Chalmers University of Technology
University of Gothenburg
Mathematical Sciences
Optimization
Ann-Brith Strömberg

MVE165
MMG631
Linear and integer optimization
with applications
Assignment information
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Assignment 2: Maintenance scheduling

Implementations of the model in Julia are found in the Canvas course room, under Files. Study the Julia files carefully to get some hints before you start solving the exercises. The file `as_run.jl` should be edited in order to solve the different instances of the model, as described in the exercises below.

The mathematical model

Given is a mathematical model for finding a maintenance schedule such that the costs of maintaining a system during a limited time period is at minimum, while ensuring that the system is functioning during the entire period. The system consists of several components with economic dependencies and limited lives. The problem background and the mathematical model are described in the article [1]; the model is also described in the notes of Lecture 9.

Sets and parameters

- $\mathcal{N}$ = the set of components in the system. `(in Julia: Components)`
- T = the number of time steps in the planning period. `(in Julia: T)`
- T_i = the life of a new component of type $i \in \mathcal{N}$ (measured in number of time steps). It is assumed that $2 \leq T_i \leq T-1$. `(in Julia: U)`
- c_{it} = the cost of a spare component of type $i \in \mathcal{N}$ at time t (measured in €). For some instances it is assumed that c_{it} is constant over time, i.e., $c_{it} = c_i$, $t = 1, \ldots, T$. `(in Julia: c)`
- d_t = the cost for a maintenance occasion at time t (measured in €). For some instances it is assumed that d_t is constant over time, i.e., $d_t = d$, $t = 1, \ldots, T$. `(in Julia: d)`

Decision variables

- $x_{it} = \begin{cases} 1 & \text{if component } i \text{ is replaced at time } t, \\ 0 & \text{otherwise,} \end{cases} \quad i \in \mathcal{N},\ t \in \{1, \ldots, T\}.$

- $z_t = \begin{cases} 1 & \text{if maintenance is made at time } t, \\ 0 & \text{otherwise,} \end{cases} \quad t \in \{1, \ldots, T\}.$

The model

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{T} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d_t z_t \right), && && \text{(1a)}\\
\text{subject to} \quad & \sum_{t=\ell+1}^{\ell+T_i} x_{it} \geq 1, && \ell = 0, \dots, T - T_i, \; i \in \mathcal{N}, && \text{(1b)}\\
& x_{it} \leq z_t, && t = 1, \dots, T, \; i \in \mathcal{N}, && \text{(1c)}\\
& x_{it}, z_t \in \{0, 1\}, && t = 1, \dots, T, \; i \in \mathcal{N}. && \text{(1d)}
\end{aligned}
$$

Description of the model

(1a) The objective is to minimize the total cost for the maintenance during the planning period (the time steps $1, \dots, T-1$) `(in Julia: Cost)`.

(1b) Each component i must be replaced at least once within each T_i time steps `(in Julia: ReplaceWithinLife)`.

(1c) Components can only be replaced at maintenance occasions `(in Julia: ReplaceOnlyAtMaintenance)`.

(1d) All the variables are required to be binary.

Exercises to perform and questions to answer

1. (a) Solve the model (1) as implemented in the file `as_mod.jl` with data from `as_dat_large.jl`, letting $T = 125$, and with integer requirements on the variables x_{it} and z_t. *Note that in this instance all costs are time independent, i.e.,* $c_{it} = c_i$ *and* $d_t = d$, $t = 1, \dots, T$. Relax the integrality requirements on the variables x_{it} and resolve the problem. Then relax the integrality on all variables and resolve the model. Compare the three solutions obtained and discuss their interpretations. Compare also the corresponding computation times (CPU) and explain their differences using mathematical properties.

   (b) Solve the model (1) as implemented in `as_mod.jl` with data from `as_dat_small.jl`. Relax the integrality constraints on all the variables and resolve. Then add the constraint given by the last function in the file `as_mod.jl` by calling (from the run-file) the function `add_cut_to_small(m)` and resolve. Compare the solutions obtained and explain their differences.

   (c) Prove mathematically that the additional constraint implemented in `as_mod.jl` is a *valid inequality* (i.e., it does not cut away any feasible solution) to the instance of (1) defined by `as_dat_small.jl`.

2. Solve the model (1) as implemented in `as_mod.jl` and `as_dat_large.jl`, letting $d = 20$, and with integer requirements on the variables z_t only.

   (a) Vary the time horizon between $T = 50$ and approximately $T = 200$ and draw a graph of the computing time (in CPU seconds) as a function of T (use a log-scale). If needed, use the options for limiting the size of the branch–and–bound tree—keeping track of upper and lower bounds on the optimal value (see the file `as_run.jl`).

(b) Make an analogous graph for the case when the integrality requirements on the variables are relaxed; vary the time horizon between $T = 50$ and approximately $T = 700$.

(c) Compare and comment on the complexity properties of the two models solved in 2a and 2b.

(d) Gurobi uses the branch–and–bound algorithm, possibly employing presolve steps including heuristics and cutting plane generation. On what does it seem to spend most of the solution time: presolve, finding an optimal (feasible) solution, or verifying its optimality?

3. The model (1a)–(1f) from [2] generalizes the model (1) above. Implement the model from [2] in Julia.

   (a) Repeat the tests made in exercises 2a–2c for this model. Illustrate the results with suitable graphs.

   (b) Compare the outcomes from the two models and make relevant conclusions. Illustrate with suitable graphs.

   (c) Discuss the characterization of the mathematical model from the article in terms of, e.g., integrality property and network flows. Explain your findings.

4. Assume that it is required that the system (including all of its components) has a remaining life which is at least $r > 0$ time steps at the end of the planning period (i.e., at time $t = T$).[1]

   (a) Add and/or modify constraints to/in the model to accomplish this and solve the resulting model. Start by the model in `as_mod.jl` with data from `as_dat_large.jl` (with $d = 20$ and $T = 100$). Verify that the solution fulfills the requirement stated.

   (b) For five different relevant values of r (these values should be chosen such that the respective optimal solutions become significantly different), compare the total cost for maintenance according to the schedule computed in 4a with that of the "original" (with $r = 0$) one. Comment on the number of maintenance occasions and the number of replaced components and compare with the corresponding numbers from the "original" model (with $r = 0$).

   (c) What values of r are relevant for this study and why?

References

[1] T. Almgren, N. Andréasson, M. Patriksson, A.-B. Strömberg, A. Wojciechowski, and M. Önnheim. The opportunistic replacement problem: theoretical analyses and numerical tests. *Mathematical Methods of Operations Research*, 76:289–319, 2012. Reachable from Chalmers' domain, `http://dx.doi.org/10.1007/s00186-012-0400-y`.

[2] E. Gustavsson, M. Patriksson, A.-B. Strömberg, A. Wojciechowski, and M. Önnheim. Preventive maintenance scheduling of multi-component systems with interval costs. *Computers & Industrial Engineering*, 76:390–400, 2014. Open access, `http://dx.doi.org/10.1016/j.cie.2014.02.009`.

[1] You may carry out this exercise using the model from [2] as well.