① $S[f](x_{i-1}, x_i) = \frac{x_i - x_{i-1}}{6}\left(f(x_{i-1}) + 4f(x_{i-1/2}) + f(x_i)\right)$

a) Må vise at $CSR[f]\left([x_{i-1}, x_i]_{i=1}^{m}\right)$

$$= \frac{h}{6}\left[f(x_0) + 4f(x_{1/2}) + 2f(x_1) + 4f(x_{3/2}) + 2f(x_2) + \dots + 2f(x_{m-1}) + 4f(x_{m-1/2}) + f(x_m)\right]$$

$$CSR[f] = \sum_{i=1}^{m} S[f](x_{i-1}, x_i)$$

$$= \sum_{i=1}^{m}\left[\frac{x_i - x_{i-1}}{6}\left(f(x_{i-1}) + 4f(x_{i-1/2}) + f(x_i)\right)\right]$$

Vi vet at $x_i - x_{i-1} = \frac{b-a}{m} = h$, hvilket vil si steglengden. Da kan vi bytte ut $x_i - x_{i-1}$ med h siden dette er konstant

$$\Rightarrow \frac{h}{6}\sum_{i=1}^{m}\left(f(x_{i-1}) + 4f(x_{i-1/2}) + f(x_i)\right)$$

Skriver man dette ut får man

$$= \frac{h}{6}\left(f(x_0) + 4f(x_{1/2}) + 2f(x_1) + 4f(x_{3/2}) + 2f(x_2) + \dots + 2f(x_{m-1}) + 4f(x_{m-\frac{1}{2}}) + f(x_m)\right) \blacksquare$$

c) $|I[f] - CSR[f]|$

$$= \left|\sum_{i=1}^{m} I[f](x_{i-1}, x_i) - \sum_{i=1}^{m} S[f](x_{i-1}, x_i)\right|$$

$$= \left|\sum_{i=1}^{m} \left(\left(I[f](x_{i-1}, x_i) - S[f](x_{i-1}, x_i)\right)\right)\right|$$

$$= \left|\sum_{i=1}^{m} -\frac{(x_i - x_{i-1})^5}{2880} f^{(4)}(\xi)\right|$$

Da kan vi velge $\xi \in [x_{i-1}, x_i]$ slik at $M_4(x_{i-1}, x_i) = \max_{\xi \in [x_{i-1}, x_i]} |f^{(4)}(\xi)|$. Deretter velger vi $M_4 = \max\{M_4(x_{i-1}, x_i)\}_{i=1}^{m}$. Siden M_4 er den største, når vi bytter og setter utenfor, må venstresiden være mindre:

$$\Rightarrow \left|\sum_{i=1}^{m} -\frac{(x_i - x_{i-1})^5}{2880} f^{(4)}(\xi)\right| \leq \frac{M_4}{2880}\left|\sum_{i=1}^{m} -(x_i - x_{i-1})^5\right|$$

Samtidig er $x_i - x_{i-1} = \frac{b-a}{m}$

$$\Rightarrow \frac{M_4}{2880}\left|\sum_{i=1}^{m} -\left(\frac{b-a}{m}\right)^5\right| = \frac{M_4}{2880}\left|\sum_{i=1}^{m} - h^4 \cdot \frac{b-a}{m}\right|$$

$$\Rightarrow \frac{M_4}{2880}\left|-h^4(b-a)\right| = \frac{M_4}{2880} h^4 (b-a)$$

Dermed er

$$\left| I[f] - CSR[f] \right| \leq \frac{M_4}{2880} h^4 (b-a)$$

Vi ser den avhengighet av h^4, som stemmer med det vi fant i b).

(2) Vi har basisene

$\varphi_0 = 1$, $\varphi_1 = x$, $\varphi_2 = x^2$ og $\varphi_3 = x^3$

Vi ser at $p_0 = 1$ og $p_1 = x + \frac{1}{2}$ og $\|p_0\|^2 = 3$

Må finne p_2 og p_3:

$$p_2 = \varphi_2 - \sum_{j=0}^{1} \frac{\langle \varphi_2, p_j \rangle}{\|p_j\|^2} \cdot p_j$$

Vi trenger da $\langle \varphi_2, p_0 \rangle$, $\langle \varphi_2, p_1 \rangle$ og $\|p_j\|^2$

For p_3 må vi ha

$$p_3 = \varphi_3 - \sum_{j=0}^{2} \frac{\langle \varphi_3, p_j \rangle}{\|p_j\|^2} p_j$$

Så vi trenger $\langle \varphi_3, p_0 \rangle$, $\langle \varphi_3, p_1 \rangle$, $\langle \varphi_3, p_2 \rangle$ og $\|p_2\|^2$

$$\langle \varphi_2, p_0 \rangle = \int_{-2}^{1} \varphi_2(x)\, p_0(x)\, dx = \int_{-2}^{1} x^2\, dx = \left[\frac{1}{3}x^3\right]_{-2}^{1} = \frac{1}{3} - \left(-\frac{8}{3}\right) = 3$$

$$\langle \varphi_2, p_1 \rangle = \int_{-2}^{1} x^2 \cdot \left(x + \frac{1}{2}\right) dx = \int_{-2}^{1} x^3 + \frac{1}{2}x^2\, dx$$

$$= \left[\frac{1}{4}x^4 + \frac{1}{6}x^3\right]_{-2}^{1} = \frac{5}{12} - \left(\frac{24}{6} - \frac{8}{6}\right) = \frac{5}{12} - \frac{16}{6}$$

$$= -\frac{27}{12} = -\frac{9}{4} = -2{,}25$$

$$\|p_1\|^2 = \int_{-2}^{1} \left(x + \frac{1}{2}\right)^2 dx = \int_{-2}^{1} x^2 + x + \frac{1}{4}\, dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x\right]_{-2}^{1}$$

$$= \frac{13}{12} - \left(-\frac{7}{6}\right) = \frac{27}{12} = \frac{9}{4} = 2{,}25$$

$$p_2 = x^2 - \frac{\langle \varphi_2, p_0 \rangle}{\|p_0\|^2} p_0 - \frac{\langle \varphi_2, p_1 \rangle}{\|p_1\|^2} p_1$$

$$= x^2 - \frac{3}{3} \cdot 1 + \frac{9/4}{9/4}\left(x + \frac{1}{2}\right) = x^2 - 1 + x + \frac{1}{2} = x^2 + x - \frac{1}{2}$$

$$\langle \varphi_3, p_0 \rangle, \langle \varphi_3, p_1 \rangle, \langle \varphi_3, p_2 \rangle$$

$$\|p_2\|^2$$

$$\langle \varphi_3, p_0 \rangle = \int_{-2}^{1} x^3 \, dx = \left[\tfrac{1}{4}x^4\right]_{-2}^{1} = \frac{1}{4} - 4 = \underline{-\frac{15}{4}}$$

$$\langle \varphi_3, p_1 \rangle = \int_{-2}^{1} x^3 \cdot (x + \tfrac{1}{2}) \, dx = \int_{-2}^{1} x^4 + \tfrac{1}{2}x^3 \, dx = \left[\tfrac{1}{5}x^5 + \tfrac{1}{8}x^4\right]_{-2}^{1}$$

$$= \frac{13}{40} + \frac{22}{5} = \underline{\frac{189}{40}}$$

$$\langle \varphi_3, p_2 \rangle = \int_{-2}^{1} x^3 (x^2 + x - \tfrac{1}{2}) \, dx = \int_{-2}^{1} x^5 + x^4 - \tfrac{1}{2}x^3 \, dx$$

$$= \left[\tfrac{1}{6}x^6 + \tfrac{1}{5}x^5 - \tfrac{1}{8}x^4\right]_{-2}^{1} = \frac{29}{120} - \frac{34}{15} = \underline{-\frac{81}{40}}$$

$$\|p_2\|^2 = \int_{-2}^{1} (x^2 + x - \tfrac{1}{2})^2 \, dx = \frac{27}{20}$$

$$p_3 = x^3 + \frac{15/4}{3} \cdot 1 - \frac{189/40}{9/4}(x + \tfrac{1}{2}) + \frac{81/40}{27/20}(x^2 + x - \tfrac{1}{2})$$

$$= x^3 + \frac{5}{4} - \frac{21}{10}x - \frac{21}{20} + \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}$$

$$= x^3 + \frac{3}{2}x^2 - \frac{3}{5}x - \frac{11}{20}$$

c) Skal finne røttene til $p_1 = x^3 + \frac{3}{2}x^2 - \frac{5}{3}x - \frac{11}{20}$

Vi vet at en av røttene er $x = -\frac{1}{2}$

Utfører polynomdivisjon

$$\left(x^3 + \frac{3}{2}x^2 - \frac{3}{5}x - \frac{11}{20}\right) : \left(x + \frac{1}{2}\right) = x^2 + x - \frac{11}{10}$$

$$-\left(x^3 + \frac{1}{2}x^2\right)$$

$$x^2 - \frac{3}{5}x$$

$$-\left(x^2 + \frac{1}{2}x\right)$$

$$-\frac{11}{10}x - \frac{11}{20}$$

$$-\left(-\frac{11}{10}x - \frac{11}{20}\right)$$

$$0$$

$$x = \frac{-1 \mp \sqrt{1 + \frac{44}{10}}}{2}$$

$$= \frac{-1 \mp \sqrt{27/5}}{2}$$

$$= \frac{-5 \mp \sqrt{135}}{10}$$

$$= \frac{-5 \mp 3\sqrt{15}}{10}$$

Så $x_1 = -\frac{1}{2}$ $\quad x_2 = \frac{-5 - 3\sqrt{15}}{10}$ $\quad x_3 = \frac{-5 + 3\sqrt{15}}{10}$

d) $$l_0(x) = \frac{(x-x_2)(x-x_3)}{(x_0-x_2)(x_0-x_3)}$$

$$l_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$l_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2}$$

$$W_1 = \frac{1}{(x_0-x_2)(x_0-x_3)} \int_{-2}^{1} x^2 - (x_2+x_3)x + x_2x_3 \, dx$$

$$= \frac{1}{(x_0-x_2)(x_0-x_3)} \cdot \left[\frac{1}{3}x^3 - \frac{(x_2+x_3)}{2}x^2 + x_2x_3x\right]_{-2}^{1}$$

$$\underline{\underline{= 4/3}}$$

$$W_2 = \frac{1}{(x_2-x_1)(x_2-x_3)} \int_{-2}^{1} x^2 - (x_1+x_2)x + x_1x_3 \, dx$$

$$\underline{\underline{= 5/6}}$$

$$W_3 = \frac{1}{(x_3-x_1)(x_3-x_2)} \int_{-2}^{1} x^2 - (x_1+x_2)x + x_1x_2 \, dx$$

$$\underline{\underline{= 5/6}}$$

e) $\tilde{x}_0 = \hat{x}_0 \cdot \frac{3}{2} - \frac{1}{2} = -\sqrt{\frac{3}{5}} \cdot \frac{3}{2} - \frac{1}{2}$

$$= \underline{\frac{-1-\sqrt{\frac{27}{5}}}{2} = x_2}$$

$\tilde{x}_1 = \hat{x}_1 \cdot \frac{3}{2} - \frac{1}{2} = 0 \cdot \frac{3}{2} - \frac{1}{2} = \underline{-\frac{1}{2} = x_1}$

$\tilde{x}_2 = \hat{x}_2 \cdot \frac{3}{2} - \frac{1}{2} = \sqrt{\frac{3}{5}} \cdot \frac{3}{2} - \frac{1}{2} = \underline{\frac{-1+\sqrt{\frac{27}{5}}}{2} = x_3}$

Spennet mellom minste og største verdi i $\{\frac{4}{3}, \frac{5}{6}, \frac{5}{6}\}$ er $\frac{1}{2}$
Spennet i $\{\frac{8}{9}, \frac{5}{9}, \frac{5}{9}\}$ er $\frac{1}{3}$
Må da skalere det siste mengden med faktor $\frac{3}{2}$ siden $\frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$. Deretter må vi forskyve ved å trekke fra/legge til en konstant. I dette tilfellet er det 0. Ganger vi $\{\frac{8}{9}, \frac{5}{9}, \frac{5}{9}\}$ med $\frac{3}{2}$ får vi $\{\frac{4}{3}, \frac{5}{6}, \frac{5}{6}\}$ som stemmer med det vi fant i a-d.

③ a) Vi har at $E(a,b) = \int_a^b f(x)\,dx - Q(a,b)$

$$= \frac{(b-a)^7}{2016000} f^{(6)}(\eta), \quad \eta \in (a,b)$$

$$E(a,b) = \int_a^b f(x)dx - Q_m(a,b)$$

$$= \int_a^b f(x)dx - \sum_{i=1}^{m} Q(x_i, x_{i+1}) \quad , x_i = a + ih \quad i = 0,1,\dots,m \quad h = \frac{b-a}{m}$$

$$= \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(x)dx - \sum_{i=1}^{m} Q(x_i, x_{i-1})$$

$$= \sum_{i=1}^{m} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - Q(x_i, x_{i-1}) \right)$$

$$= \sum_{i=1}^{m} \frac{(x_i - x_{i-1})^7}{2016000} f_i^{(6)}(\eta_i) \quad \eta_i \in (x_{i-1}, x_i)$$

Velger $M_{6i} = \max_{\eta \in (x_{i-1}, x_i)} |f_i^{(6)}(\eta_i)|$, og deretter

$$M_6 = \max\{f_1(\eta_1), \dots, f_m(\eta_m)\}$$

$$\Rightarrow \sum_{i=1}^{m} \frac{(x_i - x_{i-1})^7}{2016000} f_i^{(6)}(\eta_i) \le \frac{M_6}{2016000} m \cdot \left(\frac{b-a}{m}\right)^7$$

$$= \frac{M_6}{2016000} \cdot h^6 (b-a) \quad \blacksquare$$

b)

$$= \frac{M_6}{2016000} \cdot h^6 (b-a)$$

Ønsker at erroren er mindre enn 10^{-8}. Dvs at

$$10^{-8} > \frac{M_6}{2016000} \cdot h^6 \cdot 1 \geq \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx - Q_m(0,1)$$

Vi har at $M_6 = \max_{\eta \in (0,1)} |f^{(6)}(\eta)| = \max_{\eta \in (0,1)} \left| \frac{d^6}{dx^6} \cos \frac{\pi}{2}\eta \right|$

$$\leq 17$$

Vi har da at

$$10^{-8} > \frac{M_6}{2016000} h^6 \geq \frac{17}{2016000} \cdot \left(\frac{1-0}{m}\right)^6$$

$$10^{-8} > \frac{17}{2016000} \cdot \frac{1}{m^6}$$

$$m^6 \cdot 10^{-8} > \frac{17}{2016000}$$

$$m^6 > \frac{17}{2016000} \cdot 10^8$$

$$m > \sqrt[6]{\frac{17}{2016000} \cdot 10^8}$$

$$m > 3{,}07$$

$\underline{\underline{m \geq 4}}$ garanterer at erroren er mindre enn 10^{-8}

c) Vi vet at $\left| \int_0^1 \cos \frac{\pi}{2} x \, dx - \mathcal{S}S(a,b) \right|$

$\leq \frac{M_4}{2880} h^4 (b-a)$

Vi trenger da at

$10^{-8} > \frac{M_4}{2880} h^4 (1-0) \geq \left| \int_0^1 \cos \frac{\pi}{2} x \, dx - CS(0,1) \right|$

Tilsvarende er $M_4 = \max_{\xi \in (a,b)} |f^{(4)}(\xi)| = \max_{\xi \in (a,b)} \left| \frac{d^4}{dx^4} \cos \frac{\pi}{2} \xi \right|$
$\leq 6,1$

$10^{-8} > \frac{M_4}{2880} h^4 \geq \frac{6,1}{2880} h^4$

$10^{-8} > \frac{6,1}{2880} \cdot \frac{1-0}{m^4}$

$m^4 10^{-8} > \frac{6,1}{2880}$

$m^4 > \frac{61}{28800} \cdot 10^8$

$m > \sqrt[4]{\frac{61}{28800} \cdot 10^8}$

$m > 21,45$

$\underline{\underline{m \geq 22}}$ gir en error mindre enn 10^{-8} ☐