

VIP Refresher: Linear Algebra and Calculus

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October 6, 2018

General notations

□ **Vector** – We note $x \in \mathbb{R}^n$ a vector with n entries, where $x_i \in \mathbb{R}$ is the i^{th} entry:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

□ **Matrix** – We note $A \in \mathbb{R}^{m \times n}$ a matrix with m rows and n columns, where $A_{i,j} \in \mathbb{R}$ is the entry located in the i^{th} row and j^{th} column:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Remark: the vector x defined above can be viewed as a $n \times 1$ matrix and is more particularly called a column-vector.

□ **Identity matrix** – The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix with ones in its diagonal and zero everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Remark: for all matrices $A \in \mathbb{R}^{n \times n}$, we have $A \times I = I \times A = A$.

□ **Diagonal matrix** – A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is a square matrix with nonzero values in its diagonal and zero everywhere else:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$$

Remark: we also note D as $\text{diag}(d_1, \dots, d_n)$.

Matrix operations

□ **Vector-vector multiplication** – There are two types of vector-vector products:

- inner product: for $x, y \in \mathbb{R}^n$, we have:

$$x^T y = \sum_{i=1}^n x_i y_i \in \mathbb{R}$$

- outer product: for $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, we have:

$$xy^T = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{pmatrix} \in \mathbb{R}^{m \times n}$$

□ **Matrix-vector multiplication** – The product of matrix $A \in \mathbb{R}^{m \times n}$ and vector $x \in \mathbb{R}^n$ is a vector of size \mathbb{R}^m , such that:

$$Ax = \begin{pmatrix} a_{r,1}^T x \\ \vdots \\ a_{r,m}^T x \end{pmatrix} = \sum_{i=1}^n a_{c,i} x_i \in \mathbb{R}^m$$

where $a_{r,i}^T$ are the vector rows and $a_{c,j}$ are the vector columns of A , and x_i are the entries of x .

□ **Matrix-matrix multiplication** – The product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ is a matrix of size $\mathbb{R}^{m \times p}$, such that:

$$AB = \begin{pmatrix} a_{r,1}^T b_{c,1} & \cdots & a_{r,1}^T b_{c,p} \\ \vdots & & \vdots \\ a_{r,m}^T b_{c,1} & \cdots & a_{r,m}^T b_{c,p} \end{pmatrix} = \sum_{i=1}^n a_{c,i} b_{r,i}^T \in \mathbb{R}^{m \times p}$$

where $a_{r,i}^T, b_{r,i}^T$ are the vector rows and $a_{c,j}, b_{c,j}$ are the vector columns of A and B respectively.

□ **Transpose** – The transpose of a matrix $A \in \mathbb{R}^{m \times n}$, noted A^T , is such that its entries are flipped:

$$\forall i, j, \quad A_{i,j}^T = A_{j,i}$$

Remark: for matrices A, B , we have $(AB)^T = B^T A^T$.

□ **Inverse** – The inverse of an invertible square matrix A is noted A^{-1} and is the only matrix such that:

$$AA^{-1} = A^{-1}A = I$$

Remark: not all square matrices are invertible. Also, for matrices A, B , we have $(AB)^{-1} = B^{-1}A^{-1}$

□ **Trace** – The trace of a square matrix A , noted $\text{tr}(A)$, is the sum of its diagonal entries:

$$\text{tr}(A) = \sum_{i=1}^n A_{i,i}$$

Remark: for matrices A, B , we have $\text{tr}(A^T) = \text{tr}(A)$ and $\text{tr}(AB) = \text{tr}(BA)$

□ **Determinant** – The determinant of a square matrix $A \in \mathbb{R}^{n \times n}$, noted $|A|$ or $\det(A)$ is expressed recursively in terms of $A_{\setminus i, \setminus j}$, which is the matrix A without its i^{th} row and j^{th} column, as follows:

$$\det(A) = |A| = \sum_{j=1}^n (-1)^{i+j} A_{i,j} |A_{\setminus i, \setminus j}|$$

Remark: A is invertible if and only if $|A| \neq 0$. Also, $|AB| = |A||B|$ and $|A^T| = |A|$.

Matrix properties

□ **Symmetric decomposition** – A given matrix A can be expressed in terms of its symmetric and antisymmetric parts as follows:

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{Symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{Antisymmetric}}$$

□ **Norm** – A norm is a function $N : V \rightarrow [0, +\infty[$ where V is a vector space, and such that for all $x, y \in V$, we have:

- $N(x + y) \leq N(x) + N(y)$
- $N(ax) = |a|N(x)$ for a scalar
- if $N(x) = 0$, then $x = 0$

For $x \in V$, the most commonly used norms are summed up in the table below:

Norm	Notation	Definition	Use case
Manhattan, L^1	$\ x\ _1$	$\sum_{i=1}^n x_i $	LASSO regularization
Euclidean, L^2	$\ x\ _2$	$\sqrt{\sum_{i=1}^n x_i^2}$	Ridge regularization
p -norm, L^p	$\ x\ _p$	$\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$	Hölder inequality
Infinity, L^∞	$\ x\ _\infty$	$\max_i x_i $	Uniform convergence

□ **Linearly dependence** – A set of vectors is said to be linearly dependent if one of the vectors in the set can be defined as a linear combination of the others.

Remark: if no vector can be written this way, then the vectors are said to be linearly independent.

□ **Matrix rank** – The rank of a given matrix A is noted $\text{rank}(A)$ and is the dimension of the vector space generated by its columns. This is equivalent to the maximum number of linearly independent columns of A .

□ **Positive semi-definite matrix** – A matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite (PSD) and is noted $A \succeq 0$ if we have:

$$A = A^T \quad \text{and} \quad \forall x \in \mathbb{R}^n, \quad x^T A x \geq 0$$

Remark: similarly, a matrix A is said to be positive definite, and is noted $A \succ 0$, if it is a PSD matrix which satisfies for all non-zero vector x , $x^T A x > 0$.

□ **Eigenvalue, eigenvector** – Given a matrix $A \in \mathbb{R}^{n \times n}$, λ is said to be an eigenvalue of A if there exists a vector $z \in \mathbb{R}^n \setminus \{0\}$, called eigenvector, such that we have:

$$Az = \lambda z$$

□ **Spectral theorem** – Let $A \in \mathbb{R}^{n \times n}$. If A is symmetric, then A is diagonalizable by a real orthogonal matrix $U \in \mathbb{R}^{n \times n}$. By noting $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, we have:

$$\exists \Lambda \text{ diagonal, } A = U \Lambda U^T$$

□ **Singular-value decomposition** – For a given matrix A of dimensions $m \times n$, the singular-value decomposition (SVD) is a factorization technique that guarantees the existence of $U \in \mathbb{R}^{m \times m}$ unitary, $\Sigma \in \mathbb{R}^{m \times n}$ diagonal and $V \in \mathbb{R}^{n \times n}$ unitary matrices, such that:

$$A = U \Sigma V^T$$

Matrix calculus

□ **Gradient** – Let $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ be a function and $A \in \mathbb{R}^{m \times n}$ be a matrix. The gradient of f with respect to A is a $m \times n$ matrix, noted $\nabla_A f(A)$, such that:

$$\left(\nabla_A f(A) \right)_{i,j} = \frac{\partial f(A)}{\partial A_{i,j}}$$

Remark: the gradient of f is only defined when f is a function that returns a scalar.

□ **Hessian** – Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function and $x \in \mathbb{R}^n$ be a vector. The hessian of f with respect to x is a $n \times n$ symmetric matrix, noted $\nabla_x^2 f(x)$, such that:

$$\left(\nabla_x^2 f(x) \right)_{i,j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

Remark: the hessian of f is only defined when f is a function that returns a scalar.

□ **Gradient operations** – For matrices A, B, C , the following gradient properties are worth having in mind:

$$\nabla_A \text{tr}(AB) = B^T$$

$$\nabla_{A^T} f(A) = (\nabla_A f(A))^T$$

$$\nabla_A \text{tr}(ABA^T C) = CAB + C^T A B^T$$

$$\nabla_A |A| = |A| (A^{-1})^T$$