STA 250 Homework 1
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Problem 2

(i)

$$Y_i|\beta \sim \text{Binomial}(m_i, \text{logit}^{-1}(x_i^T\beta)), \beta \in \mathbb{R}^p$$

So we have

$$\begin{aligned} p(y_i|\beta) &= \binom{m_i}{y_i}(\frac{e^{x_i^T\beta}}{1+e^{x_i^T\beta}})^{y_i}(1-\frac{e^{x_i^T\beta}}{1+e^{x_i^T\beta}})^{m_i-y_i} \\ &= \binom{m_i}{y_i}\frac{e^{(x_i^T\beta)y_i}}{(1+e^{x_i^T\beta})^{m_i}} \end{aligned}$$

Then

$$p(\mathbf{y}|\beta) \propto \frac{\exp(\sum_{i=1}^n (x_i^T\beta)y_i)}{\prod_{i=1}^n (1+e^{x_i^T\beta})^{m_i}}$$

and

$$p(\beta) \propto \exp[-\frac{1}{2}(\beta-\mu_0)^T\Sigma_0^{-1}(\beta-\mu_0)]$$

So the posterior is, up to proportionality

$$p(\beta|\mathbf{y}) \propto \frac{\exp[\sum_{i=1}^n (x_i^T\beta)y_i - \frac{1}{2}(\beta-\mu_0)^T\Sigma_0^{-1}(\beta-\mu_0)]}{\prod_{i=1}^n (1+e^{x_i^T\beta})^{m_i}}$$

We may take logarithm of the proportional posterior for calculation, which is

$$\sum_{i=1}^n (x_i^T\beta)y_i - \frac{1}{2}(\beta-\mu_0)^T\Sigma_0^{-1}(\beta-\mu_0) - \sum_{i=1}^n m_i \log(1+e^{x_i^T\beta})$$

(ii) In this problem, I used the Metropolis-Hastings-within-Gibbs to generate the Markov chain for posterior distribution. The proposal distribution was

$$\begin{pmatrix} \beta_0^{(t)} \\ \beta_1^{(t)} \end{pmatrix} \mid \begin{pmatrix} \beta_0^{(t-1)} \\ \beta_1^{(t-1)} \end{pmatrix} \sim N(\begin{pmatrix} \beta_0^{(t-1)} \\ \beta_1^{(t-1)} \end{pmatrix}, \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix})$$

The algorithm was performed as the following:
(1) Set initial values $\beta_0^{(0)} = 0, \beta_1^{(0)} = 0, t = 1$.
(2) Let $\beta_0^{(t)} = \beta_0^{(t-1)}, \beta_1^{(t)} = \beta_1^{(t-1)}$, sample $\beta_0^{(t)}$ from $N(\beta_0^{(t-1)}, v)$ (3) Compute the

	beta_0	beta_1
p_01	0.01	0.01
p_05	0.07	0.03
p_10	0.10	0.06
p_25	0.27	0.20
p_50	0.52	0.48
p_75	0.76	0.77
p_90	0.92	0.88
p_95	0.94	0.95
p_99	0.99	0.99