STUDENT MATHEMATICAL LIBRARY
Volume 90

Discrete Morse Theory

Nicholas A. Scoville

AMS AMERICAN MATHEMATICAL SOCIETY

STUDENT MATHEMATICAL LIBRARY
Volume 90

Discrete Morse Theory

Nicholas A. Scoville

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification*. Primary 55U05, 58E05, 57Q05, 57Q10.

For additional information and updates on this book, visit
www.ams.org/bookpages/stml-90

Library of Congress Cataloging-in-Publication Data

Cataloging-in-Publication Data has been applied for by the AMS.
See http://www.loc.gov/publish/cip/.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.
Visit the AMS home page at `https://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 24 23 22 21 20 19

J.M.J.

Contents

Preface

This book serves as both an introduction to discrete Morse theory and a general introduction to concepts in topology. I have tried to present the material in a way accessible to undergraduates with no more than a course in mathematical proof writing. Although some books such as [**102**, **132**] include a single chapter on discrete Morse theory, and one [**99**] treats both smooth and discrete Morse theory together, no book-length treatment is dedicated solely to discrete Morse theory. Discrete Morse theory deserves better: It serves as a tool in applications as varied as combinatorics [**16**, **41**, **106**, **108**], probability [**57**], and biology [**136**]. More than that, it is fascinating and beautiful in its own right. Discrete Morse theory is a discrete analogue of the "smooth" Morse theory developed in Marston Morse's 1925 paper [**124**], but it is most popularly known via John Milnor [**116**]. Fields medalist Stephen Smale went so far as to call smooth Morse theory "the single greatest contribution of American mathematics" [**144**]. This beauty and utility carries over to the discrete setting, as many of the results, such as the Morse inequalities, have discrete analogues. Discrete Morse theory not only is topological but also involves ideas from combinatorics and linear algebra. Yet it is easy to understand, requiring no more than familiarity with basic set theory and mathematical proof techniques. Thus we find several online introductions to discrete Morse theory written by undergraduates. For example, see the notes of Alex Zorn for his REU project at the University of Chicago [**158**], Dominic Weiller's bachelor's thesis [**150**], and Rachel Zax's bachelor's thesis [**156**].

From a certain point of view, discrete Morse theory has its foundations in the work of J. H. C. Whitehead [**151**, **152**] from the early to mid-20th century, who made the deep connection between simple homotopy and homotopy type. Building upon this work, Robin Forman published the original paper introducing and naming discrete Morse theory in 1998 [**65**]. His extremely readable *A user's guide to discrete Morse theory* is still the gold standard in the field [**70**]. Forman published several subsequent papers [**66**, **68–71**] further developing discrete Morse theory. The field has burgeoned and matured since Forman's seminal work; it is certainly established enough to warrant a book-length treatment.

This book further serves as an introduction, or more precisely a first exposure, to topology, one with a different feel and flavor from other introductory topology books, as it avoids both the point-set approach and the surfaces approach. In this text, discrete Morse theory is applied to simplicial complexes. While restriction to only simplicial complexes does not expose the full generality of discrete Morse theory (it can be defined on regular CW complexes), simplicial complexes are easy enough for any mathematically mature student to understand. A restriction to simplicial complexes is indeed necessary for this book to act as an exposure to topology, as knowledge of point-set topology is required to understand CW complexes. The required background is only a course in mathematical proofs or an equivalent course teaching proof techniques such as mathematical induction and equivalence relations. This is not a book about smooth Morse theory either. For smooth Morse theory, one can consult Milnor's classic work [**116**] or a more modern exposition in [**129**]. A discussion of the relations between the smooth and discrete versions may be found in [**27**, **29**, **99**].

One of the main lenses through which the text views topology is homology. A foundational result in discrete Morse theory consists of the (weak) discrete Morse inequalities; it says that if K is a simplicial complex and $f\colon K \to \mathbb{R}$ a discrete Morse function with m_i critical simplices of dimension i, then $b_i \leq m_i$ where b_i is the ith Betti number. To prove this theorem and do calculations, we use $\mathbb{F}_2$-simplicial homology and build a brief working understanding of the necessary linear algebra in Chapter 3. Chapter 1 introduces simplicial complexes, collapses, and simple homotopy type, all of which are standard topics in topology.

Any book reflects the interests and point of view of the author. Combining this with space considerations, I have regrettably had to leave much more out than I included. Several exclusions are worth mentioning here. Discrete Morse theory features many interesting computational aspects, only a few of which are touched upon in this book. These include homology and persistent homology computations [**53**, **80**, **82**], matrix factorization [**86**], and cellular sheaf cohomology computations [**48**]. Mathematicians have generalized and adapted discrete Morse theory to various settings. Heeding a call from Forman at the end of *A user's guide to discrete Morse theory* [**70**], several authors have extended discrete Morse theory to certain kinds of infinite objects [**8**, **10**, **12**, **15**, **105**]. Discrete Morse theory has been shown to be a special case [**155**] of Bestvina-Brady discrete Morse theory [**34**, **35**] which has extensive applications in geometric group theory. There is an algebraic version of discrete Morse theory [**87**, **102**, **142**] involving chain complexes, as well as a version for random complexes [**130**]. E. Minian extended discrete Morse theory to include certain collections of posets [**118**], and B. Benedetti developed discrete Morse theory for manifolds with boundary [**28**]. There is also a version of discrete Morse theory suitable for reconstructing homotopy type via a certain classifying space [**128**]. K. Knudson and B. Wang have recently developed a stratified version of discrete Morse theory [**100**]. The use of discrete Morse theory as a tool to study other kinds of mathematics has proved invaluable. It has been applied to study certain problems in combinatorics and graph theory [**16**, **41**, **49**, **88**, **106**, **108**] as well as configuration spaces and subspace arrangements [**60**, **122**, **123**, **139**]. It is also worth noting that before Forman, T. Banchoff also developed a discretized version of Morse theory [**17–19**]. This, however, seemed to have limited utility. E. Bloch found a relationship between Forman's discrete Morse theory and Banchoff's [**36**].

I originally developed these ideas for a course in discrete Morse theory taught at Ursinus College for students whose only prerequisite was a proof-writing course. An introductory course might cover Chapters 1–5 and Chapter 8. For additional material, Chapters 6 and 9 are good choices for a course with students who have an interest in computer science, while Chapters 7 and 10 are better for students interested in pure math. Some of the more technical proofs in these chapters may be skipped. A more advanced course could begin at Chapter 2 and cover the rest of the book, referring back to Chapter 1 when needed. This book could also be used as a supplemental text for a course in algebraic topology or topological combinatorics, an independent study, or a directed study, or as the basis for an undergraduate research project. It is also intended for research mathematicians who need a primer in or reference for discrete Morse theory. This includes researchers in not only topology but also combinatorics who would like to utilize the tools that discrete Morse theory provides.

Exercises and Problems

The structure of the text reflects my philosophy that "mathematics is not a spectator sport" and that the best way to learn mathematics is to actively do mathematics. Scattered throughout each chapter are tasks for the reader to work on, labeled "Exercise" or "Problem." The distinction between the two is somewhat artificial. The intent is that an Exercise is a straightforward application of a definition or a computation of a simple example. A Problem is either integral to understanding, necessary for other parts of the book, or more challenging. The level of difficulty of the Problems can vary substantially.

A note on the words "easy," "obvious," etc.

In today's culture, we often avoid using words such as "easily," "clearly," "obviously," and the like. It is thought that these words can be stumbling blocks for readers who do not find it clear, causing them to become discouraged. For that reason, I have attempted to avoid using these words in the text. However, the text is not completely purged of such words, and I would like to convey what I mean when I use them. I often tell my students that a particular mathematical fact is "easy but it is very difficult to see that it is easy." By this I mean that one may need to spend a significant amount of time struggling to understand the meaning of the claim before it "clicks." So when the reader sees words like "obviously," she should not despair if it is not immediately obvious to her. Rather, the word is an indication that should alert the reader to write down an example, rewrite the argument in her own words, or stare at the definition until she gets it.

Erratum

A list of typos, errors, and corrections for the book will be kept at `http://webpages.ursinus.edu/nscoville/DMTerratum.html`.

Acknowledgements

Many people helped and supported my writing of this book. Ranita Biswas, Sebastiano Cultrera di Montesano, and Morteza Saghafian worked through the entire book and offered detailed comments and suggestions for improvement. Steven Ellis, Mark Ellison, Dominic Klyve, Max Lin, Simon Rubinstein-Salzedo, and Jonathan Webster gave helpful mathematical feedback and caught several typos and errors. I am grateful for the help of Kathryn Hess, Chris Sadowski, Paul Pollack, Lisbeth Fajstrup, Andy Putman, and Matthew Zaremsky for answering specific questions, as well as Dana Williams and Ezra Miller for their help developing the index. A very special thanks to Pawel Dłotko, Mimi Tsuruga, Chris Tralie, and David Millman for their help with Chapter 9. Of course, any errors in the book are solely *mea culpa, mea culpa, mea maxima culpa*. I would like to thank Kevin Knudson for his encouragement and support of this book, as well as Ursinus College, the Mathematics and Computer Science department, and all my students for supporting this project. I would like to express my appreciation to Ina Mette at AMS for her help and patience throughout the submission and publication process, as well as several anonymous reviewers. My family has provided amazing support for this project, through both encouragement and feedback. I am grateful for edits and suggestions from my maternal cousin Theresa Goenner and my paternal cousin Brooke Chantler who resisted the urge to change all the spellings to reflect the Queen's English. A special thanks to both my father Joseph Scoville and my aunt Nancy Mansfield, neither of whom has taken a course in calculus, but who read every single word of this book and gave detailed feedback which greatly improved its readability. Finally, this project could not have been completed without the love and support of my family: my children Gianna,

Aniela, Beatrix, Felicity, and Louisa-Marie, and most especially my wife Jennifer, *mio dolce tesoro*.

Nick Scoville
Feast of Louis de Montfort

Chapter 0

What is discrete Morse theory?

This chapter serves as a gentle introduction to the idea of discrete Morse theory and hence is free from details and technicalities. The reader who does not need any motivation may safely skip this chapter. We will introduce discrete Morse theory by looking at its two main components: discrete topology and classical Morse theory. We then combine these two ideas in Section 0.3 to give the reader a taste of discrete Morse theory. First we ought to address the question "What is topology?" I'm glad you asked!

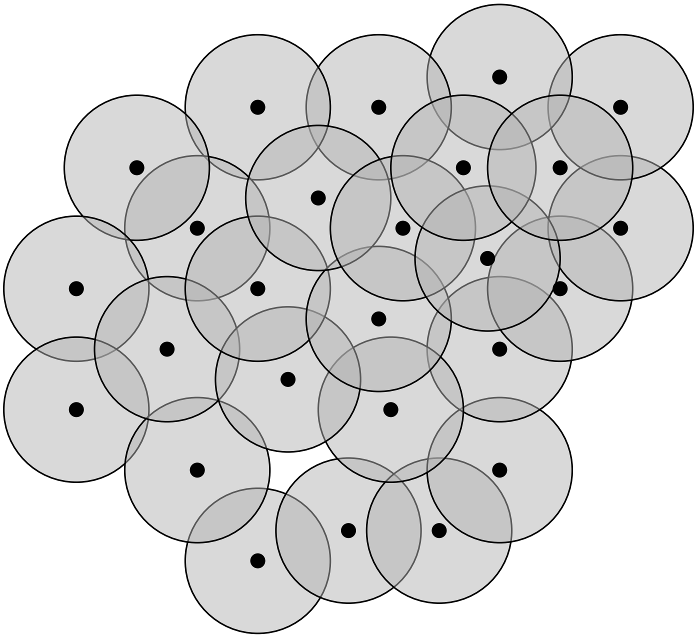
Topologists study the same objects you might study in geometry, as opposed to algebra or number theory. In algebra you study mostly equations, and in number theory you study integers. In geometry, you study mostly geometric objects: lines, points, circles, cubes, spheres, etc. While equations and integers certainly come up in the study of geometry, they are used only secondarily as tools to learn about the geometric objects. Topology, then, is like geometry in so far as it studies points, lines, circles, cubes, spheres—any physical shape you can think of—and all those objects somehow extrapolated into higher dimensions. So far so good. But it differs from geometry in the following way: topology does not have a concept of "distance," but it does have one of nearness. Now, I consider myself an amateur scholastic, and hence I am quite used to the charge of having made a distinction without a difference[1]. Below we establish a distinction between distance and nearness, a distinction that indeed makes a difference. Keep in mind, though, that not positing a distance does not imply that distance does not exist. Rather, it may simply mean we do not have access to that information. This will be a key point in what follows. Finally, topology is good at both counting and detecting holes. A hole is a somewhat nebulous beast. What exactly is it? If you think about it, a hole is defined in terms of what *isn't* there, so it can be unclear what a hole is or how to tell if you have one. Topology develops tools to detect holes. We will illustrate these aspects of topology with three examples in the following section.

0.1. What is discrete topology?

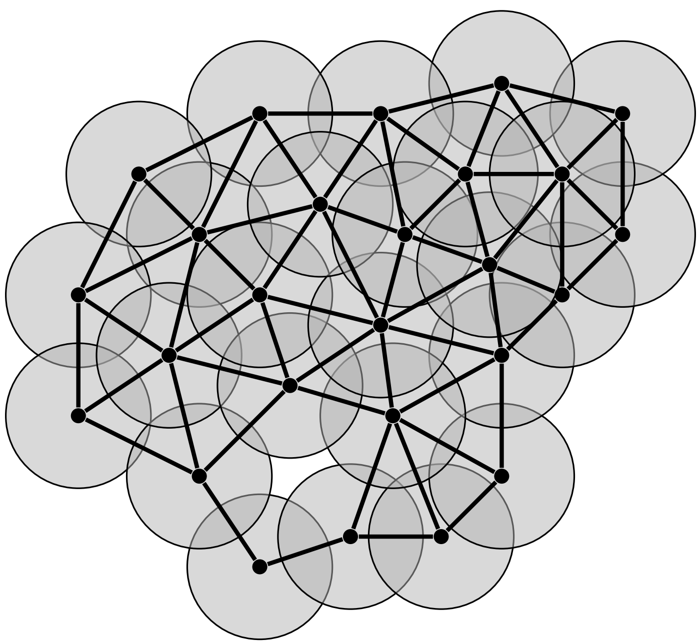
We introduce discrete topology through three applications.

0.1.1. Wireless sensor networks. Sensors surround us. They are in our cell phones, clickers, and EZ passes, to name just a few examples. They are, however, not always found in isolation. Sometimes a collection of sensors work together for a common purpose. Take, for example, cell phone towers. Each tower has a sensor to pick up cell phone signals. The whole cell phone system is designed to cover the largest area possible, and to that end, the towers communicate with each other. Given the local information that each cell phone tower provides, can we learn whether a particular region is covered, so that no matter where you are in the region you will have cell phone service?

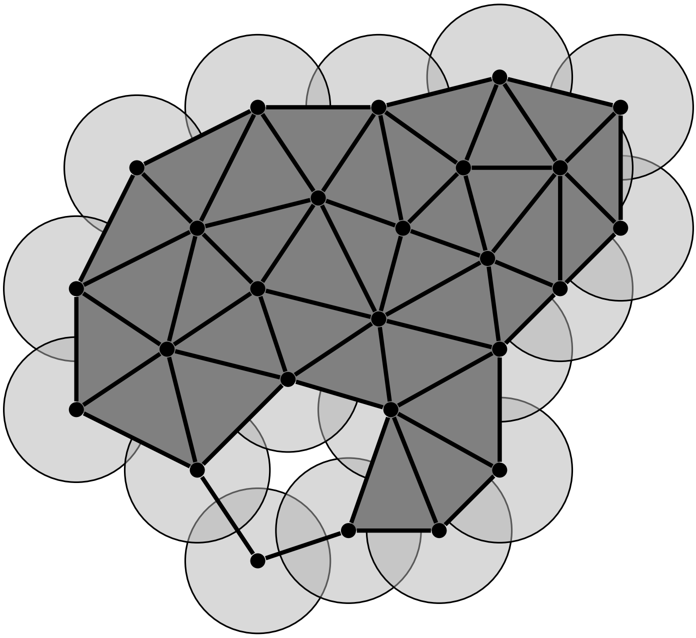
[1]Negare numquam, affirmare rare, distinguere semper.



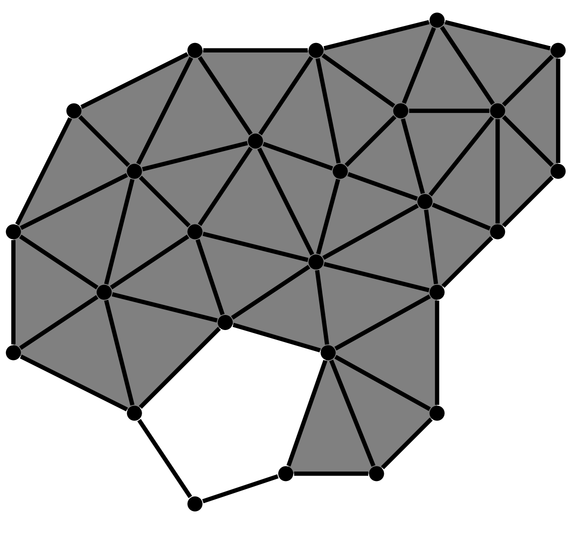
Consider the above system of cell phone towers. Each point represents a cell tower, and each circle surrounding the point is the tower's sensing area. We'll assume that all towers have the same known sensing radius along with some other technical details (see reference [**51**] or [**76**, § 5.6]). We would like each cell phone tower to send us some information and to then use that information to determine whether or not we have coverage everywhere in the region. What kind of information would we like? Perhaps each tower can tell us its location in space, its global coordinates. If it can send us this information, then we can use methods from geometry to tell whether or not we have coverage in the whole region. Not terribly exciting. But what if we don't have access to that information? Suppose that these are cheap sensors lacking a GPS, or that their coordinate information was lost in transmission. What can we do then? Here is where topology is powerful. Topology makes fewer assumptions as to what information we know or have access to. We make the much weaker assumption that each tower can only tell which other towers it is in contact with. So if tower A has tower B in its radius, it communicates that information to us. But we don't have any idea if B is just on the boundary of A's radius or if B is nearly on top of A. In other words, we know that A and B are "near" to each other, but we don't know their distance from each other. Similarly, if A and C do not communicate, it could be that C is just outside A's sensing radius or C is completely on the other side of the region from A. The only information each tower tells us is what other cell towers are in its radius. Here we reemphasize the important point mentioned above: it is not that the distance does not exist; we just do not have access to that information. Now, with this much more limited information, can we still determine if we have a hole? The answer is that we sometimes can. Using the example above, we build the communication graph, connecting any two towers that can communicate with each other.



Next, we fill in any region formed by a triangle, which represents three towers, any two of which can communicate with each other.



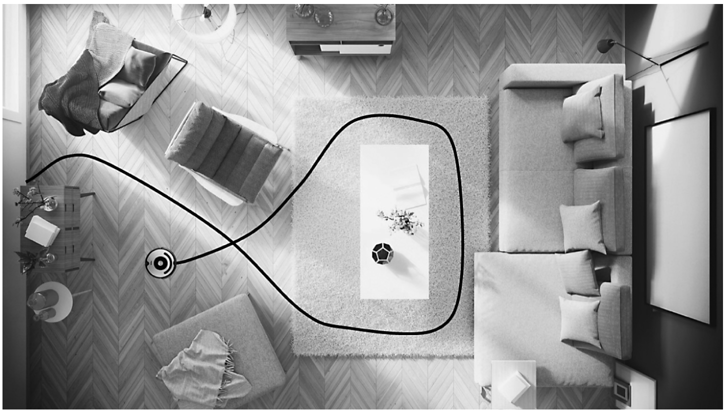
Now we remove the cell tower circles.



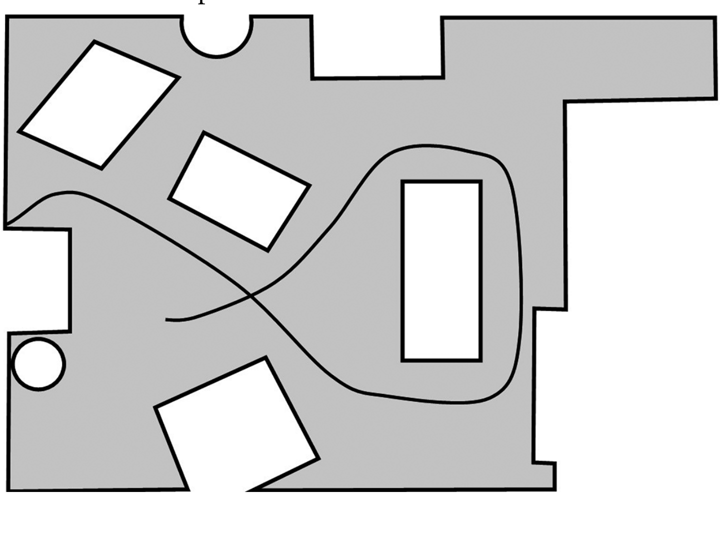
What we are left with is a pure mathematical model of this real-life situation. This object is called a **simplicial complex**, and it is this object that we study in this book. We can take this model, analyze it using the methods of topology, and determine that there is exactly one hole in it. You will learn how to do this in Section 3.2. The point for now is that we wanted to know whether or not there was a hole in a region, we assumed we had access to very limited information, and we were able to detect a hole by modeling our information with a topological object called a simplicial complex.

For the details of the mathematics behind this example, see the paper [**51**] or [**76**, § 5.6].

0.1.2. Tethered Roomba. A Roomba is a small disk-shaped automatic vacuum cleaner. It is automatic in the sense that you turn it on, set it down and leave the room, and after, say, two hours the entire room is cleaned. While this is a promising idea, it does have a major drawback: power. The idea of an upright cordless vacuum is still somewhat of a novelty, and such vacuums that do exist seem not to be terribly powerful. A vacuum must be plugged into the wall to have enough power to do a decent job in a room of any reasonable size. Hence, a battery-powered Roomba does not do a very good job of vacuuming the floor, or if it does, the batteries need to be replaced often. One way around this problem is to introduce a tethered Roomba. This is a Roomba with a cord stored inside of it and coming out the back. The cord plugs into the wall and remains taut in the sense that if the Roomba moves forward, the cord is released, and if the Roomba moves backwards, the cord is retracted.

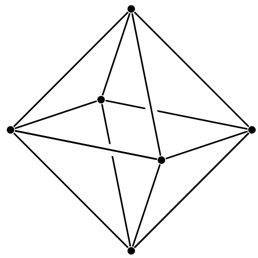


This will take care of the power problem. However, it introduces a new problem, illustrated in the above picture—namely, the cord can easily become wrapped around a piece of furniture in the room, and the Roomba could get stuck. In this scenario, obstacles in the room to be avoided, such as furniture, can be thought of as holes. When the Roomba makes a path, we want to make sure that the path does *not* loop around any holes. Topology allows us to detect holes (which corresponds to wrapping our cord around a piece of furniture) so that we can tell the Roomba to retrace its path.

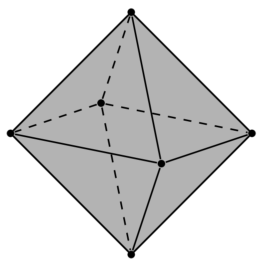


In the above picture, each piece of furniture in the room is replaced by a hole in an object, and the problem of the Roomba wrapping its cord around a piece of furniture becomes equivalent to detecting holes in this object via looping around the hole. The lesson of this example is that the size of the hole is irrelevant. It does not matter if your cord is wrapped around a huge sectional or a tiny pole—a stuck cord is a stuck cord. Topology, then, is not interested in the *size* of the hole, but the *existence* of the hole.

0.1.3. Modeling a rat brain. A Swiss team of researchers is now engaged in the Blue Brain Project, a study of brain function through computer simulations. The team has created a digital model of part of the somatosensory cortex of a rat on which neural activity can be simulated [**115**]. The model comprises 31,000 neurons and 37 million synapses, forming 8 million connections between the neurons. In addition to researchers in neuroscience, biology, and related disciplines, the team includes mathematicians, and most important for our purposes, the team employs experts in topology. We can imagine what an applied mathematician or expert in differential equations might do in a project like this, but what can a topologist contribute? The model of the rat brain can be thought of as a simplicial complex. Neurons are modeled with points, and the lines are the connections formed by synapses. Neuroscience research has made clear the importance of connectivity in the brain: if two neurons are not connected to each other, they cannot communicate directly. This connectivity tends to be directional, so that information may be able to flow from neuron A to neuron B but not necessarily vice versa. Information can also travel in loops, cycling around the same neurons multiple times. Topology can reveal not only these kinds of connections and loops, but also "higher-order notions" of connectivity, such as that illustrated below.



If we shade in each triangle, we obtain something that encloses a void.



This has interesting topological properties (see Problem 3.21), but does such a structure correspond to anything neurologically? Maybe [**135**]! If loops tell us something, perhaps this kind of structure does as well. The idea is for the mathematician to describe the higher-order connectivity in this rat brain model. If they find a 12- or 15- or 32-dimensional hole, what does this mean neurologically? If nothing else, topology can inform the neuroscientists of the existence of highly structured regions of the brain.

0.2. What is Morse theory?

Discrete Morse theory is a discretized analogue of classical Morse theory. A brief discussion of classical Morse theory, then, may help us to understand this newer field. Classical Morse theory is named after the American mathematician Marston Morse (1892–1977), who developed much of the mathematics now bearing his name in the 1920s and 1930s (e.g. [**124**, **125**]). One important result that Morse proved using Morse theory is that any two points on a sphere of a certain class are joined by an infinite number of locally shortest paths or geodesics [**126**]. However, it wasn't until the 1960s that Morse theory really shot into the limelight.

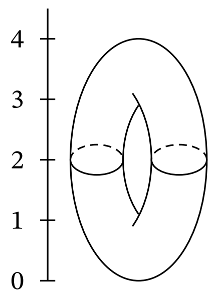
In 1961, Stephen Smale (1930–) provided a solution to a famous, unsolved problem in mathematics called the *generalized Poincaré conjecture in dimensions greater than* 4 [**143**]. This was an amazing result, causing much excitement in the mathematical community. Grigori Perelman (1966–) generated similar excitement in 2004 when he proved the *Poincaré conjecture in dimension* 3 [**121**]. A couple of years after Smale proved the generalized Poincaré conjecture, he extended his techniques to prove a result called the *h-cobordism theorem for smooth manifolds*, a result that gave profound insight into objects that look locally like Euclidean space. Smale was awarded a Fields Medal, the mathematical equivalent of the Nobel Prize, for his work. In 1965, John Milnor (1931–), another Fields Medal winner, gave a new proof of Smale's stunning result using, among other techniques, Morse theory [**117**]. Milnor's beautiful proof of Smale's result using Morse theory cemented Morse theory as, to quote Smale himself, "the single greatest contribution of American mathematics" [**144**].

Morse theory has proved invaluable not only through its utility in topology, but also through its applicability outside of topology. Another one of Smale's contributions was figuring out how to "fit Morse theory into the scheme of dynamical systems" [**37**, p. 103], a branch of mathematics studying how states evolve over time according to some fixed rules. Physicist and Fields Medal winner Ed Witten (1951–) developed a version of Morse theory to study a physics phenomenon called supersymmetry [**153**]. Raul Bott (1925–2005), the doctoral advisor of Smale, relates the following anecdote about Witten's first exposure to Morse theory:

> In August, 1979, I gave some lectures at Cargèse on equivariant Morse theory, and its pertinence to the Yang-Mills theory on Riemann surfaces. I was reporting on joint work with Atiyah to a group of very bright physicists, young and old, most of whom took a rather detached view of the lectures …Witten followed the lectures like a hawk, asked questions, and was clearly very interested. I therefore thought I had done a good

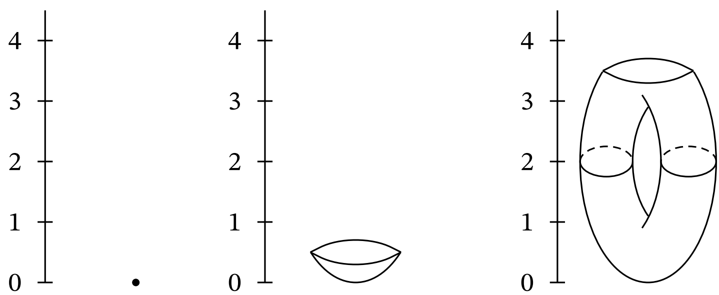
> job indoctrinating him in the rudiments of the half-space approach [to Morse theory], etc., so that I was rather nonplussed to receive a letter from him some eight months later, starting with the comment, "Now I finally understand Morse theory!" [**37**, p. 107]

So what is this Morse theory? Civil, natural, and divine law all require that the following picture of a torus (outside skin of a donut) be included in any book on Morse theory.



How can we study this object? One strategy often employed in studying anything, not just mathematics, is to break your object down into its fundamental or simple pieces. For example, in biology, you study the human organism by studying cells. In physics, you study matter by breaking it down into atoms. In number theory, you study integers by breaking them down into products of primes. What do we do in topology? We break down objects into simple pieces, such as those shown below. Think of them as the primes of topology.

After stretching, bending, pulling, and pushing these pieces, it turns out that we can glue them together to get the torus back. But how did we find these pieces? Is there a systematic way to construct them—a strategy that we can use on any such object? Let's define a "height function" f that takes any point on the torus and yields the vertical "height" of that point in space. Given any height z, we can then consider $M_z := f^{-1}(-\infty, z]$. The result is a slice through the torus at height z, leaving everything below level z. For example, M_4 is the whole torus, while $M_0, M_{0.5}$, and $M_{3.5}$ are shown below.



Notice that we can obtain M_4 from $M_{3.5}$ by gluing a "cap" onto $M_{3.5}$. The key to detecting when this gluing occurs is to find critical points of your height function. You may recall from calculus that a critical point is found when the derivative (all partial derivatives in this case) is 0. Geometrically, this corresponds to a local minimum, local maximum, or "saddle point." With this in mind, we can identify four critical points on the torus at heights 0, 1, 3, and 4. This simple observation is the starting point of Morse theory. It recovers the topology of our object and gives us information about our object, such as information about its holes.

With this in the background, Robin Forman, who received his PhD under Bott, introduced in 1998 a discretized version of Morse theory in a paper titled *Morse theory for cell complexes* [**65**]. Forman began his paper by writing:

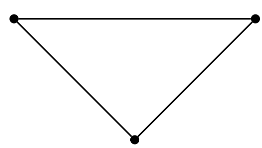
> In this paper we will present a very simple discrete Morse theory for CW complexes. In addition to proving analogues of the main theorems of Morse theory, we also present discrete analogues of such (seemingly) intrinsically smooth notions as the gradient vector field and the gradient flow associated to a Morse function. Using this, we define a Morse complex, a differential complex built out of the critical points of our

> discrete Morse function which has the same homology as the underlying manifold.

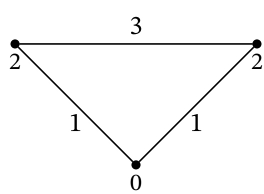
Although some of the terms may be unfamiliar, we can already see that Forman was intending to define and extend many of the concepts that we saw above.

0.3. Simplifying with discrete Morse theory

With our crash course in discrete topology and Morse theory behind us, we now illustrate how the two merge. The main idea in discrete Morse theory is to break down and study a simplicial complex in an analogous manner to that of Morse theory. To see how, consider the following simplicial version of a circle:

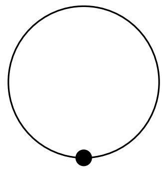


We can define an analogous notion of a "height function" on each part of the simplicial complex.



Notice that 0 is a local minimum while 3 is a local maximum.[2] But just as before, this tells us we can build the circle out of these two pieces.

[2]There are, of course, technical restrictions that any function on a simplicial complex must satisfy. See Section 2.1 for details.



Strictly speaking, this takes us out of the realm of simplicial complexes, but the idea holds nonetheless. Furthermore, we can use discrete Morse theory to reduce the complexity of an object without losing pertinent information. If we were working on the cell phone system, or the Roomba design project, or the mapping of a rat's brain, we would employ sophisticated computers, using expensive computational resources. Computing the number of holes tends to run on the order of a constant times n^3, where n is the number of lines that one inputs [**55**, Chapter IV]. Hence, reducing the number of points involved can greatly save on computational effort. For example, if we are interested in determining whether or not there are any dead zones in the cell phone example, discrete Morse theory can reduce the computational effort in Section 0.1.1 from 58^3 to 1^3. Of course, this reduction itself does have a cost, but it still makes the computation far more efficient. You will see how this reduction is accomplished in Chapter 8 and specifically in Problem 8.38. Such reductions can be invaluable for huge and unwieldy data sets. For more details, see [**82**]. The point for now is to realize that topology presents interesting questions that can help us model real-world phenomena. Discrete Morse theory can be an invaluable tool in assisting with such analysis.

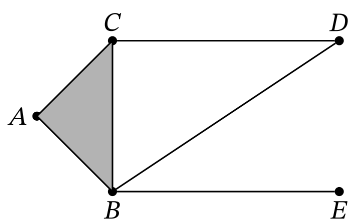
Chapter 1

Simplicial complexes

In the previous chapter, we saw three examples of applied topology problems. We modeled these problems with objects called simplicial complexes, made up of points, lines, faces, and higher-dimensional analogues. The defining characteristics of these simplicial complexes are given by the *relationships* of points, lines, etc. to one another.

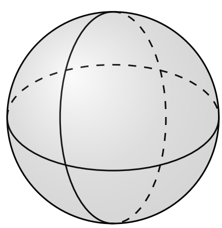
1.1. Basics of simplicial complexes

One way to communicate a simplicial complex is to draw a picture. Here is one example:

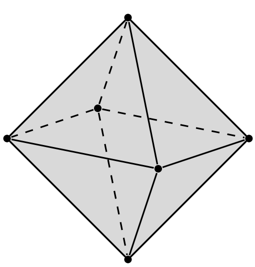


Lines between points indicate that the two points are related, and shading in the region enclosed by three mutually related points indicates that all three points are related. For example, C is related to D but C is not related to E. It is important to understand that relations do not have any geometrical interpretation. A concrete way to see this is to imagine the above simplicial complex as modeling the relation "had dinner with" among five people. So Catherine (C) has had dinner with Dominic (D), Dominic has had dinner with Beatrix (B), and Beatrix has had dinner with Catherine. Yet Catherine, Dominic, and Beatrix have not all had dinner together. Contrast this with Beatrix, Catherine, and Aniela (A). All three of them have had dinner together. This is communicated by the fact that the region enclosed by B, C, and A is shaded in. Notice that if all three of them have had dinner together, this implies that necessarily any two have had dinner together. These "face relations" are the key to understanding simplicial complexes.

We can further imagine how any "smooth" or physical object you can visualize can be approximated by these simplicial complexes. For example, a sphere like

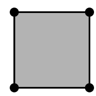


can be "approximated" or modeled by the following simplicial complex with hollow interior:

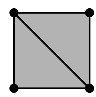


This looks to be a fairly crude approximation. If we want a better approximation of the sphere, we would use more points, more lines, and more faces.

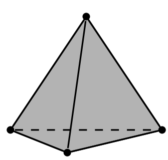
By convention, we may fill in space only when three lines are connected, not four or more. If we wish to fill in the space between four or more lines, we need to break it up into two or more triangles. So the following is not a simplicial complex:



But this is:



The advantage of viewing a simplicial complex by drawing a picture in this way is that it is easy to see the relationships between all the parts. The disadvantage is that it can be difficult to tell just from a picture what the author is trying to convey. For example, is the volume enclosed by the tetrahedron



filled in with a 3-dimensional "solid" or not? Moreover, we cannot draw a picture in more than three dimensions. But at this point, we should be able to reasonably understand the idea of building something out of 0-simplices (points), 1-simplices (lines), and n-simplices. The formal definition is given below.

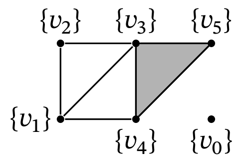
Definition 1.1. Let $n \geq 0$ be an integer and $[v_n] := \{v_0, v_1, \dots, v_n\}$ a collection of $n+1$ symbols. An **(abstract) simplicial complex** K **on** $[v_n]$ or a **complex** is a collection of subsets of $[v_n]$, excluding $\emptyset$, such that

(a) if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$;

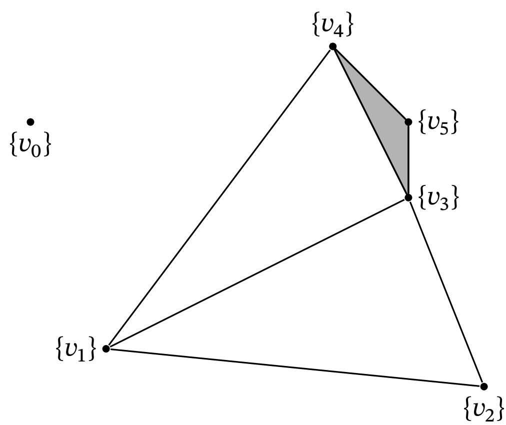
(b) $\{v_i\} \in K$ for every $v_i \in [v_n]$.

The set $[v_n]$ is called the **vertex set** of K and the elements $\{v_i\}$ are called **vertices** or 0-**simplices**. We sometimes write $V(K)$ for the vertex set of K.

Example 1.2. Let $n = 5$ and $V(K) := \{v_0, v_1, v_2, v_3, v_4, v_5\}$. Define $K := \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_3, v_4, v_5\}\}$. It is easy to check that K satisfies the definition of a simplicial complex. This simplicial complex may be viewed as follows:



The same simplicial complex may also be drawn as



In fact, there are infinitely many ways to draw this complex with different positions, distances, angles, etc. This illustrates a key conceptual point. *There is no concept of position, distance, angle, area, volume, etc. in a simplicial complex.* Rather, a simplicial complex only carries information about *relationships* between points. In our example, v_2 is related to v_3 and v_3 is related to v_4, but v_2 is not related to v_4. Any two of v_3, v_4, and v_5 are related to each other, and v_0 isn't related to anything. Again, think of the relation "had dinner with." Either two people have had dinner together or they have not. Either three people have had dinner all together or they have not. There is no geometry to the relation, only a binary yes or no.

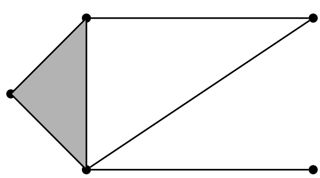
Hence, although the geometry inherent in any particular picture can be misleading, we will nevertheless often communicate a simplicial complex through a picture, keeping in mind that any geometry the picture conveys is an accident of the particular choice of drawing.

One advantage of knowing about simplicial complexes is that they can be used to model real-world phenomena. Hence, if one has a theory of abstract simplicial complexes, one can immediately use it in applications. See for instance [**58**].

To give you some practice working with simplicial complexes, the following exercise asks you to translate between the set definition of a complex and the picture, and vice versa.

Exercise 1.3. (i) Let $V(K) = [v_6]$ and $K = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_2, v_3\}, \{v_3, v_5\}, \{v_2, v_5\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_5\}\}$. Draw the simplicial complex K.

(ii) Let K be a simplicial complex on $[v_4]$ given by

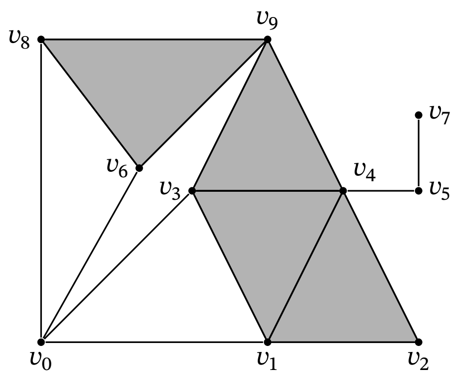


Write down the sets given by K.

Definition 1.4. A set σ of cardinality $i + 1$ is called an ***i*-dimensional simplex** or ***i*-simplex**. The **dimension** of K, denoted by $\dim(K)$, is the maximum of the dimensions of all its simplices. The ***c*-vector** of K is the vector $\vec{c}_K := (c_0, c_1, \dots, c_{\dim(K)})$ where c_i is the number of simplices of K of dimension i, $0 \le i \le \dim(K)$. A **subcomplex** L of K, denoted by $L \subseteq K$, is a subset L of K such that L is also a simplicial complex. If $\sigma \in K$ is a simplex, we write $\overline{\sigma}$ for the subcomplex generated by σ; that is, $\overline{\sigma} := \{\tau \in K : \tau \subseteq \sigma\}$.[1] If $\sigma, \tau \in K$ with $\tau \subseteq \sigma$, then τ is a **face** of σ and σ is a **coface** of τ. We sometimes use $\tau < \sigma$ to denote that τ is a face of σ. The ***i*-skeleton** of K is given by $K^i = \{\sigma \in K : \dim(\sigma) \le i\}$.

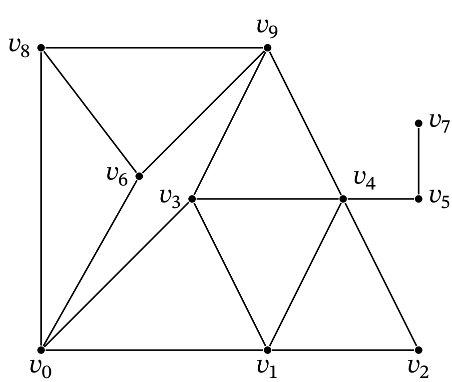
[1]Note the careful distinction between a simplex σ and a subcomplex $\overline{\sigma}$.

Example 1.5. Let K be the simplicial complex below.



To simplify notation, we describe a simplex by concatenating its 0-simplices. For example, the 2-simplex $\{v_6, v_8, v_9\}$ will be written as $v_6v_8v_9$, and the 1-simplex $\{v_3, v_4\}$ is written v_3v_4. Since the maximum dimension of the faces is 2 (e.g. the simplex $v_1v_4v_2$), $\dim(K) = 2$. By counting the number of 0-, 1-, and 2-simplices, we see that the c-vector for K is $\vec{c} = (10, 16, 4)$.

The 1-skeleton is given by



Definition 1.6. Let K be a simplicial complex. We use $\sigma^{(i)}$ to denote a simplex of dimension i, and we write $\tau < \sigma^{(i)}$ for any face of σ of dimension strictly less than i. The number $\dim(\sigma) - \dim(\tau)$ is called

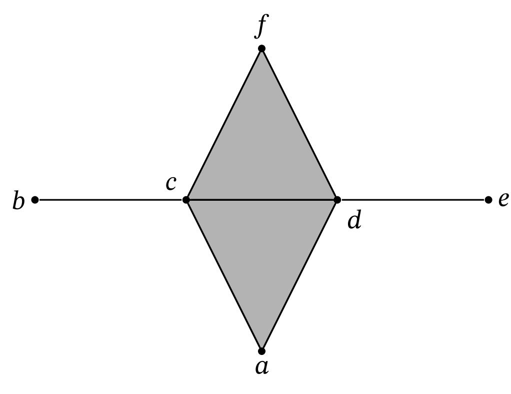
the **codimension of τ with respect to σ**. For any simplex $\sigma \in K$, we define the **boundary of σ in K** by $\partial_K(\sigma) := \partial(\sigma) := \{\tau \in K : \tau \text{ is a codimension-1 face of } \sigma\}$. A simplex of K that is not properly contained in any other simplex of K is called a **facet** of K.

Example 1.7. We continue with the simplicial complex in Example 1.5, illustrating the above definitions. The boundary of $v_3v_4v_9$ is given by $\partial_K(v_3v_4v_9) = \{v_3v_4, v_3v_9, v_4v_9\}$. The 1-simplex v_0v_6 is not contained in any other simplex of K, so v_0v_6 is a facet of K. By contrast, v_3v_1 is not a facet since it is contained in the larger simplex $v_1v_3v_4$. This may also be expressed by saying that v_3v_1 is a face of $v_1v_3v_4$ while $v_1v_3v_4$ is a coface of v_3v_1. This idea may be used to communicate a simplicial complex by considering the complex generated by a list of facets. Thus the above simplicial complex would be

$$K = \langle v_6v_8v_9, v_8v_0, v_6v_0, v_9v_4v_3, v_1v_3v_4, v_2v_1v_4, v_0v_3, v_0v_1, v_4v_5, v_5v_7\rangle$$

where $\langle\cdot\rangle$ means the simplicial complex generated by the enclosed collection of simplices (see Definition 1.10). Without extra structure, this is the most efficient way to communicate a simplicial complex. This is how simplicial complexes are stored in files, as any good program will take the facets and generate the simplicial complex.

Problem 1.8. Let K be the following simplicial complex:



(i) What is the dimension of K?

(ii) List all the facets of K.

(iii) Write down the c-vector of K.

(iv) Give an example of a subcomplex of K.

(v) Give an example of a simplex of K.

(vi) List the faces of cdf.

(vii) Compute $\partial(bc)$ and $\partial(acd)$.

(viii) Let $\sigma = acd$. Find all $\tau^{(0)}$ such that $\tau^{(0)} < \sigma$.

Exercise 1.9. Let K be a simplicial complex and $\sigma \in K$ an i-simplex of K. Prove that σ contains $i+1$ codimension-1 faces. In other words, prove that $|\partial_K(\sigma)| = \dim(\sigma) + 1$.

Definition 1.10. Let $H \subseteq \mathcal{P}([v_n]) - \{\emptyset\}$ where $\mathcal{P}([v_n])$ denotes the power set of $[v_n]$. The **simplicial complex generated by** H, denoted by $\langle H \rangle$, is the smallest simplicial complex containing H; that is, if J is any simplicial complex such that $H \subseteq J$, then $\langle H \rangle \subseteq J$.

Exercise 1.11. Prove that if H is a simplicial complex, then $\langle H \rangle = H$.

Exercise 1.12. Prove that $\langle \{\sigma\} \rangle = \bar{\sigma}$ for any simplex $\sigma \in K$.

1.1.1. A zoo of examples. Before moving further into the theory of simplicial complexes, we compile a list of some classical simplicial complexes. These complexes will be used throughout the text.

Example 1.13. The simplicial complex $\Delta^n := \mathcal{P}([v_n]) - \{\emptyset\}$ is the n-**simplex**. Be careful not to confuse the n-simplex Δ^n, which is a simplicial complex, with an n-simplex σ, which is an element of the complex K. While these two concepts share the same name, their meanings will be determined by context.

In addition, we define the **(simplicial)** n**-sphere** by

$$S^n := \Delta^{n+1} - \{[v_{n+1}]\}.$$

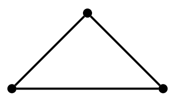
Problem 1.14. Draw a picture of $\Delta^0, \Delta^1, \Delta^2, \Delta^3$, and S^0, S^1, S^2.

Example 1.15. The simplicial complex consisting of a single vertex or point is denoted by $K = *$. We sometimes abuse notation and write $*$ for both the one-point simplicial complex and the one point *of* the simplicial complex.

Example 1.16. Any simplicial complex K such that $\dim(K) = 1$ is called a **graph**. Traditionally, such simplicial complexes are denoted by G, a convention we sometimes adopt in this text. Because they are easy to draw, many of our examples and special cases are for graphs.

Example 1.17. Let K be a simplicial complex and v a vertex not in K. Define a new simplicial complex CK, called the **cone** on K, where the simplices of CK are given by σ and $\{v\} \cup \sigma$ for all $\sigma \in K$.

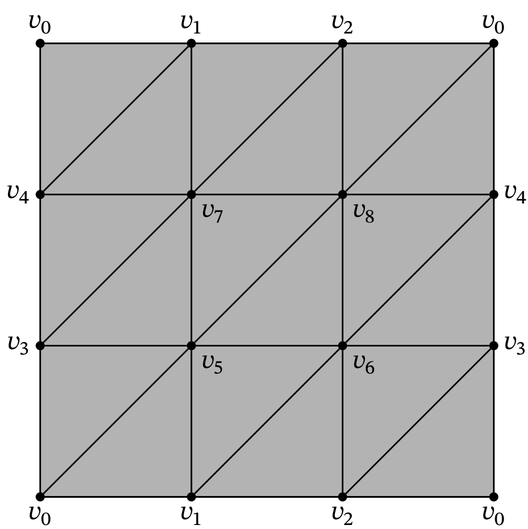
Exercise 1.18. Let K be the simplicial complex



Draw CK.

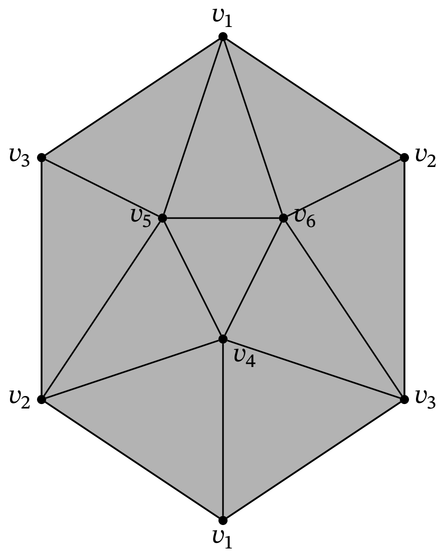
Problem 1.19. Prove that if K is a simplicial complex, then CK is also a simplicial complex.

Example 1.20. The **torus** T^2 with c-vector $(9, 27, 18)$:



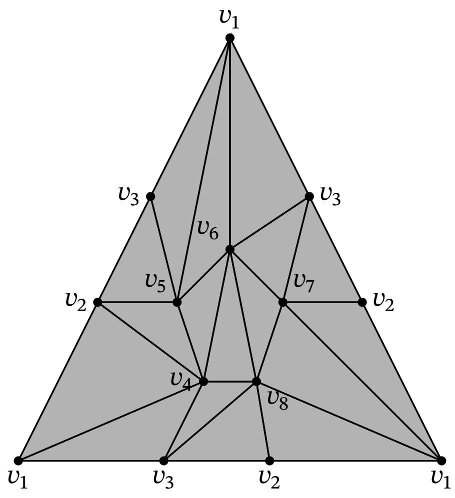
Notice that this is a 2-dimensional representation of a 3-dimensional object. That is, the horizontal sides are "glued" to each other and the vertical sides are "glued" to each other. One can glean this from the picture by noting that, for example, the horizontal edge $v_1 v_2$ appears on both the top and the bottom of the picture, but of course it is the same edge. Many of the more interesting examples of simplicial complexes are represented in this way.

Example 1.21. The **projective plane** P^2 with c-vector $(6, 15, 10)$:

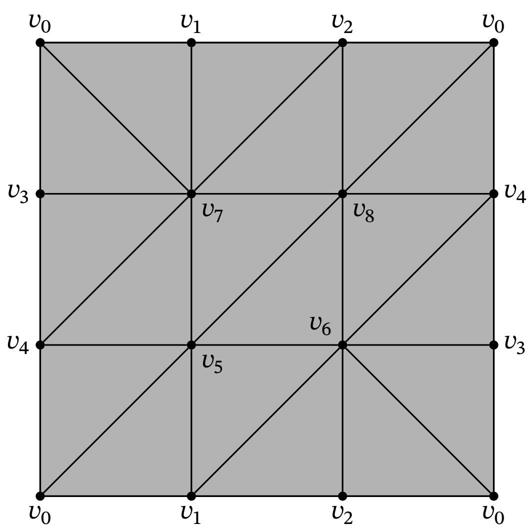


This one is more difficult to visualize. There is a good reason for this, as the projective plane needs at least four dimensions to be properly drawn.

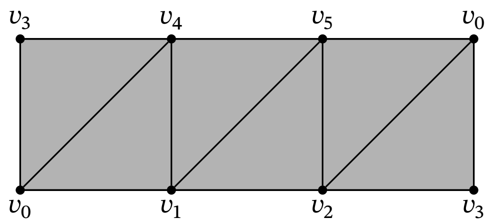
Example 1.22. The **dunce cap** D [**157**] with c-vector $(8, 24, 17)$:



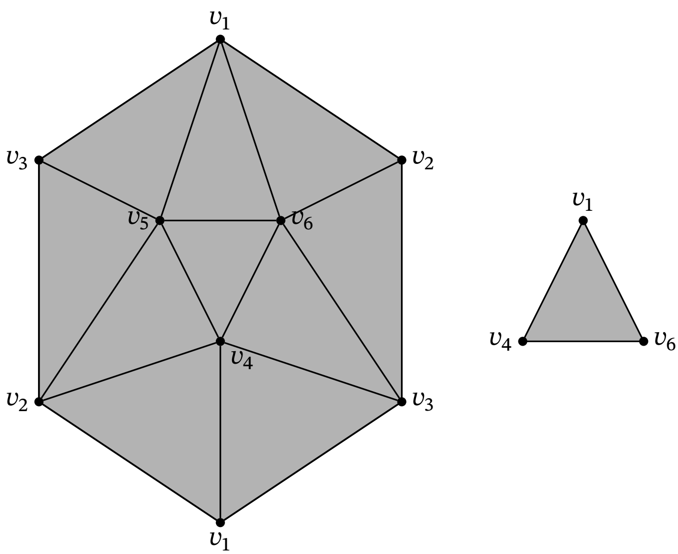
Example 1.23. The **Klein bottle** $\mathcal{K}$ with c-vector $(9, 27, 18)$:



Example 1.24. The **Möbius band** $\mathcal{M}$ with c-vector $(6, 12, 6)$:



Example 1.25. **Björner's example** with c-vector $(6, 15, 11)$, obtained by starting with the projective plane and gluing on a facet:



There are many other examples that we cannot reasonably communicate here. More information about these and other simplicial complexes, including downloadable versions, may be found online at the *Simplicial complex library* [**81**] or the *Library of triangulations* [**30**]. There is a whole branch of topology, called computational topology [**54**, **55**, **93**], devoted to the computational study of these massive objects. One subtheme of this book will be to "tell these complexes apart." What does that even mean? We will see in Section 1.2.

1.1.2. Some combinatorics. In Exercise 1.9, you were asked to show that a simplex of dimension i has exactly $i+1$ codimension-1 faces. While there are multiple ways to count such faces, one way is to let $\sigma = a_0 a_1 \cdots a_i$ be a simplex of dimension i. A face of σ with codimension 1 is by definition an $(i-1)$-dimensional subset, that is, a subset of σ with exactly i elements. To express the idea more generally, let n and k be positive integers with $n \geq k$. Given n objects, how many ways can I pick k of them? You may have seen this question in a statistics and probability or a discrete math course. This is a **combination**, sometimes denoted by ${}_nC_k$. In this book, we use the notation $\binom{n}{k}$ to denote the number of ways we can choose k objects from a set of n objects. Now $\binom{n}{k}$ is a nice looking symbol, but we don't actually know what this number is! Let us use the following example to derive a formula for $\binom{n}{k}$.

Example 1.26. Suppose we have a class of $n = 17$ students and we need to pick $k = 5$ of them to each present a homework problem to the class. How many different combinations of students can we pick? One way to do this is to count something else and then remove the extra factors. So let's first count the number of ways we could arrange all the students in a line. For the first person in line, we have 17 options. For the second person in line, we have 16 options. For the third person in line, we have 15 options. Continuing until the very last person, we obtain $17 \cdot 16 \cdot 15 \cdot \cdots \cdot 2 \cdot 1 = 17!$ possible lines. But we're not interested in a line of 17. We're interested in only the first 5 so we need to get rid of the last $17 - 5 = 12$ people. This yields $\frac{17!}{(17-5)!}$. This is the number of ways we can arrange 5 of the students, but it is counting all the different ways we can order those 5 students. Because order does not matter, we need to divide by the total number of ways to order 5 students, or 5!. This yields $\frac{17!}{(17-5)!5!} = \binom{17}{5}$.

The same argument as above shows that in general $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Definition 1.27. Let $0 \leq k \leq n$ be non-negative integers. The number of ways of choosing k objects from n objects, read n **choose** k, is given by $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. The value $\binom{n}{k}$ is also known as a **binomial coefficient**.

There are many nice combinatorial identities that the binomial coefficients satisfy. These will be helpful throughout the text.

Proposition 1.28 (Pascal's rule). Let $1 \leq k \leq n$ be positive integers. Then $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof. We compute

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} + \frac{(n-1)!}{k!\,(n-1-k)!} \\
&= \frac{(n-1)!\,k}{k!\,(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,(n-k)!} \\
&= \frac{(n-1)!\,(k+n-k)}{k!\,(n-k)!} \\
&= \frac{(n-1)!\,n}{k!\,(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

□

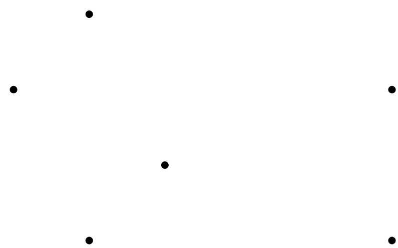
This fact can also be proved combinatorially. Suppose we are given n objects. Call one of them a. Choosing k of the n objects may be broken up into choosing k objects that include the element a and choosing k objects that do not include the element a. How many ways are there to do the former? Since we must include a, there are now $n-1$ objects to choose from, and since we have already included a in our list of chosen objects, there are $k-1$ objects left to choose; i.e., there are $\binom{n-1}{k-1}$ choices. To count the latter, we have $n-1$ total objects to choose from since we are excluding a. But this time we may choose any k of them; i.e., there are $\binom{n-1}{k}$ choices. Thus $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Problem 1.29. Prove that $\binom{n}{k} = \binom{n}{n-k}$.

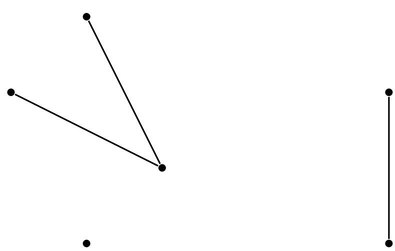
1.1.3. Euler characteristic.

In this section we associate a number called the Euler characteristic to a simplicial complex. The Euler characteristic generalizes the way that we count. The following motivation is taken from Rob Ghrist's excellent book on applied topology [**76**].

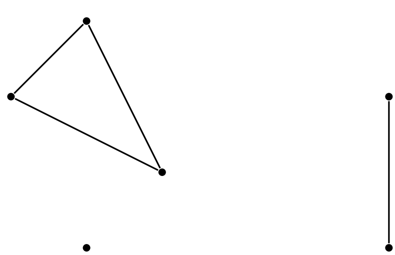
Consider a collection of points:



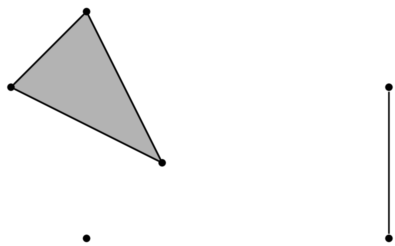
If we wish to count the number of points, we associate a weight of $+1$ to each point for a total of 6. Now if we begin to add edges between the points, we must subtract 1 for each line we add; that is, each line contributes a weight of -1.



Now there are $6 - 3 = 3$ objects. If we add an edge like



we don't lose an object. In fact, we create a hole. Nevertheless, if we continue with the philosophy that an edge has weight -1, we will still count this as $6 - 4 = 2$ for the 6 points minus the 4 edges. Now filling in this hole should "undo" the extra edge. In other words, a 2-simplex has weight $+1$.



We now have a count of $6-4+1=3$. In general, each odd-dimensional simplex has a weight of $+1$ while each even-dimensional simplex has a weight of -1. The Euler characteristic is this alternating sum of the number of simplices of a complex in each dimension.

Definition 1.30. Let K be an n-dimensional simplicial complex, and let $c_i(K)$ denote the number of i-simplices of K. The **Euler characteristic** of K, $\chi(K)$, is defined by $\chi(K) := \sum_{i=0}^{n}(-1)^i c_i(K)$.

Notice that this idea of "filling in" or "detecting" holes seems to be taken into account by this Euler characteristic. It is unclear exactly what a hole is at this point, but we should start to develop a sense of this idea.

Example 1.31. Given the c-vector of a complex, it is easy to compute the Euler characteristic. Several examples are listed in Section 1.1.1. For instance, the torus T^2 has c-vector $(9, 27, 18)$ so that $\chi(T^2) = 9-27+18 = 0$.

Example 1.32. If K is the simplicial complex defined in Problem 1.8, then $c_0 = 6, c_1 = 7$, and $c_2 = 2$. Hence $\chi(K) = c_0-c_1+c_2 = 6-7+1 = 1$.

Exercise 1.33. Compute the Euler characteristic for the simplicial complex defined in Example 1.5.

Problem 1.34. Let $n \geq 3$ be an integer, and let C_n be the simplicial complex defined on $[n-1]$ with facets given by $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_0\}$.

(i) Draw a picture of C_3, C_4, and C_5.

(ii) What is the dimension of C_n?

(iii) What is $\chi(C_n)$? Prove it by induction on n.

Problem 1.35. Recall that $\Delta^n := \mathcal{P}([n]) - \{\emptyset\}$ is the n-simplex from Example 1.13. Show that Δ^n is a simplicial complex and compute $\chi(\Delta^n)$.

Since S^n is simply Δ^{n+1} with the $(n+1)$-simplex removed, it follows that for all integers $k \geq 0$,

$$\chi(S^n) = \begin{cases} \chi(\Delta^{n+1}) + 1 & \text{if } n = 2k, \\ \chi(\Delta^{n+1}) - 1 & \text{if } n = 2k+1. \end{cases}$$

1.2. Simple homotopy

All branches of math have a notion of "sameness." For example, in group theory, two groups A and B are the "same" if there is a group isomorphism between them. In linear algebra, two vector spaces are the same if there is a vector space isomorphism between them. What would it mean for two simplicial complexes to be "the same"? In general, there is more than one answer to this question, depending on the interests of the mathematician. For our purposes, we are interested in **simple homotopy type**.

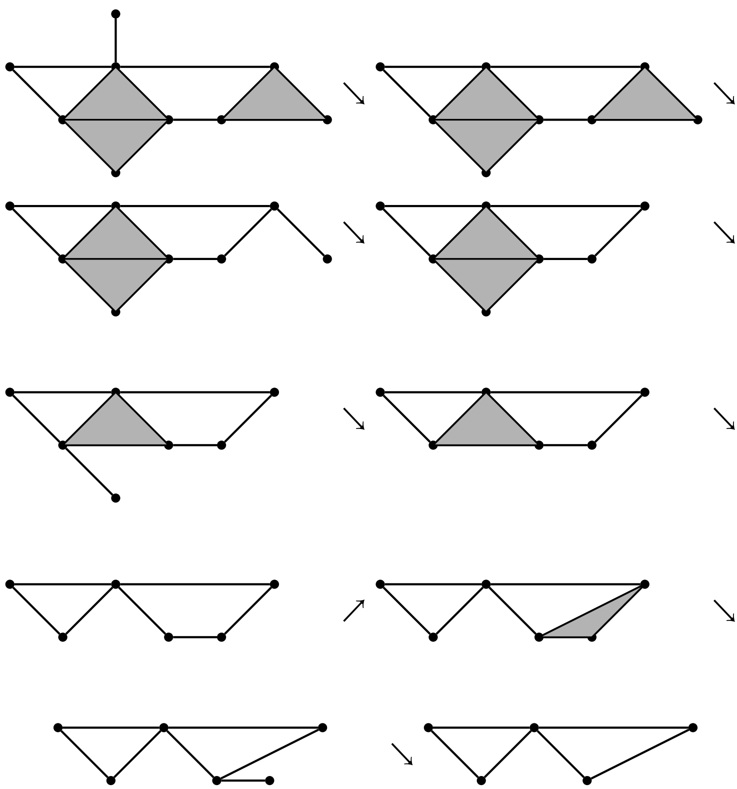
Definition 1.36. Let K be a simplicial complex and suppose that there is a pair of simplices $\{\sigma^{(p-1)}, \tau^{(p)}\}$ in K such that σ is a face of τ and σ has no other cofaces. Then $K - \{\sigma, \tau\}$ is a simplicial complex called an **elementary collapse** of K. The action of collapsing is denoted by $K \searrow K - \{\sigma, \tau\}$. On the other hand, suppose $\{\sigma^{(p-1)}, \tau^{(p)}\}$ is a pair of simplices not in K where σ is a face of τ and all other faces of τ are in K (and, consequently, all faces of σ are also in K). Then $K \cup \{\sigma^{(p-1)}, \tau^{(p)}\}$ is a simplicial complex called an **elementary expansion** of K, denoted by $K \nearrow K \cup \{\sigma^{(p-1)}, \tau^{(p)}\}$. For either an elementary collapse or an elementary expansion, such a pair $\{\sigma, \tau\}$ is called a **free pair**. We say that K and L are of the same **simple homotopy type**, denoted by $K \sim L$, if there is a series of elementary collapses and expansions from K to L. In the case where $L = \{v\} = *$ is a single point, we say that K has the **simple homotopy type of a point**.

Exercise 1.37. Let K be a simplicial complex and $\{\sigma, \tau\}$ a free pair of K. Show that $K - \{\sigma, \tau\}$ is a simplicial complex.

Simple homotopy has its origins in the work of J. H. C. Whitehead [**152**]. It has interesting connections [**21**,**22**] with Stong's theory of finite spaces [**145**], and there is even a book-length treatment of the subject [**45**].

Problem 1.38. Show that simple homotopy is an equivalence relation on the set of all simplicial complexes.

Example 1.39. We will consider a series of expansions and collapses of the simplicial complex below to find another simplicial complex which has the same simple homotopy type.



Thus the original simplicial complex has the same simple homotopy type as the above graph.

Once we have a notion of equivalence, an immediate question is "What stays the same under this notion of equivalence?"

Definition 1.40. Let K be a simplicial complex. A function α which associates a real number $\alpha(K)$ to every simplicial complex K is called a **simple homotopy invariant** or **invariant** if $\alpha(K) = \alpha(L)$ whenever $K \sim L$.

For example, suppose $K \sim L$. Do K and L have the same number of vertices? Do K and L have the same number of holes? It depends what we mean by a hole. In the above example, both complexes seem to have two holes. We will make the notion of holes precise in Chapter 3.

Problem 1.41.

(i) Is dim an invariant? That is, if $K \sim L$, is $\dim(K) = \dim(L)$? Prove it or give a counterexample.

(ii) Let $K \sim L$. Does it follow that $|V(K)| = |V(L)|$? Prove it or give a counterexample.

Another number we have associated to a simplicial complex is the Euler characteristic. The following proposition claims that the Euler characteristic is a simple homotopy invariant.

Proposition 1.42. Suppose $K \sim L$. Then $\chi(K) = \chi(L)$.

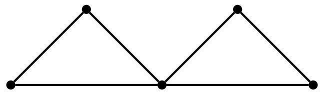
You will be asked to prove this in Problem 1.44. The following exercise may be useful in understanding how the proof of Proposition 1.42 works.

Exercise 1.43. Compute the Euler characteristics of the very first and very last simplicial complexes in Example 1.39. Compute the Euler characteristic of all the simplicial complexes in between.

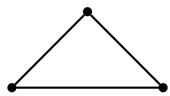
Problem 1.44. Prove Proposition 1.42.

Example 1.45. A nice use of Proposition 1.42 is that it gives us a way to show that two simplicial complexes do not have the same simple homotopy type. Consider again the simplicial complex in Example 1.39.

Through elementary collapses and expansions, we saw that it has the same simple homotopy type as the following K:



How do we know we can't perform more elementary expansions and collapses to simplify this to L given below?



Although we can't collapse any further, perhaps we could expand up to, say, a 20-dimensional simplex, and then collapse down to remove a hole.[2] We can lay all such speculations aside by simply computing the Euler characteristics. Observe that $\chi(K) = 5 - 6 = -1$ while $\chi(L) = 3 - 3 = 0$. Since the Euler characteristics are different, $K \nsim L$.

We may now distinguish some of the simplicial complexes we defined in Section 1.1.1.

Problem 1.46. Determine through proof which of the following simplicial complexes have the same simple homotopy type. Are there any complexes which you can neither tell apart nor prove have the same simple homotopy type?[3]

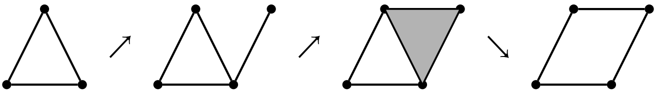
- S^1
- S^3
- S^4
- Δ^2
- Δ^3

Example 1.47. You showed in Problem 1.34 that a collection of simplicial complexes C_n all have the same Euler characteristic. This actually follows from Proposition 1.42 and the fact that $C_m \sim C_n$ for any $m, n \geq 3$.

[2] In fact, there are examples of simplicial complexes with no free pairs that can be expanded up and then collapsed back down to a point. Example 1.67 discusses one such example.

[3] Distinguishing them will have to wait until Chapters 4 and 8.

We prove this latter fact now by showing that $C_3 \sim C_4 \sim \cdots \sim C_n \sim C_{n+1} \sim \cdots$. We proceed by induction on n. We will show that $C_3 \sim C_4$ pictorially through elementary expansions and collapses:



Hence $C_3 \sim C_4$. Now assume that there is an $n \geq 4$ such that $C_i \sim C_{i-1}$ for all $3 \leq i \leq n$. We will show that $C_n \sim C_{n+1}$ using the same set of expansions and collapses going from C_3 to C_4. Formally, let $ab \in C_n$ be a 1-simplex (so that a and b are vertices) and expand $C_n \nearrow C_n \cup \{b', bb'\} =: C_n'$ where b' is a new vertex. Furthermore, expand $C_n' \nearrow C_n' \cup \{ab', abb'\} =: C_n''$. Since ab is not the face of any simplex in C_n, the pair $\{ab, abb'\}$ is a free pair in C_n'' and thus $C_n'' \searrow C_n'' - \{ab, abb'\} = C_{n+1}$. Hence $C_n \sim C_{n+1}$ and the result follows.

A special case of the Euler characteristic computation occurs when K has the simple homotopy type of a point.

Proposition 1.48. Suppose that $K \sim *$. Then $\chi(K) = 1$.

Problem 1.49. Prove Proposition 1.48.

Problem 1.50. Prove that $S^1 \sim M$ where M is the Möbius band of Example 1.24.

1.2.1. Collapsibility.

A special kind of simple homotopy type is obtained when one exclusively uses collapses to simplify K to a point.

Definition 1.51. A simplicial complex K is **collapsible** if there is a sequence of elementary collapses

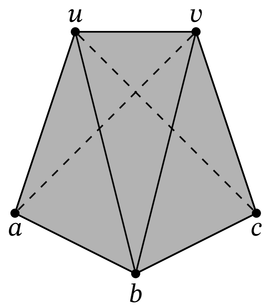
$$K = K_0 \searrow K_1 \searrow \ldots \searrow K_{n-1} \searrow K_n = \{v\}.$$

Notice the difference between a simplicial complex being collapsible and its having the simple homotopy type of a point. With the latter we allow expansions and collapses, while with the former we only allow collapses.

Definition 1.52. Let K and L be two simplicial complexes with no vertices in common. Define the **join** of K and L, written $K * L$, by

$$K * L := \{\sigma, \tau, \sigma \cup \tau : \sigma \in K, \tau \in L\}.$$

Example 1.53. Let $K := \{a, b, c, ab, bc\}$ and $L := \{u, v, uv\}$. Then the join $K * L$ is given by



Notice that there is a copy of K at the bottom of the picture and a copy of L at the top. As a set, $K * L = \langle abuv, bcuv\rangle$.

Exercise 1.54. (i) Show that $K * L$ is a simplicial complex.

(ii) Prove that $K * \{\emptyset\} = K$.

Problem 1.55. Show that if $K \searrow K'$, then $K * L \searrow K' * L$.

We have already seen one example of a join. The cone on K, which we defined in Example 1.17, is just a special case of a join with a single vertex; that is, the cone over K can be defined by $CK := K * \{v\}$. There is another special case of the join, called the suspension.

Definition 1.56. Let K be a simplicial complex with $v, w \notin K$ and let $w \neq v$. Define the **suspension** of K by $\Sigma K := K * \{v, w\}$.

Exercise 1.57. Let K be the simplicial complex from Exercise 1.18. Draw ΣK.

Exercise 1.57 helps one to see that the suspension of K is two cones on K. This, however, is not to be confused with the cone of the cone, or double cone, as the following exercise illustrates.

Exercise 1.58. Let $K = \{u, w, uw\}$. Draw ΣK and $C(CK)$, and conclude that in general $\Sigma K \neq C(CK)$.

Proposition 1.59. The cone CK over any simplicial complex is collapsible.

Proof. We prove the result by induction on n, the number of simplices in the complex K. For $n = 1$, there is only one simplicial complex with a single simplex, namely, a 0-simplex. The cone on this is clearly collapsible. Now suppose by the inductive hypothesis that $n \geq 1$ is given and that the cone on any simplicial complex with n simplices is collapsible. Let K be a simplicial complex on $n + 1$ simplices, and consider $CK = K * \{v\}$. For any facet σ of K, observe that $\{\sigma \cup \{v\}, \sigma\}$ is a free pair in CK since $\sigma \cup \{v\}$ is a facet of CK and σ is not a face of any other simplex. Hence $CK \searrow CK - \{\sigma \cup \{v\}, \sigma\}$. But $CK - \{\sigma \cup \{v\}, \sigma\} = C(K - \{\sigma\})$ (Problem 1.60) is a cone on $n-1$ simplices and, by the inductive hypothesis, collapsible. Thus $CK \searrow CK - \{\sigma \cup \{v\}, \sigma\} \searrow *$, and all cones are collapsible. □

Problem 1.60. Let K be a simplicial complex and σ a facet of K. If $CK = K * \{v\}$, prove that $CK - \{\sigma \cup \{v\}, \sigma\} = C(K - \{\sigma\})$.

Problem 1.61. Show that K is collapsible if and only if $CK \searrow K$.

Exercise 1.62. Show that the suspension is not in general collapsible but that there exist collapsible subcomplexes K_1 and K_2 (not necessarily disjoint) of ΣK such that $K_1 \cup K_2 = \Sigma K$.

Problem 1.63. Prove that $\chi(\Sigma K) = 2 - \chi(K)$.

Problem 1.64. Prove that Δ^n is collapsible for every $n \geq 1$.

Because simple homotopy involves both elementary collapses and expansions, we have the following.

Proposition 1.65. If $K \searrow H$ and $H \searrow L$, then $K \sim H \sim L$.

Remark 1.66. The above proposition along with Proposition 1.59 and Problem 1.64 combine to show that $\Delta^n \sim CK \sim *$ so that all of these have the same simple homotopy type.

What about the converse of Proposition 1.65 in the special case where $H \sim *$? That is, if $K \sim *$, is K collapsible? If not, we would need to find a simplicial complex which can't be collapsed down to a point right away—it would need to be expanded up and then collapsed down (possibly multiple times) before it could be collapsed to a point. Although the proof of the existence of such a simplicial complex is beyond the scope of this book the dunce cap is one example of a simplicial complex that has the simple homotopy type of a point but is not collapsible.

Example 1.67. Refer back to Example 1.22 for the definition of the dunce cap D. It is easy to see that $\chi(D) = 1$ but D is not collapsible. Bruno Benedetti and Frank Lutz showed that the order in which collapses are performed makes a difference. They constructed a simplicial complex with only eight vertices that can be collapsed to a point. Nevertheless, they also proved that there is a way to collapse the same complex to a dunce cap. The proof of this result is a bit technical, as great care must be taken in removing the proper sequence of free pairs in the right order. See [**31**, Theorem 1]. Hence, one can find a sequence of collapses that results in a point and another sequence of collapses that results in a dunce cap. Benedetti and Lutz have also constructed examples of complexes that "look like" a ball (simplicial 3-balls) but are not collapsible [**32**].

Exercise 1.68. Show that the dunce cap is not collapsible.

Even though the dunce cap is not collapsible, the result of Benedetti and Lutz implies that $D \sim *$. This, combined with some of our work above (including the study of the Euler characteristic), has allowed us to distinguish several of the complexes proposed in Section 1.1.1 as well as to determine which are the same.

Problem 1.69. Use the c-vectors in Section 1.1.1 as well as your work in the Exercises and Problems in this chapter to fill in the table below.

	Δ^n	S^n	$*$	CK	T^2	P^2	D	$\mathcal{K}$	M	B
$\chi =$										

Which complexes have the same simple homotopy type? Which complexes can you still not tell apart?

Further references on simplicial complexes and their generalizations are plentiful. The books [**64**, **85**, **134**] are standard references for simplicial complexes. Another excellent reference is Jakob Jonsson's unpublished notes on simplicial complexes [**89**], which include an introduction to discrete Morse theory. The material in this section is closely related to combinatorial topology [**77**, **78**] as well as piecewise-linear (PL) topology [**138**]. Other interesting uses of simplicial complexes arise in combinatorial algebraic topology [**103**] and topological combinatorics [**50**].

Chapter 2

Discrete Morse theory

In this chapter, we introduce discrete Morse theory in three sections. These sections are concerned with different ways of communicating a discrete Morse function. As we introduce these definitions, we will also build the theory by relating the definitions to one another. We will see that there are varied and diverse ways of thinking about a discrete Morse function, giving us many options in attacking a problem. In this sense, discrete Morse theory is like a Picasso painting; “looking at it from different perspectives produces completely different impressions of its beauty and applicability”[**47**].

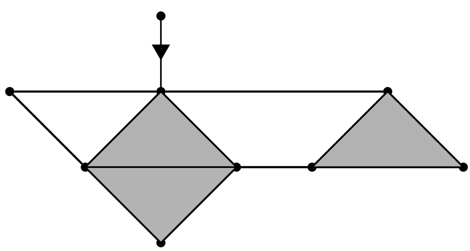
But first, we revisit the question of what discrete Morse theory can do. In this book, discrete Morse theory is a tool used to help us study simplicial complexes. It will help us

(a) detect and bound the number of holes in a simplicial complex;

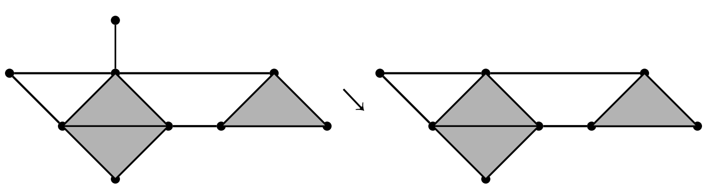
(b) replace a simplicial complex with an equivalent one that is smaller.

Now it is still not clear what we mean by a hole. But at this point, we should have some intuition and ideas about what a hole should be. For the second point, we do know what it means to replace a simplicial complex by an equivalent one—namely, a simplicial complex with the same simple homotopy type. Actually, we will be able to say something a bit stronger. We will be able to use only collapses (no expansions) to replace a complex with a different complex that has fewer simplices.[1] This is especially helpful when one is interested in only topological information of the complex. Think, for example, of the reduction mentioned in Section 0.3. If you revisit this example, you can see that the original complex we built collapses to a much smaller complex. Such a sequence of collapses can be found using discrete Morse theory.

Consider the sequence of collapses in Example 1.39. While this example was a thorough demonstration of our collapsing strategy, it was quite cumbersome to write. Indeed, it took up half a page! Is there a more succinct way to communicate this sequence of collapses? Suppose that instead of drawing each simplicial complex at each stage, we draw a single simplicial complex with some decorations on it. The decorations will indicate which simplices to collapse. These decorations will take the form of an arrow indicating the direction of the collapse. So the picture



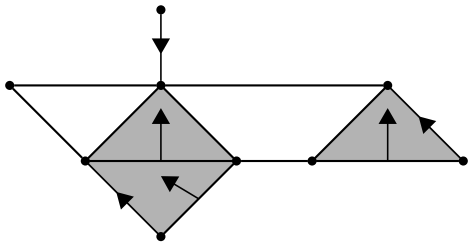
would be shorthand for



If we want to communicate more collapses, we add more arrows. In general, an arrow has its tail (or "start") in a simplex and its head in a codimension-1 face. This should be thought of as a free pair, or at

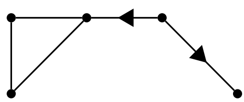
[1]In addition, we will show that one can make further reductions using the methods of Chapter 8. This is probably the most practical reduction.

least a pair that will eventually be free after other pairs are removed. The entire sequence of collapses from Example 1.39 (excluding the expansion followed by two collapses at the end) could be communicated by the decorated simplicial complex



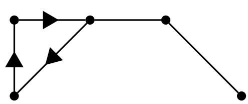
Note that no order to the collapses is specified. In Section 2.1, we'll see how to specify an exact order. But for now we'll leave the order of the collapses to the side. Another issue to consider is whether all assignments of arrows make sense or lead to a well-defined rule. You will investigate this question in the following exercises.

Exercise 2.1. Consider the simplicial complex



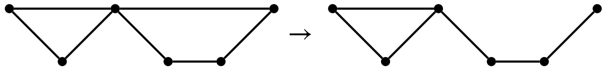
Does this determine a sequence of collapses? Why or why not?

Exercise 2.2. Also consider the following variation:

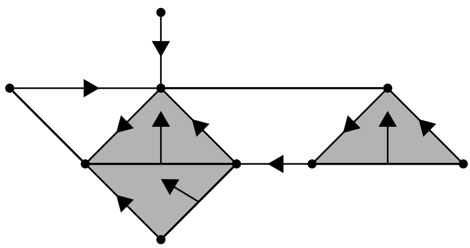


Does this determine a sequence of collapses? Why or why not?

Furthermore, recall that in Example 1.39 we eventually had to stop. We had something "blocking" us. What if we just rip out whatever is blocking us?



From there, we may continue collapsing via arrows. If we get stuck again, we rip out the obstruction. The resulting picture would still have all the same arrows on it, but the pieces that were ripped out would be unlabeled.



These are the basic ideas of discrete Morse theory. With all of these ideas in mind, we now investigate the formal definition of a discrete Morse function.

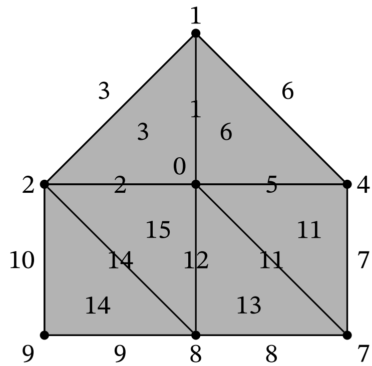
2.1. Discrete Morse functions

2.1.1. Basic discrete Morse functions. We begin by defining a simpler kind of discrete Morse function which we call a basic discrete Morse function. It is due to Bruno Benedetti.

Definition 2.3. Let K be a simplicial complex. A function $f: K \to \mathbb{R}$ is called **weakly increasing** if $f(\sigma) \leq f(\tau)$ whenever $\sigma \subseteq \tau$. A **basic discrete Morse function** $f: K \to \mathbb{R}$ is a weakly increasing function which is at most 2–1 and satisfies the property that if $f(\sigma) = f(\tau)$, then either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

A function $f: A \to B$ is 2–1 if for every $b \in B$, there are at most two values $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$. In other words, at most two elements in A are sent to any single element in B.

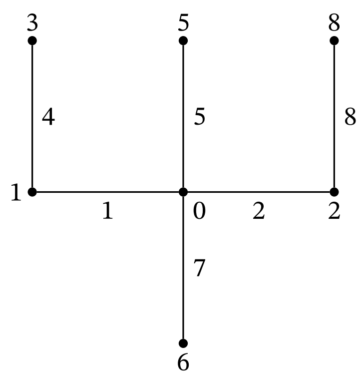
Example 2.4. The following is an example of a basic discrete Morse function.



In order to verify that this is indeed a basic discrete Morse function, three things need to be checked. First, we ask whether f is weakly increasing. Second, we check if f is 2–1. Finally, we need to check that if $f(\sigma) = f(\tau)$, then either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$. If the answer to each of these questions is "yes," then we have a basic discrete Morse function.

Exercise 2.5. Verify that the function in Example 2.4 is a basic discrete Morse function.

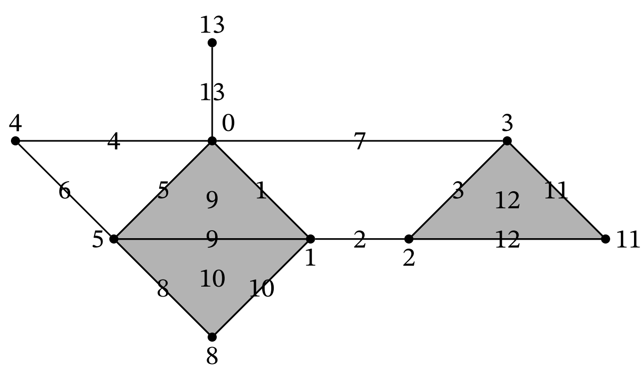
Example 2.6. Another example of a basic discrete Morse function, this time on a graph, is given by



Definition 2.7. Let $f : K \to \mathbb{R}$ be a basic discrete Morse function. A simplex σ of K is **critical** if f is injective on σ. Otherwise, σ is called **regular**. If σ is a critical simplex, the value $f(\sigma)$ is called a **critical value**. If σ is a regular simplex, the value $f(\sigma)$ is called a **regular value**.

Remark 2.8. Note the distinction between a critical simplex and a critical value. A critical simplex is a simplex, such as σ or v, which is critical. A critical value is a real number given by the labeling or output of the discrete Morse function on a simplex.

Example 2.9. Consider another example:

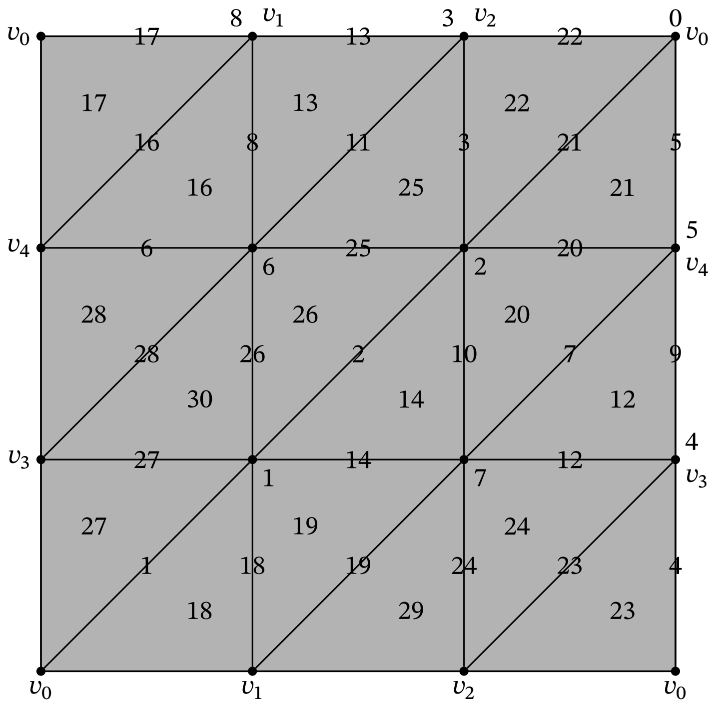


Again, one can check that this satisfies the definition of a basic discrete Morse function. The critical simplices are the ones labeled 0, 6, and 7. All other simplices are regular.

Example 2.10. The critical values in Example 2.6 are $0, 3, 4, 6$, and 7. The regular values are $1, 5, 2$, and 8.

Exercise 2.11. Identify the critical and regular simplices in Example 2.4.

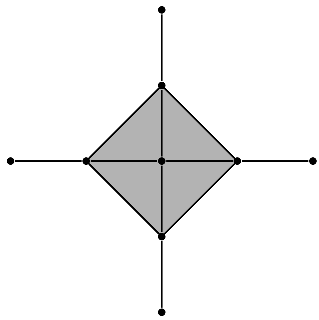
Example 2.12. For a more involved example, let us put a basic discrete Morse function on the torus from Example 1.20. Recall that the top and bottom are "glued" together, and so are the sides. In particular, the "four" corner vertices are really the same vertex v_0.



The reader should verify that this is a basic discrete Morse function and find all the critical and regular simplices.

Exercise 2.13.

Consider the following simplicial complex K:



(i) Find a basic discrete Morse function on K with one critical simplex.

(ii) Find a basic discrete Morse function on K with three critical simplices.

(iii) Find a basic discrete Morse function on K where every simplex is critical.

(iv) Can you find a basic discrete Morse function on K with two critical simplices? Zero critical simplices?

Exercise 2.14. Find a basic discrete Morse function with six critical simplices on the Möbius band (Example 1.24).

2.1.2. Forman definition. One advantage of working with a basic discrete Morse function is that the critical simplices are easy to identify. They are the simplices with a unique value. At the same time, this can be a drawback since we may want multiple critical simplices to have the same value. The original definition of a discrete Morse function, articulated by the founder of discrete Morse theory, Robin Forman [**65**], allows for this. It is the definition that tends to be used in most of the literature.

Definition 2.15. A **discrete Morse function** f on K is a function $f : K \to \mathbb{R}$ such that for every p-simplex $\sigma \in K$, we have

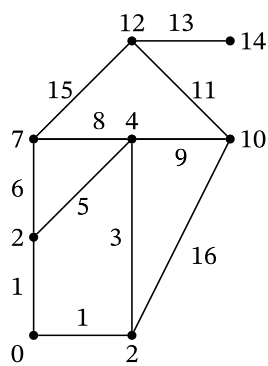
$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| \leq 1.$$

The above definition can be a bit difficult to parse. We'll illustrate with several examples, but the basic idea is that as a general rule, higher-dimensional simplices have higher values and lower-dimensional simplices have lower values; that is, the function generally increases as you increase the dimension of the simplices. But we allow at most one exception per simplex. So, for example, if σ is a simplex of dimension p, all of its $(p-1)$ faces must be given a value strictly less than the value of σ, with at most one exception. Similarly, all of the $(p+1)$-dimensional cofaces of σ must be given a value strictly greater than σ, with at most one exception. We'll see in Lemma 2.24, called the exclusion lemma, that the exceptions cannot both occur for the same simplex. But for now, the easiest way to understand the definition is to work through some examples.

Example 2.16. Consider the graph below with labeling

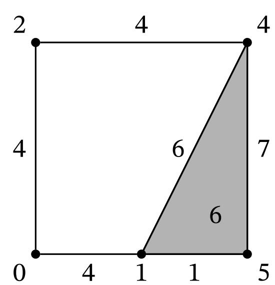


Let's investigate whether this is a discrete Morse function. We have to check that each simplex satisfies the rules of Definition 2.15. Let's start with the vertex labeled 0 in the bottom left corner. The rule is that every coface of that vertex must be labeled something greater than 0, with possibly a single exception. Both cofaces in this case happen to be labeled 1, which is greater than 0. The vertex is not the face of anything, so there is nothing more to check for this simplex. What about the edge labeled 13 at the top? It is not the coface of any simplex, but it does have two faces. Both of these faces must be given a value less than 13, with possibly one exception. One of the faces is labeled 12, which is less than 13, but the other is labeled 14, which is greater than 13. That's okay—that is our one allowed exception for that edge. Now do this for all the other simplices to see that f is indeed a discrete Morse function.

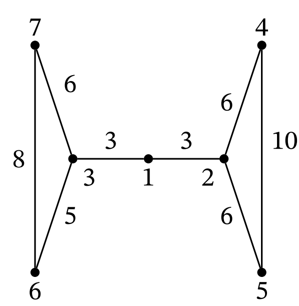
Problem 2.17. Show that every basic discrete Morse function is a discrete Morse function, but not vice versa.

Problem 2.18. Determine if each of the following labelings defines a discrete Morse function. If not, change some numbers to make it into one.

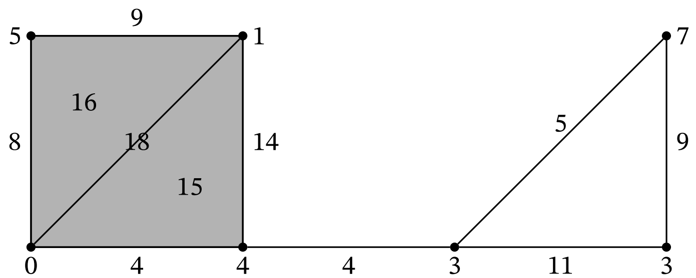
(i)



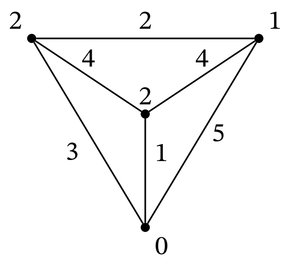
(ii)



(iii)



(iv)



Definition 2.19. A p-simplex $\sigma \in K$ is said to be **critical** with respect to a discrete Morse function f if

$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| = 0$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| = 0.$$

If σ is a critical simplex, the number $f(\sigma) \in \mathbb{R}$ is called a **critical value**. Any simplex that is not critical is called a **regular simplex**, while any output value of the discrete Morse function which is not a critical value is a **regular value**.

In other words, the critical simplices are those simplices which do not admit any "exceptions."

Example 2.20. Upon investigation, we see that the critical values in Example 2.16 are 1, 5, 8, 15, and 16.

Exercise 2.21. Let K be a simplicial complex. Show that there is a discrete Morse function on K such that every simplex of K is critical.

Exercise 2.22. Let K be a simplicial complex, $V = \{v_0, v_1, \dots, v_n\}$ the vertex set, and $f_0 : V \to \mathbb{R}^{>0}$ any function. For any $\sigma \in K$, write $\sigma := \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$.

(i) Prove that $f : K \to \mathbb{R}^{>0}$ defined by $f(\sigma) := f_0(v_{i_1}) + f_0(v_{i_2}) + \cdots + f_0(v_{i_k})$ is a discrete Morse function. What are the critical simplices? Is this discrete Morse function basic?

(ii) Let $f : K \to \mathbb{R}^{>0}$ be defined by $f(\sigma) := f_0(v_{i_1}) \cdot f_0(v_{i_2}) \cdot \cdots \cdot f_0(v_{i_k})$. Is f a discrete Morse function? If so, prove it. If not, give a counterexample.

Your work in Exercise 2.22 yields a discrete Morse function from a set of values on the vertices but it is not a very good one. To be frank, it's abysmal. A much better way to put a discrete Morse function on a complex from a set of values on the vertices is given by H. King, K. Knudson, and N. Mramor in [**97**]. We will give an algorithm for their construction in Section 9.1.

Problem 2.23. Prove that a p-simplex σ is regular if and only if either of the following conditions holds:

(i) There exists $\tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$.

(ii) There exists $\nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$.

The following, sometimes referred to as the exclusion lemma, is a very simple observation but one of the key insights into the utility of discrete Morse theory. We will refer back to it often.

Lemma 2.24 (Exclusion lemma)**.** Let $f : K \to \mathbb{R}$ be a discrete Morse function and $\sigma \in K$ a regular simplex. Then conditions (i) and (ii) in Problem 2.23 cannot both be true. Hence, exactly one of the conditions holds whenever σ is a regular simplex.

Proof. Writing $\sigma = a_0 a_1 \cdots a_{p-1} a_p$ and renaming the elements if necessary, suppose by contradiction that $\tau = a_0 \cdots a_p a_{p+1} > \sigma$ and

$\nu = a_0 \cdots a_{p_{i-1}} < \sigma$ satisfy $f(\tau) \leq f(\sigma) \leq f(\nu)$. Observe that $\tilde{\sigma} := a_0 a_1 \cdots a_{p-1} a_{p+1}$ satisfies $\nu < \tilde{\sigma} < \tau$. As $\nu < \sigma$ and $f(\nu) \geq f(\sigma)$, it follows that $f(\nu) < f(\tilde{\sigma})$ since $\nu < \tilde{\sigma}$. Similarly, $f(\tilde{\sigma}) < f(\tau)$. Hence

$$f(\tau) \leq f(\sigma) \leq f(\nu) < f(\tilde{\sigma}) < f(\tau),$$

which is a contradiction. Thus, when σ is a regular simplex, exactly one of the conditions in Problem 2.23 holds. □

Problem 2.25. Let $f : K \to \mathbb{R}$ be a discrete Morse function. Is it possible to have a pair of simplices $\tau^{(i)} < \sigma^{(p)}$ in K with $i < p - 1$ such that $f(\tau) > f(\sigma)$? If yes, give an example. If not, prove it. Does the result change if f is a basic discrete Morse function?

Problem 2.26. Prove that if $f : K \to \mathbb{R}$ is a discrete Morse function, then f has at least one critical simplex (namely, a critical 0-simplex).

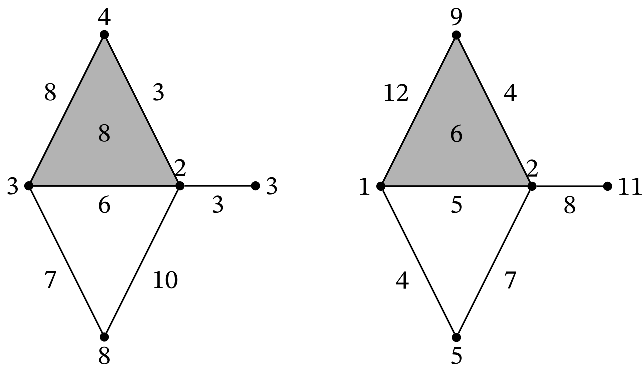
2.1.3. Forman equivalence. Upon investigation of discrete Morse functions, an immediate question arises: is there a notion of "same" discrete Morse functions? For example, if we add .01 to all of the values in any discrete Morse function, we would technically have a different function, but for all practical purposes such a function is no different. We thus need a notion of equivalence or sameness of discrete Morse functions. Since the defining characteristic of discrete Morse functions seems to be the relationship of the values to each other, we make the following definition:

Definition 2.27. Two discrete Morse functions f and g on K are said to be **Forman equivalent** if for every pair of simplices $\sigma^{(p)} < \tau^{(p+1)}$ in K, we have $f(\sigma) < f(\tau)$ if and only if $g(\sigma) < g(\tau)$.

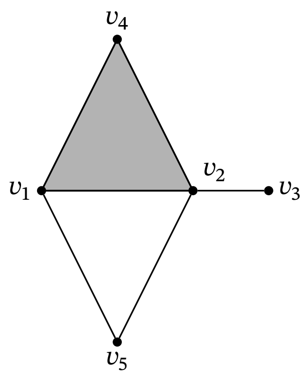
Exercise 2.28. Show that Forman equivalence is an equivalence relation on the set of all discrete Morse functions on a fixed simplicial complex K.

Exercise 2.29. Prove that f and g are Forman equivalent if and only if for every pair of simplices $\sigma^{(p)} < \tau^{(p+1)}$ in K, we have $f(\sigma) \geq f(\tau)$ if and only if $g(\sigma) \geq g(\tau)$.

Example 2.30. Consider the two discrete Morse functions $f, g \colon K \to \mathbb{R}$ with f on the left and g on the right:



For convenience, we give names to the vertices and recall the convention that a simplex can be communicated as a concatenation of its vertices.



To check if the two are Forman equivalent, we must check that f "does the same thing" as g does for any simplex/codimension-1 face pair, and vice versa. So, for example, $f(v_4) = 4 < 8 = f(v_4v_1)$. Does the same inequality hold for g? Yes, as $g(v_4) = 9 < 12 = g(v_4v_1)$. Now $f(v_5) = 8 \geq 7 = f(v_5v_1)$, and g likewise has the same relationship on this pair as $g(v_5) = 5 \geq 4 = g(v_5v_1)$. Checking this for every pair of simplices $\sigma^{(p)} < \tau^{(p+1)}$, we see that $f(\sigma) < f(\tau)$ if and only if $g(\sigma) < g(\tau)$. Hence f and g are Forman equivalent.

There is a nice characterization of Forman-equivalent discrete Morse functions which we investigate in Section 2.2.1. A special kind of discrete Morse function that is a bit more "well-behaved" is one in which all the critical values are different.

Definition 2.31. A discrete Morse function $f : K \to \mathbb{R}$ is said to be **excellent** if f is 1–1 on the set of critical simplices.

In other words, a discrete Morse function is excellent if all the critical values are distinct. We explore the controlled behavior of excellent discrete Morse functions in Section 5.1.1.

Problem 2.32. Show that every basic discrete Morse function is excellent.

The following lemma claims that up to Forman equivalence, we may assume that any discrete Morse function is excellent.

Lemma 2.33. Let $f : K \to \mathbb{R}$ be a discrete Morse function. Then there is an excellent discrete Morse function $g : K \to \mathbb{R}$ which is Forman equivalent to f.

Proof. Let $\sigma_1, \sigma_2 \in K$ be critical simplices such that $f(\sigma_1) = f(\sigma_2)$. If no such simplices exist, then we are done. Otherwise, we define $f' : K \to \mathbb{R}$ by $f'(\tau) = f(\tau)$ for all $\tau \neq \sigma_1$ and $f'(\sigma_1) = f(\sigma_1) + \epsilon$ where $f(\sigma_1) + \epsilon$ is strictly less than the smallest value of f greater than $f(\sigma_1)$. Then σ_1 is a critical simplex of f', and f' is equivalent to f. Repeat the construction for any two simplices of f' that share the same critical value. Since f has a finite number of critical values, the process terminates with an excellent discrete Morse function g that is equivalent to f. $\square$

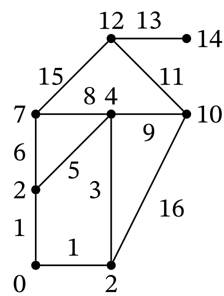
Problem 2.34. Let K be collapsible. Prove that there exists a discrete Morse function f on K with exactly one critical simplex.

Problem 2.35. Show that there exists a discrete Morse function on S^n with exactly two critical simplices—namely, a critical 0-simplex and a critical n-simplex.

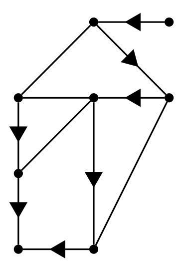
2.2. Gradient vector fields

With the Forman definition under our belts, we now show how to determine the gradient vector field or set of arrows that we saw at the beginning of the chapter. We begin with an example to see how to pass from the Forman definition of a discrete Morse function to a set of arrows.

Example 2.36. Let G be the simplicial complex from Example 2.16. The idea of the gradient vector field is to graphically show what the discrete Morse function is doing to the complex K.



Whenever a simplex σ has a value greater than or equal to one of its codimension-1 cofaces τ, we draw an arrow on τ (head of the arrow) pointing away from σ (tail of the arrow) and remove all numerical values. Thus the above graph is transformed into



Observe that in the above complex, an edge is critical if and only if it is not the head of an arrow, and a vertex is critical if and only if it is not the tail of an arrow. It is also easy to see that we do not have a discrete Morse function if either i) a vertex is the tail of more than one arrow or

ii) an edge is the head of more than one arrow. Of course, this can only be drawn up to dimension 3, but the idea generalizes to any dimension. We now make the proper general definition and discuss how the above observations hold in general.

Definition 2.37. Let f be a discrete Morse function on K. The **induced gradient vector field** V_f, or V when the context is clear, is defined by

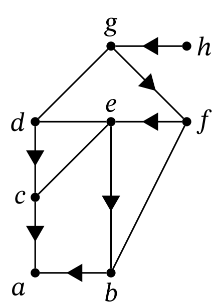
$$V_f := \{(\sigma^{(p)}, \tau^{(p+1)}) : \sigma < \tau, f(\sigma) \geq f(\tau)\}.$$

If $(\sigma, \tau) \in V_f$, (σ, τ) is called a **vector**, an **arrow**, or a **matching**. The element σ is a **tail** while τ is a **head**.

Remark 2.38. The concept of the gradient vector field is quite rich. We note that there are three ways in which to view a gradient vector field. The above definition views it as a set. In Chapter 7, we identify the gradient vector field with a discrete Morse function. Finally, Chapter 8 views the gradient vector field as a function on an algebraic structure.

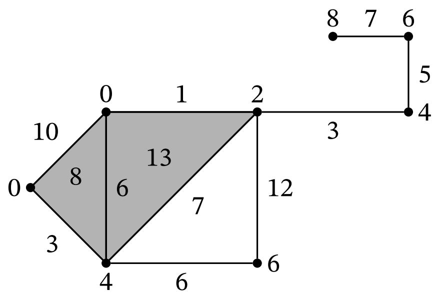
To see the connection between the arrows drawn in Example 2.36 and Definition 2.37, we give names to the simplices in Example 2.36.

Example 2.39. We label the graph in Example 2.36 as follows:



Then $V_f = \{(h, gh), (g, gf), (f, ef), (e, be), (b, ab), (c, ac), (d, dc)\}$. In other words, an element (σ, τ) of the induced gradient vector field may be thought of as an arrow induced by f on the simplicial complex with σ the tail and τ the head.

Problem 2.40. Let $f : K \to \mathbb{R}$ be the discrete Morse function given by



Find the induced gradient vector field V_f. Communicate this both by drawing the gradient vector field on K and by writing down the elements in the set V_f.

Problem 2.41. Find the induced gradient vector field of the function on the torus given in Example 2.12.

Remark 2.42. Given the induced gradient vector fields in both Example 2.39 and Problem 2.40, we see that every simplex is either exactly one tail, exactly one head, or not in the induced gradient vector field (i.e., is critical). We have already proven that this phenomenon holds in general, although not in this language. If we let σ be a simplex of K and f a discrete Morse function on K, Lemma 2.24 yields that exactly one of the following holds:

i) σ is the tail of exactly one arrow.
ii) σ is the head of exactly one arrow.
iii) σ is neither the head nor the tail of an arrow; that is, σ is critical.

Conversely, we may ask if the three conditions in Remark 2.42 always yield a gradient vector field induced by some discrete Morse function. Such a partition of the simplices of a simplicial complex K is a **discrete vector field on** K. The formal definition is given below.

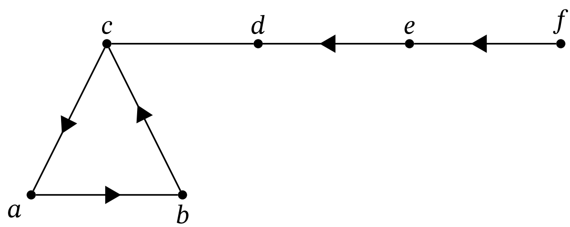
Definition 2.43. Let K be a simplicial complex. A **discrete vector field** V on K is defined by

$$V := \{(\sigma^{(p)}, \tau^{(p+1)}) : \sigma < \tau, \text{ each simplex of } K \text{ in at most one pair}\}.$$

Exercise 2.44. Show that every gradient vector field is a discrete vector field.

To state the converse of Exercise 2.44 using our new language, we ask: "Is every discrete vector field a gradient vector field of some discrete Morse function f?"

Example 2.45. Consider the discrete vector field on the following simplicial complex:



It is clear that for each simplex, exactly one of the conditions i), ii), and iii) holds; that is, the above is a discrete vector field. Can we find a discrete Morse function that induces this discrete vector field, making it a gradient vector field? Reading off the conditions that such a gradient vector field would impose, such a discrete Morse function f must satisfy

$$f(a) \geq f(ab) > f(b) \geq f(bc) > f(c) \geq f(ac) > f(a),$$

which is impossible. Thus the above discrete vector field is not induced by any discrete Morse function.

While all the arrows on the simplicial complex in Example 2.45 form the discrete vector field, it was following a particular set of arrows that led us to a contradiction. The problem is evidently that the arrows cannot form a "closed path." We will show in Theorem 2.51 below that if the discrete vector field does not contain a closed path (see Definition 2.50), then the discrete vector field is induced by a discrete Morse function. For now, we define what we mean by a path of a discrete vector field V.

Definition 2.46. Let V be a discrete vector field on a simplicial complex K. A V-**path** or **gradient path** is a sequence of simplices

$$\left(\tau_{-1}^{(p+1)},\right)\sigma_0^{(p)},\tau_0^{(p+1)},\sigma_1^{(p)},\tau_1^{(p+1)},\sigma_2^{(p)},\dots,\tau_{k-1}^{(p+1)},\sigma_k^{(p)}$$

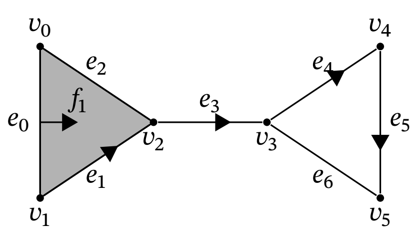
of K, beginning at either a critical simplex $\tau_{-1}^{(p+1)}$ or a regular simplex $\sigma_0^{(p)}$, such that $(\sigma_i^{(p)},\tau_i^{(p+1)})\in V$ and $\tau_{i-1}^{(p+1)}>\sigma_i^{(p)}\neq\sigma_i^{(p)}$ for $0\leq i\leq k-1$. If $k\neq 0$, then the V-path is called **non-trivial**. Note that the very last simplex, $\sigma_k^{(p)}$, need not be in a pair in V.

We sometimes use $\sigma_0\to\sigma_1\to\cdots\to\sigma_k$ to denote a gradient path from σ_0 to σ_k when there is no need to distinguish between the p- and $(p+1)$-simplices.

Remark 2.47. A V-path usually begins at a regular p-simplex, but we will sometimes need to consider a V-path beginning at a critical $(p+1)$-simplex β with $\tau>\sigma_0$ (e.g. Section 8.4).

Again, we have a very technical definition that conveys a simple concept: starting at a regular p-simplex (or critical $(p+1)$-simplex), follow arrows and end at a p-simplex (regular or critical).

Example 2.48. Consider the discrete vector field V on the simplicial complex K below:



A V-path simply starts at some regular simplex (a tail of an arrow) and follows a path of arrows. So some examples of V-paths above are

- e_0, f_1, e_2
- v_2, e_3, v_3, e_4, v_4
- v_2
- e_6, v_3, e_4, v_4

- e_0
- v_1, e_1, v_2, e_3, v_3
- $v_1, e_1, v_2, e_3, v_3, e_4, v_4, e_5, v_5$

Note that the sequence v_2, e_3, v_3, e_4, v_4 is still considered a V-path even though it is only a subset of the V-path given by $v_1, e_1, v_2, e_3, v_3, e_4, v_4, e_5, v_5$. A V-path that is not properly contained in any other V-path is called **maximal**. This example contains two maximal V-paths, and Example 2.36 also contains two maximal V-paths.

Exercise 2.49. How many maximal V-paths are in Problem 2.40?

Definition 2.50. A V-path beginning at $\sigma_0^{(p)}$ is said to be **closed** if $\sigma_k^{(p)} = \sigma_0^{(p)}$.

In Example 2.45, we see that a, ab, b, bc, c, ca, a is a closed V-path. Now we have the language and notation to characterize those discrete vector fields which are also gradient vector fields induced by some discrete Morse function.

Theorem 2.51. A discrete vector field V is the gradient vector field of a discrete Morse function if and only if the discrete vector field V contains no non-trivial closed V-paths.

The proof of this will be postponed until Section 2.2.2 on Hasse diagrams, which will allow us to grasp the concept more easily.

2.2.1. Relationship to Forman equivalence.

We saw in Example 2.30 two discrete Morse functions that turned out to be Forman equivalent. What happens if we look at their gradient vector fields?

Exercise 2.52. Compute the gradient vector fields induced by the discrete Morse functions in Example 2.30.

If you computed the gradient vector fields in the previous exercise correctly, they should be the same. This is no coincidence, as the following theorem, due to Ayala et al. [**9**, Theorem 3.1], shows that the gradient vector field characterizes Forman-equivalent discrete Morse functions.

Theorem 2.53. Two discrete Morse functions f and g defined on a simplicial complex K are Forman equivalent if and only if f and g induce the same gradient vector field.

Proof. For the forward direction, let $f, g\colon K \to \mathbb{R}$ be Forman equivalent so that if $\sigma^{(p)} < \tau^{(p+1)}$, then $f(\sigma) < f(\tau)$ if and only if $g(\sigma) < g(\tau)$. Thus $f(\sigma) \geq f(\tau)$ if and only if $g(\sigma) \geq g(\tau)$, and hence $(\sigma, \tau) \in V_f$ if and only if $(\sigma, \tau) \in V_g$.

Now suppose that f and g induce the same gradient vector field on K; i.e., $V_f = V_g =: V$. Using Lemma 2.24, any simplex of K is either critical or in exactly one pair in V. Suppose $\sigma^{(p)} < \tau^{(p+1)}$. We need to show that $f(\sigma) \geq f(\tau)$ if and only if $g(\sigma) \geq g(\tau)$. We consider the cases below.

(a) Suppose $(\sigma, \tau) \in V$. This implies that $f(\sigma) \geq f(\tau)$ and $g(\sigma) \geq g(\tau)$.

(b) Suppose σ is not in a pair in V while τ is in a pair in V. Since σ is not in a pair in V, it is critical for both functions, so it satisfies $f(\sigma) < f(\tau)$ and $g(\sigma) < g(\tau)$. The exact same conclusion follows from the supposition that σ is in a pair in V and τ is not in a pair in V.

(c) Suppose σ and τ are in different pairs in V. Then $f(\sigma) < f(\tau)$ and $g(\sigma) < g(\tau)$.

(d) Suppose neither σ nor τ are in any pair in V. Then they are both critical, so $f(\sigma) < f(\tau)$ and $g(\sigma) < g(\tau)$.

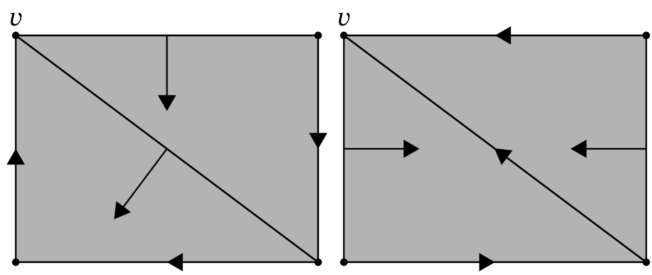
In all cases, $f(\sigma) \geq f(\tau)$ if and only if $g(\sigma) \geq g(\tau)$. $\square$

As a corollary, we have the following:

Corollary 2.54. Any two Forman-equivalent discrete Morse functions f and g defined on a simplicial complex K have the same critical simplices.

Problem 2.55. Prove Corollary 2.54.

Example 2.56. We give an example to show that the converse of Corollary 2.54 is false. Consider the complex K with two discrete vector fields given by



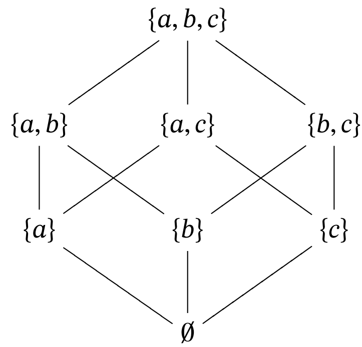
By Theorem 2.53, the two discrete Morse functions are not Forman equivalent, yet they have the same critical simplices (namely, the one 0-simplex v).

2.2.2. The Hasse diagram. Recall that a relation R on a set A is a subset of $A \times A$; i.e., $R \subseteq A \times A$. We write aRb if and only if $(a,b) \in R$. If aRa for all $a \in A$, then R is called **reflexive**. If aRb and bRc implies that aRc for all $a,b,c \in A$, then R is called **transitive**. If aRb and bRa implies that $a = b$ for all $a,b \in A$, then R is called **antisymmetric**.

Definition 2.57. A **partially ordered set** or **poset** is a set P along with a reflexive, antisymmetric, transitive relation usually denoted by $\leq$.

Example 2.58. Let $P = \mathbb{R}$ under the relation $\leq$. This is reflexive since for every $a \in \mathbb{R}$, $a \leq a$. Suppose $a \leq b$ and $b \leq a$; then, by definition, $a = b$. Finally, it is easily seen that if $a \leq b$ and $b \leq c$, then $a \leq c$.

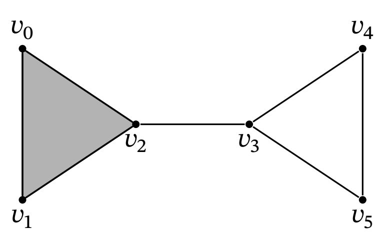
Example 2.59. Let X be a finite set. It is easy to prove that the power set of X, $\mathcal{P}(X)$, is a poset under subset inclusion. Geometrically, we can visualize the relations in a poset. We will take $X = \{a,b,c\}$. Write down all the elements of $\mathcal{P}(X)$ and draw a line between a simplex and a codimension-1 face, keeping sets of the same cardinality in the same row.



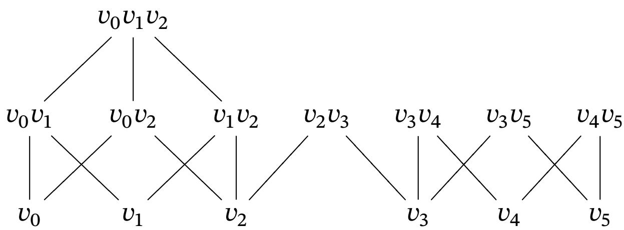
The above picture is called a Hasse diagram. To avoid confusion, we refer to a point in the Hasse diagram as a **node**. We may associate a Hasse diagram to any simplicial complex K as follows: the **Hasse diagram** [**148**] **of** K, denoted by $\mathcal{H}_K$ or $\mathcal{H}$, is defined as the partially ordered set of simplices of K ordered by the face relations; that is, $\mathcal{H}$ is a 1-dimensional simplicial complex (or graph) such that there is a 1–1 correspondence between the nodes of $\mathcal{H}$ and the simplices of K. With an abuse of notation, if $\sigma \in K$, we write $\sigma \in \mathcal{H}$ for the corresponding node. Finally, there is an edge between two simplices $\sigma, \tau \in \mathcal{H}$ if and only if τ is a codimension-1 face of σ. We organize the picture by placing nodes in rows such that every node in the same row corresponds to a simplex of the same dimension. In general, we'll let $\mathcal{H}(i)$ denote the nodes of $\mathcal{H}$ corresponding to the i-simplices of K. We refer to $\mathcal{H}(i)$ as **level** i.

Exercise 2.60. Let K be a simplicial complex. Prove that the Hasse diagram of K defines a partially ordered set.

Example 2.61. We considered the simplicial complex K given by

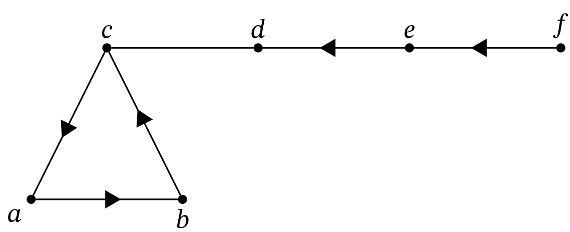


in Example 2.48. Its Hasse diagram is given by

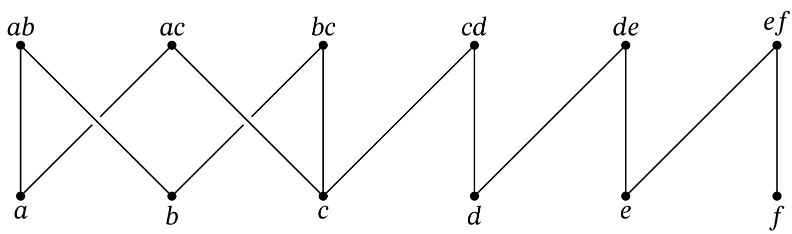


Suppose we have a discrete Morse function on K. Is there a way to put a corresponding discrete Morse function on $\mathcal{H}_K$? Conversely, is there a way to put a discrete Morse function on the Hasse diagram that yields a discrete Morse function on K? One way to figure this out is to investigate the Hasse diagram of the non-discrete Morse function from Example 2.45.

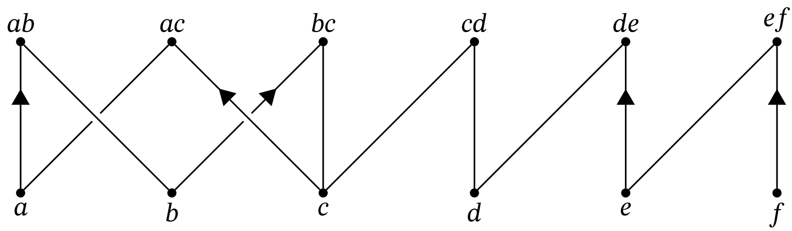
Example 2.62. Recall that Example 2.45 gave a discrete vector field on a simplicial complex K which did not correspond to a discrete Morse function.



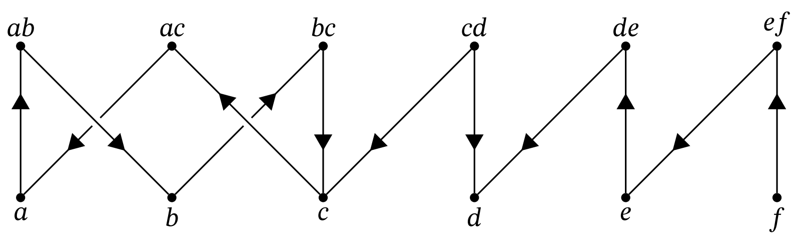
This was because of the closed V-path a, ab, b, bc, c, ac, a. The Hasse diagram of K is given by



In order to transfer this discrete vector field on to $\mathcal{H}_K$, we draw an upward-pointing arrow along an edge in the Hasse diagram between a pair in V. An arrow on a Hasse diagram is said to be **upward** if it is directed from a node in $\mathcal{H}(i)$ to a node in $\mathcal{H}(i+1)$. Since we'll need it below, we also define an arrow to be **downward** if it is directed from a node in $\mathcal{H}(i+1)$ to a node in $\mathcal{H}(i)$. If you draw all the upward arrows corresponding to pairs in V on $\mathcal{H}$, it is still unclear how you could tell from the Hasse diagram that this does not correspond to a discrete Morse function.



A slight modification will clarify this. In addition to drawing an upward-pointing arrow between pairs of elements in V, draw a downward-pointing arrow on all the other edges. The resulting Hasse diagram should look like this:



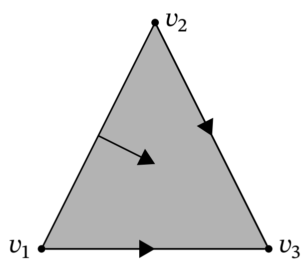
Following the direction of the arrows starting at node a, we traverse the path a, ab, b, bc, c, ac, a, bringing us back to where we started. It is this "directed cycle" that precludes this directed Hasse diagram from corresponding to a discrete Morse function.

Definition 2.63. Let K be a simplicial complex and $V_K = V$ a discrete vector field on K. The **directed Hasse diagram induced by** V, denoted by $\mathcal{H}_V$, is the Hasse diagram $\mathcal{H}_K$ of K with an arrow on every edge of

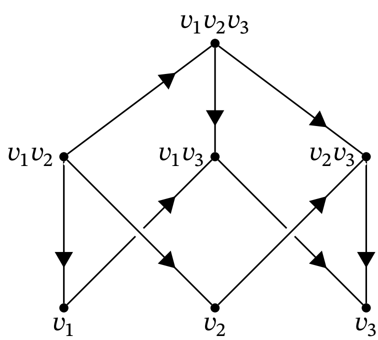
$\mathcal{H}_V$. An arrow points upward if and only if the two nodes of the edge are an ordered pair in V. Viewing the directed Hasse diagram as a 1-dimensional simplicial complex, a non-trivial closed V-path of $\mathcal{H}_V$ is called a **directed cycle.**

Remark 2.64. The directed Hasse diagram is sometimes called the modified Hasse diagram (e.g. in [**99**]).

Example 2.65. Let $K = \Delta^2$ with gradient vector field given below:

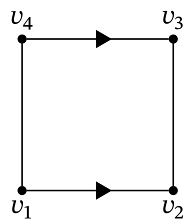


Its directed Hasse diagram is given by



Notice that the 0-, 1-, and 2-simplices of K are organized in levels.

Problem 2.66. Draw the directed Hasse diagram induced by the gradient vector field on



Does it contain a directed cycle?

Lemma 2.67. *Let K be a simplicial complex and V a discrete vector field on K. If the Hasse diagram induced by V contains a directed cycle, then the directed cycle is contained in exactly two levels.*

Problem 2.68. Prove Lemma 2.67.

Theorem 2.69. *Let K be a simplicial complex, $V_K = V$ a discrete vector field on K, and $\mathcal{H}_V$ the corresponding directed Hasse diagram. There are no non-trivial closed V-paths if and only if there are no directed cycles in $\mathcal{H}_V$.*

Proof. We start with the backward direction. Suppose by way of contradiction that V contains a closed V-path, say

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \dots, \beta_k^{(p+1)}, \alpha_{k+1}^{(p)} = \alpha_0^{(p)}.$$

We find a directed cycle in $\mathcal{H}_V$. Starting at $\alpha_0^{(p)} \in \mathcal{H}_V$, traverse the upward arrow to $\beta_0^{(p+1)}$. The arrow points upward since $(\alpha_0^{(p)}, \beta_0^{(p+1)}) \in V$. Now the arrow from $\beta_0^{(p+1)}$ to $\alpha_1^{(p)}$ points downward since otherwise $(\alpha_1^{(p)}, \beta_0^{(p+1)}) \in V$, contradicting the fact that $\beta_0^{(p+1)}$ is in at most one pair of V. Continuing in this manner, we traverse a directed path that begins and ends at $\alpha_0^{(p)}$, which is a directed cycle.

To see the forward direction, observe that if, by contradiction, there is a directed cycle in $\mathcal{H}_V$, then Lemma 2.67 guarantees that it is contained in exactly two levels. Thus, using the argument in the proceeding paragraph, following a directed cycle in the Hasse diagram will yield a non-trivial closed V-path in K. This completes the proof. $\square$

An immediate corollary of Theorem 2.69 is that Forman-equivalent discrete Morse functions have the same directed Hasse diagram.

Corollary 2.70. Let $f, g\colon K \to \mathbb{R}$ be discrete Morse functions. Then f and g are Forman equivalent if and only if $\mathcal{H}_{V_f} = \mathcal{H}_{V_g}$.

Proof. Apply Theorem 2.53 and Theorem 2.69. $\square$

We need one more lemma. It is purely graph theoretic. See [**20**, Proposition 1.4.3] for a proof.

Lemma 2.71. Let G be a 1-dimensional simplicial complex and V a discrete vector field on G. Then there is a real-valued function of the vertices that is strictly decreasing along each directed path if and only if G does not have a directed cycle.

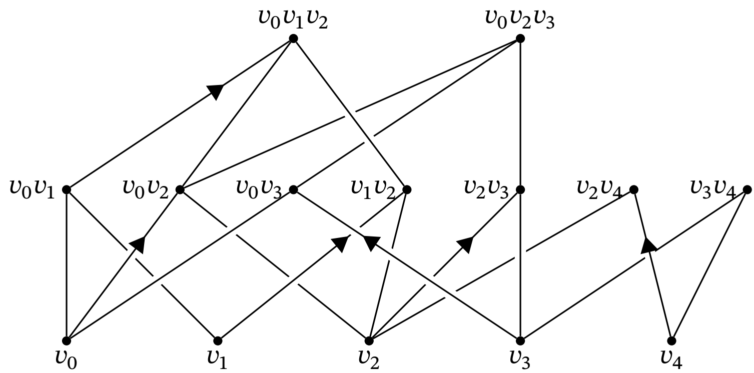
As promised, we now prove Theorem 2.51.

Proof of Theorem 2.51. A discrete vector field V is the gradient vector field of a discrete Morse function f if and only if the corresponding nodes on $\mathcal{H}_V$ with the values given by f yield a real-valued function which is strictly increasing along each directed path, and this is the case if and only if $\mathcal{H}_V$ has no directed cycle (Lemma 2.71), which is the case if and only if V contains no non-trivial closed V-paths (Theorem 2.69). $\square$

Problem 2.72. Prove Lemma 2.71.

In light of Theorem 2.51, we will use the term "gradient vector field" to mean either that which is induced by a discrete Morse function (the proper sense of the term) or a discrete vector field with no non-trivial closed V-paths.

Problem 2.73. Could the following be the directed Hasse diagram $\mathcal{H}_K$ of a simplicial complex K induced by a discrete Morse function on K? A discrete vector field on K? In both cases, justify your answer. (Note that the downward arrows are suppressed to avoid clutter.)



2.2.3. Generalized discrete Morse functions. As often happens in mathematics, an equivalent definition may allow for a more general understanding. Remark 2.42 tells us that a discrete Morse function always partitions the simplices of a simplicial complex. However, the way the partition can look is very restricted: the sets of the partition must be either of size two (regular pairs) or of size one (critical simplex). What if we allow more general partitions using sets of any size? This idea was first suggested in [**72**], and nice applications in geometric topology are found in [**25**, **57**] and generalized further by the latter authors in [**26**]. Here we content ourselves with giving the basic definitions and results. As a warmup, recall explicitly how a discrete Morse function induces a partition.

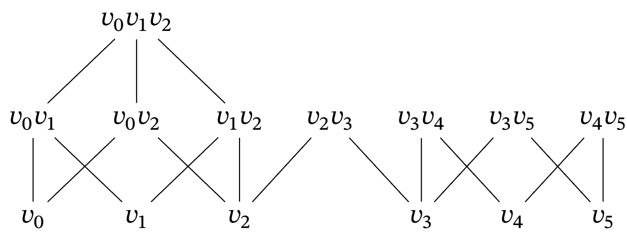
Exercise 2.74. Let f be the discrete Morse function from Example 2.39. Write down the partition induced by f.

Definition 2.75. Let K be a simplicial complex. For any $\alpha, \beta \in K$, the **interval** $[\alpha, \beta]$ is the subset of K given by

$$[\alpha, \beta] := \{\gamma \in K : \alpha \subseteq \gamma \subseteq \beta\}.$$

Exercise 2.76. Show that $[\alpha, \beta] \neq \emptyset$ if and only if $\alpha \subseteq \beta$.

Example 2.77. In Example 2.61, we found the following Hasse diagram $\mathcal{H}_K$:



We illustrate Definition 2.75 by computing intervals. For example, $[v_0, v_0v_1v_2] = \{v_0, v_0v_1, v_0v_2, v_0v_1v_2\}$, $[\emptyset, v_4v_5] = \{\emptyset, v_4, v_5, v_4v_5\}$, $[v_4v_5, v_4v_5] = \{v_4v_5\}$, and $[v_1, v_2] = \emptyset$.

Any partition W of K into intervals is called a **generalized discrete vector field**. The terminology is justified by the fact that every discrete Morse function may be viewed as a generalized discrete vector field. Indeed, if $f : K \to \mathbb{R}$ is a discrete Morse function, we know by Remark 2.42 that under f, every simplex of K is either critical or part of a free pair (and not both). For each critical simplex σ, choose $[\sigma, \sigma] = \{\sigma\}$. For each free pair $\alpha < \beta$, choose $[\alpha, \beta] = \{\alpha, \beta\}$. This forms a partition of K and hence is a generalized discrete vector field.

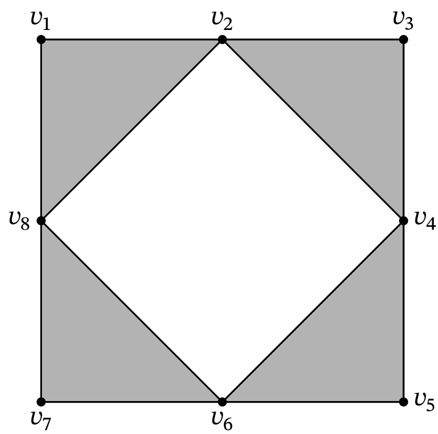
Exercise 2.78. Find a generalized discrete vector field on the simplicial complex in Example 2.77.

Problem 2.79. Give an example to show why "generalized gradient vector field" would not be an appropriate name for the above definition. You may need to recall the distinction between gradient vector field and discrete vector field.

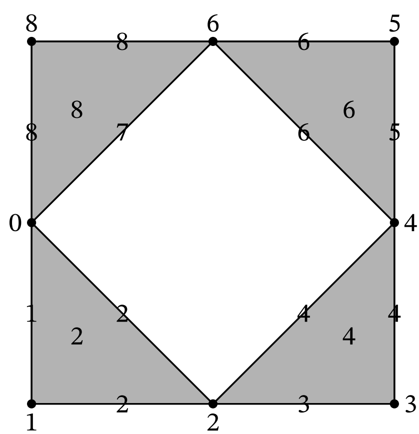
Fix a generalized discrete vector field W on K and let $f : K \to \mathbb{R}$ be a function (not necessarily a discrete Morse function) which satisfies $f(\alpha) \leq f(\beta)$ whenever $\alpha < \beta$, with $f(\alpha) = f(\beta)$ if and only if there exists an interval $I \in W$ such that $\alpha, \beta \in I$. Then f is called a **generalized discrete Morse function** and W its **generalized discrete vector field.** An interval containing only one simplex σ is **singular**, and σ is called a **critical simplex**, with $f(\sigma)$ a **critical value** of f. Note that two intervals may share the same value.

Problem 2.80. Let $K = \Delta^n$ be the n-simplex and let $f : K \to \mathbb{R}$ be defined by $f(\sigma) = 0$ for every $\sigma \in K$. Find a partition W of K that makes f a generalized discrete Morse function.

Example 2.81. Let K be the simplicial complex



and W the partition into intervals $[v_1, v_1v_2v_8], [v_2, v_2v_3v_4], [v_3, v_3v_4]$, $[v_4, v_4v_5v_6]$, $[v_5, v_5v_6]$, $[v_6, v_6v_7v_8]$, $[v_2v_8]$, $[v_7, v_7v_8]$, and $[v_8, v_8]$. Define f by



Then f is a generalized discrete Morse function, but clearly not a discrete Morse function.

Problem 2.82. Let $K, L \subseteq M$ be subcomplexes of a simplicial complex M, and let $f : K \to \mathbb{R}$ and $g : L \to \mathbb{R}$ be generalized discrete Morse functions with gradients V and W, respectively. Prove that $(f+g) : K \cap L \to \mathbb{R}$ is also a generalized discrete Morse function, with gradient given by $U := \{I \cap J : I \in V, J \in W, I \cap J \neq \emptyset\}$.

There are at least two ways in which generalized discrete Morse theory can prove helpful. One is that it can detect and perform multiple collapses at once. This will be made precise in Corollary 4.30, but the idea can be seen in Example 2.81. All simplices labeled 8 can be removed as a sequence of two elementary collapses. The edge labeled 7 is critical, but the simplices labeled 6 are again a sequence of two elementary collapses and hence may be removed. The pair labeled 5 is a free pair which may be removed, etc. The point is that we obtain a collapsing theorem by which all the simplices with the same value may be removed simultaneously. This is especially helpful for computational purposes. We will see a similar idea of how to perform simultaneous collapses in Chapter 10.

Another situation in which generalized discrete Morse theory is helpful is when the functions one is working with don't satisfy the properties of a discrete Morse function but are generalized discrete Morse functions. This was the case examined in the paper [**26**] by U. Bauer and H. Edelsbrunner. They showed that certain radius functions of Čech and Delaunay complexes are generalized discrete Morse functions and used this to prove a collapsing theorem.

2.3. Random discrete Morse theory

In this brief section, we introduce yet another way to view a discrete Morse function. This point of view is more algorithmic and lends itself well to some computational purposes.

2.3.1. Discrete Morse vectors and optimality.

Definition 2.83. Let K be an n-dimensional simplicial complex and $f\colon K \to \mathbb{R}$ a discrete Morse function, and let m_i^f (or just m_i if the function f is clear) denote the number of critical i-simplices of f. Define $\vec{f} := (m_0^f, m_1^f, m_2^f, \ldots, m_n^f)$ to be the **discrete Morse vector of** f.

Exercise 2.84. Let $f\colon K \to \mathbb{R}$ be a discrete Morse function. Recall that the c-vector of K is the vector $\vec{c} = (c_0, c_1, \ldots, c_{\dim(K)})$ where c_i is the number of simplices of K of dimension i. Show that $\vec{f} \leq \vec{c}$, meaning that $m_i^f \leq c_i$ for all i. Can it ever be the case that $\vec{f} = \vec{c}$?

Note that by Problem 2.26, we will always have $m_0 \geq 1$.

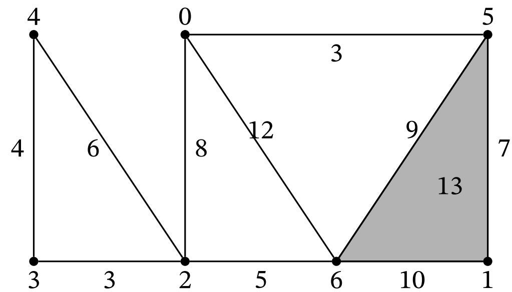
Problem 2.85. Prove that if a simplicial complex K is collapsible, then there is a discrete Morse function with discrete Morse vector $(1, 0, 0, 0, \ldots, 0)$.

Problem 2.86. If $K \nearrow L$ through a series of elementary expansions and $f\colon K \to \mathbb{R}$ is a discrete Morse function with discrete Morse vector $\vec{f}$, prove that there exists a discrete Morse function $g\colon L \to \mathbb{R}$ such that $g|_K = f$ and $\vec{g} = \vec{f}$. Here $g|_K$ denotes the **restriction** of the function g to the domain K.

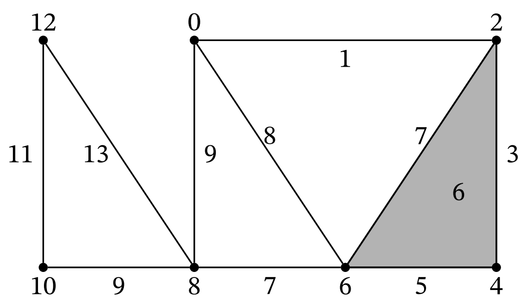
As you worked through Exercise 2.84, it may have occurred to you that for a fixed simplicial complex K, the values that $\vec{f}$ can take can vary quite a bit. This raises the question of how one might define a "best possible" f-vector. The following definition is one such attempt.

Definition 2.87. Let $f\colon K \to \mathbb{R}$ be a discrete Morse function with discrete Morse vector $\vec{f} := (m_0^f, m_1^f, m_2^f, \ldots, m_n^f)$. Then $\vec{f}$ is called **optimal** if $\sum_{i=0}^{n} m_i^f$ is minimal in the sense that if $g\colon K \to \mathbb{R}$ is any other discrete Morse function on K, then $\sum_{i=0}^{n} m_i^f \leq \sum_{i=0}^{n} m_i^g$.

Example 2.88. Let $f : K \to \mathbb{R}$ be given by



By inspection, $m_0^f = 3, m_1^f = 5$, and $m_2^f = 1$ so that the discrete Morse vector is given by $\vec{f} = (3, 5, 1)$. Can we find a discrete Morse function with fewer critical values? It isn't too hard. Let $g : K \to \mathbb{R}$ be given by



Now it is easy to check that $m_0^g = 1, m_1^g = 3$, and $m_2^g = 0$ so that $\vec{g} = (1, 3, 0)$. This is clearly better than $\vec{f}$. Can we do even better? It seems like this is the best we can do. We'll see in Section 4.1.2 that this discrete Morse vector is indeed optimal for our simplicial complex.

Although every simplicial complex has an optimal discrete Morse function, finding one can be a challenge. Lewiner et al. [**109**], for example, present a linear-time algorithm for finding optimal discrete Morse functions on a special class of 2-dimensional simplicial complexes. Note that an optimal discrete Morse vector need not be unique. Indeed, K. Adiprasito, B. Benedetti, and F. Lutz constructed a 3-dimensional simplicial complex with optimal discrete Morse vectors $(1, 1, 1, 0)$ and $(1, 0, 1, 1)$. In fact, they were able to prove the following more general result.

Theorem 2.89 ([**4**, Theorem 3.3])**.** For every $d \geq 3$ there exists a non-collapsible d-dimensional simplicial complex that has two distinct optimal discrete Morse vectors

$$(1, 0, \dots, 0, 1, 1, 0) \quad \text{and} \quad (1, 0, \dots, 0, 0, 1, 1).$$

2.3.2. Benedetti-Lutz algorithm. In this section, we study a "randomized" approach to discrete Morse theory. Let K be a simplicial complex. How would one put a random discrete Morse function on K? One could begin by labeling simplices at random, but then there is no way to guarantee that the resulting labeling is actually a discrete Morse function. Any attempt to tweak the values in order to guarantee a discrete Morse function would seem to either remove the randomness or result in something too complicated in practice. The **Benedetti-Lutz** or **B-L algorithm** [**33**] avoids these problems by viewing the discrete Morse function in terms of its geometry. The idea is simple: Given a simplicial complex K, perform one of two moves—either 1) remove a free pair or 2) remove a top-dimensional facet, with preference being given to option 1). When 2) is performed, the dimension of the removed facet is recorded in a discrete Morse vector. Formally, the B-L algorithm is given in Algorithm 1.

Algorithm 1 B-L algorithm

`Input:` A d-dimensional abstract finite simplicial complex K given by its list of facets.

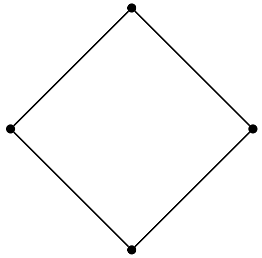
`Output:` The resulting discrete Morse vector $(c_0, c_1, \dots, c_d)$.

1 Initialize $c_0 = c_1 = \cdots = c_d = 0$.
2 If the complex is empty, STOP. Otherwise, go to Step 3.
3 If there is a free codimension-1 face, go to Step 4. If not, go to Step 5.
4 Pick a free codimension-1 coface uniformly at random and delete it with the unique face that contains it. Go back to Step 2.
5 Pick a top-dimensional i-face uniformly at random and delete it, and set $c_i = c_i + 1$. Go back to Step 2.

Problem 2.90. Let K be a simplicial complex. Show how the B-L algorithm yields a discrete Morse function on K.

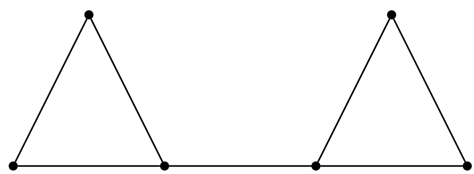
Definition 2.91. The **discrete Morse spectrum** of a simplicial complex K is the collection of all possible discrete Morse vectors produced by the B-L algorithm along with the distribution of the respective probabilities. If p_i is the probability of obtaining discrete Morse vector $\vec{c_i}$, then the discrete Morse spectrum of K is denoted by $\{p_1 - \vec{c_1}, p_2 - \vec{c_2}, \dots, p_k - \vec{c_k}\}$ for some $k \in \mathbb{N}$, where $\sum p_i = 1$ and $p_i \neq 0$ for any i.

Example 2.92. Let K be the simplicial complex of Example 2.81. The B-L algorithm removes free pairs from this complex, and it is not difficult to see that any sequence of removals of free pairs will result in the simplicial complex



Any edge is chosen with probability $\frac{1}{4}$ and the resulting output discrete Morse vector is $(1, 1)$. Thus the discrete Morse spectrum for this simplicial complex is $\{1 - (1, 1)\}$.

Problem 2.93. Compute the discrete Morse spectrum of the following simplicial complex:



Given your work in Problem 2.90, we can now see how Algorithm 1 gives a discrete Morse function. The algorithm was introduced by B. Benedetti and F. Lutz in the paper cited above. They studied this concept extensively in terms of its usefulness for testing the complexity of certain simplicial complexes. Such an endeavor is beyond the scope of this book (but see for example [**4**, **33**]). However, there are some simple observations that we can make.

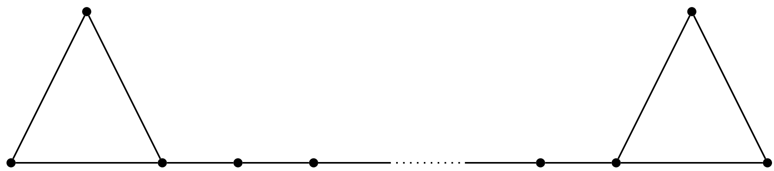
Proposition 2.94. Let K be a simplicial complex. Suppose that the discrete Morse vector output on a single run of the B-L algorithm is $(1, 0, 0, \dots, 0)$. Then K is collapsible.

Exercise 2.95. Prove Proposition 2.94.

The converse of Proposition 2.94 is false. In fact, it is worth noting that we should not always expect to find an optimal discrete Morse function this way, even after many, many runs. The next proposition spells this out explicitly, while in particular providing an example to show that the converse of Proposition 2.94 is false.

Proposition 2.96. For every $\epsilon > 0$, there exists a simplicial complex G_ϵ such that the probability that the B-L algorithm yields an optimal discrete Morse vector of G_ϵ is less than ϵ.

Proof. Let $\epsilon > 0$ be given and choose $n \in \mathbb{N}$ such that $\frac{6}{n+6} < \epsilon$. Consider the simplicial complex given by



with at least n edges between the two cycles. Now a necessary condition for a random discrete Morse vector to be optimal is that the first removal of an edge needs to be one of the six edges in the two triangles. The probability of this happening is less than $\frac{6}{n+6}$ so that the probability of failing to obtain one of the six edges is less than $\frac{6}{n+6}$. Thus the probability of obtaining an optimal discrete Morse vector is less than $\frac{6}{n+1} < \epsilon$. $\square$

Chapter 3

Simplicial homology

This chapter serves as a friendly and working introduction to simplicial homology, a theory that has been well established for many decades. The reader familiar with simplicial homology may safely skip it.

Homology is not just an extremely interesting and important tool in topology—it has a beautiful connection with discrete Morse theory as well (e.g. Sections 4.1 and 8.4). The idea of homology is to construct a rigorous theory that both counts the number and identifies the type of holes in a space. For example, a circle has one hole and a sphere has no holes. However, a sphere does seem to have a "higher-dimensional" type of hole, since the entire sphere encloses some 3-dimensional space or a void (think about the air inside of a basketball). A torus (the outside skin of a doughnut) seems to have both kinds of holes. There are many, many versions of homology, computed in different ways and for use on different kinds of objects. To give a name to what we study, for any integer $n \geq 0$, we will study a special kind of "function"[1], called **simplicial (unreduced[2]), homology** and denoted by $H_n(K)$, from the collection of finite abstract simplicial complexes to the collection of vector spaces. We will study another way to compute homology and compare it with

[1] i.e., functor

[2] There is a fairly technical distinction between "reduced" and "unreduced" homology. We won't be concerned with it, other than to note that we are using the "unreduced" kind.

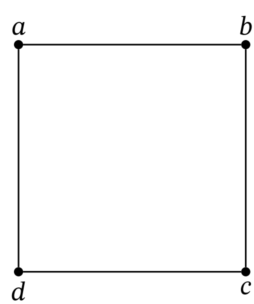
simplicial homology in Section 8.2. What makes homology a kind of "super function" is that it is not a function that simply takes in a number and outputs another number. Rather, this function takes in a simplicial complex and gives out *infinitely many* vector spaces. To illustrate, let K be the simplicial complex from Example 2.88 (this will be our running example in Section 3.2). Then under homology, K would be associated to the following sequence of vector spaces:

$$K \longmapsto [\dots, \mathbb{k}_4^0, \mathbb{k}_3^0, \mathbb{k}_2^0, \mathbb{k}_1^3, \mathbb{k}_0^1].$$

We can interpret the vector space $\mathbb{k}_j^i$ as saying that K has i holes in dimension j. So the way to interpret this sequence of vector spaces is to say that K has 0 holes in dimension 3 (and 0 holes in every dimension greater than 3) while it has 3 holes in dimension 1 and 1 hole in dimension 0. In Section 3.2 we'll see how to actually derive this sequence of vector spaces. First we will review some of the needed linear algebra.

3.1. Linear algebra

In this section, we develop a working knowledge of tools from linear algebra that allow us to define homology. It is helpful but not necessary to have taken a course in linear algebra. The needed understanding will be gained through practice with computation. We are mostly interested in vector spaces and the rank-nullity theorem. Consider the simplicial complex given below.



Now whatever our theory of holes is, it had better detect exactly one hole in the simplicial complex above. What exactly is the hole? It seems like it ought to be detected by the sequence of simplices ab, bc, cd, ad.

Now if a hole is determined by a sequence of simplices like the ones just given, we will need to somehow consider all the possible combinations of simplices of a fixed dimension. So, for example, we will get sequences like

- bc, dc, ad
- bc, bc, bc, bc
- dc, bc, ab, ad

This ends up giving us too many combinations. Furthermore, some do not even describe holes (e.g. bc, dc, ad), and some are repeats—the sequence cd, bc, ab, ad is really the same thing as ab, bc, cd, ad. A formal system that addresses these problems is a **vector space** [**107**, p. 190]. We won't give the technical definition here so much as a *working* definition. Rather than think of something like ab, bc, cd, ad as a sequence, we'll think of it as a sum: "$ab + bc + cd + ad$." Given the context, there should be no confusion between ab the 1-simplex and ab the element of a vector space. If we make this addition commutative and associative, we then see that $cd + bc + ab + ad = ab + bc + cd + ad$. Furthermore, we will use **modulo** or **mod** 2 **arithmetic**. This means that there are only two numbers, namely, 0 and 1. They obey the rules $0+1 = 1$ and $1+1 = 0$. To be technically correct[3], we would write this as $1+1 \equiv 0 \bmod 2$ and $1+0 \equiv 1 \bmod 2$. This system is called **the integers modulo** 2, denoted by $\mathbb{F}_2$. In other words, $\mathbb{F}_2 = \{0, 1\}$ along with the rule that $1 + 1 \equiv 0 \bmod 2$. The blackboard $\mathbb{F}$ stands for "field," an algebraic structure one learns about in a course in abstract algebra. Furthermore, we have $\vec{0}$ as the "zero vector." This is to be distinguished from the number 0. Using this notation, we see that, for example, $bc + bc + bc + bc = 4bc = 0 \cdot bc = \vec{0}$.

Given all the 1-simplices, we have created the vector space

$$\begin{aligned}\mathbb{k}^4 \quad = \quad & \{\vec{0}, ab, bc, cd, ad, ab + bc, ab + cd, ab + ad, bc + cd, bc + ad, \\ & cd + ad, ab + bc + cd, ab + bc + ad, ab + cd + ad, bc + cd + ad, \\ & ab + bc + cd + ad\}.\end{aligned}$$

[3]The best kind of correct.

We call it $\Bbbk^4$ because we started with 4 elements $\{ab, bc, cd, ad\}$ and generated a vector space from them—that is, all possible "sums" of those original 4 elements. The number 4 in this example is the **dimension** of the vector space $\Bbbk^4$.

Example 3.1. Consider the two symbols a and b. We can define a 2-dimensional vector space generated by a and b; specifically, if $c_1, c_2 \in \{0, 1\} = \mathbb{F}_2$, the vector space would consist of all elements of the form $c_1 a + c_2 b$. We denote this vector space by $\Bbbk^2$, meaning a vector space generated by 2 objects. For example, $0a + 1b \in \Bbbk^2$, which simplifies to b. Again, as above, we can write down all the elements of $\Bbbk^2$; that is, $\Bbbk^2 = \{\vec{0}, a, b, a + b\}$. Notice that this is "closed" under addition in the sense that the sum of any two elements in $\Bbbk^2$ is still in $\Bbbk^2$. For example, $a + a = (1 + 1)a = 0a = \vec{0}$.

The elements a and b are called **basis elements** of $\Bbbk^2$. By convention, we say that $\Bbbk^0 = \{\vec{0}\}$, the **trivial vector space** consisting of only one element, namely, the zero vector.

Definition 3.2. Let $X = \{e_1, e_2, \dots, e_n\}$ be a set of n distinct elements. The **vector space (over $\mathbb{F}_2$)** generated by X, denoted by $\Bbbk^n$, is given by $\Bbbk^n := \{c_1 e_1 + c_2 e_2 + \cdots + c_n e_n : c_i \in \{0, 1\}\}$. The elements $e_1, \dots, e_n \in \Bbbk^n$ are called **basis elements** and n is called the **dimension** of $\Bbbk^n$. It follows by definition that any particular element $x \in \Bbbk^n$ can be written as a **linear combination** of basis elements; that is, $x = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$ with $c_i \in \{0, 1\}$.

Problem 3.3. If $|X| = n$, how many elements are in the vector space $\Bbbk^n$? Prove it.

In addition to understanding vector spaces, we need to understand functions between them. A function $A : \Bbbk^n \to \Bbbk^m$ is an $\mathbb{F}_2$**-linear transformation**, or just **linear**, if for every pair of elements $v, v' \in \Bbbk^n$, A satisfies $A(v + v') = A(v) + A(v')$.[4]

Problem 3.4. Let $A : \Bbbk^n \to \Bbbk^m$ be a linear transformation as defined above. Prove that $A(\vec{0}) = \vec{0}$.

[4]In general, there is also the condition that scalars must pull through, but since we are working over $\mathbb{F}_2$, this condition is automatic.

Linear transformations can be represented by a matrix. In particular, we will be interested in the set of all elements x such that $A(x) = \vec{0}$, called the **kernel** and denoted by $\ker(A)$. That is,

$$\ker(A) := \{x \in \Bbbk^n : A(v) = \vec{0}\}.$$

This turns out to be a vector space, and we call its dimension the **nullity** of A, denoted by $\mathrm{null}(A)$. Now this is a helpful concept because when we generated $\Bbbk^4$ above, we wanted to pick out those elements representing holes. For $\Bbbk^4$, there was only one hole. This number coincides precisely with the nullity of a certain linear transformation. In general, the nullity will count all the *potential* holes in a given dimension. How is this number computed? It is found by studying a certain matrix. We will compute it in practice by using the rank-nullity theorem (Theorem 3.5) from linear algebra. Recall that the **range** or **image** of a linear transformation $A : \Bbbk^n \to \Bbbk^m$ is

$$\mathrm{Im}(A) := \{y \in \Bbbk^m : \exists x \in \Bbbk^n \text{ such that } A(x) = y\}.$$

It turns out that $\mathrm{Im}(A)$ is also a vector space, and its dimension is called the **rank** of A, denoted by $\mathrm{rank}(A)$.

Theorem 3.5 (The rank-nullity theorem)**.** Let $A : \Bbbk^n \to \Bbbk^m$ be a linear transformation so that A can be viewed as an $m \times n$ matrix. Then $\mathrm{rank}(A) + \mathrm{null}(A) = n$.

In order to utilize Theorem 3.5, we need to know how to compute the rank of a matrix. This is easily done if the matrix is in a special form called row echelon form. A **leading coefficient** or **pivot** of a non-zero row in a matrix refers to the leftmost non-zero entry. **Row echelon form** is characterized by two conditions: The first is that all non-zero rows are above any row of zeros. Second, the leading coefficient of a non-zero row is always strictly to the right of the leading coefficient of the row above it.

To transform a matrix into row echelon form, we may perform any of the following **elementary row operations**: we may replace one row by the sum of itself and another row, or we may interchange two rows. Once a matrix is in row echelon form, the rank is computed using Theorem 3.6.

Theorem 3.6. Performing elementary row operations on a matrix A does not change the rank of A. If A is in row echelon form, then the rank of A is precisely the number of non-zero rows.

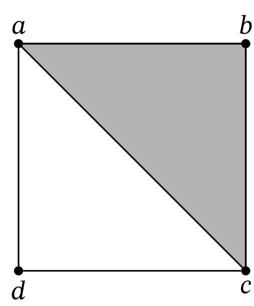
A proof of the rank-nullity theorem may be found in [**107**, p. 233], while a proof of Theorem 3.6 may be found in [**107**, p. 231].

Example 3.7. Let $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Add the second row to the third to obtain $B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Now we can't have two 1s in the first column (as it violates the second condition), so we add the first row to the second, yielding $C = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. By Theorem 3.6, $\operatorname{rank}(A) = 2$. Since A is 3×4, by Theorem 3.5, $\operatorname{null}(A) = 1$.

Problem 3.8. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Find $\operatorname{rank}(A)$ and $\operatorname{null}(A)$.

3.2. Betti numbers

We continue with the intuition behind a theory of holes or homology. Consider the simplicial complex



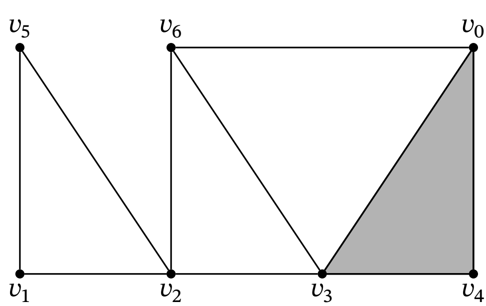
Now using the idea at the beginning of Section 3, we want to view a hole as a formal sum of elements in a vector space. But a possible problem arises. It looks like we would be counting three holes, namely, $ab+bc+ca$, $ac+cd+ad$, and $ab+bc+cd+ad$. Actually, one of these is a combination of the other two, as $(ab+bc+ca)+(ac+cd+ad) = ab+bc+cd+ad$. But that still leaves $ab+bc+ca$ as a "fake" hole, since it is filled in. How do we know it is filled in? Precisely because there is a 2-simplex abc whose boundary is the fake hole; that is, $\partial(abc) = \{ab, bc, ac\}$ (recall Definition 1.6). Viewing abc as a vector in a vector space (as opposed to a simplex), we could say that $\partial(abc) = ab + bc + ac$. So it looks like what we are doing here is counting the number of *potential* holes and then subtracting from that the number of holes that get filled in. In this case, the number of holes is $2 - 1 = 1$.

Let K be a simplicial complex on $[v_n]$. Recall that we used c_i to denote the number of i-simplices of K. Now define K_i to be the *set* of i-simplices of K. It then follows by definition that $|K_i| = c_i$.

Exercise 3.9. What are K_0 and K_j for $j > n$? Write down the values $|K_0|$ and $|K_j|$.

Let $\sigma \in K_i$. Then each $\sigma \in K_i$ is a basis element in the vector space $\Bbbk^{c_i}$ generated by all the elements of K_i.

Example 3.10. Recall the simplicial complex K from Example 2.88. We give the simplices names as follows:



Then

$$
\begin{aligned}
K \;=\; & \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_0v_3, v_0v_4, v_0v_6, v_1v_2, v_2v_3, v_3v_4, v_1v_5, \\
& v_2v_5, v_2v_6, v_3v_6, , v_0v_3v_4\}.
\end{aligned}
$$

We have

$$
\begin{aligned}
K_0 &= \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}, \\
K_1 &= \{v_0v_3, v_0v_4, v_0v_6, v_1v_2, v_2v_3, v_3v_4, v_1v_5, v_2v_5, \\
&\quad v_2v_6, v_3v_6\}, \\
K_2 &= \{v_0v_3v_4\}, \\
\cdots = K_4 = K_3 &= \emptyset,
\end{aligned}
$$

and so $c_0 = 7$, $c_1 = 10$, and $c_2 = 1$. These in turn generate vector spaces $\Bbbk^7, \Bbbk^{10}$, and $\Bbbk^1$, respectively.

Exercise 3.11. Show that if K is an n-dimensional simplicial complex, then the sets $K_0, K_1, \dots, K_n$ form a partition of K.

In other words, we partitioned K into classes where two elements of K are in the same class if and only if they have the same size. Then we created a vector space for each of those classes, and the dimension of the vector space depended on the size of the class. Any element in a vector space generated by a collection of simplices is called a **chain**. Now things get a little more tricky. For every $0 \le i < \infty$, we wish to construct linear transformations $\partial_i : \Bbbk^{c_i} \to \Bbbk^{c_{i-1}}$. To each simplex this linear transformation associates its boundary. Once we define ∂_i, this will give us the following chain complex:

$$
\cdots \xrightarrow{\partial_{i+1}} \Bbbk^{c_i} \xrightarrow{\partial_i} \Bbbk^{c_{i-1}} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} \Bbbk^{c_1} \xrightarrow{\partial_1} \Bbbk^{c_0} \xrightarrow{\partial_0} 0.
$$

A **chain complex** is a sequence of vector spaces, along with linear transformations between them, with the property that $\partial_{i-1} \circ \partial_i = 0$. We often denote a chain complex by $(\Bbbk_*, \partial_*)$ for short. Now you know immediately from your work in Problem 3.9 that $\Bbbk^{c_j} = \{\vec{0}\}$ for all $j \geq n$. We will simply write $0 = \{\vec{0}\}$ when there is no possibility of confusion. Now we define the boundary operator (again, because the boundary of the simplex is what is being computed) $\partial_i : \Bbbk^{c_i} \to \Bbbk^{c_{i-1}}$.

Definition 3.12. Let $\sigma \in K_m$ and write $\sigma = \sigma_{i_0}\sigma_{i_1}\cdots\sigma_{i_m}$. For $m = 0$, define $\partial_0 : \Bbbk^{c_0} \to 0$ by $\partial_0 = 0$, the matrix of appropriate size consisting of all zeros. For $m \geq 1$, define the **boundary operator** $\partial_m : \Bbbk^{c_m} \to \Bbbk^{c_{m-1}}$ by $\partial_m(\sigma) := \sum\limits_{0\leq j\leq m} (\sigma - \{\sigma_{i_j}\}) = \sum\limits_{0\leq j\leq m} \sigma_{i_0}\sigma_{i_1}\cdots\hat{\sigma}_{i_j}\cdots\sigma_{i_m}$ where $\hat{\sigma}_{i_j}$ excludes the value σ_{i_j}.

Remark 3.13. Note the relationship between the boundary defined in Definition 3.12 and that defined in Definition 1.6. Given a simplex σ, both definitions take into account the codimension-1 faces of σ. The difference is that the former definition produces a chain while the latter definition produces a set.

Example 3.14. We continue with Example 3.10. Since $\emptyset = K_3 = K_4 = \cdots$, we have $0 = \Bbbk^{c_3} = \Bbbk^{c_4} = \cdots$. Hence $\partial_i = 0$ for $i = 3, 4, \ldots$. We then need to compute only ∂_2 and ∂_1. Now $\partial_2 : \Bbbk^1 \to \Bbbk^{10}$ and, by the rule given above, $\partial_2(v_0v_3v_4) = \sum\limits_{j=0,3,4} (v_0v_3v_4 - v_j) = v_3v_4 + v_0v_4 + v_0v_3$. The matrix corresponding to this is

$$\partial_2 = \begin{array}{c} \\ v_0v_3 \\ v_0v_4 \\ v_0v_6 \\ v_1v_2 \\ v_2v_3 \\ v_3v_4 \\ v_1v_5 \\ v_2v_5 \\ v_2v_6 \\ v_3v_6 \end{array} \begin{array}{c} v_0v_3v_4 \\ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}.$$

Next we compute $\partial_1 : \mathbb{k}^{10} \to \mathbb{k}^7$:

$$\begin{array}{rcl}
\partial_1(v_0v_3) & = & v_3 + v_0, \\
\partial_1(v_0v_4) & = & v_4 + v_0, \\
\partial_1(v_0v_6) & = & v_6 + v_0, \\
\partial_1(v_1v_2) & = & v_2 + v_1, \\
\partial_1(v_2v_3) & = & v_3 + v_2, \\
\partial_1(v_3v_4) & = & v_4 + v_3, \\
\partial_1(v_1v_5) & = & v_5 + v_1, \\
\partial_1(v_2v_5) & = & v_5 + v_2, \\
\partial_1(v_2v_6) & = & v_6 + v_2, \\
\partial_1(v_3v_6) & = & v_6 + v_3.
\end{array}$$

This yields the matrix ∂_1 given by

$$\begin{array}{c}
\begin{array}{cccccccccc} v_0v_3 & v_0v_4 & v_0v_6 & v_1v_2 & v_2v_3 & v_3v_4 & v_1v_5 & v_2v_5 & v_2v_6 & v_3v_6 \end{array} \\
\begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}.
\end{array}$$

Let's practice using the information given in the matrix to compute $\partial\partial(v_2v_5)$ and $\partial\partial(v_0v_3v_4)$. We have

$$\begin{array}{rcl}
\partial\partial(v_2v_5) & = & \partial(v_5 + v_2) \\
 & = & \partial(v_5) + \partial(v_2) \\
 & = & \vec{0}
\end{array}$$

and

$$\begin{aligned}
\partial\partial(v_0v_3v_4) &= \partial(v_3v_4 + v_0v_4 + v_0v_3) \\
&= \partial(v_3v_4) + \partial(v_0v_4) + \partial(v_0v_3) \\
&= v_3 + v_4 + v_0 + v_4 + v_0 + v_3 \\
&= 2v_0 + 2v_4 + 2v_3 \\
&= \vec{0}.
\end{aligned}$$

In both cases, we obtain $\vec{0}$.

Proposition 3.15. With ∂_m defined as above, $\partial_{m-1}\partial_m = 0$ where 0 is the zero matrix.

Proof. Since there is no confusion, we drop the subscripts on ∂. Furthermore, we compute $\partial\partial$ on a single generator, and the general result follows by linearity. We have

$$\begin{aligned}
\partial\partial(\sigma) &= \sum_i \sum_j \sigma_0\sigma_1 \cdots \hat{\sigma_j} \cdots \sigma_m \\
&= \sum_{i \neq j} \sigma_0\sigma_1 \cdots \hat{\sigma_j} \cdots \hat{\sigma_i} \cdots \sigma_m \\
&= \vec{0},
\end{aligned}$$

where the last equality is justified since for fixed values of i and j, the value $\sigma_0\sigma_1 \cdots \hat{\sigma_j} \cdots \sigma_m$ appears twice in the sum and hence the two occurrences added together give $\vec{0}$. $\square$

We have thus defined a chain complex from a simplicial complex K. Now that we are comfortable with this new notation, we can define precisely what we mean by holes.

Definition 3.16. We define the ith **(unreduced)** $\mathbb{F}_2$**-homology** of K to be the vector space

$$H_i(K; \mathbb{F}_2) := \mathbb{k}^{\operatorname{null} \partial_i - \operatorname{rank} \partial_{i+1}}.$$

The ith $\mathbb{F}_2$**-Betti number** of K is defined to be $b_i(K; \mathbb{F}_2) = \operatorname{null} \partial_i - \operatorname{rank} \partial_{i+1}$. We usually call it the Betti number for short and use the notation $H_i(K)$ and $b_i(K)$ or even H_i and b_i when K is clear from the context.

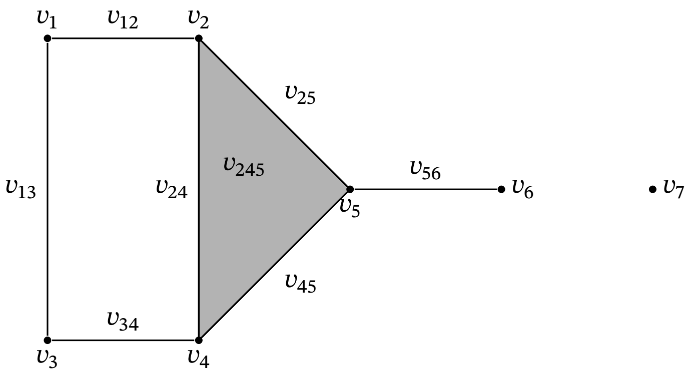
Remark 3.17. For now we will mostly use the Betti numbers without recourse to the homology vector spaces $H_i(K)$. However, in Section 8.2, we will revisit the homology vector spaces themselves and see that they can be viewed from a different perspective.

Remark 3.18. Another technical point: $\mathbb{F}_2$-Betti numbers do not always coincide with "Betti numbers" that you might encounter in other books on topology. We choose to work with $\mathbb{F}_2$-Betti numbers in order to make computations easier. The tradeoff is that we may lose some information. In particular, our $\mathbb{F}_2$-Betti numbers will differ somewhat from the "standard" Betti numbers of the Klein bottle in Example 8.34.

Example 3.19. Now using the techniques from Section 3.1, we see that $\operatorname{rank}(\partial_2) = 1, \operatorname{null}(\partial_2) = 0, \operatorname{rank}(\partial_1) = 6, \operatorname{null}(\partial_1) = 4, \operatorname{rank}(\partial_0) = 0$, and $\operatorname{null}(\partial_0) = 7$. Hence we obtain $H_2(K) = \mathbb{k}^0, H_1(K) = \mathbb{k}^{4-1} = \mathbb{k}^3$, and $H_0(K) = \mathbb{k}^{7-6} = \mathbb{k}^1$. Thus $b_0(K) = 1$, $b_1(K) = 3$, and $b_i(K) = 0$ for all $i \geq 2$. Looking at the picture of K in Example 3.10, we can identify the three holes corresponding to $b_1(K) = 3$ and furthermore see that $b_i(K) = 0$ for $i > 1$. But what does $b_0(K) = 1$ mean? The value b_0 is counting the number of vertices that are not connected by a path; i.e., it is counting the number of components or whole "pieces" of K. Since K is connected, it makes sense that $b_0(K) = 1$.

The definitions here, especially of the boundary operators, are quite technical and tricky, but practicing with several examples should help to clarify.

Example 3.20. Let $K = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{12}, v_{13}, v_{24}, v_{34}, v_{25}, v_{45}, v_{56}, v_{245}\}$. Then K is given by



We follow the same method as above. Before we begin, look at the picture of K and see if you can predict what the Betti numbers will be. First we list each K_i:

$$\begin{aligned} K_2 &= \{v_{245}\}, \\ K_1 &= \{v_{12}, v_{13}, v_{24}, v_{34}, v_{25}, v_{45}, v_{56}\}, \\ K_0 &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}. \end{aligned}$$

This induces the chain complex

$$0 \xrightarrow{0} \Bbbk^1 \xrightarrow{\partial_2} \Bbbk^7 \xrightarrow{\partial_1} \Bbbk^7 \xrightarrow{\partial_0 = 0} 0.$$

We need to compute the maps ∂_2 and ∂_1. Now $\partial_2 : \Bbbk^1 \to \Bbbk^7$, so it is realized by a 7×1 matrix. By definition of ∂_2, we see that $\partial_2(v_{245}) = v_{45} + v_{25} + v_{24}$. Thus

$$\partial_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now $\partial_1 : \Bbbk^7 \to \Bbbk^7$ and hence is a 7×7 matrix. We have

$$\begin{aligned}
\partial_1(v_{12}) &= v_2 + v_1, \\
\partial_1(v_{13}) &= v_3 + v_1, \\
\partial_1(v_{24}) &= v_4 + v_2, \\
\partial_1(v_{34}) &= v_4 + v_3, \\
\partial_1(v_{25}) &= v_5 + v_2, \\
\partial_1(v_{45}) &= v_5 + v_4, \\
\partial_1(v_{56}) &= v_6 + v_5,
\end{aligned}$$

and hence

$$\partial_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now ∂_2 is a non-zero column vector, so that $\operatorname{rank}(\partial_2) = 1$. By Theorem 3.5, $\operatorname{null}(\partial_2) = 1 - 1 = 0$. Furthermore, ∂_1 row-reduces to 5 non-zero rows, so by Theorem 3.6, $\operatorname{rank}(\partial_1) = 5$. Again using Theorem 3.5, we see that $\operatorname{null}(\partial_1) = 7-5 = 2$. Finally, $\operatorname{null}(\partial_0) = 7$ and $\operatorname{rank}(\partial_0) = 0$. Putting these pieces together and using the definition of homology above, we see that

$$\begin{aligned}
H_2(K) &= \Bbbk^0 = 0, \\
H_1(K) &= \Bbbk^{2-1} = \Bbbk^1, \\
H_0(K) &= \Bbbk^{7-5} = \Bbbk^2.
\end{aligned}$$

The Betti numbers of K are thus given by $b_2(K) = 0, b_1(K) = 1$, and $b_0(K) = 2$. As we mentioned above, b_0 counts the number of components, and since the vertex v_7 is not connected to anything else, $b_0(K) = 2$ is appropriate.

Problem 3.21. Let $K = S^2$. Compute the Betti numbers of K.

Problem 3.22. Let $\partial_i : V_i \to V_{i-1}$, $i = 1, 2, \ldots$, be a collection of vector spaces and linear transformations. Prove that $\mathrm{Im}(\partial_{i+1}) \subseteq \ker(\partial_i)$ if and only if $\partial_i \circ \partial_{i+1} = 0$.

There is a very nice relationship between the Betti numbers and the Euler characteristic.

Theorem 3.23. Let K be a simplicial complex of dimension n. Then $\chi(K) = \sum_{i=0}^{n} (-1)^i b_i$.

Proof. Observe that

$$\begin{aligned}
\chi(K) &= \sum_{i=0}^{n}(-1)^i c_i \\
&= \sum_{i=0}^{n}(-1)^i[\mathrm{rank}(\partial_i) + \mathrm{null}(\partial_i)] \\
&= \mathrm{rank}(\partial_0) + \mathrm{null}(\partial_0) - \mathrm{rank}(\partial_1) - \mathrm{null}(\partial_1) + \cdots \\
&\quad + (-1)^n \,\mathrm{rank}(\partial_n) + (-1)^n \,\mathrm{null}(\partial_n) \\
&= 0 + b_0 - b_1 + \cdots + (-1)^{n-1} b_{n-1} + (-1)^n b_n \\
&= \sum_{i=0}^{n}(-1)^n b_i,
\end{aligned}$$

where the second equality is justified by the rank-nullity theorem. □

Problem 3.24. For all integers $i \geq 0$, compute $b_i(D)$, where D is the dunce cap in Example 1.22.

3.3. Invariance under collapses

We saw in Proposition 1.42 that if two simplicial complexes have the same simple homotopy type, then they have the same Euler characteristic. The Betti numbers of a simplicial complex likewise do not change under expansions and collapses. The main result of this section is the following:

Proposition 3.25. Let K be an n-dimensional simplicial complex. Suppose $K \searrow K'$ is an elementary collapse (so that $K' \nearrow K$). Then $b_d(K) = b_d(K')$ for all $d = 0, 1, 2, \dots$.

Before proving Proposition 3.25, we need another concept from linear algebra as well as a lemma.

Definition 3.26. Let U and U' be subspaces of a vector space V. The **sum** of vector spaces is defined by $U + U' := \{u + u' : u \in U, u' \in U'\}$. Furthermore, V is said to be the **direct sum** of U and U', written as $V = U \oplus U'$, if $V = U + U'$ and $U \cap U' = \{\vec{0}\}$. If $T : U \to V$ and $T' : U' \to V'$, define $T \oplus T' : U \oplus U' \to V \oplus V'$ by $(T \oplus T')(u + u') := T(u) + T(u')$.

The following is another purely linear algebra fact. We omit the proof.

Proposition 3.27. Let $T : U \to V$ and $T' : U' \to V'$. Then $\mathrm{null}(T \oplus T') = \mathrm{null}(T) + \mathrm{null}(T')$ and $\mathrm{rank}(T \oplus T') = \mathrm{rank}(T) + \mathrm{rank}(T')$.

Definition 3.28. Given a chain complex $(\mathbb{k}_*, \partial_*)$, another chain complex $(\mathbb{k}'_*, \partial_*)$ is a **subchain complex** if for every $n \geq 1$, $\mathbb{k}'_n$ is a vector subspace of $\mathbb{k}_n$ and $\partial_n(\mathbb{k}'_n) \subseteq \mathbb{k}'_{n-1}$.

A chain complex $(\mathbb{k}_*, \partial_*)$ is said to **split** into chain complexes $(\mathbb{k}'_*, \partial_*)$ and $(\mathbb{k}''_*, \partial_*)$ if $\mathbb{k}_i = \mathbb{k}'_i \oplus \mathbb{k}''_i$ for all i.

Lemma 3.29. Suppose that the chain complex $(\mathbb{k}_*, \partial_*)$ splits as $(\mathbb{k}'_*, \partial_*)$ and $(\mathbb{k}''_*, \partial)$. Then $b_i(\mathbb{k}_*) = b_i(\mathbb{k}'_*) + b_i(\mathbb{k}''_*)$

Problem 3.30. Prove Lemma 3.29.

We are now ready to prove Proposition 3.25.

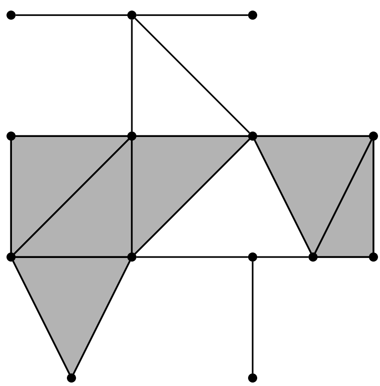
Proof of Proposition 3.25. Let K be a simplicial complex and suppose $\{\tau^{(d-1)}, \sigma^{(d)}\}$ is a free pair of K. Write $K \searrow K' := K - \{\tau, \sigma\}$. Denote by $(\mathbb{k}_*, \partial_*)$ and $(\mathbb{k}'_*, \partial_*)$ the chain complexes for K and K', respectively. Define a new chain complex $(\mathbb{k}'', \partial''_*)$ where $\mathbb{k}''_d$ is the vector space generated by σ, $\mathbb{k}''_{d-1}$ is the vector space generated by $\partial_d(\sigma)$, and all other vector spaces are the 0 vector space. Furthermore, $\partial''_d(\sigma) := \partial_d(\sigma)$, and all other boundary operators are clearly 0. We claim that $(\mathbb{k}_*, \partial_*) = (\mathbb{k}'_* \oplus \mathbb{k}''_*, \partial'_* \oplus \partial''_*)$. If so, the result follows by Lemma 3.29 since $b_i(\mathbb{k}''_*) = 0$

for all i. Now clearly $\mathbb{k}_i = \mathbb{k}'_i \oplus \mathbb{k}''_i$ for all $i \neq d-1$. It remains to show that $\mathbb{k}_{d-1} = \mathbb{k}'_{d-1} \oplus \mathbb{k}''_{d-1}$. Let $\alpha \in \mathbb{k}_{d-1}$ be a basis element. If $\alpha \in \mathbb{k}'_{d-1}$, we are done. Otherwise, $\alpha = \tau$. Write $\tau = \sigma_0 \cdots \hat{\sigma_j} \cdots \sigma_d$. Observe that $\tau = \left(\sum_{i\neq j} \sigma_0 \cdots \hat{\sigma_j} \cdots \hat{\sigma_i} \cdots \sigma_d\right) + \partial_d(\sigma) \in \mathbb{k}'_{d-1} + \langle \partial_d(\sigma)\rangle$ where $\langle \partial_d(\sigma)\rangle$ is the vector space generated by the set $\partial_d(\sigma)$. Thus $\mathbb{k}_{d-1} \subseteq \mathbb{k}'_{d-1} + \langle \partial_d(\sigma)\rangle$. Clearly we have inclusion the other way. Finally, since $\tau \notin \mathbb{k}'_{d-1}$, $\mathbb{k}'_{d-1} \cap \langle \partial_d(\sigma)\rangle = \{0\}$, and the result follows. □

Corollary 3.31. Let $K \sim L$. Then $b_i(K) = b_i(L)$ for every integer $i \geq 0$.

Problem 3.32. Prove Corollary 3.31, and then prove that if K is collapsible, then $b_i(K) = 0$ for all $i \geq 1$ and $b_0(K) = 1$.

Problem 3.33. Compute the Betti numbers of the following simplicial complex. [Hint: There is an easy way and a cumbersome way to do this.]



Problem 3.34. Complete the work you could not do in Problem 1.46. That is, show that $S^1 \not\sim S^3$.

Remark 3.35. As can be seen from your work in Problem 3.34, Corollary 3.31 is an excellent "in-theory" way to distinguish between simplicial complexes with different simple homotopy types. However, in principle it can be quite cumbersome, even impossible, to perform the computations by hand. The more simplices a complex has, the more difficult the computation of its Betti numbers will be. We will find a way to make computations of Betti numbers easier for certain cases in Section 8.4, and then we will carry out these computations in Section 8.5. We will also give an algorithm to perform these computations in Section 9.2.

Before completing this chapter, we prove another result about Betti numbers. The addition of a p-simplex will either increase b_p by 1 or decrease b_{p-1} by 1 (but not both). All other Betti numbers will be unaffected. This lemma will be utilized in Chapters 4 and 5.

Lemma 3.36. Let K be a simplicial complex and $\sigma^{(p)} \in K$ a p-dimensional facet of K, where $p \geq 1$. If $K' := K - \{\sigma\}$ is a simplicial complex, then one of the following holds:

(a) $b_p(K) = b_p(K') + 1$ and $b_{p-1}(K) = b_{p-1}(K')$;

(b) $b_{p-1}(K) + 1 = b_{p-1}(K')$ and $b_p(K) = b_p(K')$.

Furthermore, $b_i(K) = b_i(K')$ for all $i \neq p, p-1$.

Proof. Let $(\Bbbk_*, \partial_*)$ and $(\Bbbk'_*, \partial'_*)$ be the associated chain complexes for K and K', respectively. Since σ is a facet, it follows that $\partial_i = \partial'_i$ for all $i \neq p$. Hence $b_i(K) = b_i(K')$ for all i other than possibly $b_p = \text{null}(\partial_p) - \text{rank}(\partial_{p+1})$ and $b_{p-1} = \text{null}(\partial_{p-1}) - \text{rank}(\partial_p)$. We consider the cases where $\sigma \in \ker(\partial_p)$ and $\sigma \notin \ker(\partial_p)$.

Suppose that $\sigma \in \ker(\partial_p)$. Then $\text{Im}(\partial_p) = \text{Im}(\partial'_p)$ and hence $\text{rank}(\partial_p) = \text{rank}(\partial'_p)$. Since $\sigma \notin k'_*$, clearly $\sigma \notin \ker(\partial'_p)$ so that $\text{null}(\partial_p) = 1 + \text{null}(\partial'_p)$; hence

$$\begin{aligned} b_p(K) &= \text{null}(\partial_p) - \text{rank}(\partial_{p+1}) \\ &= 1 + \text{null}(\partial'_p) - \text{rank}(\partial_{p+1}) \\ &= 1 + b_p(K') \end{aligned}$$

and $b_{p-1}(K) = \text{null}(\partial_{p-1}) - \text{rank}(\partial_p) = \text{null}(\partial'_{p-1}) - \text{rank}(\partial'_p) = b_{p-1}(K')$.

Now suppose that $\sigma \notin \ker(\partial_p)$, so that $0 \neq \partial_p(\sigma) \in \text{Im}(\partial_p)$. Then $\ker(\partial_p) = \ker(\partial'_p)$ so that $b_p(K) = \text{null}(\partial_p) - \text{rank}(\partial_p) = \text{null}(\partial'_p) - \text{rank}(\partial'_p) = b_p(K')$. Furthermore, since $\partial_p(\sigma)$ is a nontrivial element of $\text{Im}(\partial_p)$ and σ is a basis element, $\text{rank}(\partial_p) = \text{rank}(\partial'_p) + 1$. Hence $b_{p-1}(K) = \text{null}(\partial_{p-1}) - \text{rank}(\partial_p) = \text{null}(\partial'_{p-1}) - \text{rank}(\partial'_p) - 1 = b_{p-1}(K') - 1$. $\square$

Problem 3.37. Using the above lemma and your knowledge of $b_i(\Delta^{n+1})$, compute $b_i(S^n)$ for every i. Conclude that if $n \neq m$, then S^n and S^m do not have the same simple homotopy type.

Problem 3.38. Let $n \geq 0$ be an integer and let $\sigma^{(n)} \in S^n$ be any simplex of dimension n. Compute $b_i(S^n - \{\sigma\})$ for all $i \geq 0$.

Chapter 4

Main theorems of discrete Morse theory

With much of the technical linear algebra machinery now behind us, we devote this chapter to two of the most utilized results in discrete Morse theory. These are the discrete Morse inequalities (Theorems 4.1 and 4.4) and the collapse theorem (Theorem 4.27). In addition to these two results, several other topics, such as perfect discrete Morse functions and level subcomplexes, are discussed.

4.1. Discrete Morse inequalities

As we have hinted, there is a strong relationship between the Betti numbers of a simplicial complex and the number of critical simplices of any discrete Morse function on that same complex. This relationship is observed in the weak discrete Morse inequalities, which we are now able to prove.

Theorem 4.1 (Weak discrete Morse inequalities)**.** Let $f : K \to \mathbb{R}$ be a discrete Morse function with m_i critical values in dimension i for $i = 0, 1, 2, \ldots, n := \dim(K)$. Then

(i) $b_i \leq m_i$ for all $i = 0, 1, \ldots, n$ and

(ii) $\sum_{i=0}^{n}(-1)^i m_i = \chi(K)$.

Proof. We prove only the first part, as the second part is Problem 4.2. Because it does not affect the values of m_i, we may assume without loss of generality that f is excellent by Lemma 2.33. We proceed by (strong) induction on ℓ, the number of simplices of K. For $\ell = 1$, the only simplicial complex with one simplex is $K = *$. By Problem 3.32, $b_0(K) = 1$ and $b_i(K) = 0$ for all $i \geq 1$. Furthermore, $m_0 \geq 1$ for any discrete Morse function on K by Problem 2.26. Thus the base case is shown.

Assume the inductive hypothesis that there is an $\ell \geq 1$ such that for every simplicial complex with $1 \leq j \leq \ell$ simplices, any discrete Morse function satisfies $m_i \leq b_i$. Suppose K is any simplicial complex with $\ell+1$ simplices and $f : K \to \mathbb{R}$ is an (excellent) discrete Morse function. Now consider $\max\{f\}$. If this is a critical value with corresponding (unique) critical p-simplex σ, we may consider $K' := K - \{\sigma\}$ and the function $f' = f|_{K'} : K' \to \mathbb{R}$. Clearly f' is a discrete Morse function satisfying $m_p(K') + 1 = m_p(K)$. Furthermore, by Lemma 3.36, removal of this critical simplex σ results in either $b_p(K) = b_p(K') + 1$ or $b_{p-1}(K) + 1 = b_{p-1}(K')$, while $b_i(K) = b_i(K')$ for all other values. Supposing the former, since K' has $\ell - 1$ simplices, it satisfies $b_i(K') \leq m_i(K')$ by the inductive hypothesis. We thus have

$$b_p - 1 = b_p(K') \leq m_p(K') = m_p(K) - 1,$$

which is the desired result. The case where $b_{p-1}(K) + 1 = b_{p-1}(K')$ is similar.

Otherwise, if σ is not critical, then σ is a regular simplex and hence part of a free pair. Removal of the free pair is an elementary collapse, and by Corollary 3.31 the resulting complex has the same Betti numbers, so by the inductive hypothesis we have that $b_i(K) = b_i(K') \leq m_i(K') = m_i(K)$ for all i. □

Problem 4.2. Prove the second part of Theorem 4.1.

4.1.1. Strong discrete Morse inequalities (optional). As the name "weak" discrete Morse inequalities implies, there are also the strong discrete Morse inequalities. Unfortunately, in order to prove them, we need theorems from homotopy theory which use techniques beyond the scope of this work.

We will state the needed results in detail using all the technical terms in Theorem 4.3 and then discuss how we will use it. The interested reader may find a proof of (i) in [**65**, Corollary 3.5], of (ii) in [**137**, Corollary 4.24], and of (iii) in [**116**, pp. 28–30].

Theorem 4.3. Let K be an n-dimensional simplicial complex with c_i simplices of dimension i, and let $f : K \to \mathbb{R}$ be a discrete Morse function. Then

(i) K is homotopy equivalent to a CW complex X where the p-cells of X are in bijective correspondence with the set of critical p-simplices of f;

(ii) $b_i(X) = b_i(K)$ for all $i = 0, 1, \ldots$;

(iii) for each $p = 0, 1, 2, \ldots, n, n+1, \ldots$ we have

$$b_p - b_{p-1} + b_{p-2} - \cdots + (-1)^p b_0 \leq c_p - c_{p-1} + c_{d-2} - \cdots + (-1)^p c_0.$$

Recall that c_i is the number of i-simplices of K, while m_i is the number of critical i-simplices of f. Theorem 4.3 says that we may replace K with a different structure (called a CW complex) in such a way that the Betti numbers do not change. See [**84, 113**] for the basics of CW complexes. The upshot of this theorem is that we may assume that given a discrete Morse function on K, we have $c_i = m_i$ for all i. Of course, this is in general impossible if we restrict ourselves to simplicial complexes. But because of the theorem, we may assume that $c_i = m_i$ and that this does not affect the Betti numbers.

Theorem 4.4 (Strong discrete Morse inequalities)**.** Let $f : K \to \mathbb{R}$ be a discrete Morse function. For each $p = 0, 1, \ldots, n, n+1$, we have

$$b_p - b_{p-1} + \cdots + (-1)^p b_0 \leq m_p - m_{p-1} + \cdots + (-1)^p m_0.$$

Proof. By Theorem 4.3, there is a CW complex X with p-cells in bijective correspondence to critical p-simplices of f. By the same theorem, we have

$$\begin{aligned} & b_p(K) - b_{p-1}(K) + \cdots + (-1)^p b_0(K) \\ = \quad & b_p(X) - b_{p-1}(X) + \cdots + (-1)^p b_0(X) \\ \leq \quad & c_p - c_{p-1} + c_{p-2} - \cdots + (-1)^p c_0 \\ = \quad & m_p - m_{p-1} + m_{d-2} - \cdots + (-1)^p m_0. \quad \square \end{aligned}$$

Problem 4.5. Use the strong discrete Morse inequalities (Theorem 4.4) to prove the weak discrete Morse inequalities (Theorem 4.1).

4.1.2. Perfect discrete Morse functions. Given Theorem 4.1, it is reasonable to ask if and when equality is ever obtained. Let K be an n-dimensional simplicial complex. Recall that if $f : K \to \mathbb{R}$ is a discrete Morse function with m_i critical simplices of dimension i, then the discrete Morse vector of f is defined by $\vec{f} = (m_0, m_1, \ldots, m_n)$.

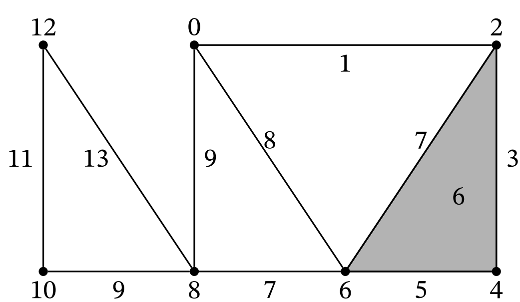
Definition 4.6. A discrete Morse vector is called **perfect** if $\vec{f} = (b_0, b_1, \ldots, b_n)$.

Problem 4.7. Let $f : K \to \mathbb{R}$ be a discrete Morse function with $\vec{f}$ a perfect discrete Morse vector. Show that

(i) $\vec{f}$ is unique;

(ii) $\vec{f}$ is optimal.

Example 4.8. Recall that in Example 2.88, we defined the discrete Morse function $g : K \to \mathbb{R}$ by



which had discrete Morse vector $\vec{g} = (1, 3, 0)$. If we compute the Betti numbers of this complex using homology, we see that $b_0 = 1, b_1 = 3$, and $b_2 = 0$. Hence $\vec{g}$ is not only optimal but also unique by Problem 4.7.

Problem 4.9. Prove that there exists a perfect discrete Morse function on Δ^n and S^n for all n.

Does every simplicial complex admit a perfect discrete Morse function? The following result was shown by Ayala et al. [**13**].

Proposition 4.10. Let K be a simplicial complex with $b_0(K) = 1$ and $b_i(K) = 0$ for all $i > 0$. If K is not collapsible, then K does not admit a perfect discrete Morse function.

We postpone the proof of this result until Section 4.2.1. It will then follow as an easy corollary.

Example 4.11. Let D be the dunce cap. In Problem 3.24 you showed that $b_0(D) = 1$ and $b_i(D) = 0$ for all $i > 0$. In addition, D is clearly not collapsible, as it has no free faces. By Proposition 4.10, D does not admit a perfect discrete Morse function.

In another paper [**14**], the same authors found examples of higher-dimensional simplicial complexes which do not admit perfect discrete Morse functions. Although Example 4.11 demonstrates the existence of a simplicial complex which does not admit a perfect discrete Morse function, many classes of simplicial complexes do admit perfect discrete Morse functions. The following problems ask you to show this is some specific cases.

Problem 4.12. Let K be the simplicial complex consisting of n isolated points, i.e., $K := \{v_0, v_1, \ldots, v_{n-1}\}$.

(i) Compute the Betti numbers of K.

(ii) Prove that every discrete Morse function on K is perfect.

Problem 4.13. Let G be a 1-dimensional simplicial complex with $b_0(G) = 1$.

(i) Prove that there are b_1 edges that may be removed from G so that the resulting graph is a tree.

(ii) Prove that there exists a perfect discrete Morse function on the resulting tree.

(iii) Now prove that there exists a perfect discrete Morse function on G.

In addition to these examples, Adiprasito and Benedetti [**3**] have shown that a certain class of 3-dimensional simplicial complexes admits perfect discrete Morse functions.

4.1.3. Towards optimal perfection. This section has raised the question of how we can improve a given discrete Morse function to obtain a better one, that is, one with fewer critical simplices and hence one step closer to optimality or perfection (if the latter exists). One method is that of **canceling critical simplices**. This method allows us to "extend" a given gradient vector field to a larger one. In order to do this, we will show how to "deform" one discrete Morse function into another. The ideas in this section are attributable to Forman [**65**] and presented in the fashion of [**150**, III.4]. We will use the method of canceling critical simplices in an algorithm in Section 9.1.

Definition 4.14. A discrete Morse function f is called **flat** if whenever (σ, τ) is a regular pair of f, we have $f(\sigma) = f(\tau)$.

Exercise 4.15. Show that every basic discrete Morse function is a flat discrete Morse function. Give an example of a flat discrete Morse function which is not basic.

The following proposition tells us that we may transform any discrete Morse function into a flat one, and the result will be Forman equivalent to the original.

Proposition 4.16. Let $f\colon K \to \mathbb{R}$ be a discrete Morse function and V_f the induced gradient vector field. Then there exists a flat discrete Morse function $g\colon K \to \mathbb{R}$ such that f and g are Forman equivalent.

Proof. Let $f\colon K \to \mathbb{R}$ be a discrete Morse function, and define a **flattening** of f by

$$g(\sigma) := \begin{cases} f(\tau) & \text{if } (\sigma,\tau) \in V_f \text{ for some } \tau, \\ f(\sigma) & \text{otherwise.} \end{cases}$$

Observe that for any $\sigma \in K$, $g(\sigma) \leq f(\sigma)$. Let c be a critical value of f. Then there is a p-simplex $\tau \in K$ with $f(\tau) = c$ such that for all $\sigma^{(p-1)} < \tau < \eta^{(p+1)}$ we have that $f(\sigma) < f(\tau) < f(\eta)$. Since τ is critical, $g(\tau) = f(\tau)$ so that $g(\sigma) \leq f(\sigma) < f(\tau) = g(\tau)$. It remains to show that $g(\tau) < g(\eta)$. If η is not the tail of a vector in V_f, then $g(\eta) = f(\eta)$, hence the result. Thus, suppose that there is $\gamma^{(p+2)} > \eta$ such that $f(\eta) \geq f(\gamma)$, and, by contradiction, suppose that $f(\tau) \geq f(\gamma)$. But Problem 4.17 implies that γ is not critical, which is a contradiction. Clearly g does not have any critical simplices that f does not, since the regular pairs of f remain regular pairs of g. Because furthermore no new regular pairs are introduced, $V_f = V_g$ and so, by Theorem 2.53, f and g are Forman equivalent. □

Problem 4.17. Let K be a simplicial complex and suppose that there exist simplices $\sigma^{(p)} < \tau^{(p+2)}$ such that $f(\sigma) \geq f(\tau)$. Prove that both σ and τ are not critical.

We can now define a family of discrete Morse functions on K that smoothly deforms a given flat discrete Morse function into another one. Generally, a deformation of one object into another is called a **homotopy**. We saw one version of "deformation" in Section 1.2, where we deformed one simplicial complex into another through a series of collapses and expansions. We started with one simplicial complex K and, through a series of intermediary simplicial complexes, obtained the simplicial complex L. Now we want to take this idea and extend it to flat discrete Morse functions; that is, given flat discrete Morse functions f and g, we want to start with f and perform a series of "deformations" to obtain intermediary flat discrete Morse functions that ultimately end with g.

Lemma 4.18. Let $f, g\colon K \to \mathbb{R}$ be flat discrete Morse functions, and define $h_t(\sigma) := (1-t)f(\sigma) + tg(\sigma)$ for all $\sigma \in K$ and $t \in [0,1]$. Then h_t is a discrete Morse function on K for all $t \in [0,1]$. Furthermore, for every $t \in (0,1)$ we have that $V_{h_t} = V_f \cap V_g$. In particular, all V_{h_t} are Forman equivalent.

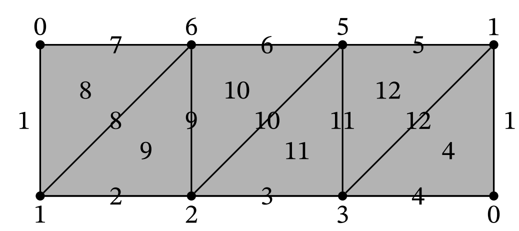
The function f_t is a standard construction in homotopy theory known as the **straight-line homotopy**.

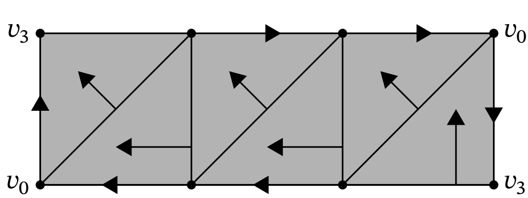
Proof. Assume without loss of generality that $f, g \colon K \to \mathbb{R}^{>0}$. That h_t is a discrete Morse function is Problem 4.19. Let $\sigma < \tau$. We use subset inclusion to show that $V_{h_t} = V_f \cap V_g$ for all $t \in (0,1)$. Let $(\sigma, \tau) \in V_{h_t}$. Then $\sigma^{(p)} < \tau^{(p+1)}$ and $h_t(\sigma) = h_t(\tau)$ using the fact that h_t is flat. Since it is always the case that $\sigma^{(p)} < \tau^{(p+1)}$, we only need to show that $f(\sigma) = f(\tau)$ and $g(\sigma) = g(\tau)$. Now since $\sigma^{(p)} < \tau^{(p+1)}$, $f(\sigma) \not> f(\tau)$ and $g(\sigma) \not> g(\tau)$ by definition of f and g being flat. Hence $f(\sigma) \leq f(\tau)$ and $g(\sigma) \leq g(\tau)$. Suppose by contradiction that at least one of $f(\sigma) < f(\tau)$ and $g(\sigma) < g(\tau)$ holds. Then $h_t(\sigma) = (1-t)f(\sigma) + tg(\sigma) < (1-t)f(\tau) + tg(\tau) = h_t(\tau)$, a contradiction. Thus $f(\sigma) = f(\tau)$ and $g(\sigma) = g(\tau)$, so that $(\sigma, \tau) \in V_f, V_g$.

Now we show that if $(\sigma, \tau) \in V_f \cap V_g$, then $(\sigma, \tau) \in V_{h_t}$. Since $(\sigma, \tau) \in V_f \cap V_g$ and f and g are flat, $f(\sigma) = f(\tau) = g(\sigma) = g(\tau)$; hence it easily follows that $h_t(\sigma) = (1-t)f(\sigma) + tg(\sigma) = (1-t)f(\tau) + tg(\tau) = h_t(\tau)$. We conclude that $V_{h_t} = V_f \cap V_g$ and all V_{h_t} are Forman equivalent. $\square$

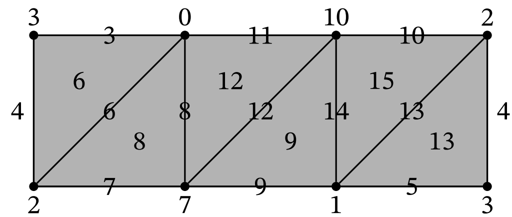
Problem 4.19. Prove that the function h_t defined in Lemma 4.18 is a discrete Morse function.

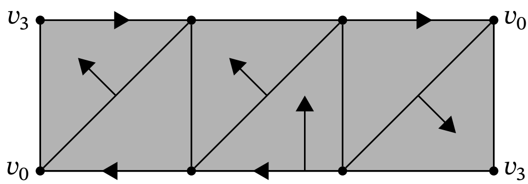
Example 4.20. To illustrate Lemma 4.18, we will find a homotopy between two very different discrete Morse functions on the Möbius band M. The discrete Morse function f along with its induced gradient vector field is shown below.



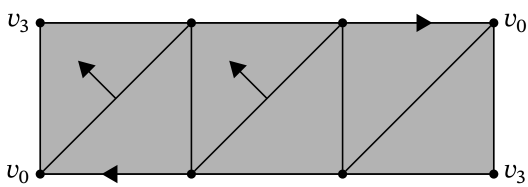


Let g along with its gradient vector field be as follows:





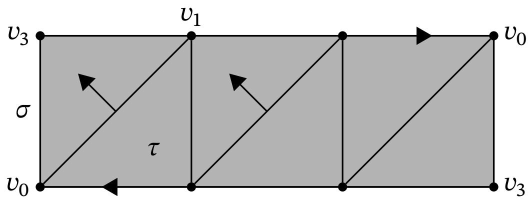
As in Lemma 4.18, the value of a simplex σ for any $t \in (0,1)$ is given by $(1-t)f(\sigma)+tg(\sigma)$. It can then be checked that when M is given these labels, the resulting gradient vector field is



which is precisely $V_f \cap V_g$.

Recall that given a gradient vector field V on K, a V-path γ is a sequence of simplices $\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \dots, \beta_k^{(p+1)}, \alpha_{k+1}^{(p)}$ in V such that $(\alpha_i^{(p)}, \beta_i^{(p+1)}) \in V$ for $0 \le i \le k$ and $\beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}$. Furthermore, we may view a gradient vector field V as being induced by a discrete Morse function f or abstractly as a discrete vector field with no closed V-paths (Theorem 2.51).

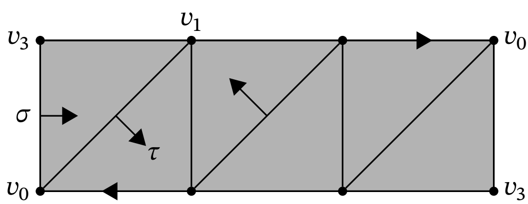
Example 4.21. Let us again consider the gradient vector field $V_f \cap V_g$ found in Example 4.20, with σ and τ labeled as follows:



Now both σ and τ are critical and there is furthermore a path

$$\tau, v_0v_1, v_0v_1v_3, \sigma$$

between them. We can extend this path and simultaneously eliminate these critical simplices by reversing the arrows of the path as follows:



In general, given the right set-up, it is always possible to reverse arrows in a path, thereby turning two critical simplices into non-critical simplices. The precise meaning of this statement is given in Proposition 4.22 below.

Proposition 4.22 (Canceling critical simplices)**.** Let V be a gradient vector field on K, and suppose that there are two critical simplices $\sigma^{(p)} = \sigma$ and $\tau^{(p+1)} = \tau$ with the property that there exists a unique V-path $\gamma := [\gamma_0^{(p)}, \tau_0^{(p+1)}, \gamma_1, \dots, \gamma_{n-1}^{(p)}, \tau_{n-1}^{(p+1)}, \gamma_n = \sigma]$ where $\gamma_0^{(p)} < \tau$. Define $\bar{V}$ to satisfy the following three properties:

(a) $\bar{V} - \gamma = V - \gamma$;

(b) $(\gamma_0, \tau) \in \bar{V}$;

(c) $(\gamma_{i+1}, \tau_i) \in \bar{V}$ for $i = 0, \dots, n-1$.

Then $\bar{V}$ is a gradient vector field. Moreover, there exists a unique $\bar{V}$-path from σ to γ_0.

Proof. First, observe that by construction, the critical simplices of V are exactly the critical simplices of $\bar{V}$ other than τ and σ. It is clear that $\bar{V}$ is a discrete vector field. By Theorem 2.51, it remains to show that $\bar{V}$ does not contain any closed V-paths. By (a), $\bar{V}$ and V differ only on γ so that $\bar{V}$ cannot contain a closed V-path on $\bar{V}-\gamma$ (otherwise it would also be a closed V-path in V). Hence if $\bar{V}$ does have a closed path, it must contain a segment $\gamma_i, \delta_0, \dots, \delta_r, \gamma_j$ where $\delta_k \notin \gamma$. Since $(\gamma_{i-1}, \tau_{i-1}) \in V$ and $(\gamma_i, \tau_{i-1}) \in \bar{V}$, we have that $\gamma_0, \dots, \gamma_{i-1}, \delta_0, \dots, \delta_r, \gamma_j, \dots, \gamma_n$ is a V-path from τ to σ, contradicting the fact that γ is unique.

To see that the path $\sigma = \gamma_n, \tau_{n-1}, \dots, \gamma_0$ is the unique $\bar{V}$-path between σ and γ_0, suppose there is another such $\bar{V}$-path. Then, as above, it must contain a segment of the form $\gamma_i, \epsilon_0, \dots, \epsilon_\ell, \gamma_j$ with $\epsilon_k \notin \gamma$ and $i < j$ (otherwise $\gamma_j, \dots, \epsilon_0, \dots, \epsilon_\ell, \gamma_i, \dots, \gamma_j$ would be a closed $\bar{V}$-path). But now $\epsilon_0, \dots, \epsilon_\ell, \gamma_j, \gamma_{j+1}, \dots, \gamma_{i-1}, \epsilon_0$ is a closed V-path, a contradiction. Hence the $\bar{V}$-path between σ and γ_0 is unique. □

The method of Proposition 4.22 is known as **canceling critical simplices**, and it is much easier to understand than the proposition lets on. The criterion for detecting when canceling is possible is the existence of a unique V-path between critical simplices. Then you simply reverse the directions of the arrows in the V-path, add one extra arrow, and voilá—one fewer critical simplex! Remember that it is necessary that the V-path be unique, as Problem 4.23 illustrates.

Problem 4.23. Why can we not cancel two critical simplices if there is more than one V-path between them? Give an example.

4.2. The collapse theorem

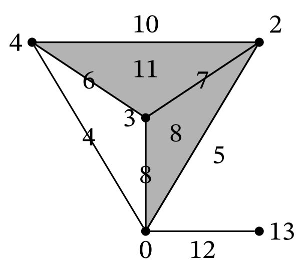
Another foundational result with many applications is the collapse theorem. At the beginning of Chapter 2, we saw how a collection of arrows on a simplicial complex could be thought of as encoding a sequence of collapses. Sometimes, after a sequence of collapses, we would get stuck and have to "rip out" a simplex before we could continue collapsing. Given a discrete Morse function, the collapse theorem tells us when we can perform these collapses and when we will "get stuck." Before we state and prove this theorem, we introduce level subcomplexes. In addition to giving us a language to make the collapse theorem precise, level subcomplexes allow us to define a new notion of equivalence of discrete Morse functions in Section 5.1.1.

4.2.1. Level subcomplexes. Let $f: K \to \mathbb{R}$ be a discrete Morse function. For any $c \in \mathbb{R}$, the **level subcomplex** $K(c)$ is the subcomplex of K consisting of all simplices τ with $f(\tau) \leq c$, as well as their faces; i.e.,

$$K(c) = \bigcup_{f(\tau) \leq c} \bigcup_{\sigma \leq \tau} \sigma.$$

Usually we are interested in studying the level subcomplexes induced by the critical values of a discrete Morse function, as the following example illustrates.

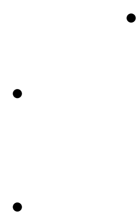
Example 4.24. Let K be the simplicial complex with discrete Morse function given below:



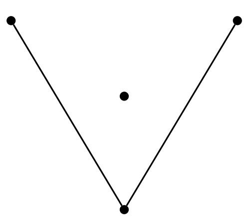
We are interested in the level subcomplexes induced by critical values. The critical values in increasing order are easily seen to be $0, 2, 3, 5, 6, 7, 10$, and 11. Each of these critical values induces a level subcomplex. Think of them as building K in stages. The level subcomplex $K(0)$ is everything labeled 0 or less:

i.e., a single vertex. The level subcomplex $K(2)$ is almost as uninteresting, as it consists of only two isolated vertices:

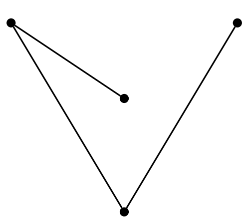
Next is level subcomplex $K(3)$:



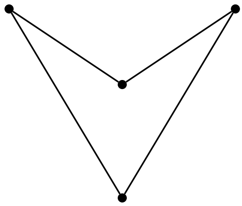
Now for $K(5)$:



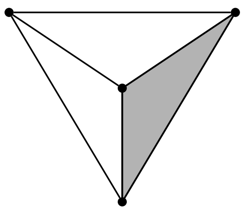
On to $K(6)$:



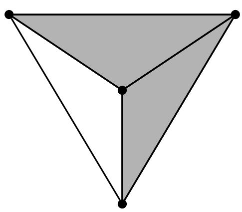
The level subcomplex $K(7)$ then completes a cycle:



We see a 2-simplex come in at $K(10)$:



We finish with $K(11)$:



The following lemma tells us that given a discrete Morse function f, we may perturb f slightly to make it 1–1 without changing a couple of specified level subcomplexes.

Lemma 4.25. Let $f : K \to \mathbb{R}$ be a discrete Morse function and $[a, b] \subseteq \mathbb{R}$ an interval that contains no critical values. Then there is a discrete Morse function $f' : K \to \mathbb{R}$ satisfying the following properties:

(i) f' is 1–1 on $[a, b]$;

(ii) f' has no critical values in $[a, b]$;

(iii) $K_f(b) = K_{f'}(b)$ and $K_f(a) = K_{f'}(a)$;

(iv) $f = f'$ outside of $[a, b]$.

Problem 4.26. Prove Lemma 4.25.

The next theorem tells us that there is "nothing interesting" (topologically speaking) happening in between level subcomplexes. In other words, we are justified in only considering the level subcomplexes induced by the critical values (as opposed to regular values or values not in the range of the discrete Morse function).

Theorem 4.27 (Collapse theorem)**.** Let $f : K \to \mathbb{R}$ be a discrete Morse function and $[a, b] \subseteq \mathbb{R}$ an interval that contains no critical values. Then $K(b) \searrow K(a)$.

Proof. Applying Lemma 4.25 and with an abuse of notation, we may assume that f is 1–1. If $f(\sigma) \notin [a, b]$ for all $\sigma \in K$, then $K(a) = K(b)$ and we are done. Otherwise, since f has discrete image and was assumed to be 1–1, we may break $[a, b]$ up into subintervals such that each interval contains exactly one regular value. Again, with an abuse of notation, we will assume σ is the simplex such that $f(\sigma)$ is the unique regular value of f in $[a, b]$. By Lemma 2.24, exactly one of the following holds:

- there exists $\tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$;
- there exists $\nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$.

For the second case, suppose that there exists $\nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$. We claim that $\{\sigma, \nu\}$ is a free pair in $K(b)$. Suppose to the contrary that there exists a second coface $\tilde{\sigma}^{(p)} > \nu$ with $\tilde{\sigma} \in K(b)$. Because $f(\nu) \geq f(\sigma)$ and f is a discrete Morse function, $f(\nu) < f(\tilde{\sigma})$. By definition of $\tilde{\sigma} \in K(b)$, we have that either $f(\tilde{\sigma}) \leq b$ or there exists $\alpha > \tilde{\sigma}$ such that $f(\alpha) \leq b$. If $f(\tilde{\sigma}) \leq b$, then $a \leq f(\sigma) \leq f(\nu) < f(\tilde{\sigma}) \leq b$. Since $f(\tilde{\sigma})$ cannot be a critical value by hypothesis, $f(\tilde{\sigma})$ must be a regular value, contradicting the supposition that $f(\sigma)$ is the only regular value in $[a, b]$. The same argument shows that such an α would also yield an additional regular value in $[a, b]$. Thus $\{\sigma, \nu\}$ is a free pair in $K(b)$, so $K(b) \searrow K(b) - \{\sigma, \nu\}$ is an elementary collapse. Doing this over the subintervals, we see that $K(b) \searrow K(a)$. The first case is identical. $\square$

Problem 4.28. Let $f : K \to \mathbb{R}$ be a discrete Morse function with exactly one critical simplex. Prove that K is collapsible. (Note that this is the converse of Problem 2.34.)

We may now prove Proposition 4.10, which says that there exist simplicial complexes that do not admit perfect discrete Morse functions.

Proof of Proposition 4.10. If K admits a perfect discrete Morse vector $\vec{f}$, then by definition $\vec{f} = (1, 0, 0, \dots, 0)$. By Problem 4.28, K being not collapsible implies that any discrete Morse vector on K has at least two critical values. Thus K cannot admit a perfect discrete Morse vector. $\square$

We may also generalize Theorem 4.27 to the case where f is a generalized discrete Morse function as in Section 2.2.3. We first state and prove an easy lemma.

Lemma 4.29. For every generalized discrete vector field, there is a (standard) discrete vector field that refines every non-singular, non-empty interval into pairs.

Proof. Let $[\alpha, \beta]$ be a non-singular and non-empty interval. Then $\alpha < \beta$. Hence, choose a vertex $v \in \beta - \alpha$ and for every $\gamma \in [\alpha, \beta]$ partition $[\alpha, \beta]$ into the pairs $\{\gamma - \{v\}, \gamma \cup \{v\}\}$. $\square$

The technique employed in the proof of Lemma 4.29 is known as a **vertex refinement** of the partition. This is simply a way to break each interval of a generalized discrete vector field into elementary collapses.

The following corollary is immediate from Lemma 4.29 and Theorem 4.27.

Corollary 4.30 (Generalized collapse theorem)**.** Let K be a simplicial complex with generalized discrete vector field V, and let $K' \subseteq K$ be a subcomplex. If $K - K'$ is a union of non-singular intervals in V, then $K \searrow K'$.

Exercise 4.31. Give an example to show that in general the vertex refinement of Lemma 4.29 need not be unique.

Chapter 5

Discrete Morse theory and persistent homology

This chapter introduces persistent homology, a powerful computational tool with a wealth of applications [**40**, **44**, **141**, **149**], including many in data analysis [**133**]. Persistent homology was originally introduced by Edelsbrunner, Letscher, and Zomorodian [**56**] in 2002, although certain ideas in persistent homology can be found earlier [**73**], even in the work of Morse himself [**125**].

In Section 5.1, we compute persistent homology without worrying about its theoretical foundations. For those who would like to delve more deeply into the theoretical relationship between discrete Morse theory and persistent homology, we offer Section 5.2.4, where we follow U. Bauer's doctoral thesis [**24**] by using discrete Morse theory to develop a theoretical framework for persistent homology.

5.1. Persistence with discrete Morse functions

Before we can begin performing persistent homology computations, we introduce a new notion of equivalence of discrete Morse functions. We saw in Section 2.1 one such notion of equivalence of discrete Morse functions, defined in terms of their induced gradient vector field. Another notion of equivalence is based on the homology of the induced level subcomplexes.

5.1.1. Homological equivalence.

Example 5.1. Consider the discrete Morse function f in Example 4.24. This is an excellent discrete Morse function with critical values $0, 2, 3, 5, 6, 7, 10$, and 11. For each of the level subcomplexes $K(0), K(2), K(3), K(5), K(6), K(7), K(10)$, and $K(11)$, recording the corresponding Betti numbers yields the following **homological sequence**:

$$\begin{array}{rcccccccc} B_0: & 1 & 2 & 3 & 2 & 1 & 1 & 1 & 1 \\ B_1: & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ B_2: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

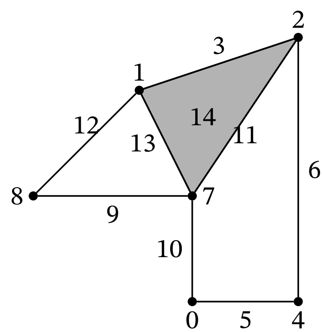
Notice that only one value changes when moving from column to column and that the last column is the homology of the original simplex K even though $K \neq K(11)$. These observations and others are true of the homological sequence of any excellent discrete Morse function. We prove this in Theorem 5.9.

Definition 5.2. Let f be a discrete Morse function with m critical values on an n-dimensional simplicial complex K. The **homological sequence of** f is given by the $n + 1$ functions

$$B_0^f, B_1^f, \dots, B_n^f : \{0, 1, \dots, m-1\} \to \mathbb{N} \cup \{0\}$$

defined by $B_k^f(i) := b_k(K(c_i))$ for all $0 \leq k \leq n$ and $0 \leq i \leq m - 1$. We usually write $B_k(i)$ for $B_k^f(i)$ when the discrete Morse function f is clear from the context.

Problem 5.3. Let $f : K \to \mathbb{R}$ be the discrete Morse function given by

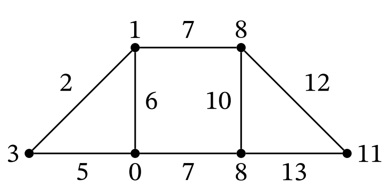


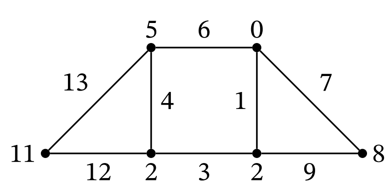
Find the homological sequence of f.

Exercise 5.4. Suppose $f : K \to \mathbb{R}$ is a perfect discrete Morse function. Prove that for any k, $B_k(i) \leq B_k(i+1)$ for all $0 \leq i \leq m-2$.

Definition 5.5. Two discrete Morse functions $f, g : K \to \mathbb{R}$ with m critical values are **homologically equivalent** if $B_k^f(i) = B_k^g(i)$ for all $0 \leq k \leq m-1$ and $0 \leq i$.

Example 5.6. Consider the two discrete Morse functions f and g on the complex K shown respectively on the left and right below.

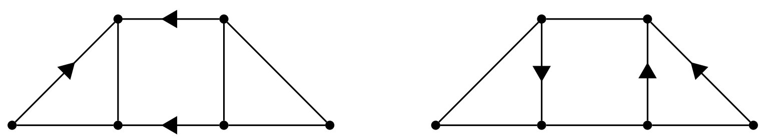




It is an easy exercise to write down the level subcomplexes and compute that both discrete Morse functions have the homological sequence

$$\begin{array}{cccccccccc} B_0: & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ B_1: & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{array}$$

Hence f and g are homologically equivalent. However, if we pass to their gradient vector fields



we see that $V_f \neq V_g$ and hence, by Proposition 2.53, f and g are not Forman equivalent.

Exercise 5.7. Prove that if f is a discrete Morse function, the flattening g of f is homologically equivalent to f. See the proof of Proposition 4.16 for the definition of the flattening.

Problem 5.8. Let K be a simplicial complex with excellent discrete Morse function f and suppose that a is the minimum value of f. Prove that there exists a unique critical 0-simplex σ such that $f(\sigma) = a$.

Homologically equivalent discrete Morse functions were first introduced and studied by Ayala et al. [**6**, **12**] in the context of graphs with infinite rays. They were further studied for orientable surfaces [**11**] and for 2-dimensional collapsible complexes [**5**]. There is a version for persistent homology [**112**] and for graph isomorphisms [**1**].

Excellent discrete Morse functions are particularity well behaved.

Theorem 5.9. Let f be an excellent discrete Morse function on a connected n-dimensional simplicial complex K with m critical values $c_0, c_1, \dots, c_{m-1}$. Then each of the following holds:

(i) $B_0(0) = B_0(m-1) = 1$ and $B_d(0) = 0$ for all $d \in \mathbb{Z}^{\geq 1}$.

(ii) For all $0 \leq i < m-1$, $|B_d(i+1) - B_d(i)| = 0$ or 1 whenever $0 \leq d \leq n$ and $B_d(i) = 0$ whenever $d \geq n+1$.

(iii) $B_d(m-1) = b_d(K)$ for all $d \in \mathbb{Z}^{\geq 0}$.

(iv) For each $i = 0, 1, \dots, m-2$ and all $p \geq 1$, either
$B_{p-1}(i) = B_{p-1}(i+1)$ or
$B_p(i) = B_p(i+1)$.
In either case, $B_d(i) = B_d(i+1)$ for any $d \neq p, p-1$ and $1 \leq d \leq n$.

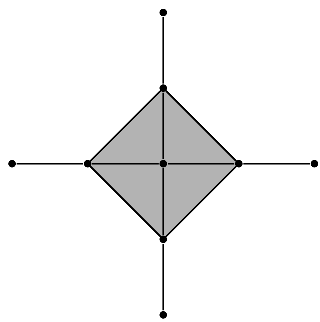
Proof. We proceed in order. For (i), choose $y \in \mathbb{N}$ such that $K(c_{m-1} + y) = K$. By Theorem 4.27, $b_0(K_{c_{m-1}}) = b_0(K(c_{m-1} + y)) = b_0(K)$. Since K is connected, $b_0(K(c_{m-1})) = B_0(m-1) = 1$. By Problem 5.8, $K(0)$ consists of a single 0-simplex. Thus $B_d(0) = 0$ for all $d \in \mathbb{Z}^{\geq 1}$. This proves the first assertion.

For (ii), we note that by Theorem 4.27, $b_d(K(c_i)) = b_d(K(x))$ for any $x \in [c_i, c_{i+1})$. Since f is excellent, there exists $\epsilon > 0$ such that $K(c_{i+1}) = K(c_{i+1} - \epsilon) \cup \sigma^{(p)}$ where $\sigma^{(p)}$ is a critical p-simplex such that $f(\sigma^{(p)}) = c_{i+1}$. We now apply Lemma 3.36 to each of the following cases: If $p = d$, then $B_d(i+1) - B_d(i) = 0$ or 1. If $p = d+1$, then $B_d(i+1) - B(i) = -1$ or 0. Otherwise, $B_d(i+1) - B_d(i) = 0$, which proves (ii).

For (iii), observe that $m-1$ is the maximum critical value. By Theorem 4.27, B_d is constant for all values $x > c_{m-1}$. Since there is a $y \in \mathbb{N}$ such that $K(c_{m-1} + y) = K$, we see that $B_d(m-1) = b_d(K)$.

Finally, we apply Theorem 4.27 to see that $b_d(K(c_i)) = b_d(K(x))$ for all $x \in [c_i, c_{i+1})$. Since f is excellent, there exists $\epsilon > 0$ such that $K(c_{i+1}) = K(c_{i+1} - \epsilon) \cup \sigma_p$ as in the proof of (ii). Observe that, by Lemma 3.36, the addition of a p-dimensional simplex will change either B_p or B_{p-1}, leaving all other values fixed. $\square$

Exercise 5.10. Find a discrete Morse function on the simplicial complex



that induces the homological sequence

$$\begin{array}{lccccccc} B_0: & 1 & 2 & 3 & 3 & 2 & 1 & 1 \\ B_1: & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}$$

Exercise 5.11. Give an example of a simplicial complex K and a discrete Morse function on K that induces the following homological sequence:

$$B_0: \quad 5 \quad 10 \quad 20 \quad 30 \quad 36 \quad 43$$

Why does this example not contradict Theorem 5.9?

5.1.2. Classical persistence. While the homological sequence does give us important information about how the topology of the simplicial complex changes with respect to a fixed discrete Morse function, we may want more information. The following example shows what kind of other information we may want.

Example 5.12. Suppose you are told that a discrete Morse function f on some simplicial complex yields the following homological sequence:

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}
B_0 :	1	2	3	2	3	2	3	4	3	2	3	2	2	2	1	1	1
B_1 :	0	0	0	0	0	0	0	0	0	0	0	0	1	2	2	3	2
B_2 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The homological sequence is meant to give us summary information about f. But we might want more information. We can see that b_0 is fluctuating tremendously for quite a while. For example, at c_9 we see that we lose—or, more precisely, merge—a component. But *which* component did we merge? The one introduced at critical value c_7? Or the one introduced at critical value c_4? A similar question can be asked of the cycles. The last critical value killed a cycle, but which one? The one introduced at c_{12}, c_{13}, or c_{15}? A homological sequence, while a nice summary of a discrete Morse function, does not include any of this information. These questions lead to the idea of persistent homology. We want to know not only what the homological sequence is, but also *which* topological information seems to persist and which is just "noise."

In Section 5.1.1, we saw how the sequence of level subcomplexes induced by the critical values builds a simplicial complex in stages. This is a special kind of **filtration**; if K is a simplicial complex, a **filtration** of K is a sequence of subcomplexes

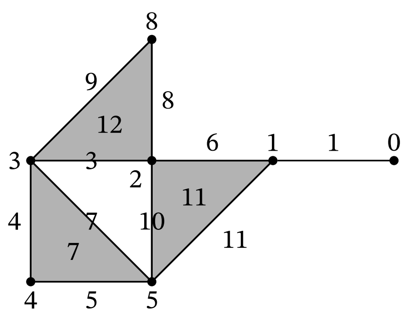
$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{m-1}.$$

In order to perform persistent homology computations, we work with a filtration induced by a basic discrete Morse function (see Definition 2.3). Let $f : K \to \mathbb{R}$ be a basic discrete Morse function. We will forgo the definitions of persistent homology for now, as the technicalities will be shown in all their gory detail in Section 5.2.4. For now, it suffices to know that we can store *all* needed information about topology change in a single, albeit giant, matrix. First, we will put a total ordering induced by the basic discrete Morse function on the simplices. For any two simplices $\sigma, \tau \in K$, if $f(\sigma) < f(\tau)$, define $\sigma < \tau$. If $f(\sigma) = f(\tau)$, then define $\sigma < \tau$ if $\dim(\sigma) < \dim(\tau)$.[1]

Exercise 5.13. Show that the above defines a total ordering on K; that is, for every $\sigma, \tau \in K$, either $\sigma < \tau$ or $\tau < \sigma$.

We will use this total ordering to organize our matrix. Let's look at an example.

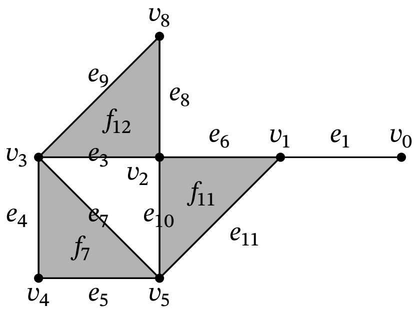
Example 5.14. Let K be the simplicial complex along with the basic discrete Morse function below:



The reader can check that this discrete Morse function is basic. We now construct our linear transformation that encodes all the persistent information. The size of the matrix is determined by the number of simplices of K, and since K has 20 simplices, this will be a 20×20 matrix.

[1]Note that if we associate to the simplex σ the ordered pair $(f(\sigma), \dim(\sigma))$, then essentially we are defining a lexicographic ordering.

Let σ_i denote the simplex labeled i:



We place a 1 in entry a_{i_j} if and only if σ_i is a codimension-1 face of τ_j. All other entries are 0. In practice, we have

$$
\begin{array}{c}
\\
v_0\\ v_1\\ e_1\\ v_2\\ v_3\\ e_3\\ v_4\\ e_4\\ v_5\\ e_5\\ e_6\\ e_7\\ f_7\\ v_8\\ e_8\\ e_9\\ e_{10}\\ e_{11}\\ f_{11}\\ f_{12}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccccccc}
v_0 & v_1 & e_1 & v_2 & v_3 & e_3 & v_4 & e_4 & v_5 & e_5 & e_6 & e_7 & f_7 & v_8 & e_8 & e_9 & e_{10} & e_{11} & f_{11} & f_{12}
\end{array}\\
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

For each j, define $\mathrm{low}(j)$ to be the row index of the 1 in column j with the property that any other 1 in column j has row index strictly less than $\mathrm{low}(j)$. Otherwise, if column j consists of all 0s, then $\mathrm{low}(j)$ is undefined. For example, $\mathrm{low}(6) = 5$ since the lowest 1 in the column 6 (labeled e_3) is found in row 5 (labeled by v_3); $\mathrm{low}(9)$ (in the column labeled v_5) would be undefined. We will reduce the matrix above so that it has the following property: whenever $j \neq i$ are two non-zero columns, $\mathrm{low}(i) \neq \mathrm{low}(j)$. This is easy in practice by working left to right. Working left to right, we see that everything is a-okay until we reach column 12 (labeled e_7), in which case $\mathrm{low}(12) = \mathrm{low}(10) = 9$. We then simply add columns 10 and 12 modulo 2, replacing column 12 with the result. This yields

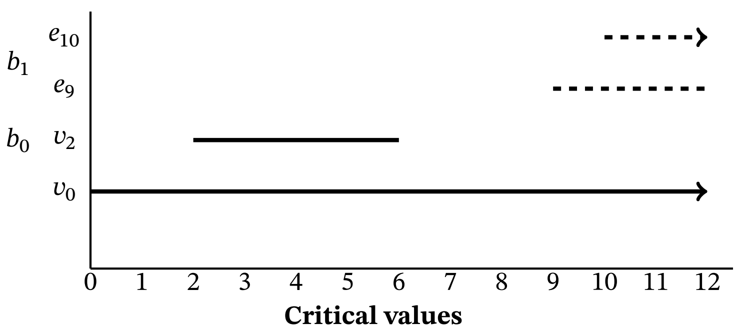
$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccccccc} v_0 & v_1 & e_1 & v_2 & v_3 & e_3 & v_4 & e_4 & v_5 & e_5 & e_6 & e_7 & f_7 & v_8 & e_8 & e_9 & e_{10} & e_{11} & f_{11} & f_{12} \end{array} \\
\begin{array}{c} v_0 \\ v_1 \\ e_1 \\ v_2 \\ v_3 \\ e_3 \\ v_4 \\ e_4 \\ v_5 \\ e_5 \\ e_6 \\ e_7 \\ f_7 \\ v_8 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ f_{11} \\ f_{12} \end{array} &
\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&1&0&0&0 \\
0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\end{array}
$$

But now notice that low(12) = low(8). Okay, repeat this same process, this time adding columns 12 and 8, then replacing column 12 with the result. Now column 12 is the zero column, so low(12) is undefined. That's no problem, so we move on. After reducing the matrix, you should obtain

$$
\begin{array}{c}
\\
v_0\\ v_1\\ e_1\\ v_2\\ v_3\\ e_3\\ v_4\\ e_4\\ v_5\\ e_5\\ e_6\\ e_7\\ f_7\\ v_8\\ e_8\\ e_9\\ e_{10}\\ e_{11}\\ f_{11}\\ f_{12}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccccccc}
v_0 & v_1 & e_1 & v_2 & v_3 & e_3 & v_4 & e_4 & v_5 & e_5 & e_6 & e_7 & f_7 & v_8 & e_8 & e_9 & e_{10} & e_{11} & f_{11} & f_{12}
\end{array}\\
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Now we need to interpret the matrix. The value low(i) is key. It identifies the point at which a simplex generated a Betti number, and it also tells us when that Betti number died. It will help to think of the values of the basic discrete Morse function as units of time. Take the situation low(11) = 4, corresponding to the column and row labeled e_6 and v_2, respectively. This means that at time 2, v_2 generated a new Betti number. Since v_2 is a vertex, it generated a new component. However, its life was cut short: the edge e_6 killed it at time 6. Thus, a component was born at time 2 and died at time 6. Consider another example where low(20) = 16, corresponding to f_{12} and e_9. Using the same interpretation scheme, e_9 generated homology at time 9 and died at time 12 at the hands of the face f_{12}. We can make this analysis for any column for which low(i) exists.

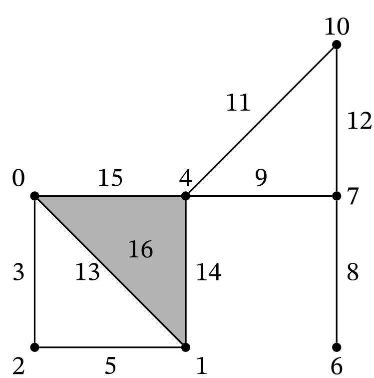
What about a column where low(i) is not defined, such as the column for v_2? We need not worry about that particular column, since we saw above that v_2 was already born and killed. There are also columns that satisfy low(i) $= i$, such as column 6. This corresponds to a regular pair, i.e., a simultaneous birth and death. Since there is no topology change at such instances, we can ignore them. Finally, there will be some columns (in this case v_0 and e_{10}) for which low(i) is not defined and which we never detected being born through an investigation of the low function. These correspond to homology that is born but never dies—it persists until the end. Such homology is born at its index and never dies. Thus column v_0 corresponds to a component generated by v_0 that never dies, and column e_{11} corresponds to a cycle generated by e_{11} that never dies. This makes perfect sense, as in the end we expect homology that persists to be precisely the homology of the original complex. All of the birth and death information is summed up in the following **bar code**:



The solid bars represent a component (i.e., b_0), while the dashed lines represent a cycle (i.e., b_1).

Notice that given a bar code induced by a basic discrete Morse function f, we may recover the homological sequence of f by drawing a vertical line at each critical value c_j. The number of times the vertical line intersects a horizontal bar corresponding to b_i (other than a death time) is precisely $B_i(j)$.

Example 5.15. Now we will use persistent homology to investigate the discrete Morse function that induced the homological sequence in Example 5.12. The homological sequence was induced by the following discrete Morse function:



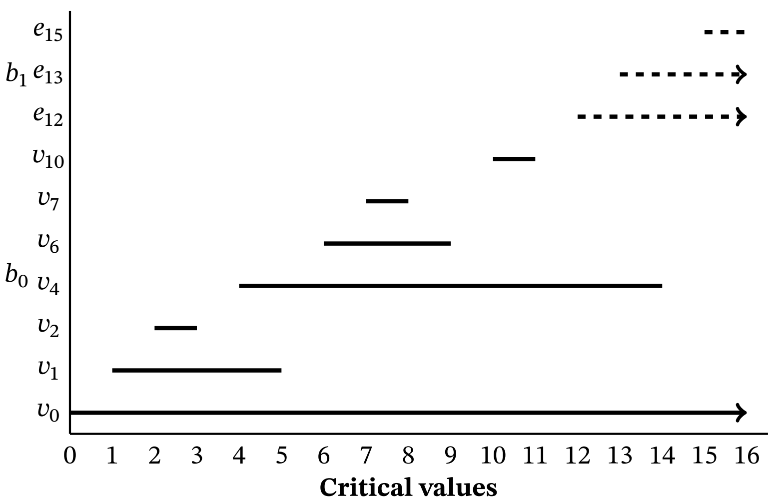
Note that this discrete Morse function is basic but has no regular simplices. As above, we determine its boundary matrix according to the rule that if σ_i is a simplex of K representing the column indexed by i, we place a 1 in entry a_{i_j} if and only if σ_i is a proper face of τ_j. All other entries are 0.

$$
\begin{array}{c}
\\
v_0 \\ v_1 \\ v_2 \\ e_3 \\ v_4 \\ e_5 \\ v_6 \\ v_7 \\ e_8 \\ e_9 \\ v_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \\ e_{15} \\ f_{16}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccc}
v_0 & v_1 & v_2 & e_3 & v_4 & e_5 & v_6 & v_7 & e_8 & e_9 & v_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & f_{16}
\end{array}
\\
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Now reduce this matrix as specified above:

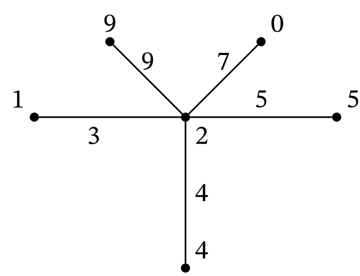
$$
\begin{array}{c}
 \\ v_0 \\ v_1 \\ v_2 \\ e_3 \\ v_4 \\ e_5 \\ v_6 \\ v_7 \\ e_8 \\ e_9 \\ v_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \\ e_{15} \\ f_{16}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccc} v_0 & v_1 & v_2 & e_3 & v_4 & e_5 & v_6 & v_7 & e_8 & e_9 & v_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & f_{16} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

The bar code is thus

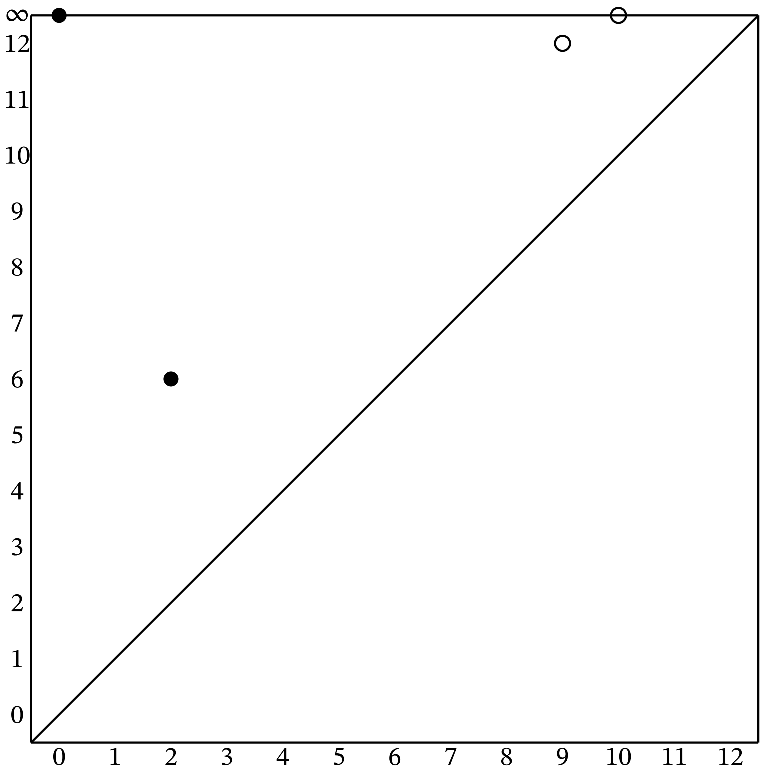


Problem 5.16. Find the barcode for the discrete Morse function in Problem 5.3.

Problem 5.17. Compute the barcode of



Remark 5.18. In the field of statistics, we know that different modes of presentation can yield varying insights into the same data. This data can be displayed as a histogram, pie chart, or some other graphical form, and sometimes you can see a trend in one graphic display that you can't see in another. Likewise, a bar code is one nice graphical representation of the data from our reduced matrix, but this is not our only option. Another option is to display our data as a **persistence diagram**. To do this, we work in the first quadrant of $\mathbb{R}^2$. We begin by drawing the diagonal consisting of all points (x, x). The data points given here have first coordinate the birth time and second coordinate the death time. A Betti number that is born and never dies is a **point at infinity** and is plotted at the very top of the diagram. For example, the bar code found in Example 5.14 has corresponding persistence diagram given below, where a solid dot is a component and an open dot is a cycle.



Part of the interpretation of the persistence diagram is that points farther away from the diagonal are more significant, while points closer to the diagonal tend to be considered noise. We will look at persistence diagrams in more detail in Section 5.2.4.

Exercise 5.19. In a persistence diagram, is it ever possible to plot a point below the diagonal? Why or why not?

Problem 5.20. Find the persistence diagram for the discrete Morse function in Problem 5.3.

Problem 5.21. Compute the persistence diagram of the complex in Problem 5.17.

5.2. Persistent homology of discrete Morse functions

In the previous section, we used persistent homology to study the induced filtration of a discrete Morse function. In this section, we are interested in building the framework of persistent homology from discrete Morse theory. I am indebted to Uli Bauer for this concept. The material in this section is based on part of his masterful PhD thesis [**24**].

5.2.1. Distance between discrete Morse functions. There are several natural notions of a distance between two functions into $\mathbb{R}$. The one we will use is called the uniform norm. Let $f\colon K \to \mathbb{R}$ be any function on a finite set K (in particular, a discrete Morse function on a simplicial complex). The **uniform norm** of f, denoted by $||f||_\infty$, is defined by

$$||f||_\infty := \max\{|f(\sigma)| : \sigma \in K\}.$$

Exercise 5.22. Compute $||f||_\infty$ for f the discrete Morse functions defined in Problems 5.3 and 5.17.

Using the uniform norm, we can define a **distance** between discrete Morse functions on a fixed simplicial complex in the following way: Let $f, g\colon K \to \mathbb{R}$ be discrete Morse functions. The **distance** between f and g is defined by $d(f,g) := ||f - g||_\infty$. Any good theory of distance will satisfy the four properties found in the following proposition:

Proposition 5.23. Let $f, g\colon K \to \mathbb{R}$ be discrete Morse functions. Then

(i) $d(f,g) = 0$ if and only if $f = g$;

(ii) $d(f,g) \geq 0$;

(iii) $d(f,g) = d(g,f)$;

(iv) $d(f,h) + d(h,g) \geq d(f,g)$.

Proof. Suppose that $d(f,g) = 0$. Then $0 = \max\{|f(\sigma) - g(\sigma)|\colon \sigma \in K\}$ so that $|f(\sigma) - g(\sigma)| = 0$ for all $\sigma \in K$. Hence $f = g$. Conversely, if $f = g$, then $d(f,g) = \max\{|f(\sigma) - f(\sigma)|\colon \sigma \in K\} = 0$. For (ii), since

$|f(\sigma)-g(\sigma)| \geq 0$, $d(f,g) \geq 0$. For (iii), since $|f(\sigma)-g(\sigma)| = |g(\sigma)-f(\sigma)|$, we have that $d(f,g) = d(g,f)$. Finally, the standard triangle inequality tells us that

$$|f(\sigma) - g(\sigma)| \leq |f(\sigma) - h(\sigma)| + |h(\sigma) - g(\sigma)|;$$

hence $d(f,g) \leq d(f,h) + d(h,g)$. □

We will use distances between discrete Morse functions at the end of Section 5.2.3 and in the main results of Section 5.2.4.

Example 5.24. Let $f, g\colon K \to \mathbb{R}$ be two discrete Morse functions, and define $f_t := (1-t)f + tg$ for all $t \in [0,1]$. To gain a little bit of practice manipulating this distance, we will show by direct computation that $||f_r - f_s||_\infty = |s-r|||f-g||$ for all $s, t \in [0,1]$. We have

$$\begin{aligned}
||f_r - f_s|| &= \max\{|f_r(\sigma) - f_s(\sigma)| : \sigma \in K\} \\
&= \max\{|f(\sigma)[(1-r) - (1-s)] - g(\sigma)[s-r]| : \sigma \in K\} \\
&= \max\{|s-r||f(\sigma) - g(\sigma)| : \sigma \in K\} \\
&= |s-r| \max\{|f(\sigma) - g(\sigma)| : \sigma \in K\} \\
&= |s-r|||f-g||_\infty.
\end{aligned}$$

5.2.2. Pseudo-discrete Morse functions. Given a discrete Morse function f, we know from Section 2.2 that f induces a unique gradient vector field V_f. On the one hand, this is a good thing, since there is no ambiguity about what gradient vector field is associated with a particular discrete Morse function. On the other hand, the fact that we only have one gradient vector field implies a certain strictness of the definition of a discrete Morse function. In an attempt to relax the definition of a discrete Morse function so that it can induce multiple gradient vector fields, we introduce the concept of a **pseudo-discrete Morse function**.

Definition 5.25. A function $f\colon K \to \mathbb{R}$ is called a **pseudo-discrete Morse function** if there is a gradient vector field V such that whenever $\sigma^{(p)} < \tau^{(p+1)}$, the following conditions hold:

- $(\sigma, \tau) \notin V_f$ implies $f(\sigma) \leq f(\tau)$, and
- $(\sigma, \tau) \in V_f$ implies $f(\sigma) \geq f(\tau)$.

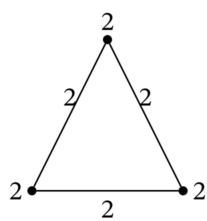
Any such V and f are called **consistent**.

A good strategy for putting a pseudo-discrete Morse function on a simplicial complex is to start with the gradient vector field you want and then label the simplices accordingly.

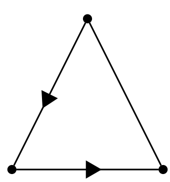
Exercise 5.26. Give a definition of a discrete Morse function in the style of Definition 5.25.

As we will see, pseudo-discrete Morse functions can look quite different from discrete Morse functions.

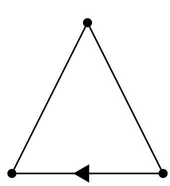
Example 5.27. Let G be the simplicial complex below and $f : G \to \mathbb{R}$ the labeling



To confirm that this is a pseudo-discrete Morse function, we need to find a gradient vector field V such that $(\sigma, \tau) \notin V_f$ implies $f(\sigma) \leq f(\tau)$ and $(\sigma, \tau) \in V_f$ implies $f(\sigma) \geq f(\tau)$ whenever $\sigma^{(p)} < \tau^{(p+1)}$. The following gradient vector field satisfies this property:



So does this gradient vector field:



In fact, any gradient vector field on G is consistent with f. More generally, for any simplicial complex K, the function $f: K \to \mathbb{R}$ defined by $f(\sigma) = 2$ for all $\sigma \in K$ is a pseudo-discrete Morse function consistent with all possible gradient vector fields on K. In this case, it is interesting to ask how many such gradient vector fields are consistent with the constant function. We will give a partial answer to this question in Chapter 7 in the case where K is 1-dimensional or a graph.

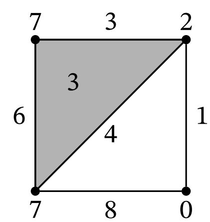
By contrast, Problem 5.29 has you investigate when a vector field consistent with a pseudo-discrete Morse function is unique.

Problem 5.28. Show that a pseudo-discrete Morse function f is flat if and only if it is consistent with the empty vector field.

Problem 5.29. Find and prove a characterization for when a pseudo-discrete Morse function has a unique gradient vector field.

What would something that is *not* a pseudo-discrete Morse function look like?

Problem 5.30. Consider the following simplicial complex K with labeling $f: K \to \mathbb{R}$:



Show that this is not a pseudo-discrete Morse function.

In Section 4.1.3, we saw that a linear combination of flat discrete Morse functions yielded another flat discrete Morse function. This also works for pseudo-discrete Morse functions, so that we may obtain new pseudo-discrete Morse functions from old.

Lemma 5.31. Let f and g be pseudo-discrete Morse functions consistent with a gradient vector field V, and let $t_1, t_2 \geq 0$ be real numbers. Then $t_1 f + t_2 g$ is a pseudo-discrete Morse function consistent with V.

Problem 5.32. Prove Lemma 5.31.

One thing that a pseudo-discrete Morse function does for us, similar to a basic discrete Morse function, is to induce a **strict total order** on the set of simplices, that is, a relation $\prec$ on K satisfying the following three properties:

(a) Irreflexive: for all $\sigma \in K$, $\sigma \nprec \sigma$.

(b) Asymmetric: if $\sigma \prec \tau$, then $\tau \nprec \sigma$.

(c) Transitive: $\sigma \prec \tau$ and $\tau \prec \gamma$ implies $\sigma \prec \gamma$.

The order is total in the sense that for any $\sigma, \tau \in K$, either $\sigma \prec \tau$ or $\tau \prec \sigma$. Simply put, all simplices are comparable. We will build the total order from a partial order.

Definition 5.33. Let V be a gradient vector field on K. Define a relation $\leftarrow_V$ on K such that whenever $\sigma^{(p)} < \tau^{(p+1)}$, the following hold:

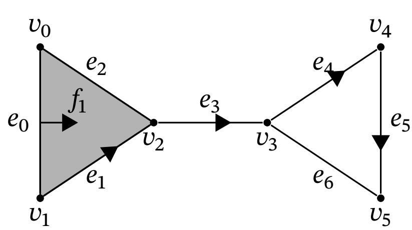
(a) if $(\sigma, \tau) \notin V$, then $\sigma \leftarrow_V \tau$;

(b) if $(\sigma, \tau) \in V$, then $\tau \leftarrow_V \sigma$.

Let $\prec_V$ be the transitive closure of $\leftarrow_V$. Then $\prec_V$ is the **strict partial order induced by** V.

The transitive closure of a relation is a new relation that forces transitivity by fiat; i.e., if $a < b$ and $b < c$, the transitive closure $<'$ has, by definition, $a <' c$.

Example 5.34. Consider the gradient vector field V from Example 2.48:



Let's begin to investigate $\prec_V$ by taking pairs of simplices and asking if they are related. For example, consider v_3 and e_4. Since $(v_3, e_4) \in V$, this means that $e_4 \leftarrow_V v_3$ so that $e_4 \prec_V v_3$. We also have that $(v_4, e_4) \notin V$ so $v_4 \leftarrow_V e_4$, and hence $v_4 \prec_V e_4$. Since $\prec_V$ is defined to be the transitive closure of $\leftarrow_V$, we get transitive relations for free; i.e., $v_4 \prec_V v_3$. Continuing in this manner, we obtain a partial order on all of K (See Problem 5.35).

Problem 5.35. Draw the Hasse diagram showing the partial order relations induced by V from Example 5.34.

Problem 5.36. Show that Definition 5.33 yields a strict partial order on the set of simplices of a fixed simplicial complex K.

As you might have realized, the way to think about the partial order $\prec_V$ is that as the partial ordering increases, so do the values of f.

Proposition 5.37. If $\alpha \prec_V \beta$, then $f(\alpha) \leq f(\beta)$.

Proof. Suppose $\alpha \prec_V \beta$. Then there exists a sequence

$$\alpha = \alpha_n \leftarrow_V \alpha_{n-1} \leftarrow_V \cdots \leftarrow_V \alpha_0 = \beta.$$

For any pair $\alpha_i \leftarrow_V \alpha_{i-1}$ above, the definition of $\leftarrow_V$ implies that either

(a) $(\alpha_i, \alpha_{i-1}) \notin V$ with $\alpha_i < \alpha_{i-1}$ or

(b) $(\alpha_{i-1}, \alpha_i) \in V$ with $\alpha_{i-1} < \alpha_i$.

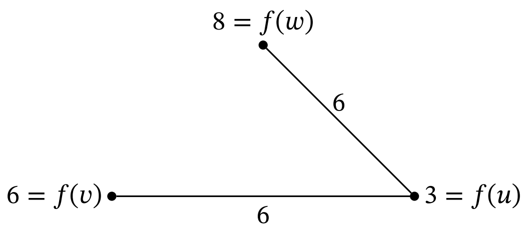
If the former case, then the definition of a discrete Morse function implies that $f(\alpha_i) < f(\alpha_{i-1})$. If the latter, then the definition of being an element of the gradient vector field implies that $f(\alpha_{i-1}) \geq f(\alpha_i)$. In either case, we have that $f(\alpha_{i-1}) \geq f(\alpha_i)$ for all i, and hence $f(\alpha) \leq f(\beta)$. □

In addition to $\prec_V$, we have another partial order, denoted by $\prec_f$. This one is induced by a pseudo-discrete Morse function $f : K \to \mathbb{R}$ and is defined by $\alpha \prec_f \beta$ if and only if $f(\alpha) < f(\beta)$ for any simplices (not necessarily codimension-1 pairs) $\alpha, \beta \in K$. If there are no two simplices σ and τ such that $\sigma \prec_V \tau$ and $\tau \prec_f \sigma$, we say that the two orders are **consistent**. We may now use the consistent orders as the basis of a strict total order on the simplices of K.

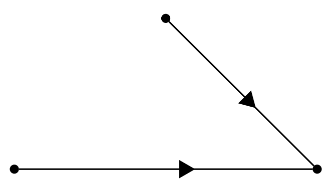
Definition 5.38. A **linear extension** of a poset (P, V) is a permutation of all the elements $p_1, p_2, \dots, p_m \in P$ such that if $p_i < p_j$, then $i < j$. Let $f : K \to \mathbb{R}$ be a pseudo-discrete Morse function and V a gradient vector field consistent with f. A **strict total order $\prec$ consistent with f and V** is a linear extension of $\prec_V$ (and hence $\prec_f$).

A linear extension of $\prec_V$ is really just a choice of total ordering that respects the partial ordering of $\prec_V$. We illustrate with an example.

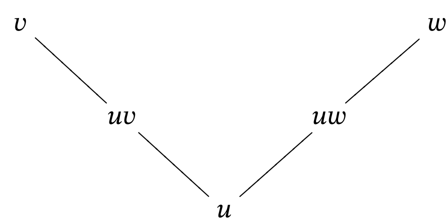
Example 5.39. Let $f : K \to \mathbb{R}$ be the pseudo-discrete Morse function below.



One possible V consistent with f is



and the corresponding partial order is seen in the Hasse diagram



A linear extension $\prec$ is simply a total ordering that respects the above. Several are listed below:

- $u \prec uv \prec v \prec uw \prec w$
- $u \prec uw \prec w \prec uv \prec v$

- $u \prec uv \prec uw \prec w \prec v$
- $u \prec uw \prec uv \prec w \prec v$

Using our new language of a total order $\prec$ consistent with a vector field V and pseudo-discrete Morse function f, we will find it useful to rephrase Theorem 4.27 (the collapse theorem). If $f(\alpha) = c$, write $K(\alpha) := K(c)$, the level subcomplex containing all simplices whose values are less than or equal to that of the value of α under f.

Theorem 5.40 (Theorem 4.27 rephrased)**.** Let V be a gradient vector field consistent with a pseudo-discrete Morse function f, and let $\prec$ be a linear extension of $\prec_V$. If $\alpha \prec \beta$ and there are no critical simplices γ with $\alpha \prec \gamma \prec \beta$, then $K(\beta) \searrow K(\alpha)$.

5.2.3. Flat pseudo-discrete Morse functions. Recall from Section 4.1.3 that a discrete Morse function f is called flat if whenever (σ, τ) is a regular pair, $f(\sigma) = f(\tau)$. We now define a flat pseudo-discrete Morse function in terms of the gradient vector field.

Definition 5.41. Let $f : K \to \mathbb{R}$ be a pseudo-discrete Morse function consistent with a gradient vector field V. We call f **flat** if whenever $\sigma^{(p)} < \tau^{(p+1)}$, we have that

- if $(\sigma, \tau) \notin V$, then $f(\sigma) \leq f(\tau)$;
- if $(\sigma, \tau) \in V$, then $f(\sigma) = f(\tau)$.

The solution to Problem 5.30 required you to compute a "minimal" set of arrows on the simplicial complex. In general, we can define this for any pseudo-discrete Morse function.

Definition 5.42. Let f be a pseudo-discrete Morse function. Define

$$V := \{(\sigma, \tau) : \sigma^{(p)} < \tau^{(p+1)} \text{ and } f(\sigma) > f(\tau)\}$$

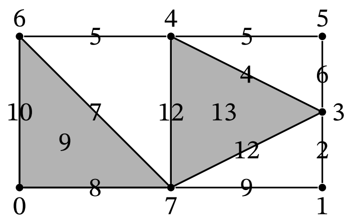
to be the **minimal gradient vector field consistent with** f.

The adjective "minimal" is justified by the following:

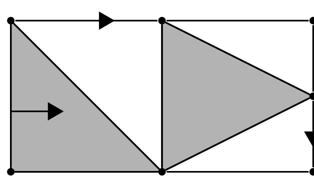
Proposition 5.43. The minimal gradient vector field consistent with f is minimal in the sense that it is the intersection of all gradient vector fields consistent with f.

Proof. Let $V' = \bigcap V_i$ be the intersection over all gradient vector fields V_i consistent with f. Clearly $V' \subseteq V$ for any gradient vector field V consistent with f. Suppose $(\sigma, \tau) \in V$. Then by definition, $f(\sigma) > f(\tau)$. Hence, $(\sigma, \tau) \in V_i$ for all gradient vector fields V_i consistent with f, using the contrapositive of the first line of Definition 5.25. Thus the result follows. $\square$

Example 5.44. If f is the pseudo-discrete Morse function defined by



then its minimal consistent gradient vector field is given by



The vectors above are precisely the ones that must be there. In other words, they correspond to regular pairs (σ, τ) satisfying $f(\sigma) > f(\tau)$.

Exercise 5.45. Find all gradient vector fields consistent with the pseudo-discrete Morse function from Example 5.44. Verify that the minimal gradient vector field computed in that example is contained in all the vector fields you find.

Equipped with a well-defined gradient vector field consistent with a pseudo-discrete Morse function, we can now define a "flattening" operation on any pseudo-discrete Morse function, that is, an operation yielding a flat pseudo-discrete Morse function with the same level subcomplexes as the original. If f is a pseudo-discrete Morse function and V the minimal gradient vector field consistent with f, the **flattening of** f, denoted by $\bar{f}$, is defined by

$$\bar{f}(\sigma) := \begin{cases} f(\tau) & \text{if } (\sigma, \tau) \in V \text{ for some } \tau, \\ f(\sigma) & \text{otherwise.} \end{cases}$$

Proposition 5.46. Let $\bar{f}$ be the flattening of the pseudo-discrete Morse function $f : K \to \mathbb{R}$. Then f and $\bar{f}$ have the same set of critical values $c_0, \dots, c_n$. Furthermore, $K_f(c_i) = K_{\bar{f}}(c_i)$ for all critical values c_i. In particular, f and $\bar{f}$ are homologically equivalent.

Problem 5.47. Prove Proposition 5.46.

Exercise 5.48. Call f a **pure** pseudo-discrete Morse function if f is a pseudo-discrete Morse function and not a discrete Morse function. If f is a pure pseudo-discrete Morse function, is $\bar{f}$ always a pure pseudo-discrete Morse function, a discrete Morse function, or neither?

Flattening makes discrete Morse functions "closer" to each other in the following sense.

Theorem 5.49. Let $f, g : K \to \mathbb{R}$ be pseudo-discrete Morse functions, and let $\bar{f}$ and $\bar{g}$ be their respective flattenings. Then $||\bar{f} - \bar{g}||_\infty \leq ||f - g||_\infty$.

Proof. Let $\tau \in K$ and let V be a gradient vector field consistent with f. We will show that $||\bar{f}(\tau) - \bar{g}(\tau)||_\infty \leq ||f - g||_\infty$. If τ is critical for V, set $\sigma = \tau$. Otherwise, let σ be such that $(\sigma, \tau) \in V$. In a similar way to σ, define ϕ for g.

By definition of flattening, we have $\bar{f}(\tau) = f(\tau)$ and $\bar{f}(\phi) = f(\phi)$. By definition of a pseudo-discrete Morse function, $\bar{f}(\tau) \geq \bar{f}(\phi)$. Then $\bar{f}(\tau) = \max\{f(\sigma), f(\phi)\}$. Similarly, we have $\bar{g}(\tau) = \max\{g(\sigma), g(\phi)\}$. By Problem 5.50, we have

$$\begin{aligned} |\bar{f}(\tau) - \bar{g}(\tau)| &\leq |\max\{f(\sigma), f(\phi)\} - \max\{g(\sigma), g(\phi)\}| \\ &\leq \max\{|f(\sigma) - g(\sigma)|, |f(\phi) - g(\phi)|\} \\ &\leq ||f - g||_\infty. \end{aligned}$$ □

Problem 5.50. Prove that if $a, b, c, d \in \mathbb{R}$, then

$$|\max\{a, b\} - \max\{c, d\}| \leq \max\{|a - c|, |b - d|\}.$$

5.2.4. Persistence diagrams of pseudo-discrete Morse functions. In Section 5.1.2, we examined a theory of persistent homology applicable to certain kinds of filtrations of simplicial complexes. From this point of view, there is nothing particularly "discrete Morse theory" about persistent homology. Certain discrete Morse functions happen to give us the right kind of filtration, so finding a discrete Morse function is just one of many ways that we could perform a persistent homology computation. By contrast, this section is devoted to constructing a theory of persistence based exclusively on a pseudo-discrete Morse function. We begin by recasting the language of persistent homology for a pseudo-discrete Morse function.

We know by Theorem 5.40 that, topologically speaking, nothing interesting happens between regular simplices or, to put it in a positive light, the action happens precisely at the critical simplices. Hence, let σ and τ be critical simplices with $\sigma \prec \tau$ (there may be other critical simplices between them). Then there is an **inclusion** function $i^{\sigma,\tau}: K(\sigma) \to K(\tau)$ defined by $i^{\sigma,\tau}(\alpha) = \alpha$. Now we know from Section 2.53 that to any $K(\sigma)$ we can associate the vector space $H_p(K(\sigma))$. Again, as in Section 2.53, if $\alpha^{(p)} \in K(\sigma)$, we think of α as a basis element in $\Bbbk^{|K_p(\sigma)|}$. Then $[\alpha] \in H_p(K(\sigma))$. Furthermore, we may use $i^{\sigma,\tau}$ to define a linear transformation $i_p^{\sigma,\tau}: H_p(K(\sigma)) \to H_p(K(\tau))$ by $i_p^{\sigma,\tau}([\alpha]) = [i_p^{\sigma,\tau}(\alpha)]$.

The p**th persistent homology vector space**, denoted by $H_p^{\sigma,\tau}$, is defined by $H_p^{\sigma,\tau} := \mathrm{Im}(i_p^{\sigma,\tau})$. The p**th persistent Betti numbers** are the corresponding Betti numbers, $\beta_p^{\sigma,\tau} := \mathrm{rank}\, H_p^{\sigma,\tau}$. Denote by $\sigma\_$ the predecessor in the total ordering given by $\prec$. If $[\alpha] \in H_p(K(\sigma))$, we say that $[\alpha]$ is **born at the positive simplex** σ if $[\alpha] \notin H_p^{\sigma_-,\sigma}$, and that it **dies at negative simplex** σ if $i_p^{\sigma,\tau_-}([\alpha]) \notin H_p^{\sigma_-\tau_-}$ but $i_p^{\sigma,\tau}([\alpha]) \in H_p^{\sigma_-,\tau}$. If the class $[\alpha]$ is born at σ and dies at τ, we call (σ,τ) a **persistence pair**. The difference $f(\sigma) - f(\tau)$ is the **persistence** of (σ,τ). If a positive simplex σ is not paired with any negative simplex τ (i.e., σ is born and never dies), then σ is called **essential** or a **point at infinity**.

Although these definitions are quite technical, we have done all of this before in Section 5.1.2. At this point, it might help to go back to one of the concrete computations in that section to see if you can understand how the above definition is precisely that computation.

We will start by defining the **persistence diagram of a pseudo-discrete Morse function**. Recall that a **multiset** on a set S is an ordered pair (S, m) where $m\colon\ S \to \mathbb{N}$ is a function recording the **multiplicity** of each element of S. In other words, a multiset is a set that allows multiple occurrences of the same value.

Definition 5.51. Let $f\colon K \to \mathbb{R}$ be a pseudo-discrete Morse function, and write $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$. The **persistence diagram of** f, denoted by $D(f)$, is the multiset on $\overline{\mathbb{R}}^2$ containing one instance of $(f(\sigma), f(\tau))$ for each persistence pair (σ, τ) of f, one instance of $(f(\sigma), \infty)$ for each essential simplex σ, and each point on the diagonal with countably infinite multiplicity.

We saw an explicit example of a persistence diagram in Remark 5.18. Again, it may help to refer back to see how the remark and Definition 5.51 relate to each other.

One concern that might be raised at this point is that it seems that the persistence diagram $D(f)$ depends not only on the choice of pseudo-discrete Morse function f and gradient vector field V, but also on the choice of total order $\prec$. Happily, $D(f)$ is dependent only on f, as the following theorem shows:

Theorem 5.52. The persistence diagram $D(f)$ is well-defined; that is, it is independent of both the chosen gradient vector field V consistent with f and the total order $\prec$ consistent with f and V.

Proof. We need to show that the information recorded in the definition of the persistence diagram is the same regardless of the choice of V or of $\prec$, i.e., that we obtain the same values for the persistence pairs, essential simplices, and points on the diagonal. By definition, $D(f)$ must always contain each point on the diagonal with countably infinite multiplicity, regardless of the chosen f.

Fix a total order $\prec$ consistent with f, and suppose there are k positive d-simplices σ_i with non-zero persistence and $f(\sigma_i) = s$. We show that for any other total order $\prec'$ consistent with f, we must have k positive d-simplices σ_i with non-zero persistence and $f(\sigma_i) = s$. To see this, define $s\_ := \max\{f(\alpha) : \alpha \in K, f(\alpha) < s\}$. Then $\beta_d(K(s)) = \beta_d(K(s\_)) + k$. These Betti numbers, however, are independent of $\prec$, so for any other total order $\prec'$ consistent with f, there must be k positive d-simplices σ_i with non-zero persistence and $f(\sigma_i) = s$.

Now suppose that there are k persistence pairs $(\sigma_i^{(d)}, \tau_i^{(d+1)})$ such that $(f(\sigma_i), f(\tau_i)) = (s,t)$. Let $\beta_d^{s,t}$ be the dth persistent Betti numbers of $K(s)$ in $K(t)$. There are exactly $\beta_d^{s,t} - \beta^{s_-,t}$ classes born at s, and by definition of a persistence pair, the rank of the subvector space of classes born at s decreases at t by k from the next smaller value $t_- = \max\{f(\phi) : \phi \in K, f(\phi) < t\}$. Hence we have that

$$\beta_d^{s,t} - \beta_d^{s_-,t} = (\beta_d^{s,t} - \beta_d^{s_-,t_-}) - k.$$

The persistence Betti numbers are independent of the total order $\prec$, so the persistence diagram depends only on f. $\square$

We now define a way to measure how "far apart" or different two persistence diagrams are from each other. If $a \in X$ has multiplicity $m(a)$, we view this as $a_1, a_2, \ldots, a_{m(a)}$. These terms will be referred to as **individual elements**.

Definition 5.53. Let X and Y be two multisets of $\overline{\mathbb{R}}^2$. The **bottleneck distance** is $d_B(X,Y) := \inf_{\gamma} \sup_{x\in X} ||x - \gamma(x)||_\infty$ where γ ranges over all bijections from the individual elements of X to the individual elements of Y.

By convention, we define $(a,\infty)-(b,\infty) = (a-b,0)$, $(a,\infty)-(b,c) = (a-b,\infty)$, and $||(a,\infty)||_\infty = \infty$ for all $a,b,c \in \mathbb{R}$.

The bottleneck distance provides another measure of how far apart two pseudo-discrete Morse functions are. In this case, it measures how far away their corresponding diagrams are. One nice property of the bottleneck distance is that if two pseudo-discrete Morse functions are close, then their corresponding diagrams are close. This is the so-called **stability theorem** below. It is called a stability theorem because a slight change in a pseudo-discrete Morse function will not result in a wildly different persistence diagram; in other words, a slight change in the pseudo-discrete Morse function results in a slight change in the persistence diagram. In that sense, the bottleneck distance is stable. First we present a lemma.

Lemma 5.54. Let f and g be two flat pseudo-discrete Morse functions consistent with gradient vector fields V_f and V_g, respectively. Then $f_t := (1-t)f + tg$ is a flat pseudo-discrete Morse function consistent with gradient vector field $V := V_f \cap V_g$.

Notice the similarity to Lemma 4.18.

Proof. We know by Lemma 5.31 that f_t is a pseudo-discrete Morse function. It remains to show that it is consistent with the gradient vector field $V = V_f \cap V_g$. Since f and g are flat, for any pair $\sigma^{(p)} < \tau^{(p+1)}$ we have that $f(\sigma) \leq f(\tau)$ and $g(\sigma) \leq g(\tau)$. The same argument used for Lemma 4.18 then shows that f_t is consistent with V. □

We are now able to prove the stability theorem.

Theorem 5.55 (Stability theorem)**.** Let $f, g\colon K \to \mathbb{R}$ be two pseudo-discrete Morse functions. Then $d_B(D(f), D(g)) \leq ||f - g||_\infty$.

Proof. Without loss of generality, it suffices to prove the result in the case where f and g are flat pseudo-discrete Morse functions (see Problem 5.56). Consider the family $f_t := (1-t)f + tg$ for $t \in [0,1]$. By Lemma 5.54, f_t is a flat pseudo-discrete Morse function. However, the total orders $\prec_{f_t}$ may be different for different t. We will partition $[0,1]$ into subintervals and find a total order that is consistent on a fixed interval. To that end, let $\sigma^{(p)} < \tau^{(p+1)}$ with $f(\sigma) - g(\sigma) \neq f(\tau) - g(\tau)$. Then there exists a unique t such that $f_t(\sigma) = f_t(\tau)$. This value is given by

$$t_{\sigma,\tau} := \frac{f(\sigma) - f(\tau)}{f(\sigma) - f(\tau) - g(\sigma) + g(\tau)}.$$

If $f(\sigma) - g(\sigma) = f(\tau) - g(\tau)$, then $f_t(\sigma) = f_t(\tau)$ for all t if and only if $f(\sigma) = f(\tau)$. Hence the order $\prec_{f_t}$ only changes at the values $t_{\sigma,\tau}$. Since there are only finitely many of these and the total ordering remains constant in between these $t_{\sigma,\tau}$, we obtain the desired partition along with the total order $\prec_i$ on K that is consistent with f_t for all $t \in [t_i, t_{i+1}]$. Since these all have the same total order, they all have the same gradient vector field so that, in particular, all such f_t in the interval $[t_i, t_{i+1}]$ have the same persistence pairs P_i. Hence there is a natural correspondence of points between each of the persistence diagrams $D(f_t)$. If σ is essential, write $(\sigma, \infty) \in P_i$ and $f_t(\infty) = \infty$. Let $r, s \in [t_i, t_{i+1}]$. Because we have identified persistence pairs, the bottleneck distance satisfies

$$\begin{aligned} d_B(D(f_r), D(f_s)) &\leq \max_{(\sigma,\tau)\in P_i} \|(f_r(\sigma), f_r(\tau)) - (f_s(\sigma), f_s(\tau))\| \\ &\leq \|f_r - f_s\|_\infty = |s - r|\|f - g\|_\infty, \end{aligned}$$

where the last equality was derived in Example 5.24. Using this fact, we then sum over the partition and use the triangle inequality for the bottleneck distance to obtain

$$\begin{aligned} d_B(D(f), D(g)) &\leq \sum_{i=0}^{k-1} d_B(D(f_{t_i}), D(f_{t_{i+1}})) \\ &\leq \sum_{i=0}^{k-1} (t_{i+1} - t_i)\|f - g\|_\infty \\ &= \|f - g\|_\infty, \end{aligned}$$

which is the desired result. □

Problem 5.56. Prove that it is sufficient to consider flat pseudo-discrete Morse functions in the proof of Theorem 5.55.

Exercise 5.57. Verify that the value $t_{\sigma,\tau}$ given in the proof of Theorem 5.55 is the correct value and is unique.

Problem 5.58. Let K be a complex. For any $\epsilon > 0$, find two persistence diagrams $D(f)$ and $D(g)$ on K such that $0 < d_B(D(f), D(g)) < \epsilon$.

Chapter 6

Boolean functions and evasiveness

This chapter is devoted to a fascinating application to a computer science search problem involving "evasiveness," originally due to Forman [**67**]. Along the way, we will introduce Boolean functions and see how they relate to simplicial complexes and the evasiveness problem. For another interesting use of discrete Morse theory in the type of application discussed in this chapter, see [**59**].

6.1. A Boolean function game

Let's play a game[1]. For any integer $n \geq 0$, we'll pick a function f whose input is an $(n+1)$-tuple of 0s and 1s, usually written in vector notation as $\vec{x}$. The output of f is either 0 or 1. In other words, we are looking at a function of the form $f : \{0,1\}^{n+1} \to \{0,1\}$ with $f(\vec{x}) = 0$ or 1. Such a function is called a **Boolean function**. The **hider** creates a strategy for creating an input $\vec{x} := (x_0, x_1, \dots, x_n)$. The **seeker** does not know the hider's input, and has to guess the output by asking the hider to reveal any particular input value (i.e., entry of the input vector). The seeker may only ask yes or no questions of the form "Is $x_i = 0$?" or "Is $x_i = 1$?" The chosen Boolean function f is known to both players. The object

[1] Not the kind that Jigsaw plays, thank goodness.

of the game is for the seeker to guess the output in as few questions as possible. The seeker wins if she can correctly predict the output without having to reveal every single input value. The seeker is said to have a **winning strategy** if she can always predict the correct output without revealing every input value, regardless of the strategy of the hider. The hider wins if the seeker needs to ask to reveal all the entries of the input vector.

In the following examples and exercises, it may help to actually "play" this game with a partner.

Example 6.1. Let $n = 99$ and let $f : \{0,1\}^{100} \to \{0,1\}$ be defined by $f(\vec{x}) = 0$ for all $\vec{x} \in \{0,1\}^{100}$. Then the seeker can always win this game in zero questions; that is, it does not matter what strategy the hider picks for creating an input since the output is always 0.

Example 6.2. Let $P_3^4 : \{0,1\}^{4+1} \to \{0,1\}$ by $P_3^4(x_0, x_1, x_2, x_3, x_4) = x_3$. The seeker can always win in one question by asking "Is $x_3 = 1$?" In general, we define the **projection** function $P_i^n : \{0,1\}^{n+1} \to \{0,1\}$ by $P_i^n(\vec{x}) = x_i$ for $0 \le i \le n$.

Exercise 6.3. Prove that the seeker can always win in one question for the Boolean function P_i^n.

Note that in Example 6.2, it is true that the seeker could ask an unhelpful question like "Is $x_4 = 1$?" but we are trying to come up with the best possible strategy.

Exercise 6.4. Let $P_{j,\ell}^n : \{0,1\}^{n+1} \to \{0,1\}$ be defined by $P_{j,\ell}^n(x_0, \dots, x_n) = x_j + x_\ell \bmod 2$ for $0 \le j \le \ell \le n$. Prove that the seeker has a winning strategy and determine the minimum number of questions the seeker needs to asks in order to win.

Although we can think of the above as simply a game, it is much more important than that. Boolean functions are used extensively in computer science, logic gates, cryptography, social choice theory, and many other applications. See for example [**131**], [**92**], and [**154**]. You can imagine that if a computer has to compute millions of outputs of Boolean functions, tremendous savings in time and memory are possible if the computer can compute the output by processing only minimal input. As a silly but illustrative example, consider the Boolean function in Example 6.1. This Boolean function takes in a string of one hundred 0s and 1s and will always output a 0. If a computer program had to compute this function for 200 different inputs, it would be horribly inefficient for the computer to grab each of the 200 inputs from memory and then do the computation, because who cares what the inputs are? We know the output is always 0. In a similar manner, if we had to compute P_3^{999} on 1000 inputs, all we would need to do is call the third entry of each input, not all 1000 entries of each input. Hence, we can possibly save time and many resources by determining the minimum amount of information needed in order to determine the output of a Boolean function.

On the other hand, a Boolean function that requires knowledge of every single input value is going to give us some trouble. A Boolean function $f : \{0,1\}^{n+1} \to \{0,1\}$ for which there exists a strategy for the hider to always win is called **evasive**. In other words, such a function requires knowledge of the value of every single entry of the input in order to determine the output.

Define $T_2^4 : \{0,1\}^{4+1} \to \{0,1\}$ by $T_2^4(x_0, x_1, x_2, x_3, x_4) = 1$ if there are two or more 1s in the input and 0 otherwise. Let's simulate one possible playing of this game.

SEEKER: Is $x_1 = 1$?

HIDER: Yes.

SEEKER: Is $x_3 = 1$?

HIDER: Yes.

SEEKER: Now I know the input has at least two 1s, so the output is 1. I win!

While the seeker may have won this game, the hider did not use a very good strategy. Letting $x_3 = 1$ on the second question guaranteed two 1s in the input, so the output was determined. Rather, the hider should have had a strategy that did not reveal whether or not the input had two 1s until the very end. Let's try again.

SEEKER: Is $x_1 = 1$?

HIDER: Yes.

SEEKER: Is $x_3 = 1$?

HIDER: No.

SEEKER: Is $x_2 = 1$?

HIDER: No.

SEEKER: Is $x_0 = 1$?

HIDER: No.

SEEKER: Is $x_4 = 1$?

HIDER: Yes.

SEEKER: Now I know that the output is 1, but I needed the whole input. I lose.

Hence, the hider can always win if the Boolean function is T_2^4. In other words, T_2^4 is evasive.

Problem 6.5. The **threshold function** $T_i^n = T_i : \{0,1\}^{n+1} \to \{0,1\}$ is defined to be 1 if there are i or more 1s in the input and 0 otherwise. Prove that the threshold function is evasive.

6.2. Simplicial complexes are Boolean functions

Although our game may seem interesting, what does it have to do with simplicial complexes? There is a natural way to associate a simplicial complex to a Boolean function that satisfies an additional property called monotonicity.

Definition 6.6. Let $f : \{0,1\}^{n+1} \to \{0,1\}$ be a Boolean function. Then f is called **monotone** if whenever $\vec{x} \leq \vec{y}$, we have $f(\vec{x}) \leq f(\vec{y})$.

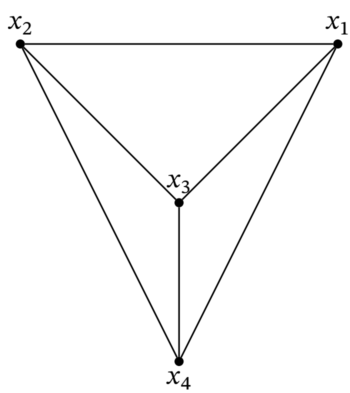
Exercise 6.7. Prove that the two constant Boolean functions $\mathbf{0}(\vec{x}) = 0$ and $\mathbf{1}(\vec{x}) = 1$ are monotone.

Exercise 6.8. Prove that the threshold function $T_i^n : \{0,1\}^{n+1} \to \{0,1\}$ defined in Problem 6.5 is monotone.

As we will see, the condition of being monotone will ensure that the following construction is closed under taking subsets. The construction is attributed to Saks et al. [**95**].

Definition 6.9. Let f be a monotone Boolean function. The **simplicial complex** Γ_f **induced by** f is the set of all sets of coordinates (excluding $(1, 1, \dots, 1)$) such that if $\vec{x} \in \{0, 1\}^{n+1}$ is 0 exactly on these coordinates, then $f(\vec{x}) = 1$. We adopt the convention that if $x_{i_0} = \cdots = x_{i_k} = 0$ are the 0 coordinates of an input, then the corresponding simplex in Γ_f is denoted by $\sigma_{\vec{x}} := \{x_{i_0}, \dots, x_{i_k}\}$.

Example 6.10. You proved in Exercise 6.8 that the threshold function is monotone. Hence, it corresponds to some simplicial complex. Let us investigate $T_2^3 = T_2$. A simplex in Γ_{T_2} corresponds to the 0 coordinates of a vector $\vec{x}$ such that $T_2(\vec{x}) = 1$. We thus need to find all $\vec{x}$ such that $T_2(\vec{x}) = 1$. For example, $T_2((1, 0, 0, 1)) = 1$. This has a 0 in x_1 and x_2, so this vector corresponds to the simplex x_1x_2. Also, $T_2((0, 1, 1, 1)) = 1$, which corresponds to the simplex x_0. Finding all such simplices, T_2 produces the simplicial complex



Exercise 6.11. Why do we exclude vector $(1, 1, \dots, 1)$ in Definition 6.9?

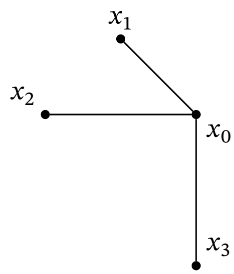
As mentioned above, we need monotonicity to ensure that Γ_f is a simplicial complex.

Problem 6.12. Show that the Boolean function $f : \{0, 1\}^{n+1} \to \{0, 1\}$ defined by $f(\vec{x}) = x_0 + x_1 + \cdots + x_n \mod 2$ is not monotone. Then attempt to construct the corresponding Γ_f and show why it is not a simplicial complex.

Exercise 6.13. Compute the simplicial complexes $\Gamma_{\mathbf{0}}$ and $\Gamma_{\mathbf{1}}$ induced by the constant functions defined in Exercise 6.7.

Problem 6.14. Show that the projection $P_i^n : \{0,1\}^{n+1} \to \{0,1\}$ defined by $P_i^n(\vec{x}) = x_i$ is monotone, and then compute $\Gamma_{P_i^n}$.

Problem 6.15. Find a Boolean function f such that Γ_f is the simplicial complex below:



We now show that monotone Boolean functions and simplicial complexes are in bijective correspondence. In other words, a monotone Boolean function and a simplicial complex are two sides of the same coin.

Proposition 6.16. Let $n \geq 0$ be a fixed integer. Then there is a bijection between monotone Boolean functions $f : \{0,1\}^{n+1} \to \{0,1\}$ and simplicial complexes on $[n]$.

Proof. Let $f : \{0,1\}^{n+1} \to \{0,1\}$ be a monotone Boolean function. If $f(\vec{x}) = 1$ with 0 coordinates $x_{i_0}, \dots, x_{i_k}$, write $\sigma_{\vec{x}} := \{x_{i_0}, \dots, x_{i_k}\}$ and let Γ_f be the collection of all such $\sigma_{\vec{x}}$. To see that Γ_f is a simplicial complex, let $\sigma_{\vec{x}} \in \Gamma_f$ and suppose $\sigma_{\vec{y}} := \{y_{j_0}, y_{j_1}, \dots, y_{j_\ell}\}$ is a subset of $\sigma_{\vec{x}}$. Define $\vec{y}$ to be 0 exactly on the coordinates of $\sigma_{\vec{y}}$. We need to show that $f(\vec{y}) = 1$. Suppose by contradiction that $f(\vec{y}) = 0$. Since $\sigma_{\vec{x}} \in \Gamma_f$, $f(\vec{x}) = 1$. Observe that since $\sigma_{\vec{y}} \subseteq \sigma_{\vec{x}}$, $\vec{y}$ can be obtained from $\vec{x}$ by switching all the coordinates in $\sigma_{\vec{x}} - \sigma_{\vec{y}}$ from 0 to 1. But then if $f(\vec{y}) = 0$, we have switched our inputs from 0 to 1 while our output has switched from 1 to 0, contradicting the assumption that f is monotone. Thus $f(\vec{y}) = 1$ and Γ_f is a simplicial complex.

Now let K be a simplicial complex on $[n]$. Let $\sigma \in K$ with $\sigma = \{x_{i_0}, \dots, x_{i_k}\}$. Define $\vec{x}_\sigma \in \{0,1\}^{n+1}$ to be 0 on coordinates $x_{i_0}, \dots, x_{i_k}$ and 1 on all other coordinates. Define $f : \{0,1\}^{n+1} \to \{0,1\}$ by $f(\vec{x}_\sigma) = 1$ for all $\sigma \in K$ and 0 otherwise. To see that f is monotone, let $f(\vec{x}_\sigma) = 1$ with 0 coordinates $x_0, \dots, x_k$ and suppose $\vec{x'}$ has 0 coordinates on a subset S of $\{x_0, \dots, x_k\}$. Since K is a simplicial complex, $S \in K$. Hence the vector with 0 coordinates S has output 1. But this vector is precisely $\vec{x'}$.

It is clear that these constructions are inverses of each other, hence the result.

$\square$

Problem 6.17. Let $T_i^n : \{0,1\}^{n+1} \to \{0,1\}$ be the threshold function. Compute $\Gamma_{T_i^n}$.

6.3. Quantifying evasiveness

By Proposition 6.16, there is a bijective correspondence between monotone Boolean functions and simplicial complexes. Hence any concept we can define for a monotone Boolean function can be "imported" to a simplicial complex. In particular, we can define what it means for a simplicial complex to be evasive. Furthermore, we will generalize the notion of evasiveness by associating a number to the complex, quantifying "how close to being evasive is it?"

Let Δ^n be the n-simplex on vertices $v_0, v_1, \dots, v_n$. Suppose $M \subseteq \Delta^n$ is a subcomplex known to both players. Let $\sigma \in \Delta^n$ be a simplex known only to the hider. The goal of the seeker is to determine whether or not $\sigma \in M$ using as few questions as possible. The seeker is permitted questions of the form "Is vertex v_i in σ?" The hider is allowed to use previous questions to help determine whether to answer "yes" or "no" to the next question. The seeker uses an algorithm A based on all previous answers to determine which vertex to ask about. Any such algorithm A is called a **decision tree algorithm**.

Example 6.18. Let us see how this game played with simplicial complexes is the same game played with Boolean functions by starting with the Boolean function from Example 6.1, defined by $f : \{0,1\}^{99+1} \to \{0,1\}$ where $f(\vec{x}) = 0$ for all $\vec{x} \in \{0,1\}^{99+1}$. Then $n = 99$ so that we have Δ^{99}.

The subcomplex M known to both players is then the simplicial complex induced by the Boolean function, which in this case is again Δ^{99}. Now think about this. It makes no difference at all what hidden face $\sigma \in \Delta^{99}$ is. Since $\sigma \in \Delta^{99}$ and $M = \Delta^{99}$, σ is most certainly in M. So the seeker will immediately win without asking any questions, just as in Example 6.1.

Example 6.19. Now we will illustrate with the threshold function T_2^3, a more interesting example. In this case, $\Delta^n = \Delta^3$ and $M \subseteq \Delta^3$ is the simplicial complex from Example 6.10, the complete graph on four vertices, denoted by K_4, or the 1-skeleton of Δ^3. We can consider the same mock versions of the game as in the dialogue preceding Problem 6.5.

SEEKER: Is $x_1 \in \sigma$?

HIDER: No.

SEEKER: Is $x_3 \in \sigma$?

HIDER: No.

SEEKER: Now I know that $\sigma \in M$ because everything that is left over is contained in the 1-skeleton, which is precisely M.

Again, as before, this illustrates a poor choice of strategy by the hider (following the algorithm "always say no"[2]). Let's show a better strategy for the hider.

SEEKER: Is $x_1 \in \sigma$?

HIDER: No.

SEEKER: Is $x_3 \in \sigma$?

HIDER: Yes.

SEEKER: Is $x_2 \in \sigma$?

HIDER: Yes.

SEEKER: Is $x_0 \in \sigma$?

HIDER: No.

SEEKER: Now I know that $\sigma = x_2x_3 \in M$, but it is too late. I needed the whole input.

[2]Much to the displeasure of Joe Gallian.

Notice that the hider keeps the seeker guessing until the very end. If you follow along, you realize that everything depends on the answer to the question “Is $x_0 \in \sigma$?”

Denote by $Q(\sigma, A, M)$ the number of questions the seeker asks to determine if σ is in M using algorithm A. The **complexity** of M, denoted by $c(M)$, is defined by

$$c(M) := \min_A \max_\sigma Q(\sigma, A, M).$$

If $c(M) = n+1$ for a particular M, then M is called **evasive**, and it is called **nonevasive** otherwise. For the examples above, we saw that Δ^{99} is nonevasive while K_4 is evasive. Of course, this language is intentionally consistent with the corresponding Boolean function language. Furthermore, notice how evasiveness is just the special case where $c(M) = n+1$, while for nonevasive complexes $c(M)$ can have any value from 0 (extremely nonevasive) to n (as close to being evasive as possible without actually being evasive).

Any σ such that $Q(\sigma, A, M) = n + 1$ is called an **evader** of A. In Example 6.19, we saw that the hider had chosen $\sigma = x_2x_3$, so x_2x_3 is an evader. However, the hider could have just as easily answered “yes” to the seeker’s last question, and in that case $\sigma = x_2x_3x_0$ would have been an evader. Hence, evaders come in pairs in the sense that by the time the seeker gets down to question $n + 1$, she is trying to distinguish between two simplices σ_1 and σ_2. If $\sigma_1 < \sigma_2$ with $\dim(\sigma_1) = p$, we call p the **index** of the evader pair $\{\sigma_1, \sigma_2\}$.

Exercise 6.20. Show that for a pair of evaders $\{\sigma_1, \sigma_2\}$, we have $\sigma_1 < \sigma_2$ (without loss of generality), $\dim(\sigma_2) = \dim(\sigma_1) + 1$, and that $\sigma_1 \in M$ while $\sigma_2 \notin M$. What does this remind you of?[3]

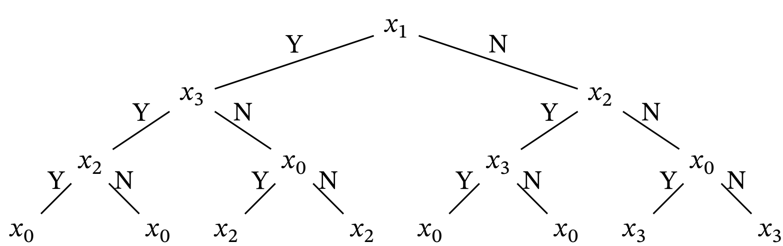
[3] I suppose in theory this could remind you of anything, from Sabellianism to pop tarts, but try to give a coherent answer.

6.4. Discrete Morse theory and evasiveness

You may have observed in Exercise 6.20 that a pair of evaders is a lot like an elementary collapse. Yet while a collapse represents something simple, a pair of evaders represents something complex, almost like a critical simplex. We will use discrete Morse theory to prove the following:

Theorem 6.21. Let A be a decision tree algorithm and let $e_p(A)$ denote the number of pairs of evaders of A having index p. For each $p = 0, 1, 2, \ldots, n-1$, we have $e_p(A) \geq b_p(M)$ where $b_p(M)$ denotes the pth Betti number of M.

In order to prove Theorem 6.21, we need to carefully investigate the relationship between the decision tree algorithm A chosen by the seeker and discrete Morse theory. To do this, let us explicitly write down a possible decision tree algorithm A that the seeker could execute in the case of our running Example 6.19. A node x_i in the tree below is shorthand for the question "Is vertex $x_i \in \sigma$?"



Once the seeker is on the very last row, she knows whether or not the previous vertices are in σ but cannot be sure if the current vertex is in σ. For example, suppose the seeker finds herself on the third value from the left, x_2. Working our way back up, the seeker knows that $x_0x_1 \in \sigma$ but isn't sure if $x_2 \in \sigma$. In other words, she cannot distinguish between x_0x_1 and $x_0x_2x_1$. The key to connecting this algorithm to discrete Morse theory is to use the observation of Exercise 6.20 and view $(x_0x_1, x_0x_2x_1)$ as a vector of a gradient vector field. In fact, we can do this for all nodes in the last row to obtain a gradient vector field on Δ^3. Actually, we have to be a little careful. Notice that the rightmost node on the bottom row corresponds to "vector" $(\emptyset, x_3)$, which we do not consider part of a gradient vector field. But if we throw it out, we obtain a gradient vector field that collapses Δ^3 to the vertex x_3.

Problem 6.22. Write down explicitly a possible decision tree algorithm for the Boolean projection function $P_3^4 : \{0,1\}^{4+1} \to \{0,1\}$. (Note that this requires you to translate this into a question of simplicial complexes.) Compute the corresponding induced gradient vector field on Δ^4.

In general, any decision tree algorithm A induces a gradient vector field $V_A = V$ on Δ^n as follows: For each path in a decision tree, suppose the seeker has asked n questions and has determined that $\alpha \subseteq \sigma$. The $(n+1)$st (and final) question is "Is $v_i \in \sigma$?" Let $\beta = \alpha \cup \{v_i\}$, and include $\{\alpha, \beta\} \in V$ if and only if $\alpha \neq \emptyset$. We saw in Section 5.2.2 both a partial and a total order on the simplices of a simplicial complex. In the proof of the following lemma, we will construct another total order on the simplices of Δ^n.

Lemma 6.23. Let A be a decision tree algorithm. Then $V = V_A$ is a gradient vector field on Δ^n.

Proof. By Theorem 2.51, it suffices to show that V is a discrete vector field with no closed V-paths. That V is a discrete vector field is clear, since each path in the decision tree algorithm corresponds to a unique simplex.

To show that there are no closed V-paths, we put a total order $\prec$ on the simplices of Δ^n and show that if $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ is a V-path, then $\alpha_0 \succ \beta_0 \succ \alpha_1 \succ \beta_1 \succ \cdots$. That this is sufficient is Exercise 6.24. Assign to each edge labeled Y the depth of its sink node (lower node). For each path from the root vertex to a **leaf** (vertex with a single edge), construct a tuple whose entries are the values assigned to Y each time an edge labeled Y is traversed. In this way, we associate to each simplex $\alpha^{(p)}$ a tuple $n(\alpha) := (n_0(\alpha), n_1(\alpha), \ldots, n_p(\alpha))$ where $n_i(\alpha)$ is the value of the ith Y answer and $n_0(\alpha) < \cdots < n_p(\alpha)$. For any two simplices $\alpha^{(p)}$ and $\beta^{(q)}$ we define $\alpha \succ \beta$ if there is a k such that $n_k(\alpha) < n_k(\beta)$ and $n_i(\alpha) = n_i(\beta)$ for all $i < k$. If there is no such k and $q > p$, then $n_i(\alpha) = n_i(\beta)$ for all $0 \leq i \leq p$, so set $\alpha \succ \beta$. This order is shown to be transitive in Problem 6.25.

Now we show that the total order we defined above is preserved by a V-path. To that end, suppose $\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}$ is a segment in a V-path. We show that $\alpha_0^{(p)} \succ \beta_0^{(p+1)} \succ \alpha_1^{(p)}$. Since $(\alpha_0, \beta_0) \in V$, we have that $\alpha_0 \neq \alpha_1$ and $\alpha_1 \subseteq \beta_0$. Now α_0 and β_0 differ by only a single vertex, a difference which was not determined until the very bottom row of the decision tree. Hence, by definition of our total order, $n_i(\alpha) = n_i(\beta)$ for all $1 \leq i \leq p$ while $n_{p+1}(\beta) = n + 1$ and $n_{p+1}(\alpha_0)$ is undefined. Thus $\alpha \succ \beta$.

It remains to show that $\beta_0^{(p+1)} \succ \alpha_1^{(p)}$. Write $\beta_0 = u_0u_1 \cdots u_pu_{p+1}$. For $\sigma = \alpha_0$ or β_0, we then have that the question $n_i(\beta_0)$ is "Is $u_i \in \sigma$?" Since $\alpha_1 \subseteq \beta_0$, there is a k such that $\alpha_1 = u_0u_1 \cdots u_{k-1}u_{k+1} \cdots u_{p+1}$. Observe that the first $n_k(\beta_0) - 1$ questions involve $u_0, \ldots, u_{k-1}$ as well as vertices not in β_0, which also means that they are not in α_0 or α_1. Hence, the first $n_k(\beta_0) - 1$ answers will all be the same whether $\sigma = \alpha_0, \alpha_1$, or β_0; i.e., the corresponding strings all agree until n_k. For n_k, the question is "Is $u_k \in \sigma$?" If $\sigma = \beta_0$, the answer is "yes." If $\sigma = \alpha_1$, the answer is "no." Thus, the next value in the string for α_1 will be larger than the value just placed in $n_k(\beta_0)$. We conclude that $n_i(\alpha_1) = n_i(\beta_0)$ for $i < k$ and $n_k(\alpha_1) > n_k(\beta_0)$, hence $\beta_0 \succ \alpha_1$. $\square$

It may aid understanding of the construction in Lemma 6.23 to draw an example and explicitly compute some of these strings.

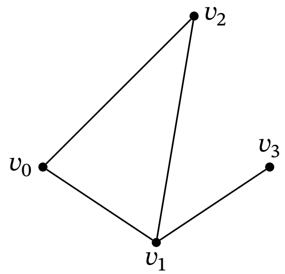
Problem 6.24. Suppose that there is a total order $\prec$ on Δ^n such that if $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ is a V-path, then $\alpha_0 \succ \beta_0 \succ \alpha_1 \succ \beta_1 \succ \cdots$. Prove that V does not contain any closed V-paths.

Problem 6.25. Prove that the order defined in Lemma 6.23 is transitive.

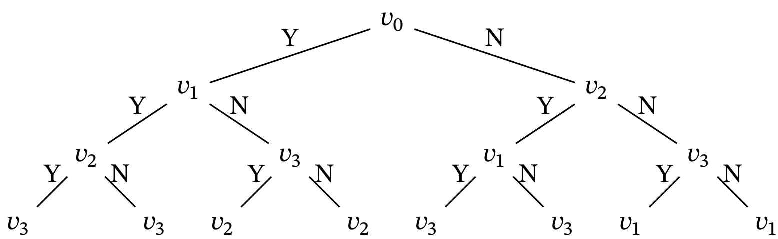
Exercise 6.26. Let A be a decision tree algorithm and V the corresponding gradient vector field. How can you tell which vertex V collapses onto immediately from the decision tree A?

There is one more fairly involved result that we need to combine with Lemma 6.23 in order to prove Theorem 6.21. The upshot is that after this result, not only will we be able to prove the desired theorem, but we will also get several other corollaries for free. We motivate this result with an example.

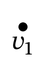
Example 6.27. Let $M \subseteq \Delta^3$ be given by



and consider the decision tree

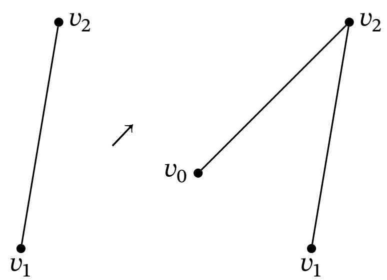


We know from Lemma 6.23 that A induces a gradient vector field on Δ^3, which in this case is a collapse onto v_1. Now the pairs of evaders are easily seen to be $\{v_0v_1, v_0v_1v_3\}, \{v_1v_2, v_1v_2v_3\}$, and $\{v_2, v_2v_3\}$, and by definition one element in each pair lies in M while the other lies outside of M. Hence, if we think of the evaders as critical elements, note that we can start with v_1 and perform elementary expansions and additions of the three evaders in M to obtain M. Furthermore, starting from M, we can perform elementary expansions and attachments of the three evaders not in M to obtain Δ^3. What are these elementary expansions? They are precisely the elements of the induced gradient vector field which are not the evaders. In the decision tree above, these are $\{v_0v_1v_2, v_0v_1v_2v_3\}, \{v_0v_3, v_0v_3v_2\}, \{v_0, v_0v_2\}$, and $\{v_3, v_3v_1\}$. Note that the excluded $\{\emptyset, v_1\}$ tells us where to begin. Explicitly, the series of expansions and attachments of evaders is given by starting with the single vertex

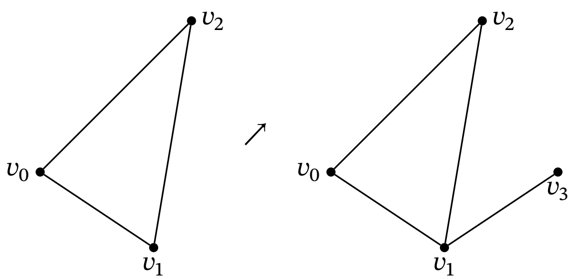


Next we attach the evader v_2:

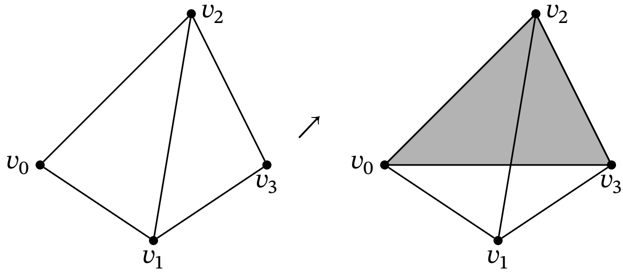
Now we attach the evader $v_1 v_2$ and perform the elementary expansion $\{v_0, v_0 v_2\}$:



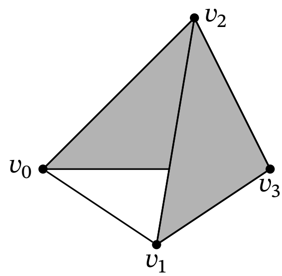
To obtain M, we attach the evader $v_0 v_1$ and make the expansion $\{v_3, v_3 v_1\}$:



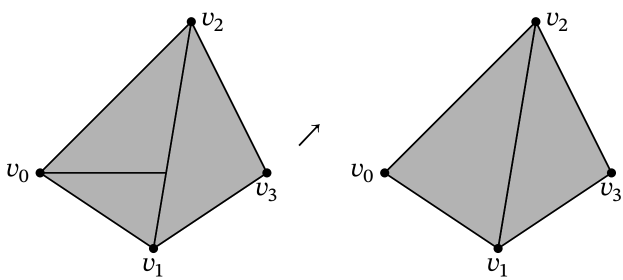
We continue to build from M all the way up to Δ^3. This begins with the attachment of the evader v_2v_3 followed by the expansion $\{v_0v_3, v_0v_3v_2\}$:



Next we have the attachment of evader $v_1v_3v_2$:



And finally we have the attachment of $v_0v_1v_3$ followed by the expansion $\{v_0v_1v_2, v_0v_1v_2v_3\}$:



This completes the construction of Δ^3 from A.

We now take the ideas from Example 6.27 and generalize to construct Δ^n out of expansions and attachments of evaders using the information in a decision tree A.

Theorem 6.28. Let A be a decision tree algorithm and k the number of pairs of evaders of A. If $\emptyset$ is not an evader of A, then there are evaders $\sigma_1^1, \sigma_1^2, \dots, \sigma_1^k$ of A in M, and evaders $\sigma_2^1, \sigma_2^2, \dots, \sigma_2^k$ of A not in M, along with a nested sequence of subcomplexes of Δ^n,

$$v = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{k-1} \subseteq M_k \subseteq M = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k \subseteq \Delta^n$$

where v is a vertex of M which is not an evader of A, such that

$$\begin{array}{rcl} v & \nearrow & M_1 - \sigma_1^1 \\ M_1 & \nearrow & M_2 - \sigma_1^2 \\ M_2 & \nearrow & M_3 - \sigma_1^3 \\ & \vdots & \\ M_{k-1} & \nearrow & M_k - \sigma_1^k \\ M_k & \nearrow & M = S_0 \\ S_0 & \nearrow & S_1 - \sigma_2^1 \\ & \vdots & \\ S_{k-1} & \nearrow & S_k - \sigma_2^k \\ S_k & \nearrow & \Delta^n \end{array}$$

If $\emptyset$ is an evader of A, the theorem holds if $\sigma_1^1 = \emptyset$ and $M_0 = M_1 = \emptyset$. This requires $M_2 = \sigma_1^2$.

Proof. First suppose that $\emptyset$ is not an evader of A. We will construct a discrete Morse function on Δ^n whose critical simplices are the evaders of A and induced gradient vector field $V_A = V$ minus the evader pairs. Let $W \subseteq V$ be the set of pairs $(\alpha, \beta) \in V$ such that either $(\alpha, \beta) \in M$ or $(\alpha, \beta) \notin M$. By Lemma 6.23, W is a gradient vector field on Δ^n. We now determine the critical simplices of W. By construction, a pair (α, β) of simplices of V is not in W if and only if $\alpha \in M$ and $\beta \notin M$ or $\alpha \notin M$ and $\beta \in M$; i.e., all evaders of A are critical simplices of W. Furthermore, the vertex v, which is paired with $\emptyset$, is also critical. This vertex and the evaders of A form all the critical simplices of W. It remains to ensure that the ordering of expansions and attachments given in the statement of the theorem is respected.

To that end, let $f: \Delta^n \to \mathbb{R}$ be any discrete Morse function with induced gradient vector field W. By definition, if $\alpha^{(p)} \in M$ and $\gamma^{(p+1)} \notin M$ with $\alpha < \gamma$, then $(\alpha, \gamma) \notin W$ so that $f(\gamma) > f(\alpha)$. Define

$$\begin{aligned} a &:= \sup_{\alpha \in M} f(\alpha), \\ b &:= \inf_{\alpha \notin M} f(\alpha), \\ c &:= 1 + a - b, \\ d &:= \inf_{\alpha \in \Delta^n} f(\alpha), \end{aligned}$$

and a new discrete Morse function

$$g: \Delta^n \to \mathbb{R}$$

by

$$g(\alpha) := \begin{cases} f(\alpha) & \text{if } \alpha \in M - v, \\ f(\alpha) + c & \text{if } \alpha \notin M, \\ d - 1 & \text{if } \alpha = v. \end{cases}$$

To see that g is a discrete Morse function with the same critical simplices as f, observe that for each $\alpha \in M$ and $\beta \notin M$, $g(\beta) \geq c + 1 > c \geq g(\alpha)$. For each pair $\alpha^{(p)} < \beta^{(p+1)}$, we have $g(\beta) > g(\alpha)$ if and only if $f(\beta) > f(\alpha)$. Then g is a discrete Morse function with the same critical simplices as f.

The case where $\emptyset$ is an evader of A is similar. $\square$

Exercise 6.29. Explain why the function f in Theorem 6.28 may not be the desired discrete Morse function. In other words, why was it necessary to construct the function g?

The proof of Theorem 6.21 follows immediately from Theorem 4.1; that is, for any decision tree A, $e_i(K) \geq b_i(K)$ where e_i is the number of pairs of evaders of index i. An immediate corollary is the following:

Corollary 6.30. For any decision tree algorithm A, the number of pairs of evaders of A is greater than or equal to $\sum_{i=0}^{n} b_i$.

Another corollary of Theorem 6.28 is the following:

Theorem 6.31. If M is nonevasive, then M is collapsible.

Problem 6.32. Prove Theorem 6.31.

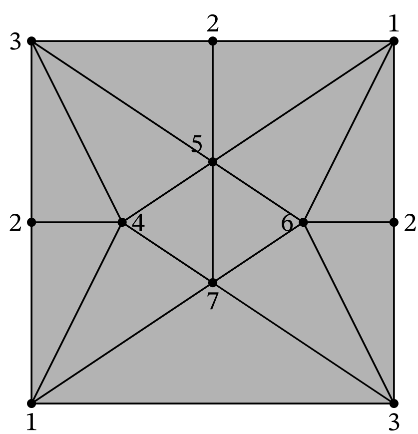
An interesting question is whether or not the converse of Theorem 6.31 is true. That is, is nonevasive the same thing as collapsible? The following example answers this question in the negative. One concept that will be helpful is that of the **link** of a vertex. We define it below and will study it in more detail in Chapter 10.

Definition 6.33. Let K be a simplicial complex and $v \in K$ a vertex. The **star of** v **in** K, denoted by $\text{star}_K(v)$, is the simplicial complex induced by the set of all simplices of K containing v. The **link of** v **in** K is the set $\text{link}_K(v) := \text{star}_K(v) - \{\sigma \in K : v \in \sigma\}$.

The following lemma gives a sufficient condition for detecting evasiveness.

Lemma 6.34. If $M \subseteq \Delta^n$ is nonevasive, then there exists a vertex $v \in M$ such that $\text{link}_{\Delta^n}(v)$ is nonevasive.

Example 6.35. Let C be the simplicial complex below.



Note that there is only one free pair (namely, $\{13, 137\}$) and that starting from this free pair, we may collapse C. Showing that C is evasive is Problem 6.36.

Problem 6.36. Show that C in Example 6.35 is evasive.

Chapter 7

The Morse complex

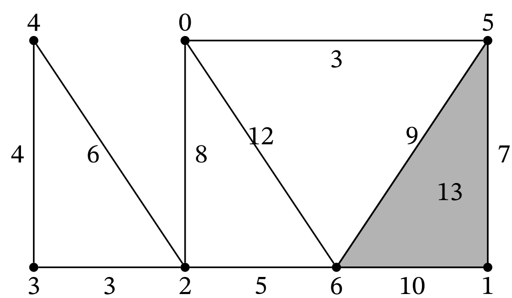
The Morse complex was first introduced and studied in a paper by Chari and Joswig [**42**] and was studied further by Ayala et al. [**7**], Capitelli and Minian [**39**], and Lin et al. [**110**], among others. Like much of what we have seen so far, the Morse complex has at least two equivalent definitions. We will give these definitions of the Morse complex and leave it to you in Problem 7.7 to show that they are equivalent. Because we will be working up to Forman equivalence in this chapter, we will often identify a discrete Morse function f with its induced gradient vector field V_f and use the notations interchangeably.

7.1. Two definitions

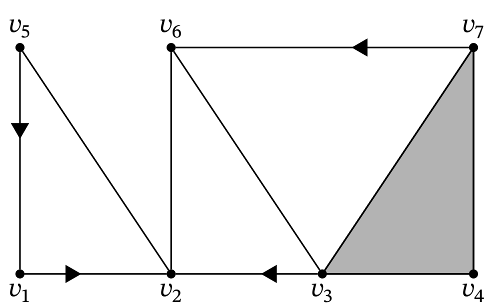
Recall from Section 2.2.2 that given a discrete Morse function $f : K \to \mathbb{R}$, we construct the induced gradient vector field (which is also a discrete vector field) $V_f = V$ and use this to form the directed Hasse diagram $\mathcal{H}_V$. This is accomplished by drawing an upward arrow on the edge connecting each regular pair while drawing a downward arrow on all remaining edges. We then saw in Theorem 2.69 that the resulting directed graph will contain no directed cycles. Any Hasse diagram drawn in this way (or more generally a directed graph with no directed cycles) we will call **acyclic**. Furthermore, we know from Lemma 2.24 that the set of upward-pointing arrows forms a **matching** on the set of edges of

$\mathcal{H}_V$, that is, a set of edges without common nodes. Hence, to use our new jargon, a discrete Morse function always yields an acyclic matching of the induced directed Hasse diagram. An acyclic matching of the directed Hasse diagram is called a **discrete Morse matching**.

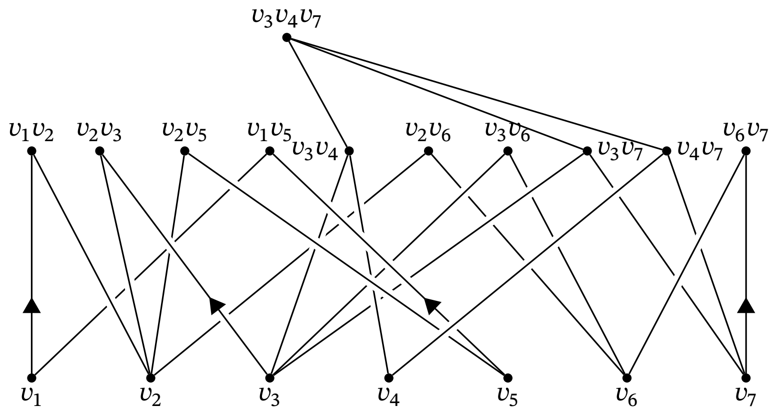
Example 7.1. To remind yourself of this construction, let $f : K \to \mathbb{R}$ be the discrete Morse function from Example 2.88 given by



Giving names to the vertices, the induced gradient vector field V_f is given by



while the directed Hasse diagram $\mathcal{H}_V$ is as shown below (note that the downward arrows are not drawn in order to avoid cluttering the picture).

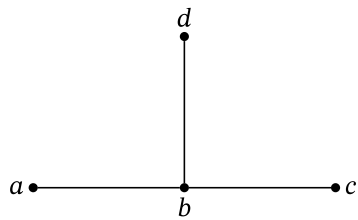


If $f : K \to \mathbb{R}$ is a discrete Morse function, we will sometimes use $\mathcal{H}_f$ to denote the directed Hasse diagram induced by V_f. Because we are primarily interested in $\mathcal{H}_f$ in this section, we define two discrete Morse functions $f, g : K \to \mathbb{R}$ to be **Hasse equivalent** if $\mathcal{H}_f = \mathcal{H}_g$. Recall from Corollary 2.70 that f and g are Hasse equivalent if and only if they are Forman equivalent.

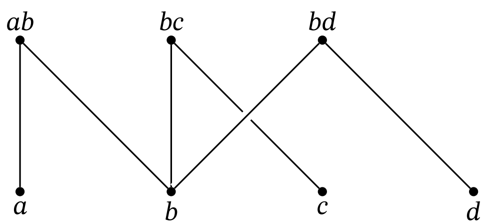
It is not difficult to see that one could include more edges in the discrete Morse matching in Example 7.1. Moreover, there are many different discrete Morse matchings one could put on the Hasse diagram, ranging from the empty discrete Morse matching (no edges) to a "maximal" discrete Morse matching consisting of as many edges as possible without a cycle. It turns out that the collection of all discrete Morse matchings has the added structure of itself being a simplicial complex.

Definition 7.2. Let K be a simplicial complex. The **Morse complex** of K, denoted by $\mathcal{M}(K)$, is the simplicial complex on the set of edges of $\mathcal{H}(K)$ defined as the set of subsets of edges of $\mathcal{H}(K)$ which form discrete Morse matchings (i.e., acyclic matchings), excluding the empty discrete Morse matching.

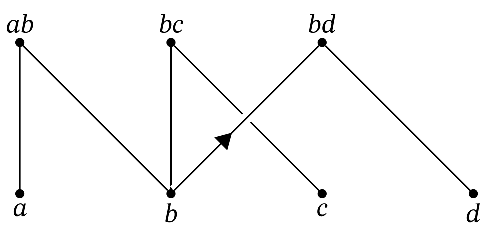
Example 7.3. Let G be the simplicial complex given by

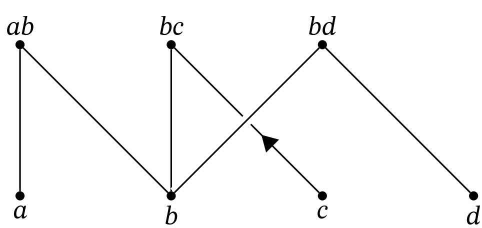


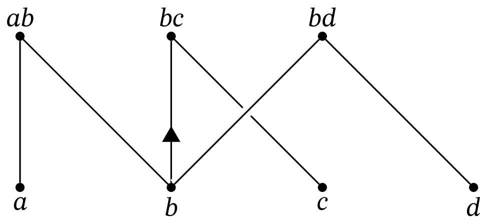
Its Hasse diagram $\mathcal{H}_G = \mathcal{H}$ is

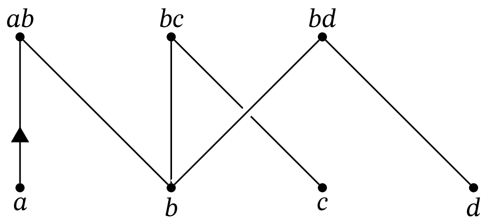


We wish to find all discrete Morse matchings on $\mathcal{H}$. Four such matchings are given below.

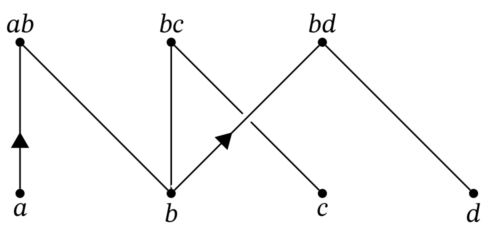




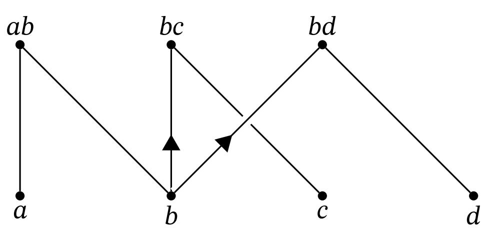




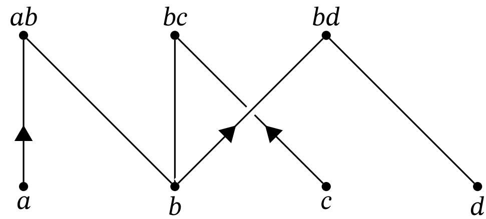
Call these matchings bd, cb, bc, and ab, respectively. Then these four matchings correspond to four vertices in the Morse complex. Higher-dimensional simplices then correspond to more arrows on the Hasse diagram. For example, there is an edge between bd and ab in the Morse complex since



is a discrete Morse matching. There is not, however, an edge between bc and bd in the Morse complex since



is not a discrete Morse matching. We obtain a 2-simplex in the Morse complex by observing that

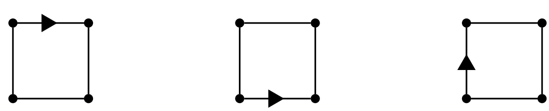


is a discrete Morse matching. Continuing in this way, one constructs the Morse complex $\mathcal{M}(G)$.

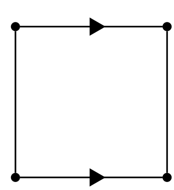
Problem 7.4. Compute the Morse complex of the simplicial complex given in Example 7.3.

We now give an alternative definition of the Morse complex. This definition builds the Morse complex out of gradient vector fields from the ground up. If a discrete Morse function f has only one regular pair, we call f **primitive**. Given multiple primitive discrete Morse functions, we wish to combine them to form a new discrete Morse function. This will be accomplished by viewing each discrete Morse function as a gradient vector field and "overlaying" all the arrows. Clearly such a construction may or may not yield a gradient vector field.

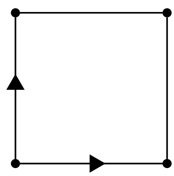
Example 7.5. Let primitive gradient vector fields f_0, f_1, and f_2 be given by



respectively. Then f_0 and f_1 combine to form a new gradient vector field f:



But clearly combining f_1 and f_2 gives



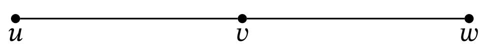
which does not yield a gradient vector field.

If $f, g\colon K \to \mathbb{R}$ are two discrete Morse functions, write $g \leq f$ whenever the regular pairs of g are also regular pairs of f. In general, we say that a collection of primitive discrete Morse functions $f_0, f_1, \ldots, f_n$ is **compatible** if there exists a discrete Morse function f such that $f_i \leq f$ for all $0 \leq i \leq n$.

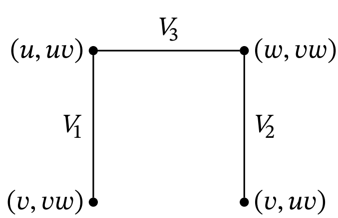
Definition 7.6. The **Morse complex** of K, denoted by $\mathcal{M}(K)$, is the simplicial complex whose vertices are given by primitive discrete Morse functions and whose n-simplices are given by gradient vector fields with $n + 1$ regular pairs. A gradient vector field f is then associated with all primitive gradient vector fields $f\ := \{f_0, \ldots, f_n\}$ with $f_i \leq f$ for all $0 \leq i \leq n$.

Problem 7.7. Prove that Definitions 7.2 and 7.6 are equivalent.

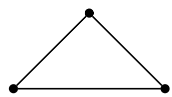
Example 7.8. We construct the Morse complex for the graph given by



There are four primitive pairs, namely, $(u, uv), (w, vw), (v, uv)$, and (v, vw). These pairs correspond to four vertices in the Morse complex. The only primitive vectors that are compatible are $V_1 = \{(u, uv), (v, vw)\}$, $V_2 = \{(w, vw), (v, uv)\}$, and $V_3 = \{(u, uv), (w, vw)\}$. Hence the Morse complex is given by



Exercise 7.9. Construct the Morse complex for the graph

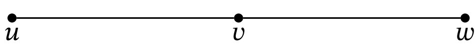


Problem 7.10. Let K be a 1-dimensional, connected simplicial complex with e 1-simplices and v 0-simplices. Prove that $\dim(\mathcal{M}(G)) = v - 2$. How many vertices does $\mathcal{M}(K)$ have?

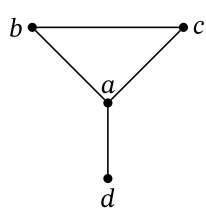
Problem 7.11. Prove that there does not exist a simplicial complex K such that $\mathcal{M}(K) = \Delta^n$ for any $n \geq 1$.

Capitelli and Minian have shown that a simplicial complex is uniquely determined by its Morse complex [**39**] up to isomorphism. The proof is beyond the scope of this text, but we can show that a weakening is false. Specifically, we show that this is not true up to simple homotopy type.

Example 7.12 (Capitelli and Minian)**.** Consider the graphs G given by



and G' given by



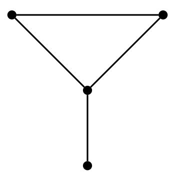
We saw in Example 7.8 that $\mathcal{M}(G)$ is given by a path on four vertices and it can be checked whether $\mathcal{M}(G')$ is collapsible. Clearly then G and G' do not have the same simple homotopy type, but $\mathcal{M}(G) \searrow * \swarrow \mathcal{M}(G')$ so the Morse complexes have the same simple homotopy type.

At this point, it should be clear that the Morse complex gets out of hand very, very quickly. Hence we will study two special cases of the Morse complex.

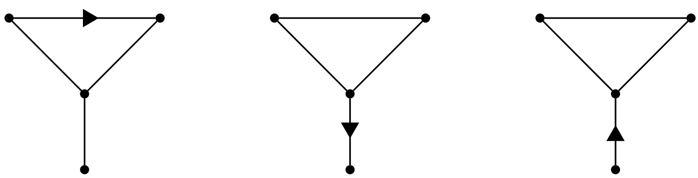
7.2. Rooted forests

In this section we limit ourselves to the study of 1-dimensional simplicial complexes or **graphs**. Hence, we will employ some terminology from graph theory. Our reference for graph theory basics and more information is [**43**], but we give a quick review of commonly used terms and notation here. Any 1-dimensional simplicial complex is also referred to as a **graph**. The 0-simplices are **vertices** and the 1-simplices are called **edges**. A graph G such that $b_0(G) = 1$ is called **connected**. A connected graph G with $b_1(G) = 0$ is called a **tree**.

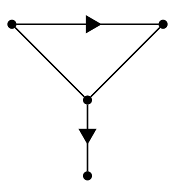
Example 7.13. Let G be the graph



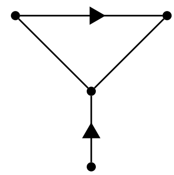
and consider the three primitive gradient vector fields



denoted by f_0, f_1, and f_2, respectively. Then f_0 and f_1 are compatible, and f_0 and f_2 are compatible, since there exist the gradient vector fields f and g given by



and



respectively. Notice that by removing all critical edges from f and g, we obtain two graphs, each consisting of two trees. A graph that is made up of one or more components, each of which is a tree, is called a **forest**, appropriately enough. Furthermore, each tree "flows" via the gradient vector field to a unique vertex, called the root. The **root** of a tree equipped with a gradient vector field without critical edges is the unique critical vertex. Such a graph is called a **rooted forest**.

Definition 7.14. Let G be a graph. A subset F of edges of G along with a choice of direction for each edge is called a **rooted forest** of G if F is a forest as an undirected graph and each component of F has a unique root with respect to the given gradient vector field.

In Example 7.13, we obtained a rooted forest from a gradient vector field, i.e., an element of the Morse complex. Conversely, you can imagine that given a rooted forest, we can construct an element of the Morse complex. This is summed up in the following theorem.

Theorem 7.15. Let G be a graph. Then there is a bijection between the simplices of $\mathcal{M}(G)$ and rooted forests of G.

Proof. Let $f = \sigma \in \mathcal{M}(G)$. We construct a rooted forest $\mathcal{R}(\sigma)$ of G by orienting the edges according to the gradient of f and removing the critical edges. By definition, f can be viewed as giving an orientation to a subset of edges in G. This subset of edges (along with the corresponding vertices) is clearly a forest. To see that it is rooted, consider any tree in the forest. If the tree does not have a unique root, then it contains either a directed cycle or multiple roots. Clearly a directed cycle is impossible, since it is a tree. Multiple roots would occur if there were a vertex with two arrows leaving it, i.e., a vertex which was the head of two arrows. This is impossible by Lemma 2.24. Hence the gradient vector field of f with the critical edges removed is a rooted forest of G.

Now let R be a rooted forest of G. We construct a simplex $\mathcal{S}(\mathcal{R}) \in \mathcal{M}(G)$. This is accomplished by starting with the rooted forest with each edge oriented, and then adding in the edges of G not in R. This provides an element $\mathcal{S}(R) \in \mathcal{M}(G)$ since each tree has a unique root. Hence, each vertex and edge appears in at most one pair, creating a gradient vector field. □

Problem 7.16. Show that the two operations in Theorem 7.15 are inverses of each other. That is, show that if $f \in \mathcal{M}(G)$, then $\mathcal{S}(\mathcal{R}(f)) = f$, and if R is a rooted forest of G, then $\mathcal{R}(\mathcal{S}(R)) = R$.

Given the result of Theorem 7.15, we will use "gradient vector field" and "rooted forest" interchangeably (again, in the context of graphs).

Simplicial complexes of rooted forests were first studied by Kozlov [**101**]. However, Kozlov's work was not in the context of discrete Morse theory. It was Chari and Joswig, cited above, who first worked within the general framework.

Just counting the number of simplices in a Morse complex can be a huge job. Although Theorem 7.15 gives us a way to count the number of simplices of the Morse complex, it simply reduces the problem to one in graph theory. Fortunately, much is known. For details on this and other interesting counting problems in graph theory, see J. W. Moon's classic book [**120**].

7.3. The pure Morse complex

Another special class of simplicial complexes are the pure complexes.

Definition 7.17. An abstract simplicial complex K is said to be **pure** if all its facets are of the same dimension.

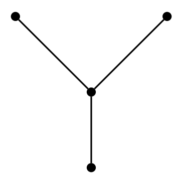
As mentioned above, the Morse complex is quite large and difficult to compute. We can make our work easier by "purifying" any Morse complex.

Definition 7.18. Let $n := \dim(\mathcal{M}(K))$. The **pure Morse complex of discrete Morse functions**, denoted by $\mathcal{M}_P(K)$, is the subcomplex of $\mathcal{M}(K)$ generated by the facets of dimension n.

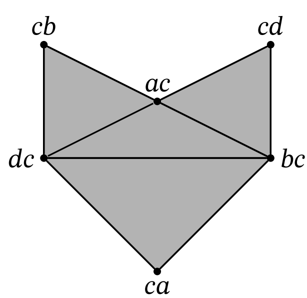
Problem 7.19. Give an example to show that if K is a pure simplicial complex, then $\mathcal{M}(K)$ is not necessarily pure (and thus Definition 7.18 is not redundant).

7.3.1. The pure Morse complex of trees. In Theorem 7.15, we found a bijection between the Morse complex of a graph and rooted forests of that same graph. We now investigate this connection more deeply. First, we look at the pure Morse complex of a tree. The results in this section are originally due to R. Ayala et al. [**7**, Proposition 2].

Example 7.20. Let G be the graph



To simplify notation, we will denote a 0-simplex (v, vu) of $\mathcal{M}(G)$ by vu, indicating that the arrow points away from vertex v towards vertex u (this notation only works when G is a graph). It is easily verified that $\mathcal{M}(G)$ is given by



Notice that there is a 1–1 correspondence between the vertices of G and the facets of $\mathcal{M}_P(G) = \mathcal{M}(G)$. Furthermore, two facets in $\mathcal{M}(G)$ share a common edge if and only if the corresponding vertices are connected by an edge in G.

Problem 7.21. Let T be a tree with n vertices.

(i) Show there is a bijection between the vertices of T and the facets of $\mathcal{M}_P(G)$. [Hint: Given a vertex u of G, how many rooted forests in u that use every edge can be constructed?]

(ii) If $v \in G$, denote by σ_v the unique corresponding facet in $\mathcal{M}_P(G)$. Prove that σ_v is an $(n-2)$-simplex.

(iii) Prove that if uv is an edge in G, then σ_u and σ_v share an $(n-3)$-face. [Hint: If uv is an edge in G, consider the gradient vector field $\sigma_v - \{uv\}$.]

You have proved the following proposition.

Proposition 7.22. Let T be a tree on n vertices. Then $\mathcal{M}_P(G)$ is the simplicial complex constructed by replacing a vertex of T with a copy of Δ^{n-2} and decreeing that two such facets share an $(n-3)$-face if and only if the corresponding vertices are connected by an edge.

As an important corollary, we obtain the following result:

Corollary 7.23. If T is a tree with at least three vertices, then $\mathcal{M}_P(T)$ is collapsible.

Proof. We proceed by induction on n, the number of vertices of T. For $n = 3$, there is only one tree, and the Morse complex of this tree we computed in Example 7.8 to be K_1, which is clearly collapsible. Now suppose that any tree on n vertices has the property that its Morse complex is collapsible for some $k \geq 3$, and let T be any tree on $n + 1$ vertices. Then T has a leaf by Lemma 7.24, call it v. Consider the $(n-1)$-dimensional simplex $\sigma_v \in \mathcal{M}_P(T)$. By Proposition 7.22, all the $(n-2)$-dimensional faces of σ_v are free, except for one (namely, the one corresponding to the edge connecting v to its unique vertex in T). Hence σ_v collapses onto its non-free facet. It is easy to see that the resulting complex is precisely the Morse complex of $T - v$, which is a tree on n vertices and hence collapsible by the inductive hypothesis. Thus the result is proved. □

Lemma 7.24. Every tree with at least two vertices has at least two leaves.

Problem 7.25. Prove Lemma 7.24.

7.3.2. The pure Morse complex of graphs. We now take some of the above ideas and begin to extend them to arbitrary (connected) graphs. The goal of this section will then be to count the number of facets in the pure Morse complex of any graph G.

Definition 7.26. Let G be a graph. A gradient vector field (rooted forest) $f = \{f_0, \dots, f_n\}$ on G is said to be **maximum** if $n + 1 = e - b_1$ where e is the number of edges of G and b_1 is the first Betti number of G.

In other words, a gradient vector field is maximum if "as many edges as possible have arrows." Clearly f is a maximum gradient vector field of G if and only if f is a facet in $\mathcal{M}_P(G)$.

Exercise 7.27. Is a maximum gradient vector field on a graph always unique? Prove it or give a counterexample.

The following exercise gives a characterization of maximum gradient vector fields on graphs. It is simply a matter of staring at both definitions and realizing that they are saying the same thing.

Exercise 7.28. Let G be a connected graph. Prove that a gradient vector field f on G is maximum if and only if f is a perfect discrete Morse function.

It then immediately follows from this and Problem 4.13 that every connected graph has a maximum gradient vector field.

If G is a connected graph on v vertices, we call any subgraph of G on v vertices that is a tree a **spanning tree**. We now determine a relationship between the maximum gradient vector fields of G and its spanning trees. First we state a lemma. If G is a graph, let $v(G)$ denote the number of vertices of G and $e(G)$ the number of edges of G.

Lemma 7.29. Let T be a spanning tree of G. Then $e(G) - e(T) = b_1(G)$.

Proof. By Theorem 3.23, $v - e = b_0 - b_1$ for any graph. In particular, for the spanning tree T and graph G, we have $v(T) - e(T) = 1 - 0$ and $v(G) - e(G) = 1 - b_1(G)$. Subtracting the latter equation from the former and noting that $v(T) = v(G)$ (since T is a spanning tree), we obtain $e(G) - e(T) = b_1(G)$. $\square$

Theorem 7.30. Let G be a connected graph. Then f is a maximum gradient vector field on G if and only if f is a maximum gradient vector field on T for some spanning tree $T \subseteq G$.

Proof. We show the backward direction and leave the forward direction as Problem 7.31. Suppose that $f = \{f_0, \dots, f_n\}$ is a maximum gradient vector field on T for some spanning tree T of G so that $n + 1 = e(T) - b_1(T) = e(T)$. We need to show that f is a maximum gradient vector field on G. Suppose by contradiction that f is not maximum on G so that $n + 1 < e(G) - b_1(G)$. This implies that $e(T) < e(G) - b_1(G)$. But T is a spanning tree of G, so by Lemma 7.29 we have $e(G) - e(T) = b_1(G)$, a contradiction. Thus a maximum gradient vector field on a spanning tree T of G gives rise to a maximum gradient vector field of G. □

Problem 7.31. Prove the forward direction of Theorem 7.30.

Translating this into the language of the Morse complex, we obtain the following corollary.

Corollary 7.32. Let G be a connected graph. Then

$$\mathcal{M}_P(G) = \bigcup_{T_i \in S(G)} \mathcal{M}_P(T_i)$$

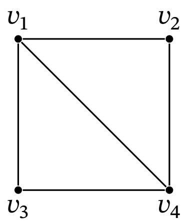
where $S(G)$ is the set of all spanning trees of G.

Problem 7.33. Prove Corollary 7.32.

Finally, we can count the number of facets in $\mathcal{M}_P(G)$. In order to do so, we need a few other ideas from graph theory. Two vertices of a graph are said to be **adjacent** if they are joined by a single edge. The number of edges of any vertex v is known as the **degree** of v, denoted by $\deg(v)$. Given a graph G with n vertices, we then form the $n \times n$ matrix $L(G)$, the **Laplacian**, whose entries are given by

$$L_{i,j}(G) = L := \begin{cases} \deg(v_i) & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.34. Let G be given by



Then it can be easily checked that $L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$.

Equipped with the Laplacian, we may utilize the following theorem to count the number of spanning trees of G, which in turn will allow us to count the number of facets of $\mathcal{M}_P(G)$.

Theorem 7.35 (Kirchhoff's theorem)**.** Let G be a connected graph on n vertices with $\lambda_1, \ldots, \lambda_{n-1}$ the non-zero eigenvalues of $L(G)$. Then the number of spanning trees of G is given by $\frac{1}{n}\lambda_1 \cdots \lambda_{n-1}$.

A proof of this fact goes beyond the scope of this text but may be found in [**43**, Theorem 4.15].

Theorem 7.36. Let G be a connected graph on n vertices. Then the number of facets of $\mathcal{M}_P(G)$ is $\lambda_1 \cdots \lambda_{n-1}$. Equivalently, there are $\lambda_1 \cdots \lambda_{n-1}$ maximum gradient vector fields of G.

Proof. By definition, the facets of $M_P(G)$ are in 1–1 correspondence with the maximum gradient vector fields on G. By Theorem 7.30, the maximum gradient vector fields on G are precisely those of all the spanning trees of G. Given a spanning tree of G and a vertex of G, there exists exactly one forest rooted in v. Hence, each spanning tree determines n different gradient vector fields. Since there are $\frac{1}{n}\lambda_1 \cdots \lambda_{n-1}$ by Theorem 7.35, there are exactly $n \cdot \left(\frac{1}{n}\lambda_1 \cdots \lambda_{n-1}\right) = \lambda_1 \cdots \lambda_{n-1}$ maximum gradient vector fields on G and hence $\lambda_1 \cdots \lambda_{n-1}$ facets of $\mathcal{M}_P(G)$. $\square$

Problem 7.37. Count the number of facets of $\mathcal{M}_P(G)$ for G in Example 7.34 (if you know how to compute eigenvalues).

Problem 7.38. Compute the Laplacian of the graph in Example 7.20 and use Theorem 7.36 to compute the number of facets of $\mathcal{M}_P(G)$. Does your answer match that computed by hand in the example?

Chapter 8

Morse homology

We saw in Chapter 4 how the theory of simplicial homology relates to discrete Morse theory. This relationship is most clearly seen in Theorem 4.1, where we proved that the number of critical i-simplices always bounds from above the ith $\mathbb{F}_2$-Betti number. But simplicial homology is not the only means by which to compute homology. The discrete Morse inequalities just mentioned point to a much deeper and more profound relationship between discrete Morse theory and homology. The goal of this chapter will be to develop a theory to compute simplicial homology using discrete Morse theory. Once we develop Morse homology, we will use it in Section 8.5 to make computing Betti numbers of certain simplicial complexes easier. We will furthermore utilize this theory to perform computations algorithmically in Section 9.2.

Let f be a discrete Morse function on K. In order to define Morse homology, we undertake a deeper study of $V = V_f$, the induced gradient vector field. In particular, we will view it as a function. Using this, we will construct a collection of $\mathbb{F}_2$-vector spaces $\Bbbk_p^{\Phi}(K)$, along with linear transformations between them, just as we did in Section 3.2. The chain complex consisting of these vector spaces $\Bbbk_p^{\Phi}(K)$ taken together with linear transformations between them is referred to as the **flow complex**. It will then be shown that this chain complex is the same as another chain complex, called the **critical complex**[1]. The vector spaces in this chain complex are not only much smaller than the ones in the standard chain complex, but the boundary maps can be computed using a gradient vector field. From there, just as before, we can construct homology. We will then prove that this way of constructing homology is the same as our construction of homology in Section 3.2. Thus the critical complex has the distinct advantage that since the matrices and vector spaces are much smaller, homology is easier to compute.

8.1. Gradient vector fields revisited

We first view the gradient vector field as a function. Recall from Section 3.2 that if K is a simplicial complex, K_p denotes the set of all p-dimensional simplices of K, $c_p := |K_p|$ is the size of K_p, and $\mathbb{k}^{c_p}$ is the unique vector space of dimension c_p with each element of K_p corresponding to a basis element of $\mathbb{k}^{c_p}$. If f is a discrete Morse function on K, the induced gradient vector field was defined in Chapter 2 by $V = V_f := \{(\sigma^{(p)}, \tau^{(p+1)}) : \sigma < \tau, f(\sigma) \geq f(\tau)\}$.

Definition 8.1. Let $f : K \to \mathbb{R}$ be a discrete Morse function with $V = V_f$ the induced gradient vector field. Define a function $V_p : \mathbb{k}^{c_p} \to \mathbb{k}^{c_{p+1}}$ by

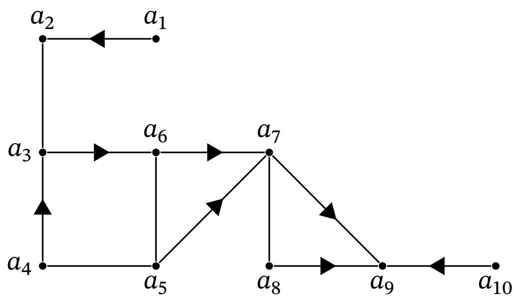
$$V_p(\sigma) := \begin{cases} \tau & \text{if } \exists\, (\sigma, \tau) \in V, \\ 0 & \text{otherwise} . \end{cases}$$

Abusing language and notation, the set of all functions $V := \{V_p\}$ for $p = 0, 1, \ldots, \dim(K)$ will be referred to as the **gradient vector field** induced by f (or V).

Exercise 8.2. Prove that if σ is critical, then $V(\sigma) = 0$. Does the converse hold?

[1]In the literature, this is sometimes referred to as the Morse complex, so be sure not to confuse it with the object defined in Chapter 7.

Example 8.3. Let V be the gradient vector field



Then V is simply associating the corresponding head to any vertex that is a tail; for example, $V_0(a_4) = a_4a_3$, $V_0(a_3) = a_3a_6$, and $V_1(a_4a_3) = V_0(a_2) = 0$.

We can also recast some known facts about the gradient vector field discussed in Chapter 2 from the function viewpoint.

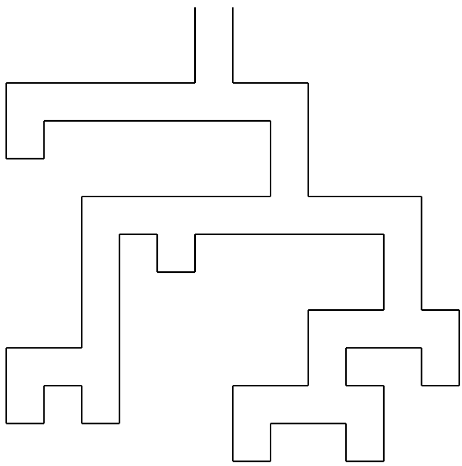
Proposition 8.4. Let V be a gradient vector field on a simplicial complex K and $\sigma^{(p)} \in K$. Then

(i) $V_{i+1} \circ V_i = 0$ for all integers $i \geq 0$;
(ii) $|\{\tau^{(p-1)} : V(\tau) = \sigma\}| \leq 1$;
(iii) σ is critical if and only if $\sigma \notin \mathrm{Im}(V)$ and $V(\sigma) = 0$.

Proof. We suppress the subscripts when there is no confusion or need to refer to dimension. If τ is not the tail of an arrow (the first element in an ordered pair in V), then $V(\tau) = 0$. Hence, assume $(\tau, \sigma) \in V$ so that $V(\tau) = \sigma$. By Lemma 2.24, since σ is the head of an arrow in V, it cannot be the tail of any arrow in V; hence $V(V(\tau)) = V(\sigma) = 0$. The same lemma immediately yields (ii). Part (iii) is Problem 8.5. $\square$

Problem 8.5. Prove part (iii) of Proposition 8.4.

8.1.1. Gradient flow. Look back at Example 8.3. We would now like to use our new understanding of V to help create a description of the "flow" at a vertex, say a_4; that is, starting at a_4, we would like an algebraic way to determine where one is sent after following a sequence of arrows for a fixed number of steps and, ultimately, determine if one ends up flowing to certain simplices and staying there indefinitely. Think, for example, of a series of vertical pipes with a single inflow but emptying in several locations. Water poured into the opening flows downward through several paths and ultimately ends in those several locations.



Following the arrows in Example 8.3, we see that the flow starting at a_4 ultimately leads us to vertex a_9. But what would the flow mean in higher dimensions? The next definition will define such a flow in general.

Recall that for $p \geq 1$, the boundary operator $\partial_p : \mathbb{k}^{c_p} \to \mathbb{k}^{c_{p-1}}$ is defined by $\partial_p(\sigma) := \sum_{0 \leq j \leq p}(\sigma - \{\sigma_{i_j}\}) = \sum_{0 \leq j \leq p} \sigma_{i_0}\sigma_{i_1} \cdots \hat{\sigma_{i_j}} \cdots \sigma_{i_p}$ where $\hat{\sigma}_{i_j}$ excludes the value $\hat{\sigma}_{i_j}$ (Definition 3.12).

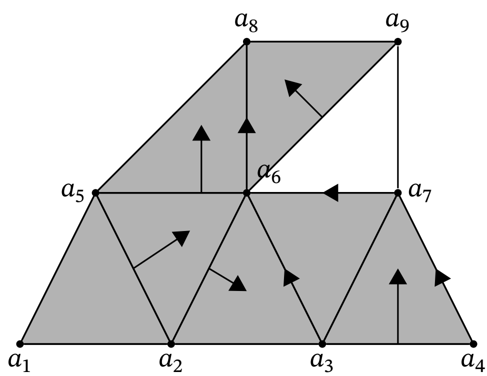
Definition 8.6. Let V be a gradient vector field on K. Define $\Phi_p : \mathbb{k}^{c_p} \to \mathbb{k}^{c_p}$ given by $\Phi_p(\sigma) := \sigma + \partial_{p+1}(V_p(\sigma)) + V_{p-1}(\partial_p(\sigma))$ to be the **gradient flow** or **flow** of V. We write $\Phi(\sigma) = \sigma + \partial(V(\sigma)) + V(\partial(\sigma))$ when p is clear from the context.

Remark 8.7. It is important not to confuse the flow of a simplex with a V-path at that simplex. One important distinction is that while V-paths only look at arrows in dimensions p and $p+1$, flows may take into account arrows in many dimensions.

Let $f : A \to A$ be any function and write $f^n := f \circ f \circ \cdots \circ f$ for the composition of f with itself n times. Then f is said to **stabilize** if there is an integer m and an $x \in A$ such that $f \circ f^m(x) = f^m(x)$. We are interested in determining whether or not a flow Φ stabilizes, and hence in computing Φ^n for $n \geq 1$. Since the values of Φ depend on the chosen gradient vector field V, we sometimes say that V stabilizes if Φ stabilizes.

Problem 8.8. Apply the definition of Φ iteratively starting with a_4 from Example 8.3 until it stabilizes.

Example 8.9. It can be a little tricky to intuitively grasp what the flow is doing in higher-dimensional simplicial complexes, especially when there are multiple ways in which to flow. Let us look at a more complex example. Let V be the gradient vector field below.



We will compute the flow starting at a_2a_5. Before we do the computation, take a moment to think what you expect the answer to be. To make the computation easier, note that $\Phi(a_2a_5) = a_2a_6 + a_5a_6, \Phi(a_2a_6) = a_2a_3 + a_3a_6 + a_6a_8, \Phi(a_5a_6) = a_5a_8, \Phi(a_2a_3) = a_2a_3 + a_3a_6, \Phi(a_3a_6) = a_6a_8, \Phi(a_6a_8) = 0$, and $\Phi(a_5a_8) = a_5a_8$. We thus compute

$$\begin{aligned}
\Phi(a_2a_5) &= a_2a_6 + a_5a_6, \\
\Phi(a_2a_6 + a_5a_6) &= a_2a_3 + a_3a_6 + a_6a_8 + a_5a_8, \\
\Phi(a_2a_3 + a_3a_6 + a_6a_8 + a_5a_6) &= a_2a_3 + a_3a_6 + a_6a_8 + a_5a_8.
\end{aligned}$$

After the first flow, we find ourselves on a_2a_6 and a_5a_6, with two 2-faces to flow into. From there, we find ourselves "stuck" on each of their boundaries. In other words, $\Phi^2(a_2a_5) = \Phi^n(a_2a_5)$ for every $n \geq 2$ so that Φ stabilizes at a_2a_5 with $\Phi^n(a_2a_5) = a_2a_3 + a_3a_6 + a_6a_8 + a_5a_8$ for $n \geq 2$.

Problem 8.10. Verify the computations in Example 8.9; that is, prove that

(i) $\Phi(a_2a_5) = a_2a_6 + a_5a_6$;

(ii) $\Phi(a_2a_6) = a_2a_3 + a_3a_6 + a_6a_8$;

(iii) $\Phi(a_5a_6) = a_5a_8$;

(iv) $\Phi(a_2a_3) = a_2a_3 + a_3a_6$;

(v) $\Phi(a_3a_6) = a_6a_8$;

(vi) $\Phi(a_6a_8) = 0$;

(vii) $\Phi(a_5a_8) = a_5a_8$.

Not only can we compute the flow of a single simplex iteratively, but we can also compute the flow of an entire simplicial complex or subcomplex by decreeing that if $\sigma, \tau \in K$ are simplices, then $\Phi(\sigma \cup \tau) := \Phi(\sigma) + \Phi(\tau)$ where the sum is modulo 2. For example, using the simplicial complex from Example 8.9, we can compute

$$\begin{aligned}\Phi(\langle a_2a_5\rangle) &= \Phi(a_2 \cup a_2a_5 \cup a_5)\\ &= 0 + a_2a_6 + a_5a_6 + 0\\ &= a_2a_6 + a_5a_6.\end{aligned}$$

Problem 8.11. Let K be a simplicial complex and let $\Phi(K) := \{\sum_{\sigma\in K}\Phi(\sigma)\}$. Viewing each summand element as a simplex, is $\Phi(K)$ necessarily a simplicial complex? Prove it or give a counterexample.

We compared stabilization of a flow to water flowing through pipes and ultimately emptying at the bottom. But perhaps this comparison is misleading. Maybe the water loops in a cycle in the pipes or flows within the pipes forever in an erratic way. Turning to mathematical flow, this raises the natural question: if V is a gradient vector field on K, does V stabilize for every $\sigma \in K$? Since K is finite, if V does not stabilize, then it eventually "loops," becomes periodic, or maybe does something else. Applying what we know about discrete Morse functions, we would be surprised if V does become periodic.

Exercise 8.12. Give an example to show that a discrete vector field need not stabilize at every simplex.

Problem 8.13. Let Φ be the flow induced by a gradient vector field V on a simplicial complex K. Prove that Φ stabilizes at every vertex.

Problem 8.13 is a precursor to the more general result that Φ stabilizes at all simplices whenever we have a gradient vector field. Before we show this, we need a few other results that tell us more about what the flow looks like. First we present a lemma.

Lemma 8.14. Let K be a simplicial complex, ∂ the boundary operator, and Φ a flow on K. Then $\Phi\partial = \partial\Phi$.

Problem 8.15. Prove Lemma 8.14.

The following technical result gives us a better understanding of the flow of a p-simplex in terms of a linear combination of all p-simplices.

Proposition 8.16. Let $\sigma_1, \ldots, \sigma_r$ be the p-dimensional simplices of K and write $\Phi(\sigma_i) = \sum_j a_{ij}\sigma_j$ (i.e., as a linear combination of p-simplices) where a_{ij} is either 0 or 1. Then $a_{ii} = 1$ if and only if σ_i is critical. Furthermore, if $a_{ij} = 1$, then $f(\sigma_j) < f(\sigma_i)$.

Proof. By Proposition 8.4, the p-simplex σ_i satisfies exactly one of the following: σ_i is critical, $\sigma_i \in \mathrm{Im}(V)$, or $V(\sigma_i) \neq 0$. We proceed by considering the different cases.

Suppose σ_i is critical. We show that σ_i appears in $\Phi(\sigma_i)$ and that $f(\sigma_j) < f(\sigma_i)$ whenever σ_j appears in $\Phi(\sigma_i)$. Since σ_i is critical, $V(\sigma_i) = 0$ and for any codimension-1 face $\tau < \sigma_i$, $f(\tau) < f(\sigma_i)$. By definition, $V(\tau) = 0$ or $V(\tau) = \tilde{\sigma}_\tau$ with $f(\tilde{\sigma}_\tau) \leq f(\tau) < f(\sigma_i)$. Using these facts, we see that

$$\begin{aligned}\Phi(\sigma_i) &= \sigma_i + V(\partial\sigma_i) + 0\\ &= \sigma_i + V\sum_\tau \tau\\ &= \sigma_i + \sum_\tau V\tau\\ &= \sigma_i + \sum_\tau \tilde{\sigma}_\tau\end{aligned}$$

where all values $\tilde{\sigma}_\tau$ satisfy $f(\tilde{\sigma}_\tau) < f(\sigma_i)$.

Now suppose that $\sigma_i \in \mathrm{Im}(V)$. Then there exists $\eta \in V$ such that $V(\eta) = \sigma_i$. By Proposition 8.4(i), $V \circ V = 0$, yielding

$$\Phi(\sigma_i) = \sigma_i + V(\partial\sigma_i) + \partial(V(V(\eta))) = \sigma_i + \sum_\tau V(\tau)$$

where, as before, $\tau < \sigma_i$ is a codimension-1 face of σ_i. Next, by Proposition 8.4(ii), η is the unique codimension-1 face of σ_i satisfying $V(\eta) = \sigma_i$, and hence

$$\Phi(\sigma_i) = \sigma_i + V(\eta) + \sum_{\tau,\tau\neq\eta} V(\tau) = \sigma_i + \sigma_i + \sum_{\tau,\tau\neq\eta} V(\tau) = \sum_{\tau,\tau\neq\eta} V(\tau).$$

Furthermore, any other codimension-1 face $\tau < \sigma_i$ satisfies $V(\tau) = 0$ or $V(\tau) = \widetilde{\sigma_i}$ with $f(\widetilde{\sigma_i}) \leq f(\eta) < f(\sigma_i)$. Thus

$$\Phi(\sigma_i) = \sum \widetilde{\sigma_i}.$$

Finally, suppose that $V(\sigma_i) = \tau, \tau \neq 0$. For each codimension-1 face $\eta < \sigma_i$, either $V(\eta) = 0$ or $V(\eta) = \widetilde{\sigma_i}$ where $f(\widetilde{\sigma_i}) \leq f(\eta) \leq f(\sigma_i)$ so that $V(\partial\sigma_i) = \sum \widetilde{\sigma_i}$. Furthermore, $\partial(V\sigma_i) = \partial(\tau) = \sigma_i + \sum \widetilde{\sigma_i}$. Combining these facts, we obtain

$$\Phi(\sigma_i) = \sigma_i + V\partial\sigma_i + \partial V\sigma_i = \sigma_i + \left(\sum \widetilde{\sigma_i}\right) + \sigma_i = \sum \widetilde{\sigma_i}.$$

Note that the only time in the above three cases at which $a_{ii} = 1$ is when σ_i is critical. $\square$

8.2. The flow complex

Now we are ready to define the flow complex discussed at the beginning of this chapter. Let $f : K \to \mathbb{R}$ be a discrete Morse function. For K an n-dimensional complex, let $\Bbbk_p^\Phi(K) = \{c \in \Bbbk^{c_p} : \Phi(c) = c\}$. We write $\Bbbk_p^\Phi$ when K is clear from the context. Note that while c_p indicates the dimension of the vector space $\Bbbk^{c_p}$, the p in $\Bbbk_p^\Phi$ indicates that it is the vector space induced by linear combinations of certain p-simplices (and consequently the dimension of the vector space is not obvious).

Problem 8.17. Show that the boundary operator $\partial_p : \Bbbk^{c_p} \to \Bbbk^{c_{p-1}}$ can be restricted to $\partial_p : \Bbbk_p^\Phi \to \Bbbk_{p-1}^\Phi$.

Given the result of Problem 8.17, we then obtain the chain complex

$$0 \longrightarrow \Bbbk_n^\Phi \xrightarrow{\partial} \Bbbk_{n-1}^\Phi \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Bbbk_0^\Phi \longrightarrow 0,$$

which is called the **flow complex** of K and denoted by $\Bbbk_*^\Phi(K) = \Bbbk_*^\Phi$. Let $c \in \Bbbk_p^\Phi(K)$. Then c is an $\mathbb{F}_2$-linear combination of p-simplices from K; i.e., $c = \sum_{\sigma \in K_p} a_\sigma \sigma$ where $a_\sigma \in \mathbb{F}_2$. For a fixed $c \in \Bbbk_p^\Phi$, call any $\sigma^* \in \Bbbk_p^\Phi$ a **maximizer** of c if $f(\sigma^*) \geq f(\sigma)$ for all $\sigma \in \Bbbk_p^\Phi$ with $a_\sigma \neq 0$.

Lemma 8.18. If σ^* is a maximizer of some $c \in \Bbbk_p^\Phi(K)$, then σ^* is critical.

Proof. Write $c = \sum_{\sigma \in K_p} a_\sigma \sigma$ and apply Φ to both sides, yielding $\Phi(c) = \sum_{\sigma \in K_p} a_\sigma \Phi(\sigma)$. Since $c \in \Bbbk_p^\Phi(K)$, we have that $c = \Phi(c) = \sum_{\sigma \in K_p} a_\sigma \Phi(\sigma)$. Now σ^* is a maximizer of c, so $f(\sigma^*) \geq f(\sigma)$ whenever $a_\sigma \neq 0$. We thus apply the contrapositive of the last statement in Proposition 8.16 to $\Phi(\sigma^*)$ to see that all coefficients are 0 other than a_{σ^*}, the coefficient in front of σ^*, which is equal to 1. By the same proposition, σ^* is critical. $\square$

We are now ready to prove that the gradient flow stabilizes for any $\sigma \in K$.

Theorem 8.19. Let $c \in \Bbbk^{c_p}$. Then the flow Φ stabilizes at c; i.e., there is an integer N such that $\Phi^i(c) = \Phi^j(c)$ for all $i, j \geq N$.

Proof. It suffices to show the result for any $\sigma \in K$, as the general statement follows by linearity. Hence let $\sigma \in K$ and $r_\sigma = r := |\{\tilde{\sigma} \in K : f(\tilde{\sigma}) < f(\sigma)\}|$. We proceed by induction on r. For $r = 0$, $f(\tilde{\sigma}) \geq f(\sigma)$ for all $\tilde{\sigma} \in K$, and hence $\Phi(\sigma) = \sigma$ or $\Phi(\sigma) = 0$, yielding the base case.

Now suppose that $r \geq 0$. We consider σ regular and σ critical. If σ is regular, then $\Phi(\sigma) = \sum_{f(\tilde{\sigma})<f(\sigma)} a_{\tilde{\sigma}}\tilde{\sigma}$ by Proposition 8.16. Now $r_{\tilde{\sigma}} \subseteq r_\sigma$ with $\tilde{\sigma} \in r_\sigma$ and $\tilde{\sigma} \notin r_{\tilde{\sigma}}$, so $|r_{\tilde{\sigma}}| < |r_\sigma|$ for all $\tilde{\sigma}$ with $f(\tilde{\sigma}) < f(\sigma)$. By the inductive hypothesis, there exists $N_{\tilde{\sigma}}$ such that for all $i, j \geq N_{\tilde{\sigma}}$, $\Phi^i(\tilde{\sigma}) = \Phi^j(\tilde{\sigma})$. Since Φ is linear, $\tilde{\sigma}^N$ is fixed for $N > \max_{f(\tilde{\sigma})<f(\sigma)}\{N_{\tilde{\sigma}}\}$.

Next, assume σ is critical, and write $c := V(\partial\sigma)$. Then $c = \sum_{f(\tilde{\sigma})<f(c)} a_{\tilde{\sigma}}\tilde{\sigma}$ by the proof of Proposition 8.16. By Problem 8.20 (see below), we may write $\Phi^m(\sigma) = \sigma + c + \Phi(c) + \cdots + \Phi^{m-1}(c)$. It thus suffices to show that there is an N such that $\Phi^N(c) = 0$. Since $c = \sum_{f(\tilde{\sigma})<f(c)} a_{\tilde{\sigma}}\tilde{\sigma}$, the inductive hypothesis applied to each $\tilde{\sigma}$ implies there exists $\widetilde{N}$ such that $\Phi^{\widetilde{N}}(c) \in \Bbbk_*^\Phi$ is stable. Observe that if $\tau \in \operatorname{Im}(V)$, then $\Phi(V\tau) = V\tau + \partial VV\tau + V\partial V\tau = V\tau + V\partial V\tau = V(\tau + \partial V\tau)$. In other words, Φ sends elements in $\operatorname{Im}(V)$ to $\operatorname{Im}(V)$. Since $c \in \operatorname{Im}(V)$, $\Phi^{\widetilde{N}}(c) \in \operatorname{Im}(V)$. Write $\Phi^{\widetilde{N}}(c) = V(w)$ where $w = \sum_\tau a_\tau\tau$. Then $\Phi^{\widetilde{N}}(c) = V(w) = \sum_\tau a_\tau V\tau$. It follows by Proposition 8.4(iii) that $a_\tau = 0$ whenever τ is critical. Furthermore, any maximizer of $A := \{f(\tau) : a_\tau \neq 0\}$ is critical by Lemma 8.18. But if $\tau \in A$ is a maximizer, then τ is critical and $a_\tau \neq 0$, a contradiction. Thus $A = \emptyset$ so that $a_\tau = 0$ for all $\tau \in K_p$; i.e., $\Phi^{\widetilde{N}}(c) = 0$. As mentioned above, this is the desired result. $\square$

Problem 8.20. Let $\sigma \in K$ and write $c := V(\partial\sigma)$. Prove that if σ is critical, then $\Phi^m(\sigma) = \sigma + c + \Phi(c) + \Phi^2(c) + \cdots + \Phi^{m-1}(c)$.

8.3. Equality of homology

We have just shown that for any $c \in \Bbbk^{C_p}(K)$, the flow starting at c eventually stabilizes to some element in $\Bbbk_p^\Phi$. Denote this element by $\Phi^\infty(c)$ so that we obtain a function $\Phi^\infty : \Bbbk^{C_p} \to \Bbbk_p^\Phi$. Using this, we relate the homology of the chain complex $\Bbbk_*^\Phi(K)$ with simplicial homology defined in Section 3.2. In order to do this, we need a few more ideas from linear algebra. Refer back to Section 3.1 if needed. First, recall that a **chain complex**

$$\cdots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is a sequence of vector spaces C_i where each ∂_i is a linear transformation with the property that $\partial_{i-1} \circ \partial_i = 0$ for all i. The chain complex is denoted by (C_*, ∂_*) for short, and any element $c \in C_i$ is called a **chain**. By your work in Problem 3.22, this implies that $\mathrm{Im}(\partial_{i+1}) \subseteq \ker(\partial_i)$, and hence we defined the ith homology vector space of the chain complex C_* by $H_i(C_*) := \Bbbk^{\mathrm{null}\,\partial_i - \mathrm{rank}\,\partial_{i+1}}$. To see how $v \in C_i$ relates to an element in $H_i(C_*)$, we give a more precise definition of $H_i(C_*)$. Let $z \in \ker(\partial_i)$ and define $[z] := \{z + w : w \in \mathrm{Im}(\partial_{i+1})\}$. Since $\mathrm{Im}(\partial_{i+1}) \subseteq \ker(\partial_i)$, such a definition makes sense. Then we may define $H_i(C_*) := \{[z] : z \in \ker(\partial_i)\}$ with a vector space structure $[z] + [v] := [z + v]$. It is easy to show that this is a vector space of dimension $\mathrm{null}\,\partial_i - \mathrm{rank}\,\partial_{i+1}$ and hence we recover the same definition as before. The advantage here is that $H_i(C_*)$ is now defined in terms of elements of C_i. Notice that elements of H_i are *sets*. Viewing homology in this way, suppose there is another chain complex (C'_*, ∂'_*) along with linear transformations $f_i : C_i \to C_i$ such that $\partial_i \circ f_{i-1} = f_i \circ \partial'_i$ for all i:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{i+1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r, "\partial_{i-1}"] \arrow[d, "f_{i-1}"] & \cdots \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & 0 \\
\cdots \arrow[r, "\partial'_{i+1}"] & C'_i \arrow[r, "\partial'_i"] & C'_{i-1} \arrow[r, "\partial'_{i-1}"] & \cdots \arrow[r, "\partial'_2"] & C'_1 \arrow[r, "\partial'_1"] & C'_0 \arrow[r, "\partial'_0"] & 0
\end{tikzcd}$$

In this case, we say that we have a **commutative diagram**. Such a sequence of f_i is called a **chain map**. The chain map induces a map $(f_i)_*$ on the homology vector spaces, $(f_i)_* : H_i(C_*) \to H_i(C'_*)$, by defining $(f_i)_*([z]) := [f_i(z)]$. We sometimes write f instead of f_i.

Exercise 8.21. Show that $[z] = [z']$ if and only if $z - z' = w$ for some $w \in \mathrm{Im}(\partial)$.

Proposition 8.22. The function $(f)_*$ is a **well-defined** linear transformation. That is, if $[z] = [z']$, then $[f(z)] = [f(z')]$.

Proof. Let $[z] = [z']$. By Exercise 8.21, $z - z' = \partial(v)$ for some v. We have

$$\begin{aligned} z - z' &= \partial(v), \\ f(z - z') &= f \circ \partial(v), \\ f(z) - f(z') &= \partial \circ f(v), \end{aligned}$$

where the last line is justified since f is a chain map.

Now we show that f_* is linear. Let $[c], [c'] \in H_*(C_*)$. Then $f_*([c] + [c']) = f_*([c + c']) = [f(c + c')] = [f(c) + f(c')] = [f(c)] + [f(c')] = f_*([c]) + f_*([c'])$ so that f_* is linear. □

For any set A, let $\mathrm{id}_A : A \to A$ be the function defined by $\mathrm{id}_A(a) = a$ for every $a \in A$. Such a function is called the **identity map on** A.

Problem 8.23. Let $f_i : V_i \to W_i$ and $g_i : W_i \to Z_i$ be two sets of chain maps. Prove that

(i) $(g_i \circ f_i)_* = (g_i)_* \circ (f_i)_*$;

(ii) if $\mathrm{id}_{V_i} : V_i \to V_i$ is the identity, then $(\mathrm{id}_{V_i})_* = \mathrm{id}_{H_*(V_i)}$.

Now let K be a simplicial complex with $(\mathbb{k}^*, \partial_*)$ the chain complex defined in Section 3.2 and $(\mathbb{k}_*^\Phi, \partial_*)$ the flow complex. Then for each p, we have the inclusion map $i_p : \mathbb{k}_p^\Phi \to \mathbb{k}^{c_p}$ defined by $i_p(c) = c$. This is a chain map as the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r, "\partial_{p+1}"] & \mathbb{k}_p^\Phi \arrow[r, "\partial_p"] \arrow[d, "i_p"] & \mathbb{k}_{p-1}^\Phi \arrow[r, "\partial_{p-1}"] \arrow[d, "i_{p-1}"] & \cdots \arrow[r, "\partial_2"] & \mathbb{k}_1^\Phi \arrow[r, "\partial_1"] \arrow[d, "i_1"] & \mathbb{k}_0^\Phi \arrow[r, "\partial_0"] \arrow[d, "i_0"] & 0 \\ \cdots \arrow[r, "\partial_{p+1}"] & \mathbb{k}^{c_p} \arrow[r, "\partial_p"] & \mathbb{k}^{c_{p-1}} \arrow[r, "\partial_{p-1}"] & \cdots \arrow[r, "\partial_2"] & \mathbb{k}^{c_1} \arrow[r, "\partial_1"] & \mathbb{k}^{c_0} \arrow[r, "\partial_0"] & 0 \end{tikzcd}$$

A linear transformation $f : V \to W$ is a **vector space isomorphism** if there is a linear transformation $g : W \to V$ such that $g \circ f = \mathrm{id}_V$

and $f \circ g = \mathrm{id}_W$. In this case, we say that V and W are **isomorphic**, denoted by $V \cong W$. Our main result in this section is that the homology obtained from the chain complex $(\Bbbk^*, \partial_*)$ and the homology obtained from the flow complex are isomorphic.

Theorem 8.24. For all $p \geq 0$, we have $H_p(\Bbbk_*^\Phi) \cong H_p(K)$.

Proof. In order to show that $H_p(k_*^\Phi) \cong H_p(K)$, we will show that Φ_*^∞ is an isomorphism. Since $\Phi^\infty \circ i_p = \mathrm{id}_{\Bbbk_p^\Phi}$, we have $\Phi_*^\infty \circ i_* = \mathrm{id}_{H_p(\Bbbk_*^\Phi)}$ by Problem 8.23. It thus remains to show that $i_* \circ \Phi_*^\infty = \mathrm{id}_{H_p(K)}$. In order to do this, we construct a function $D\colon \Bbbk_p^\Phi \to \Bbbk_{p-1}^\Phi$ with the property that $\mathrm{id}_{\Bbbk^{c_p}} - i_p \circ \Phi^\infty = \partial \circ D + D \circ \partial$. If we can construct such a function, then the result will follow since $\mathrm{id}_{H_p(K)} - i_* \circ \Phi_*^\infty = \partial_* \circ D_* + D_* \circ \partial_* = 0$. Now there exists $N > 0$ such that $\Phi^\infty = \Phi^N$ by Theorem 8.19. Using the algebraic fact that $(1 - a^n) = (1 - a)(1 + a + a^2 + \cdots + a^{n-1})$, we have

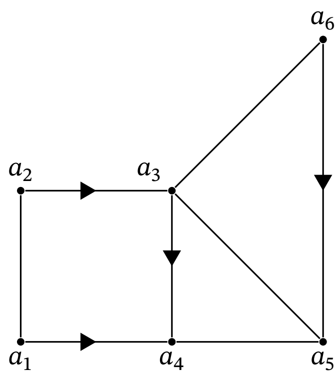
$$\begin{aligned}
\mathrm{id}_{\Bbbk^{c_p}} - i_p \circ \Phi^\infty &= \mathrm{id}_{\Bbbk^{c_p}} - \Phi^N \\
&= (\mathrm{id}_{\Bbbk^{c_p}} - \Phi)(\mathrm{id}_{\Bbbk^{c_p}} + \Phi + \Phi^2 + \cdots + \Phi^{N-1}) \\
&= (-\partial \circ V - V \circ V)(\mathrm{id}_{\Bbbk^{c_p}} + \Phi + \Phi^2 + \cdots + \Phi^{N-1}) \\
&= \partial[(-V(\mathrm{id}_{\Bbbk^{c_p}} + \Phi + \cdots + \Phi^{N-1})] \\
&\quad + [-V(\mathrm{id}_{\Bbbk^{c_p}} + \Phi + \cdots + \Phi^{N-1})]\partial.
\end{aligned}$$

Hence, define $D := -V(\mathrm{id}_{\Bbbk^{c_p}} + \Phi + \Phi^2 + \cdots + \Phi^{N-1})$. This is the desired result. $\square$

8.4. Explicit formula for homology

Example 8.25. The last section was mathematically intense, so let's get back to a concrete example. Let us investigate the function $\Phi^\infty : \Bbbk^{c_p} \to \Bbbk_p^\Phi$. Consider the discrete Morse function with gradient vector field V

given with arrows below:



Although tedious, it is not difficult to compute that

$$\begin{aligned}
\Phi^\infty(a_1) = \Phi^\infty(a_2) = \Phi^\infty(a_3) = \Phi^\infty(a_4) &= a_4, \\
\Phi^\infty(a_5) = \Phi^\infty(a_6) &= a_5, \\
\Phi^\infty(a_1a_2) &= a_1a_2 + a_1a_4 \\
&\quad + a_2a_3 + a_3a_4, \\
\Phi^\infty(a_3a_5) &= a_3a_5 + a_3a_4, \\
\Phi^\infty(a_4a_5) &= a_4a_5, \\
\Phi^\infty(a_3a_6) &= a_3a_6 + a_6a_5 \\
&\quad + a_3a_4, \\
\Phi^\infty(a_1a_4) = \Phi^\infty(a_2a_3) = \Phi^\infty(a_3a_4) = \Phi^\infty(a_6a_5) &= 0.
\end{aligned}$$

Hence $\Bbbk_0^\Phi$ is the vector space generated by $\{a_4, a_5\}$ while $\Bbbk_1^\Phi$ is the vector space generated by $\{a_1a_2 + a_1a_4 + a_2a_3 + a_3a_4, a_3a_5 + a_3a_4, a_4a_5, a_3a_6 + a_6a_5 + a_3a_4\}$. It is not difficult to see that these elements are linearly independent. More interestingly, consider that if we restrict the simplices in $\Bbbk^{c_0}$ and $\Bbbk^{c_1}$ to the critical simplices in these vector spaces, we obtain a 1–1 correspondence with the non-zero elements in $\Bbbk_0^\Phi$ and

$\Bbbk_1^\Phi$, respectively. That is, we have the correspondence

$$\begin{array}{rcl} a_4 & \leftrightarrow & a_4, \\ a_5 & \leftrightarrow & a_5, \\ a_1a_2 & \leftrightarrow & a_1a_2 + a_1a_4 + a_2a_3 + a_3a_4, \\ a_3a_5 & \leftrightarrow & a_3a_5 + a_3a_4, \\ a_4a_5 & \leftrightarrow & a_4a_5, \\ a_3a_6 & \leftrightarrow & a_3a_6 + a_6a_5 + a_3a_4. \end{array}$$

Exercise 8.26. Verify the Φ^∞ computations in Example 8.25.

In other words, if we restrict the function $\Phi^\infty : \Bbbk^{c_p} \to \Bbbk_p^\Phi$ to just critical simplices, we obtain a bijection and furthermore a vector space isomorphism (at least in the above example). This suggests that it might be worth studying the following:

Definition 8.27. Let K be a simplicial complex with gradient vector field V. Let $\mathcal{M}_p$ denote the vector space generated by the critical p-simplices of V. Then $\mathcal{M}_p$ is a vector subspace of $\Bbbk^{c_p}$ called the **critical complex of K with respect to V.**

The function $\Phi^\infty : \Bbbk^{c_p} \to \Bbbk_p^\Phi$ restricts to the function $\Phi^\infty : \mathcal{M}_p \to \Bbbk_p^\Phi$, which we refer to by the same name with an abuse of notation. If $\dim(\mathcal{M}_p) = i$, we write $\mathcal{M}_p^i$. As hinted above, we now have the following:

Theorem 8.28. The function $\Phi^\infty : \mathcal{M}_p \to \Bbbk_p^\Phi$ is a vector space isomorphism.

Proof. We show that Φ^∞ is both surjective and injective. To see that it is surjective, let $c \in \Bbbk_p^\infty$, and write $c = \sum_{\sigma \in K_p} a_\sigma \sigma$ where $a_\sigma \in \mathbb{F}_2$ is the coefficient of σ in the expansion of c. Consider the new element $\tilde{c}$ given by restricting σ to only critical simplices in the expansion of c; i.e.,

$\tilde{c} := \sum_{\sigma \text{ is critical}} a_\sigma \sigma$. We claim that $\Phi^\infty(\tilde{c}) = c$. We have

$$\begin{aligned}\Phi^\infty(\tilde{c}) &= \sum_{\sigma \text{ is critical}} a_\sigma \Phi^\infty(\sigma) \\ &= \sum_{\sigma \text{ is critical}} a_\sigma(\sigma + V_{a_\sigma}) \\ &= \sum_{\sigma \text{ is critical}} a_\sigma \sigma = \tilde{c},\end{aligned}$$

where V_{a_σ} is an element in the image of V. Since σ is critical and V_{a_σ} is in the image of V, the expansion of σ as a combination of basis elements will have none of the same basis elements as the expansion of V_{a_σ} as a combination of basis elements. It follows that $a_\sigma V_{a_\sigma} = 0$. Applying this fact, one shows that $\Phi^\infty(\tilde{c}) - c \in \Bbbk^\Phi$ (Problem 8.29). By Lemma 8.18, $\Phi^\infty(\tilde{c}) - c = 0$ so that Φ^∞ is surjective.

To see that Φ^∞ is injective, suppose that $\Phi^\infty(c) = 0$ for $c \in \mathcal{M}_p$. Then $\Phi^\infty(c) = \sum_{\sigma \in K_p} a_\sigma \Phi^\infty(\sigma) = 0$, and using the same observation made in the proof of surjectivity, the coefficients of any critical simplex in the expansion of c are all 0, i.e., $c = 0$. Thus Φ^∞ is injective. □

Problem 8.29. Show that $\Phi^\infty(\tilde{c}) - c \in \Bbbk^\Phi$, where c and $\tilde{c}$ are as defined in the proof of Theorem 8.28.

Theorem 8.28 tells us that the vector spaces in the flow complex are the same as the vector spaces generated by all the critical simplices. So, for example, the simplicial complex K in Example 8.25 would have the following chain complex induced by its critical simplices:

$$\mathcal{M}_1^4 \longrightarrow \mathcal{M}_0^2 \longrightarrow 0$$

where $\mathcal{M}_1^4$ is the vector space generated by the four critical 1-simplices a_1a_2, a_3a_5, a_4a_5, and a_3a_6, while $\mathcal{M}_0^2$ is generated by the two critical 0-simplices a_4 and a_5. Now we know from Theorem 8.24 that the homology of K can be computed using the flow complex. Since the flow complex is the same as the critical complex with critical simplices as the basis, this raises the question: what are the boundary operators ∂_p for $\mathcal{M}_p$?

Example 8.30. Refer to the simplicial complex and gradient vector field in Example 8.25. We seek to define a linear transformation from $\mathbb{k}^4$ to $\mathbb{k}^2$; that is, to each pair consisting of a critical 1-simplex and a critical 0-simplex we need to associate either a 0 or a 1. How can we do this? Take, for example, a_3a_6 and a_5. We want to use the gradient vector field, so starting at a_3a_6, why not count the number of V-paths (mod 2) to a_5? In this case, there is just one. Let's try it for another pair, a_1a_2 and a_4. Here there are two such V-paths, starting at a_1a_2 and ending at a_4. Note that we are using the definition of a V-path which allows us to start at a critical simplex. We seem to want, then, to start at a maximal face of a_1a_2 and count the number of paths to a_4. If we continue in this way, we obtain the linear transformation $\partial : \mathcal{M}_1^4 \to \mathcal{M}_0^2$ given by

$$\partial = \begin{array}{c} \\ a_1a_2 \\ a_4a_5 \\ a_3a_5 \\ a_3a_6 \end{array} \begin{array}{c} \begin{array}{cc} a_4 & \quad a_5 \end{array} \\ \begin{pmatrix} 2=0 & 2=0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}.$$

Sure enough, this has a rank of 1 and a nullity of 3 so that $b_1(K) = 3$ and $b_0(K) = 2 - 1 = 1$, precisely what we would expect (see Definition 3.16 for a reminder on how to compute homology).

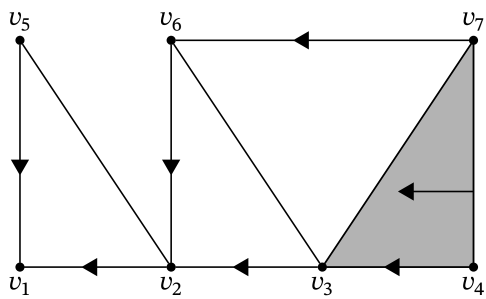
We are thus led to the following theorem, which tells us that we can compute the boundary operators of $\partial_p : \mathcal{M}_p \to \mathcal{M}_{p-1}$ by counting V-paths. Because it is a bit technical, we omit the proof here. The interested reader may find a proof in [**65**, Section 8] or [**99**, Section 7.4].

Theorem 8.31. Let K be a simplicial complex and V a gradient vector field on K. For each $\sigma \in \mathcal{M}_p$,

$$\partial(\sigma) = \sum_{\text{critical } \beta^{(p-1)}} \delta_{\sigma,\beta}\beta$$

where $\delta_{\sigma,\beta} = 0$ if the number of V-paths starting from a maximal face of σ to β is even, and $\delta_{\sigma,\beta} = 1$ if the number of V-paths is odd. Then $H_p(\mathcal{M}_p) \cong H_p(K)$.

With an abuse of language, we will call the pair $(\mathcal{M}_*, \partial_*)$ the **critical complex of K with respect to the gradient vector field** V.[2] For those of you keeping track, we have the "standard" chain complex introduced in Section 3.2, the flow complex from Section 8.2, and now the critical complex. All three yield the same homology for a fixed simplicial complex K. Of course, one of the main advantages of using the critical complex is that the matrices and vector spaces are much smaller! For example, let us compute the homology of the simplicial complex in Example 3.10. In order to do this, we need a discrete Morse function on the simplicial complex, and of course the fewer critical simplices the better. An optimal one is given below.



The set of critical 1- and 0-simplices are $\{v_2v_5, v_3v_6, v_3v_7\}$ and $\{v_1\}$, respectively (note that there are no critical 2-simplices). We thus have

$$\mathcal{M}_1^3 \longrightarrow \mathcal{M}_0^1 \longrightarrow 0$$

as our critical chain complex. Counting the number of V-paths from a maximal face of a critical 1-simplex to a critical 0-simplex yields

$$\partial = \begin{array}{c c} & v_1 \\ \begin{array}{c} v_2v_5 \\ v_3v_6 \\ v_3v_7 \end{array} & \begin{pmatrix} 2 = 0 \\ 2 = 0 \\ 2 = 0 \end{pmatrix} \end{array}.$$

[2]Again, it is known as the Morse complex in certain parts of the literature, so be careful not to confuse it with our object of study from Chapter 7.

Clearly null(∂) $= 3$ so that $b_1 = 3$ and $b_0 = 1$. Now compare the computation in this example with what we did in Example 3.10. Wasn't this much easier?

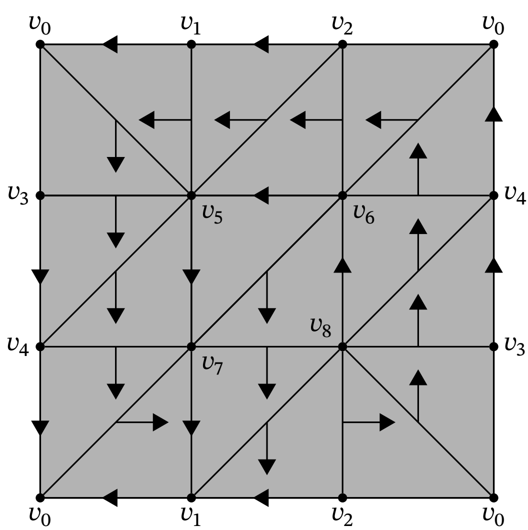
Problem 8.32. Use Theorem 8.31 to prove the strong discrete Morse inequalities (Theorem 4.4).

Problem 8.33. Show that if one puts an empty gradient vector field on K (i.e., everything is critical), then the formula for the boundary operator in Theorem 8.31 coincides with the definition of the boundary operator in Chapter 3.

8.5. Computation of Betti numbers

In this section, we will finally be able to distinguish between many of the simplicial complexes posed to us in Section 1.1.1.

Example 8.34. Let us compute the Betti numbers of the Klein bottle $\mathcal{K}$. We put a gradient vector field on $\mathcal{K}$, attempting to minimize the number of critical simplices. The gradient vector field we will use is below.



The critical simplices are v_0, v_0v_2, v_0v_3, and $v_5v_6v_7$ so that $m_0 = 1, m_1 = 2$, and $m_2 = 1$. This yields the critical chain complex

$$\mathcal{M}_2^1 \xrightarrow{\partial_2} \mathcal{M}_1^2 \xrightarrow{\partial_1} \mathcal{M}_0^1 \longrightarrow 0.$$

Applying Theorem 8.31, we see that

$$\partial_2 = \begin{array}{c c} & \begin{array}{cc} v_0v_2 & v_0v_3 \end{array} \\ v_5v_6v_7 & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{array}$$

and

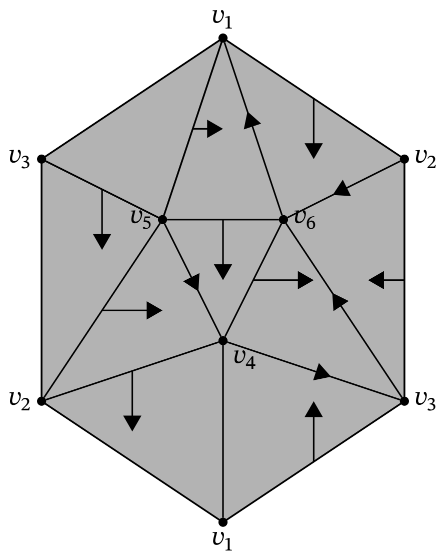
$$\partial_1 = \begin{array}{c c} & v_0 \\ \begin{array}{c} v_0v_2 \\ v_0v_3 \end{array} & \begin{pmatrix} 2 = 0 \\ 2 = 0 \end{pmatrix} \end{array}.$$

Hence the $\mathbb{F}_2$-Betti numbers of K are $b_2 = 1, b_1 = 2$, and $b_0 = 1$. We thus conclude that neither S^1 nor the Möbius band M have the same simple homotopy type as K, even though they have the same Euler characteristic.

Problem 8.35. Use the discrete Morse function you constructed in Problem 2.41 (or a better one if you can find one) to show that the $\mathbb{F}_2$-Betti numbers of the torus T^2 are $b_2 = 1, b_1 = 2$, and $b_0 = 1$.

Remark 8.36. In Problem 8.35, you showed that the torus T^2 has the exact same $\mathbb{F}_2$-Betti numbers as the Klein bottle from Example 8.34. Unfortunately, determining whether or not these two complexes have the same simple homotopy type is beyond the scope of this book. It turns out that $T^2 \not\sim \mathcal{K}$. We could have distinguished them using homology with coefficients in $\mathbb{Z}$, but we have chosen to trade precision for computability. One can distinguish between these two complexes by using the techniques in [**134**, Section 1.5], for example.

Example 8.37. We compute the $\mathbb{F}_2$-Betti numbers of the projective plane P^2 from Example 1.21. Consider the following gradient vector field on P^2:



We obtain the critical chain complex

$$\mathcal{M}_2^1 \xrightarrow{\partial_2} \mathcal{M}_1^1 \xrightarrow{\partial_1} \mathcal{M}_0^1 \longrightarrow 0.$$

By Theorem 8.31, we count the number of paths from the boundary of the critical 2-simplex $v_1v_3v_5$ to the critical 1-simplex v_1v_4, yielding the boundary operator

$$\partial_2 = \begin{array}{cc} & v_1v_4 \\ v_1v_3v_5 & \begin{pmatrix} 2=0 \end{pmatrix} \end{array}.$$

Similarly, we have

$$\partial_1 = \begin{array}{cc} & v_1 \\ v_1v_4 & \begin{pmatrix} 2=0 \end{pmatrix} \end{array}.$$

Hence, the $\mathbb{F}_2$-Betti numbers of P^2 are $b_0 = b_1 = b_2 = 1$ and $b_i = 0$ for all $i > 2$. In particular, even though $\chi(P^2) = 1$, $P^2 \nsim \Delta^n$.

Problem 8.38. Compute the Betti numbers of the simplicial complex approximating the cell phone towers in Section 0.1.1.

Problem 8.39. Compute the Betti numbers of Björner's complex in Example 1.25.

Chapter 9

Computations with discrete Morse theory

The purpose of this chapter is to present and briefly explain algorithms in pseudo-code which may be implemented to perform computations using discrete Morse theory. We assume the reader is familiar with some of the basics of coding such as data structures.

9.1. Discrete Morse functions from point data

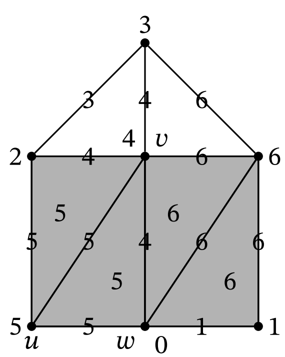
When we were first learning about discrete Morse functions, we saw in Exercise 2.22 that if we have a positive function on the vertex set V of a simplicial complex K, then we could create a discrete Morse function on K by declaring that the value on any simplex is the sum of the values of its vertices. While this does yield a discrete Morse function, every simplex is critical, which is not at all helpful. In this section we give an algorithm, due to King et al. [**97**], that uses the values on the vertices to construct a gradient vector field on K with "few" critical simplices. More details about this algorithm may be found in the original paper cited above or [**99**, Section 8.3]. Before presenting the algorithm, we fix some notation and terminology.

Let K be a simplicial complex, V the vertex set of K, and $f_0 : V \to \mathbb{R}$ an injective function. For any $\sigma \in K$, write $\sigma = v_0 v_1 \cdots v_i$. Define the **lower star filtration on** K **induced by** f_0 by

$$\mathrm{maxf}_0(\sigma) := \max_{0 \le j \le i}\{f_0(v_j)\}.$$

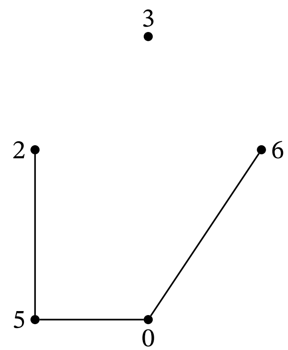
In Definition 6.33, we defined the link of a vertex $v \in K$. The star of v in K, denoted by $\mathrm{star}_K(v)$, is the simplicial complex induced by the set of all simplices of K containing v. The link of v in K is the set $\mathrm{link}_K(v) := \mathrm{star}_K(v) - \{\sigma \in K : v \in \sigma\}$. We now define the **lower link** of v to be the maximal subcomplex of $\mathrm{link}_K(v)$ whose vertices have f_0-value less than $f_0(v)$. Note that the lower link depends on the chosen f_0 function while the link does not.

Example 9.1. We illustrate the above definitions. Let K be the simplicial complex below and f_0 the function on the vertices, with values on the other simplices induced by the lower star filtration.

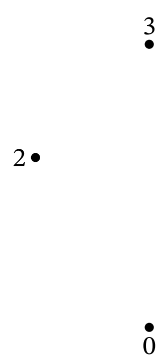


Write $u := f_0^{-1}(5), v := f_0^{-1}(4)$, and $w := f_0^{-1}(0)$ where u, v, and w are the unique vertices with those values. Then $\mathrm{maxf}_0(uvw) = 5$ while

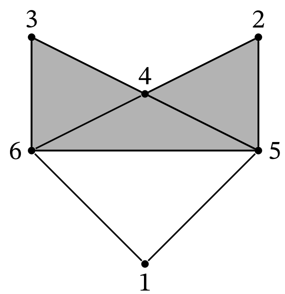
$\max f_0(vw) = 4$. The link of v is given by



The lower link of v is the maximal subcomplex of the above complex with values less than $f_0(v) = 4$, which is then seen to be



Exercise 9.2. Let K be the following simplicial complex and $f_0 : V \to \mathbb{R}$ the function on the vertices:



Compute the lower link of $f_0^{-1}(6)$ and of $f_0^{-1}(1)$.

Exercise 9.3. Let $f_0 : V \to \mathbb{R}$ be a function. Suppose that $f_0(u) = a$ where $a := \min_{v \in V}\{f_0(v)\}$. Prove that the lower link of u is empty.

We constructed the join of two simplicial complexes in Definition 1.52. Now we define the join of two simplices. Let $\sigma^{(i)}$ and $\tau^{(j)}$ be two disjoint simplices in K. The **join** of σ and τ, denoted by $\sigma * \tau$, is either undefined or the $(i+j+1)$-simplex whose vertices are the union of those of σ and those of τ; that is, $\sigma * \tau = \sigma \cup \tau \in K$. The join is undefined when $\sigma \cup \tau \notin K$. In Example 9.2, $f_0^{-1}(2) * f_0^{-1}(3)$ would be undefined.

Problem 9.4. Prove that the lower link of v is also given by the set of all simplices $\tau \in K$ such that $v * \tau$ is defined and $\max \mathrm{f}_0(\tau) < f_0(v)$.

We now state our main algorithm, which produces a gradient vector field from a set of values on the vertices. Note that it calls two other algorithms, `ExtractRaw` and `ExtractCancel`, which we will define later in this section.

Algorithm 2 (King et al.) Extract

`Input:` Simplicial complex K, injective $f_0 : V(K) \to \mathbb{R}$, $p \geq 0$

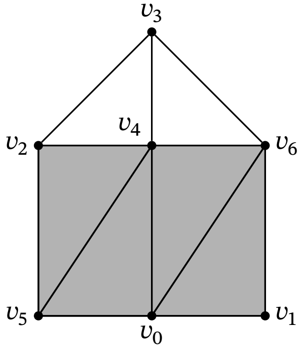
`Output:` Gradient vector field $(A, B, C, r : B \to A)$ on K

`ExtractRaw`(K, f_0) (Algorithm 3)
for $j = 1, \dots, \dim(K)$ **do**
  `ExtractCancel`(K, h, p, j) (Algorithm 5)
end for

Algorithm 2 takes in a simplicial complex K, an injective function f_0, and a parameter p called the **persistence**. The output is a gradient vector field on K. To see how to view a gradient vector field as $(A, B, C, r : B \to A)$, recall from Section 2.2.2 that a gradient vector field on K yields a discrete Morse matching on the directed Hasse diagram of K. Those simplices which are unmatched form a set C. They are precisely the critical simplices. For any matched pair (σ, τ), we may break the pair up by placing the tail σ in a set A and the head τ in a set B. This induces a bijection $r : B \to A$. It follows immediately from Lemma 2.24 that $\{A, B, C\}$ forms a partition of the simplices of K.

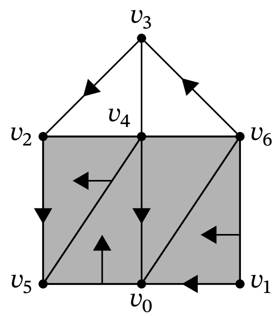
Exercise 9.5. Write down the sets $A, B,$ and C and the bijection $r : B \to A$ of the directed Hasse diagram in Example 7.1.

We will use Example 9.1 as a running example to illustrate Algorithm 2. We will use the same f_0 function on the vertices as in Example 9.1 and label the vertices according to their value under f_0, i.e.,

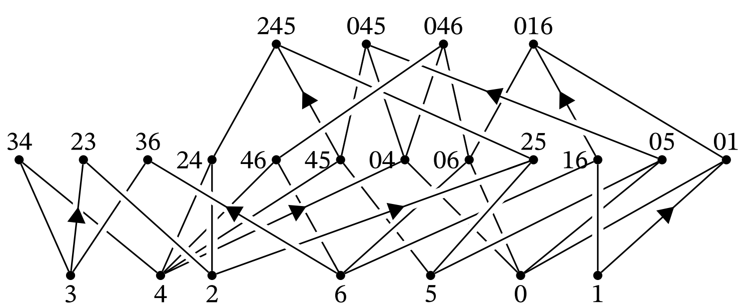


This will be our input simplicial complex. Before we can illustrate the algorithm, we need some more preliminaries. Clearly the bulk of the work in Algorithm 2 is happening in two other algorithms (one of which calls a third, but simple, algorithm). In order to understand these algorithms, we define a subgraph R_i, $i = 1, \dots, \dim(K)$, of the directed Hasse diagram induced by a gradient vector field. For each i, let $A_i := A \cap K_i, B_i := B \cap K_i$, and $C_i := C \cap K_i$ where $K_i := \{\sigma \in K : \dim(\sigma) = i\}$. Vertices of R_i are of two types: either $(i-1)$-simplices in $A_{i-1} \cup C_{i-1}$ or the i-simplices in $B_i \cup C_i$. An edge is directed from σ to all of its codimension-1 faces, unless $\sigma \in B_i$, in which case the direction is from $r(\sigma)$ to σ.

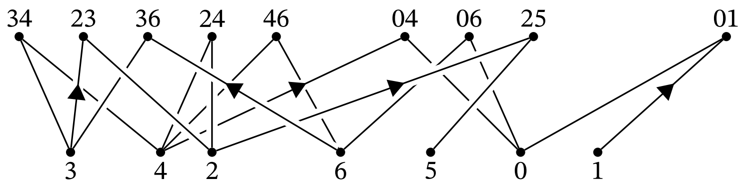
Example 9.6. Let K be the simplicial complex from Example 9.1 with gradient vector field given below:



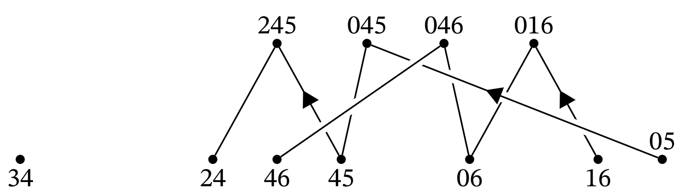
To save space, we write a simplex in the Hasse diagram $\mathcal{H}_K$ by concatenating its subscripts, e.g. ij stands for the simplex v_iv_j. We have that $\mathcal{H}_V$ is given by



where an unmarked edge implicitly has a downward arrow (suppressed to avoid cluttering the picture). Now $A_0 = \{v_3, v_4, v_2, v_6, v_1\}$, $B_1 = \{v_2v_3, v_0v_4, v_2v_5, v_3v_6, v_0v_1\}$, $C_0 = \{v_0, v_5\}$, and $C_1 = \{v_3v_4, v_2v_4, v_4v_6, v_0v_6\}$. Hence R_1 is the subgraph



We also compute $A_1 = \{v_4v_5, v_1v_6, v_0v_5\}$, $B_2 = \{v_2v_4v_5, v_0v_4v_5, v_0v_1v_6\}$, $C_1 = \{v_3v_4, v_2v_4, v_4v_6, v_0v_6\}$, and $C_2 = \{v_0v_4v_6\}$ so that R_2 is given by



Algorithm 3 below works inductively on the link of a vertex. A vertex v is chosen in Step 2 and its lower link computed. Then v is determined to be either critical or part of a regular pair. If it is critical, we move on and another vertex is chosen. Otherwise, Step 8 passes this link to Algorithm 2, which then passes it back to Algorithm 3. This gives a gradient vector field on the link of v. From this information, Steps 9–15 determine a vector in the gradient vector field.

Algorithm 3 (King et al.) ExtractRaw

`Input`: Simplicial complex K, injective $f_0 : V(K) \to \mathbb{R}$
`Output`: Gradient vector field $(A, B, C, r : B \to A)$ on K

1 Initialize A, B, C to be empty
2 **for all** $v \in K_0$ **do**
3   Let $K' := $ the lower link of v
4   **if** $K' = \emptyset$ **then** add v to C
5   **else**
6   Add v to A
7   Let $f_0' : K_0' \to \mathbb{R}$ be the restriction of f_0
8     $(A', B', C', r') \leftarrow \texttt{Extract}(K', f_0', \infty)$
9   Find the $w_0 \in C_0'$ such that $f_0'(w_0)$ is the smallest
10   Add vw_0 to B
11   Define $r(vw_0) := v$
12   for each $\sigma \in C' - \{w_0\}$ add $v * \sigma$ to C
13   for each $\sigma \in B'$ add $v * \sigma$ to B
14   Add $v * r'(\sigma)$ to A
15   Define $r(v * \sigma) = v * r'(\sigma)$
16   **end if**
17 **end for**

Example 9.7. We will illustrate the steps of Algorithm 3 on the simplicial complex given immediately after Exercise 9.5. To begin, set $A = B = C = \emptyset$. For Step 2, let $v := v_3$. Then the lower link of v is given by $K' = \{v_2\}$, so by Step 6 we have $A = \{v_3\}$ and $f_0' : \{v_2\} \to \mathbb{R}$ given by $f_0'(v_2) = 2$. At this point we are on Step 8 of Algorithm 3, and we pass (K', f_0', ∞) into Algorithm 2, which immediately sends us back to Algorithm 3 but this time with a different input. Our only choice for a vertex in Step 2 is v_2. We see that the lower link of v_2 is empty, so that by Step 4 we have $C' = \{v_2\}$ (note that this is the set C' for K', not for K). Since K' is a single point, we have completed Step 2, which in turn completes this run of Algorithm 3 called in Step 1 of Algorithm 2. Moving to Step 2 of Algorithm 2, we observe that $\dim(K) = 0$, so Step 2 is skipped and we complete this run of Algorithm 2 with output $A' = B' = \emptyset$, $C' = \{v_2\}$, and $r' = \emptyset$. Recall that we are still on Step 8 of Algorithm 3, and now we have the output of Algorithm 2. There is only one element in C', so $w_0 := v_2$, and we add $v_3v_2 \in B$, defining $r(v_3v_2) := v_3$. In other words, we have created an arrow (v_3, v_3v_2) in a gradient vector field on K. Since $C' - \{w_0\} = \emptyset$, we skip Step 12, and since $B' = \emptyset$, we also skip Steps 13–15. Moving back to Step 2, we choose a vertex in K other than v_2 and repeat.

The `ExtractRaw` algorithm (Algorithm 3) takes in a simplicial complex K and injective function on the vertex set. It outputs a "raw" gradient vector field (given by A, B, C, r) in the sense that it may have many critical simplices. One way to improve a given gradient vector field is to cancel out superfluous critical simplices by the method of canceling critical simplices described in Proposition 4.22. This will be done via two more algorithms. Algorithm 4, or `Cancel`, reverses the arrows in a unique gradient path between critical simplices τ and σ. This algorithm is utilized in Algorithm 5, and Algorithm 5 is utilized in our main algorithm, Algorithm 2. Recall that we use $\sigma_0 \to \sigma_1 \to \cdots \to \sigma_n$ to denote a gradient path from σ_0 to σ_n. Note that our gradient paths below start at a critical simplex.

Algorithm 4 (King et al.) Cancel

`Input:` Simplicial complex K, injective $f_0 : V(K) \to \mathbb{R}$, $\tau \in C_{j-1}, \sigma \in C_j, 1 \leq j \leq \dim(K)$

`Output:` Gradient vector field $(A, B, C, r : B \to A)$ on K

Find unique gradient path $\sigma = \sigma_1 \to \tau_1 \to \sigma_2 \to \tau_2 \to \cdots \to \tau_k = \tau$
Delete τ and σ from C, add σ to B, and add τ to A
for $i = 1, \ldots, k$ **do**
    Redefine $r(\sigma_i) = \tau_i$
end for

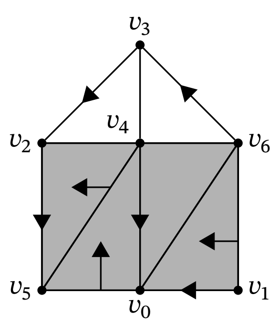
Algorithm 5 (King et al.) ExtractCancel

`Input:` Simplicial complex K, injective $f_0 : V(K) \to \mathbb{R}$, $p \geq 0$, $1 \leq j \leq \dim(K)$

`Output:` Gradient vector field $(A, B, C, r : B \to A)$ on K

1 **for all** $\sigma \in C_j$ **do**
2   Find all gradient paths $\sigma = \sigma_{i1} \to \sigma_{i2} \to \cdots \to \sigma_{i\ell_i} \in C_{j-1}$ with $\mathrm{maxf}_0(\sigma_{i\ell_i}) > \mathrm{maxf}_0(\sigma) - p$
3   **for all** i **do**
4   if $\sigma_{i\ell_i}$ does not equal any other $\sigma_{j\ell_j}$ let $m_i := \mathrm{maxf}_0(\sigma_{i\ell_i})$
5   **if** at least one m_i is defined **then**
6   Choose j with $m_j = \min\{m_i\}$
7   `Cancel`$(K, f_0, \sigma_{j\ell_j}, \sigma, j)$
8   **end if**
9   **end for**
10 **end for**

Example 9.8. To illustrate Algorithms 4 and 5, we again use the simplicial complex K, $f_0 : K_0 \to \mathbb{R}$, and the gradient vector field below, along with $j = 1$, for our input to Algorithm 5.



We computed that $C_1 = \{v_3v_4, v_2v_4, v_4v_6v_0v_6\}$ above. For Step 1 of Algorithm 5, let $\sigma := v_3v_4$. We have two gradient paths starting at v_3v_4 to elements of C_0, namely,

$$v_3v_4 \to v_3 \to v_2v_3 \to v_2 \to v_2v_3 \to v_5$$

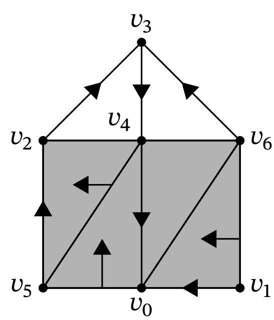
and

$$v_3v_4 \to v_4 \to v_4v_0 \to v_0.$$

Now $\mathrm{maxf}_0(v_5) = 5$ in the first gradient path while $\mathrm{maxf}_0(v_0) = 0$ for the second one. For $0 \leq p < 4$, only the first gradient path satisfies $\mathrm{maxf}_0(\sigma_{i\ell_i}) > \mathrm{maxf}_0(\sigma) - p$. Since $\sigma_{1\ell_1} = v_5$ does not equal any other $\sigma_{j\ell_j}$, we define $m_1 := \mathrm{maxf}_0(v_5) = 5$. Moving to Step 6 of Algorithm 5, we must choose $j = 1$. Thus we pass $(K, f_0, v_5, v_3v_4, 1)$ into Algorithm 4. For Step 1, the unique gradient path is precisely

$$v_3v_4 \to v_3 \to v_2v_3 \to v_2 \to v_2v_3 \to v_5,$$

which we had already found. Now delete v_5 and v_3v_4 from C and add $v_3v_4 \in B$ and $v_5 \in A$. Finally, in Step 3 of Algorithm 4, we redefine $r(v_3v_4) := v_3, r(v_2v_3) := v_2$, and $r(v_2v_5) := v_5$. This has the effect of reversing the path, yielding



which has one less critical simplex.

We now verify that our algorithms produce the desired output; that is, we need to show that we produce a Morse matching on the directed Hasse diagram. By Theorem 2.51, this is equivalent to showing that the directed Hasse diagram has no directed cycles.

Proposition 9.9. The output (A, B, C, r) produced by Algorithm 3 has the property that there are no directed cycles in the resulting directed Hasse diagram.

Proof. We first claim that the function maxf_0 is non-increasing along any directed path in the directed Hasse diagram. We first note that

$\mathrm{maxf}_0(r(\sigma)) = \mathrm{maxf}_0(\sigma)$ since v is the vertex in all mentioned simplices with the highest value of f_0. Furthermore, for any face $\sigma < \tau$, we have that $\mathrm{maxf}_0(\sigma) \leq \mathrm{maxf}_0(\tau)$, which proves the claim that maxf_0 is non-increasing along a directed path. It follows that if $\sigma_0 \to \sigma_1 \to \cdots \to \sigma_k = \sigma_0$ is a directed cycle in the directed Hasse diagram, then maxf is constant on all of σ_i. Let v be the unique vertex such that $f_0(v) = \mathrm{maxf}(\sigma_j)$ for all j. Since $v \in \sigma_j$ for all j, we have that $\sigma_j = v * \tau_j$ for some simplices τ_j in the lower link of v. This yields that $\tau_0 \to \tau_1 \to \cdots \to \tau_k = \tau_0$ is a directed cycle in the directed Hasse diagram of the lower link of v. This is a contradiction by induction on the dimension and Problem 9.10. □

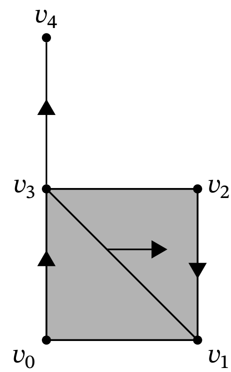
Problem 9.10. Prove that Algorithm 4 does not produce directed cycles, and thus the output of Algorithm 2 also does not contain directed cycles.

9.2. Iterated critical complexes

We will follow an algorithm due to P. Dłotko and H. Wagner to compute the Betti numbers of a simplicial complex [**52**]. There are many other algorithms to compute homology with discrete Morse theory [**74**, **82**, **83**, **104**]. Other resources for computing persistent homology with discrete Morse theory are given at the end of the chapter. Algorithms to compute homology that do not rely on discrete Morse theory may be found in [**55**, IV.2] and [**54**, Chapter 11], while a more advanced algorithm is given in [**127**].

Let K be a simplicial complex. We know from our work in Chapter 8 that we can greatly reduce the size of our vector spaces in a chain complex by finding a gradient vector field on K. The critical simplices in each dimension generate vector spaces, so the fewer critical simplices the better, and counting V-paths between pairs of critical simplices of codimension 1 determines the boundary operator. This gives the critical complex constructed in Section 8.4. But sometimes this process does not work as well as we would hope. An algorithm may produce an optimal discrete Morse function, a discrete Morse function where every simplex is critical, or anything in between. One way to help address this problem is to iteratively compute critical complexes, that is, compute the critical complex of the critical complex of the critical complex, etc. If the boundary operator eventually becomes 0, then we can simply read off the Betti numbers from the vector space dimensions without having to do any kind of matrix reductions or complicated counting of paths. The idea is illustrated in the following example.

Example 9.11. Let K be the simplicial complex along with the gradient vector field V given below:



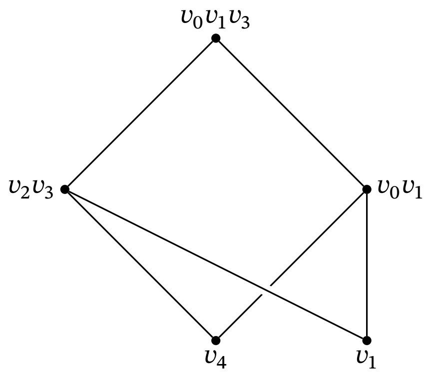
Suppose we wish to compute the homology of K. Of course, we can clearly see that K is collapsible so that $b_0(K) = 1$ and all its other Betti numbers are 0. But we wish to illustrate the iterative critical complex construction. The c-vector for this complex is $\vec{c}_K = (5, 6, 2)$. Hence if we used the techniques of Chapter 3, we would obtain the chain complex

$$\mathbb{k}^2 \longrightarrow \mathbb{k}^6 \longrightarrow \mathbb{k}^5.$$

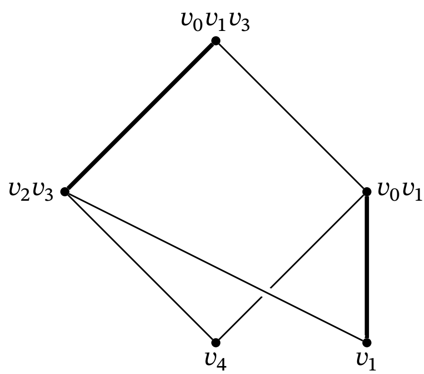
Although this gradient vector field is far from optimal, the number of critical simplices in each dimension is $m_0 = m_1 = 2$ and $m_2 = 1$. Applying Theorem 8.31 yields the critical complex

$$\mathbb{k}^1 \longrightarrow \mathbb{k}^2 \longrightarrow \mathbb{k}^2,$$

which is certainly an improvement over the first chain complex. But we wish to reduce this even further. We can do this by pairing critical simplices that have an odd number of V-paths between them. We will draw the directed Hasse diagram of the critical simplices with an edge between nodes if and only if there are an odd number of paths between them:



Can we find a discrete Morse matching on this directed Hasse diagram? There are several. One is given below, where matched nodes are connected by a thickened edge.



With $v_0v_1v_3$ matched to v_2v_3 and v_0v_1 matched to v_1, the only simplex left unmatched is v_4. In other words, a second iteration of the critical complex construction produces the critical complex

$$\mathbb{k}^0 \longrightarrow \mathbb{k}^0 \longrightarrow \mathbb{k}^1,$$

which is the best we can do for this complex.

In Section 9.2.2, we will refer to the above structure as a discrete Morse graph. The critical simplices in a fixed dimension may be viewed as generating a vector space, while the gradient vector field may be viewed as the boundary operator. Hence a discrete Morse graph is also a critical complex. Strictly speaking, the difference between the discrete Morse graph and the critical complex is that the critical complex also contains boundary operator information. The notion of taking the critical complex of a critical complex is made precise using algebraic discrete Morse theory, which was discovered independently by Kozlov [**102**], Sköldberg [**142**], and Joellenbeck and Welker [**87**]. Other treatments of algebraic discrete Morse theory may be found in [**99**, Section 9.4] and [**103**, Section 11.3].[1] We will make use of the boundary operator in Section 9.2.3 without discussing the details. The interested reader may consult any one of the above sources for more details on algebraic Morse theory.

9.2.1. Computing the upward Hasse diagram. Before we can do any computing, we need a data structure to encode a simplicial complex. Given a list of facets which generate a simplicial complex, we need to store this as a Hasse diagram. We give a simple method to accomplish this in Algorithm 6.[2] A more refined but also more complicated algorithm may be found in the paper by V. Kaibel et al. [**96**] and is discussed further in [**90**]. Alternatively, one may utilize free online software equipped with packages to compute the Hasse diagram, such as `polymake` [**75**].

The Hasse diagram we construct with our algorithm differs from the Hasse diagram introduced in Section 2.2.2 in two respects. First, we will include the empty set in the very bottom row of the Hasse diagram, along with edges from the empty set to each node. Second, every edge in the Hasse diagram will feature an upward arrow. Hence we will call the result of our algorithm the **upward Hasse diagram of** K. The algorithm in Section 9.2.2 will then reverse some of these arrows, yielding a discrete Morse matching.[3] Freely downloadable files which contain facets of many interesting and complicated simplicial complexes may be found

[1] As mentioned before, note that these authors refer to the critical complex as the Morse complex.

[2] Thanks to Pawel Dłotko for suggesting this algorithm along with Example 9.12.

[3] This is opposite to the convention of computing the directed Hasse diagram in Section 2.2.2, but the choice is arbitrary.

at the Simplicial Complex Library [**81**] or the Library of Triangulations [**30**].

The data structure for $\mathcal{H}$ will be given by $\mathcal{H} = (G, V)$ where G is the set of vertices of $\mathcal{H}$ and V consists of ordered pairs (σ, τ) of vertices in G, indicating an arrow from σ to τ.

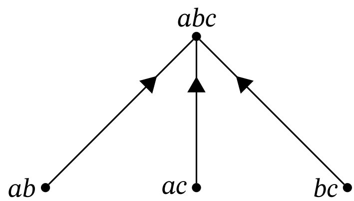
Algorithm 6 Upward Hasse diagram $\mathcal{H}$

`Input:` List of facets S

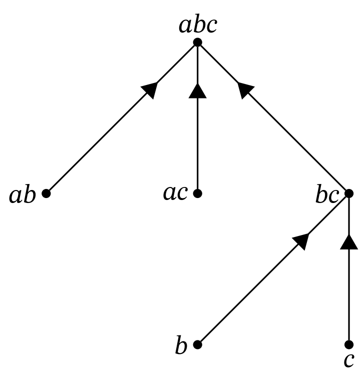
`Output:` Upward Hasse diagram $\mathcal{H} = (G, V)$ of simplicial complex K generated by S

1 Initialize $\mathcal{H} = G = V = \emptyset$
2 **for** every $\emptyset \neq \sigma \in S$
3   **if** $\sigma \notin G$
4         $G \leftarrow G \cup \{\sigma\}$
5   **end if**
6   **for** every vertex $v \in \sigma$
7         $\sigma' := \sigma - \{v\}$
8   **if** $\sigma' \notin G$
9             $G \leftarrow G \cup \{\sigma'\}$ and $S \leftarrow S \cup \{\sigma'\}$
10   **else** replace σ' with the instance of σ' already in G
11         $V \leftarrow V \cup \{(\sigma', \sigma)\}$
12   **end if**
13   **end for**
14 **end for**
15 Return $\mathcal{H} = (G, V)$

Example 9.12. Let $K = \Delta^2$ on vertices a, b, and c. Then $S = \{abc\}$; i.e., $\langle abc \rangle = \Delta^2$. We illustrate Algorithm 6. For Step 2, $\sigma = abc \in S$ is the only option, and since $G = \emptyset$, we add abc to G in Step 4. For Step 6 pick $v = a$, and construct $\sigma' := abc - \{a\} = bc$ in Step 7. Now $bc \notin G$, so we add $bc \in G$ and $bc \in S$ in Step 9. For Step 11, we add the directed edge $(bc, abc) \in V$. Moving back to Step 6, we repeat with both $\sigma' = b$ and $\sigma' = c$. After coming out of the first iteration of Step 2 with $\sigma = abc$, we have constructed

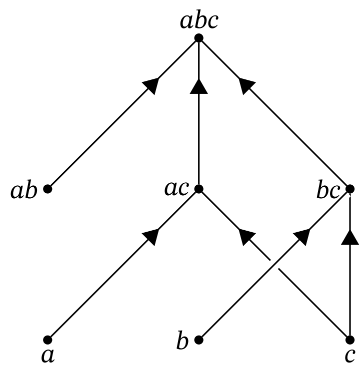


with $S = \{abc, ab, ac, bc\}$. Now in Step 2, pick $\sigma = bc$. We see that $bc \in G$, so we skip to Step 6. As before, we run through $v = b$ and $v = c$ to obtain

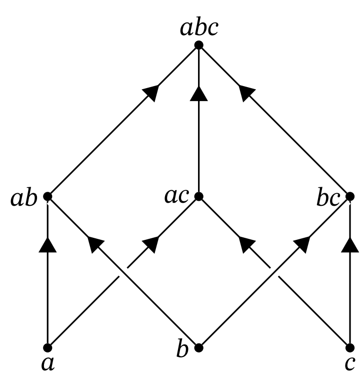


with $S = \{abc, ab, ac, bc, b, c\}$. Going back to Step 2 and choosing $\sigma = ac$, consider $\sigma' = ac - \{a\} = c$ in Step 7. Since $c \in G$ already, we must view this c as the same c already in G via Step 10. In particular, there is a directed edge $(c, bc) \in V$. Next we add the directed edge (c, ac) in

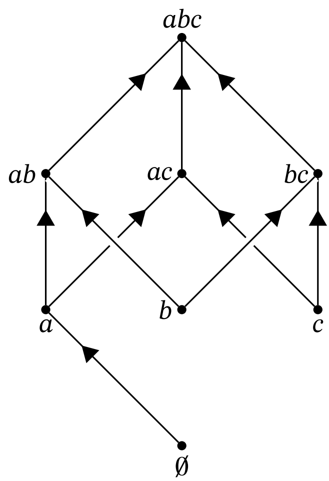
Step 11. After we consider $\sigma' = a$, we have



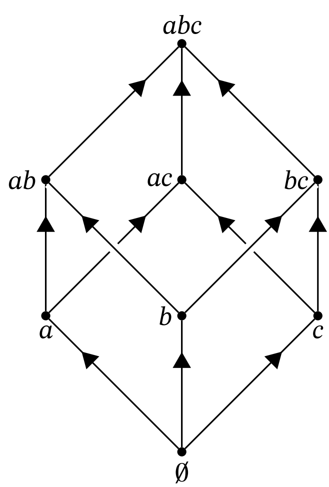
with $S = \{abc, ab, ac, bc, b, c, a\}$. Returning to Step 2 with $\sigma = ab$, we see that no new nodes are created, but we do obtain two new directed edges (a, ab) and (b, ab). This yields



with the same S as before. Now with $\sigma = a$ we construct the empty set, adding the directed edge from $\emptyset$ to a:



After running through b and c in Step 2, we have run through all nonempty $\sigma \in S$, yielding the directed Hasse diagram



9.2.2. Computing a discrete Morse graph. Suppose we have constructed the upward Hasse diagram $\mathcal{H}$. We give an algorithm to construct a discrete Morse matching M. In fact, we can think of the upward Hasse diagram as already communicating a discrete Morse matching—namely, the trivial one where there are no pairings so that every simplex is critical.

Strictly speaking, a discrete Morse matching does not contain information about the critical simplices even though the critical simplices are the ones which are not matched. Hence, given a discrete Morse matching M on a directed Hasse diagram $\mathcal{H}$, we define the **discrete Morse graph** G to be the directed Hasse diagram $\mathcal{H}_M$ along with the matching M and set of critical simplices C. In the case of the upward Hasse diagram, $M = \emptyset$ while C consists of all nodes of $\mathcal{H}$. We write $G = (M, C)$, viewing G as a directed graph along with matching and critical simplex information. Note that there is no substantive difference between $\mathcal{H}_M$ and G. Rather, as mentioned just before Section 9.2.1, the technical distinction is that G explicitly contains the information M and C, while this information can be deduced from $\mathcal{H}_M$.

Recall that if $\sigma \in K$ is a simplex, the node corresponding to σ in $\mathcal{H}$ is also called σ.

Algorithm 7 (Dłotko and Wagner) Discrete Morse graph

`Input:` Discrete Morse graph $G = (M, C)$

`Output:` Discrete Morse graph $G = (M, C)$

while there exists unmatched element $\alpha \in C$ with unique unmatched element $\beta \in \partial(\alpha)$ **do**
  Match α and β; i.e., reverse arrow on edge $\alpha\beta$ to be upward.

The output discrete Morse graph will have at most as many critical simplices as the input discrete Morse graph, if not fewer. Algorithm 7 is similar to Algorithm 4 from Section 9.1 in the sense that both algorithms attempt to cancel critical simplices. However, it should be noted that Joswig and Pfetsch have shown that the number of critical cells is NP-complete and MAXSNP-hard [**91**]. In other words, there is no fully polynomial-time algorithm that minimizes the number of critical simplices.

9.2.3. Computing the boundary operator. Let $G = (M, C)$ be a discrete Morse graph. For any nodes $s, t \in G$, let $P_s(t)$ be the number of distinct directed paths from node s to node t and let $\mathrm{prev}(v) := \{x : xv \text{ is an edge in } G\}$. We then have the recurrence relation

$$P_s(u) := \begin{cases} 1 & \text{for } u = s, \\ \sum_{v \in \mathrm{prev}(u)} P_s(v) & \text{for } u \neq s. \end{cases}$$

A **topological sorting** of a directed acyclic graph G (e.g. a discrete Morse graph or directed Hasse diagram) is a total ordering $\prec$ of the vertices such that for every directed edge uv from vertex u to vertex v, we have that $u \prec v$. Several well-known algorithms exist to topologically sort the vertices of a directed graph. See for instance Kahn's algorithm [**94**] or depth-first search in [**46**, Section 22.4].

Recall from Section 3.2 that we compute homology over $\mathbb{F}_2 := \{0, 1\}$ so that $1 + 1 \equiv 0 \bmod 2$.

Algorithm 8 (Dłotko and Wagner) Boundary operator

`Input:` Discrete Morse graph $G = (M, C)$

`Output:` Boundary operator ∂

Topologically sort the vertices of G
for each critical vertex $s \in G$ **do**
  Assign $P_s(v) := 0$ for each vertex $v \neq s$
  Assign $P_s(s) := 1$
  for each vertex c following s in topological order **do**
    if c is critical **then**
      $\partial(s, c) := P_s(c) \mod 2$
    else
      for each v such that cv is an edge **do**
        $P_s(v) += P_s(c)$

Algorithm 8 works in $O(|C|(V + E))$ time, where V is the number of vertices and E the number of edges of G. A proof that the algorithm is correct may be found in [**52**, Theorem 5.1].

9.2.4. Computing homology with the iterated critical complex. At this point, we have assembled all the necessary ingredients, and now it is simply a matter of piecing them together. After building the upward Hasse diagram (viewed as the trivial discrete Morse graph on K) of a simplicial complex K, we compute the critical complex of K; in other words, we build a better discrete Morse graph on K. Given this critical complex, we build a new critical complex, one with hopefully fewer critical simplices. From this new critical complex, we build yet another critical complex, etc. This process of building a new critical complex from an existing critical complex is known as an **iterated discrete Morse decomposition**. It is guaranteed that at each iteration we will always obtain a critical complex with fewer or the same number of critical simplices. Furthermore, we are guaranteed that this process will eventually stabilize by returning the same critical complex as the previous input after a finite number of steps. Once this happens, the iteration breaks and the Betti numbers are computed. See [**52**, Section 6] for details.

Algorithm 9 (Dłotko and Wagner) Homology via iterated discrete Morse decomposition

`Input:` Simplicial complex K

`Output:` Betti numbers $\beta_i(K)$

Compute upward Hasse diagram $\mathcal{H}$ of K
$\mathcal{H} =: G$, trivial critical complex
while true **do**
  $\mathcal{C} :=$ build critical complex of G
    $G := (M, C)$ (Algorithm 7)
    $\partial :=$ compute boundary operator (Algorithm 8)
  if $\mathcal{C} = G$ **then**
    break
  $G := \mathcal{C}$
for $i := 0$ to d **do**
  $\beta_i :=$ number of i-dimensional simplices of G

In the same paper introducing Algorithm 9, the authors give an algorithm to compute persistent homology using the iterated discrete Morse decomposition approach [**52**, Section 7-8]. Other algorithms that utilize discrete Morse theory in the service of persistent homology are found in [**38**, **86**, **98**, **111**, **119**].

Chapter 10

Strong discrete Morse theory

This final chapter discusses another kind of discrete Morse theory and, along the way, spends some effort developing other types of discrete topology that one can pursue. In that sense, this chapter shows how discrete Morse theory may act as a catalyst in exposing one to other kinds of topology. The chapter reflects the personal taste of the author. One could pursue many other kinds of topology as well as other kinds of mathematics using discrete Morse theory. See the preface for a brief survey of several such directions.

10.1. Strong homotopy

10.1.1. Simplicial maps. When encountering a new area of mathematics, one should ask the question "Where are the functions?" Functions are how we study structure in modern mathematics. They carry the structure of one object into the structure of another object. For example, in linear algebra, the structure of a vector space V is carried into the structure of a vector space W by a function which is a linear transformation. The properties of a linear transformation guarantee that the vector space structure is preserved. In algebra, group, ring, and field homomorphisms are functions respecting the structure of groups, rings, and fields, respectively. In all cases, each kind of function must satisfy certain properties to preserve the structure of the object.

What is the right notion of a function between simplicial complexes? We desire a function $f: K \to L$ between simplicial complexes that preserves the simplicial structure.

Definition 10.1. A **simplicial function** or **simplicial map** $f: K \to L$ is a function $f_V: V(K) \to V(L)$ on the vertex sets of K and L with the property that if $\sigma = v_{i_0}v_{i_1}\cdots v_{i_m}$ is a simplex in K, then $f(\sigma) := f_V(v_{i_0})f_V(v_{i_1})\cdots f_V(v_{i_m})$ is a simplex in L.

In other words, a simplicial map is one induced by a map on the vertices that takes simplices to simplices; i.e., it preserves the simplicial structure, even if it may take a larger simplex to a smaller simplex. The simplicial structure is preserved in the sense that if $\alpha \subseteq \beta$, then $f(\alpha) \subseteq f(\beta)$.

Problem 10.2. Let $f: K \to L$ be a simplicial map. Prove that if $\alpha \subseteq \beta$, then $f(\alpha) \subseteq f(\beta)$.

Exercise 10.3. Determine whether the following functions are simplicial maps. If so, prove it. If not, show why not.

(i) For any simplicial complexes K and L and a fixed vertex $u \in L$, $f: K \to L$ defined by $f(v_i) = u$ for every $v_i \in V(K)$.

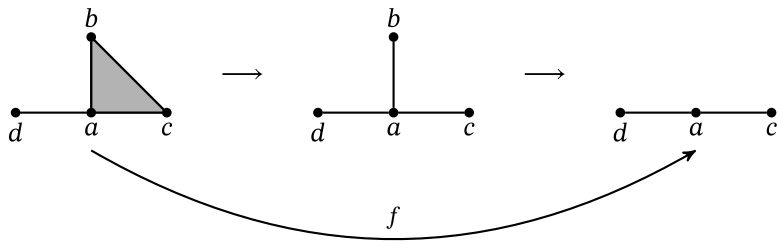
(ii) For a subcomplex $U \subseteq K$, the inclusion $i_U: U \to K$ defined by $i_U(v) = v$.

(iii) For $K = \langle abc \rangle$ and $L = \langle ab, bc \rangle$, $f: K \to L$ defined by $f(x) = x$ for every $x \in K$.

Hopefully, you showed that the very last example in Exercise 10.3 is not a simplicial map. You might realize that this function corresponds to the elementary collapse obtained by removing the free pair $\{ac, abc\}$. In fact, an elementary collapse is almost never a simplicial map (see Problem 10.4).

Problem 10.4. Let $\{\sigma^{(p)}, \tau^{(p+1)}\}$ be a free pair for a simplicial complex K^n, $0 \leq p \leq n-1$. Determine with proof when the induced function $f: K \to K - \{\sigma, \tau\}$ is a simplicial map. For every vertex $v \in K$, the induced function f is defined by $f(v) = v$ when $p > 0$ and $f(\sigma) = \tau$ with all other vertices being sent to themselves when $p = 0$.

The prospect of combining discrete Morse theory with simplicial maps thus seems bleak. However, if we start with an elementary collapse and continue collapsing, as in the following example, we do notice something.



While the initial collapse is not a simplicial map, the composition of the two elementary collapses, that is, the composition defined by $f(a) = f(b) = a$ and $f(c) = c$, is a simplicial map.

Exercise 10.5. Check that the above composition is a simplicial map.

We thus desire to find a condition under which a composition of elementary collapses corresponds to a simplicial map. This will be the goal of the next subsection.

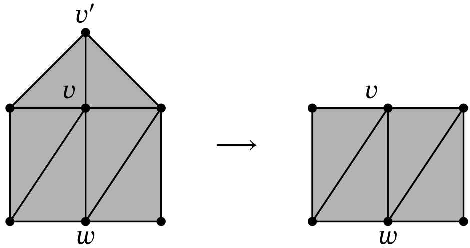
10.1.2. Dominating vertices.

In the previous subsection, we found a sequence of elementary collapses that induced a simplicial map. The property that allowed us to do this concerned the relationship between the vertices a and b in our example. More generally, we define the following:

Definition 10.6. Let K be a simplicial complex. A vertex v is said to **dominate** v' (it is also said that v' is **dominated** by v) if every maximal simplex (facet) of v' also contains v.

The idea behind this definition is that v' “can't get away from” v.

Example 10.7. Borrowing an example from [**63**], we have the following simplicial map where v' is sent to v and all other vertices are sent to themselves.



Note here that v dominates v'. In general, we may ask of any pair of vertices "does a dominate b?" We see that v' does not dominate v since at least one facet of v does not contain v'. Similarly, v does not dominate w, nor does w dominate v.

Exercise 10.8. Is it possible to have a pair of vertices u and v such that v dominates u and u dominates v?

For any vertex $v \in K$, let $K-\{v\} := \{\sigma \in K : v \notin \sigma\}$, i.e., the largest subcomplex of K not containing v. The key observation is that almost by definition, a dominated vertex corresponds to a simplicial map. Many of the results in this subsection are due to J. Barmak [**21**].

Proposition 10.9. Let K be a simplicial complex, and suppose that v dominates v'. Then the function $r\colon K \to K - \{v'\}$ defined by $r(x) = x$ for all $x \neq v'$ and $r(v') = v$ is a simplicial map called an **elementary strong collapse**. Moreover, an elementary strong collapse is a sequence of elementary (standard) collapses.

The simplicial map r in this proposition is called a **retraction**.

Proof. That r is a simplicial map is Problem 10.10. To show that an elementary strong collapse is a sequence of elementary collapses, suppose v dominates v' and let $\sigma_1, \ldots, \sigma_k$ be the facets of v'. Then $v \in \sigma_i$ for all i. Write $\sigma_i = vv'v_{i_2} \cdots v_{i_n}$. We claim that $\{\sigma_i, \tau_i\}$ is a free pair, where $\tau_i := \sigma_i - \{v\}$. Clearly $\tau_i \subseteq \sigma_i$. If τ_i is the coface of another simplex β, then $v' \in \beta$ and $v \notin \beta$ since σ_i is maximal. Thus $\{\sigma_i, \tau_i\}$ is a free pair, and it may be removed. Do this over all facets of v' to see that the elementary strong collapse is a sequence of elementary collapses. $\square$

Problem 10.10. Prove that r in the statement of Proposition 10.9 is a simplicial map.

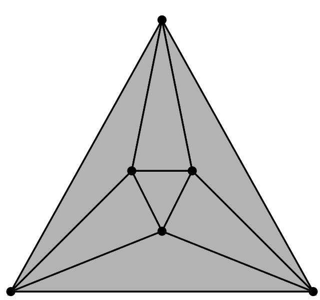
Exercise 10.11. According to Proposition 10.9, every elementary strong collapse is a sequence of standard collapses. For a fixed elementary strong collapse, is such a sequence unique up to choice of collapses?

Formally, we now make the following definition.

Definition 10.12. If v dominates v', then the removal of v' from K (without necessarily referencing the induced map) is called an **elementary strong collapse** and is denoted by $K \searrow\!\!\searrow K - \{v'\}$. The addition of a dominated vertex is an **elementary strong expansion** and is denoted by $\nearrow\!\!\nearrow$. A sequence of elementary strong collapses or elementary strong expansions is also called a strong collapse or strong expansion, respectively, and also denoted by $\nearrow\!\!\nearrow$ or $\searrow\!\!\searrow$, respectively. If there is a sequence of strong collapses and expansions from K into L, then K and L are said to have the same **strong homotopy type**. In particular, if $L = *$, then K is said to have the **strong homotopy type of a point**. If there is a sequence of elementary strong collapses from K to a point, K is called **strongly collapsible**.

The reader will immediately recognize the similarity to simple homotopy type from Section 1.2.

Example 10.13. The above definition immediately raises the question of whether strongly collapsible is the same as collapsible. Clearly strongly collapsible implies collapsible, but the following example shows that the converse is false.



Because this simplicial complex was first used in this context by Barmak and Minian [**23**], it is affectionately referred to as the **Argentinian complex**. This simplicial complex is clearly collapsible, but it is not strongly collapsible since there is no dominating vertex. This latter fact can be observed by brute force. Thus collapsible does not imply strongly collapsible.

Remark 10.14. At this point, it is still theoretically possible that the Argentinian complex could have the strong homotopy type of a point. As we mentioned in Example 1.67, it is possible to construct collapsible simplicial complexes for which one can get stuck while performing a series of collapses. In other words, just because a simplicial complex is collapsible does not mean any sequence of collapses will bring you down to a point. Perhaps we could perform several strong expansions followed by a clever choice of strong collapses to show that the Argentinian complex has the strong homotopy type of a point. The advantage of working with strong collapses and expansions is that these kinds of bizarre pathologies do not occur. We will prove this as a special case in Corollary 10.32 in Section 10.1.3.

10.1.3. Contiguity.

We seemingly shift gears now but will make the connection with the previous section at the end.

Definition 10.15. Let $f, g\colon K \to L$ be maps between simplicial complexes. Then f and g are said to be **contiguous**, denoted by $f \sim_c g$, if for every simplex $\sigma \in K$ we have that $f(\sigma) \cup g(\sigma)$ is a simplex of L.

Contiguity *almost* forms an equivalence relation on the set of simplicial functions from K to L, but not quite.

Problem 10.16. Show that contiguity is reflexive and symmetric. Give a counterexample to show that it is not in general transitive.

Like the relation $\prec$ from Definition 5.33, we need to force transitivity by taking the transitive closure.

Definition 10.17. We say that simplicial maps $f, g\colon K \to L$ are in the same **contiguity class** or **strongly homotopic**, denoted by $f \sim g$, if there is a sequence

$$f = f_0 \sim_c f_1 \sim_c \cdots \sim_c f_{n-1} \sim_c f_n = g$$

of contiguous maps joining f and g.

Combining this with Problem 10.16, we immediately obtain the following:

Proposition 10.18. Being strongly homotopic is an equivalence relation on simplicial maps from K to L.

Because we are now working with equivalence classes of simplicial maps (i.e, not individual simplicial maps), we may investigate the equivalence class of special simplicial maps. Call a simplicial complex K **minimal** if it contains no dominating vertices.

Proposition 10.19. Let K be minimal, and suppose that $\phi\colon K \to K$ is a simplicial map satisfying $\phi \sim \mathrm{id}_K$. Then $\phi = \mathrm{id}_K$.

Proof. Let $\phi \sim \mathrm{id}_K$. We need to show that $\phi(v) = v$ for every $v \in K$. Let σ be a facet containing v. Then $\phi(\sigma) \cup \sigma$ is a simplex of K, and since σ is a facet, we have $\phi(v) \in \phi(\sigma) \cup \sigma = \sigma$. Hence every facet containing v also contains $\phi(v)$. But K has no dominating vertices by hypothesis. Hence $\phi(v) = v$. $\square$

Definition 10.20. Let $*\colon K \to L$ be any map defined by $*(v) = u$ for all $v \in K$ and a fixed $u \in L$. If $f \sim *$, we say that f is **null-homotopic**.

Using the generic $*$ to denote the simplicial map sending everything to a single point is justified by the following:

Proposition 10.21. Suppose K and L are (connected) simplicial complexes. Let $u, u' \in L$ be two fixed vertices, and define $u, u'\colon K \to L$ by $u(v) = u$ and $u'(v) = u'$. Then $u \sim u'$.

Problem 10.22. Prove Proposition 10.21.

By Proposition 10.21, the choice of vertex is irrelevant, so we simply write $*$ for the map sending each vertex to a fixed vertex.

Definition 10.23. Let $f : K \to L$ be a simplicial map. Then f is a **strong homotopy equivalence** if there exists $g : L \to K$ such that $g \circ f \sim \mathrm{id}_K$ and $f \circ g \sim \mathrm{id}_L$. If $f : K \to L$ is a strong homotopy equivalence, we write $K \approx L$.

Problem 10.24. Prove that $K \approx *$ if and only if id_K is null-homotopic.

What is the relationship between strong homotopy equivalence and having the same strong homotopy type? As you may have guessed, these are two sides of the same coin. First, a lemma.

Lemma 10.25. If $f : K \to L$ is a strong homotopy equivalence between minimal simplicial complexes, then there exists a simplicial map $g : L \to K$ such that $g \circ f = \mathrm{id}_K$ and $f \circ g = \mathrm{id}_L$.

Problem 10.26. Prove Lemma 10.25. [Hint: Use Proposition 10.19.]

Any $f : K \to L$ that satisfies the conditions found in the conclusion of Lemma 10.25 is called a **simplicial complex isomorphism**. If f is a simplicial complex isomorphism, we say that K and L are **isomorphic**.

As we know, a strong collapse may be viewed as a simplicial map. It is not difficult to imagine that such a map is a strong homotopy equivalence. If so, we may view a sequence of strong collapses as a sequence of strong homotopy equivalences, which itself is a strong homotopy equivalence.

Proposition 10.27. Let K be a simplicial complex and v' a vertex dominated by v. Then the inclusion map $i : K - \{v'\} \to K$ is a strong homotopy equivalence. In particular, if K and L have the same strong homotopy type, then $K \approx L$.

Proof. Define $r : K \to K - \{v'\}$ to be the retraction from Proposition 10.9. If $\sigma \in K$ is a simplex without v', then clearly $ir(\sigma) = \mathrm{id}_K(\sigma)$. Otherwise, let σ be a simplex with $v' \in \sigma$, and let τ be any maximal simplex containing σ. Then $ir(\sigma) \cup \mathrm{id}_K(\sigma) = \sigma \cup \{v\} \subseteq \tau$. Hence $ir(\sigma) \cup \mathrm{id}_K(\sigma)$ is a simplex of K. Thus $ir \sim \mathrm{id}_K$. Clearly $ri = \mathrm{id}_{K-\{v'\}}$ so that both i and r are strong homotopy equivalences and hence $K \approx K - \{v'\}$. Inductively, $K \approx L$ whenever K and L have the same strong homotopy type. $\square$

Definition 10.28. Let K be a simplicial complex. The **core** of K is the minimal subcomplex $K_0 \subseteq K$ such that $K \searrow\searrow K_0$.

Remark 10.29. The notation K_0 for the core of K should not be confused with the set of all 0-simplices of K, which is also denoted by K_0.

The use of the article “the” above is justified by the following:

Proposition 10.30. Let K be a simplicial complex. Then the core of K is unique up to isomorphism.

Proof. Let K_1 and K_2 be two cores of K. Now these are both obtained by removing dominating points, so $K_1 \approx K_2$ by Proposition 10.27. Furthermore, because these cores are minimal, Lemma 10.25 implies that K_1 and K_2 are isomorphic. □

We may now summarize all of this work in the following theorem:

Theorem 10.31. [**23**] Let K and L be simplicial complexes. Then K and L have the same strong homotopy type if and only if there exists a strong homotopy equivalence $f\colon K \to L$.

Proof. The forward direction is Proposition 10.27. For the backward direction, assume $K \approx L$. Since $K \approx K_0$ and $L \approx L_0$, $K_0 \approx L_0$. As before, these strongly homotopic minimal cores are then isomorphic. Since they are isomorphic, they clearly have the same strong homotopy type. Hence K and L also have the same strong homotopy type. □

As a special case of Theorem 10.31, we are able to address the point taken up in Remark 10.14. Unlike the case of standard collapses, we will never “get stuck” in a strongly collapsible complex.

Corollary 10.32. Let K be a simplicial complex. Then K is strongly collapsible if and only if every sequence of strong collapses of K collapses to a single point. In other words, a simplicial complex K has the strong homotopy type of a point if and only if K is strongly collapsible.

Remark 10.33. The beautiful thing about Theorem 10.31 is that, like many other results in mathematics, it bridges two different viewpoints, showing that they are in essence the same. Strong homotopy type is a combinatorial definition where one checks a combinatorial condition. Strong homotopy equivalence is more along the lines of a topological definition, one involving interactions between several simplicial maps.

Yet Theorem 10.31 tells us that they are two sides of the same coin, one from a combinatorial viewpoint, the other from a topological viewpoint. This surprising juxtaposition of Theorem 10.31 is simply enchanting!

Problem 10.34. In Example 1.47, we showed that all cycles C_n have the same simple homotopy type. Do they have the same strong homotopy type?

10.2. Strong discrete Morse theory

As is now apparent, the idea behind discrete Morse theory is extremely simple: every simplicial complex can be broken down (or, equivalently, built up) using only two moves: 1) perform an elementary collapse, and 2) remove a facet. This idea has led to many insights and relationships. Given our work in Section 10.1, we now have a third move—namely, a strong elementary collapse. Although this is a sequence of elementary standard collapses, a strong collapse, if found, does digest a complex quite quickly. Hence, it is worth investigating what happens when we add this "third move" into discrete Morse theory. This was first accomplished by Fernández-Ternero et al. in [**62**]. We present a slightly different framework than the one utilized in the cited paper. We begin by investigating the Hasse diagram, similar to our approach in Section 2.2.3.

Definition 10.35. Let K be a simplicial complex and $\mathcal{H}_K$ the Hasse diagram of K. For any 1-simplex $uv = e \in \mathcal{H}_K$, define $F_e := \{\sigma \in \mathcal{H}_K : e \in \sigma\}$. Let $S \subseteq F_e$ be any subset. The (v,e)-**strong vector generated by** S, denoted by $B^S(v,e)$, is defined by

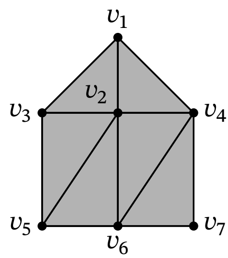
$$B^S(v,e) := \{\sigma \in \mathcal{H}_K : v \leq \sigma \leq \tau \text{ for some } \tau \in S\}.$$

We sometimes write $B(v,e)$ when S is clear from the context, and we sometimes refer to it as a strong vector.

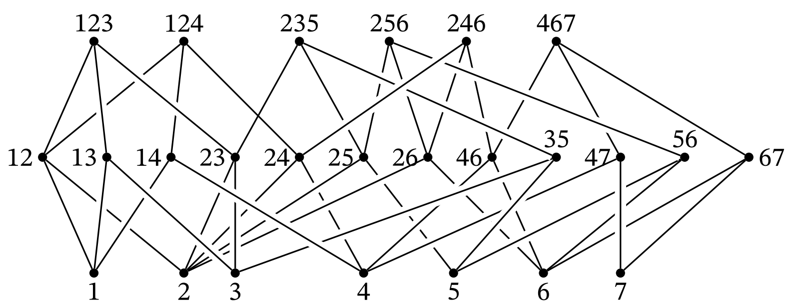
Exercise 10.36. Show that if $S \neq \emptyset$, then $v, e \in B^S(v,e)$.

Any strong vector satisfying $B^S(v,e) = \{v,e\}$ is called a **Forman vector**. A pair $\sigma^{(p)} < \tau^{(p+1)}$ for which $p > 0$ (i.e., not a Forman vector) is a **critical pair**.

Example 10.37. Let K be the simplicial complex



To save space, we write a simplex in the Hasse diagram $\mathcal{H}_K$ by concatenating its subscripts, e.g. ij stands for the simplex v_iv_j.



Let $e = v_1v_2$. Then $F_e = \{v_1v_2, v_1v_2v_3, v_1v_2v_4\}$. Pick $v = v_1$, and define $S_1 := \{v_1v_2\}, S_2 := \{v_1v_2v_3\}, S_3 := \{v_1v_2v_4\}$, and $S_4 := \{v_1v_2v_3, v_1v_2v_4\}$. We then generate all possible strong vectors on F_e, namely,

$$\begin{aligned}
B^{S_1}(v,e) &= \{v_1, v_1v_2\},\\
B^{S_2}(v,e) &= \{v_1, v_1v_2, v_1v_3, v_1v_2v_3\},\\
B^{S_3}(v,e) &= \{v_1, v_1v_2, v_1v_4, v_1v_2v_4\},\\
B^{S_4}(v,e) &= \{v_1, v_1v_2, v_1v_3, v_1v_4, v_1v_2v_3, v_1v_2v_4\}.
\end{aligned}$$

We will show in Proposition 10.48 that a strong collapse is precisely a strong vector. First, we give the definition of a strong discrete Morse matching.

Definition 10.38. A **strong discrete Morse matching** $\mathcal{M}$ is a partition of $\mathcal{H}_K$ into strong vectors, critical pairs, and singletons. A singleton is called a **critical simplex**. The **index** of a critical pair $\{\alpha^{(p)}, \beta^{(p+1)}\}$ is $p+1$. We refer to the set of all critical pairs and critical simplices as **critical objects**. The number of critical simplices of dimension i is denoted by m_i, while the number of critical pairs of index i is denoted by p_i. The set of all critical objects of $\mathcal{M}$ is denoted by $\mathrm{scrit}(\mathcal{M})$.

Example 10.39. We give two examples of strong Morse matchings on K from Example 10.37. Consider the sets $F_{v_1v_2}, F_{v_2v_5}, F_{v_4v_6}, F_{v_2v_3}, F_{v_6v_7}$, and $F_{v_2v_6}$. Then there exist subsets that generate the following strong vectors:

$$
\begin{aligned}
B(v_1, v_1v_2) &= \{v_1, v_1v_2, v_1v_3, v_1v_4, v_1v_2v_3, v_1v_2v_4\},\\
B(v_5, v_2v_5) &= \{v_5, v_2v_5, v_5v_6, v_3v_5, v_2v_5v_6, v_2v_3v_5\},\\
B(v_4, v_4v_6) &= \{v_4, v_4v_6, v_4v_7, v_2v_4, v_4v_6v_7, v_2v_4v_6\},\\
B(v_3, v_2v_3) &= \{v_3, v_2v_3\},\\
B(v_7, v_6v_7) &= \{v_7, v_6v_7\},\\
B(v_2, v_2v_6) &= \{v_2, v_2v_6\}.
\end{aligned}
$$

These strong vectors, along with the singleton $\{v_6\}$, form a strong Morse matching $\mathcal{M}_1$. The only critical object is the critical simplex v_6 (the last three strong vectors are Forman vectors).

We define another strong Morse matching $\mathcal{M}_2$. Using the same F as above, there exist subsets that generate the following strong vectors:

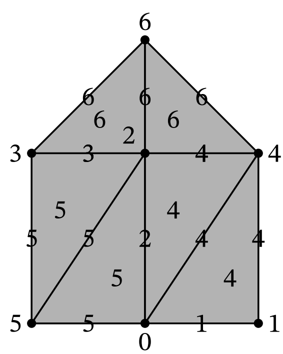
$$
\begin{aligned}
B(v_1, v_1v_2) &= \{v_1, v_1v_2, v_1v_3, v_1v_2v_3\},\\
B(v_5, v_2v_5) &= \{v_5, v_2v_5, v_5v_6, v_2v_5v_6\},\\
B(v_4, v_4v_6) &= \{v_4, v_4v_6, v_4v_7, v_2v_4, v_4v_6v_7, v_2v_4v_6\},\\
B(v_7, v_6v_7) &= \{v_7, v_6v_7\},\\
B(v_2, v_2v_6) &= \{v_2, v_2v_6\}.
\end{aligned}
$$

Critical pairs $\{v_1v_4, v_1v_2v_4\}$ and $\{v_3v_5, v_2v_3v_5\}$ are then combined with these strong vectors, along with the singletons $\{v_3\}, \{v_2v_3\}$, and $\{v_6\}$, to yield the strong Morse matching $\mathcal{M}_2$. Here we have five critical objects, consisting of two critical pairs and three critical simplices.

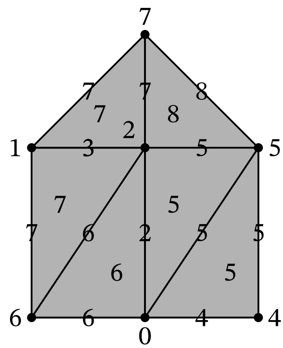
Definition 10.40. A **strong discrete Morse function** f **on** K **with respect to a strong Morse matching** $\mathcal{M}$ is a function $f: K \to \mathbb{R}$ satisfying $f(\alpha) \leq f(\beta)$ whenever $\alpha < \beta$, with $f(\alpha) = f(\beta)$ if and only if there is a set I in the partition $\mathcal{M}$ such that $\alpha, \beta \in I$. We set $\mathrm{scrit}(f) := \mathrm{scrit}(\mathcal{M})$.

Note that one obtains a flat discrete Morse function in the sense of Forman by taking a partition consisting of only Forman vectors, critical pairs, and singletons.

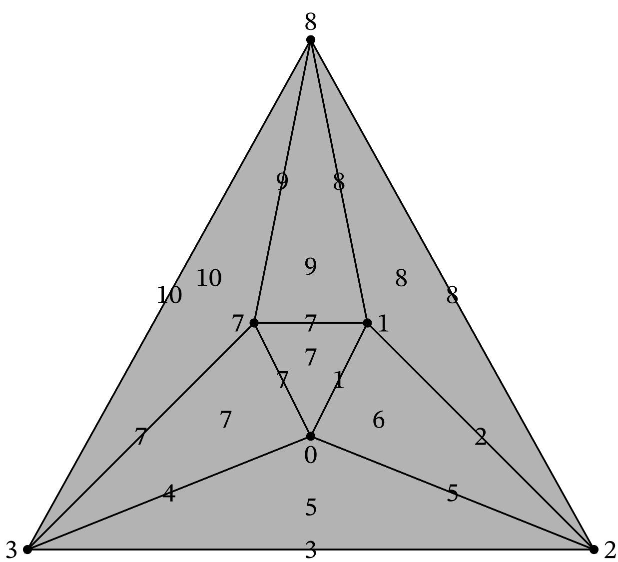
Example 10.41. We now put a strong discrete Morse function on the simplicial complex K using the strong Morse matchings $\mathcal{M}_1$ and $\mathcal{M}_2$ from Example 10.39. A strong discrete Morse function with respect to $\mathcal{M}_1$ is given by



while a strong discrete Morse function with respect to $\mathcal{M}_2$ is given by



Exercise 10.42. Verify that the following labeling is a strong discrete Morse function on the Argentinian complex A. Identify the critical simplices and critical pairs.



We may also combine strong discrete Morse functions to obtain new ones.

Problem 10.43. Let $f_1 : K_1 \to \mathbb{R}$ and $f_2 : K_2 \to \mathbb{R}$ be strong discrete Morse functions with strong Morse matchings $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Prove that $(f_1 + f_2) : K_1 \cap K_2 \to \mathbb{R}$ is also a strong discrete Morse function with strong Morse matching given by $\mathcal{M} := \{I \cap J : I \in \mathcal{M}_1, J \in \mathcal{M}_2, I \cap J \neq \emptyset\}$. You may need to adjust $f_1 + f_2$ by adding a small amount ϵ to some values to ensure that simplices in different sets of the partition have different values.

Problem 10.44. Find the strong discrete Morse function obtained by adding together the two strong discrete Morse functions in Example 10.41. What is the corresponding strong Morse matching?

Strong discrete Morse functions are inspired by and similar to generalized discrete Morse functions.

Problem 10.45. Give an example to show that a strong vector need not be an interval (Definition 2.75) and an interval need not be a strong vector.

Problem 10.46. Let K be a simplicial complex, and let $v \in K$ with $v < \sigma$. If $S := \{\sigma\}$, show that $[v, \sigma] = B^S(v, e)$ for all edges $v < e < \sigma$.

We now prove a collapsing theorem for strong discrete Morse functions.

Theorem 10.47. Let $f : K \to \mathbb{R}$ be a strong discrete Morse function, and suppose that $(a, b]$, $a < b$, is a real interval containing no critical values. Then $K(b) \searrow\searrow K(a)$.

Proof. Let $\mathcal{M}$ be the strong Morse matching for f. If $(a, b]$ contains no regular values, then we are done. By partitioning $(a, b]$ into subintervals, we may assume without loss of generality that $(a, b]$ contains exactly one regular value $c \in (a, b]$. Then there is a set I in $\mathcal{M}$ such that $c = f(\alpha)$ for every $\alpha \in I$. Furthermore, since c is a regular value by supposition, $I = B(v, e)$ for some v and $e = vu$. We claim that v is dominated by u in $K(b)$. If so, we may perform a strong collapse by removing v (and all simplices containing it) from $K(b)$, yielding $K(a)$. To see that v is dominated by u in $K(b)$, let σ be any facet of v in $K(b)$. Since $v < \sigma$, $f(v) \leq f(\sigma)$, and since there is only one regular value in $(a, b]$, it must be that $f(v) = f(\sigma) = c$. Hence $\sigma \in B(v, e)$ so that $v \leq \sigma \leq \tau$ for some

τ satisfying $e < \tau$. As above, we must have $f(\tau) = c$, so $\tau \in K(b)$. By supposition, σ is a facet in $K(b)$, so $\sigma = \tau$. Since $uv = e < \tau$, we have $u \in \sigma$. Thus u dominates v in $K(b)$, and $K(b) \searrow\searrow K(a)$. $\square$

We then obtain a characterization for strongly collapsible simplicial complexes in terms of a strong matching on the Hasse diagram.

Proposition 10.48. Let K be a simplicial complex. There is a strong discrete Morse function $f : K \to \mathbb{R}$ with only one critical value if and only if K is strongly collapsible.

Proof. The forward direction is simply Theorem 10.47. For the reverse direction, let K be strongly collapsible via a sequence of n strong collapses. It suffices to show that the simplices in a single strong collapse correspond to a strong vector in the Hasse diagram. The result will then follow by induction on the strong collapses by associating a strong vector to each strong collapse (and the unique critical vertex with a singleton). Since K is strongly collapsible, there exists a vertex v'_n dominated by a vertex v_n along edge $e_n = v_n v'_n$. Label the simplices of the strong vector $B^{S_n}(v'_n, e_n)$ with $S_n := F_{e_n}$ with value n; i.e., define $f(\sigma) = n$ for every $\sigma \in B^{S_n}(v'_n, e_n)$. Perform the strong collapse and let $K - \{v'_n\}$ be the resulting complex. Since $K - \{v'_n\}$ is strongly collapsible, there exists a vertex v'_{n-1} dominated by a vertex v_{n-1} along edge $e_{n-1} = v_{n-1}v'_{n-1}$. Taking $S_{n-1} = F_{e_{n-1}} \cap (K - \{v_n\})$, we see that $B^{S_{n-1}}(v'_{n-1}, e_{n-1})$ yields another strong vector. Label these values $n-1$. Continuing in this manner, we obtain a strong discrete Morse matching on $\mathcal{H}_K$.

It remains to show that the labeling specified satisfies the properties of a strong discrete Morse function. Clearly two simplices are given the same label if and only if they are in the same set of the partition. Let $\alpha < \beta$ and suppose by contradiction that $k = f(\alpha) > f(\beta) = \ell$. Then, given the order specified by the strong collapses, α was part of a strong collapse, other strong collapses were performed, and then β was part of a strong collapse. But this is impossible since $\alpha < \beta$ and results of strong collapses need to remain simplicial complexes. $\square$

10.3. Simplicial Lusternik-Schnirelmann category

By now, you have discerned a principled theme of this book: once we have a notion of what it means for two objects to be the "same," we desire ways to tell them apart. In Chapter 1, we introduced simple homotopy type and have spent much effort developing tools such as the Euler characteristic and Betti numbers, allowing us to tell simplicial complexes apart up to simple homotopy. For example, Proposition 1.42 tells us that if K and L have the same simple homotopy type, then they have the same Euler characteristic. The contrapositive of this is most useful for telling simplicial complexes apart; that is, if $\chi(K) \neq \chi(L)$, then K and L do not have the same simple homotopy type. This section is devoted to developing not only a strong homotopy invariant (Corollary 10.65), but one that has a nice relationship with the number of critical objects of a strong discrete Morse function (Theorem 10.70). We first define our object of study.

Definition 10.49. Let K be a simplicial complex. We say that a collection of subcomplexes $\{U_0, U_1, \ldots, U_n\}$ is a **covering** or **covers** K if $\bigcup\limits_{i=0}^{n} U_i = K$.

Definition 10.50. Let $f\colon K \to L$ be a simplicial map. The **simplicial Lusternik-Schnirelmann category**, **simplicial LS category**, or **simplicial category of** f, denoted by $\mathrm{scat}(f)$, is the least integer n such that K can be covered by subcomplexes $U_0, U_1, \ldots, U_n$ with $f|_{U_j} \sim *$ for all $0 \leq j \leq n$. We call $U_0, \ldots, U_n$ a **categorical cover of** f.

Remark 10.51. The original Lusternik-Schnirelmann category of a topological space was defined in 1934 by the Soviet mathematicians L. Lusternik and L. Schnirelmann [**114**]. It has generated a great deal of research, several variations, and many related invariants. See, for example, the book-length treatment [**47**]. However, it was not until 2015 that Fernández-Ternero et al. defined a simplicial version of the LS category (Definition 10.52), one which was defined purely for simplicial complexes [**61**, **63**]. Our definition of the simplicial category of a map, first studied in [**140**], is a slight generalization of the original definition, as we will show in Proposition 10.53. Other versions of the Lusternik-Schnirelmann category for simplicial complexes are studied in [**2**, **79**] and [**146**, **147**].

Definition 10.52. A subcomplex $U \subseteq K$ is called **categorical in** K if $i_U : U \to K$ is null-homotopic, where i_U is the inclusion. The **simplicial category of** K, denoted by $\mathrm{scat}(K)$, is the least integer n such that K can be covered by $n+1$ categorical sets in K.

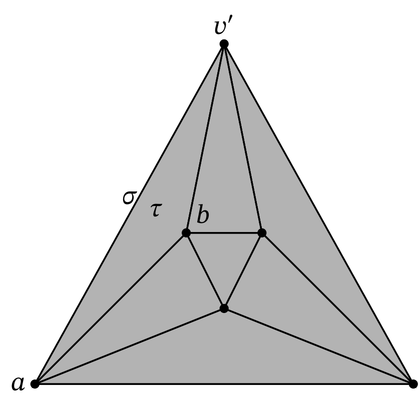
We now show that $\mathrm{scat}(K)$ is just a special case of Definition 10.50,

Proposition 10.53. Let K be a simplicial complex. Then $\mathrm{scat}(K) = \mathrm{scat}(\mathrm{id}_K)$.

Proof. Assume that $\mathrm{scat}(\mathrm{id}_K) = n$. Then K can be covered by subcomplexes $U_0, U_1, \ldots, U_n$ with $\mathrm{id}_K|_{U_j} \sim *$. Since $\mathrm{id}_K|_{U_j} = i_{U_j}$ and $\mathrm{id}_K|_{U_j} \sim *$, $i_{U_j} \sim *$ so that $U_0, U_1, \ldots, U_n$ forms a categorical cover of K. The same reasoning shows that a categorical cover of K yields a categorical cover of id_K. Thus $\mathrm{scat}(\mathrm{id}_K) = \mathrm{scat}(K)$. $\square$

We compute a few examples.

Example 10.54. Let A be the Argentinian complex from Example 10.13. We will cover A with two categorical sets. These will be given by $\{\bar{\tau}, A - \{\sigma, \tau\}\}$ where σ and τ are the free pair below and $\bar{\tau}$ is the subcomplex of A generated by τ (Definition 1.4).



We will explicitly show that the inclusion $i\colon \bar{\tau} \to A$ is null-homotopic. We claim that $i \sim *$ where $*\colon \bar{\tau} \to A$ is given by sending every vertex of $\bar{\tau}$ to v'. We need to check that if $\alpha \in \bar{\tau}$ is a simplex, then $i(\alpha) \cup *(\alpha)$ is a simplex in A. We have

$$\begin{aligned} i(a) \cup *(a) &= av', \\ i(b) \cup *(b) &= bv', \\ i(v') \cup *(v') &= v', \\ i(ab) \cup *(ab) &= abv', \\ i(av') \cup *(av') &= av', \\ i(bv') \cup *(bv') &= bv', \\ i(abv') \cup *(abv') &= abv'. \end{aligned}$$

Checking over all seven simplices, we obtain a simplex of A. Hence $i \sim_c *$ so that $i \sim *$. Another straightforward but more tedious calculation shows that the inclusion of $A - \{\sigma, \tau\}$ into A is null-homotopic. Hence $\mathrm{scat}(A) \leq 1$.

That $\mathrm{scat}(A) > 0$ will follow from Corollary 10.67 below.

Example 10.55. An interesting example is given by a 1-dimensional simplicial complex on n vertices with an edge between every pair of vertices, i.e., the **complete graph on n vertices**, denoted by K_n. A subcomplex U of K_n is categorical if and only if U is a forest (Problem 10.56). Hence if U is categorical in K_n, the maximum number of edges that U can have is $n-1$. This follows from the fact that $v - e = b_0 - b_1$ by Theorem 3.23. Since K_n has a total of $\frac{n(n-1)}{2}$ edges, K_n needs at least $\lceil \frac{n}{2} \rceil$ categorical sets in a cover.

Problem 10.56. Prove that if a subgraph U of a connected graph G is a forest, then U is categorical in G.

The converse of Problem 10.56 is proved in [**61**].

Problem 10.57. Let K be a simplicial complex and ΣK the suspension of K. Prove that $\mathrm{scat}(\Sigma K) \leq 1$.

There are several simple yet key properties of scat.

Proposition 10.58. If $f: K \to L$ and $g: L \to M$, then

$$\mathrm{scat}(g \circ f) \leq \min\{\mathrm{scat}(g), \mathrm{scat}(f)\}.$$

Proof. Suppose that $f: K \to L$ and $g: L \to M$. We will show that $\mathrm{scat}(g \circ f)$ is less than or equal to both $\mathrm{scat}(g)$ and $\mathrm{scat}(f)$. The result then follows. We first show that $\mathrm{scat}(g \circ f) \leq \mathrm{scat}(f)$. Write $\mathrm{scat}(f) = n$ so that there exist $U_0, U_1, \ldots, U_n \subseteq K$ covering K with $f|_{U_j} \sim *$. We claim that $(g \circ f)|_{U_j} \sim *$. Observe that $(g \circ f)|_{U_j} = g \circ (f|_{U_j}) \sim g \circ * \sim *$. Thus, $\mathrm{scat}(g \circ f) \leq \mathrm{scat}(f) = n$.

Now write $\mathrm{scat}(g) = m$. Then there exist $V_0, V_1, \ldots, V_m \subseteq L$ such that $g|_{V_j} \sim *$. Define $U_j := f^{-1}(V_j)$ for all $0 \leq j \leq m$. Then each U_j is a subcomplex of K and hence forms a cover of K. The diagram

$$\begin{tikzcd} U_j \arrow[r, "(g\circ f)|_{U_j}"] \arrow[d, "f|_{U_j}"'] & M \\ V_j \arrow[ur, "g|_{V_j}"'] & \end{tikzcd}$$

commutes up to strong homotopy; that is, $g|_{V_j} \circ f|_{U_j} \sim g \circ f|_{U_j}$. Since $g|_{V_j} \sim *$, we have $(g \circ f)|_{U_j} \sim *$. □

We now show that simplicial category is well-defined up to strong homotopy. First we give a lemma.

Lemma 10.59. Let $f, g: K \to L$ and let $U \subseteq K$ be a subcomplex. If $g|_U \sim *$ and $f \sim g$, then $f|_U \sim *$.

Proof. We may assume without loss of generality that $f \sim_c g$, as the general result follows by induction. By definition of $f \sim_c g$, $f(\sigma) \cup g(\sigma)$ is a simplex in L, so $f|_{U_j}(\sigma) \cup g|_{U_j}(\sigma)$ is also a simplex in L. Thus, $f|_{U_j} \sim_c g|_{U_j} \sim *$ and the result follows. □

Proposition 10.60. Let $f, g: K \to L$. If $f \sim g$, then $\mathrm{scat}(f) = \mathrm{scat}(g)$.

Problem 10.61. Prove Proposition 10.60.

A simple consequence of Propositions 10.58 and 10.60 is the following lemma:

Lemma 10.62. Suppose

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d] & L \\ J \arrow[r, "g"] & M \arrow[u] \end{tikzcd}$$

commutes up to strong homotopy. Then $\mathrm{scat}(f) \leq \mathrm{scat}(g)$.

It then follows that the simplicial category of a map bounds from below the simplicial category of either simplicial complex.

Proposition 10.63. Let $f : K \to L$. Then

$$\mathrm{scat}(f) \leq \min\{\mathrm{scat}(K), \mathrm{scat}(L)\}.$$

Proof. Apply Lemma 10.62 to the diagrams

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d] & L \\ K \arrow[r, "\mathrm{id}_K"] & K \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d] & L \\ L \arrow[r, "\mathrm{id}_L"] & L \arrow[u] \end{tikzcd}$$

□

Exercise 10.64. Give an example to show that the inequality in Proposition 10.63 can be strict.

It now follows that scat remains the same under strong homotopy type.

Corollary 10.65. If $f : K \to L$ is a strong homotopy equivalence, then $\mathrm{scat}(f) = \mathrm{scat}(K) = \mathrm{scat}(L)$.

Problem 10.66. Prove Corollary 10.65.

An immediate corollary of Corollary 10.65 is the following:

Corollary 10.67. Let K be a simplicial complex. Then $\mathrm{scat}(K) = 0$ if and only if K is strongly collapsible.

Hence, scat acts as a kind of quantification to tell us how close or far a simplicial complex is from being strongly collapsible.

10.3.1. Simplicial LS theorem. This short and final subsection is devoted to proving the simplicial Lusternik-Schnirelmann theorem, a theorem relating $\mathrm{scat}(K)$ to the number of critical objects of any strong discrete Morse function on K.

The next lemma follows by using the covers of K and L to cover $K \cup L$.

Lemma 10.68. Let K and L be two simplicial complexes. Then $\mathrm{scat}(K \cup L) \leq \mathrm{scat}(K) + \mathrm{scat}(L) + 1$.

Problem 10.69. Prove Lemma 10.68.

Theorem 10.70. Let $f : K \to \mathbb{R}$ be a strong discrete Morse function. Then

$$\mathrm{scat}(K) + 1 \leq |\,\mathrm{scrit}(f)|.$$

Proof. For any natural number n, define

$$c_n := \min\{a \in \mathbb{R} : \mathrm{scat}(K(a)) \geq n - 1\}.$$

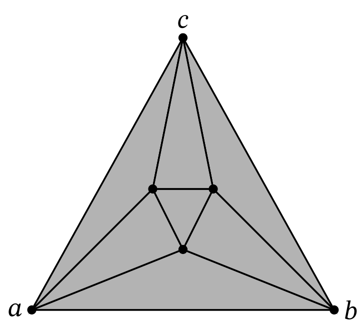
We claim that c_n is a critical value of f. Suppose by contradiction that c_n is a regular value. Since there are only finitely many values of $\mathrm{im}(f)$, there exists a largest value $b \in \mathrm{im}(f)$ such that $b < c_n$. Since there are no critical values in $[b, c_n]$, by Theorem 10.47 we have $K(c_n) \searrow\searrow K(b)$. By Corollary 10.65, $\mathrm{scat}(K(c_n)) = \mathrm{scat}(K(b))$, contradicting the fact that c_n is minimum. Thus c_n must be critical.

It remains to show that the addition of a critical object increases scat by at most 1. That is, we need to show that if σ is a critical simplex, then

$\mathrm{scat}(K(a)\cup\{\sigma\}) \leq \mathrm{scat}(K(a))+1$. This follows from Lemma 10.68 since $\mathrm{scat}(\bar{\sigma}) = 0$ where $\bar{\sigma}$ is the smallest simplicial complex containing σ. The same argument shows the result when attaching a critical pair. □

Example 10.71. We show that the bound in Theorem 10.70 is sharp. Let A be the Argentinian complex along with the discrete Morse function from Exercise 10.42. This satisfies $|\mathrm{scrit}(f)| = 2$. Since A has no dominating vertex, $\mathrm{scat}(A) > 0$; hence $\mathrm{scat}(A)+1 = |\mathrm{scrit}(f)| = 2$.

Now we will show that the inequality can be strict. Consider the labeling of the vertices of A below.



Let $A' := A \cup \{abc\}$, gluing a 2-simplex abc to A. It is easy to show that $b_2(A') = 1$, so by the discrete Morse inequalities (Theorem 4.1), it follows that every discrete Morse function f defined on A' (and in particular every strong discrete Morse function) has at least two critical simplices: a critical vertex and a critical 2-simplex. Clearly A' does not contain any dominated vertex. Moreover, the removal of any critical 2-simplex does not result in any dominating vertices, so at least one critical pair arises in order to collapse to a subcomplex containing dominated vertices. Hence, any strong discrete Morse function $f : A' \to \mathbb{R}$ must have at least three critical objects. Furthermore, we can cover A' with two categorical sets so that $\mathrm{scat}(A') = 1$. Hence $\mathrm{scat}(A')+1 = 2 < 3 \leq |\mathrm{scrit}(f)|$.

Problem 10.72. Find a categorical cover of A' of size 2.

Problem 10.73. Verify that $b_2(A') = 1$.

Bibliography

[1] S. E. Aaronson, M. E. Meyer, N. A. Scoville, M. T. Smith, and L. M. Stibich, *Graph isomorphisms in discrete Morse theory*, AKCE Int. J. Graphs Comb. **11** (2014), no. 2, 163–176. MR3243115

[2] S. Aaronson and N. A. Scoville, *Lusternik-Schnirelmann category for simplicial complexes*, Illinois J. Math. **57** (2013), no. 3, 743–753. MR3275736

[3] K. Adiprasito and B. Benedetti, *Tight complexes in 3-space admit perfect discrete Morse functions*, European J. Combin. **45** (2015), 71–84, DOI 10.1016/j.ejc.2014.10.002. MR3286622

[4] K. A. Adiprasito, B. Benedetti, and F. H. Lutz, *Extremal examples of collapsible complexes and random discrete Morse theory*, Discrete Comput. Geom. **57** (2017), no. 4, 824–853, DOI 10.1007/s00454-017-9860-4. MR3639606

[5] M. Agiorgousis, B. Green, A. Onderdonk, N. A. Scoville, and K. Rich, *Homological sequences in discrete Morse theory*, Topology Proc. **54** (2019), 283–294.

[6] R. Ayala, L. M. Fernández, D. Fernández-Ternero, and J. A. Vilches, *Discrete Morse theory on graphs*, Topology Appl. **156** (2009), no. 18, 3091–3100, DOI 10.1016/j.topol.2009.01.022. MR2556069

[7] R. Ayala, L. M. Fernández, A. Quintero, and J. A. Vilches, *A note on the pure Morse complex of a graph*, Topology Appl. **155** (2008), no. 17-18, 2084–2089, DOI 10.1016/j.topol.2007.04.023. MR2457993

[8] R. Ayala, L. M. Fernández, and J. A. Vilches, *Critical elements of proper discrete Morse functions*, Math. Pannon. **19** (2008), no. 2, 171–185. MR2553730

[9] R. Ayala, L. M. Fernández, and J. A. Vilches, *Characterizing equivalent discrete Morse functions*, Bull. Braz. Math. Soc. (N.S.) **40** (2009), no. 2, 225–235, DOI 10.1007/s00574-009-0010-3. MR2511547

[10] R. Ayala, L. M. Fernández, and J. A. Vilches, *Discrete Morse inequalities on infinite graphs*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 38, 11 pp. MR2491640

[11] R. Ayala, D. Fernández-Ternero, and J. A. Vilches, *Counting excellent discrete Morse functions on compact orientable surfaces*, Image-A: Applicable Mathematics in Image Engineering **1** (2010), no. 1, 49–56.

[12] R. Ayala, D. Fernández-Ternero, and J. A. Vilches, *The number of excellent discrete Morse functions on graphs*, Discrete Appl. Math. **159** (2011), no. 16, 1676–1688, DOI 10.1016/j.dam.2010.12.011. MR2825610

[13] R. Ayala, D. Fernández-Ternero, and J. A. Vilches, *Perfect discrete Morse functions on* 2*-complexes*, Pattern Recognition Letters **33** (2012), no. 11, pp. 1495–1500.

[14] R. Ayala, D. Fernández-Ternero, and J. A. Vilches, *Perfect discrete Morse functions on triangulated 3-manifolds*, Computational topology in image context, Lecture Notes in Comput. Sci., vol. 7309, Springer, Heidelberg, 2012, pp. 11–19, DOI 10.1007/978-3-642-30238-1_2. MR2983410

[15] R. Ayala, J. A. Vilches, G. Jerše, and N. M. Kosta, *Discrete gradient fields on infinite complexes*, Discrete Contin. Dyn. Syst. **30** (2011), no. 3, 623–639, DOI 10.3934/dcds.2011.30.623. MR2784612

[16] E. Babson and P. Hersh, *Discrete Morse functions from lexicographic orders*, Trans. Amer. Math. Soc. **357** (2005), no. 2, 509–534, DOI 10.1090/S0002-9947-04-03495-6. MR2095621

[17] T. Banchoff, *Critical points and curvature for embedded polyhedra*, J. Differential Geometry **1** (1967), 245–256. MR0225327

[18] T. F. Banchoff, *Critical points and curvature for embedded polyhedral surfaces*, Amer. Math. Monthly **77** (1970), 475–485, DOI 10.2307/2317380. MR0259812

[19] T. F. Banchoff, *Critical points and curvature for embedded polyhedra. II*, Differential geometry (College Park, Md., 1981/1982), Progr. Math., vol. 32, Birkhäuser Boston, Boston, MA, 1983, pp. 34–55. MR702526

[20] J. Bang-Jensen and G. Gutin, *Digraphs. Theory, algorithms and applications*, 2nd ed., Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2009. MR2472389

[21] J. A. Barmak, *Algebraic topology of finite topological spaces and applications*, Lecture Notes in Mathematics, vol. 2032, Springer, Heidelberg, 2011. MR3024764

[22] J. A. Barmak and E. G. Minian, *Simple homotopy types and finite spaces*, Adv. Math. **218** (2008), no. 1, 87–104, DOI 10.1016/j.aim.2007.11.019. MR2409409

[23] J. A. Barmak and E. G. Minian, *Strong homotopy types, nerves and collapses*, Discrete Comput. Geom. **47** (2012), no. 2, 301–328, DOI 10.1007/s00454-011-9357-5. MR2872540

[24] U. Bauer, *Persistence in discrete Morse theory*, Ph.D. thesis, Georg-August-Universität Göttigen, 2011.

[25] U. Bauer and H. Edelsbrunner, *The Morse theory of Čech and Delaunay filtrations*, Computational geometry (SoCG'14), ACM, New York, 2014, pp. 484–490. MR3382330

[26] U. Bauer and H. Edelsbrunner, *The Morse theory of Čech and Delaunay complexes*, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3741–3762, DOI 10.1090/tran/6991. MR3605986

[27] B. Benedetti, *Discrete Morse Theory is at least as perfect as Morse theory*, arXiv e-prints (2010), arXiv:1010.0548.

[28] B. Benedetti, *Discrete Morse theory for manifolds with boundary*, Trans. Amer. Math. Soc. **364** (2012), no. 12, 6631–6670, DOI 10.1090/S0002-9947-2012-05614-5. MR2958950

[29] B. Benedetti, *Smoothing discrete Morse theory*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **16** (2016), no. 2, 335–368. MR3559605

[30] B. Benedetti and F. Lutz, *Library of triangulations*, Available at `http://page.math.tu-berlin.de/~lutz/stellar/library_of_triangulations/`.

[31] B. Benedetti and F. H. Lutz, *The dunce hat and a minimal non-extendably collapsible* 3*-ball*, Electronic Geometry Models **16** (2013), Model 2013.10.001.

[32] B. Benedetti and F. H. Lutz, *Knots in collapsible and non-collapsible balls*, Electron. J. Combin. **20** (2013), no. 3, Paper 31, 29 pp. MR3104529

[33] B. Benedetti and F. H. Lutz, *Random discrete Morse theory and a new library of triangulations*, Exp. Math. **23** (2014), no. 1, 66–94, DOI 10.1080/10586458.2013.865281. MR3177457

[34] M. Bestvina, *PL Morse theory*, Math. Commun. **13** (2008), no. 2, 149–162. MR2488666

[35] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470, DOI 10.1007/s002220050168. MR1465330

[36] E. D. Bloch, *Polyhedral representation of discrete Morse functions*, Discrete Math. **313** (2013), no. 12, 1342–1348, DOI 10.1016/j.disc.2013.02.020. MR3061119

[37] R. Bott, *Morse theory indomitable*, Inst. Hautes Études Sci. Publ. Math. **68** (1988), 99–114 (1989). MR1001450

[38] B. Brost, *Computing persistent homology via discrete Morse theory*, Master's thesis, University of Copenhagen, December 2013.

[39] N. A. Capitelli and E. G. Minian, *A simplicial complex is uniquely determined by its set of discrete Morse functions*, Discrete Comput. Geom. **58** (2017), no. 1, 144–157, DOI 10.1007/s00454-017-9865-z. MR3658332

[40] E. Chambers, E. Gasparovic, and K. Leonard, *Medial fragments for segmentation of articulating objects in images*, Research in shape analysis, Assoc. Women Math. Ser., vol. 12, Springer, Cham, 2018, pp. 1–15. MR3859054

[41] M. K. Chari, *On discrete Morse functions and combinatorial decompositions* (English, with English and French summaries), Discrete Math. **217** (2000), no. 1-3, 101–113, DOI 10.1016/S0012-365X(99)00258-7. Formal power series and algebraic combinatorics (Vienna, 1997). MR1766262

[42] M. K. Chari and M. Joswig, *Complexes of discrete Morse functions*, Discrete Math. **302** (2005), no. 1-3, 39–51, DOI 10.1016/j.disc.2004.07.027. MR2179635

[43] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, 6th ed., Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2016. MR3445306

[44] Y.-M. Chung and S. Day, *Topological fidelity and image thresholding: a persistent homology approach*, J. Math. Imaging Vision **60** (2018), no. 7, 1167–1179, DOI 10.1007/s10851-018-0802-4. MR3832139

[45] M. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics, vol. 10, Springer-Verlag, New York-Berlin, 1973. MR0362320

[46] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to algorithms*, 3rd ed., MIT Press, Cambridge, MA, 2009. MR2572804

[47] O. Cornea, G. Lupton, J. Oprea, and D. Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, vol. 103, American Mathematical Society, Providence, RI, 2003. MR1990857

[48] J. Curry, R. Ghrist, and V. Nanda, *Discrete Morse theory for computing cellular sheaf cohomology*, Found. Comput. Math. **16** (2016), no. 4, 875–897, DOI 10.1007/s10208-015-9266-8. MR3529128

[49] T. K. Dey, J. Wang, and Y. Wang, *Graph reconstruction by discrete Morse theory*, arXiv e-prints (2018), arXiv:1803.05093.

[50] M. de Longueville, *A course in topological combinatorics*, Universitext, Springer, New York, 2013. MR2976494

[51] V. de Silva and G. Ghrist, *Coordinate-free coverage in sensor networks with controlled boundaries*, Intl. J. Robotics Research **25** (2006), no. 12, 1205–1222.

[52] P. Dłotko and H. Wagner, *Computing homology and persistent homology using iterated Morse decomposition*, arXiv e-prints (2012), arXiv:1210.1429.

[53] P. Dłotko and H. Wagner, *Simplification of complexes for persistent homology computations*, Homology Homotopy Appl. **16** (2014), no. 1, 49–63, DOI 10.4310/HHA.2014.v16.n1.a3. MR3171260

[54] H. Edelsbrunner, *A short course in computational geometry and topology*, SpringerBriefs in Applied Sciences and Technology, Springer, Cham, 2014. MR3328629

[55] H. Edelsbrunner and J. L. Harer, *Computational topology. An introduction*, American Mathematical Society, Providence, RI, 2010. MR2572029

[56] H. Edelsbrunner, D. Letscher, and A. Zomorodian, *Topological persistence and simplification*, Discrete Comput. Geom. **28** (2002), no. 4, 511–533, DOI 10.1007/s00454-002-2885-2. Discrete and computational geometry and graph drawing (Columbia, SC, 2001). MR1949898

[57] H. Edelsbrunner, A. Nikitenko, and M. Reitzner, *Expected sizes of Poisson-Delaunay mosaics and their discrete Morse functions*, Adv. in Appl. Probab. **49** (2017), no. 3, 745–767, DOI 10.1017/apr.2017.20. MR3694316

[58] S. P. Ellis and A. Klein, *Describing high-order statistical dependence using "concurrence topology," with application to functional MRI brain data*, Homology Homotopy Appl. **16** (2014), no. 1, 245–264, DOI 10.4310/HHA.2014.v16.n1.a14. MR3211745

[59] A. Engström, *Discrete Morse functions from Fourier transforms*, Experiment. Math. **18** (2009), no. 1, 45–53. MR2548985

[60] D. Farley and L. Sabalka, *Discrete Morse theory and graph braid groups*, Algebr. Geom. Topol. **5** (2005), 1075–1109, DOI 10.2140/agt.2005.5.1075. MR2171804

[61] D. Fernández-Ternero, E. Macías-Virgós, E. Minuz, and J. A. Vilches, *Simplicial Lusternik-Schnirelmann category*, Publ. Mat. **63** (2019), no. 1, 265–293, DOI 10.5565/PUBLMAT6311909. MR3908794

[62] D. Fernández-Ternero, E. Macías-Virgós, N. A. Scoville, and J. A. Vilches, *Strong discrete Morse theory and simplicial L-S category: A discrete version of the Lusternik-Schnirelmann Theorem*, Discrete Comput. Geom. (2019), DOI 10.1007/s00454-019-00116-8.

[63] D. Fernández-Ternero, E. Macías-Virgós, and J. A. Vilches, *Lusternik-Schnirelmann category of simplicial complexes and finite spaces*, Topology Appl. **194** (2015), 37–50, DOI 10.1016/j.topol.2015.08.001. MR3404603

[64] D. L. Ferrario and R. A. Piccinini, *Simplicial structures in topology*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011. Translated from the 2009 Italian original by Maria Nair Piccinini. MR2663748

[65] R. Forman, *Morse theory for cell complexes*, Adv. Math. **134** (1998), no. 1, 90–145, DOI 10.1006/aima.1997.1650. MR1612391

[66] R. Forman, *Combinatorial differential topology and geometry*, New perspectives in algebraic combinatorics (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ., vol. 38, Cambridge Univ. Press, Cambridge, 1999, pp. 177–206. MR1731817

[67] R. Forman, *Morse theory and evasiveness*, Combinatorica **20** (2000), no. 4, 489–504, DOI 10.1007/s004930070003. MR1804822

[68] R. Forman, *Combinatorial Novikov-Morse theory*, Internat. J. Math. **13** (2002), no. 4, 333–368, DOI 10.1142/S0129167X02001265. MR1911862

[69] R. Forman, *Discrete Morse theory and the cohomology ring*, Trans. Amer. Math. Soc. **354** (2002), no. 12, 5063–5085, DOI 10.1090/S0002-9947-02-03041-6. MR1926850

[70] R. Forman, *A user's guide to discrete Morse theory*, Sém. Lothar. Combin. **48** (2002), Art. B48c. MR1939695

[71] R. Forman, *Some applications of combinatorial differential topology*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 281–313, DOI 10.1090/pspum/073/2131018. MR2131018

[72] R. Freij, *Equivariant discrete Morse theory*, Discrete Math. **309** (2009), no. 12, 3821–3829, DOI 10.1016/j.disc.2008.10.029. MR2537376

[73] P. Frosini, *A distance for similarity classes of submanifolds of a Euclidean space*, Bull. Austral. Math. Soc. **42** (1990), no. 3, 407–416, DOI 10.1017/S0004972700028574. MR1083277

[74] U. Fugacci, F. Iuricich, and L. De Floriani, *Computing discrete Morse complexes from simplicial complexes*, Graph. Models **103** (2019), 1–14, DOI 10.1016/j.gmod.2019.101023. MR3936806

[75] E. Gawrilow and M. Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR1785292

[76] R. W. Ghrist, *Elementary applied topology*, edition 1.0, ISBN 978-1502880857, Createspace, 2014.

[77] L. C. Glaser, *Geometrical combinatorial topology. Vol. I*, Van Nostrand Reinhold Mathematics Studies, vol. 27, Van Nostrand Reinhold Co., New York, 1970. MR3309564

[78] L. C. Glaser, *Geometrical combinatorial topology. Vol. II*, Van Nostrand Reinhold Mathematics Studies, vol. 28, Van Nostrand Reinhold Co., London, 1972. MR3308955

[79] B. Green, N. A. Scoville, and M. Tsuruga, *Estimating the discrete geometric Lusternik-Schnirelmann category*, Topol. Methods Nonlinear Anal. **45** (2015), no. 1, 103–116, DOI 10.12775/TMNA.2015.006. MR3365007

[80] D. Günther, J. Reininghaus, H. Wagner, and I. Hotz, *Efficient computation of 3D Morse–Smale complexes and persistent homology using discrete Morse theory*, The Visual Computer **28** (2012), no. 10, 959–969.

[81] M. Hachimori, *Simplicial complex library*, Available at `http://infoshako.sk.tsukuba.ac.jp/~hachi/math/library/index_eng.html`.

[82] S. Harker, K. Mischaikow, M. Mrozek, and V. Nanda, *Discrete Morse theoretic algorithms for computing homology of complexes and maps*, Found. Comput. Math. **14** (2014), no. 1, 151–184, DOI 10.1007/s10208-013-9145-0. MR3160710

[83] S. Harker, K. Mischaikow, M. Mrozek, V. Nanda, H. Wagner, M. Juda, and P. Dłotko, *The efficiency of a homology algorithm based on discrete Morse theory and coreductions*, Proceedings of the 3rd International Workshop on Computational Topology in Image Context (CTIC 2010), vol. 1, 2010, pp. 41–47.

[84] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354

[85] M. Henle, *A combinatorial introduction to topology*, Dover Publications, Inc., New York, 1994. Corrected reprint of the 1979 original [Freeman, San Francisco, CA; MR0550879 (81g:55001)]. MR1280460

[86] G. Henselman and R. Ghrist, *Matroid filtrations and computational persistent homology*, arXiv e-prints (2016), arXiv:1606.00199.

[87] M. Jöllenbeck and V. Welker, *Minimal resolutions via algebraic discrete Morse theory*, Mem. Amer. Math. Soc. **197** (2009), no. 923, vi+74 pp, DOI 10.1090/memo/0923. MR2488864

[88] J. Jonsson, *Simplicial complexes of graphs*, Lecture Notes in Mathematics, vol. 1928, Springer-Verlag, Berlin, 2008. MR2368284

[89] J. Jonsson, *Introduction to simplicial homology*, 2011, Available at `https://people.kth.se/~jakobj/homology.html`.

[90] M. Joswig, F. H. Lutz, and M. Tsuruga, *Heuristic for sphere recognition*, Mathematical software—ICMS 2014, Lecture Notes in Comput. Sci., vol. 8592, Springer, Heidelberg, 2014, pp. 152–159, DOI 10.1007/978-3-662-44199-2_26. MR3334760

[91] M. Joswig and M. E. Pfetsch, *Computing optimal Morse matchings*, SIAM J. Discrete Math. **20** (2006), no. 1, 11–25, DOI 10.1137/S0895480104445885. MR2257241

[92] S. Jukna, *Boolean function complexity*, Algorithms and Combinatorics, vol. 27, Springer, Heidelberg, 2012. Advances and frontiers. MR2895965

[93] T. Kaczynski, K. Mischaikow, and M. Mrozek, *Computational homology*, Applied Mathematical Sciences, vol. 157, Springer-Verlag, New York, 2004. MR2028588

[94] A. B. Kahn, *Topological sorting of large networks*, Communications of the ACM **5** (1962), no. 11, 558–562.

[95] J. Kahn, M. Saks, and D. Sturtevant, *A topological approach to evasiveness*, Combinatorica **4** (1984), no. 4, 297–306, DOI 10.1007/BF02579140. MR779890

[96] V. Kaibel and M. E. Pfetsch, *Computing the face lattice of a polytope from its vertex-facet incidences*, Comput. Geom. **23** (2002), no. 3, 281–290, DOI 10.1016/S0925-7721(02)00103-7. MR1927137

[97] H. King, K. Knudson, and N. Mramor, *Generating discrete Morse functions from point data*, Experiment. Math. **14** (2005), no. 4, 435–444. MR2193806

[98] H. King, K. Knudson, and N. Mramor Kosta, *Birth and death in discrete Morse theory*, J. Symbolic Comput. **78** (2017), 41–60, DOI 10.1016/j.jsc.2016.03.007. MR3535328

[99] K. P. Knudson, *Morse theory: Smooth and discrete*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. MR3379451

[100] K. Knudson and B. Wang, *Discrete stratified Morse theory: a user's guide*, 34th International Symposium on Computational Geometry, LIPIcs. Leibniz Int. Proc. Inform., vol. 99, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, Art. No. 54, 14 pp. MR3824298

[101] D. N. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A **88** (1999), no. 1, 112–122, DOI 10.1006/jcta.1999.2984. MR1713484

[102] D. N. Kozlov, *Discrete Morse theory for free chain complexes* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **340** (2005), no. 12, 867–872, DOI 10.1016/j.crma.2005.04.036. MR2151775

[103] D. Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, Berlin, 2008. MR2361455

[104] M. Krčál, J. Matoušek, and F. Sergeraert, *Polynomial-time homology for simplicial Eilenberg-MacLane spaces*, Found. Comput. Math. **13** (2013), no. 6, 935–963, DOI 10.1007/s10208-013-9159-7. MR3124946

[105] M. Kukieła, *The main theorem of discrete Morse theory for Morse matchings with finitely many rays*, Topology Appl. **160** (2013), no. 9, 1074–1082, DOI 10.1016/j.topol.2013.04.025. MR3049255

[106] F. Larrión, M. A. Pizaña, and R. Villarroel-Flores, *Discrete Morse theory and the homotopy type of clique graphs*, Ann. Comb. **17** (2013), no. 4, 743–754, DOI 10.1007/s00026-013-0204-7. MR3129782

[107] D. Lay, *Linear algebra and its applications*, 4th ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2012.

[108] I.-C. Lazăr, *Applications to discrete Morse theory: the collapsibility of CAT(0) cubical complexes of dimension 2 and 3*, Carpathian J. Math. **27** (2011), no. 2, 225–237. MR2906606

[109] T. Lewiner, H. Lopes, and G. Tavares, *Optimal discrete Morse functions for 2-manifolds*, Comput. Geom. **26** (2003), no. 3, 221–233, DOI 10.1016/S0925-7721(03)00014-2. MR2005300

[110] M. Lin and N. A. Scoville, *On the automorphism group of the Morse complex*, arXiv e-prints (2019), arXiv:1904.10907.

[111] N. Lingareddy, *Calculating persistent homology using discrete Morse theory*, 2018, Available at `http://math.uchicago.edu/~may/REU2018/REUPapers/Lingareddy.pdf`.

[112] Y. Liu and N. A. Scoville, *The realization problem for discrete Morse functions on trees*, Algebra Colloquium, to appear.

[113] A. T. Lundell and S. Weingram, *The topology of CW complexes*, The University Series in Higher Mathematics, Van Nostrand Reinhold Co., New York, 1969. MR3822092

[114] L. Lusternik and L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Hermann, Paris, 1934.

[115] H. Markram et al., *Reconstruction and simulation of neocortical microcircuitry*, Cell **163** (2015), no. 2, 456–492, DOI 10.1016/j.cell.2015.09.029.

[116] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. MR0163331

[117] J. Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press, Princeton, N.J., 1965. MR0190942

[118] E. G. Minian, *Some remarks on Morse theory for posets, homological Morse theory and finite manifolds*, Topology Appl. **159** (2012), no. 12, 2860–2869, DOI 10.1016/j.topol.2012.05.027. MR2942659

[119] K. Mischaikow and V. Nanda, *Morse theory for filtrations and efficient computation of persistent homology*, Discrete Comput. Geom. **50** (2013), no. 2, 330–353, DOI 10.1007/s00454-013-9529-6. MR3090522

[120] J. W. Moon, *Counting labelled trees*, From lectures delivered to the Twelfth Biennial Seminar of the Canadian Mathematical Congress (Vancouver, 1969), Canadian Mathematical Congress, Montreal, Que., 1970. MR0274333

[121] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. MR2334563

[122] F. Mori and M. Salvetti, (*Discrete*) *Morse theory on configuration spaces*, Math. Res. Lett. **18** (2011), no. 1, 39–57, DOI 10.4310/MRL.2011.v18.n1.a4. MR2770581

[123] F. Mori and M. Salvetti, *Discrete topological methods for subspace arrangements*, Arrangements of hyperplanes—Sapporo 2009, Adv. Stud. Pure Math., vol. 62, Math. Soc. Japan, Tokyo, 2012, pp. 293–321.

[124] M. Morse, *Relations between the critical points of a real function of n independent variables*, Trans. Amer. Math. Soc. **27** (1925), no. 3, 345–396, DOI 10.2307/1989110. MR1501318

[125] M. Morse, *Functional topology and abstract variational theory*, Ann. of Math. (2) **38** (1937), no. 2, 386–449, DOI 10.2307/1968559. MR1503341

[126] M. Morse, *The calculus of variations in the large*, American Mathematical Society Colloquium Publications, vol. 18, American Mathematical Society, Providence, RI, 1996. Reprint of the 1932 original. MR1451874

[127] M. Mrozek and B. Batko, *Coreduction homology algorithm*, Discrete Comput. Geom. **41** (2009), no. 1, 96–118, DOI 10.1007/s00454-008-9073-y. MR2470072

[128] V. Nanda, D. Tamaki, and K. Tanaka, *Discrete Morse theory and classifying spaces*, Adv. Math. **340** (2018), 723–790, DOI 10.1016/j.aim.2018.10.016. MR3886179

[129] L. Nicolaescu, *An invitation to Morse theory*, 2nd ed., Universitext, Springer, New York, 2011. MR2883440

[130] A. Nikitenko, *Discrete Morse theory for random complexes*, Ph.D. thesis, Institute of Science and Technology Austria, 2017.

[131] R. O'Donnell, *Analysis of Boolean functions*, Cambridge University Press, New York, 2014. MR3443800

[132] P. Orlik and V. Welker, *Algebraic combinatorics*, Universitext, Springer, Berlin, 2007. Lectures from the Summer School held in Nordfjordeid, June 2003. MR2322081

[133] S. Y. Oudot, *Persistence theory: from quiver representations to data analysis*, Mathematical Surveys and Monographs, vol. 209, American Mathematical Society, Providence, RI, 2015. MR3408277

[134] V. V. Prasolov, *Elements of homology theory*, Graduate Studies in Mathematics, vol. 81, American Mathematical Society, Providence, RI, 2007. Translated from the 2005 Russian original by Olga Sipacheva. MR2313004

[135] M. Reimann et al., *Cliques of neurons bound into cavities provide a missing link between structure and function*, Front. Comput. Neurosci. (2017), 12 June, DOI 10.3389/fncom.2017.00048.

[136] R. Reina-Molina, D. Díaz-Pernil, P. Real, and A. Berciano, *Membrane parallelism for discrete Morse theory applied to digital images*, Appl. Algebra Engrg. Comm. Comput. **26** (2015), no. 1-2, 49–71, DOI 10.1007/s00200-014-0246-z. MR3320905

[137] J. J. Rotman, *An introduction to algebraic topology*, Graduate Texts in Mathematics, vol. 119, Springer-Verlag, New York, 1988. MR957919

[138] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer Study Edition, Springer-Verlag, Berlin-New York, 1982. Reprint. MR665919

[139] A. Sawicki, *Discrete Morse functions for graph configuration spaces*, J. Phys. A **45** (2012), no. 50, 505202, 25, DOI 10.1088/1751-8113/45/50/505202. MR2999710

[140] N. A. Scoville and W. Swei, *On the Lusternik-Schnirelmann category of a simplicial map*, Topology Appl. **216** (2017), 116–128, DOI 10.1016/j.topol.2016.11.015. MR3584127

[141] N. A. Scoville and K. Yegnesh, *A persistent homological analysis of network data flow malfunctions*, Journal of Complex Networks **5** (2017), no. 6, 884–892.

[142] E. Sköldberg, *Morse theory from an algebraic viewpoint*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 115–129, DOI 10.1090/S0002-9947-05-04079-1. MR2171225

[143] S. Smale, *The generalized Poincaré conjecture in higher dimensions*, Bull. Amer. Math. Soc. **66** (1960), 373–375, DOI 10.1090/S0002-9904-1960-10458-2. MR0124912

[144] S. Smale, *Marston Morse (1892–1977)*, Math. Intelligencer **1** (1978/79), no. 1, 33–34, DOI 10.1007/BF03023042. MR0490936

[145] R. E. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc. **123** (1966), 325–340, DOI 10.2307/1994660. MR0195042

[146] K. Tanaka, *Lusternik-Schnirelmann category for cell complexes and posets*, Illinois J. Math. **59** (2015), no. 3, 623–636. MR3554225

[147] K. Tanaka, *Lusternik-Schnirelmann category for categories and classifying spaces*, Topology Appl. **239** (2018), 65–80, DOI 10.1016/j.topol.2018.02.031. MR3777323

[148] H. Vogt, *Leçons sur la résolution algébrique des équations*, Nony, Paris, 1895.

[149] P. Škraba, *Persistent homology and machine learning* (English, with English and Slovenian summaries), Informatica (Ljubl.) **42** (2018), no. 2, 253–258. MR3835054

[150] D. Weiller, *Smooth and discrete Morse theory*, Bachelor's thesis, Australian National University, 2015.

[151] J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math. Soc. (2) **45** (1939), no. 4, 243–327, DOI 10.1112/plms/s2-45.1.243. MR1576810

[152] J. H. C. Whitehead, *Simple homotopy types*, Amer. J. Math. **72** (1950), 1–57, DOI 10.2307/2372133. MR0035437

[153] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983). MR683171

[154] C.-K. Wu and D. Feng, *Boolean functions and their applications in cryptography*, Advances in Computer Science and Technology, Springer, Heidelberg, 2016. MR3559545

[155] M. C. B. Zaremsky, *Bestvina-Brady discrete Morse theory and Vietoris-Rips complexes*, arXiv e-prints (2018), arXiv:1812.10976.

[156] R. E. Zax, *Simplifying complicated simplicial complexes: discrete Morse theory and its applications*, Bachelor's thesis, Harvard University, 2012.

[157] E. C. Zeeman, *On the dunce hat*, Topology **2** (1964), 341–358, DOI 10.1016/0040-9383(63)90014-4. MR0156351

[158] A. Zorn, *Discrete Morse theory on simplicial complexes*, 2009, Available at `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Zorn.pdf`.

Notation and symbol index

Index

Photo credit: Jennifer Scoville

Discrete Morse theory is a powerful tool combining ideas in both topology and combinatorics. Invented by Robin Forman in the mid 1990s, discrete Morse theory is a combinatorial analogue of Marston Morse's classical Morse theory. Its applications are vast, including applications to topological data analysis, combinatorics, and computer science.

This book, the first one devoted solely to discrete Morse theory, serves as an introduction to the subject. Since the book restricts the study of discrete Morse theory to abstract simplicial complexes, a course in mathematical proof writing is the only prerequisite needed. Topics covered include simplicial complexes, simple homotopy, collapsibility, gradient vector fields, Hasse diagrams, simplicial homology, persistent homology, discrete Morse inequalities, the Morse complex, discrete Morse homology, and strong discrete Morse functions. Students of computer science will also find the book beneficial as it includes topics such as Boolean functions, evasiveness, and has a chapter devoted to some computational aspects of discrete Morse theory. The book is appropriate for a course in discrete Morse theory, a supplemental text to a course in algebraic topology or topological combinatorics, or an independent study.

ISBN 978-1-4704-5298-8

9 781470 452988

STML/90

For additional information and updates on this book, visit
www.ams.org/bookpages/stml-90

AMS
www.ams.org