Digital Filter Design

(Subject Code - EET 3134)

Dr. Kundan Kumar
Ph.D. (IIT Kharagpur)
Assistant Professor
ECE Department (Cabin - E139)

Institute of Technical Education & Research (ITER)
SOA Deemed to be University, Bhubaneswar, Odisha, India-751030

Digital Filters

Digital Filtering System

- Types of Filters
  1. Time-domain
  2. Frequency-domain
- Simplest kind of filters are non-recursive filters defined as

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{1}$$

where, $h(k)$ are the constants of the filter, the $x(n-k)$ are the input data, and the $y(n)$ are the outputs.

Moving Averaging Filter

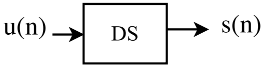
$$y(n) = \sum_{k=0}^{M-1} x(k)h(n-k) \tag{2}$$

	0	0	0	0	x_0	x_1	x_2	x_3	x_4	x_5	$\cdots$	
	h_5	h_4	h_3	h_2	h_0							y_0
		h_5	h_4	h_3	h_2	h_0						y_1
			h_5	h_4	h_3	h_2	h_0					y_3
				h_5	h_4	h_3	h_2	h_0				y_4
					h_5	h_4	h_3	h_2	h_0			y_5

- Implement a de-noising filter for a noisy sinusoidal signal using moving averaging Filter for different size of rectangular gates.

Introduction to Digital Filters

- In general, digital filters are used for
  1. Separation
  2. Restoration
- Every linear filter has
  1. Impulse response
  2. Unit step response
  3. Frequency response
- All these representation are important because they describe that how the filter will react under different circumstances.



Representation of Signals

- Information can be represented in
  - Time-domain
  - Frequency-domain
- Types of filters
  - Finite impulse response (FIR) filter

$$h(n) \neq 0 \qquad N_1 \leq n \leq N_2 \tag{3}$$
$$= 0 \qquad everywhereelse \tag{4}$$

  - Infinite impulse response (IIR) filter

$$h(n) \neq 0 \qquad \infty \leq n \leq \infty \tag{5}$$

Advantage of FIR over IIR filters

- We can easily design the FIR filter to meet the required magnitude response in such a way that it achieves a constant group delay, i.e., they can have exactly linear phase.
- Always stable, even when quantized
- Design methods are generally linear
- They can be realized efficiently in hardware.
- The filter startup transients have finite duration.

Disadvantages:

- Often require higher filter order than IIR filters to achieve a good level of performance.
- The delay of these filters is often much greater than IIR filters having equal performance.

z Transform

The z transform of a sequence $x(n)$ is simply defined as

$$\mathcal{Z}[x(n)] = x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{6}$$

and the inverse z transform defined as

$$\mathcal{Z}[x(z)] = x(n) = \frac{1}{2\pi j} \int_C x(z) z^{n-1} dz \tag{7}$$

This is the double-sided or bilateral z transform of sequence $x(n)$ defined for $-\infty < n < \infty$.

Group Delay

- Group delay is defined as

$$\tau = -(d\theta/d\omega) \tag{8}$$

where ω is the phase response of the filter.

- The phase response of a filter with a constant group delay is therefore a linear function of frequency.
- It transmits all frequencies with the same amount of delay, which means that there will not be any phase distortion and the input signal will be delayed by a constant when it is transmitted to the output.
- A filter with a constant group delay is highly desirable in the transmission of digital signals.

Notation

- $$H\left(z^{-1}\right) = \sum_{n=0}^{N} h(n)z^{-n} \tag{9}$$

  The order of the FIR filter or degree of the polynomial $= N$
  The length of the filter or the number of coefficient $= N + 1$
- Example: $H(z^{-1}) = 0.3z^{-4} + 0.1z^{-5} + 0.5z^{-6}$, order and length?
- Notation often used in MATLAB

$$\begin{aligned} H(z^{-1}) =& h(1) + h(2)z^{-1} + h(3)z^{-2} + \cdots + h(N+1)z^{-N} \\ =& \sum_{n=0}^{N} h(n+1)z^{-n} \end{aligned}$$

Linear time-invariant causal system

Difference Equation

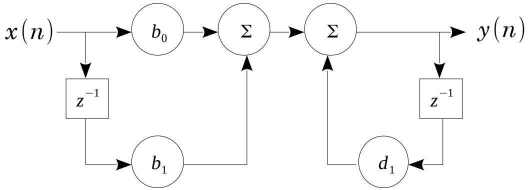
$$y(n) = -\sum_{k=1}^{N} a(k)y(n-k) + \sum_{k=0}^{M} b(k)x(n-k) \tag{10}$$

Transfer function

$$H(z^{-1}) = \frac{b(0) + b(1)z^{-1} + b(2)z^{-2} + \cdots + b(M)z^{-M}}{1 + a(1)z^{-1} + a(2)z^{-2} + \cdots + a(N)z^{-N}} \tag{11}$$

Example

$$y(n) = b_0 x(n) + b_1 x(n-1) + d_1 y(n-1) \tag{12}$$



Recursive and non-recursive filters

$$\left.\begin{array}{r}\text{FIR} \rightarrow \text{non-recursive} \\ \text{IIR} \rightarrow \text{recusive}\end{array}\right\} \begin{array}{l}\text{not necessarily,} \\ \text{it depends on representation} \\ \text{of output of the system}\end{array} \tag{13}$$

$$y(n) = x(n) + x(n-1) \rightarrow \text{non-recursive}$$
$$\Rightarrow y(n-1) = x(n-1) + n(n-2)$$
$$\text{So, } y(n) - y(n-1) = x(n) - x(n-2)$$
$$y(n) = y(n-1) + x(n) - x(n-2) \rightarrow \text{recursive}$$

Ideal filters

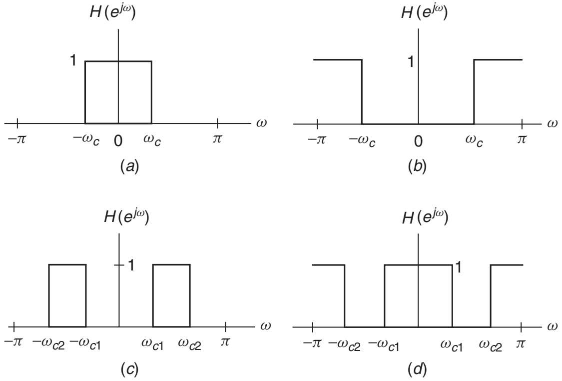


Figure: Basic filters (a) Lowpass filter, (b) Highpass filter, (c) Bandpass filter, and (d) Bandstop filter

Impulse responses of ideal filters

$$h(n) = \begin{cases} \dfrac{\omega_c}{\pi}; & n = 0 \\ \dfrac{\sin(\omega_c n)}{\pi n}; & |n| > 0 \end{cases}$$

(a) Lowpass filter

$$h(n) = \begin{cases} \dfrac{\omega_{c2} - \omega_{c1}}{\pi}; & n = 0 \\ \dfrac{1}{\pi n}[\sin(\omega_{c2} n) - \sin(\omega_{c1} n)]; & |n| > 0 \end{cases}$$

(b) Bandpass filter

$$h(n) = \begin{cases} 1 - \dfrac{\omega_c}{\pi}; & n = 0 \\ -\dfrac{\sin(\omega_c n)}{\pi n}; & |n| > 0 \end{cases}$$

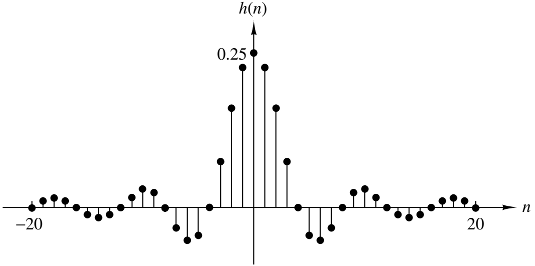
(c) Highpass filter

$$h(n) = \begin{cases} 1 - \dfrac{(\omega_{c2} - \omega_{c1})}{\pi}; & n = 0 \\ \dfrac{1}{\pi n}[\sin(\omega_{c1} n) - \sin(\omega_{c2} n)]; & |n| > 0 \end{cases}$$

(d) Bandstop filter

Ideal lowpass filter realization

$$H(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < \omega \leq \pi \end{cases} \qquad h(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin \omega_c n}{\omega_c n}, & n \neq 0 \end{cases}$$



- Ideal filters are not realizable due its non-causal property of their impulse response.

What we can realize?

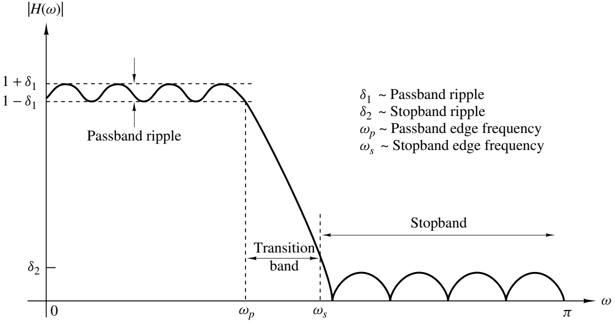


Figure: Magnitude characteristics of physically realizable filters

$$\Delta\omega = \omega_s - \omega_p$$

$$\omega_c = \omega_p + 0.5\Delta\omega$$

$$A_p = 20\log_{10}(1+\delta_p)$$

$$A_s = -20\log_{10}(\delta_s)$$

Linear Phase FIR Filters

Linear Phase FIR Filters

Consider the special types of FIR filters in which the coefficients $h(n)$ of the transfer function

$$H(z^{-1}) = \sum_{n=0}^{N} h(n) z^{-n} \tag{14}$$

Type I. The coefficients are symmetric, and N is even.

Type II. The coefficients are symmetric, and N is odd.

Type III. The coefficients are antisymmetric, and N is even.

Type IV. The coefficients are antisymmetric, and N is odd.

Type I Linear phase FIR Filter

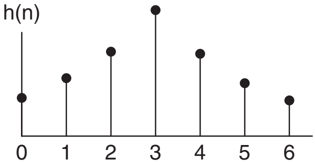
The coefficients are symmetric [i.e., $h(n) = h(N - n)$], and the order N is even.

Type I Linear phase FIR Filter

The coefficients are symmetric [i.e., $h(n) = h(N-n)$], and the order N is even.

Examples: $N = 6$

$$H(z^{-1}) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$



$$h(0) = h(6)$$
$$h(1) = h(5)$$
$$h(2) = h(4)$$

Type I Linear phase FIR Filter

$$\begin{aligned} H(z^{-1}) &= \{h(0)[1+z^{-6}] + h(1)[z^{-1}+z^{-5}] + h(2)[z^{-2}+z^{-4}] + h(3)z^{-3}\} \\ &= z^{-3}\{h(0)[z^{3}+z^{-3}] + h(1)[z^{2}+z^{-2}] + h(2)[z+z^{-1}] + h(3)\} \end{aligned}$$

Type I Linear phase FIR Filter

$$\begin{aligned} H(z^{-1}) &= \{h(0)[1+z^{-6}] + h(1)[z^{-1}+z^{-5}] + h(2)[z^{-2}+z^{-4}] + h(3)z^{-3}\} \\ &= z^{-3}\{h(0)[z^{3}+z^{-3}] + h(1)[z^{2}+z^{-2}] + h(2)[z+z^{-1}] + h(3)\} \end{aligned}$$

put $z = e^{jw}$ to find frequency response (DTFT), we get

Type I Linear phase FIR Filter

$$
\begin{aligned}
H(z^{-1}) &= \{h(0)[1+z^{-6}] + h(1)[z^{-1}+z^{-5}] + h(2)[z^{-2}+z^{-4}] + h(3)z^{-3}\} \\
&= z^{-3}\{h(0)[z^{3}+z^{-3}] + h(1)[z^{2}+z^{-2}] + h(2)[z+z^{-1}] + h(3)\}
\end{aligned}
$$

put $z = e^{jw}$ to find frequency response (DTFT), we get

$$
\begin{aligned}
H(e^{-j\omega}) =& e^{-3j\omega}\{h(0)[e^{3j\omega}+e^{-3j\omega}] + h(1)[e^{2j\omega}+e^{-2j\omega}] \\
&+ h(2)[e^{j\omega}+e^{-j\omega}] + h(3)\}
\end{aligned}
$$

Type I Linear phase FIR Filter

$$H(z^{-1}) = \{h(0)[1 + z^{-6}] + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} + z^{-4}] + h(3)z^{-3}\}$$
$$= z^{-3}\{h(0)[z^{3} + z^{-3}] + h(1)[z^{2} + z^{-2}] + h(2)[z + z^{-1}] + h(3)\}$$

put $z = e^{jw}$ to find frequency response (DTFT), we get

$$\begin{aligned} H(e^{-j\omega}) =& e^{-3j\omega}\{h(0)[e^{3j\omega} + e^{-3j\omega}] + h(1)[e^{2j\omega} + e^{-2j\omega}] \\ & + h(2)[e^{j\omega} + e^{-j\omega}] + h(3)\} \\ =& e^{-3j\omega}\{2h(0)\cos(3\omega) + 2h(1)\cos(2\omega) + 2h(2)\cos(\omega) + h(3)\} \\ =& e^{j\theta(\omega)}\{H_R(\omega)\} \end{aligned}$$

Type I Linear phase FIR Filter

$$H(z^{-1}) = \{h(0)[1+z^{-6}] + h(1)[z^{-1}+z^{-5}] + h(2)[z^{-2}+z^{-4}] + h(3)z^{-3}\}$$
$$= z^{-3}\{h(0)[z^{3}+z^{-3}] + h(1)[z^{2}+z^{-2}] + h(2)[z+z^{-1}] + h(3)\}$$

put $z = e^{jw}$ to find frequency response (DTFT), we get

$$\begin{aligned} H(e^{-j\omega}) =& e^{-3j\omega}\{h(0)[e^{3j\omega}+e^{-3j\omega}] + h(1)[e^{2j\omega}+e^{-2j\omega}] \\ &+ h(2)[e^{j\omega}+e^{-j\omega}] + h(3)\} \\ =& e^{-3j\omega}\{2h(0)\cos(3\omega) + 2h(1)\cos(2\omega) + 2h(2)\cos(\omega) + h(3)\} \\ =& e^{j\theta(\omega)}\{H_R(\omega)\} \end{aligned}$$

$$\text{phase angle, } \theta(\omega) = -3\omega \Rightarrow \text{linear function of } \omega$$

$$\text{group delay} = -\frac{d\theta(\omega)}{d\omega} = 3 \text{ samples} = 3T \text{ seconds} \tag{15}$$

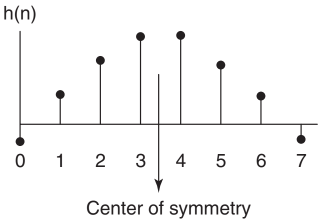
Total group delay is a constant $= \frac{N}{2}$ in the general case for a type I FIR filters.

Type II Linear phase FIR Filter

The coefficients are symmetric [i.e., $h(n) = h(N-n)$], and the order N is odd.

Examples: $N = 7$

$$H(z^{-1}) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$



$$h(0) = h(7)$$
$$h(1) = h(6)$$
$$h(2) = h(5)$$
$$h(3) = h(4)$$

Type II Linear phase FIR Filter

$$
\begin{aligned}
H(e^{-j\omega}) =& e^{-3.5j\omega}\{2h(0)\cos(3.5\omega) + 2h(1)\cos(2.5\omega) + 2h(2)\cos(1.5\omega) \\
&+ 2h(3)\cos(0.5\omega)\} \\
=& e^{j\theta(\omega)}\{H_R(\omega)\}
\end{aligned}
$$

$$\text{phase angle, } \theta(\omega) = -3.5\omega \Rightarrow \text{linear function of } \omega$$

$$\text{group delay} = -\frac{d\theta(\omega)}{d\omega} = 3.5 \text{ samples} = 3.5T \text{ seconds} \tag{16}$$

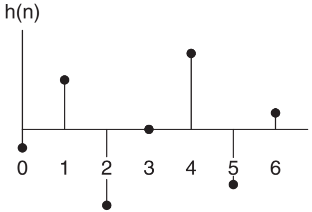
- In general, a linear phase $\theta(\omega) = -[\frac{N}{2}]\omega$
- Constant group delay $= \frac{N}{2}$

Type III Linear phase FIR Filter

The coefficients are antisymmetric [i.e., $h(n) = -h(N-n)$], and the order N is even. Here, $h(\frac{N}{2}) = 0$ for antisymmetric

Examples: $N = 6$

$$H(z^{-1}) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$



$$h(0) = -h(6)$$
$$h(1) = -h(5)$$
$$h(2) = -h(4)$$
$$h(3) = 0$$

Type III Linear phase FIR Filter

$$\begin{aligned} H(z^{-1}) &= \{h(0)[1 - z^{-6}] + h(1)[z^{-1} - z^{-5}] + h(2)[z^{-2} - z^{-4}]\} \\ &= z^{-3}\{h(0)[z^{3} - z^{-3}] + h(1)[z^{2} - z^{-2}] + h(2)[z - z^{-1}]\} \end{aligned}$$

put $z = e^{jw}$ to find frequency response (DTFT), we get

$$\begin{aligned} H(e^{-j\omega}) =& e^{-3j\omega}\{h(0)[e^{3j\omega} - e^{-3j\omega}] + h(1)[e^{2j\omega} - e^{-2j\omega}] + h(2)[e^{j\omega} - e^{-j\omega}]\} \\ =& e^{-3j\omega} e^{j\frac{\pi}{2}}\{2h(0)\sin(3\omega) + 2h(1)\sin(2\omega) + 2h(2)\sin(\omega)\} \\ =& e^{j(-3\omega+\frac{\pi}{2})}\{2h(0)\sin(3\omega) + 2h(1)\sin(2\omega) + 2h(2)\sin(\omega)\} \\ =& e^{j\theta(\omega)}\{H_R(\omega)\} \end{aligned}$$

$$\text{phase angle, } \theta(\omega) = -3\omega + \frac{\pi}{2} \Rightarrow \text{linear function of } \omega$$

$$\text{group delay} = -\frac{d\theta(\omega)}{d\omega} = 3 \text{ samples} = 3T \text{ seconds} \tag{17}$$

Type III Linear phase FIR Filter

$$H(z^{-1}) = \{h(0)[1 - z^{-6}] + h(1)[z^{-1} - z^{-5}] + h(2)[z^{-2} - z^{-4}]\}$$
$$= z^{-3}\{h(0)[z^{3} - z^{-3}] + h(1)[z^{2} - z^{-2}] + h(2)[z - z^{-1}]\}$$

put $z = e^{jw}$ to find frequency response (DTFT), we get

$$\begin{aligned} H(e^{-j\omega}) =& e^{-3j\omega}\{h(0)[e^{3j\omega} - e^{-3j\omega}] + h(1)[e^{2j\omega} - e^{-2j\omega}] + h(2)[e^{j\omega} - e^{-j\omega}]\} \\ =& e^{-3j\omega} e^{j\frac{\pi}{2}}\{2h(0)\sin(3\omega) + 2h(1)\sin(2\omega) + 2h(2)\sin(\omega)\} \\ =& e^{j(-3\omega+\frac{\pi}{2})}\{2h(0)\sin(3\omega) + 2h(1)\sin(2\omega) + 2h(2)\sin(\omega)\} \\ =& e^{j\theta(\omega)}\{H_R(\omega)\} \end{aligned}$$

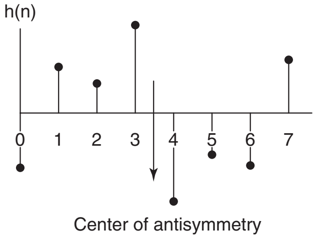
- In general, a linear phase $\theta(\omega) = -\left(\frac{N\omega - \pi}{2}\right)$
- Constant group delay $\tau = \frac{N}{2}$

Type IV Linear phase FIR Filter

The coefficients are symmetric [i.e., $h(n) = -h(N-n)$], and the order N is odd.

Examples: $N = 7$

$$H(z^{-1}) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$



$$h(0) = -h(7)$$
$$h(1) = -h(6)$$
$$h(2) = -h(5)$$
$$h(3) = -h(4)$$

Type IV Linear phase FIR Filter

$$
\begin{aligned}
H(e^{-j\omega}) =& e^{-3.5j\omega}\{2jh(0)\sin(3.5\omega) + 2jh(1)\sin(2.5\omega) + 2jh(2)\sin(1.5\omega) \\
&+ 2jh(3)\sin(0.5\omega)\} \\
=& e^{-3.5j\omega}e^{j\frac{\pi}{2}}\{2h(0)\sin(3.5\omega) + 2h(1)\sin(2.5\omega) + 2h(2)\sin(1.5\omega) \\
&+ 2h(3)\sin(0.5\omega)\} \\
=& e^{j\theta(\omega)}\{H_R(\omega)\}
\end{aligned}
$$

$$\text{phase angle, } \theta(\omega) = -3.5\omega + \frac{\pi}{2} \Rightarrow \text{linear function of } \omega$$

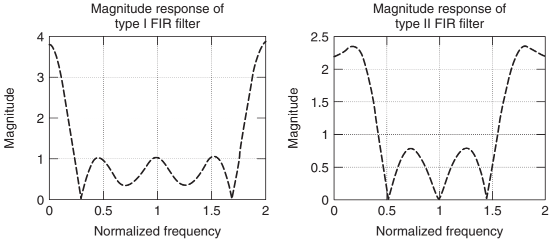
$$\text{group delay} = -\frac{d\theta(\omega)}{d\omega} = 3.5 \text{ samples} = 3.5T \text{ seconds} \tag{18}$$

- In general, a linear phase $\theta(\omega) = -\left(\frac{N\omega-\pi}{2}\right)$
- Constant group delay $\tau = \frac{N}{2}$

Frequency Response and Group Delay of FIR Filters

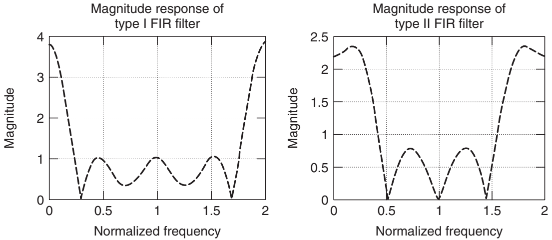
	Frequency Response	Group Delay
Type I (Symmetric and Even)	$H(e^{j\omega}) = e^{-j[(N/2)\omega]}\left\{h\left(\frac{N}{2}\right) + 2\sum_{n=1}^{N/2} h\left[\frac{N}{2}-n\right]\cos(n\omega)\right\}$	$N/2$
Type II (Symmetric and Odd)	$H(e^{-j\omega}) = e^{-j[(N/2)\omega]}\left\{2\sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2}-n\right]\cos\left(\left(n-\frac{1}{2}\right)\omega\right)\right\}$	$N/2$
Type III (Antisymmetric and Even)	$H(e^{-j\omega}) = e^{-j[(N\omega-\pi)/2]}\left\{2\sum_{n=1}^{N/2} h\left[\frac{N}{2}-n\right]\sin(n\omega)\right\}$	$N/2$
Type IV (Antisymmetric and Odd)	$H(e^{-j\omega}) = e^{-j[(N\omega-\pi)/2]}\left\{2\sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2}-n\right]\sin\left(\left(n-\frac{1}{2}\right)\omega\right)\right\}$	$N/2$

Magnitude Response of Linear Phase FIR Filters



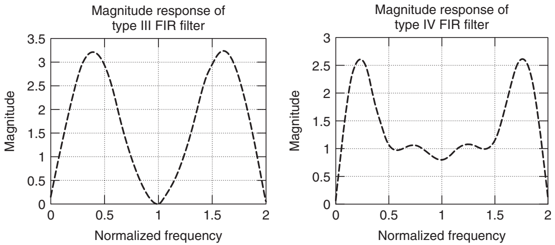
- Type I filters have a non-zero magnitude at $\omega = 0$ and also a non-zero value at the normalized frequency $\omega/\pi = 1$.
- Type II filters have non-zero magnitude at $\omega = 0$ but a zero value at the Nyquist frequency.

Magnitude Response of Linear Phase FIR Filters



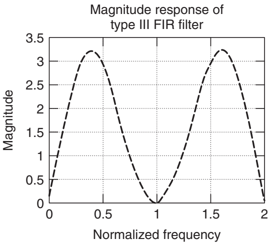
- Type I filters have a non-zero magnitude at $\omega = 0$ and also a non-zero value at the normalized frequency $\omega/\pi = 1$.
- Type II filters have non-zero magnitude at $\omega = 0$ but a zero value at the Nyquist frequency.
- So it is obvious that these filters are not suitable for designing bandpass and highpass filters, where both of them are suitable for low-pass filters.

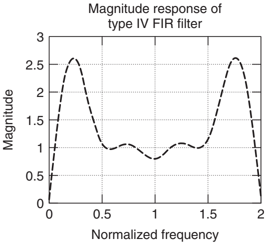
Magnitude Response of Linear Phase FIR Filters



- Type III filters have zero magnitude at $\omega = 0$ and also at $\omega/\pi = 1$ (Nyquist frequency).
- Type IV filters have zero magnitude at $\omega = 0$ and a non-zero magnitude at $\omega/\pi = 1$.

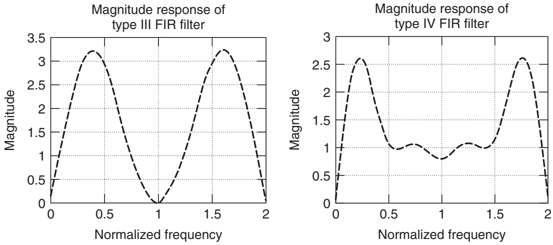
Magnitude Response of Linear Phase FIR Filters





- Type III filters have zero magnitude at $\omega = 0$ and also at $\omega/\pi = 1$ (Nyquist frequency). So they are suitable for designing bandpass filters but not lowpass and bandstop filters.
- Type IV filters have zero magnitude at $\omega = 0$ and a non-zero magnitude at $\omega/\pi = 1$.

Magnitude Response of Linear Phase FIR Filters



- Type III filters have zero magnitude at $\omega = 0$ and also at $\omega/\pi = 1$ (Nyquist frequency). So they are suitable for designing bandpass filters but not lowpass and bandstop filters.
- Type IV filters have zero magnitude at $\omega = 0$ and a non-zero magnitude at $\omega/\pi = 1$. They are not suitable for designing lowpass and bandstop filters but are candidates for bandpass and highpass filters.

Properties of Linear Phase FIR Filters

- An FIR filter with symmetric or antisymmetric coefficients has a linear phase and therefore a constant group delay.
- The reverse statement, an FIR filter with a constant group delay must have symmetric or antisymmetric coefficients.

Thank You

Queries?