The excluded minors for embeddability into a compact surface

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Abstract

We determine the excluded minors characterising the class of countable graphs that embed into some compact surface.

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1 Introduction

The main aim of this paper is to provide the excluded minors characterising the class of countable graphs that embed into a compact surface, whereby we put no restriction on the genus. We will prove

Theorem 1.1. *A countable graph G embeds into a compact (orientable) surface if and only if it does not have one of the 8 graphs of Figure 1 as a minor.*[1]

It is an exercise to show that none of these graphs is a minor of another. Since none of these graphs embeds into a closed surface, orientable or not, our theorem remains valid if we remove the word 'orientable'.

All graphs in this paper are countable. In the locally finite case, only the first two obstructions $\Sigma_1 = \omega \cdot K_5, \Sigma_2 = \omega \cdot K_{3,3}$ are needed (see Corollary 4.6).

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[1] Every non-trivial infinite graph has many 'minor-twins'; for example, for each pair x, y of vertices of infinite degree in $\Sigma_i, i \geq 5$, we could add or remove the x–y edge.

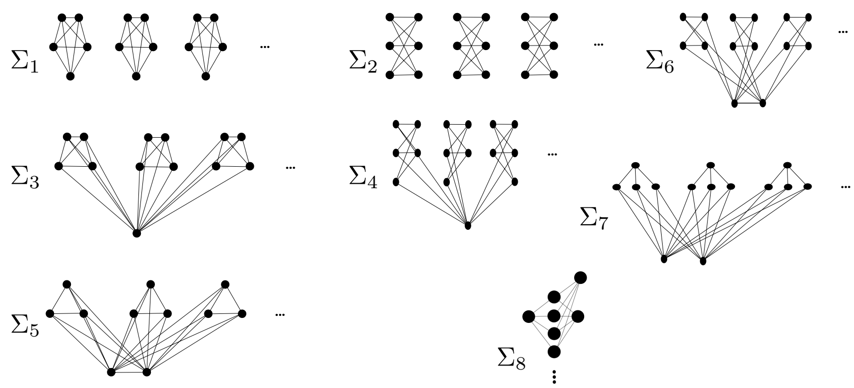

Figure 1: The excluded minors of Σ as provided by Theorem 1.1.
$\mathbf{\Sigma_1}$ (respectively $\mathbf{\Sigma_2}$): the disjoint union of infinitely many copies of K_5 (resp. $K_{3,3}$);
$\mathbf{\Sigma_3}$ (resp. $\mathbf{\Sigma_4}$): the graph arising from Σ_1 (resp. Σ_2) by picking one vertex from each component and identifying them;
$\mathbf{\Sigma_5}$ (resp. $\mathbf{\Sigma_6}$): the graph arising from Σ_1 (resp. Σ_2) by picking one edge from each component and identifying them;
$\mathbf{\Sigma_7}$: the graph arising from Σ_2 by picking one pair of non-adjacent vertices from each component and identifying these pairs; and $\mathbf{\Sigma_8} = K_{3,\omega}$.

An analogous statement for embeddings into a fixed surface is the following theorem of Robertson & Seymour [27] (previously announced in [5, 8, 13]). For every $n \in \mathbb{N}$, there is g, such that for every graph G of genus at least g, there is some Σ_i as in Figure 1, such that any subgraph of size n of Σ_i is a minor of G. I do not see a way to deduce this from Theorem 1.1 or vice-versa. An important difference between these two results is that if we restrict to the orientable case, then we need to allow the $n \times n$ projective grid as a further obstruction to embeddability into a fixed surface, but Theorem 1.1 proves that we have the same excluded minors with or without the orientability restriction when allowing arbitrarily high genus.

One of the tools for our proof of Theorem 1.1 is the following result of independent interest, saying that a graph embeds into a compact surface if and only if it can be decomposed into finitely many planar subgraphs with finite pairwise intersections.

Theorem 1.2. *A countable graph embeds into a compact (orientable) surface if and only if it admits a finitary decomposition into planar pieces.*

See Section 3 for the precise definitions. Theorem 1.2 supports the meta-conjecture that any result proved for infinite planar graphs generalises, rather easily, to graphs embeddable into a compact surface. Examples of such results include the main results of [4, 15, 17, 21, 23]. See Section 3.1 for more.

Part of the motivation for Theorem 1.1 comes from a well-known conjecture of Thomas [32] postulating that the countable graphs are well-quasi-ordered under the minor relation. The analogous statement for finite graphs is the celebrated Graph Minor Theorem of Robertson & Seymour [31]. A positive answer to Thomas's conjecture would imply that every minor-closed class $\mathcal{C}$ of countable graphs is characterised by forbidding a finite list $\mathrm{Ex}(\mathcal{C})$ of *excluded minors*. This is in general hard to show even for a concrete class of graphs like the class Σ of Theorem 1.1; indeed, it is a-priori not even clear that $\mathrm{Ex}(\Sigma)$ is finite. Apart from the fact that Σ is a natural class to consider, another reason why the finiteness of $\mathrm{Ex}(\Sigma)$ is a pressing question if one is interested in Thomas's conjecture is the important role played by classes of finite graphs embeddable in a fixed surface in the proof of the Graph Minor Theorem.

Many natural minor-closed graph classes $\mathcal{C}$, e.g. the graphs embeddable into a fixed surface, have the property that a graph is in $\mathcal{C}$ as soon as every finite subgraph is. This has the consequence that $\mathrm{Ex}(\mathcal{C})$ coincides with the list of excluded minors of the subclass of $\mathcal{C}$ comprising its finite elements. Apart from such classes, there are very few classes $\mathcal{C}$ of infinite graphs for which $\mathrm{Ex}(\mathcal{C})$ is explicitly known. The only example I am aware of are the graphs with accumulation-free embeddings in the plane [20].

Additional motivation for Theorem 1.1 comes from a question raised by Christian, Richter & Salazar [8], asking for a characterisation of the Peano continua that embed into a closed surface analogous to Claytor's [9] characterisation of the Peano continua embeddable into $\mathbb{S}^2$. The special case of graph-like continua was handled in [8], and the characterisation obtained is similar to Theorem 1.1. But the lack of compactness does not allow using that characterisation to deduce Theorem 1.1. Using Theorem 1.1 and the star-comb lemma (Lemma 2.2 below) it is not difficult to determine the excluded *topological* minors for embeddability into a compact surface, and this could be a first step towards answering the aforementioned question of Christian et al. [8].

Our proof of Theorem 1.1 is elementary (but involved), relying only on Kuratowski's theorem, and a classical result of Youngs about cellular embeddings of finite graphs. It is carried out mostly in Sections 3, 5 and 6. On the way to Theorem 1.1 we will develop techniques that allow us to find the excluded minors of families of infinite graphs that satisfy a property up to finitely many flaws: we will characterize the graphs that become forests after deleting, or contracting, finitely many edges (Sections 4 and 8), as well as the graphs that are outerplanar up to deleting finitely many edges (Section 7).

The star-comb lemma is one of the most useful tools in infinite graph theory. In Section 9 we obtain the following strengthening for 2-connected graphs:

Theorem 1.3. *Let G be a countable, 2-connected, graph, and $U \subseteq V(G)$ infinite. Then G contains a subdivision of an infinite ladder, or of an infinite fan, or of $K_{2,\infty}$, having infinitely many vertices in U.*

If G is locally finite, then this results in a ray in G containing an infinite subset of U.

The aforementioned conjecture of Thomas was studied by Robertson, Seymour & Thomas [28, 29], and they concluded that *there is not much chance of proving* it, as it would have implications about the ordering of finite graphs. It is therefore natural to try to extend the Graph Minor Theorem to an intermediate level covering all finite graphs but not necessarily all countable ones. A concrete approach for doing so is offered by Conjecture 10.2 and other questions in Section 10 arising from our results and methods.

2 Preliminaries

We follow the terminology of Diestel [10]. We use $V(G)$ to denote the set of vertices, and $E(G)$ the set of edges of a graph G. For $S \subseteq V(G)$, the subgraph $G[S]$ of G *induced* by S has vertex set S and contains all edges of G with both end-vertices in S.

The *degree* $d(v) = d_G(v)$ of a vertex v in a graph G, is the number of edges of G incident with v.

A *ray* is a one-way infinite path. We say that G is *locally finite*, if no vertex of G lies in infinitely many edges.

Let G, H be graphs. An H *minor* of G is a collection of disjoint connected subgraphs $B_v, v \in V(H)$ of G, called *branch sets*, and edges $E_{uv}, uv \in E(H)$ of G such that each E_{uv} has one end-vertex in B_u and one in B_v. We write $H < G$ to express that G has an H minor.

Given a set X of graphs, we write $\mathrm{Forb}(X)$ for the class of graphs H such that no element of X is a minor of H.

A *subdivision* of a graph G is a graph obtained by replacing some of the edges of G by paths with the same end-vertices.

A *surface* is a connected 2-manifold without boundary. An *embedding* of a countable graph G into a surface S is a map $f : G \to S$ from the 1-complex obtained from G when identifying each edge with the interval $[0, 1]$ to S such that the restriction of f to each finite subgraph of G is an embedding in the topological sense, i.e. a homeomorphism onto its image. (The reason why we restrict to finite subgraphs here is that the 1-complex topology of G is not metrizable when G is not locally finite, and so such G cannot have an embedding into a metrizable space S. For example, a star with infinitely many leaves admits an embedding into $\mathbb{R}^2$ in our sense but it does not admit a topological embedding. Let $\gamma(G)$ denote the minimum genus of an orientable surface into which a graph G embeds.

The following is perhaps folklore, but we sketch a proof for completeness. The locally finite case has been proved by Mohar [25, §5].

Lemma 2.1. *Let G be a countable graph, and S an orientable surface. Then G admits an embedding into S if each of its finite subgraphs does.*

When S is the sphere, Dirac & Shuster [11] provide a proof by an elementary compactness argument (which they atribute to Erdős). Our proof is a combination of this with Youngs' Theorem 3.4 below.

Proof. If S has infinite orientable genus $\gamma(S)$, then it is easy to embed any countable graph in it, so let us assume $\gamma(S)$ is finite. We may assume that G is connected, for otherwise we can apply the result to each component of G, using the well-known fact that the genus of a finite (disconnected) graph equals the sum of the genuses of its components [3, Corollary 2]. Suppose every finite subgraph of G embeds into S. Let $\gamma := \max_{H \text{ is a finite subgraph of } G} \gamma(H)$. We may assume without loss of generality that $\gamma = \gamma(S)$, because if G embeds into a surface S' with $\gamma(S') < \gamma(S)$, then G also embeds into S by the classification of closed surfaces.

Thus we can pick a finite subgraph H of G with $\gamma(H) = \gamma(S)$. We may assume that H is connected since G is, because adding vertices and edges to H cannot decrease its genus. Let $H = G_1 \subset G_2 \subset \ldots$ be a sequence of finite subgraphs of G, such that $\bigcup G_i = G$. Let $g_i : G_i \to S$ be an embedding. By a standard compactness argument (see e.g. [25, §5]), there is a subsequence $\{g_{a_i}\}_{i\in\mathbb{N}}$ along which the restriction of g_{a_i} to H coincides, up to an automorphism of S, with a fixed embedding $g : H \to S$. By Youngs' Theorem 3.4, each face of g is homeomorphic to an open disc. Let us first assume for simplicity that the closure $\overline{F}$ of each face F of g is homeomorphic to a closed disc, and then make the necessary modifications to our arguments. For each such F, note that the subgraph F_i of G_{a_i} that g_{a_i} maps to $\overline{F}$ is planar. Let $G_F := \bigcup_{i\in\mathbb{N}} F_i$, and note that G_F is planar by the aforementioned result of Dirac & Shuster [11]. Even more, it follows from the arguments of [11] that G_F admits an embedding g_F into $\mathbb{R}^2$ the outer face of which coincides with the boundary of F in g. Thus by combining these embeddings g_F with our embedding g of H we obtain an embedding of G into S.

It remains to remove our assumption that $\overline{F}$ is homeomorphic to a closed disc for each face F of g, but this is not difficult. If this is not the case, then the closed walk W_F of H bounding F is not a cycle, but traverses some vertices at least twice, and this may prevent G_F, defined as above, from being planar. In this case, we modify G_F into a planar auxiliary graph G'_F as follows. For each vertex v of H that W_F visits $i > 1$ times, we introduce 'copies' $v_1, \ldots, v_i$ of v, and replace W_F by a cycle visiting each $v_j, j \leq i$ exactly once. Using the result of [11] as above it follows that the resulting graph G'_F is planar if we distribute the edges of each v to the v_j appropriately; we leave some straightforward details to the reader. We can then identify all v_j to a single vertex to deduce that G_F embeds into $\overline{F}$, and as above combine those embeddings with g to obtain an embedding of G into S. □

We let Σ denote the class of countable graphs that embed into a compact orientable surface. (We will prove that every graph embeddable into a compact non-orientable surface also embeds into a compact orientable one.)

We let ω denote the smallest infinite ordinal.

2.1 The star-comb lemma

Given a graph G, and an infinite set $U \subseteq V(G)$, we define a *U-star* to be a subdivision of the infinite star $K_{1,\omega}$ in G all leaves of which lie in U. We define a *U-comb* to be the union of a ray R of G with infinitely many pairwise disjoint, possibly trivial, U–R paths. We call R the *spine* of C, and the U–R paths its *teeth*.

Lemma 2.2 (Star-comb lemma [10, Lemma 8.2.2])**.** *Let U be an infinite set of vertices in a connected graph G. Then G contains either a U-star or a U-comb.*

3 Decomposing into planar graphs

The aim of this section is to prove the orientable case of Theorem 1.2, which will be used as a tool for the proof of Theorem 1.1. (The non-orientable case will follow after we have proved Theorem 1.1.)

Definition 3.1. *A* decomposition *of a graph G is a family $(G_i)_{i\in\mathcal{I}}$ of subgraphs of G, called the* pieces*, such that $G = \bigcup_{i\in\mathcal{I}} G_i$. We say that a decomposition $(G_i)_{i\in\mathcal{I}}$ is* finitary*, if $\mathcal{I}$ is finite, and the intersection of any two distinct pieces is finite. Note that this means that at most finitely many vertices of G lie in more than one piece.*

Let us collect a few lemmas for the proof of Theorem 1.2, and for later use.

Lemma 3.2. *Let $G \in \Sigma$ and suppose G' is obtained from G by identifying two vertices $v, w \in V(G)$. Then $G' \in \Sigma$.*

Proof. Let Γ be a compact orientable surface into which G embeds. If G is finite, then it is easy to embed G' into a surface Γ' obtained from Γ by adding a handle. By combining this observation with Lemma 2.1 we deduce that G' is embeddable into Γ', and hence $G' \in \Sigma$. □

The power of Lemma 3.2 lies in our ability to apply it repeatedly. This way we obtain

Corollary 3.3. *Let G be a countable graph admitting a finitary decomposition $G_1, \ldots, G_k$. If each G_i lies in Σ, then so does G.*

Proof. Starting from the disjoint union of copies of each G_i, we can repeatedly apply Lemma 3.2 to identify pairs of vertices corresponding to the same vertex of G_i, remaining in Σ after each step. Since the decomposition is finitary, we end up with G after finitely many such identifications. □

Our last lemma is a classical result of Youngs about cellular embeddings. A *face* of an embedding $g : G \to \Gamma$ of a graph into a surface is a component of $\Gamma \backslash g(G)$.

Theorem 3.4 ([34])**.** *Let Γ be a closed orientable surface, and let G be a finite connected graph that embeds into Γ but does not embed into a closed orientable surface of smaller genus. Then for every embedding $g : G \to \Gamma$, each face of g is homeomorphic to an open disc.*

We are now ready for the proof of the main result of this section, which we restate for convenience:

Theorem 3.5. *A countable graph G embeds into a compact, orientable, surface if and only if it admits a finitary decomposition into planar pieces.*

Proof. The backward direction follows immediately from Corollary 3.3.

For the forward direction, it suffices to consider the case where G is connected: at most finitely many of the components of G can have positive genus by additivity of the genus [3], and so we can work with each such component separately.

Let H be a finite subgraph of G with $\gamma(H) = \gamma(G)$. Such an H exists by Lemma 2.1. We may assume that H is connected since G is.

Let $g : G \to \Gamma$ be an embedding into the closed orientable surface of genus $\gamma(G)$, and recall that we may assume g to be generous. Note that g induces an embedding $g_H : H \to \Gamma$ by restriction. By Youngs' Theorem 3.4, every face F of g_H is homeomorphic to an open disc. Therefore, the subgraph $G_F := g^{-1}(F)$ of G embedded into F is planar. Let $\overline{G_F}$ denote the subgraph of G induced by G_F and all its neighbours, which must lie on $\partial F \subset H$. If each $\overline{G_F}$ was planar, which would be the case if the closure of each F was homeomorphic to a closed disc, then we could decompose G into the G_F's and be done. But the existence of such an embedding g is Jaeger's strong embedding conjecture [22], which is open. Instead, we will find a finitary decomposition of each G_F into planar subgraphs as follows.

Let $(D_v)_{v \in V(H)}$ be open topological discs witnessing the generosity of g. We can assume that these discs are pairwise disjoint, by choosing sub-discs if needed. Note that $g(H)$ separates each D_v into a finite number of components, each itself an open topological disc, which components we will call D_v-sectors. Moreover, $D_v \cap F$ is the union of some of these D_v-sectors for every $v \in V(H)$ and every face F of H. Define a plane supergraph G'_F of G_F as follows. For each $v \in V(H) \cap \partial F$, and each D_v-sector $s \subset F$, we introduce a new vertex v_s of G'_F, embed v_s into s, and reroute the edges incident with s so that they attach to v_s instead of v. We call these new vertices v_s the *boundary vertices* of G'_F, and denote their set by $\mathcal{B}_F$.

Note that G'_F is planar by construction. It is easy to see that

$$G'_F \text{ remains planar if we identify any one pair of boundary vertices.} \tag{1}$$

Indeed, all the boundary vertices of G'_F lie on its outer face, and identifying a pair of vertices on the outer face of any plane graph results in a planar graph.

By definition, $\overline{G_F}$ can be obtained from G'_F by identifying each set of boundary vertices corresponding to D_v-sectors of the same $v \in V(H) \cap \partial F$ to one vertex. As (1) applies to one pair only, it is not enough to guarantee that $\overline{G_F}$ is planar. Therefore, we decompose it further as follows.

For every face F of g as above, we pick a minimal forest T_F in G'_F such that any two boundary vertices that lie in the same component of G'_F also lie in the same component of T_F. Since $\mathcal{B}_F$ is finite, the existence of T_F is easily guaranteed. Recall that each boundary vertex v_s corresponds to some vertex v of H; we obtain T'_F by replacing each $v_s \in \mathcal{B}_F$ by its corresponding v. Let $H' := H \cup \bigcup_F T'_F$. Note that T'_F is a subgraph of G, hence so is H'. Moreover, H' is connected since H is. Easily, any face F' of H' is contained in a face F of H. We repeat the above constructions to define $G'_{F'}$ and $\overline{G_{F'}}$ in analogy with G'_F and $\overline{G_F}$.

We claim that each component C of $G'_{F'}$ contains at most two vertices of $\mathcal{B}_F$. Indeed, any triple of vertices of $\mathcal{B}_F \cap C$ would be contained in a subtree of T_F, which subtree would separate F into three regions, none of which can contain the whole triple.

If no pair of vertices in $G'_{F'} \cap \mathcal{B}_F$ corresponds to the same $v \in V(H)$, then $G'_{F'}$ is a planar subgraph of G (and $\overline{G_F}$). In this case we just accept $G^*_{F'} := G'_{F'}$ as a piece of our decomposition. If there is such a pair v_{s_1}, v_{s_2}, then by the previous remark this pair is unique for each component C of $G'_{F'}$. By identifying each such pair into a vertex we thus obtain a subgraph $G^*_{F'}$ of G, which by (1) is planar. Here we used the fact that a graph is planar if each of its components is.

Note that the union of all the $G^*_{F'}$, where F' ranges over all faces of H', contains all edges in $E(G) \backslash E(H')$. Since H' is finite, its edge-set forms a finitary decomposition of H'. Thus we obtain the desired decomposition of G as the set of graphs $G^*_{F'}$ united with the set of edges of H'. This is a finitary decomposition, since the intersection of any two of its elements is contained in H'. □

3.1 Implications of Theorem 3.5

We remark that Theorem 3.5 allows us to extend many results obtained for planar graphs, e.g. those of [4, 15, 17], to graphs in Σ. Motivated by the fact that some such results (e.g. [21, 23]) only apply to graphs with vertex-accumulation-free embeddings into the plane, we will now formulate and prove a refinement of Theorem 3.5 that takes accumulation points into account. This refinement is not needed for the proof of Theorem 1.1, and the reader may skip the rest of this section.

We let Σ^* denote the class of countable graphs G that embed into a compact orientable surface Γ so that there are at most finitely points of Γ that are accumulation points of vertices of G. We can always choose our embeddings so that no such accumulation point lies in the image of G. We define Planar* analogously, with Γ replaced by $\mathbb{S}^2$. Finally, we say that G is Vertex-Accumulation-Free, or *VAP-free* for short, if it admits an embedding in $\mathbb{R}^2$ with no accumulation point of vertices. We will prove

Corollary 3.6. *A countable graph G lies in Σ^* if and only if it admits a finitary decomposition into VAP-free pieces.*

Our proof will be a combination of Theorem 3.5 with the following basic fact about VAP-free graphs:

Lemma 3.7 ([33, LEMMA 7.1])**.** *A countable graph H is VAP-free, if and only if some embedding $g : H \to \mathbb{S}^2$ has the property that for every cycle C one of the two sides of $g(C)$ contains only finitely many vertices.*

Proof of Corollary 3.6. The proof of the backward direction follows the lines of that of Corollary 3.3, the only difference being that we start by embedding each piece into $\mathbb{S}^2$ with at most one accumulation point of vertices.

For the forward direction, given $G \in \Sigma^*$, we first apply Theorem 3.5 to obtain a finitary decomposition of G into planar pieces, and our proof guarantees that each piece lies in Planar*. Thus it now suffices to prove that each $H \in$ Planar* admits a finitary decomposition into VAP-free pieces. This is easy using Lemma 3.7: if H has k accumulation points in some embedding g, but is not itself VAP-free, then some cycle C decomposes it into two pieces each embedded with fewer than k accumulation points, and our result follows by induction on k. □

4 Graphs that have a property up to finitely many flaws

This section introduces classes of graphs that have a property up to finitely many 'flaws', and basic techniques for finding their excluded minors. This will suffice to prove the analogue of Theorem 1.1 for locally finite graphs.

Given a minor-closed family $\mathcal{C}$ of infinite graphs, one can define classes of graphs that are *almost* in $\mathcal{C}$ in the following sense.

Definition 4.1. *Let $\mathcal{C}_{\mathrm{V}}$ (respectively, $\mathcal{C}_{\mathrm{E}}$) denote the class of graphs G, such that by removing finitely many vertices (resp. edges) from G we obtain a graph belonging to $\mathcal{C}$. Similarly, we let $\mathcal{C}_{/\mathrm{E}}$ denote the graphs that belong to $\mathcal{C}$ after contracting finitely many edges.*

It is easy to see that $(\mathcal{C}_{\mathrm{E}})_{/\mathrm{E}} = (\mathcal{C}_{/\mathrm{E}})_{\mathrm{E}}$ for every $\mathcal{C}$, and we will simply write $\mathcal{C}_{\mathrm{E}/\mathrm{E}}$ instead.

The following examples show that neither of $\mathcal{C}_{/\mathrm{E}}, \mathcal{C}_{\mathrm{E}}$ is contained in the other in general.

Example 1: Let M denote the graph consisting of a ray emanating from the centre of an infinite star $K_{1,\omega}$ (we could call M the infinite mop), and let $\mathcal{C} := \mathrm{Forb}(M)$. Then $\mathcal{C}_{/\mathrm{E}} = \mathcal{C} \subsetneq \mathcal{C}_{\mathrm{E}}$, because $\mathcal{C}_{\mathrm{E}}$ contains M while $\mathcal{C}$ does not.

Example 2: It is easy to prove $\Sigma = \Sigma_{\mathrm{E}}$ similarly to Lemma 3.2. But $\Sigma \subsetneq \Sigma_{/\mathrm{E}}$, because $K_{3,\omega} \in \Sigma_{/\mathrm{E}}$.

Definition 4.2. *We write $\omega \cdot H$ for the disjoint union of countably infinitely many copies of a graph H. If H is vertex-transitive, we let $\bigvee H$ denote the graph obtained from $\omega \cdot H$ by picking one vertex from each copy of H and identifying them (by vertex-transitivity, it does not matter which vertices we pick).*

For example, $\bigvee K_3$ is a *bouquet of triangles*, i.e. an infinite union of triangles having exactly one vertex in common.

The next proposition provides the excluded minors of the class $\mathcal{F}_{\mathrm{E}}$ of 'almost forests' (the analogous result for $\mathcal{F}_{/\mathrm{E}}$ is given in Section 8). Although it is not formally needed, we include it here as a gentle introduction to the techniques we will later use to prove our main result (Theorem 1.1).

Proposition 4.3. *Let $\mathcal{F}$ denote the class of countable forests. Then $\mathcal{F}_{\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_3, \bigvee K_3, K_{2,\omega})$.*

Proof. Suppose $G \notin \mathcal{F}_{\mathrm{E}}$. Then the set $\mathcal{C}$ of cycles of G is infinite. If $\mathcal{C}$ contains an infinite subset consisting of pairwise disjoint cycles, then we obtain a $\omega \cdot K_3$ minor and we are done. Otherwise, we claim that there is a vertex v having infinitely many incident edges in $\bigcup \mathcal{C}$. In particular, v is contained in each cycle of an infinite subset $\mathcal{C}' \subseteq \mathcal{C}$. Indeed, if this fails for every $v \in V(G)$, then we can greedily find an infinite sequence $(C_n)_{n\in\mathbb{N}}$ of pairwise vertex-disjoint cycles, by removing, at each step $n \in \mathbb{N}$ all the edges of all the cycles intersecting $\bigcup_{j<n} C_j$; since the edges we thereby remove are finitely many, and $G \notin \mathcal{F}_{\mathrm{E}}$, we can always find a new cycle C_n avoiding $\bigcup_{j<n} C_j$.

If v is the only vertex incident with infinitely many cycles in $\mathcal{C}'$, then by a similar greedy construction we obtain a $\bigvee K_3$ minor in $\bigcup \mathcal{C}'$, with v being the vertex of infinite degree. Otherwise, let $w \neq v$ be a vertex contained in each cycle of an infinite subset $\mathcal{C}'' \subseteq \mathcal{C}'$. Note that $G' := \bigcup \mathcal{C}'' - v$ is a connected graph, because $C - v$ contains w for every $C \in \mathcal{C}''$. We apply the star-comb Lemma 2.2 to G' with U being the set of neighbours of v, to obtain a U-star or a U-comb X. If X is a U-star, then attaching v to X we obtain a subdivision of $K_{2,\omega}$ in G. If X is a U-comb, then contracting its ray, and attaching v, we again obtain a $K_{2,\omega}$ minor in G.

In all cases we have obtained one of $\omega \cdot K_3, \bigvee K_3, K_{2,\omega}$ as a minor of G. $\square$

The class $\mathcal{F}_{\mathrm{V}}$ is easier to characterise in terms of excluded minors: we have $\mathcal{F}_{\mathrm{V}} = \mathrm{Forb}(\omega \cdot K_3)$. This is a special case of the following helpful fact. Its main argument is well-known in the context of Andreae's ubiquity conjecture [2].

Proposition 4.4. *Let $\mathcal{P} = \mathrm{Forb}(H_1, \ldots, H_k)$ be a minor-closed class of countable graphs, where the H_i are finite. Then $\mathcal{P}_V = \mathrm{Forb}(\omega \cdot H_1, \ldots, \omega \cdot H_k)$.*

Proof. If $G \notin \mathcal{P}_V$, then G has an H_i minor $M = M_0$ for some i. We may assume that each branch set of M is finite since H_i is finite. Notice that $G_1 := G - M$ does not belong to $\mathcal{P}_V$ either. By repeating the argument to G_1, and continuing recursively, we obtain a sequence $(M_n)_{n\in\mathbb{N}}$ of minors of G, with pairwise disjoint branch sets, each being an H_i minor. By passing to a subsequence we may assume that i is fixed. Thus we have found $\omega \cdot H_i$ as a minor of G, establishing $\mathcal{P}_V \supseteq \mathrm{Forb}(\omega \cdot H_1, \dots, \omega \cdot H_k)$.

The converse is straightforward: if $\omega \cdot H_i < G$, then $G - F \notin \mathcal{P}$ for any finite $F \subset V(G)$ as $H_i < G - F$ holds. $\square$

As an example application of Proposition 4.4, we deduce that $\mathrm{Planar_V} = \mathrm{Forb}(\omega \cdot K_5, \omega \cdot K_{3,3})$, where we write Planar for the class of planar graphs. Since $\mathrm{Planar_V} \subset \Sigma_V$, and $\omega \cdot K_5, \omega \cdot K_{3,3} \notin \Sigma_V$ ([3]; see also the proof of Theorem 1.1) this yields

Corollary 4.5. $\Sigma_V = \mathrm{Planar_V} = \mathrm{Forb}(\omega \cdot K_5, \omega \cdot K_{3,3})$.

An alternative proof of Corollary 4.5 can be obtained by using Theorem 3.5: the latter implies that $\Sigma \subset \mathrm{Planar_V}$, by removing the intersections of the pieces of any finitary planar decomposition of $G \in \Sigma$. This in turn implies $\Sigma_V \subset (\mathrm{Planar_V})_V = \mathrm{Planar_V}$, and so $\Sigma_V = \mathrm{Planar_V}$ as the converse inclusion is trivial.

As another corollary of Proposition 4.4, we obtain an easy proof of the analogue of Theorem 1.1 for locally finite graphs:

Corollary 4.6. *Let G be a locally finite graph. Then $G \in \Sigma$ unless $\omega \cdot K_5 < G$ or $\omega \cdot K_{3,3} < G$.*

Proof. As in the proof of Proposition 4.4, we can find either an $\omega \cdot K_5$ or an $\omega \cdot K_{3,3}$ minor in G, or a finite set of vertices $F \subset V(G)$ such that $G - F \in \Sigma$. But in the latter case we easily deduce $G \in \Sigma$, since F is incident with finitely many edges (we could for example apply Corollary 3.3). $\square$

Remark 1. The aforementioned fact that $\mathcal{F}_\mathrm{V} = \mathrm{Forb}(\omega \cdot K_3)$ can be thought of as the infinite version of the classical result of Erdős & Pósa [12] saying that every finite graph has either a $k \cdot K_3$ minor or a set of at most $f(k)$ vertices the removal of which results into a forest. I do not expect there to be an easy way to deduce the one from the other, as this Erdős-Pósa property fails for non-planar graphs in the finite case [30], while the infinite version holds for every finite graph by Proposition 4.4.

4.1 Almost planar graphs

The following is another consequence of Theorem 3.5 of independent interest. It is not needed for the proof of our other results, and the reader may skip to the next section.

Proposition 4.7. $\Sigma_{/\mathrm{E}} = \mathrm{Planar}_{/\mathrm{E}} = \mathrm{Planar}_{\mathrm{E}/\mathrm{E}}$.

Proof. The following inclusions follow immediately from the definitions:

$$\mathrm{Planar}_{/\mathrm{E}} \subseteq \mathrm{Planar}_{\mathrm{E}/\mathrm{E}} \subseteq \Sigma_{\mathrm{E}/\mathrm{E}} = \Sigma_{/\mathrm{E}},$$

where for the last equality we used the fact that $\Sigma_{\mathrm{E}} = \Sigma$. Thus it only remains to show that $\Sigma_{/\mathrm{E}} \subseteq \mathrm{Planar}_{/\mathrm{E}}$. For this, suppose $G \in \Sigma_{/\mathrm{E}}$, and let $F \subset E(G)$ be finite and such that $G' := G/F \in \Sigma$. Note that at most finitely many of the components of G' are non-planar [24]. For each such component C, apply Theorem 3.5 to obtain a finitary decomposition $G_1, \ldots, G_k$ of C into planar pieces. Let S denote the set of vertices of C that lie in at least two of the pieces, and let T be a finite subtree of C containing S, which exists since C is connected and S finite. We claim that $C/E(T)$ is planar. Indeed, each $G_i/(E(T) \cap E(G_i))$ is planar since G_i is, and $G_i \cap G_j$ contains at most one vertex after contracting $E(T)$. This proves that $C \in \mathrm{Planar}_{/\mathrm{E}}$ for each non-planar component of G', and therefore $G' \in \mathrm{Planar}_{/\mathrm{E}}$. It follows that G too lies in $\mathrm{Planar}_{/\mathrm{E}}$. □

5 Outerplanar graphs and related classes

By Proposition 4.4 and Corollary 4.5, if a graph G does not lie in $\Sigma_{\mathrm{V}} = \mathrm{Planar}_{\mathrm{V}}$ then it has one of the desired minors, and so the most challenging part of our proof of Theorem 1.1 is to handle the case where G becomes planar after removing a finite vertex set W. When W is a single vertex, the latter is tantamount to saying that $G - W$ is planar but not outerplanar *relative to* the neighbourhood of W. The aim of this section is to characterise graphs that are outerplanar relative to a vertex set U, and the analogous class for embeddings in any compact surface, in terms of forbidden minors that are *marked* by U. The precise definitions follow.

Let G be a graph and $U \subset V(G)$. Define the *U-cone* $C_U(G)$ of G to be the graph obtained from G by adding a new vertex u, the *cone vertex*, and joining it to each vertex in U with an edge. We say that G is *U-outerplanar*, if $C_U(G)$ is planar. If G is finite, it is easy to see that G is U-outerplanar if and only if it admits an embedding into $\mathbb{S}^2$ such that all vertices in U lie on a common face-boundary. Moreover, by letting $U = V(G)$ we recover the standard notion of outerplanarity: G is outerplanar if and only if it is $V(G)$-outerplanar.

A *marked graph* is a pair consisting of a graph G and a subset U of $V(G)$, called the *marked vertices*. Given two marked graphs $(G, U), (H, U')$, an H *marked minor* of G is defined just like an H minor of G (see Section 2), except that for each marked vertex v of H, we require that the corresponding branch set B_v contains at least one marked vertex of G. We write $(G, U) < (H, U')$ when this is possible.

Our next lemma adapts the well-known fact that the finite outerplanar graphs coincide with $\mathrm{Forb}(K_4, K_{2,3})$.

Lemma 5.1. *Suppose G is a countable planar graph, and let $U \subset V(G)$. Then G is U-outerplanar if and only if (G, U) does not contain one of the marked graphs $(\Theta_i, U_i), 1 \leq i \leq 4$ of Figure 2 as a marked minor.*

Proof. Since a graph is planar if and only if each of its finite subgraphs is [11], we may assume for simplicity that G is finite.

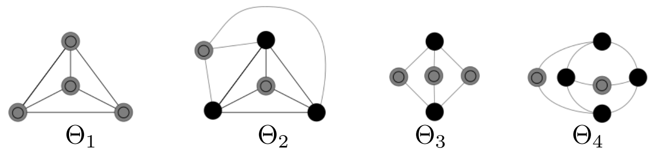

Figure 2: The excluded marked minors for relative outerplanarity. The sets of marked vertices U_i are shown in grey.

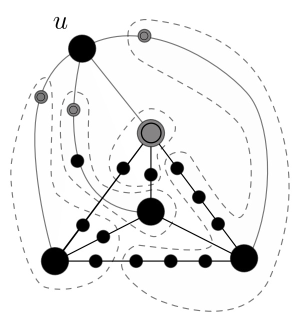

Figure 3: Turning K into a (Θ_1, U_1) marked-minor in Case 1. The dashed enclosures represent the branch sets B_i, and the grey vertices represent the v_i.

Let $G' := C_U(G)$, as defined above. If $(\Theta_i, U_i) < (G, U)$ for some i, then it is easy to find a K_5 or $K_{3,3}$ minor in G', hence G is not U-outerplanar.

For the other direction, if G' is non-planar, i.e. G is not U-outerplanar, then by Kuratowski's theorem G' contains a subdivision K of K_5 or $K_{3,3}$. Since G is planar, K contains the cone vertex u. There are four simple cases depending on which of $K_5, K_{3,3}$ it coincides with, and on whether u is a branch vertex or a subdivision vertex:

Case 1: K is a subdivision of K_5, and $d_K(u) = 4$. In this case we obtain (Θ_1, U_1) as a U-marked minor of G as follows. We remove u from K, and for each of the four neighbours $v_i \in U, 1 \leq i \leq 4$ of u, we define a branch set B_i containing the (possibly trivial) path of K joining v_i to a vertex $x_i \neq u$ with $d_K(x_i) = 4$. We extend the B_i so that each vertex of $K - u$ lies in exactly one B_i (Figure 3).

Case 2: K is a subdivision of K_5, and $d_K(u) = 2$. Similarly to the previous case, we obtain (Θ_2, U_2) as a U-marked minor of G. Indeed, Θ_2 is isomorphic to K_5 with the edge joining the two marked vertices removed.

Case 3: K is a subdivision of $K_{3,3}$, and $d_K(u) = 3$. Similarly to Case 1, we obtain (Θ_3, U_3) as a U-marked minor of G.

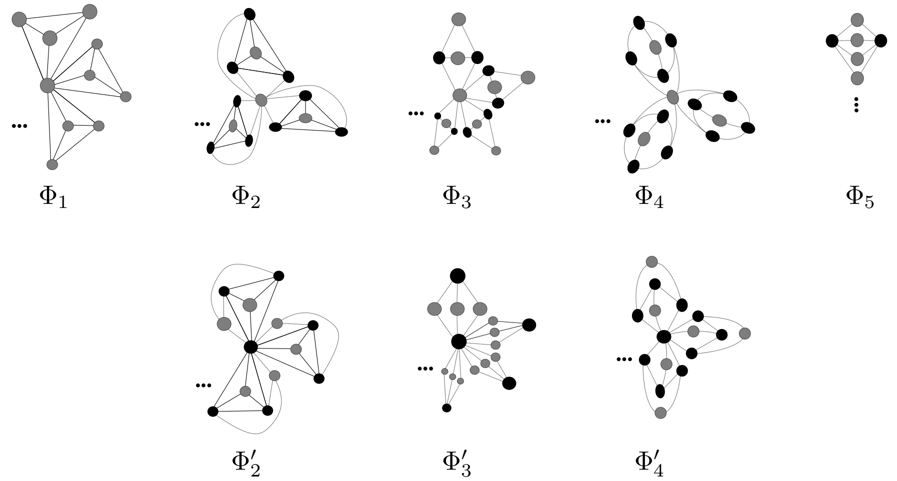

Figure 4: Some of the excluded marked minors of $\Sigma_\bullet$. The grey vertices represent the marked ones.

Case 4: K is a subdivision of $K_{3,3}$, and $d_K(u) = 2$. Similarly to Case 2, we obtain (Θ_4, U_4) as a U-marked minor of G, whereby we use the fact that Θ_4 is isomorphic to $K_{3,3}$ with the edge joining the two marked vertices removed.

Thus in each case we have found a (Θ_i, U_i) as a marked minor in G. □

Remark 2. *It follows from Lemma 5.1 that G is U-outerplanar as soon as each of its finite subgraphs is.*

We now introduce a generalisation of outerplanarity to arbitrary surfaces that will play an important role in the proof of Theorem 1.1:

Definition 5.2. *We say that a marked graph (G, U) lies in $\Sigma_\bullet$, and write $(G, U) \in \Sigma_\bullet$, if $C_U(G) \in \Sigma$.*

Lemma 5.3. *Suppose $G \in \Sigma$ is a countable graph, and (G, U) is not in $\Sigma_\bullet$ for some $U \subset V(G)$. Then (G, U) contains one of the following marked graphs as a marked minor:*
the graphs $\Phi_i, 1 \leq i \leq 5$, $\Phi'_i, 2 \leq i \leq 4$ of Figure 4, or
the graphs $\omega \cdot \Theta_i, 1 \leq i \leq 4$, with Θ_i as in Figure 2.

Lemma 5.3 is the technically most challenging part of the proof of Theorem 1.1. We prepare its proof with a number of lemmas. The first one is similar to Corollary 3.3.

Lemma 5.4. *Let G be a countable graph admitting a finitary decomposition $G_1, \ldots, G_k$. If each $(G_i, U \cap V(G_i))$ lies in $\Sigma_\bullet$ for some $U \subseteq V(G)$, then $G \in \Sigma_\bullet$.*

Proof. Our assumption says that $C_U(G_i)$ lies in Σ for every i. (To simplify notation we write $C_U(G_i)$ instead of $C_{U\cap V(G_i)}(G_i)$ throughout this proof.) Let G' be the graph obtained from the disjoint union of the $C_U(G_i), i \leq k$, by identifying each set of vertices corresponding to the same vertex of G into one vertex. By applying Corollary 3.3 to G', decomposed into the $C_U(G_i), i \leq k$, we deduce that that $G' \in \Sigma$.

Note that $C_U(G)$ can be obtained from G' by identifying the cone vertices of each $C_U(G_i)$ into one cone vertex. Thus by applying Lemma 3.2 $k-1$ times, we deduce that $C_U(G) \in \Sigma$, i.e. $G \in \Sigma_\bullet$. □

Using this, we can now extend Lemma 5.1 from planar graphs to graphs in Σ:

Lemma 5.5. *Let G be a graph in Σ, and let $U \subset V(G)$. Then G lies in $\Sigma_\bullet$ if and only if it does not contain one of the marked graphs $(\Theta_i, U_i), 1 \leq i \leq 4$ of Figure 2 as a marked minor.*

Proof. Let $G_1, \ldots, G_k$ be a finitary decomposition of G into planar pieces, as provided by Theorem 3.5. The backward directions is straightforward, so assume $G \notin \Sigma_\bullet$. If each G_i is $(U \cap V(G_i))$-outerplanar, then Lemma 5.4 implies that $G \in \Sigma_\bullet$ contrary to our assumption. Thus some G_i is —planar and— not $(U \cap V(G_i))$-outerplanar, and plugging it into Lemma 5.1 yields the desired minor. □

Given a marked graph (G, U) with $G \in \Sigma$, and a vertex $x \in V(G)$, we say that x is *$\Sigma_\bullet$-critical*, if (G, U) is not in $\Sigma_\bullet$ but $(G-x, U-x)$ is. The most difficult part of the proof of Lemma 5.3 lies in finding a Φ_5 minor in the case where G contains at least two $\Sigma_\bullet$-critical vertices. This is achieved (and refined) by the following two lemmas. Recall the definition of a U-star from Section 2.1.

Lemma 5.6. *Suppose $G \in \Sigma$, and $x \in V(G)$ is $\Sigma_\bullet$-critical for some $U \subseteq V(G)$. Then G contains a U-star with x as the infinite-degree vertex.*

Proof. Finding the desired U-star is tantamount to finding an infinite set of pairwise disjoint $N(x)$–U paths, where $N(x)$ stands for the set of vertices sending an edge to x. This is possible unless there is a finite set S of vertices separating $N(x)$ from U in $G-x$. Indeed, if there is a finite and maximal set of pairwise disjoint $N(x)$–U paths, then their union forms such an S. (We have just used a trivial infinite version of Menger's theorem.) Assuming such an S exists, let D_x denote the component of $G-S$ containing x, and let $\overline{\overline{D_x}} := G[D_x \cup S]$ denote the subgraph of G induced by $D_x \cup S$. Note that $N(x) \subseteq V(\overline{\overline{D_x}})$ by the definitions. Let D_U denote the union of all other components of $G-S$, and let $\overline{\overline{D_U}} := G[D_U \cup S]$. Note that $U \subseteq V(\overline{\overline{D_U}})$.

Thus $\overline{\overline{D_x}}, \overline{\overline{D_U}}$ is a finitary decomposition of G, as $V(\overline{\overline{D_x}} \cap \overline{\overline{D_U}}) \subset S$. We will use Lemma 5.4 to deduce that $G \in \Sigma_\bullet$, contradicting our assumptions. Indeed, both $\overline{\overline{D_x}}, \overline{\overline{D_U}}$ lie in Σ being subgraphs of G. Since $\overline{\overline{D_x}}$ contains at most finitely many elements of U (those in $S \cap U$), it lies in $\Sigma_\bullet$. Since $\overline{\overline{D_U}}$ is a subgraph of

G that avoids the $\Sigma_\bullet$-critical vertex x, we have $\overline{\overline{D_U}} \in \Sigma_\bullet$ too. Thus $G \in \Sigma_\bullet$ by Lemma 5.4. This contradicts the existence of S, thus proving the existence of the desired U-star.

□

We use Lemma 5.6 in order to prove

Lemma 5.7. *Suppose $G \in \Sigma$, and G has two $\Sigma_\bullet$-critical vertices x, y for some $U \subseteq V(G)$. Then G contains the marked double-star Φ_5 as a marked minor.*

Proof. By Lemma 5.6 G contains a U-star T_x centred at x. By Zorn's lemma we can choose T_x to be maximal, i.e. such that there is no x–U path avoiding $T_x - x$. By repeating with x replaced by y, we obtain a maximal, infinite, U-star T_y centred at y.

If G contains an infinite set $\mathcal{P}$ of pairwise disjoint T_x–T_y paths, e.g. when $T_x \cap T_y$ is infinite, then it is easy to construct the double-star as a marked minor of $T_x \cup \bigcup \mathcal{P} \cup T_y$ greedily.

Thus from now on we may assume that no such $\mathcal{P}$ exists, and therefore there is a finite set of vertices S separating T_x from T_y in G by (the trivial version of) Menger's theorem. Similarly to the proof of Lemma 5.6, we let C_x denote the union of the components of $G - S$ intersecting T_x, and let $\overline{\overline{C_x}} := G[C_x \cup S]$. We let $\overline{\overline{C_y}}$ be the subgraph of G induced by S and all other components of $G - S$, and note that $T_y \subseteq \overline{\overline{C_y}}$. Since G decomposes into $\overline{\overline{C_x}}, \overline{\overline{C_y}}$, and $G \notin \Sigma_\bullet$, Lemma 5.4 implies that at least one of $\overline{\overline{C_x}}, \overline{\overline{C_y}}$, say $\overline{\overline{C_y}}$, is not in $\Sigma_\bullet$. Note that x must lie in $\overline{\overline{C_y}}$ (hence in S) because $G - x \in \Sigma_\bullet$ by assumption. Thus x must be $\Sigma_\bullet$-critical in $\overline{\overline{C_y}}$ since it is $\Sigma_\bullet$-critical in G. Applying Lemma 5.6 to x in $\overline{\overline{C_y}}$, we thus obtain a U-star T in $\overline{\overline{C_y}}$ with x as the centre. Only finitely many vertices of T lie in S, and so almost all of T is disjoint from T_x, contradicting the maximality of the latter. This contradiction proves that $\mathcal{P}$ exists, hence so does the desired double-star. □

We can now prove the main result of this section:

Proof of Lemma 5.3. We claim that

$$\text{G has either an $\omega \cdot \Theta_i$ marked minor for some $1 \leq i \leq 4$, or a subgraph} \atop \text{$G' \in \Sigma$ with $(G', U \cap V(G')) \notin \Sigma_\bullet$ containing a $\Sigma_\bullet$-critical vertex.} \tag{2}$$

Indeed, by Lemma 5.5 we can find a marked Θ_i minor Θ in G for some $1 \leq i \leq 4$. Easily, we can choose the branch sets of Θ to be finite. We can now recursively remove the vertices in the branch sets of Θ one by one, and we will either encounter a $\Sigma_\bullet$-critical vertex of $G - F$ for some finite $F \subset V(G)$, or deduce that $G - \Theta$ is still not in $\Sigma_\bullet$, in which case we can apply Lemma 5.5 again to find another Θ_i marked minor disjoint from Θ. Continuing like this ad infinitum, we achieve one of our two aims of (2).

If we thereby obtain an $\omega \cdot \Theta_i$ marked minor our task is complete, so we can assume from now on that some subgraph $G' \subseteq G$ in $\Sigma \backslash \Sigma_\bullet$ has a $\Sigma_\bullet$-critical vertex x. We may assume without loss of generality that $G' = G$, since both are graphs in $\Sigma \backslash \Sigma_\bullet$. By modifying (2) slightly, we will next prove

$$G \text{ has either a } \Phi_i \text{ or } \Phi'_i \text{ marked minor for some } 1 \leq i \leq 4 \text{, or a subgraph } G' \in \Sigma \backslash \Sigma_\bullet \text{ containing two } \Sigma_\bullet\text{-critical vertices.} \tag{3}$$

The proof is similar to that of (2): we recursively apply Lemma 5.5 to find a marked Θ_i minor Θ in G, and remove the vertices inside Θ other than x one by one, until we encounter a second $\Sigma_\bullet$-critical vertex $y \neq x$ of $G - F$ for some finite $F \subset V(G)$. If this happens, we note that x remains $\Sigma_\bullet$-critical in $G - F$ by the definitions, and so the second option of (3) is satisfied. If we never encounter such a vertex y, then we continue ad infinitum to find an infinite set $\mathcal{C}$ of Θ_i marked minors of G intersecting at x only.

We can easily turn $\mathcal{C}$ into a Φ_i or Φ'_i minor as follows. By the pigeonhole principle, we can find an infinite subset $\mathcal{C}' \subseteq \mathcal{C}$ all elements of which coincide with Θ_i for a fixed i, and moreover their branch set containing x always corresponds to a marked vertex of Θ_i or always corresponds to an unmarked one. We form a minor M of $\bigcup \mathcal{C}' \subseteq G$ as follows. For each $C \in \mathcal{C}'$ we let B_x denote the branch set of C containing x, and we declare their union $Y := \bigcup_{C \in \mathcal{C}'} B_y$ to be a branch set of M. Every other branch set of any $C \in \mathcal{C}'$ is declared to be a branch set of M too. The latter are pairwise disjoint since x is the only common vertex of any two elements of $\mathcal{C}'$. This M is a Φ_i or Φ'_i minor of G as in our statement: if each element of $\mathcal{C}'$ is a Θ_i minor, then M is a Φ_i or Φ'_i minor; it is the former if the branch set containing x always corresponds to a marked vertex, and the latter if it always corresponds to an unmarked one.

This completes the proof of (3), from which our statement immediately follows: if (3) returns a Φ_i or Φ'_i minor then our task is complete, and if it returns a pair of $\Sigma_\bullet$-critical vertices we apply Lemma 5.7 to obtain a Φ_5 minor. Thus in every possible case we have found one of the desired U-marked graphs as a minor of G. □

Remark 3. The converse of Lemma 5.3 holds too, that is, if G has one of the U-marked minors as in the statement, then G is not in $\Sigma_\bullet$. Indeed, the U-cone of each of these graphs is not in Σ, because such a cone contains one of the forbidden structures of Theorem 1.1 as we will see in the proof of Theorem 1.1.

6 The excluded minors of Σ

We can now prove our main result.

Proof of Theorem 1.1. Let us start with the easy direction, that if G has a minor as in the statement, then $G \notin \Sigma$. It suffices to show that none of these graphs lies in Σ. This is indeed the case, as finite subgraphs of those graphs are well-known to have unbounded Euler genus; see e.g. [26] for $\Sigma_8 = K_{3,\omega}$, and [24] for the other seven Σ_i.

In particular, none of the Σ_i embed in a closed non-orientable surface either, and so we can drop the word 'orientable' in the statement of Theorems 1.1 and 1.2.

For the other direction, suppose $G \notin \Sigma$. If $G \notin \Sigma_V$, then by Corollary 4.5 we can find $\omega \cdot K_5$ or $\omega \cdot K_{3,3}$ as a minor of G, and so our statement holds in this case.

So assume $G \in \Sigma_V$, and choose a smallest set of vertices $F = \{v_1, \ldots, v_k\}$ such that $G - F \in \Sigma$. We may assume without loss of generality that $|F| = 1$, i.e. $G - v_1 \in \Sigma$, because we could replace G by $G - \{v_2, \ldots, v_k\}$.

Let $U = N(v_1)$ be the set of vertices of $G' := G - v_1$ sending an edge to v_1. If (G', U) lies in $\Sigma_\bullet$, then G lies in Σ by the definitions, and we have a contradiction.

Thus we can assume from now on that (G', U) does not lie in $\Sigma_\bullet$, and apply Lemma 5.3, to obtain $(G', U) > X$ where X is one of the excluded marked graphs of that Lemma. We use use the singleton v_1 as an additional branch set to obtain the U-cone $X' := C_U(X)$ as a minor of G. Ignoring the marking, we will thus obtain one of the graphs of Figure 1 as a minor, depending on which graph X coincides with, as follows.

The easiest case is where $X = \Phi_1$, in which case we have $X' = \Sigma_5$.
If $X = \Phi_2$, we obtain $X' > \Sigma_3$ by forming a branch set comprising the two infinite-degree vertices of X'.
If $X = \Phi_2'$, we have $X' > \Sigma_6$; to see this, notice that after removing the central vertex of Φ_2', each component has a 4-cycle in alternating colours.
If $X = \Phi_3$, we clearly have $X' = \Sigma_6$.
If $X = \Phi_3'$, we clearly have $X' = \Sigma_7$.
If $X = \Phi_4$, then similarly to the $X = \Phi_2$ case, we obtain $X' > \Sigma_4$ by forming a branch set comprising the two infinite-degree vertices of X'.
If $X = \Phi_4'$, we have $X' > \Sigma_6$; to see this, remove the edge from the central vertex w of Φ_4' to each grey vertex, and contract the other edge incident with that grey vertex. Each component of $\Phi_4' - w$ thus becomes a 4-cycle with alternating colours.
If $X = \Phi_5$, we have $X' = \Sigma_8 = K_{3,\omega}$.

If $X = \omega \cdot \Theta_1$, we have $X' = \Sigma_3 = \bigvee K_5$ (Figure 2).
If X is $\omega \cdot \Theta_3$, we have $X' = \Sigma_4 = \bigvee K_{3,3}$.
If $X = \omega \cdot \Theta_2$, we obtain $X' > \Sigma_3$ by contracting one of the two marked vertices u_i of each copy T_i of Θ_2 onto the cone vertex v_1; note that this contracted vertex inherits the three edges from u_i to T_i, as well as a fourth edge from v_1 to the remaining vertex of T_i.
Finally, if $X = \omega \cdot \Theta_4$, then similarly we obtain $X' > \Sigma_4$ by contracting one of the two marked vertices of each copy of Θ_4 onto v_1.

To summarize, assuming $G \notin \Sigma$, we obtained one of the graphs of Figure 1 as a minor of G. □

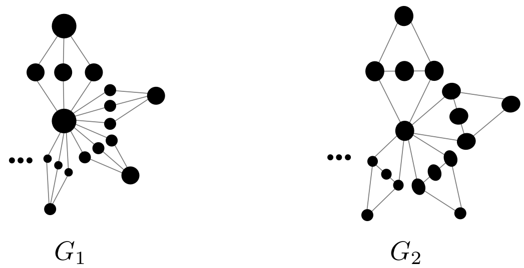

Figure 5: Two of the excluded minors of Proposition 7.1, arising by combining infinitely many copies of $K_{2,3}$.

7 Almost outerplanar graphs

The aim of this section is to prove the analogue of Proposition 4.3 for outerplanar graphs. This is included as a result of independent interest, proved using some of the techniques developed above.

Let OP denote the class of countable outerplanar graphs. Let G_1 (respectively, G_2) be the graph obtained from $\omega \cdot K_{2,3}$ by choosing a vertex of degree 3 (resp. degree 2) from each copy of $K_{2,3}$ and identifying them (Figure 5).

Proposition 7.1. $OP_{\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_4, \omega \cdot K_{2,3}, \bigvee K_4, G_1, G_2, K_{2,\omega})$.

Proof. Suppose $G \notin OP_{\mathrm{E}}$. If $G \notin OP_{\mathrm{V}}$, then by Proposition 4.4, and the well-known fact that $\mathrm{OP} = \mathrm{Forb}(K_4, K_{2,3})$, we deduce that G contains one of $\omega \cdot K_4, \omega \cdot K_{2,3}$ as a minor, and we are done. Thus we may assume that $G \in OP_{\mathrm{V}}$. Choose a finite, minimal $W \subset V(G)$ such that $G - W \in OP_{\mathrm{E}}$, which exists since we can even achieve $G - W \in OP$. Pick $v \in W$, and let $G' := G - (W \backslash \{v\})$. Then $G' \notin OP_{\mathrm{E}}$, but $G' - v \in OP_{\mathrm{E}}$ by the choice of W. We will find one of the desired minors in G'.

Note that there must be infinitely many subdivisions of K_4 or $K_{2,3}$ in G' containing v. Even more,

> there is an infinite set $\mathcal{C}$ of subdivisions of K_4 or $K_{2,3}$ in G' containing v, no two of which share an edge of v. (4)

Indeed, if not, then there is a finite set F of edges of v such that $G' - F$ contains no subdivision of K_4 or $K_{2,3}$ containing v, which contradicts $G' \notin OP_{\mathrm{E}}$.

We distinguish two cases. If v is the only vertex contained in infinitely many elements of $\mathcal{C}$, then we can greedily find an infinite subset $\mathcal{C}' \subseteq \mathcal{C}$ such that $C \cap D = \{v\}$ holds for every $C, D \in \mathcal{C}'$. We may further assume that all elements of $\mathcal{C}'$ are copies of K_4 or they are all copies of $K_{2,3}$, and that v has the same degree —either 2 or 3— in each of them. Thus we find one of $\bigvee K_4, G_1, G_2$ as a minor of $\bigcup \mathcal{C}'$, with v being the vertex of infinite degree.

Otherwise, there is a vertex $w \neq v$ contained in each element of an infinite subset $\mathcal{C}' \subseteq \mathcal{C}$. Note that $G'' := \bigcup \mathcal{C}' - v$ is a connected graph, containing infinitely many neighbours of v by (4). We apply the star-comb lemma to G'' with U being the set of neighbours of v, to obtain a $K_{2,\omega}$ minor in G' as in the proof of Proposition 4.3.

In all cases we have obtained one of $\omega \cdot K_4, \omega \cdot K_{2,3}, \bigvee K_4, G_1, G_2, K_{2,\omega}$ as a minor of G. $\square$

8 Almost forests revisited

The aim of this section is to prove the following result, which complements Proposition 4.3.

Proposition 8.1. *Let $\mathcal{F}$ denote the class of countable forests. Then* $\mathcal{F}_{/\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_3, \bigvee K_3) = \mathcal{F}_{\mathrm{E}/\mathrm{E}}$.

Combined with Proposition 4.3, it follows that $\mathcal{F}_{\mathrm{E}} \subsetneq \mathcal{F}_{/\mathrm{E}}$. For our proof we will need the following extension of the star-comb lemma.

A *2-star* is a graph obtained from the star $K_{1,\omega}$ by subdividing each edge at least once. In other words, a 2-star is obtained from the disjoint union of infinitely many paths of length at least 2 by identifying their first vertices.

We say that a vertex-set D *dominates* another vertex-set U of a graph, if $\overline{N}(D) \supseteq U$, where $\overline{N}(D)$ consists of D and all vertices sending an edge to D.

Lemma 8.2. *Let G be a connected graph, and $U \subseteq V(G)$. Then G contains at least one of the following:*

(i) A U-comb;

(ii) a 2-star with leaves in U;

(iii) a finite vertex-set dominating U.

Proof. We say that a vertex v is *U-dominant*, if $|N(v) \cap U| = \infty$, where $N(v)$ stands for the set of neighbours of v. Let D denote the set of U-dominant vertices of G.

If $|D| = \infty$, we apply the star-comb lemma to G, D, to obtain either a D-star S or a D-comb K. In the former case, we construct a 2-star with leaves in U as follows. Let $(w_n)_{n \in \mathbb{N}}$ be an enumeration of the leaves of S, and let s be the centre of S. Pick a neighbour $u_0 \neq s \in U$ of w_0, and add the w_0–u_0 edge to S. If $u_0 \in V(S)$, then we also *delete* the subpath of $S - s$ containing u_0. For $i = 1, 2, \ldots$, we proceed similarly: we let w_{a_i} be the next leaf of S that has not been deleted, we pick a neighbour $u_i \in U$ of w_{a_i} that does not coincide with s or any $u_j, j < i$, attach it to w_{a_i}, and delete the subpath of $S - s$ containing u_i if it exists. Such a u_j always exists, because $w_{a_i} \in D$ has infinitely many neighbours in U. After ω steps we have transformed S into the desired 2-star S' witnessing (ii).

In the other case where we obtain a comb K, then either $|K \cap U| = \infty$, and we are done as (i) holds, or way may assume that $|K \cap U| = \emptyset$ by replacing K with a sub-comb. In this case we construct a comb K' by attaching leaves in U to infinitely many vertices of $K \cap D$ by imitating the way we obtained S' from S (except that now we never have to delete anything).

Thus we have settled the case $|D| = \infty$, and we now proceed to the case where $|D| < \infty$. Suppose first that $U' := U \backslash \overline{N}(D)$ is finite. Then $D \cup U'$ is a finite set dominating U, i.e. option (iii) holds.

So suppose U' is infinite. We apply the star-comb Lemma 2.2 to G and U'. If it returns a U'-comb then option (i) holds. If it returns a U'-star S, then as no vertex is U'-dominant by the definition of U', it follows that S contains a 2-star with leaves in U', which means that option (ii) holds.

Thus in every case we have obtained one of the desired sub-structures. □

Proof of Proposition 8.1. Suppose $G \notin \mathcal{F}_{/\mathrm{E}}$. If $G \notin \mathcal{F}_{\mathrm{V}}$, then by Proposition 4.4 we deduce that G has a $\omega \cdot K_3$ minor, and we are done. Thus we may assume that $G \in \mathcal{F}_{\mathrm{V}}$. Choose a finite $W \subset V(G)$ such that $G - W \in \mathcal{F}$, minimal with this property. Assume first that G is connected. Thus we can find a finite subtree T of G containing W. Let $G' := G/T$ be the minor of G obtained by contracting T onto a vertex v. Then $G' \notin \mathcal{F}_{/\mathrm{E}}$ since $G \notin \mathcal{F}_{/\mathrm{E}}$ and T is finite. Moreover, $F := G' - v \in \mathcal{F}$. We will prove that G', and hence G, has a $\bigvee K_3$ minor.

If there is an infinite set $\{C_n\}_{n \in \mathbb{N}}$ of components of F such that v sends at least two edges e_n, f_n to C_n, then by appending an e_n–f_n path path through C_n for each n we obtain a $\bigvee K_3$ minor in $\{v\} \cup \bigcup_{n \in \mathbb{N}} C_n$. Thus we may assume that there is a finite set $\{C_1, \ldots, C_k\}$ of components of F sending at least two edges to v, and all other components of F send at most one edge to v. Note that $G' - (C_1 \cup \ldots \cup C_k)$ is a forest.

For each $i = 1, \ldots, k$, we let G'_i be the subgraph of G' induced by $C_i \cup \{v\}$, and let U_i be the set of vertices of C_i sending an edge to v. We apply Lemma 8.2 to $G'_i - v, U_i$, and obtain a U_i-comb, or a 2-star with leaves in U_i, or a finite set W_i of vertices dominating U_i. If this yields a U_i-comb C, then we easily obtain a subdivision of $\bigvee K_3$ in $G'_i[C \cup \{v\}]$. If instead we obtain a 2-star S, with centre s and leaves in U_i, then by contracting a v–s path we obtain a subdivision of $\bigvee K_3$ in $S \cup \{v\}$, centred at the contracted path. Finally, if Lemma 8.2 returns a finite dominating set W_i, then we let T_i be a finite subtree of G'_i containing $W_i \cup \{v\}$. Note that G'_i/T_i is a tree. To summarize, either we have obtained a $\bigvee K_3$ minor in G, or a finite $T_i \subset G'_i$ for each $1 \leq i \leq k$ such that G'_i/T_i is a tree. But the latter implies that contracting each T_i turns G' into a forest, contradicting the fact that $G' \notin \mathcal{F}_{/\mathrm{E}}$.

It remains to handle the case where G is disconnected. In this case we apply the same reasoning to one of the components of G intersecting W, and find a $\bigvee K_3$ minor there.

To deduce that $\mathcal{F}_{/\mathrm{E}} = \mathcal{F}_{\mathrm{E}/\mathrm{E}}$, note that $\mathcal{F}_{/\mathrm{E}} \subseteq \mathcal{F}_{\mathrm{E}/\mathrm{E}}$, and that none of the excluded minors $\omega \cdot K_3, \bigvee K_3$ of the former lies in the latter. Thus $\mathcal{F}_{\mathrm{E}/\mathrm{E}} \backslash \mathcal{F}_{/\mathrm{E}}$

is empty. □

9 A star-comb lemma for 2-connected graphs

While trying to prove Theorem 1.1 I came up with the following strengthening of the star-comb lemma for 2-connected graphs. Although it is not used for any of our proofs, I decided to include as it might become useful elsewhere. The star-comb lemma is one of the most useful tools in infinite graph theory. Some other strengthenings were obtained in a recent series of 4 papers by Bürger & Kurkofka [6]–[7]. A related result determining unavoidable induced subgraphs for infinite 2-connected graphs is obtained by Allred, Ding & Oporowski [1].

In analogy with U-stars and U-combs as in the statement of the star-comb lemma, we introduce the following structures. A *double-star* is a subdivision of $K_{2,\omega}$. A *ladder* consists of two disjoint rays R, L and an infinite collection of pairwise disjoint R–L paths. A *fan* consists of a ray R, a vertex $d \notin V(R)$, and an infinite collection of d–R paths having only d in common. For each of these three terms, adding the prefix *U-* means that the structure has infinitely many of its vertices in U. With this terminology, Theorem 1.3 from the introduction can be formulated as follows.

Theorem 9.1. *Let G be a 2-connected graph, and $U \subseteq V(G)$ infinite. Then G contains a U-double-star, or a U-ladder, or a U-fan.*

The following follows from the statement of Theorem 9.1, but we need to prove it first as a first step towards the proof of the latter.

Lemma 9.2. *Let G be a 2-connected, locally finite graph, and $U \subseteq V(G)$ infinite. Then G has a ray containing an infinite subset of U.*

In this section we assume that the reader is familiar with the basics about the end-compactification of a graph, and normal spanning trees; we refer to [10] therefor.

Proof. Let χ be an accumulation point of U in the end-compactification of G, and choose $U' \subseteq U$ converging to χ. Let R be a ray converging to χ, such that each component of $G - R$ sends only finitely many edges to R; we could for example choose R inside a normal spanning tree of G; see [10, Exercise 8.27] or [14, Lemma 11]. If $R \cap U'$ is infinite then we are done, so assume this is not the case.

Note that no component of $G - R$ can contain an infinite subset of U'. Thus there is an infinite set $(C_n)_{n\in\mathbb{N}}$ of components of $G - R$ each intersecting U'. For each C_n, pick $u_n \in C_n \cap U'$, and two disjoint u_n–R paths. The union of these two paths is a path P_n through C_n with end-vertices x_n, y_n on R. Since G is locally finite, we can easily choose an infinite subset $(C'_n)_{n\in\mathbb{N}}$ such that the subpaths R_n of R from x_n to y_n are pairwise disjoint. By replacing each R_n with P_n we thus transform R into a ray R' containing infinitely many elements of U'.

□

Proof of Theorem 9.1. Suppose first that G is locally finite. By Lemma 9.2, G has a ray R with $R \cap U$ infinite. Let χ be the end of G containing R. Halin [19] proved that we can find two disjoint rays belonging to χ. It is proved in [14, Lemma 10] that we can choose these two rays X, Y so that they intersect every ray of χ infinitely often. In particular, $R \cap (X \cup Y)$ is infinite. Since X, Y belong to the same end, we can find a sequence $(P_n)_{n\in\mathbb{N}}$ of pairwise disjoint X–Y paths such that $\bigcup_{n\in\mathbb{N}} P_n \cup X \cup Y$ forms a ladder L. By replacing some subpaths of L by subpaths of R we can easily obtain a ladder L' containing infinitely many elements of U. This settles the case where G is locally finite.

If G is not locally finite, then we can still assume it is countable. For if not, then we can find a countable 2-connected subgraph G' with $V(G) \cap U$ infinite, and find the desired structure in G'. Indeed, we can pick a countably infinite subset $\{u_1, u_2, \dots\}$ of U, choose a pair of independent paths from each u_i to the two end-vertices of a fixed edge xy of G, which exist by Menger's theorem, and let G' be the union of all these paths and xy.

Let T be a normal spanning tree of G. Let r be the root of T, and define the *height* $h(v)$ of each $v \in V(G) = V(T)$ to be the distance between v and r in T. Apply the star-comb lemma to T, U to obtain a subgraph Z which is either a U-star or a U-comb.

If Z is a U-star, let c be its centre. We claim that G contains a U-double-star, with c being one of its two infinite degree vertices. To prove this claim we introduce the following terms. Given an edge $e \in E(T)$, define the *branch of e* to be the subgraph B_e of G induced by the vertices in the component of $T - e$ that does not contain r. Note that all neighbours of B_e lie on a subpath I_e of T because T is normal, and the top end-vertex I_e^{t} of I_e lies in e. We choose I_e to be minimal with these properties. Define the *closed branch $\overline{B_e}$* as the subgraph of G induced by $B_e \cup I_e$. Easily, $\overline{B_e}$ is 2-connected since G is. A *U-ear* of B_e is a path E_e in $\overline{B_e}$ joining I_e^{t} to some other vertex of I_e, such that E_e is otherwise disjoint from I_e and contains a vertex of $U \cap B_e$. It is easy to find a U-ear for each branch B_e intersecting U using the 2-connectedness of $\overline{B_e}$; indeed, note that $\overline{B_e}$ remains 2-connected after contracting $I_e - I_e^{\mathrm{t}}$.

To prove our claim, note that there are infinitely many such branches B_e with e incident with c, and these branches are pairwise disjoint. Since there are only finitely many vertices below c in T, we can find an infinite subset of those branches with U-ears (starting at c and) ending at the same vertex c'. The union of those U-ears is the desired U-double-star.

This completes the case where Z is a U-star, and we now proceed to the case where Z is a U-comb. We further distinguish two cases according to whether the spine R of Z is *dominated* in G or not. Here, we say that R is dominated, if there is a vertex d sending infinitely many independent paths to R.

Suppose first that R is dominated, by a vertex d say. By extending or shortening R as needed, we may assume that R starts at r. (It follows that $d \in V(R)$, but we will not need this.) If $R \cap U$ is infinite then we immediately obtain a U-fan, with d being the infinite degree vertex, and we are done. So assume $R \cap U$ is finite, and note that this means that there is an infinite set

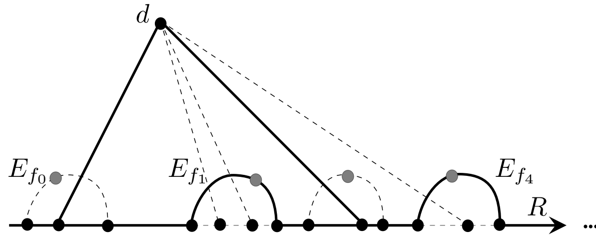

Figure 6: Obtaining a U-fan (bold lines) from a dominated comb R of T . The dashed lines represent paths that are not contained in the U-fan. The grey vertices represent elements of U.

$\{B_{e_n}\}_{n\in\mathbb{N}}$ of pairwise disjoint branches each containing a tooth of Z; indeed, we can choose e_n to be the nth edge of Z incident with R but not contained in it. Recall from above that each $\overline{B_{e_n}}$ contains a U-ear E_{e_n}, and note that each E_{e_n} has both end-vertices on R. We distinguish two cases according to how these end-vertices are distributed along R:

Case 1: No vertex of R is incident with infinitely many E_{e_n}'s. In this case we can greedily choose an infinite subsequence $\{E_{f_n}\}$ of $\{E_{e_n}\}$ such that the subpaths R_n of R between the two end-vertices of each E_{f_n} are pairwise disjoint. By replacing infinitely many of these subpaths R_n by E_{f_n}, we can modify R into a ray R' such that $R' \cap U$ is infinite and d still dominates R' (Figure 6), hence obtaining a U-fan in G. Indeed, we can greedily alternate between choosing a d–R path and a U-ear E_{f_n} to include in our U-fan; each time we choose a d–R path, we delete the finitely many U-ears it meets, and conversely, each time we choose a U-ear E_{f_n}, we replace R_n by E_{f_n}, and delete the finitely many d–R paths it meets. Here, we assume that we have pre-selected an infinite family of pairwise independent d–R paths, which we can since d dominates R.

Case 2: Some $v \in V(R)$ is incident with infinitely many E_{e_n}'s. Note that v must be the lower of the two end-vertices of these E_{e_n}'s, since the top end-vertex lies in e_n by the definition of E_{e_n}, with just one possible exception in case v happens to be an end-vertex of some e_n. By the same argument, the other end-vertices of the E_{e_n}'s incident with v can be assumed to be pairwise distinct. Thus the union of all these E_{e_n}'s with a sub-ray of R forms a U-fan in G, with v being the infinite degree vertex.

This completes the case where R is dominated, and we now proceed to the case where it is not. Again, we may assume that R starts at the root r. In this case we claim that

there is a locally finite, 2-connected, subgraph H of G containing R with infinite $H \cap U$. (5)

To construct H, for each vertex $w \neq r \in V(R)$, let P_w be a minimal path connecting the two components of $R - w$, which exists since G is 2-connected. Suppose first that for some w, there are such paths P_w with end-vertices of arbitrarily large height. In this case we apply the star-comb lemma to their union, to obtain a $V(R)$-star or $V(R)$-comb X in that subgraph. If X was a $V(R)$-star then this would contradict the fact that R is not dominated, and so X is a $V(R)$-comb. If $(X \cup R) \cap U$ is infinite, then we just let $H := X \cup R$ and have achieved (5). If not, then similarly to the case where R is dominated, we can find infinitely many edges e incident with R such that their branch B_e contains a tooth of Z lying in U. For each such B_e, we pick a U-ear E_e, and add it to $X \cup R$ to obtain H. Note that H must be locally finite, because the B_e are pairwise disjoint, and if infinitely many ears E_e share an end-vertex v (on R), then v dominates R, contradicting our assumption.

If, on the contrary, no such w exists, then we choose each P_w so as to maximise the height of its top end-vertex on R. This choice of P_w allows us to ensure that for $w \neq w' \in V(R)$, the paths $P_w, P_{w'}$ either coincide, or they are disjoint, or the top end-vertex of one of them coincides with the bottom end-vertex of the other; for otherwise we could find a path P' in $P_w \cup P_{w'}$ that can serve as both P_w and $P_{w'}$. It follows that $H' := R \cup \bigcup_{w \in V(R)} P_w$ is locally finite. Easily, H' is also 2-connected. If $H' \cap U$ is infinite we set $H := H'$ and have proved our claim. If it is finite, then again we can find infinitely many edges e incident with R such that their branch B_e contains a tooth of Z lying in U. For each such B_e, we pick a U-ear E_e, and add it to H' to obtain H. Easily, H is still 2-connected, and $H \cap U$ is infinite. As above, H is locally finite because R is not dominated. This completes the proof of (5).

We can now reduce our problem to the locally finite case, by replacing G by H. But we have handled the locally finite case above, obtaining a U-ladder. $\square$

Problem 9.1. *Is it possible to generalise Theorem 9.1 to k-connected graphs, obtaining a finite list of subdivisions of k-connected graphs as unavoidable structures?*

Results of similar flavour have been obtained by Gollin & Heuer [18].

10 Final remarks

It would be interesting to find the excluded minors for the classes $\Sigma_{/\mathrm{E}}$, $\mathrm{Planar}_{\mathrm{E}}$, $\mathrm{Planar}_{/\mathrm{E}}$ and $\mathrm{Planar}_{\mathrm{E}/\mathrm{E}}$, and this should be within reach with the above methods and a little bit more work. I suspect that

$$\Sigma_{/\mathrm{E}} = \mathrm{Planar}_{/\mathrm{E}} = \mathrm{Planar}_{\mathrm{E}/\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_5, \omega \cdot K_{3,3}, \bigvee K_5, \bigvee K_{3,3}).$$

The first two equalities have been proved in Proposition 4.7. I also suspect that $\mathrm{Planar}_{\mathrm{E}} = \mathrm{Forb}(\mathrm{Ex}(\Sigma) \cup \{K_5^{\oslash}, K_{3,3}^{\oslash}\})$, where $K^{\oslash}$ is obtained from a graph K by replacing each edge uv by infinitely many u–v paths of length 2.

Let us say that a minor-closed class $\mathcal{C}$ of graphs is *good*, if $\mathcal{C} = \text{Forb}(X)$ for a finite set X of (possibly infinite) graphs. A well-know conjecture of Thomas [32] postulates that the countable graphs are well-quasi-ordered under the minor relation. A positive answer would imply that all minor-closed classes of countable graphs are good, but as mentioned in the introduction, this seems out of reach at the moment. Still, we could seek to extend the Graph Minor Theorem [31] by finding sufficient conditions for classes of infinite graphs to be good. The following questions suggest a possible direction, and the methods of this paper could be helpful. For further questions in a similar vein see [16].

Question 10.1. *Suppose $\mathcal{C}$ is a good minor-closed class of countable graphs. Must each of $\mathcal{C}_{\text{V}}, \mathcal{C}_{\text{E}}, \mathcal{C}_{/\text{E}}, \mathcal{C}_{\text{E}/\text{E}}$ be good?*

We say that a class $\mathcal{C}$ of graphs is *co-finite*, if $\mathcal{C} = \text{Forb}(S)$ for a set S of finite graphs (which set can be chosen to be finite by the Graph Minor Theorem [31]). Note that a graph G belongs to such a class $\mathcal{C}$ if and only if every finite minor of G does. Question 10.1 is open in general even if $\mathcal{C}$ is co-finite, except that $\mathcal{C}_{\text{V}}$ is covered by Proposition 4.4 in this case. This papers provides some techniques for attacking it. In a similar spirit, one can ask whether the class of graphs admitting a finitary decomposition into graphs in $\mathcal{C}$ is good whenever $\mathcal{C}$ is good/co-finite.

We say that a class $\mathcal{C}$ of graphs is *UNCOF* (Union of Nested Co-finite classes), if there is a sequence $(C_n)_{n\in\mathbb{N}}$ of co-finite classes such that $\mathcal{C} = \bigcup_{n\in\mathbb{N}} C_n$ and $C_n \subseteq C_{n+1}$ holds for every $n \in \mathbb{N}$. The classes studied in this paper $(\Sigma, \mathcal{F}_{\text{E}}, \mathcal{F}_{/\text{E}}, OP_{\text{E}}$, etc.) are easily seen to be UNCOF. Our results support

Conjecture 10.2. *Every UNCOF class of countable graphs is good.*

Another interesting example of an UNCOF class $\mathcal{C}$ comprises the graphs G of finite Colin de Verdière invariant $\mu(G)$, whereby for infinite G we define $\mu(G)$ to be the supremal m such that every finite subgraph $H \subset G$ satisfies $\mu(H) \leq m$. Is this $\mathcal{C}$ good? Can we determine $\text{Ex}(\mathcal{C})$?

Not every proper minor-closed class is UNCOF. For example, $\text{Forb}(K_\omega)$ is not, because it contains the disjoint union of $K_n, n \in \mathbb{N}$, which no proper co-finite class contains. Thus Conjecture 10.2 is weaker than Thomas' conjecture. Beware however that Conjecture 10.2 implies the Graph Minor Theorem: any minor-closed class of finite graphs is shown to be UNCOF by letting C_n be its sub-class comprising the elements with at most n vertices.

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