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Exercise 6

Problem 1 a)

$$F = m \cdot a$$

$m=1$, $F=u$, $x_1 = \text{position}$

$x_2 = \dot{x}_1 = \text{velocity}$

$$\underline{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_c} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B_c} u$$

b) $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$A = e^{A_c T} = \sum_{k=0}^{\infty} \frac{(A_c T)^k}{k!} \approx 1 + A_c T + \frac{(A_c T)^2}{2}$$

$$= 1 + \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \cancel{\begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

$$b = \left(\int_0^{T} e^{A_c \tau} d\tau\right) b_c$$

$$= \left(\int_0^{T} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} d\tau\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.125 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}}}$$

c) $$f(z) = \frac{1}{2} \sum_{t=0}^{N-1} \left\{ x_{t+1}^T Q x_{t+1} + u_t^T R u_t \right\}$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = 2$$

The solution to $\min_{z \in \mathbb{R}^N} f(z)$

$$u_t = -K_t x_t$$

(1) $$K_t = R_t^{-1} B_t^T P_{t+1} (I + B_t R_t^{-1} B_t^T P_{t+1})^{-1} A_t$$

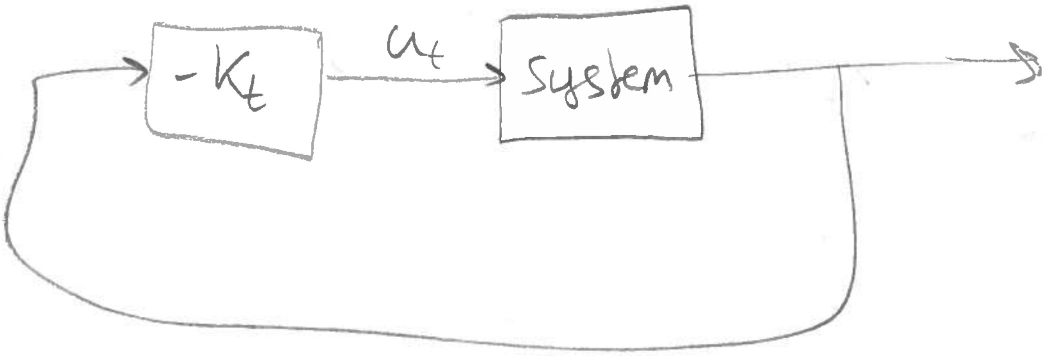
$$P_t = Q_t + A_t^T P_{t+1} (I + B_t R_t^{-1} B_t^T P_{t+1})^{-1} A_t$$

$$P_N = Q_N$$

We can omit the subscript on Q, R, A, B since they are constant.

- Since P_N is known ($= Q$) we start here and calculate $P_{N-1}, P_{N-2}, \dots, P_1$

And then calculate $K_1, \dots, K_N$.



d) Let $N \to \infty$ $\Rightarrow$ We solve the LQR problem with:

$$K = R^{-1} B^T P (I + B R^{-1} B^T P)^{-1} A$$

$$P = Q + A^T P (I + B R^{-1} B^T P)^{-1} A$$

$$P = P^T \geq 0$$ (symmetric and positive semidefinite)

$K = [0.6514,\ 1.3142]$

eig Closed loop system:

$$A - BK = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.6514 & 1.3142 \end{bmatrix}$$

$$\text{eig}(A-BK): \quad \begin{aligned} \lambda_1 &= 0.6307 + 0.1628i \\ \lambda_2 &= 0.6307 - 0.1628i \end{aligned}$$

$|\lambda_1| = |\lambda_2| = 0.6509 < 1 \Rightarrow$ system is stable.

e) That is generally true. if ~~A~~ the system (A, B) is controllable

Problem 2

$$x_{t+1} = 3x_t + 2u_t \qquad x_t \in R^1,\ u_t \in R^1$$

cost function:

$$f^{\infty}(z) = \frac{1}{2}\sum_{t=0}^{\infty}\{q x_{t+1}^2 + u_t^2\} \qquad q>0$$

a) The Riccati equation would then be:

$$P = Q + A^T P (I + BR^{-1}B^T P)^{-1} A$$

$$p = q + ap\left(1 + \frac{b}{r}bp\right)^{-1} \cdot a = q + \frac{a^2 p \cdot r}{r + b^2 p}$$

Set $q=2,\ a=3,\ b=2,\ r=1$

$$p = 2 + \frac{9p}{1+4p} \Rightarrow \quad p + 4p^2 - 2 - 8p - 9p = 0$$

$$4p^2 - 16p - 2 = 0$$

$$p^2 - 4p - \frac{1}{2} = 0$$

$$P_1 = 4{,}12$$

$$P_2 = -0{,}12$$

$$k = \frac{bp}{r}\frac{a}{\left(1 + \frac{b}{r}\cdot b \cdot p\right)}$$

$$= \frac{2 \cdot 4{,}12}{1} \cdot \frac{3}{(1 + 2^2 \cdot 4{,}12)} = \underline{1.414}$$

$$\Rightarrow \quad u_t = -1.414 \cdot x_t$$

2c) This gives an asymptotically stable system iff A, B is controllable.

Problem 3

$$x_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1 & -0.79 & 1.78 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix} u_t$$

$$y_t = [0 \; 0 \; 1] x_t$$

$$x_0 = [0 \; 0 \; 1]^T.$$

$$f(y_1, \ldots, y_N, u_0, \ldots, u_{N-1}) = \sum_{t=0}^{N-1} \{ y_{t+1}^2 + r \cdot u_t^2 \} \qquad r > 0$$

$N = 30$. Input constraint: $-1 \leq u_t \leq 1$, $t \in \langle 0, N-1 \rangle$

b) With $N = 30$ and $u_N = 6$ we will have 6 columns:

$$\begin{matrix} -B & 0 & 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 & 0 & \\ -B & 0 & & \vdots & & \\ -B & 0 & & \vdots & & \\ -B & 0 & & & & \\ -B & & & & & \\ -B & & & & & \\ -B & & & & & \\ -B & & & & & \\ B & & & & & \\ B & & & & & \end{matrix} \qquad \text{osv.}$$

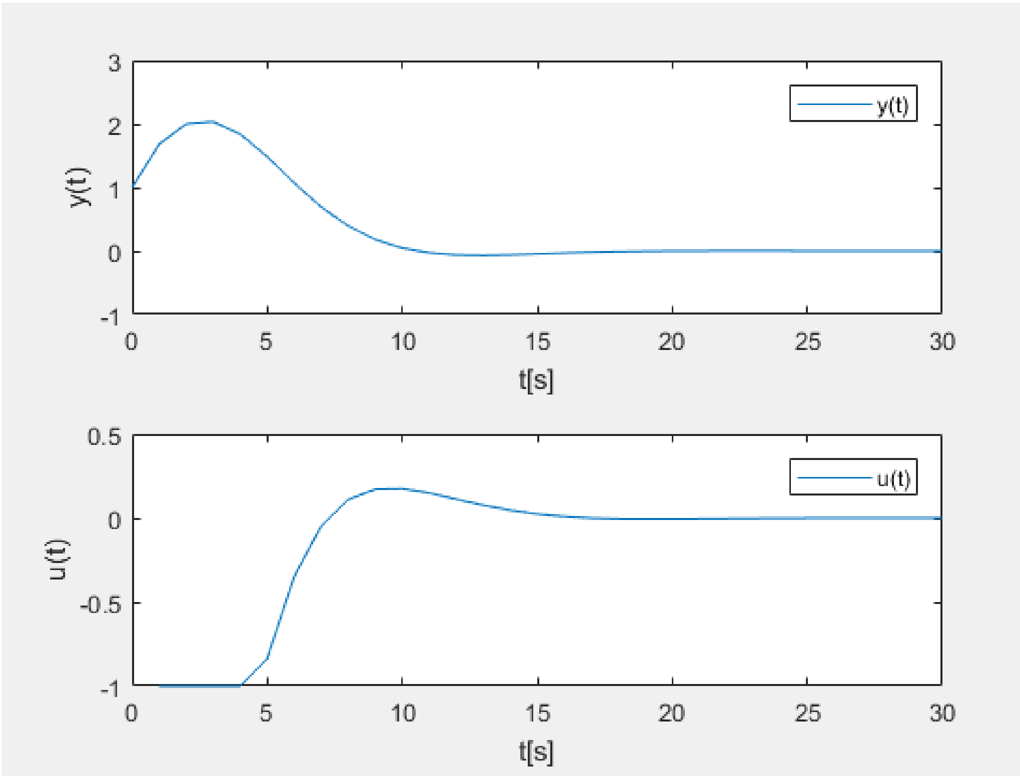
Problem 3 c)

Now I use 5 iterations.

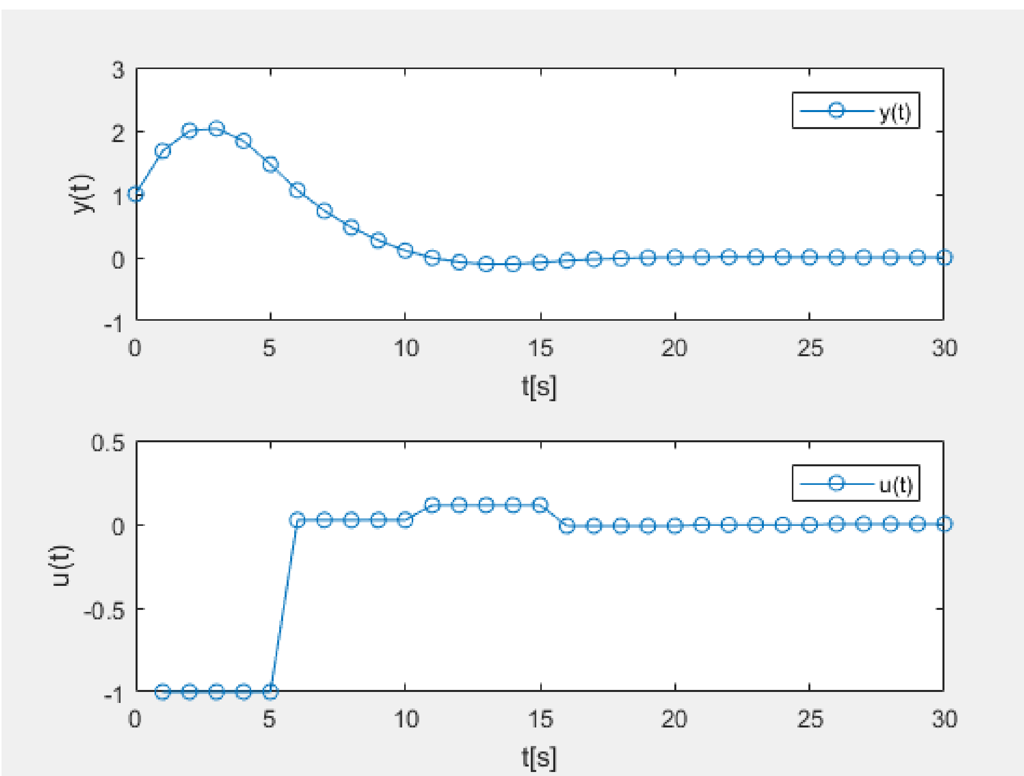
e)

f) To reduce number of iterations. We acheive almost the same efficiency but with ~~greatly~~ reduced cost.

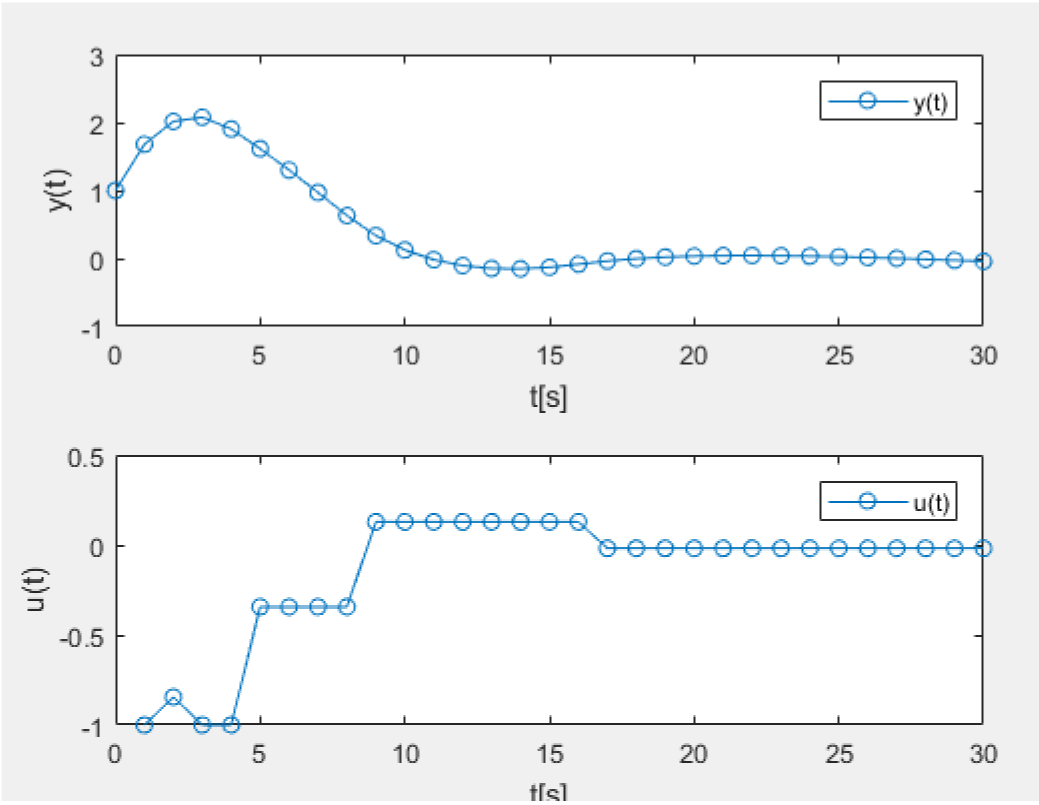
3a)



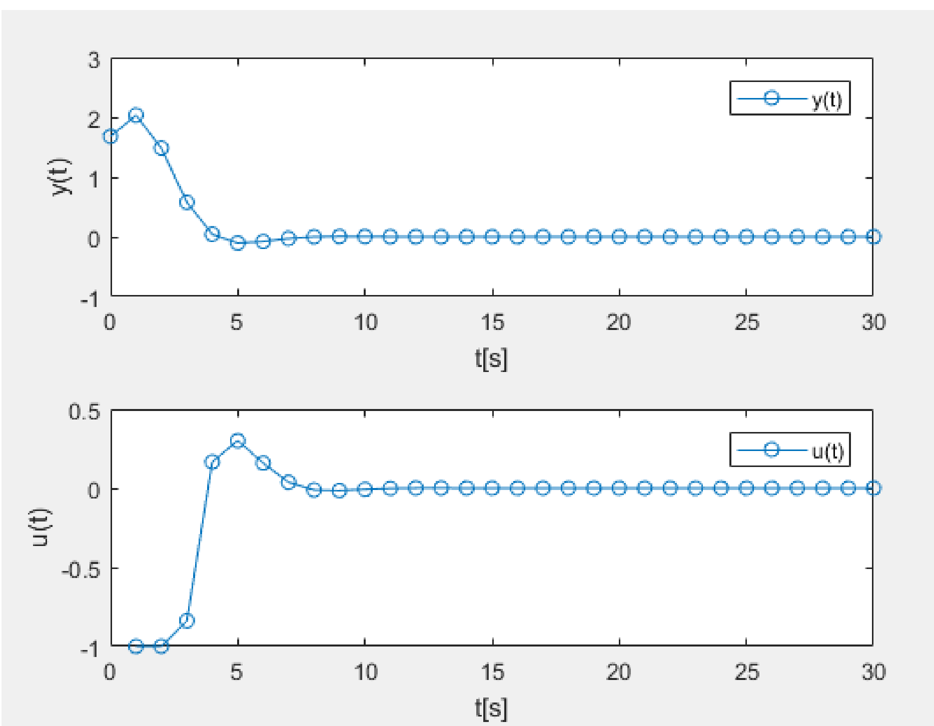
3b)



3c)



3d)



3e)