquad -M \le n \le M$$

- where $I_0\{\}$ is the modified zero-order Bessel function. It is a power series of the form

$$I_0\{x\} = 1 + \sum_{k=1}^{\infty}\left[\frac{(x/2)^k}{k!}\right]^2$$

Computation of Kaiser window in three steps

1. The parameter β requires to achieve the desired attenuation $\alpha_s = -20\log_{10}(\delta_s)$ in the stopband is calculated form the following empirical formula derived by Kaiser (the ripple in the passband in nearly the same as δ_s)

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7) & \text{for} \quad \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) & \text{for} \quad 21 \leq \alpha_s \leq 50 \\ 0 & \text{for} \quad \alpha_s < 21 \end{cases}$$

2. Next the order of the filter $N(= 2M)$ is estimated from the another empirical formula derived by Kaiser

$$N = \frac{(\alpha_s - 8)}{2.285(\Delta\omega)} \qquad (\Delta\omega = \omega_s - \omega_p)$$

Computation of Kaiser window in three steps

3. The third step is to compute $I_0\{x\}$. In practice adding a finite number of terms, say, 20 terms of the infinite series gives a sufficiently accurate value $I_0\{x\}$. The parameters x in the numerator represents $\beta\sqrt{1-(n/M)^2}$ in the numerator of Kaiser window, so the value of x takes different values as n changes.

Digital Filter Design Lab

Lab Session 02

Assignment 1

- Design a low-pass filter using FIR windowing technique that satisfies the specification of $\omega_p = 0.4\pi$ and $\omega_s = 0.6\pi$ and exhibits a minimum attenuation greater than 50dB in the stop-band.

Lab Session 02

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Assignment 2

- A FIR low-pass filter is required to have the following specifications:
  - Pass-band edge frequency $f_p = 2kHz$
  - Transition band $\Delta f = 200Hz$
  - Pass-band ripple $A_p = 0.1dB$
  - Minimum stop-band attenuation $A_s = 50dB$
  - Sampling frequency of $f_s = 10kHz$

  Using the window method, determine an appropriate window function and calculate the required number of filter coefficients to design this filter. Furthermore, ascertain the corresponding filter coefficient values $h[n]$ for $-10 \leq n \leq 10$.

Lab Session 02

Assignment 3

- Design an FIR filter using windowing method that need to adhere to the following specifications.
    - Passband 8-12 kHz
    - Stopband ripple 0.001
    - Peak passband ripple 0.01
    - Sampling frequency 44.14 kHz
    - Transition width 3 kHz

Lab Session 02

Assignment 3

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  - Peak passband ripple 0.01
  - Sampling frequency 44.14 kHz
  - Transition width 3 kHz

Assignment 4

- Design an optimal FIR filter for the same specifications of Assignment 3 and compare the performances of both.

Thank You

Queries?