Assignment 1

Linear and integer optimization with applications

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1

Our objective function is:

$$maximize\ z = \sum_{i}^{3} B_i p_i,\ i \in \{1, 2, 3\} \tag{1}$$

where B_i represents litres biodiesel for the 3 products B5, B30, B100 and p_i is the profit margin for each product. Our objective function gives us how much money [euro] we can earn in total for selling the 3 products. The product flows are illustrated in figure 1.

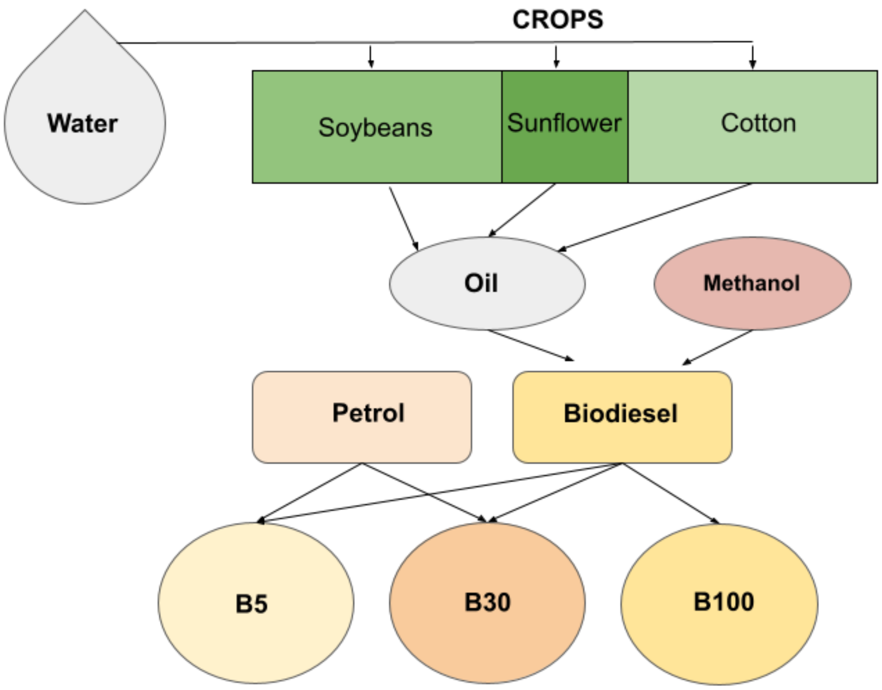

Figure 1: Product flow

Our variables for the model are:

$$B_i,\ i \in \mathcal{I} = \{1, 2, 3\}$$
$$V_j,\ j \in \mathcal{J} = \{1, 2, 3\}$$

where V_j represents litres vegetable oil farmed from 3 different crops: soybeans, sunflower seeds, cotton seeds. We got the following crop data from the assignment description:

Crop	Biomass yield [t/ha]	Water demand [Ml/ha]	Oil content [l/kg]	Oil yield [l/ha]
Soybeans	2.6	5.0	0.178	462.8
Sunflower seeds	1.4	4.2	0.216	302.4
Cotton seeds	0.9	1.0	0.433	389.7

Table 1: Crop data.

We introduce the following notation:

$$\begin{cases} y_j \; biomassyield \\ w_j \; water\ demand \\ c_j \; oil\ content \\ o_j \; oil\ yield \end{cases} \quad j \in \{1,2,3\}$$

The data for *oil yield* was calculated using $o_j = 1000 y_j c_j$.

The following data for the 3 biodiesel products were also obtained from the assignment description.

Product	Biodiesel content	Sale price [euro/l]	Tax [%]	Profit margin [euro/l]	Veg.oil usage [l/l biodiesel]
B5	5%	1.43	20	0.18	0.056
B30	30%	1.29	5	0.43	0.33
B100	100%	1.16	0	0.83	1.11

Table 2: Biodiesel data.

where *veg.oil usage* is calculated as the biodiesel content divided by 0.9 (1 L veg.oil per 0.9 L biodiesel), and *profit margin* was calculated per litre biodiesel as the revenue after tax minus expenses.

Profit margin = sale price after tax - petrol costs - methanol costs

where the petrol costs 1 euro/l and the methanol costs from the transesterification process is $0.2 \cdot 1.5/0.9$ [euro/l], since every 0.9 litre biodiesel requires 0.2 litres of methanol which costs 1.5 euro/l.

Our constraints for the model are:

Name	Value
Biodiesel demand	280000 litres
Available farming area	1600 ha
Available water	5000 Ml
Available petrol	150000 litres

Table 3: Constraints

Finally we have our model constraints:

$$\left.\begin{aligned} \sum_i^3 B_i &\geq Biodiesel\ demand \\ \sum_j^3 V_j/o_j &\leq \ Available\ farming\ area \\ \sum_j^3 V_j w_j/o_j &\leq \ Available\ water \\ 0.95B_1 + 0.7B_2 &\leq Available\ petrol \\ \sum_i^3 B_i v_i &\leq \sum_j^3 V_j \end{aligned}\right\} \tag{2}$$

where v_i in the last constraint is the veg.oil usage per litre biodiesel (described in Table 2 above). The last constraint ensures that at least as much vegetable oil as used is produced.

2

For the implementation of the model described in section 1, view the appended files *run.jl*, *dat.jl* and *mod.jl*. When running the model the optimal solution in Table 4 was reached:

B_i [k litres]	V_j [k litres]
0	393.38
214.29	0
552.80	292.28

Table 4: Results.

The corresponding data about the area and water usage for each crop can be found in Table 6:

Crop	Area usage [ha]	Water usage [Ml]
Soybeans	850	4250
Sunflower seeds	0	0
Cottons seeds	750	750

Table 5: Total area and water usage for the 3 different crops.

This maximizes the cost function, giving a profit of $\approx$ 548.16k euros. It can be seen that no B5 fuel is produced and no sunflower crops are grown. This is a reasonable result as the sunflower seeds need far more water than the cotton seeds, while yielding only just over half the vegetable oil of the soybeans. This means that a combination of soybeans and cotton exists that will produce more oil for less water per hectare compared to sunflowers. As to why no production of B5 is optimal comes down to the profit margins.

3

a.

By reducing the availability of each resource until no feasible could be found the following limits were found:

Resource	Min resource usage [%]	Min resource usage
Water	7.4 %	370 ML
Area	19.5 %	31.2 Ha
Petrol	0 %	0 L

Table 6: Minimum resource utilization while staying feasible.

One can see that no petrol is needed. This is because B100 can be produced exclusively whilst still meeting all demands. Note that each resource were reduced independently to find the given limit, however there exists feasible solutions with one resource at the limit and others also reduced. E.g. the solution minimizing the profit whilst still producing the required fuel uses 7.4% of the water, 100% of the petrol and 23.2% of the area. This solution gives a profit of 128.94k euros. The solution minimizing the sum of ratios used of each resource uses 16% of the water and 50% of the area. This solution gives a profit of 231.47k euros.

From the constraints on the model from equation set 2 we can see that the constraint of the maximum petrol usage only depend on B_1 and B_2 (corresponding to B5 and B30). We can choose to only produce B_3 corresponding to B100, so we can decrease/increase the maximal petrol usage without affecting the water and area constraints. However, when it comes to the constraints of the maximal farming area and the maximum water usage, both includes $\frac{V_1}{o_1}$, $\frac{V_2}{o_2}$ and $\frac{V_3}{o_3}$. This means that the constraints of the maximal farming area and the maximal water usage are dependent. If both the area constraint and the water constraint are binding this means that if we for instance want to change the maximal farming area we may also have to change the maximal water usage.

b.

Figure 2, 3 and 4 illustrates how much the profit can be increased by making marginal increments of the maximal water usage, maximal farming area and maximal petrol usage respectively. The data were gained by iteratively increasing the original right hand side of the constraint by one percent and updating the objective value. This was done considering one constraint at a time. The results show that a marginal increase of the maximal area lead to a significant increase of the profit, compared to a marginal increase of the maximal water and petrol usage respectively which only has a small effect on the objective value. By increasing the area limit one percent we get an increase of the objective value by around 4420 euros, and if we divide that by 16ha (which is 1% of the initial area limit) we get a dual price of 276,34. The objective value increases by around 680 and 380 euros respectively, by increasing the water or petrol limit one percent. This corresponds to a dual price for the water limit of 1,36e-5 and a dual price for the petrol limit of 0.25. By using the dual function in Julia we get values of -276.34, -1.36e-5 and -0.25 for the area, water and petrol limits respectively. We can state that these values, except for the signs, are the size of the dual price. However, we can not be sure of that these signs are correct and if we increase the rhs of a $\leq$-constraint we get a larger feasible region (relaxation) so the signs should be positive. Our calculations therefore give very similar results compared to the solution in Julia.

By a further increase of the right hand side of the constraints we aim to find the point of which a higher value of the rhs does not contribute to an increasing objective value. This is illustrated in figure 2, 3 and 4.

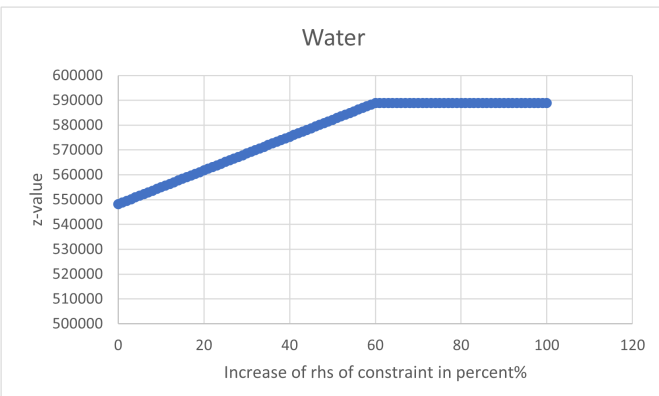

Figure 2: Change of profit from increasing water availability

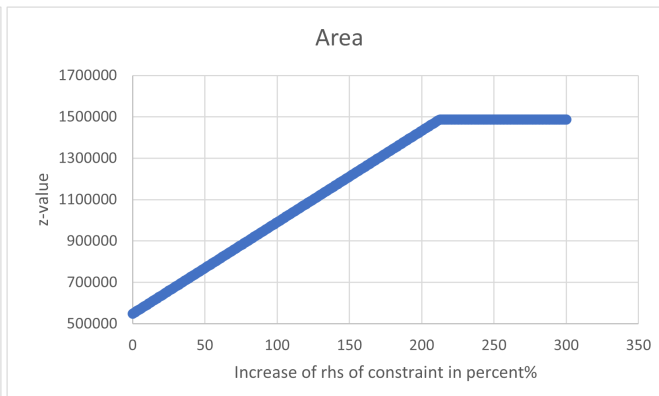

Figure 3: Change of profit from increasing farming area availability

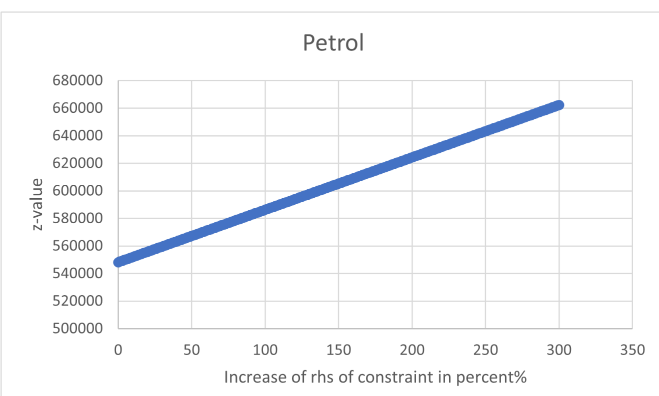

Figure 4: Change of profit from increasing petrol availability

We can see that considering the increase of the maximal water usage we reach a point around 60% where a further increase does not affect the objective value. This corresponds to a value of 8e9 litres. When it comes to the maximal farming area the point where a further increase would not affect the objective value is around 212% (corresponding to a value of 4992 ha). In figure 4 the maximal petrol usage is examined up to 3 times the original value, and we can conclude that the point when the increase of the petrol limit no longer lead to increase of the objective value is beyond that. By further investigation of this we found that the petrol limit need to be increased 78 times before a further increase would not affect the objective value.

c.

To make sunflowers a part of the optimal solution, its oil-per-hectare property needs to be improved. Per the given data a combination of cotton and soy exists that yields more oil for less water per hectare than sunflowers. When increasing the oil yield of sunflowers it will replace the soy. For there to be included more than two types of crops, there can be introduced some kind of constraint on how many hectare of each crop is available. The area and water constraint determines the distribution between the two types of crops selected. The less water is available the more cotton will be part of the solution since it is the most water-efficient. Similarly, if land is the limiting constraint more soy will be planted, as it is the most oil per hectare efficient. Usually, a combination of both constraints will be limiting our freedom and thus affect the crop distribution, therefore the optimal solution will consist of two types of crops. Alternatively, the sunflower can also be modified to be far more water efficient so that it replaces cotton instead of soy. To make this switch the water availability has to be reduced as well, or the area increased.
Some experimentation revealed that the oil per hectare property of the sunflowers has to be increased almost 50%, to 448.2 L/Ha with the given constraints for it to be included instead of soy.

d.

The same results as in Table 4 and 6 were reached but the increased petrol cost reduced our profit margin so the total profit was $\approx$ 518.16k euros in this case.

e.

In this subtask we call the tax for B5, B30 and B100 for tax_1, tax_2 and tax_3 respectively. By fixing $tax_2 = 20\%$ and $tax_3 = 0\%$ (as in the original problem) we investigate how the optimal solution changes when varying tax_1 between 0 and 100%. The optimal base changes as $tax_1 = 14$, as can be seen in table 7.

B_i [k litres] ($tax_1 < 14\%$)	B_i [k litres] ($tax_1 \geq 14\%$)
157.89	0
0	214.29
609.19	552.80

Table 7: Change of base depending on tax_1

When it comes to the objective value it starts at $\approx$ 576.76k euros when $tax_1 = 0$, and decreases until reaching $tax_1 = 14$ where the objective value becomes constant at $\approx$ 548.16k euros for all larger taxes.

By doing the same thing for tax_2 and fixing $tax_1 = 20\%$ and $tax_3 = 0\%$ we get a change of the optimal basis at $tax_2 = 12\%$, illustrated in table 8.

B_i [k litres] ($tax_2 < 12\%$)	B_i [k litres] ($tax_2 \geq 12\%$)
0	157.89
214.29	0
552.80	609.19

Table 8: Change of base depending on tax_2

The objective value it starts at $\approx$ 561.98k euros when $tax_2 = 0$, and decreases until reaching $tax_2 = 12$ where the objective value becomes constant at $\approx$ 531.60k euros for all larger taxes.

We also investigate how varying tax_3 affects the optimal basis and the objective value. By fixing $tax_1 = 20\%$ and $tax_2 = 5\%$ we find that the base does not change but the optimal amount of the production of B100 changes at $tax_3 = 1\%$, as can be seen in table 9.

B_i [k litres] ($tax_3 < 1\%$)	B_i [k litres] ($tax_3 \geq 1\%$)
0	0
214.29	214.29
552.80	657.14

Table 9: Base depending on tax_3

The objective value starts at $\approx$ 548.16k, becomes negative at $tax_3 = 1\%$ and keeps decreasing until reaching the lowest objective value -7.48e6 as $tax_3 = 100\%$.

f.

In this subtask we started by varying the water demand for all crops by the same values at the same times. We started with the original values, generating an objective value of 548.16k euros. By increasing the water demand more and more we get a lower and lower objective value. On the contrary, by decreasing the water demand we get a higher and higher objective value. This makes sense since an increase of the water demand corresponds to a restriction of the constraint, and a decrease corresponds to a relaxation of the constraint.

For the soybeans, the water demand are not allowed to increase anything before the optimal solution changes. The soybean crops have the highest water demand in the original problem which is why an increase of that demand affects the optimal solution immediately. However, the base does not change, so we still only produce B30 and B100 in the optimal solution. When starting to increase the water demand for the soybeans we get small reductions in the objective value. We can increase the water demand 2000 times, from 5e6 litres/ha to 1e10 litres/ha and still get an objective value larger than 500k euros. In that solution we get values of B = [0.0, 214.29, 496.90]k litres which is not so different from the values we get in the original solution. The largest change when increasing only the water demand for soybeans we see in the values of the vector V (representing the litres of vegetable oil farmed from the 3 different crops). We can see that by increasing the water demand for soybeans we get reductions in the production of vegetable oil from soybeans and increments in the production of vegetable oil from cotton. We get V-values of V = [0.16, 0.0, 623.39]k litres by increasing the soybean water demand 2000 times.

g.

The basic variables in our optimal solution are B_2, B_3, V_1, V_3 and s_1, where s_1 is the surplus variable corresponding to the constraint involving biodiesel demand, where $\sum_i^3 B_i - s_1 = \textit{Biodiesel demand}$. Therefore $x_B = [214.29, 552.80, 393.38, 292.28, 487.09]$k litres. For the vector of objective function coefficients for the basic variables, c_B, we have that $c_B = [0.42, 0.83, 0, 0, 0]$. By creating a linear system $Ax = b$ of our constraints, we let b be the right-hand side data of our constraints and B be the basis matrix (the columns in A that belongs to the basic variables). We have that $x_B = B^{-1}b$, and in terms of the basic variables we can write the objective function as $z = c_B^T B^{-1} b$, since all of our non-basic variables are zero. For a small change in b the following relation can be found:

$$\lim_{\Delta b \to 0} \Delta z = \lim_{\Delta b \to 0} c_B^T B^{-1} \Delta b = 0 \tag{3}$$

Equation 3 assumes that the basis is still valid after the perturbing the right hand side of the constraints by Δb. As $\Delta b \to 0$ this is reasonable. However, changing a binding constraint might yield the need for a new basis.

In order for the basic solution to be feasible, we need for our basic variables to be non-negative, namely: $x_B = B^{-1}b \geq 0$. For that basis to also be optimal we need, in a maximization problem like ours, the relation: $\bar{c_N}^T = c_N^T - c_B^T B^{-1} N \leq 0$ to hold, according to [1]. Here c_N are the objective function coefficients for the non-basic variables and N are the columns in A that belongs to the non-basic variables.

h.

One aspect which the original objective function does not account for is *how* the crops are being planted and harvested. This task is usually done using heavy machinery with large diesel engines. If we know for example the operational costs for a tractor [euro/ha] we can add an extra cost to our objective function like this:

$$maximize\ z = \sum_{i}^{3} p_i B_i - t \sum_{j}^{3} V_j / o_j$$

where t is the cost per area for operating a tractor and $\sum_j^3 V_j/o_j$ is the total area used for all crops. This should decrease the overall profit but also try optimize area usage even more, which in turn should minimize tractor usage meaning less pollution from the diesel engines. According to [2] a crop planting and harvesting operation has the following average costs:

Operation	Diesel consumption [gal/ac]
First tillage	1.36
Second tillage	0.78
Fertilizer	0.30
Planting	0.51
Harvest	1.23
Total	**4.18**

Table 10: Fuel consumption for different tractor operations.

The source also suggests that diesel costs are usually around 30% of the total operational cost of a tractor. So in total our planting and harvesting operations would require:

$$4.18\ gal/ac = 39.10\ litres/ha$$

As the time of writing the diesel price in Sweden is 2.37 $euro/litre$ which gives us:

$$diesel = 92.67\ euro/ha$$
$$t = diesel/0.3 = 308.89\ euro/ha$$

Implementing our new objective function in our model gives us a total area usage of 1000ha and a profit of $\approx$ 73.47k euros. So accounting for the costs for planting and harvesting will reduce our profits quite a bit.

References

[1] Lundgren, L., Rönnqvist, M., Värbrand, P. (2012). *Optimization*(1). Studentlitteratur.

[2] Grisso, R. (2020). *Predicting Tractor Diesel Fuel Consumption*. Virginia Cooperative Extension. 442-073. https://vtechworks.lib.vt.edu/bitstream/handle/10919/98875/BSE-328.pdf?sequence=1

Appendix

run.jl

```
using JuMP
using Gurobi

include("mod.jl")
m, B, V = build_biodiesel_model("dat.jl")

print(m) # prints the model instance

set_optimizer(m, Gurobi.Optimizer)
set_optimizer_attribute(m, "OutputFlag", 0)
optimize!(m)

println("Optimal profit: z = ", value(objective_function(m)))
println("Optimal biofuel distribution [L]: B = ", value.(B.data))
println("Optimal crop distribution [ha]: V = ", value.(V.data./oil_per_hectar))
println("Water usage per crop [ML]: W = ", value.(V.data./oil_per_hectar.*water_per_hectar)/1e6)
println("Water used: ", sum(value.(V.data./oil_per_hectar).*water_per_hectar)/water_max*100, "%")
println("Petrol used: ", [0.95, 0.7, 0]'*value.(B.data)/petrol_max*100, "%")
println("Area used: ", sum(value.(V.data)./oil_per_hectar)/area_max*100, "%")

function get_slack(constraint::ConstraintRef)::Float64
    con_func = constraint_object(constraint).func
    interval = MOI.Interval(constraint_object(constraint).set)
    row_val = value(con_func)
    return min(interval.upper - row_val, row_val - interval.lower)
end

println("veg_oil slack = ", get_slack(enough_veg_oil))
println("area slack = ", get_slack(area_limited))
println("water slack = ", get_slack(water_limited))
println("biodiesel surplus = ", get_slack(biodiesel_demand))
println("petrol slack = ", get_slack(petrol_limited))
```

mod.jl

```
function build_biodiesel_model(data_file::String)
  include(data_file)
  m = Model()

  @variable(m, B[I] >= 0) # 3 different products [litre biofuel]
  @variable(m, V[J] >= 0) # 3 different crops [litre veg.oil]

  # total profit
  @objective(m, Max, B5_profit*B[1]+B30_profit*B[2]+B100_profit*B[3])

  @constraint(m, biodiesel_demand, sum(B) >= biodiesel_min)
  @constraint(m, area_limited, sum(V./oil_per_hectar) <= area_max)
```

```
  @constraint(m, water_limited, sum(V./oil_per_hectar.*water_per_hectar) <= water_max)
  @constraint(m, petrol_limited, [0.95, 0.7, 0]'*B <= petrol_max)
  @constraint(m, enough_veg_oil, sum(B.*veg_oil_used) <= sum(V))

  return m, B, V
end
```

dat.jl

```
# sets
I = 1:3 # 3 different products
J = 1:3 # 3 different crops

# labels
products = ["B5", "B30", "B100"]
crops = ["Soy", "Sunflower", "Cotton"]

# creating veg.oil
oil_per_hectar = [462.8, 302.4, 389.7] # [L/ha]
water_per_hectar = [5e6, 4.2e6, 1e6] # [L/ha]

# creating biodiesel
methanol_cost = 0.2*1.5/0.9 # per litre biodiesel
veg_oil_used = [0.05, 0.3, 1]./0.9 # per litre biodiesel

# profits after taxes per litre biodiesel
petrol_cost = 1
B5_profit = 1.43*0.8 - 0.95*petrol_cost - 0.05*methanol_cost
B30_profit = 1.29*0.95 - 0.7*petrol_cost - 0.3*methanol_cost
B100_profit = 1.16 - 1*methanol_cost

# constraints
biodiesel_min = 280e3
area_max = 1600
water_max = 5000e6# L
petrol_max = 150e3
```