Project 3 Autumn 2021

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The link to the GitHub repository is: https://github.uio.no/jonathel/Project_2.

PROBLEM 1

Exercise: *Show that the differential equations governing the time evolution of the particle's position can be written as*

$$\ddot{x} - \omega_0 \dot{y} - \frac{1}{2}\omega_z^2 x = 0 \tag{1}$$

$$\ddot{y} + \omega_0 \dot{x} - \frac{1}{2}\omega_z^2 y = 0 \tag{2}$$

$$\ddot{z} + \omega_z^2 z = 0 \tag{3}$$

We can write out N2L firstly with each component. As noted in the text, this deduction is for a single particle.

$$m\ddot{\mathbf{r}} = \sum_i \mathbf{F}_i \tag{4}$$

$$= \sum_i q\vec{E} + q\vec{v} \times \vec{B} \tag{5}$$

where we have the following variables

$$\vec{E} = -\nabla V = -\left(-\frac{V_0 x}{d^2}, -\frac{V_0 y}{d^2}, \frac{2V_0 z}{d^2}\right) = \left(\frac{V_0 x}{d^2}, \frac{V_0 y}{d^2}, -\frac{2V_0 z}{d^2}\right) \tag{6}$$

$$\vec{v} = (v_x, v_y, v_z), \quad \vec{B} = (0, 0, B_0) \tag{7}$$

$$\vec{v} \times \vec{B} = v_y B_0 \hat{i} - v_x B_0 \hat{j} \tag{8}$$

Now none of these variables are in iteration, so we can remove the sum sign and proceed

$$m\ddot{\mathbf{r}} = q\vec{E} + q\vec{v} \times \vec{B} \tag{9}$$

$$= q \cdot \left(\frac{V_0 x}{d^2}, \frac{V_0 y}{d^2}, -\frac{2V_0 z}{d^2}\right) + q\left(v_y B_0 \hat{i} - v_x B_0 \hat{j}\right) \tag{10}$$

Now in x-, y- and z- direction respectively

$$\ddot{x} - \frac{qB_0}{m} \cdot \dot{y} - \frac{qV_0}{md^2} \cdot x = 0 \tag{11}$$

$$\ddot{y} + \frac{qB_0}{m} \cdot \dot{x} - \frac{qV_0}{md^2} \cdot y = 0 \tag{12}$$

$$\ddot{z} + \frac{2qV_0}{md^2} \cdot z = 0 \tag{13}$$

we remind of the relation

$$\omega_o = \frac{|q|B_0}{m}, \qquad \omega_z = \sqrt{\frac{2|q|B_0}{md^2}} \tag{14}$$

using these we get

$$\ddot{x} - \omega_0 \cdot \dot{y} - \frac{1}{2}\omega_z^2 \cdot x = 0 \tag{15}$$

$$\ddot{y} + \omega_0 \cdot \dot{x} - \frac{1}{2}\omega_z^2 \cdot y = 0 \tag{16}$$

$$\ddot{z} + \omega_z^2 \cdot z = 0 \tag{17}$$

Then we will show the general solution for z. We can interpret this as an ODE of second order and set up the characteristic equation utilizing imaginary numbers

$$\ddot{z} + \omega_z^2 z = 0 \quad \rightarrow r^2 + \omega_z^2 = 0 \tag{18}$$

$$\frac{0 \pm \sqrt{0 - 4 \cdot 1 \cdot \frac{2aV_0}{md^2}}}{2} = \pm i\sqrt{\frac{8qV_0}{md^2}} \cdot \frac{1}{2} \tag{19}$$

$$z_1 = i\sqrt{\frac{8qV_0}{md^2}} \cdot \frac{1}{2}, \quad z_2 = -i\sqrt{\frac{8qV_0}{md^2}} \cdot \frac{1}{2} \tag{20}$$

now we integrate eq. 17. Firstly we assume a solution on the form $r(z) = e^{\alpha z}$ for some constant α. By substitution we get

$$\frac{\partial^2 r}{\partial z^2} + \omega_z^2 r_z = \frac{\partial^2 (e^{\alpha z})}{\partial z^2} + \omega_z^2 e^{\alpha z} \tag{21}$$

$$\alpha^2 e^{\alpha z} \omega_z^2 e^{\alpha z} = e^{\alpha z}(\alpha^2 + \omega_z^2) \tag{22}$$

Now we recognize that eq 22 has a solution on the form

$$r(z) = r(z)_1 + r(z)_2 = c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z} \tag{23}$$

where α_1 and α_2 will be

$$\alpha_1 = i\sqrt{\frac{8qV_0}{md^2}} \cdot \frac{1}{2} = i\omega_z \tag{24}$$

$$\alpha_2 = -i\sqrt{\frac{8qV_0}{md^2}} \cdot \frac{1}{2} = -i\omega_z \tag{25}$$

so we get

$$r(z) = c_1 e^{i\omega_z z} + c_2 e^{-i\omega_z z} \tag{26}$$

We can apply Eulers Identity which states

$$e^{\gamma + i\beta} = e^{\gamma}\cos(\beta) + ie^{\gamma}\sin(\beta) \tag{27}$$

So in our case we get

$$r(z) = c_1(\cos(\omega_z z) + i\sin(\omega_z z) + c_2(\cos(\omega_z z) - i\sin(\omega_z z) \tag{28}$$

c_1 and c_2 are arbitrary and i is a constant, so by regrouping cos and sin we get

$$(c_1 + c_2)\cos(\omega_z z) + i(c_1 - c_2)\sin(\omega_z z) = c_1\cos(\omega_z z) + c_2\sin(\omega_z z) \tag{29}$$

so our general solution is

$$r(z) = c_1\cos(\omega_z z) + c_2\sin(\omega_z z) \tag{30}$$

PROBLEM 2

We have

$$f(t) = x(t) + iy(t) \tag{31}$$

and the equation

$$\ddot{f} + i\omega_0\dot{f} - \frac{1}{2}\omega_z^2 f = 0 \tag{32}$$

being a superposition of the two equations

$$\ddot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x = 0 \tag{33}$$

$$\ddot{y} + \omega_0\dot{x} - \frac{1}{2}\omega_z^2 y = 0 \tag{34}$$

If we do analyze each term in 32 we have

$$\ddot{f} = \ddot{x} + i\ddot{y} \tag{35}$$

$$i\omega_0\dot{f} = i\omega_0\dot{x} - \omega_0\dot{y} \tag{36}$$

$$-\frac{1}{2}w_z^2 f = -\frac{1}{2}\omega_z^2\left[x(t) + iy(t)\right] \tag{37}$$

Now since the function f consists of a real part and a imaginary part, we will seperate the two as well respectively

$$\mathrm{Re}f = \ddot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x(t) \tag{38}$$

$$\mathrm{Im}f = \ddot{y} + \omega_0\dot{x} - \frac{1}{2}\omega_z^2 y(t) \tag{39}$$

Considering the hint, that the following applies

$$\mathcal{F}(f, \dot{f}, \ddot{f}) = 0 \tag{40}$$

$$\mathcal{G}(g, \dot{g}, \ddot{g}) = 0 \tag{41}$$

$$\mathcal{F} + c\cdot\mathcal{G} = 0 \tag{42}$$

we can set up the equation in a similar manner for the functions in our problem

$$\mathcal{X}(x, \dot{x}, \ddot{x}) = 0 \tag{43}$$
$$\mathcal{Y}(y, \dot{y}, \ddot{y}) = 0 \tag{44}$$
$$\mathcal{X} + i \cdot \mathcal{Y} = 0 \tag{45}$$

then as for $\mathcal{F}+c\cdot\mathcal{G}=0$, we have $\mathcal{X}+i\cdot\mathcal{Y}=0$ since equations 33 and 34 both equals zero and fulfills the requirement. This means that we can write f as a superposition of the real and imaginary part since in essence, 33 and 34 does not change its mapping when multiplied with any constant.

PROBLEM 3

General solution to eq. 16 is

$$f(t) = A_+ e^{-i\omega_+ t} + A_- e^{-i\omega_- t} \tag{46}$$

where

$$w_\pm = \frac{\omega_0 \pm \sqrt{w_0^2 - 2w_z^2}}{2} \tag{47}$$

We also have the definitions

$$x(t) = Re\ f(t), \quad y(t) = Im\ f(t) \tag{48}$$

We firstly assume that $A_\pm$ are constants. To hinder $|f(t)| \to \infty$ we need

$$w_0^2 \geq 2w_z^2 \tag{49}$$
$$\frac{q^2 B_0^2}{m^2} \geq \frac{4qV_0}{md^2} \tag{50}$$
$$\frac{q}{m} \geq \frac{4V_0}{d^2 B_0^2} \tag{51}$$

where $d^2 > 0$, $m > 0$, $|q| > 0$ and $B_0^2 > 0$. So the inequality is always held. This is the constraint we need inside the root-sign. Now we need to examine if ω_- is positive or negative.
For this, we will check

$$\omega_0 > \sqrt{\frac{d^2 q B_0^2}{V_0 m}} > 0 \tag{52}$$

which holds for all variables. Thus the constraint we need is that the denominator is smaller than the numerator.

PROBLEM 4

For us to get a better understanding of the particles movement, it can be useful to check a upper and lower bound for the particles distance from the origin in the xy-plane. Remember that the general solution is

$$f(t) = A_+ e^{-i\omega_+ t} + A_- e^{-i\omega_- t} \tag{53}$$

We need to write this out on a form without imaginary numbers to get a better understanding of the distance. We do this with the formula

$$e^{it} = cos(t) + sin(t) \tag{54}$$

So now we transform our general solution

$$f(t) = A_+(cos(w_+t) - isin(w_+t)) + A_-(cos(w_-t) - isin(w_-t)) \tag{55}$$
$$= A_+cos(w_+t) + A_-cos(w_-t) - i(A_+sin(w_+t) + A_-sin(w_-t)) \tag{56}$$

We then want to use the formula for distance

$$r = \sqrt{x^2 + y^2} = \sqrt{(Ref)^2 + (Imf)^2} \tag{57}$$

where we have that

$$x(t) = Ref(t) = A_+cos(w_+t) + A_-cos(w_-t) \tag{58}$$
$$y(t) = Imf(t) = -(A_+sin(w_+t) + A_-sin(w_-t)) \tag{59}$$

so we get

$$r = [A_+^2cos^2(w_+t) + 2A_+A_-cos(w_+t)cos(w_-t) + A_-^2cos^2(w_-t)+ \tag{60}$$
$$A_+^2sin^2(w_+t) + 2A_+A_-sin(w_+t)sin(w_-t) + A_+^2sin^2(w_-t)]^{1/2} \tag{61}$$

using the relations

$$cos^2(a * t) + sin^2(a * t) = 1 \tag{62}$$
$$cos(X)cos(Y) - sin(X)sin(Y) = cos(X - Y) \tag{63}$$

we then get the result

$$r = [A_+^2 + A_-^2 + 2A_+A_-cos(t(w_+ - w_-))]^{1/2} \tag{64}$$

to find the upper and lower bounds, we look at what scenarios maximize and minimize the resulting distance. To maximize we set

$$cos(t(w_+ - w_-)) = 1 \tag{65}$$
$$\implies r = [A_+^2 + A_-^2 + 2A_+A_-]^{1/2} \tag{66}$$
$$= [(A_+ + A_-)^2]^{1/2} = \pm A_+ + A_- \tag{67}$$

since we are after the maximized value, we get

$$r \leq A_+ + A_- \tag{68}$$

To minimize r, we set

$$cos(t(w_+ - w_-)) = -1 \tag{69}$$
$$\implies r = [A_+^2 + A_-^2 - 2A_+A_-]^{1/2} \tag{70}$$
$$= [(A_+ - A_-)^2]^{1/2} = \pm(A_+ - A_-) \tag{71}$$

since we don't know which of A_+ or A_- is larger, we change this to the absolute value, so we get

$$|A_+ - A_-| \leq r \tag{72}$$

we then have

$$|A_+ - A_-| \leq r \leq A_+ + A_- \tag{73}$$
$$R_- \leq r \leq R_+ \tag{74}$$

where

$$R_- = |A_+ - A_-| \tag{75}$$
$$R_+ = A_+ + A_- \tag{76}$$

PROBLEM 5

To test an implementation of the penning trap, it can be smart to have a specific analytical solution to test against. We therefore want to find a specific solution for $z(t)$, and also for $f(t)$. We set up a few initial conditions to help us find these

$$x(0) = x_0, \quad \dot{x}(0) = 0, \tag{77}$$
$$y(0) = 0, \quad \dot{y}(0) = v_0, \tag{78}$$
$$z(0) = z_0, \quad \dot{z}(0) = 0, \tag{79}$$

we remember that

$$x(t) = A_+ cos(w_+ t) + A_- cos(w_- t) \tag{80}$$
$$y(t) = -(A_+ sin(w_+ t) + A_- sin(w_- t)) \tag{81}$$

and the derivatives are therefore

$$\dot{x}(t) = -A_+ w_+ sin(w_+ t) - A_- w_- sin(w_- t) \tag{82}$$
$$\dot{y}(t) = -(A_+ w_+ cos(w_+ t) + A_- w_- cos(w_- t)) \tag{83}$$

we then put in the initial conditions

$$x(0) = A_+ cos(0) + A_- cos(0) = A_+ + A_- = x_0 \tag{84}$$
$$\dot{x}(0) = -A_+ w_+ sin(0) - A_- w_- sin(0) = 0 \tag{85}$$
$$y(0) = -(A_+ sin(0) + A_- sin(0)) = 0 \tag{86}$$
$$\dot{y}(0) = -(A_+ w_+ cos(0) + A_- w_- cos(0)) = -A_+ w_+ - A_- w_- = v_0 \tag{87}$$

from equation (84) we have that

$$A_+ = x_0 - A_- \tag{88}$$

we put this into equation (87) and get

$$-(x_0 - A_-)w_+ - A_- w_- = v_0 \tag{89}$$
$$A_- w_+ - x_0 w_+ - A_- w_- = v_0 \tag{90}$$
$$A_-(w_+ - w_-) = v_0 + x_0 w_+ \tag{91}$$
$$A_- = \frac{v_0 + x_0 w_+}{w_+ - w_-} = -\frac{v_0 + x_0 w_+}{w_- - w_+} \tag{92}$$
$$\tag{93}$$

so we get

$$A_+ = x_0 - A_- = \frac{x_0(w_- - w_+) + v_0 + x_0 w_+}{w_- - w_+} = \frac{v_0 + x_0 w_-}{w_- - w_+} \tag{94}$$

and then to find the specific solution for $z(t)$ we need to remember that

$$\ddot{z} + w_z^2 z = 0 \tag{95}$$

we then set $t = 0$, and use the initial condition $z(0) = z_0$

$$\ddot{z}(0) + w_z^2 z(0) = 0 \tag{96}$$
$$\ddot{z}(0) + w_z^2 z_0 = 0 \tag{97}$$
$$\ddot{z}(0) = -w_z^2 z_0 \tag{98}$$

We see that this is a linear differential equation, which means that

$$z(t) = c_2 sin(w_z t) + c_1 cos(w_z t) \tag{99}$$
$$\dot{z}(t) = c_2 w_z cos(w_z t) - c_1 w_z sin(w_z t) \tag{100}$$

setting in initial conditions

$$z(0) = c_2 sin(0) + c_1 cos(0) = c_1 = z_0 \tag{101}$$
$$\dot{z}(0) = c_2 w_z cos(0) - c_1 w_z sin(0) = c_2 w_z = 0 \tag{102}$$

since $w_z > 0$, then $c_2 = 0$, and we end up with the specific solution

$$z(t) = z_0 cos(w_z t) \tag{103}$$

to test that it is correct, we take the double derivative of the function

$$\ddot{z}(t) = -w_z^2 z_0 cos(w_z t) = -w_z^2 z(t) \tag{104}$$

which is correct