2019 Putnam A1 We claim that our solution set is

$$\boxed{\{x \text{ such that } x \in \mathbb{Z}^{\geq 0}, x \not\equiv 0 \pmod 3 \text{ or } x \equiv 0 \pmod 9\}}$$

First, note that by AM-GM, we have $A^3+B^3+C^3 \geq 3ABC$, so $A^3+B^3+C^3-3ABC \geq 0$. Now, we will perform some transformations. Since the equation is symmetric, we can assume WLOG that $A \leq B \leq C$. Since $A, B, C \in \mathbb{Z}^{\geq 0}$, we can let $B = A + x$ and $C = A + y$, where $x, y \in \mathbb{Z}^{\geq 0}$, and also $0 \leq x \leq y$.

Now, we can substitute the values into our original expression, which yields our new expression $x^3 + y^3 + 3Ax^2 + 3Ay^2 - 3Axy$. Furthermore, this is factorable as $(x^2 + y^2 - xy)(3A + x + y)$. We will also rewrite the first factor as $(x + y)^2 - 3xy$.

Next, we will show that all values described in the solution set are achievable. If we let $x = y = 0$, then our expression evaluates to 0. If we let $x = 0, y = 1$, then our expression evaluates to $3A + 1$. If we let $x = 1, y = 1$, then our expression evaluates to $3A + 2$. Lastly, if we let $x = 1, y = 2$, then our expression evaluates to $9A + 9$. All that remains to be shown is that if an element is divisible by 3, then it is divisible by 9.

Consider both factors mod 3. Specifically, note that $3xy \equiv 0 \pmod 3$, and $3A \equiv 0 \pmod 3$. Thus, $(x^2 + y^2 - xy) \equiv (x + y)^2 \pmod 3$, and $(3A + x + y) \equiv x + y \pmod 3$. In order for the expression to be divisible by 3, both terms must be divisble by 3. Since we are multiplying two terms divisible by 3, the final expression is divisible by 9. Thus, we are done. ■

2013 Putnam A1 Consider any vertex V on the icosahedron. By definition, exactly 5 faces share that vertex. Assume, FTSOC, that none of the 5 faces contain the same nonnegative integer. Thus, the minimum sum of the faces that meet at V is $0 + 1 + 2 + 3 + 4 = 10$. Since we have 12 vertices, this leads to a minimum total sum of $10 \cdot 12 = 120$.

Now however, note that every face was counted 3 times in our calculation (since all faces are equilateral triangles, and thus have 3 vertices). Thus, this sum should equal $3 \cdot$ total face sum $= 3 \cdot 39 = 117 < 120$, giving us our contradiction. ■

2009 Putnam A2 Note that we can multiply all given equations by the denominator of their second term. This gives

$$f'gh = 2f^2g^2h^2 + 1$$
$$fg'h = f^2g^2h^2 + 4$$
$$fgh' = 3f^2g^2h^2 + 1$$

Now, we can sum all three equations, noting that the LHS evaluates to $f'gh + fg'h + fgh' = (fgh)'$. So, we get

$$(fgh)' = 6f^2g^2h^2 + 6 = 6(fgh)^2 + 6$$
$$\implies \int \frac{1}{(fgh)^2 + 1} d(fgh) = \int 6dx$$
$$\implies \arctan(fgh) = 6x + C,$$

where $C = \frac{\pi}{4}$ can easily be computed using our initial conditions of $f(0) = g(0) = h(0) = 1$. This further gives $fgh = \tan(6x + \frac{\pi}{4})$. Now, we can use this result in our original equation for f', which yields

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh} \\ &= 2f\tan\left(6x + \frac{\pi}{4}\right) + f\cot\left(6x + \frac{\pi}{4}\right) \\ \implies \frac{f'}{f} &= 2\tan\left(6x + \frac{\pi}{4}\right) + \cot\left(6x + \frac{\pi}{4}\right) \end{aligned}$$

Integrating both sides of the above equation yields

$$\begin{aligned} \ln(f) &= \frac{1}{6}\left(\ln(\sin(6x + \frac{\pi}{4})) - 2\ln(\cos(6x + \frac{\pi}{4}))\right) + C \\ &= \frac{1}{6}\left(\ln\left(\frac{\sin(6x + \frac{\pi}{4})}{\cos^2(6x + \frac{\pi}{4})}\right)\right) + C \\ \implies f &= C\left(\frac{\sin(6x + \frac{\pi}{4})}{\cos^2(6x + \frac{\pi}{4})}\right)^{\frac{1}{6}} \end{aligned}$$

Lastly, we can compute $C = 2^{-\frac{1}{12}}$ using $f(0) = 1$, which makes our final explicit formula for f be

$$\boxed{f(x) = 2^{-\frac{1}{12}}\left(\frac{\sin(6x + \frac{\pi}{4})}{\cos^2(6x + \frac{\pi}{4})}\right)^{\frac{1}{6}}}.$$

2001 Putnam B2 The most obvious first step is to both add and subtract the two given equations, yielding a new system of equations

$$\begin{aligned} \frac{2}{x} &= x^4 + 10x^2y^2 + 5y^4 \\ \frac{1}{y} &= 5x^4 + 10x^2y^2 + y^4 \\ \implies 2 &= x^5 + 10x^3y^2 + 5xy^4 \quad (1) \\ 1 &= 5x^4y + 10x^2y^3 + y^5 \quad (2) \end{aligned}$$

Now, notice that the coefficients, as well as powers of x and y are that of the expressions $(x+y)^5$, and $(x-y)^5$. In fact, adding equation (1) and (2) yields the former, and subtracting equation (2) from equation (1) yields the latter. Thus,

$$(x+y)^5 = 3, (x-y)^5 = 1 \implies x + y = \sqrt[5]{3}, x - y = 1$$

$$\implies \boxed{(x, y) = \left(\frac{\sqrt[5]{3}+1}{2}, \frac{\sqrt[5]{3}-1}{2}\right)}$$

2021 Putnam A1 We claim that the answer is $\boxed{578}$. First, we will show that this is achievable. Note that all moves are of the form $(\pm 5, 0), (\pm 3, \pm 4), (\pm 4, \pm 3), (0, \pm 5)$. To achieve our 578 move solution, we will use 288 moves of the form $(3, 4)$ and $(4, 3)$ each, and 1 move of the form $(0, 5)$ and $(5, 0)$ each. This gets the grasshopper from $(0, 0)$ to $(2021, 2021)$, and it does so in $2 \cdot 288 + 2 = 578$ moves.

Now, we will show that this is the smallest number of hops required. To do so, note that the taxicab distance between $(0,0)$ and $(2021, 2021)$ is 4042. Our goal is to decrease this distance as much as possible on any given move in order to reach our end point in the fewest moves. As a result, we only use moves with both components positive.

Clearly, in any one move, the distance decreases by either 5 (if $(0,5)$ or $(5,0)$ is used), or 7 (if $(3,4)$ or $(4,3)$ is used). However, note that $7 \cdot 577 = 4039 < 4042$. Thus, it is impossible to reach $(2021, 2021)$ in any less than 578 moves. ■

2021 Putnam A2 To begin, we will rewrite $g(x)$ by taking the natural logarithm of both sides to get rid of the annoying $\frac{1}{r}$ exponent, which yields

$$g(x) = \exp\left(\lim_{r\to 0} \ln\left(((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}\right)\right).$$

Now, we will evaluate the limit, which can be rewritten as

$$\lim_{r\to 0} \frac{\ln((x+1)^{r+1} - x^{r+1})}{r}.$$

However, this limit is indeterminate (specifically, it evaluates to $\frac{0}{0}$), so we use L'Hopital's rule. Thus, we have

$$\begin{aligned}
\lim_{r\to 0} \frac{\ln((x+1)^{r+1} - x^{r+1})}{r} &= \lim_{r\to 0}\left(\frac{d}{dr}\ln((x+1)^{r+1} - x^{r+1})\right) \\
&= \lim_{r\to 0} \frac{(x+1)^{r+1}\ln(x+1) - x^{r+1}\ln(x)}{(x+1)^{r+1} - x^{r+1}} \\
&= (x+1)\ln(x+1) - x\ln(x) \\
&= \ln\left(\left(\frac{x+1}{x}\right)^x \cdot (x+1)\right).
\end{aligned}$$

Now, we have $g(x) = \left(\frac{x+1}{x}\right)^x \cdot (x+1)$. Thus,

$$\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to\infty}\left(\left(\frac{x+1}{x}\right)^x \cdot \frac{x+1}{x}\right).$$

Finally, note that the second terms tends to 1 as $x \to \infty$, and rewrite the insides of the parentheses of the first term as $1 + \frac{1}{x}$. So, our final limit is

$$\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \boxed{e},$$

where our final limit is simply the limit defintion of e.