① La $f, g, h : \mathbb{R}^n \to \mathbb{R}$ være gitt ved

I $f(x,y) := xy + x + 2y^2$

II $g(x,y) := e^{-2xy} - 4$

III $h(x,y,z) := \arctan(x) \cdot \cos(y) - 3z\sin(x)$

IV $p(x,y,z) := \ln(x+y^3) - (x^2y + y^2) \cdot \ln(2z^2)$

I $\frac{\partial f}{\partial x^2} = \frac{\partial}{\partial x}(y+1) = \underline{\underline{0}}$

$\frac{\partial f}{\partial y^2} = \frac{\partial}{\partial y}(x+4y) = \underline{\underline{4}}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(y+1) = \underline{\underline{1}}$

II $\frac{\partial g}{\partial x^2} = \frac{\partial}{\partial x}(-2ye^{-2xy}) = \underline{\underline{4y^2 e^{-2xy}}}$

$\frac{\partial g}{\partial y^2} = \frac{\partial}{\partial y}(-2xe^{-2xy}) = \underline{\underline{4x^2 e^{-2xy}}}$

$\frac{\partial g}{\partial x \partial y} = \frac{\partial}{\partial y}(-2ye^{-2xy}) = \underline{\underline{4xye^{-2xy}}}$

III $\frac{\partial h}{\partial x} = \underline{\underline{\frac{\cos(y)}{1+x^2} - 3z\cos(x)}}$

$\frac{\partial h}{\partial y} = \underline{\underline{-\arctan(x)\cdot\sin(y)}}$

$\frac{\partial h}{\partial z} = \underline{\underline{-3\sin(x)}}$

$\frac{\partial^2 h}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\cos(y)}{1+x^2} - 3z\cos(x)\right) = \underline{\underline{-\frac{\sin(y)}{1+x^2}}}$

IV $\frac{\partial p}{\partial x} = \underline{\underline{\frac{1}{x+y^3} - 2xy\ln(2z^2)}}$

$\frac{\partial p}{\partial y} = \frac{1}{x+y^3}\cdot 3y^2 - (x^2+2y)\ln(2z^2)$

$= \underline{\underline{\frac{3y^2}{x+y^3} - (x^2+2y)\ln(2z^2)}}$

$\frac{\partial p}{\partial z} = -\frac{1}{\cancel{2}z^{\cancel{2}}}\cdot\cancel{4}^{2}\cancel{z}\cdot(x^2y+y^2) = \underline{\underline{-2\cdot\frac{x^2y+y^2}{z}}}$

$$\frac{\partial^3 \varphi}{\partial x \partial y \partial z} = \frac{\partial^2}{\partial y \partial z}\left(\frac{1}{x+y^3} - 2xy\ln(2z^2)\right)$$

Teorem sier at $\frac{\partial^2 i}{\partial y \partial z} = \frac{\partial^2 i}{\partial z \partial y}$. Deriverer altså nå først på $z$, for å få lettere utregning.

$$\Rightarrow \frac{\partial}{\partial y}\left(-2xy \cdot \frac{1}{\cancel{2}z^{\cancel{2}}} \cdot \overset{2}{\cancel{4}}\cancel{z}\right):$$

$$= \frac{\partial}{\partial y}\left(-4 \cdot \frac{xy}{z}\right) = \underline{\underline{-\frac{4x}{z}}}$$

(2) Gitt $\{(x_i, y_i)\}_{i=0}^{n} = \{(3,2), (4,6), (5,12)\}$

der $y_i = f(x_i)$

a) Lagrange:

Vet at $f(x) = \sum_{i=0}^{n} y_i l_i(x) = \sum_{i=0}^{n} \frac{y_i}{q_i(x_i)} q_i(x)$

der $q_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - x_j)$

Vi er gitt $y_i$-ene, og må da finne $q_i(x_i)$ og $q_i(x)$.

$$q_0(x) = (x - x_1)(x - x_2) = \underline{(x-4)(x-5)}$$
$$q_0(x_0) = (3-4)(3-5) = (-1)\cdot(-2) \underline{= 2}$$

$$q_1(x) = (x - x_0)(x - x_2) = \underline{(x-3)(x-5)}$$
$$q_1(x_1) = (4-3)(4-5) = 1 \cdot (-1) \underline{= -1}$$

$$q_2(x) = (x - x_0)(x - x_1) = \underline{(x-3)(x-4)}$$
$$q_2(x_2) = (5-3)(5-4) \underline{= 2}$$

Da er $f(x)$:

$$f(x) = \sum_{i=0}^{2} \frac{y_i}{q_i(x_i)} \cdot q_i(x):$$

$$= \frac{2}{2}(x-4)(x-5) - \frac{6}{1}(x-3)(x-5) + \frac{\cancel{12}^{6}}{\cancel{2}}(x-3)(x-4)$$

$$= x^2 - 9x + 20 - \cancel{6x^2} + 48x - 90 + \cancel{6x^2} - 42x + 72$$
$$\underline{\underline{= x^2 - 3x + 2}}$$

Newton:

Ønsker at $f(x) = \sum_{i=0}^{n} c_i w_i(x)$,

der $w_i(x) = \prod_{k=0}^{i-1} x - x_k$.

Vi må altså finne $c_0$, $c_1$ og $c_2$

$$y_0 = f(x_0) = \sum_{i=0}^{n} c_i w_i(x_0) = \underline{c_0 = 2}$$

$$y_1 = f(x_1) = \sum_{i=0}^{n} c_i w_i(x_1) = c_0 + c_1(x_1 - x_0)$$

$$\Rightarrow \quad c_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{6-2}{4-3} = \underline{4}$$

$$y_2 = f(x_2) = \sum_{i=1}^{n} c_i w_i(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow \quad c_2 = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

For min egen del vil jeg gå fra uttrykket over til

$$c_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

$$\frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - y_0 - y_1\frac{x_2 - x_0}{x_1 - x_0} + y_0\frac{x_2 - x_0}{x_1 - x_0}}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{y_2 + y_0\left(\frac{x_2 - x_0}{x_1 - x_0} - 1\right) - y_1\frac{x_2 - x_0}{x_1 - x_0}}{(x_2 - x_0)(x_2 - x_1)}$$

$$\frac{x_2 - x_0}{x_1 - x_0} - 1 = \frac{x_2 - x_0 - x_1 + x_0}{x_1 - x_0} = \frac{x_2 - x_1}{x_1 - x_0}$$

Da er $\frac{x_2 - x_0}{x_1 - x_0} = \frac{x_2 - x_1}{x_1 - x_0} + 1$

$$\Rightarrow \frac{y_2 + y_0\frac{x_2 - x_1}{x_1 - x_0} - y_1 \cdot \frac{x_2 - x_1}{x_1 - x_0} - y_1}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{y_2 - y_1 - \left(\frac{x_2 - x_1}{x_1 - x_0}\right)(y_1 - y_0)}{(x_2 - x_0)(x_2 - x_i)}$$

$$c_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

Da har vi at

$$c_2 = \frac{\frac{12-6}{5-4} - \frac{6-2}{4-3}}{5-3} = \frac{6-4}{2} = \underline{1}$$

Vi kan så lage polynomet

$$f(x) = \sum_{i=0}^{2} c_i w_i(x) = 2 + 4\cdot(x-3) + (x-3)(x-4)$$

$$= 2 + 4x - 12 + x^2 - 7x + 12$$

$$= \underline{\underline{x^2 - 3x + 2}}$$

som stemmer med det vi fant da vi brukte Lagrange-metoden

Metoden viser også at for $m+1$ verdier $y_{i_0}, y_{i_1}, \dots, y_{i_m}$ så er

$$\underline{c_k = [y_0, y_1, \dots, y_k]} = \frac{[y_0, \dots, y_k] - [y_0, \dots, y_{k-1}]}{x_k - x_0}$$

b) Siden Newtons ide bruker foregående verdier for å regne ut neste koeffisient bruker vi denne metoden når vi legger til et datapunkt.

La $f(6) = 26$, dvs $x_3 = 6$ og $y_3 = 26$

Da er

$$c_3 = [y_0, y_1, y_2, y_3] = \frac{[y_1, y_2, y_3] - [y_0, y_1, y_2]}{x_3 - x_0}$$

Vi har $c_2 = [y_0, y_1, y_2] = 1$

$$[y_1, y_2, y_3] = \frac{[y_2, y_3] - [y_1, y_2]}{x_3 - x_1} = \frac{\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}$$

$$= \frac{\frac{26-12}{6-5} - \frac{12-6}{5-4}}{6-4} = \frac{14-6}{2} = \underline{4}$$

Da er $c_3 = \frac{4-1}{6-3} = \frac{3}{3} = \underline{1}$

Så trenger vi $c_3 w_3(x) = w_3(x)$

$= (x - x_0)(x - x_1)(x - x_2) = (x-3)(x-4)(x-5)$

$= (x^2 - 7x + 12)(x - 5) = x^3 - 12x^2 + 47x - 60$

og så legger vi dette sammen med resultatet vårt fra a:

$$f(x) = x^3 - 12x^2 + 47x - 60 + x^2 - 3x + 2 = \underline{\underline{x^3 - 11x^2 + 44x - 58}}$$

3 Gjør oppgaven i jupyter. Brukte en kanskje ikke like effektiv måte å regne divided difference, men det funker.

4) a) Vet at

$$x_i = \frac{b-a}{2}\tilde{x}_i + \frac{a+b}{2}$$ , der $\tilde{x}_i$ er

Chebyshev-nodene $\tilde{x}_i \in [-1, 1]$

Samtidig er

$$\tilde{x}_i = \cos\left(\frac{(2i+1)\pi}{2(n+1)}\right) \quad , i = 0, 1, \dots, n$$

Vi finner da nodene på $[-1, 1]$:

$$\tilde{x}_0 = \cos\left(\frac{\pi}{10}\right) \approx 0{,}951$$

$$\tilde{x}_1 = \cos\left(\frac{3\pi}{10}\right) \approx 0{,}588$$

$$\tilde{x}_2 = \cos\left(\frac{5\pi}{10}\right) \approx 0$$

$$\tilde{x}_3 = \cos\left(\frac{7\pi}{10}\right) \approx -0{,}588$$

$$\tilde{x}_4 = \cos\left(\frac{9\pi}{10}\right) \approx -0{,}951$$

Vi kan deretter finne nodene på $[2, 5]$

$$x_0 = \frac{5-2}{2}\cdot 0{,}951 + \frac{5+2}{2} = \underline{4{,}927}$$

$$x_1 = \frac{3}{2}\cdot 0{,}588 + \frac{7}{2} = \underline{4{,}382}$$

$$x_2 = \underline{3{,}500}$$

$$x_3 = \frac{3}{2}\cdot(-0{,}588) + \frac{7}{2} = \underline{2{,}618}$$

$$x_4 = \frac{3}{2}(-0{,}951) + \frac{7}{2} = \underline{2{,}074}$$

Chebyshev-nodene er altså

$$\underline{\underline{\{4.927,\ 4.382,\ 3.500,\ 2.618,\ 2.074\}}}$$

b) $f(x) = 2^{x^2 - 6x + 9}$

$f(x_0) = 2^{4.927^2 - 6 \cdot 4{,}927 + 9} = 13{,}12$

$f(x_1) = 2^{4{,}382^2 - 6 \cdot 4{,}382 + 9} = 3{,}758$

$f(x_2) = 2^{3{,}500^2 - 6 \cdot 3{,}500 + 9} = 1{,}189$

$f(x_3) = 2^{2{,}616^2 - 6 \cdot 2.618 + 9} = 1{,}106$

$f(x_4) = 2^{2{,}074^2 - 6 \cdot 2{,}074 + 9} = 1.812$

Interpolerer oppg. b og c i python, samt plotte forskjellen (d)

c) Siden vi har funnet 5 Chebyshev noder, antar jeg vi skal interpolere f på de 5 ~~likedist~~ uniforme nodene, ikke 4 som det står i oppg.teksten.

$\bar{x}_i = a + i \cdot \frac{b-a}{n-1} = 2 + i \cdot \frac{5-2}{5} = 2 + i \cdot \frac{3}{4}$

$\bar{x}_0 = 2 + 0 \cdot 0{,}75 = 2$

$\bar{x}_1 = 2 + 1 \cdot 0{,}75 = 2{,}75$

$\bar{x}_2 = 2 + 2 \cdot 0{,}75 = 3{,}50$

$\bar{x}_3 = 2 + 3 \cdot 0{,}75 = 4{,}25$

$\bar{x}_4 = 2 + 4 \cdot 0{,}75 = 5$

$$f(\bar{x}_0) = 2^{2^2-6\cdot 2+9} = 2{,}000$$
$$f(\bar{x}_1) = f(2{,}75) = 1{,}044$$
$$f(\bar{x}_2) = f(3{,}50) = 1{,}189$$
$$f(\bar{x}_3) = f(4{,}25) = 2{,}454$$
$$f(\bar{x}_4) = f(5) = 16{,}00$$

d) Gjores i python sammen med resten av oppg. 4