Reachability in 3-VASS is Elementary

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Abstract

The reachability problem in 3-dimensional vector addition systems with states (3-VASS) is known to be PSpace-hard, and to belong to Tower. We significantly narrow down the complexity gap by proving the problem to be solvable in doubly-exponential space. The result follows from a new upper bound on the length of the shortest path: if there is a path between two configurations of a 3-VASS then there is also one of at most triply-exponential length. We show it by introducing a novel technique of approximating the reachability sets of 2-VASS by small semi-linear sets.

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1 Introduction

Petri nets are an established model of concurrent systems with extensive applications in various areas of theoretical computer science. For most algorithmic questions, the model is equivalent to *vector addition systems* with states (VASS in short). A d-dimensional VASS (d-VASS in short) is a finite automaton equipped additionally with a finite number d of nonnegative integer counters that are updated by transitions, under the proviso that the counter values can not drop below zero. Importantly, VASS have no capability to zero-test counters, and hence the model is not Turing-complete.

One of the central algorithmic problems for VASS is the *reachability problem* which asks, if in a given VASS there is a path (a sequence of executions of transitions) from a given source configuration (consisting of a state together with counter values) to a given target configuration:

VASS REACHABILITY PROBLEM
Input: VASS V, source and target configurations s, t.
Question: Is there a path from s to t?

Already in 1976, the problem was shown to be ExpSpace-hard by Lipton [21], and few years later decidability was shown by Mayr [22]. Later improvements [15, 16] simplified the construction, but no complexity upper bound was given until Leroux and Schmitz showed that the problem can be solved in Ackermannian complexity [18, 19]. At the same time the lower bound was lifted to Tower-hardness [8], and soon afterwards, in 2021, improved to Ackermann-hardness [9, 17].

Although the complexity of the reachability problem is now settled to be Ackermann-complete, various complexity questions remain still widely open. One of them is the complexity of the reachability problem parametrised by dimension, namely in d-dimensional VASS (d-VASS) for fixed $d \in \mathbb{N}$. Although the question has been investigated for a few decades, exact bounds are only known for dimensions 1 and 2 (both for unary or binary representations of numbers in counter updates). In the case of binary encoding, the reachability problem is NP-complete for 1-VASS [12] and PSpace-complete for 2-VASS [2], and in case of unary encoding the problem is NL-complete both for 1-VASS (folklore) and for 2-VASS [10]. For higher dimension almost nothing is known.

All the upper complexity bounds for dimension 1 or 2 were obtained by estimating the length of the shortest path, or of its representation. For unary 1-VASS, it is a folklore that if there is a path between two configurations then there is also one of polynomial length (see [3] for more demanding quadratic upper bound), which implies NL-completeness. For binary 1-VASS, a polynomial-size representation of the shortest path was provided by [12]. Concerning binary 2-VASS, already in 1979 Hopcroft and Pansiot showed that the reachability sets of 2-VASS are effectively semi-linear, and therefore the reachability problem is decidable [13]. Subsequently, 2-ExpTime upper complexity bound for binary 2-VASS was established by [14]. In [20] Leroux and Sutre showed that even the reachability relation is semi-linear, and that the relation is flattable, namely that it can be described by a finite number of simple expressions called *linear path schemes*. Only in 2015, careful examination of these linear path schemes led to exponential upper bound on the length of the shortest path, and consequently to PSpace upper bound [2]. Concerning unary 2-VASS, polynomial upper bound on the length of the shortest path was shown [1, 10], thus yielding NL-completeness.

Our understanding of the model drops drastically for dimensions larger than 2, as most of good properties admitted in dimension 1 or 2 vanish. For instance, already since the seminal paper [13] it is known that reachability sets of 3-VASS are not necessarily semi-linear. Investigation of 3-VASS was advocated by many papers cited above, e.g. [1, 2, 13], but until now no specific complexity bounds for 3-VASS are known, except for generic parametric bounds known for d-VASS in arbitrary fixed dimension $d \geq 3$. By [19], the reachability problem in d-VASS is in $\mathcal{F}_{d+4}$,[1] later improved to $\mathcal{F}_d$ [11]. In case of 3-VASS, this yields membership in $\mathcal{F}_3 = $ Tower. On the other hand, no lower bound is known for binary 3-VASS except for the PSpace lower bound inherited from binary 2-VASS (for unary 3-VASS, NP-hardness has been recently shown by [4]). The complexity gaps remains thus huge, namely between PSpace and Tower. As our main result, we narrow down this gap significantly.

Contribution. In this paper we investigate complexity of the reachability problem in 3-VASS. Our main result is the first elementary upper complexity bound for the problem:

▶ **Theorem 1.** *The reachability problem in* 3-VASS *is in 2-ExpSpace, under binary encoding.*

In particular, this refutes the natural conjecture that for every $d \geq 3$, the reachability problem for d-VASS is $\mathcal{F}_d$-complete and provides the first algorithm, which solves the problem for VASS with finite reachability sets faster than exhaustive search.

Our way to prove Theorem 1 is by bounding triply-exponentially the length of the shortest path between the given source and target configurations. This main technical result, formulated in Lemma 2 below, applies to *sequential* 3-VASS, which are sequences of strongly connected components $V_1, \ldots, V_k$ linked by single transitions $u_1, \ldots, u_{k-1}$ (see Figure 1; the rigorous definition is given in Section 2).

Given a VASS V and source and target configurations s, t, by SIZE(V, s, t) we mean the sum of absolute values of all the numbers occurring in transitions of V, s and t, plus the number thereof.

[1] The complexity class $\mathcal{F}_i$ corresponds to the i-th level of Grzegorczyk's fast-growing hierarchy [25].

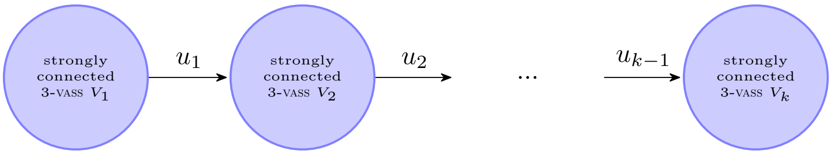

Figure 1 A sequential 3-VASS.

▶ **Lemma 2.** *If there is a path from s to t in a sequential* 3-VASS V*, then there is one of length at most* $\textsc{size}(V,s,t)^{2^{2^{\mathcal{O}(k)}}}$*, where k is the number of components of V.*

Therefore the length of the shortest path in a k-component 3-VASS is bounded by $M^{2^{2^{\mathcal{O}(k)}}}$, where $M = \textsc{size}(V,s,t)$ is the size of input, under unary encoding. This is the first bound on the shortest path in VASS of dimension higher than 2, that is not based on the size of (finite) reachability sets. Indeed, in a 3-VASS of size M, the size of finite reachability sets may be an arbitrary high tower of exponentials.

In consequence of Lemma 2, Theorem 1 follows immediately: the upper bound of Lemma 2 is triple-exponential in the size of input, irrespectively whether unary or binary encoding is used, which implies the same bound on norms of configurations along the shortest path. This yields a nondeterministic double-exponential space algorithm that first guesses a sequence of components leading from the source state to the target one, and then searches for a witnessing path. Note that the complexity bound under unary encoding is not better than under binary encoding.

Lemma 2 immediately yields further upper bounds for the reachability problem, when the number of components is fixed:

▶ **Corollary 3.** *For every fixed $k \geq 1$, the reachability problem in k-component* 3-VASS *is:*
- *in NL, under unary encoding,*
- *in PSPACE, under binary encoding.*

Indeed, for every fixed $k \geq 1$, the bound of Lemma 2 is polynomial with respect to unary input size. Therefore the length of the shortest path, as well as the norm of configurations along this path, are polynomially bounded in case of unary encoding, and exponentially bounded in case of binary encoding. This bounds yield membership in NL and PSPACE, respectively. Thus for every fixed $k \geq 1$, the complexity of k-component 3-VASS matches the complexity of 2-VASS.

Organisation of the paper. We start by introducing notation and basic facts in Section 2. Overview of the proof of our main result, Lemma 2, is presented in Section 3. In Section 4 we focus on 1-component 3-VASS, thus establishing the base of induction for Lemma 2. Next, in Section 5 we introduce the fundamental concept of polynomially approximable sets, and formulate our core technical result: reachability sets of 2-VASS are polynomially approximable. The result is then applied in the inductive proof of Lemma 2 in Section 6. We conclude in Section 7.

2 Preliminaries

Let $\mathbb{Q}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{>0}$ denote the set of all, nonnegative, and positive rationals, respectively, and likewise let $\mathbb{Z}, \mathbb{N}, \mathbb{N}_{>0}$ denote the respective sets of integers. For $a, b \in \mathbb{Z}$, $a \leq b$, let $[a,b]$ denote the set $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$. The jth coordinate of a vector $w \in \mathbb{Q}^d$ we write as w_j. Thus $w = (w_1, \ldots, w_d)$. For $q \in \mathbb{Q}$, by $\overrightarrow{q}$ we denote the constant vector $(q, \ldots, q) \in \mathbb{Q}^d$.

Vector addition systems with states. Let $d \in \mathbb{N}_{>0}$. A d-dimensional *vector addition system with states* (d-VASS in short) $V = (Q, T)$ consists of a finite set Q of states, and a finite set of transitions $T \subseteq Q \times \mathbb{Z}^d \times Q$. A *configuration* c of V consists of a state $q \in Q$ and a nonnegative vector $w \in \mathbb{N}^d$, and is written as $c = q(w)$. A transition $u = (q, v, q')$ induces *steps* $q(w) \xrightarrow{u} q'(w')$ between configurations, where $w' = w + v$. We refer to the vector $v \in \mathbb{Z}^d$ as the *effect* of the transition (q, v, q') or of an induced step. A *path* π in V is a sequence of steps with the proviso that the target configuration of every step matches the source configuration of the next one:

$$\pi \;=\; c_0 \xrightarrow{u_1} c_1 \longrightarrow \ldots \xrightarrow{u_n} c_n. \tag{1}$$

The *effect* $\text{EFF}(\pi) \in \mathbb{Z}^d$ of a path is the sum of effect of all steps, and its *length* is the number n of steps. We say that the path is *from c_0 to c_n*, call c_0, c_n source and target configuration, respectively, of the path, and write $c_0 \xrightarrow{\pi} c_n$. We also write $c \xrightarrow{*} c'$ if there is some path from c to c'. A path $q(v) \xrightarrow{*} q(v')$ in V with the same source and target state we call a *cycle*. A cycle is *simple* if the only equality of states along the cycle is the equality of source and target states. When dimension d is irrelevant, we write VASS instead of d-VASS.

By the *geometric* dimension of a d-VASS we mean the dimension of the vector space $\text{LIN}(V) \subseteq \mathbb{Q}^d$ spanned by effects of all its simple cycles. In the sequel we most often consider 2-VASS and 3-VASS, but also *geometrically 2-dimensional* 3-VASS, i.e., 3-VASS of geometric dimension at most 2.

Two paths π and π' can be *concatenated* (composed), written $\pi; \pi'$, if the target configuration of π equals the source one of π'. As long as it does not lead to confusion, we adopt a convention that when concatenating paths $\pi; \pi'$, the latter path π' is silently *moved* so that its source matches the target of π, under assumption that the source state of π' is the same as the target state of π. For instance, we write π^m to denote the m-fold concatenation of a cycle π, even if $\text{EFF}(\pi) \neq \overrightarrow{0}$.

We use the following notation for measuring size of representation of a VASS. By *norm* of a vector $v = (v_1, \ldots, v_d) \in \mathbb{Q}^d$, denoted $\text{NORM}(v)$, we mean the sum of absolute values of all numbers appearing in it: $\text{NORM}(v) = |v_1| + \ldots + |v_d|$; and by norm of a set of vectors P we mean the sum of norms of its elements: $\text{NORM}(P) = \sum\{\text{NORM}(v) \mid v \in P\}$. By *norm* of a configuration $q(w)$, or of a transition (q, w, q'), we mean the norm of its vector w. By *size* of a VASS V, denoted $\text{SIZE}(V)$, we mean the sum of norms of all its transitions, plus the number of transitions $|T|$. The *norm* of a VASS is the maximal norm of its transition.

We often implicitly extend a VASS V with source configuration s, or with a pair of source and target configurations s, t. Slightly overloading terminology, a pair (V, s) and a triple (V, s, t) we call a VASS too. For convenience we overload further and put $\text{SIZE}(V, s) = \text{SIZE}(V) + \text{NORM}(s)$ and $\text{SIZE}(V, s, t) = \text{SIZE}(V) + \text{NORM}(s) + \text{NORM}(t)$. The *reverse* of a VASS $V = (Q, T)$ is defined as $V^{\text{rev}} = (Q, T')$, where T' is obtained by reversing all transitions in T: $T' = \{(q', -v, q) \mid (q, v, q') \in T\}$. Overloading the notation again, we put $(V, s, t)^{\text{rev}} := (V^{\text{rev}}, t, s)$.

Given a VASS together with an initial configuration (V, s), we write $\text{REACH}(V, s)$ to denote the set of configurations t such that V has a path from s to t. For every state $q \in Q$ we write $\text{REACH}_q(V, s)$ to denote the set of vectors $w \in \mathbb{N}^d$ such that $q(w) \in \text{REACH}(V, s)$. We write shortly $\text{REACH}(s)$ and $\text{REACH}_q(s)$ if the VASS V is clear from the context. If $t \in \text{REACH}(s)$ we say that t is *reachable* from s. If $t + \Delta \in \text{REACH}(s)$ for some $\Delta \in \mathbb{N}^d$, we say that t is *coverable* from s.

We consider a variant of VASS, called $\mathbb{Z}$-VASS, where the nonnegativeness constraint is dropped. Syntactically, $\mathbb{Z}$-VASS is the same as VASS, namely consists of a finite set of states and a finite set of transitions (Q, T). Semantically, configurations of a $\mathbb{Z}$-VASS are $Q \times \mathbb{Z}^d$, while all definitions (path, reachability set, etc.) are the same as in case of VASS. Equivalently, we may also speak of $\mathbb{Z}$-configurations and $\mathbb{Z}$-paths of a VASS, i.e., configurations and paths where the nonnegativeness constraint is dropped. Note that every $\mathbb{Z}$-path $q(w) \xrightarrow{\pi} q'(w')$ may be *lifted* to become a path $q(w + \Delta) \xrightarrow{\pi} q'(w' + \Delta)$, for some $\Delta \in \mathbb{N}^d$.

Sequential VASS. We define the state graph of a VASS $V = (Q, T)$: nodes are states Q, and there is an edge (q, q') if T contains a transition (q, v, q') for some $v \in \mathbb{Z}^d$. A VASS is called *strongly connected* if its state graph is so. A VASS $V = (Q, T)$ is called *sequential*, if it can be partitioned into a number of strongly connected VASS $V_1 = (Q_1, T_1), \ldots, V_k = (Q_k, T_k)$ with pairwise disjoint state spaces, and

$k-1$ transitions $u_i = (q_i, v_i, q_i')$, for $i \in [1, k-1]$, where $q_i \in Q_i$ and $q_i' \in Q_{i+1}$ (recall Figure 1). Thus $Q = Q_1 \cup \ldots \cup Q_k$ and $T = T_1 \cup \ldots \cup T_k \cup \{u_1, \ldots, u_{k-1}\}$. We call V a *k-component* sequential VASS, or k-component VASS in short, and write down succinctly as $V = (V_1)u_1(V_2)u_2 \ldots u_{k-1}(V_k)$. The VASS $V_1, \ldots, V_k$ are called *components*, and transitions $u_1, \ldots, u_{k-1}$ *bridges*. By definition, a 1-component sequential VASS is just a strongly connected VASS.

Integer solutions of linear systems. We will intensively use the following immediate corollary of [23] (see also [5, Prop. 4]):

▶ **Lemma 4** ([5], Prop. 4). *Consider a system $A \cdot x = b$ of m Diophantine linear equations with n unknowns, where absolute values of coefficients are bounded by N. Every pointwise minimal nonnegative integer solution has norm at most $\mathcal{O}(nN)^m$.*

Diagonal property. We prove that if there is a path in a VASS achieving a large value on every coordinate, then there is a path of bounded length that achieves *simultaneously* large values on all coordinates.[2] We consider a general case of arbitrary dimension, as we believe it is of an independent interest, but in the sequel we will use it only for dimension $d = 3$.

▶ **Lemma 5.** *For every $d \in \mathbb{N}$ there are nondecreasing polynomials P_d, R_d such that for every d-VASS (V, s) of norm N, with n states, and for every $U \in \mathbb{N}$, if V has a path from s that for every $i \in [1, d]$ contains a configuration $q(w_1, \ldots, w_d)$ with $w_i \geq P_d(n, N, U)$, then V has also a path $s \xrightarrow{*} q(w_1, \ldots, w_d)$ of length at most $R_d(n, N, U)$ such that $w_i \geq U$ for every $i \in [1, d]$.*

Length-bound on shortest path. A function $f : \mathbb{N} \to \mathbb{N}$ is called *nondecreasing* if $f(n) \geq n$ for every $n \in \mathbb{N}$, and $f(n) < f(m)$ for all $n < m$. Functions used in the sequel are most often nondecreasing. We say that a class $\mathcal{C}$ of VASS or $\mathbb{Z}$-VASS is *length-bounded* by a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ if for every (V, s, t) in $\mathcal{C}$, if $s \xrightarrow{*} t$ then $s \xrightarrow{\pi} t$ for some path π of length at most $f(\textsc{size}(V, s, t))$. A class $\mathcal{C}$ which is length-bounded by some nondecreasing polynomial we call *polynomially length-bounded*. It is known that 2-VASS have this property:

▶ **Lemma 6** ([1], Theorem 3.2). 2-VASS *are polynomially length-bounded.*[3]

As a corollary of Lemmas 6 and 4, respectively, we derive the property for geometrically 2-dimensional 3-VASS (using [27, Lemma 5.1]) and 3-$\mathbb{Z}$-VASS, respectively:

▶ **Lemma 7.** *Geometrically 2-dimensional* 3-VASS *are polynomially length-bounded.*

▶ **Lemma 8.** 3-$\mathbb{Z}$-VASS *are polynomially length-bounded.*

Lemma 2 states that there exists a constant $C \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, the k-component 3-VASS are length-bounded by the function $M \mapsto M^{2^{2^{C \cdot k}}}$. Therefore for every fixed $k \in \mathbb{N}$, the k-component 3-VASS are polynomially length-bounded, even if the degree of polynomial grows doubly exponentially in k.

3 Overview

In this section we present an overview of the proof of our main result, namely of Lemma 2. The proof proceeds by an induction on the number k of components in a sequential 3-VASS. The main idea is that either the situation is easy (a short path can be obtained by lifting up a $\mathbb{Z}$-path) or the

[2] Lemma 5 is inspired by [19, Lemma 4.13], but the statement and the proof are different.
[3] [1] adopts a slightly different, but equivalent up to a constant multiplicative factor, definition of norm and size.

first component can be transformed into a finite union of essentially two-dimensional VASS (more precisely, finite union of geometrically 2-dimensional 3-VASS), each of size bounded polynomially. This transformation is shown in Lemma 28. The induction base is shown in Section 4. We present the proof of the one component case in detail, as it illustrates the main concepts of the proof of Lemma 28, but in a much simpler setting. When the first component is transformed into essentially a 2-VASS, we can use the fact that reachability sets in 2-VASS are semi-linear and the size of the semi-linear representation is at most exponential [2] (the result is true as well in geometrically 2-dimensional 3-VASS). This fact can be exploited to reduce reachability for k-component 3-VASS to reachability for $(k-1)$-component 3-VASS of exponentially larger size, the details of this reduction are explained in Section 5 in the paragraph about the idea of the proof of Lemma 2. However, if we use semi-linear sets, the exponential blowup in unavoidable and this approach gives us a TOWER algorithm resulting from a linear number of exponential blowups (thus not better than [11]). In order to improve the complexity we introduce a novel notion of suitable over- and under-approximations of semi-linear sets. One of our key technical contributions is Lemma 20 stating that reachability sets of 2-VASS can be well approximated. Intuitively speaking, the precision of the approximation has to be good enough for correctness of the inductive proof; the better the precision, the bigger the representation of approximants gets. This approach allows us to reduce reachability for k-component 3-VASS to reachability in $(k-1)$-component 3-VASS of size which is not anymore exponential, but is polynomial in B, where B is the minimal length of a path in $(k-1)$-component 3-VASS. This means that n^m bound on the minimal path length for $(k-1)$-component 3-VASS implies roughly a n^{m^2} bound on the minimal path length for k-component 3-VASS. The transformation $n^m \mapsto n^{m^2}$ applied linear number of times results in triply-exponential upper bound for the minimal length of a path in 3-VASS.

4 1-component 3-VASS are polynomially length-bounded

This section is devoted to the proof of the induction base for the proof of Lemma 2:

▶ **Lemma 9.** *1-component 3-VASS are polynomially length-bounded.*

We also develop a framework to be exploited in the induction step in Section 6. We may safely restrict to 3-VASS of geometric dimension 3, as otherwise Lemma 7 immediately implies Lemma 9.

Case distinction. A 3-VASS (V, s, t), where $s = p(w)$ and $t = p'(w')$, is *forward-diagonal* if $p(w) \xrightarrow{*} p(w + \Delta)$ in V for some $\Delta \in (\mathbb{N}_{>0})^3$. Symmetrically, (V, s, t) is *backward-diagonal* if $(V, t, s)^{\text{rev}}$ is diagonal, i.e., if $p'(w' + \Delta') \xrightarrow{*} p'(w')$ in V for some $\Delta' \in (\mathbb{N}_{>0})^3$. Finally, V is *diagonal* if it is both forward- and backward-diagonal. Obviously, the vectors Δ and Δ' need not be equal in general.

Let $E = \{e_1, \ldots, e_n\} \subseteq \mathbb{Z}^3$ be the effects of simple cycles of V. We define the (rational) open cone generated by this set to contain all positive rational combinations of vectors from E:

$$\textsc{Cone}(V) \;=\; \{r_1 \cdot e_1 + \ldots + r_n \cdot e_n \mid r_1, \ldots, r_n \in \mathbb{Q}_{>0}\} \;\subseteq\; \textsc{Lin}(V).$$

$\textsc{Cone}(V)$ is thus an open cone inside $\textsc{Lin}(V)$. A 1-component 3-VASS V is called *wide* if $(\mathbb{Q}_{>0})^3 \subseteq \textsc{Cone}(V)$, i.e., if $\textsc{Cone}(V)$ includes the whole positive orthant.

Let $\textsc{Len}(V, s, t)$ denote the set of lengths of paths $s \xrightarrow{*} t$ in V. We need to argue that there is a nondecreasing polynomial Q such that every 1-component 3-VASS (V, s, t) with a path $s \xrightarrow{*} t$, has such path of length at most $Q(M)$, where $M = \textsc{size}(V, s, t)$. We split the proof into three cases:

1. If (V, s, t) is diagonal and wide, we exploit the fact that 3-$\mathbb{Z}$-VASS are polynomially length-bounded, and use diagonality and wideness to lift a short $\mathbb{Z}$-path into a path.
2. If (V, s, t) is diagonal but non-wide, we show that (V, s, t) is *length-equivalent* to a geometrically 2-dimensional 3-VASS $(\overline{V}, \overline{s}, \overline{t})$ of polynomially larger size, namely $\textsc{Len}(V, s, t) = \textsc{Len}(\overline{V}, \overline{s}, \overline{t})$.

3. Finally, if (V, s, t) is non-diagonal, we show that (V, s, t) is *length-equivalent* to a set of three geometrically 2-dimensional 3-VASS $\{(V_1, s_1, t_1), (V_2, s_2, t_2), (V_3, s_3, t_3)\}$, namely $\textsc{Len}(V, s, t) = \textsc{Len}(V_1, s_1, t_1) \cup \textsc{Len}(V_2, s_2, t_2) \cup \textsc{Len}(V_3, s_3, t_3)$ of polynomially larger size.

In the two latter cases we rely on the fact that geometrically 2-dimensional 3-VASS are polynomially length-bounded (Lemma 7). In consequence, Q is to be the sum of polynomials claimed in the respective cases. In the sequel let (V, s, t) be a fixed 1-component 3-VASS with $s \xrightarrow{*} t$, where $s = p(w)$ and $t = p'(w')$.

Case 1. (V, s, t) is diagonal and wide. By diagonality, $p(w) \xrightarrow{\pi} p(w + \Delta)$ and $p'(w' + \Delta') \xrightarrow{\pi'} p'(w')$ for some $\Delta, \Delta' \in (\mathbb{N}_{>0})^3$.

Let P be a nondecreasing polynomial witnessing Lemma 8, i.e., 3-$\mathbb{Z}$-VASS are length-bounded by P. As there is a path $s \xrightarrow{*} t$ in V, there is also a $\mathbb{Z}$-path $s \xrightarrow{*} t$, and by Lemma 8 there is a $\mathbb{Z}$-path $s \xrightarrow{\sigma} t$ of length at most $P(M)$. The maximal norm N of $\mathbb{Z}$-configurations along σ is thus bounded by $M \cdot P(M)$, as every step may update counters by at most M.

By diagonality, the configuration $p(w + \overrightarrow{1})$ is coverable in V from s, and symmetrically the configuration $p'(w' + \overrightarrow{1})$ is coverable in V^{rev} from t. Due to the upper bound of Rackoff [24, Lemma 3.4], there is a nondecreasing polynomial R such that in every 3-VASS of size m, the length of a covering path is at most $R(m)$. Therefore the lengths of both paths $p(w) \xrightarrow{\pi} p(w + \Delta)$ and $p'(w' + \Delta') \xrightarrow{\pi'} p'(w')$ in V, where $\Delta, \Delta' \in (\mathbb{N}_{>0})^3$, may be assumed to be at most $R(M)$. We argue that there is a cycle from the source configuration $p(w)$ that increases w by some multiplicity of Δ':

▶ **Lemma 10.** *There is a path $p(w) \xrightarrow{\delta} p(w + \ell \cdot \Delta')$ of length $R(M)^{\mathcal{O}(1)}$, for some $\ell \in \mathbb{N}_{>0}$.*

Before proving the lemma we use it to complete Case 1. We build a path $p(w) \xrightarrow{*} p'(w')$ by concatenating 3 paths given below. The first one is δ given by Lemma 10. Note that ℓ is necessarily also bounded by $R(M)^{\mathcal{O}(1)}$. We replace ℓ by its sufficiently large multiplicity to enforce $\ell \geq M \cdot P(M)$, which makes the length of δ and ℓ only bounded by $P(M) \cdot R(M)^{\mathcal{O}(1)}$. The multiplicity guarantees that the $\mathbb{Z}$-path $p(w) \xrightarrow{\sigma} p'(w')$, lifted by $\ell \cdot \Delta'$, becomes a path:

$$p(w + \ell \cdot \Delta') \xrightarrow{\sigma} p'(w' + \ell \cdot \Delta'),$$

The length of σ is bounded by $P(M)$. Finally, let $\delta' = (\pi')^\ell$ be the ℓ-fold concatenation of the cycle π':

$$p'(w' + \ell \cdot \Delta') \xrightarrow{\delta'} p'(w').$$

The length of this path is bounded by $\ell \cdot R(M) \leq P(M) \cdot R(M)^{\mathcal{O}(1)}$. We concatenate the three paths, $\tau := \delta; \sigma; \delta'$, to get a required path

$$p(w) \xrightarrow{\tau} p'(w')$$

of length bounded by $P(M) \cdot R(M)^{\mathcal{O}(1)}$. It thus remains to prove Lemma 10 in order to complete Case 1.

Proof of Lemma 10. Let π_1 be a cycle that visits all states (it exists since the considered VASS is strongly connected), and let $\Delta_1 \in \mathbb{Z}^3$ be its effect. Relying on $\Delta \in (\mathbb{N}_{>0})^3$, take a sufficiently large multiplicity $m \in \mathbb{N}_{>0}$ so that $\widetilde{\pi} = \pi^m; \pi_1$ is a path with nonnegative effect. The path $\widetilde{\pi}$ is a cycle and its effect is $\widetilde{\Delta} = m \cdot \Delta + \Delta_1 \in \mathbb{N}^3$.

As $\Delta' \in (\mathbb{N}_{>0})^3$, there is $\ell' \in \mathbb{N}_{>0}$ such that $\ell' \cdot \Delta' - \widetilde{\Delta} \in (\mathbb{Q}_{>0})^3$, and hence, by wideness of (V, s), we have $\ell' \cdot \Delta' - \widetilde{\Delta} \in \textsc{Cone}(V)$, namely

$$\ell' \cdot \Delta' - \widetilde{\Delta} = r_1 \cdot e_1 + \ldots + r_n \cdot e_n$$

for some positive rationals $r_1, \ldots, r_n \in \mathbb{Q}_{>0}$. By Carathéodory's Theorem [26, p.94], $\ell' \cdot \Delta' - \widetilde{\Delta}$ is a combination of some 3 vectors among $e_1, \ldots, e_n$, say e_1, e_2, e_3:

$$\ell' \cdot \Delta' - \widetilde{\Delta} \;=\; r_1 \cdot e_1 + r_2 \cdot e_2 + r_3 \cdot e_3,$$

for some positive rationals $r_1, r_2, r_3 \in \mathbb{Q}_{>0}$. Therefore, the system of 3 equations

$$\ell \cdot (\ell' \cdot \Delta' - \widetilde{\Delta}) \;=\; r_1 \cdot e_1 + r_2 \cdot e_2 + r_3 \cdot e_3,$$

with unknowns ℓ, r_1, r_2, r_3, has a positive integer solution. We rewrite the system to:

$$\ell\ell' \cdot \Delta' - \ell m \cdot \Delta \;=\; \ell \cdot \Delta_1 + r_1 \cdot e_1 + r_2 \cdot e_2 + r_3 \cdot e_3. \tag{2}$$

Let σ_i be simple cycle of effect e_i, for $i \in [1, 3]$. Let σ be a $\mathbb{Z}$-path that starts (and ends) in state p and consists of ℓ-fold concatenation of the cycle π_1, with attached (r_1)-fold concatenation of σ_1, (r_2)-fold concatenation of σ_2, and (r_3)-fold concatenation of σ_3 (since π_1 visits all states, this is possible). The effect of σ is the right-hand side of (2), and therefore σ is a $\mathbb{Z}$-path from $p(w + \ell m \cdot \Delta)$ to $p(w + \ell\ell' \cdot \Delta')$:

$$p(w + \ell m \cdot \Delta) \xrightarrow{\sigma} p(w + \ell\ell' \cdot \Delta').$$

It need not be a path in general, and therefore we are going to lift it. Let $k \in \mathbb{N}_{>0}$ be a multiplicity large enough so that σ becomes a path when lifted by $(k-1)\ell m \cdot \Delta$, i.e., when starting in $p(w + k\ell m \cdot \Delta)$, and also becomes a path when lifted by $(k-1)\ell\ell' \cdot \Delta'$, i.e., when ending in $p(w + k\ell\ell' \cdot \Delta')$. In this case, the k-fold concatenation of σ is also a path:

$$p(w + k\ell m \cdot \Delta) \xrightarrow{\sigma^k} p(w + k\ell\ell' \cdot \Delta'), \tag{3}$$

since all points visited in the inner iterations of σ are bounded from both sides by corresponding points visited in the first and the last iteration of σ. Precomposing this path with $p(w) \xrightarrow{\pi^{k\ell m}} p(w + k\ell m \cdot \Delta)$ yields a path $p(w) \xrightarrow{*} p(w + k\ell\ell' \cdot \Delta')$, as required.

We now (roughly) bound the magnitudes of all items involved in the above reasoning by a constant power of $R(M)$. W.l.o.g. we may assume that the cycle π_1 uses every transition at most $|Q| \leq M$ times, and thus both the length and norm of the effect of π_1 are bounded by M^2. In consequence, π_1 may decrease counters by at most M^2. Therefore $m \leq M^2$, and norms of vectors $\Delta, \Delta', \widetilde{\Delta}$ are all bounded by $\mathcal{O}(M^3 \cdot R(M))$. The effects e_1, e_2, e_3 of simple cycles $\sigma_1, \sigma_2, \sigma_3$ are at most M, as no transition repeats along a simple cycle. Therefore by Lemma 4, the system (2) has a solution (ℓ, r_1, r_2, r_3) of norm at most $D = \mathcal{O}(M^3 \cdot R(M))^3 = \mathcal{O}(M^9 \cdot R(M)^3)$, and σ has length at most $M^2 \cdot D$ (since π_1 has length at most M^2). Therefore we deduce $k \leq M^3 \cdot D$, and hence the path (3) has length at most $M^5 \cdot D^2$. In consequence of the above bounds, $k\ell m \leq M^5 \cdot D^2$, and the final path $p(w) \xrightarrow{*} p(w + k\ell v)$ has length at most $R(M) \cdot M^5 \cdot D^2 \leq \mathcal{O}(R(M)^{30}) \leq R(M)^{\mathcal{O}(1)}$. ◀

Case 2. (V, s, t) **is non-wide.** Every non-zero vector $a = (a_1, a_2, a_3) \in \mathbb{Z}^3$ defines an open half-space

$$H_a \;=\; \{x \in \mathbb{Q}^3 \mid a \diamond x > 0\},$$

where $a \diamond x = a_1x_1 + a_2x_2 + a_3x_3$ stands for the inner product of $x = (x_1, x_2, x_3)$ and a. As V is assumed to be of geometric dimension 3, $\textsc{Cone}(V)$ is an intersection of open half-spaces:

▷ Claim 11. $\textsc{Cone}(V)$ is an intersection of finitely many open half-spaces H_a, with $\textsc{norm}(a) \leq D := \mathcal{O}(M^2)$.

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Proof. Norms of vectors generating $\text{CONE}(V)$ — i.e., effects of simple cycles — are at most M, as no transition repeats along a simple cycle. Consider vectors a orthogonal to some of the facets of $\text{CONE}(V)$, i.e., orthogonal to two of the vectors generating $\text{CONE}(V)$. The vector a is thus an integer solution of a system of 2 linear equations with 3 unknowns, where absolute values of coefficients are bounded by M. By Lemma 4, there is such an integer solution with $\text{NORM}(a) \leq \mathcal{O}(M^2)$. This completes the proof. ◀

As V is non-wide, due to Claim 11 we know that $\text{CONE}(V)$ is a *non-empty* intersection of half-spaces H_a. Therefore for some of these H_a we have $\text{CONE}(V) \subseteq H_a$, i.e., all points $x \in \text{CONE}(V)$ have positive inner product $a \diamond x > 0$. This implies that the value of inner product with a may not decrease along any cycle in V:

▷ Claim 12. The effect $\delta \in \mathbb{Z}^3$ of every simple cycle has nonnegative inner product $a \diamond \delta \geq 0$.

In consequence, on every path $s \xrightarrow{*} t$ the value of inner product with a is polynomially bounded:

▷ Claim 13. Every configuration $q(x)$ on a path from s to t satisfies $-B \leq a \diamond x \leq B$, where $B := \mathcal{O}(M \cdot D)$.

Proof. Let $s = p(w)$ and $t = p'(w')$, and let $b = a \diamond w$ and $b' = a \diamond w'$. Every path $s \xrightarrow{*} t$ may be decomposed into simple cycles, whose effect may only preserve or increase the inner product with a, plus a short path without cycles, and hence without repetitions of a transitions. The effect of the latter path is thus in $[-M, M]$. Therefore, as inner product may at most multiply norms, every configuration $q(x)$ on a path from s to t satisfies $b - M \cdot D \leq a \diamond x \leq b' + M \cdot D$. Knowing that $\text{NORM}(a) \leq D$, $\text{NORM}(w), \text{NORM}(w') \leq M$, and that inner product may at most multiply norms, by Claim 11 we deduce

$$-M \cdot D \leq b, b' \leq M \cdot D,$$

and therefore $-2 \cdot M \cdot D \leq a \diamond x \leq 2 \cdot M \cdot D$, which implies the claim. ◀

We define a geometrically 2-dimensional 3-VASS $\overline{V} = (\overline{Q}, \overline{T})$ by extending states with the possible values of inner product with a (bounded polynomially by Claim 13). We call $\overline{V}$ the *(a, B)-trim* of V. The set of states $\overline{Q}$ contains states of the form q_b, where $q \in Q$ and $-B \leq b \leq B$, with the intention that every configuration $c = q(x)$ of V has a corresponding configuration $\overline{c} = q_b(x)$ in $\overline{V}$, where $a \diamond x = b$. Therefore, for each transition $(q, v, q') \in T$ and for all $b, b' \in [-B, B]$ such that $b + a \diamond v = b'$, we add to $\overline{T}$ the transition

$$\left(q_b, v, q'_{b'}\right). \tag{4}$$

▷ Claim 14. $\overline{V}$ is a geometrically 2-dimensional 3-VASS.

Proof. By construction, the effect of each cycle in $\overline{V}$ is orthogonal to a, and therefore $\text{LIN}(\overline{V})$ is included in a 2-dimensional vector space. ◀

Relying on Claim 13, paths $s \xrightarrow{*} t$ in V have corresponding paths in $\overline{V}$, and hence we get:

▷ Claim 15. $\text{LEN}(V, s, t) = \text{LEN}(\overline{V}, \overline{s}, \overline{t})$.

Finally, we argue that the size of $\overline{V}$ is bounded polynomially with respect to the size of V:

▷ Claim 16. $\text{SIZE}(\overline{V}) \leq R(M) = \mathcal{O}(M \cdot B)$.

Proof. Recalling Claims 11 and 13, namely $\text{NORM}(a) \leq D$, $B = \mathcal{O}(M \cdot D)$, we deduce that transitions (4) contribute at most $(2B+1)M \leq \mathcal{O}(M \cdot B)$ to the size of $\overline{V}$. ◀

We are now prepared to complete Case 2. Let P be the polynomial witnessing Lemma 7, i.e., geometrically 2-dimensional 3-VASS are length-bounded by P. As V has a path $s \xrightarrow{*} t$, By Claim 15, $\overline{V}$ has a path $\overline{s} \xrightarrow{*} \overline{t}$. By Lemma 7, $\overline{V}$ has thus a path $\overline{s} \xrightarrow{*} \overline{t}$ of length at most $P(\textsc{size}(\overline{V}))$, i.e., relying on Claim 16, of length at most $P(\mathcal{O}(M^2 \cdot D)) = \mathcal{O}(P(M^4))$. By Claim 15 again, we get a path $s \xrightarrow{*} t$ in V of length $\mathcal{O}(P(M^4))$. This completes Case 2.

Case 3. (V, s, t) is non-diagonal. W.l.o.g. assume that (V, s) is not forward-diagonal (otherwise replace V by V^{rev}). Therefore for all states q the configuration $s' = q(w + (\overrightarrow{M+1}))$ is not coverable from $s = p(w)$. Indeed, using strong-connectedness of V, if s' were coverable from s then the configuration $p(w + \overrightarrow{1})$ would be coverable form $p(w)$, by extending the covering path of s' with an arbitrary shortest path back to state p (that cannot decrease a counter by more than M), which would contradict forward non-diagonality.

By Lemma 5, in every path from s, some coordinate $j \in [1, 3]$ is bounded by $B := P_3(M, M, M+1)$ (we take M as an upper bound for n and N, relying on P_3 being nondecreasing, and take $U = M+1$). This property allows us, intuitively speaking, to describe all the paths of V by paths of three geometrically 2-dimensional 3-VASS V_j, for $j \in [1, 3]$, where V_j behaves exactly like V except that dimension j is additionally kept in state. Formally, let $V_j := (Q_j, T_j)$, where

$$
\begin{aligned}
Q_j &= \{q_b \mid q \in Q,\ b \in [0, B]\} \\
T_j &= \{(q_b, v, q'_{b'}) \mid (q, v, q') \in T,\ b' = b + v_j\}.
\end{aligned}
$$

The source and target configurations in V_j are $s_j = p_{w_j}(w)$ and $t_j = p'_{w'_j}(w')$, and there is a tight correspondence between paths in V and paths in V_1, V_2, V_3:

▷ Claim 17. $\textsc{Len}(V, s, t) = \textsc{Len}(V_1, s_1, t_1) \cup \textsc{Len}(V_2, s_2, t_2) \cup \textsc{Len}(V_3, s_3, t_3)$.

The size of each of V_j is bounded polynomially with respect to the size of V:

▷ Claim 18. The size of each of V_j is at most $R(M) = \mathcal{O}(M \cdot B)$.

Let P be the polynomial witnessing Lemma 7. As $p(w) \xrightarrow{*} p'(w')$ in V, by Claim 17 there is a path $p_{w_j}(w) \xrightarrow{*} p'_{w'_j}(w')$ in V_j for some $j \in [1, 3]$. Therefore, by Lemma 7 there is such a path of length at most $P(R(M))$ which, again using Claim 17, implies a path $p(w) \xrightarrow{*} p'(w')$ in V of the same length. This polynomial bound completes Case 3, and hence also the proof of Lemma 9.

5 Polynomially approximable reachability sets

In this section we introduce the crucial concept of *polynomially approximable* sets. In order to motivate it, we start by sketching the overall idea of the proof of Lemma 2 (given in Section 6).

Given a finite set $P \subseteq \mathbb{N}^d$ and $B \in \mathbb{N}$, we set:

$$
\begin{aligned}
P^* &= \{p_1 + \ldots + p_k \mid k \geq 0,\ p_1, \ldots, p_k \in P\} \\
P^{\leq B} &= \{p_1 + \ldots + p_k \mid B \geq k \geq 0,\ p_1, \ldots, p_k \in P\}.
\end{aligned}
$$

Sets of the form $b + P^* = \{b + p \mid p \in P^*\}$, for $b \in \mathbb{N}^d$ and finite $P \subseteq \mathbb{N}^d$, are called *linear*, and finite unions of linear sets are called *semi-linear*.

Idea of the proof of Lemma 2. Let $V = (V_1)u_1(V_2)u_2 \ldots u_{k-1}(V_k)$ be a k-component 3-VASS that has a path $s = q(w) \xrightarrow{*} q'(w') = t$. If V is diagonal and wide, we use the pumping cycles $q(w) \xrightarrow{*} q(w + \Delta)$ in V_1 and $q'(w' + \Delta') \xrightarrow{*} q'(w')$ in V_k to lift a $\mathbb{Z}$-path $s \xrightarrow{*} t$, polynomially length-bounded due to Lemma 8, until it becomes a path (as in Case 1 in the proof of Lemma 9).

On the other hand, if V is non-diagonal or non-wide, our strategy is to reduce the number of components by 1, and to rely on the induction assumption for $k-1$, by replacing the first component

V_1 by one of finitely many geometrically 2-dimensional 3-VASS (as in Cases 2 and 3 of the proof of Lemma 9). Relying on the fact that the reachability sets in a geometrically 2-dimensional 3-VASS are semi-linear [11], the proof could go as follows (yielding however only the already known TOWER upper bound [11]). Using any of the linear sets $L = a + P^*$ describing the set $\text{REACH}_{q_2}(V, s)$, where q_2 is the source state of the second component V_2, transform V into a $(k-1)$-component 3-VASS V' by dropping the first component V_1 and the first bridge u_1, and by adding to the remaining $(k-1)$-component 3-VASS $(V_2)u_2 \dots u_{k-1}(V_k)$ the self-looping transitions (q_2, r, q_2), one for every period $r \in P$. The source configuration of V' is $s' = q_2(a)$, i.e., its vector is the base of L. The transformation preserves behaviour of V. In one direction, a path $s \xrightarrow{*} q_2(x) \xrightarrow{*} t$ in V crossing through $q_2(x)$ for some $x \in L$ has a corresponding path $q_2(a) \xrightarrow{*} q_2(x) \xrightarrow{*} t$ in V'. Conversely, each path $q_2(a) \xrightarrow{*} q_2(x) \xrightarrow{*} t$ in V' gives rise to a path $s \xrightarrow{*} t$ in V, by replacing executions of the self-looping transitions $q_2(a) \xrightarrow{*} q_2(x)$ (w.l.o.g. executed in the beginning), by a path $s \xrightarrow{*} q_2(x)$ in V_1, bounded polynomially due to Lemma 9. However, $\text{SIZE}(V')$ may blow-up exponentially with respect to $\text{SIZE}(V)$, as bases and periods of L are only bounded exponentially, and therefore this approach could only yield a k-fold exponential bound on the length of the shortest path in k-component 3-VASS.

Polynomially approximable sets are designed as a remedy against the k-fold exponential blowup. The idea is to measure the norms of base and periods of a semi-linear set L parametrically with respect to, intuitively speaking, the prospective length B of a path $s' \xrightarrow{*} t$ in V'. This allows us to control the blow-up of size of V', also parametrically with B, but requires going outside of semi-linear sets and considering their *B-approximations*, namely sets sandwiched between $a + P^{\leq B}$ and $a + P^*$, good enough for correctness of the above-described transformation of V to V'. As the outcome, the exponent of our bound on length of the shortest path in k-component 3-VASS is, roughly speaking, square of the exponent of the respective bound in $(k-1)$-component 3-VASS. For k-component 3-VASS this yields exponent doubly-exponential in k, and hence the bound triply-exponential in k. The rigorous reasoning is given in the proof of Lemma 25 in Section 6.

Polynomially approximable sets. Let $A, B \in \mathbb{N}$. By a *B-approximation* of a linear set $a + P^* \subseteq \mathbb{N}^d$ we mean any set $S \subseteq \mathbb{N}^d$ satisfying $a + P^{\leq B} \subseteq S \subseteq a + P^*$. A set $X \subseteq \mathbb{N}^d$ is *(A, B)-approximately semi-linear* if it is a finite union of:

- linear sets $a + P^* \subseteq \mathbb{N}^d$ with $\text{NORM}(a) \leq B \cdot A$ and $\text{NORM}(P) \leq A$; and
- B-approximations of linear sets $a + P^* \subseteq \mathbb{N}^d$ with $\text{NORM}(a) \leq A$ and $\text{NORM}(P) \leq A$.

Thus X either includes B-approximation of a linear set $a + P^*$, whose norm of base is bounded by A, or X includes a whole linear set $a + P^*$, whose norm of base is only bounded by $B \cdot A$. In both cases, norms of periods are bounded by A.

We say that a class $\mathcal{C}$ of d-VASS is *F-approximable* for a function $F : \mathbb{N} \to \mathbb{N}$, if for every VASS (V, s) in $\mathcal{C}$, its state q, and $B \in \mathbb{N}$, the set $\text{REACH}_q(V, s)$ is $(F(M), B)$-approximately semi-linear, where $M = \text{SIZE}(V, s)$. The class $\mathcal{C}$ is *polynomially approximable* if it is F-approximable, for some nondecreasing polynomial F.

▶ **Example 19.** For $k \geq 1$, let V_k be a $(2k)$-component 2-VASS, where each component has just one state q_i and one transition: $(q_i, (-1, 2), q_i)$ for odd i, and $(q_i, (2, -1), q_i)$ for even i. Bridge transitions are $(q_i, (0, 0), q_{i+1})$. Figure 2 shows V_2 (left) and a path in V_2 from $s = q_1(1, 0)$ to $t = q_4(16, 0)$ together with the reachability set $\text{REACH}_{q_4}(V_2, s)$ (middle). In general,

$$X_k := \text{REACH}_{q_{2k}}(V_k, s) \;=\; \{(x_1, x_2) \mid x_1 + 2x_2 \leq 4^k,\ x_1 + 2x_2 \equiv 1 \mod 3\}. \tag{5}$$

Even if the size of the reachability set is exponential in k, for small (x_1, x_2) it is periodic and the periods are small. The set X_k can be over-approximated by $A + P^*$ for $A = \{(1, 0), (2, 1), (0, 2)\}$ and $P = \{(0, 3), (3, 0)\}$ (shown on the right of Figure 2), namely for every $k \geq 1$ and $B \in \mathbb{N}$, the

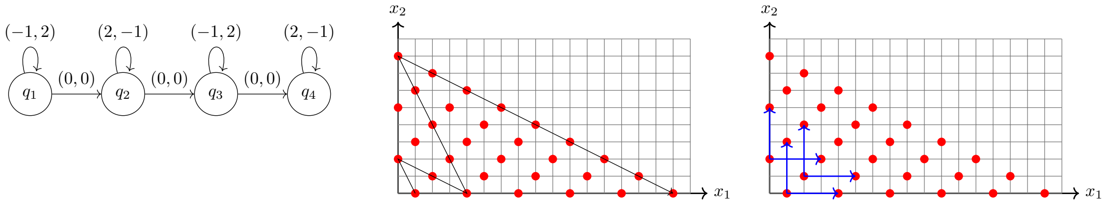

Figure 2 Left: 4-component 2-VASS V_2. Middle: the set $\text{REACH}_{q_4}(V_2, q_1(1,0))$ and a path $q_1(1,0) \xrightarrow{*} q_4(16,0)$. Right: bases and periods of an over-approximating semi-linear set $A + P^*$.

set X_k is (8, B)-approximately semi-linear. For illustration, consider $Y := X_k \cap ((1,0) + P^*)$. If $(1,0) + P^{\leq B} \subseteq X_k$ then Y is a B-approximation of $(1,0) + P^*$ with $\text{NORM}((1,0)), \text{NORM}(P) \leq 3 \leq 8$. Otherwise, there is some $(v_1, v_2) \in \big((1,0) + P^{\leq B}\big) \setminus X_k$, and then B is larger than 4^k:

$$4^k < v_1 + 2v_2 \leq 2(v_1 + v_2) \leq 2(1 + 3B) \leq 8B.$$

Therefore by (5), each $(x_1, x_2) \in Y$ satisfies $\text{NORM}(x_1, x_2) = x_1 + x_2 \leq x_1 + 2x_2 \leq 4^k < 8B$, and thus Y, seen as a union of singletons, is a union of linear sets with norm of base bounded by $8B$ and empty set of periods. In both cases, Y is (8, B)-approximately semi-linear.

In our subsequent reasoning we rely on the core technical fact:

▶ **Lemma 20.** 2-VASS *are polynomially approximable.*

We sketch here the intuition behind Lemma 20, since it is one of our main technical contributions. We first show that it is enough to show Lemma 20 for a simple class of 2-VASS, called 2-SLPS, which are of the form $\alpha_0 \beta_1^* \alpha_2 \ldots \alpha_{k-1} \beta_k^* \alpha_k$, where α_i are fixed sequences of transitions and the loops β_i are single transitions. This reduction uses standard techniques, namely Theorem 1 in [2] stating that the reachability relation of a 2-VASS can be expressed as a union of reachability relations of a 2-LPS (2-LPS are 2-SLPS without the assumption that β_i are single transitions) and Theorem 15 in [10] providing the reduction from 2-LPS to 2-SLPS. Next, we simplify the 2-SLPS even more, using Theorem 4.16 in [4], which states that any two vectors reachable by an 2-SLPS can be reached also by a path of the 2-SLPS of a special form: except a short prefix and suffix it zigzags all the time between configurations close to vertical axis to configurations close to horizontal axis. Thus, to prove Lemma 20 it essentially remains to show polynomial approximability for zigzagging paths. To achieve that, we roughly speaking investigate how application of two consecutive loops $\beta \in \mathbb{N}_+ \times \mathbb{N}_-$ and $\beta' \in \mathbb{N}_- \times \mathbb{N}_+$ affects the set of reachable configurations on some vertical line close to the vertical axis. We show that arithmetic sequence is transformed into a finite union of arithmetic sequences such that the difference is kept at most polynomial in size of the 2-SLPS and the first term grows additively by at most a polynomial value. All that allows us to conclude that the set of vectors reachable by zigzagging paths is a union of sets of a form similar to $a + Q^* + P^{\leq T}$ for some a, Q, P and T. This quite easily implies polynomial approximability.

In the proof of Lemma 2 we actually need polynomial approximability not only for 2-VASS, but also for its generalisation, geometrically 2-dimensional 3-VASS. It is stated below and shown in the Appendix using Lemma 20.

▶ **Lemma 21.** *Geometrically 2-dimensional* 3-VASS *are polynomially approximable.*

6 Proof of Lemma 2

In this section we prove Lemma 2, by induction on k. The base of induction, when $k = 1$, follows by Lemma 9: 1-component 3-VASS are polynomially length-bounded. Before engaging in the induction

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step we need to generalise wideness, defined up to now for 1-component 3-VASS only, to all sequential 3-VASS.

Sequential cones. Consider a k-component 3-VASS $V = (V_1)u_1(V_2)u_2 \ldots u_{k-1}(V_k)$. By a *cascade* we mean a tuple of k vectors $(v_1, \ldots, v_k)$ such that the partial sum $v_1 + \ldots + v_i \in (\mathbb{Q}_{>0})^3$ for every $i \in [1, k]$. Then the *sequential cone* of V, denoted $\textsc{SeqCone}(V)$, is the set of sums of all cascades $(v_1, \ldots, v_k)$ whose every ith vector v_i belongs to $\textsc{Cone}(V_i)$:

$$\textsc{SeqCone}(V) = \{v_1 + \ldots + v_k \mid (v_1, \ldots, v_k) \in \textsc{Cone}(V_1) \times \ldots \times \textsc{Cone}(V_k) \text{ is a cascade}\}.$$

▷ Claim 22. If $\textsc{Lin}(V)$ is 3-dimensional then $\textsc{SeqCone}(V)$ is a finitely generated open cone.

We prove the fundamental property: all reachable configurations are at close distance to the sequential cones. We focus on 3-VASS, but actually the same proof works for VASS in any other fixed dimension. Below, let $d(x, y)$ denote Euclidean distance between x and y, and let $d(x, S)$ denote the distance between x and a set S, that is $d(x, S) = \inf\{d(x, y) \mid y \in S\}$.

▶ **Lemma 23.** *There exists a nondecreasing polynomial P such that each reachable configuration $q(w)$ in a forward-diagonal sequential 3-VASS (V, s), satisfies $d(w, \textsc{SeqCone}(V)) \leq P(\textsc{size}(V, s))$.*

Proof. Let V be a k-component sequential 3-VASS, and $M := \textsc{size}(V, s)$. Let $s = p_1(w)$ and suppose $s \xrightarrow{\pi} q(x)$. W.l.o.g. we assume that $q(x)$ is in the last component V_k; indeed, if $q(x)$ is in V_ℓ for some $\ell < k$, we prove that claim for $V' = (V_1)u_1(V_2)u_2 \ldots u_{\ell-1}(V_\ell)$ and rely on the inclusion $\textsc{SeqCone}(V') \subseteq \textsc{SeqCone}(V)$. Our aim is to define a polynomial P and a point $y \in \textsc{SeqCone}(V)$ such that $d(x, y) \leq P(M)$. The path π decomposes into $\pi = \pi_1; u_1; \ldots; \pi_{k-1}; u_{k-1}; \pi_k$ where $u_1, u_2, \ldots, u_{k-1}$ are bridge transitions. Let p_i be the source state of π_i and p'_i be the target state of π_i. Let σ_i be a shortest path from p'_i to p_i (there is such a path because they are in the same strongly connected component V_i). Let $v_i \in \mathbb{Z}^3$ be the effect of $\pi_i; \sigma_i$, which is a cycle in V_i.

As $\textsc{Cone}(V_i)$ is an open cone, we are not guaranteed that $v_i \in \textsc{Cone}(V_i)$. In order to ensure this property, we add to v_i some small multiples of all the simple cycles in V_i. For each $i \in [1, k]$, let $c_i \in \mathbb{Z}^3$ be the sum of effects of all simple cycles in V_i and let $\varepsilon \in \mathbb{Q}_{>0}$ be a small positive rational such that for all $i \in [1, k]$ we have $\textsc{norm}(\varepsilon \cdot c_i) < 1$. Then $v_i + \varepsilon \cdot c_i \in \textsc{Cone}(V_i)$.

By forward-diagonality, the configuration $p_1(w + \overrightarrow{1})$ is coverable in V_1 from s. Due to the upper bound of Rackoff [24, Lemma 3.4], instantiated to the fixed dimension 3, for some nondecreasing polynomial R the length of such a covering path $p_1(w) \longrightarrow p_1(w + \Delta)$, and the norm of Δ, are both at most $R(M)$. For some $m \in \mathbb{N}$, the vector $m \cdot \Delta + v_1 + \varepsilon \cdot c_1$ belongs to $\textsc{Cone}(V_1)$ and the k-tuple

$$(m \cdot \Delta + v_1 + \varepsilon \cdot c_1,\ v_2 + \varepsilon \cdot c_2,\ \ldots,\ v_k + \varepsilon \cdot c_k) \tag{6}$$

is a cascade. Therefore the sum $y = m \cdot \Delta + \Sigma_{j=1}^{k}(v_j + \varepsilon \cdot c_j) \in \textsc{SeqCone}(V)$. We argue that is is enough to use polynomially bounded value of m. Since π starts in $s = p_1(w)$, each its prefix can drop by at most $\textsc{norm}(w) \leq M$ on any coordinate. Thus, for each $j \in [1, k]$ we have $\textsc{eff}(\pi_1; u_1; \ldots; \pi_{k-1}; u_{j-1}; \pi_j) \geq -\overrightarrow{M}$. As u_i are single transitions, we also have $\textsc{eff}(u_1) + \ldots + \textsc{eff}(u_{j-1}) \leq \overrightarrow{M}$. In consequence, $\textsc{eff}(\pi_1) + \ldots + \textsc{eff}(\pi_j) \geq -2\overrightarrow{M}$. Moreover, the sum of norms of effects of all the paths σ_j is at most $-M$, which implies that $v_1 + \ldots + v_j \geq -3\overrightarrow{M}$. Finally, for each $j \in [1, k]$ we have $\textsc{norm}(\varepsilon \cdot (c_1 + \ldots + c_j)) \leq j \leq M$. Therefore, setting $m = 4M$ guarantees that the tuple (6) is a cascade.

Let $C = \varepsilon \cdot (c_1 + \ldots + c_k)$, $S = \textsc{eff}(\sigma_1) + \ldots + \textsc{eff}(\sigma_k)$ and $U = \textsc{eff}(u_1) + \ldots + \textsc{eff}(u_k)$. We have $\textsc{norm}(C), \textsc{norm}(S), \textsc{norm}(U) \leq M$, and $y = x + m \cdot \Delta + S + C - U$. Therefore $\delta = y - x$ satisfies $\textsc{norm}(\delta) \leq P(M) := 4M \cdot R(M) + 3M$, and we get the bound

$$d(x, y)^2 \ = \ d(x, x + \delta)^2 \ = \ \delta \diamond \delta \ \leq \ \textsc{norm}(\delta)^2$$

that implies $d(x, y) \leq \textsc{norm}(\delta)$, and hence $d(x, y) \leq P(M)$ as required. ◀

We say that a k-component 3-VASS V is *wide* if $(\mathbb{Q}_{>0})^3 \subseteq \textsc{SeqCone}(V)$ or $(\mathbb{Q}_{>0})^3 \subseteq \textsc{SeqCone}(V^{\mathrm{rev}})$. For $k = 1$, the definition relaxes the definition of Section 4.

Proof of Lemma 2. Lemma 26, formulated below, is a refinement of Lemma 2 suitable for inductive reasoning. When proving the lemma, we distinguish 2 cases, depending on whether a 3-VASS is diagonal and wide (we call the 3-VASS *easy* in this case), or not (we call it *non-easy* then), and rely on the following two facts:

▶ **Lemma 24.** *Easy sequential* 3-VASS *are length-bounded by* $P_k(M) = M^{\mathcal{O}(k)}$*, where* k *is the number of components.*

▶ **Lemma 25.** *There is a nondecreasing polynomial* H *such that for every* $k > 1$*, if* $(k-1)$*-component* 3-VASS *are length-bounded by a function* h *then non-easy* k*-component* 3-VASS *are length-bounded by the function* $H \circ h \circ H \circ h \circ H$*.*

For stating Lemma 26, we define inductively a sequence of polynomials $(h_i)_{i\in\mathbb{N}}$, where h_1 is the polynomial witnessing Lemma 9 (thus 1-component 3-VASS are length-bounded by h_1), and

$$h_{j+1} = H \circ h_j \circ H \circ h_j \circ H.$$

W.l.o.g. we also assume that h_j dominates the polynomial P_j of Lemma 24, namely $P_j(m) \le h_j(m)$ for every $j, m \ge 1$.

▶ **Lemma 26.** *For every* $k \ge 1$*,* k*-component* 3-VASS *are length-bounded by* h_k*.*

Proof. The induction base is given by Lemma 9. For the induction step, suppose $(k-1)$-component 3-VASS are length-bounded by h_{k-1}. Easy k-component 3-VASS are length-bounded by h_k due to Lemma 24 and the above-assumed domination, without referring to induction assumption, while non-easy k-component 3-VASS are length-bounded by h_k due to Lemma 25. ◀

Let $c \in \mathbb{N}$ be large enough so that $H(m), h_1(m) \le m^c$ for every $m > 1$, and let $C = c^3$.

▷ Claim 27. $h_k(m) \le m^{C^{2^k-1}}$ for all $m > 1$.

Lemma 26 implies Lemma 2, as the right-hand side of the inequality in Claim 27 is bounded by $m^{2^{2^{\mathcal{O}(k)}}}$. Lemma 2 is thus proved (once we prove Lemmas 24 and 25). ◀

The proof of Lemma 24 generalises Case 1 of the proof of Lemma 9. The proof of Lemma 25 makes crucial use of polynomially approximable sets introduced in Section 5, and builds on Lemma 28, stated below, whose proof generalises Cases 2 and 3 of the proof of Lemma 9.

For stating and proving Lemma 28 we need a variant of sequential 3-VASS: a *good-for-induction* k-component 3-VASS $V = (V_1)u_1(V_2)u_2 \ldots u_{k-1}(V_k)$ is defined exactly like k-component sequential 3-VASS, except that the first component V_1 is an arbitrary geometrically 2-dimensional 3-VASS, not necessarily being strongly connected.

▶ **Lemma 28.** *There is a nondecreasing polynomial* R *such that every non-easy* k*-component* 3-VASS (V, s, t) *is length-equivalent to a finite set* S *of good-for-induction* k*-component* 3-VASS *of size at most* $R(\textsc{size}(V, s, t))$*, namely* $\textsc{Len}(V, s, t) = \bigcup_{(V',s',t')\in S} \textsc{Len}(V', s', t')$*.*

Proof of Lemma 25. Relying on Lemma 28, assume w.l.o.g. that (V, s, t) is a good-for-induction k-component 3-VASS of size $\textsc{size}(V, s, t) = R(M)$, where R is a polynomial of Lemma 28. Let $V = (V_1)u_1(V_2)u_2 \ldots u_{k-1}(V_k)$, $u_i = (p'_i, \delta_i, p_{i+1})$, $s = p_1(w)$ in V_1 and $t = p'_k(w')$ in V_k. Let F be a nondecreasing polynomial witnessing Lemma 21, and let $f(x) = 2R(x) \cdot F(R(x))$. Let G be a nondecreasing polynomial witnessing Lemma 7, and let $g(x) = G(2x^2) + x$.

Let $B := h(f(M))$, namely the length-bound for $(k-1)$-component VASS of size $f(M)$. Assuming $s \xrightarrow{\pi} t$ in V, we aim at proving that there is such a path of length at most $g(h(f(h(f(M)))))$. Decompose the path $s \xrightarrow{*} t$ into

$$s \xrightarrow{\pi_1} p_1'(v') \xrightarrow{u_1} p_2(v) \xrightarrow{\pi'} t, \tag{7}$$

where $p_2(v)$ is the first configuration of V_2 appearing on π. As V_1 is a geometrically 2-dimensional 3-VASS, so is the *lollypop* 3-VASS V_1' obtained by adding to V_1 the first bridge transition u_1 (indeed, adding a bridge transition does not create any new cycles). As $\text{SIZE}(V, s, t) \leq R(M)$ and F is the polynomial witnessing Lemma 21, by Lemma 21 we get:

▷ Claim 29. The set $\text{REACH}_{p_2}(V, s)$ is $(F(R(M)), B)$-approximately semi-linear.

Thus $v \in L = a + P^*$, where L is a linear set $(*)$ or a B-approximation thereof $(**)$. In both cases $v = a + r$, for $r \in P^*$. We construct a $(k-1)$-component 3-VASS (V', s', t) as follows. V' is obtained from $(V_2)u_2 \ldots u_{k-1}(V_k)$ by adding, for every period vector $r \in P$, a self-looping transition (p, r, p) to V_2. As the source configuration we take $s' = p_2(a)$, and keep t as the target configuration.

As R is nondecreasing, $B = h(f(M))$ and $f \geq R$ we have $M \leq R(M) \leq B$. As (V', s', t') is obtained from of (V, s, t) by removing from V the part V_1, adding transitions of total norm at most equal $\text{NORM}(P)$ and setting $s' = p_2(a)$ therefore we estimate $M' = \text{SIZE}(V', s', t)$ as $M' \leq R(M) + \text{NORM}(a) + \text{NORM}(P)$. Recall that in the case when L is a linear set $(*)$ we have $\text{NORM}(a) \leq B \cdot A$, $\text{NORM}(P) \leq A$, while in the case when L is a B-approximation $(**)$ we have $\text{NORM}(a), \text{NORM}(P) \leq A$, in our case $A = F(R(M))$. Thus the estimations look as follows:

$$(*) \quad M' \leq R(M) + (B+1) \cdot F(R(M)) \leq 2 \cdot R(B) \cdot F(R(B)) = f(B)$$
$$(**) \quad M' \leq R(M) + 2 \cdot F(R(M)) \leq 2 \cdot R(M) \cdot F(R(M)) = f(M),$$

as f was defined exactly as $f(x) = 2R(x) \cdot F(R(x))$. Note that the latter bound is dominated by the former one, thus in both cases $M' \leq f(B) = f(h(f(M)))$.

There is a path $s' \xrightarrow{*} t$ in V', namely the concatenation of a path $p_2(a) \xrightarrow{*} p_2(v)$ using the just added self-looping transition in the state p, with the suffix π' of (7):

$$p_2(a) \xrightarrow{*} p_2(v) \xrightarrow{\pi'} t.$$

By induction assumption, there is a path $s' \xrightarrow{*} t$ in V' of length at most

$$(*) \quad \ell \; = \; h(M') \; \leq \; h(f(B)) \; = \; h(f(h(f(M))))$$
$$(**) \quad \ell \; = \; h(M') \; \leq \; h(f(M)).$$

W.l.o.g. we may assume that the path executes all just added self-looping transitions in the beginning, and therefore it splits into:

$$p_2(a) \xrightarrow{\rho''} p_2(v'') \xrightarrow{\rho'} t$$

such that the suffix ρ' is actually a path in $(V_2)u_2 \ldots u_{k-1}(V_k)$. Since the length of the prefix ρ'' is at most ℓ, we bound $\text{NORM}(v'') \leq M \cdot \ell$. If L is a linear set, the following claim is obvious since $L \subseteq \text{REACH}_{p_2}(V, s)$. On the other hand, the claim requires a proof in case when L is a B-approximation of a linear set:

▷ Claim 30 $(**)$. There is a path $s \xrightarrow{*} p_2(v'')$ in the lollypop 3-VASS V_1'.

Indeed, since the length ℓ of the path $s' \xrightarrow{*} t$ in V' is at most $h(f(M))$ and $B = h(f(M))$, we deduce that $v'' \in a + P^{\leq B}$, and therefore $s \xrightarrow{*} p_2(v'')$ in V_1'.

By Lemma 7, there is a path $s \xrightarrow{\rho} p_2(v'')$ in V_1' of length at most $G(M + \textsc{norm}(v'')) \leq G(M \cdot (\ell+1)) \leq G(2\ell^2)$. Concatenating the two paths ρ and ρ' we get a path

$$s \xrightarrow{\rho} p_2(v'') \xrightarrow{\rho'} t$$

in V, of length at most $G(2\ell^2) + \ell = g(\ell) \leq g(h(f(h(f(M)))))$. Taking $H = f + g$, the sum of polynomials f and g, we get the bound $H(h(H(h(H(M)))))$, as required. ◀

7 Future research

Below we list a few research questions, which we find interesting and particularly promising directions after our contribution.

Exact complexity for 3-VASS. We have shown that shortest paths in binary 3-VASS are of at most triply-exponential length. It is tempting to conjecture that actually the upper bound for the length of the paths is shorter, at most doubly-exponential. We conjecture that it is possible with techniques similar to the developed ones, but with more focus on polynomials growing linearly with respect to norms of source and target. We leave proving this conjecture to the future research.

Example of a 3-VASS with doubly-exponential path. We have shown that shortest paths in binary 3-VASS are of at most triple-exponential length. However, currently we still do not know any example in which even a path of doubly-exponential length is needed, it might be that paths of exponential length are sufficient leading to PSpace-completeness for binary 3-VASS. It would be very interesting to find an example of a binary 3-VASS with shortest path between two configurations being doubly exponential. An example of binary 4-VASS of doubly-exponential shortest path is known (see Section 5 in [7]). Maybe some modification of this 4-VASS would allow to design a 3-VASS with similar properties.

Reachability for d-VASS with $d \geq 4$. It is a natural question whether our techniques extend to higher dimensions. The answer is: possibly yes, but we would need a few other structural results for 3-VASS to make a similar approach to 4-VASS possible. In the proof of Lemma 2 we do not only use 2-VASS reachability as a black box, but we use a deep understanding of the reachability relation in 2-VASS from [4]. Probably a similar understanding of the reachability relation for 3-VASS would be needed to advance understanding of 4-VASS along our lines.

In general it is very interesting to determine the complexity of the reachability problem for d-VASS. We have excluded that for each $d \geq 3$ the problem is $\mathcal{F}_d$-completely, but it is still possible that the problem is $\mathcal{F}_{d-C}$-complete for some constant $C \in \mathbb{N}$ and d big enough. Recall that in [6] it was shown that the reachability problem for $(2d+4)$-VASS is $\mathcal{F}_d$-hard for any $d \geq 3$ and this is the best currently known lower bound for arbitrary dimension. Therefore the other natural possibility is that the reachability problem for $(2d+C)$-VASS is $\mathcal{F}_d$-complete for some constant $C \in \mathbb{N}$.

Applications of the approximation technique. Another natural research direction is to search for other applications of the technique of approximating the reachability sets, which allows to lower the complexity down, below the size of the reachability set. One particular case, which seems to be prone to such techniques is the 2-VASS with some number of $\mathbb{Z}$-counters, namely counters, which can take values below zero. The best complexity lower bound for the reachability problem in this model is PSpace-hardness inherited from [2], while the best upper bound is Ackermann membership inherited from VASS reachability [19]. The reachability sets for that systems are not necessarily semilinear. This disqualifies most of the techniques relying on the semilinearity of reachability sets, but our techniques seem to be promising for that model.

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A Proofs for Section 2 (Preliminaries)

Proof of Lemma 5. Below, we will refer to two inductively defined sequences $(H_i)_{i\in\mathbb{N}_{>0}}$, $(L_i)_{i\in\mathbb{N}_{>0}}$, (implicitly) parametrised by numbers $n, N, U \in \mathbb{N}$: let $H_1 := U$, $L_1 = nU$, and for $i > 1$ let $H_i = U + NL_{i-1}$ and $L_i = n(H_i)^i + L_{i-1}$.

▶ **Lemma 31.** *Let $d \in \mathbb{N}$, and let (V, s) be a d-VASS of norm N, with n states. If V has a path from s that for every $i \in [1, d]$ contains a configuration $q(w_1, \ldots, w_d)$ with $w_i \geq H_d$, then V has also a path $s \xrightarrow{*} q(w_1, \ldots, w_d)$ of length at most L_d such that $w_i \geq U$ for every $i \in [1, d]$.*

Proof. The proof proceeds by induction over the dimension d, and follows the idea of Rackoff [24, Lemma 3.4]. Let V be a fixed d-VASS.

When $d = 1$, as $H_1 = U$ the first claim is immediate. The bound on length $L_1 = nU$ is also obtained immediately, by removing repetitions of configurations.

For the induction step, we assume that Lemma 31 holds for dimension $d - 1$, and show it for dimension d. Consider the shortest path $s \xrightarrow{\rho} u$ from s in V such that for every $i \in [1, d]$, the path contains a configuration $q(w_1, \ldots, w_d)$ with $w_i \geq H_d$. Let $q(w_1, \ldots, w_d)$ be the first configuration on ρ with $(w_1, \ldots, w_d) \notin [0, H_d - 1]^d$:

$$s \xrightarrow{\rho_1} q(w_1, \ldots, w_d) \xrightarrow{\rho_2} u.$$

W.l.o.g. we may assume that $w_d \geq H_d$. The length of ρ_1 is at most $n(H_d)^d$, as the configurations along ρ_1 are bounded by $H_d - 1$ and do not repeat.

Let $\overline{V}$ denote the $(d-1)$-VASS obtained by dropping the dth coordinate of V. By the induction assumption, $\overline{V}$ has a path

$$q(w_1, \ldots, w_{d-1}) \xrightarrow{\rho_3} p(v_1, \ldots, v_{d-1})$$

of length at most L_{d-1}, whose target vector satisfies $v_i \geq U$ for all $i \in [1, d-1]$. Steps of $\overline{V}$ are steps of V where the dth coordinate is dropped, and each such step may decrease the value on dth coordinate by at most N. Therefore $w_d \geq H_d = U + NL_{d-1}$ is large enough so that L_{d-1} steps of ρ_3 yield at least U on the dth coordinate. Therefore ρ_3 can be traced back to a path

$$q(w_1, \ldots, w_d) \xrightarrow{\rho_4} p(v_1, \ldots, v_d)$$

in V, of the same length as ρ_3, where $v_d \geq U$. The concatenated path $\rho_1; \rho_4$ has length at most $L_d = n(H_d)^d + L_{d-1}$ and hence satisfies the claim of Lemma 31. ◀

We show that H_i and L_i are bounded by a polynomial in n, N, U, of degree doubly exponential in dimension d. We concentrate on H_i, as $H_i \leq L_i \leq H_{i+1}$.

▶ **Proposition 32.** $H_i \leq (4Nn)^{2i!}U^{i!}$.

Proof. Let $C = 4Nn$. As $H_i \leq 2NL_{i-1}$ and $L_i \leq 2n(H_i)^i$, we have:

$$H_i \leq C \cdot (H_{i-1})^{i-1}. \tag{8}$$

Instead of showing $H_i \leq C^{2i!}U^{i!}$, we prove a slightly stronger inequality:

$$H_i \leq C^{2(i-1)!-1}U^{(i-1)!},$$

by induction on i. When $i = 1$ we have $H_1 = U \leq CU$, and when $i = 2$, by (8) we have $H_2 \leq CH_1 = CU$, as required. The induction step follows by:

$$H_{i+1} \leq C \cdot (H_i)^i \leq C \cdot (C^{2(i-1)!-1}U^{(i-1)!})^i \leq C^{2i!-1}U^{i!},$$

where first inequality is (8), the second one is the inductive assumption, and the latter one follows since $1 + (2(i-1)! - 1) \cdot i \leq 2i! - 1$ when $i \geq 2$. This completes the proof. ◀

Lemma 31, Proposition 32 and the inequality $L_i \leq H_{i+1}$ entail Lemma 5. ◀

Proof of Lemma 7. We rely on the construction of [27, Lemma 5.1], which transforms a given geometrically 2-dimensional 3-VASS (V, s, t) into a 2-VASS $(\overline{V}, \overline{s}, \overline{t})$ such that[4]

$$\textsc{Len}(\overline{V}, \overline{s}, \overline{t}) \;=\; \{3 \cdot n \mid n \in \textsc{Len}(V, s, t)\},$$

and the size of the 2-VASS is only polynomially larger than the size of the original geometrically 2-dimensional 3-VASS. Therefore, as 2-VASS are polynomially length-bounded due to Lemma 6, so are also geometrically 2-dimensional 3-VASS. ◀

Proof of Lemma 8. Let (V, s, t) be a 3-$\mathbb{Z}$-VASS such that $s \xrightarrow{\sigma} t$. Let $V = (Q, T)$, $s = q(w)$ and $t = q'(w')$. Let $M = \textsc{size}(V, s, t) = \textsc{size}(V) + \textsc{norm}(s) + \textsc{norm}(t)$. We express a path as a solution of a Diophantine system of linear equations, and rely on Lemma 4.

Let $Q' \subseteq Q$ and $T' \subseteq T$ be the subsets of states and transitions that appear in σ. Simple cycles that use only transitions from T' we call T'-cycles. The $\mathbb{Z}$-path $s \xrightarrow{\sigma} t$ decomposes into a $\mathbb{Z}$-path σ_0 that visits all states of Q', plus a number of T'-cycles. Choose the shortest such σ_0. The $\mathbb{Z}$-path σ_0 visits each state at most $|Q|$ times, as otherwise it could be shortened, and therefore its effect has norm at most M^2. The effect Δ of each T'-cycle has norm at most M, as it contains no repetition of a transition. We choose, for each such vector Δ, one of T'-cycles with effect Δ. Let $\mathcal{C}$ be the set of chosen T'-cycles. Its size is at most $(2M+1)^3 \leq \mathcal{O}(M^3)$. We define a system $\mathcal{U}$ of 3 linear equations (one for each dimension), whose unknowns x_δ correspond to T'-cycles δ from $\mathcal{C}$:

$$\sum_{\delta \in \mathcal{C}} x_\delta \cdot e_\delta \;+\; e_{\sigma_0} \;=\; t - s,$$

where $e_\delta \in \mathbb{Z}^3$ is the effect of δ, and $e_{\sigma_0} \in \mathbb{Z}^3$ is the effect of σ_0. The system has a nonnegative integer solution, namely the one obtained from decomposition of σ into σ_0 and simple cycles. As all coefficients of $\mathcal{U}$ are bounded by M^2, by Lemma 4 the system has a solution of norm $\mathcal{O}(M^3 \cdot M^2)^3 = \mathcal{O}(M^{15})$. The solution $(x_\delta)_\delta$ yields a $\mathbb{Z}$-path $s \xrightarrow{*} t$ of length $\mathcal{O}(M^{16})$, consisting of σ_0 with attached all cycles $\delta \in \mathcal{C}$ (this is possible, as σ_0 visits all states used by the cycles), each δ iterated x_δ times. This completes the proof. ◀

B Proofs for Section 5 (Polynomially approximable reachability sets)

Proof of Lemma 20. We extend the definition of nondecreasing functions to many-argument ones: a function $f : \mathbb{N}^k \to \mathbb{N}$ is *nondecreasing* if it is monotonic in every argument and $f(n_1, \ldots, n_k) \geq n_1 + \ldots + n_k$. In the sequel we often bound certain quantities polynomially, but an exact polynomial is irrelevant. We thus say that a value n is *polynomially bounded* in $n_1, \ldots, n_k$ if there exists a nondecreasing polynomial $P : \mathbb{N}^k \to \mathbb{N}$ such that $n \leq P(n_1, \ldots, n_k)$ for all $n_1, \ldots, n_k \in \mathbb{N}$. We also write $n \leq \mathrm{poly}(n_1, \ldots, n_k)$.

Linear path schemes. A d-dimensional *linear path scheme* (d-LPS in short, or LPS if dimension is irrelevant) is a sequential VASS where every component is either trivial (a singleton) or a simple cycle, i.e., a VASS whose control graph is a simple path with disjoint simple cycles attached to some states of the path. We write down LPS in the following form

$$\alpha_0 \beta_1^* \alpha_1 \cdots \alpha_{k-1} \beta_k^* \alpha_k,$$

[4] In [27] only zero source and target vectors are considered, but the construction routinely extends to arbitrary such vectors.

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where each α_i and β_i is a fixed sequence of transitions. Thus the cycles β_i of an LPS may be repeated arbitrarily many times (possibly zero). An LPS is *simple* (SLPS) when all β_i are single transitions, i.e., each component is either trivial or a single self-loop transition. When considering the reachability relation in a d-LPS, we often implicitly take the first and the last state of the LPS as the source and target state, respectively, and consider the reachability $w \xrightarrow{*} w'$ between vectors $w, w' \in \mathbb{N}^d$ only.

▶ **Lemma 33.** *For every 2-VASS V and two its states q, q', there is a finite set Γ of 2-SLPS of size polynomially bounded in $\text{SIZE}(V)$, such that for all $w, w' \in \mathbb{N}^2$,*

$$q(w) \xrightarrow{*} q'(w') \text{ in } V \iff w \xrightarrow{*} w' \text{ in some 2-SLPS in } \Gamma.$$

Proof. The claim follows by combination of Theorem 3.1 from [1], due to which we get a finite set Γ of 2-LPS of size polynomially bounded in $\text{SIZE}(V)$ that satisfies the claim, with Lemma 5.2 from [1] (or Theorem 15 from [10]), due to which we can transform every 2-LPS λ into an 2-SLPS of size polynomially bounded in $\text{SIZE}(\Lambda)$. ◀

▶ **Lemma 34.** *2-SLPS are polynomially approximable.*

Before proving Lemma 34, we use it together with Lemma 33 to prove Lemma 20. To this aim consider a 2-VASS (V, s) where $s = p(w)$, and its state q, and let $M = \text{SIZE}(V, s)$. By Lemma 33 we get a finite set Γ of 2-SLPS such that:

$$\text{REACH}_q(V, s) \;=\; \bigcup_{\Lambda \in \Gamma} \text{REACH}(\Lambda, w).$$

Moreover, for some nondecreasing polynomial F, every $\Lambda \in \Gamma$ satisfies $\text{SIZE}(\Lambda) \leq F(M)$. By Lemma 34, there is a nondecreasing polynomial G such that for every $\Lambda \in \Gamma$ and $B \in \mathbb{N}$, the set $\text{REACH}(\Lambda, w)$ is $(G(\text{SIZE}(\Lambda, w)),\ B)$-approximately semi-linear. Combining the last two statements, we deduce that $\text{REACH}_q(V, s)$ is $(G(F(M)),\ B)$-approximately semi-linear for every $B \in \mathbb{N}$, as required. Lemma 20 is thus proved (once we prove Lemma 34). ◀

Proof of Lemma 34. We generalise finite prefixes $P^{\leq B}$ of P^* as follows. Let $P = \{p_1, \ldots, p_m\}$. For a positive vector $c \in (\mathbb{N}_{>0})^m$ and $T \in \mathbb{N}$ we define

$$P^{x \cdot c \leq T} := \{\Sigma_{i=1}^m n_i \cdot p_i \mid \Sigma_{i=1}^m n_i \cdot c_i \leq T\}.$$

In particular, $P^{\leq T} = P^{x \cdot \vec{1} \leq T}$. In the sequel, sets of the form

$$a + P^{x \cdot c \leq T} + Q^*, \tag{9}$$

for $a \in \mathbb{N}^2$, $c \in (\mathbb{N}_{>0})^{|P|}$ and $P, Q \subseteq_{\text{fin}} \mathbb{N}^2$, we call *hybrid* sets.

▶ **Lemma 35.** *For every 2-SLPS (Λ, s), the set $\text{REACH}(\Lambda, s)$ is a finite union of hybrid sets (9), where $\text{NORM}(a), \text{NORM}(c), \text{NORM}(P \cup Q)$ are bounded polynomially in $\text{SIZE}(\Lambda, s)$.*

Before proving Lemma 35 we use it to prove Lemma 34. We need to argue that there is a nondecreasing polynomial F such that for every 2-SLPS (Λ, s) and $B \in \mathbb{N}$, the set $\text{REACH}(\Lambda, s)$ is $(F(M),\ B)$-approximately semi-linear, where $M = \text{SIZE}(\Lambda, s)$. We fix the nondecreasing polynomial $F(x) = R(x) + R^2(x)$, where R is a polynomial witnessing Lemma 35, and some arbitrary $B \in \mathbb{N}$, and prove that each hybrid set H (9) of Lemma 35 is $(F(M),\ B)$-approximately semi-linear. We distinguish two cases. If $T \geq R(M) \cdot B$ then, since $\text{NORM}(c) \leq R(M)$, we have

$$a + (P \cup Q)^{\leq B} \;\subseteq\; H \;\subseteq\; a + (P \cup Q)^*,$$

and $\textsc{norm}(a), \textsc{norm}(P \cup Q) \leq R(M) \leq F(M)$, as required. On the other hand, if $T < R(M) \cdot B$ then H is a finite union of linear sets of the form $u + Q^*$ for $u \in b + P^{c \cdot x \leq T}$, where

$$\textsc{norm}(u) \ \leq \ \textsc{norm}(a) + \textsc{norm}(P) \cdot T \ \leq \ R(M) + R(M)^2 \cdot B \leq F(M) \cdot B,$$

as required. As before, $\textsc{norm}(Q) \leq R(M) \leq F(M)$. This completes the proof of Lemma 34 (once we prove Lemma 35). ◀

Proof of Lemma 35. The proof occupies the rest of this section. We rely on an insightful characterisation of paths of SLPS [4, Theorem 4.16], which we state below using a slightly different terminology. Speaking informally, a *detailing* of an SLPS $\Lambda = \alpha_0\beta_1^*\alpha_1 \ldots \alpha_{k-1}\beta_k^*\alpha_k$ is any SLPS obtained by fixing exponents of some of the cycles of Λ. Formally, a detailing of Λ is any Λ' obtained by choosing a subset $S \in [1,k]$ and, for all $i \in S$, by replacing the cycle β_i by a path $\beta_i^{n_i}$, for some $n_i \in \mathbb{N}$, which becomes an infix of the simple path of Λ'. The number of cycles of Λ' is thus $k - |S|$. An 2-SLPS is *zigzagging* if the effect of its every cycle β_i belongs either to the quadrant $\mathbb{N}_{>0} \times (-\mathbb{N}_{>0})$, or to the quadrant $(-\mathbb{N}_{>0}) \times \mathbb{N}_{>0}$, and additionally effects of every two consecutive cycles belong to different quadrants (the effects of cycles $\beta_1, \ldots, \beta_k$ thus alternate between quadrants), and the effect of the first cycle belongs to $\mathbb{N}_{>0} \times (-\mathbb{N}_{>0})$ and the effect of the last cycle belongs to $(-\mathbb{N}_{>0}) \times \mathbb{N}_{>0}$. Finally, an SLPS is *short* if it contains at most three cycles, $k \leq 3$. For $B \in \mathbb{N}$, a path

$$s_0 \xrightarrow{\alpha_0} t_0 \xrightarrow{\sigma_1} s_1 \xrightarrow{\alpha_1} t_1 \ \cdots \ s_{k-1} \xrightarrow{\alpha_{k-1}} t_{k-1} \xrightarrow{\sigma_k} s_k \xrightarrow{\alpha_k} t_k$$

of an 2-SLPS, where $\sigma_i \in \beta^*$ for $i \in [1,k]$, is called *B-close* if all the vectors $x \in \{s_0, t_0, s_1, t_1, \ldots, s_k, t_k\}$, called *midpoints* below, are *B-close* to some axis, namely either $x \in [0,B] \times \mathbb{N}$ or $x \in \mathbb{N} \times [0,B]$.

▶ **Theorem 36** (Thm 4.16 in [4]). *For every 2-SLPS Λ there is $B \leq \mathrm{poly}(\textsc{size}(\Lambda))$ such that for every path $s \xrightarrow{*} t$ in Λ there is a detailing $\Lambda' = \Lambda_1\Lambda_2\Lambda_3$ of Λ of $\textsc{size}(\Lambda') \leq \mathrm{poly}(\textsc{size}(\Lambda))$ and $u, u' \in [0,B] \times \mathbb{N}$ such that*
1. *Λ_1 and Λ_3 are short,*
2. *Λ_2 is zigzagging,*
3. *there are paths $s \xrightarrow{*} u$ in Λ_1, a B-close path $u \xrightarrow{*} u'$ in Λ_2, and $u' \xrightarrow{*} t$ in Λ_3.*

Fix in the sequel B given by Theorem 36. There are only finitely many detailings Λ' of Λ of a bounded size, only finitely many possible decompositions of Λ' into Λ_1, Λ_2 and Λ_3, and only finitely many values of $u_1, u_1' \in [0,B]$. By Theorem 36, vectors t reachable from s in Λ are exactly those reachable from s in some of detailing Λ'. Therefore it is enough to show Lemma 35 for the set of vectors $t \in \mathbb{N}^2$ reachable by paths as in point 3 above, in a fixed SLPS (Λ', s), where $\Lambda' = \Lambda_1\Lambda_2\Lambda_3$ satisfies points 1, 2 above, and where $u_1 = b$ and $u_1' = b'$ for some fixed $b, b' \in [0,B]$.

In a path $u \xrightarrow{*} u'$, every second midpoint is B-close to one axis, say $x \in [0,B] \times \mathbb{N}$, while the remaining midpoints are B-close to the other one. We relax this requirement slightly, by dropping the latter condition. A path $u \xrightarrow{*} u'$ in the zigzagging SLPS Λ_2 is *B-vertical-close* if u, u' and every second midpoint x are B-close to the vertical axis, namely $x \in [0,B] \times \mathbb{N}$ (thus the remaining endpoints on the path do not have to be B-close to the horizontal axis). In order to have more flexibility in the proof of Lemma 35, in the sequel we consider those paths in Λ', as in point 3 in Theorem 36, where the infix $u \xrightarrow{*} u'$ is B-vertical-close but not necessarily B-close. Notice that by relaxing the condition to B-vertical-closeness we enlarge the set of considered paths, but do not enlarge the set of reachable points, as every t such that $s \xrightarrow{*} t$ is already reachable by the paths with the middle path being even B-close. We denote by $\textsc{Reach}(\Lambda', s)$ the set of all vectors $t \in \mathbb{N}^2$ reachable by such a path $s \xrightarrow{*} t$ with the middle part being B-vertical-close.

We formulate below Claims 37, 38 and 39 (taking care of a prefix, infix, and suffix, respectively, of a path $s \xrightarrow{*} t$), show how they imply Lemma 35, and finally proceed with the proofs of the three claims. To this aim we introduce some more notation. Given a start $a \in \mathbb{N}$, a difference $r \in \mathbb{N}$, and a bound $T \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, the set $a + \{r\}^{\leq T}$ is called (a, r, T)-*arithmetic*. We omit brackets and write $a + r^{\leq T}$. In particular, $a + r^{\leq 0} = \{a\}$. A 2-SLPS $\alpha_1 \beta_1^* \alpha_2 \beta_2^*$ is *one-turn* if $\text{EFF}(\beta_1) \in \mathbb{N}_{>0} \times -\mathbb{N}_{>0}$ and $\text{EFF}(\beta_2) \in -\mathbb{N}_{>0} \times \mathbb{N}_{>0}$ (it is thus a special case of short zigzagging 2-SLPS). For a set $S \subseteq \mathbb{N}^2$, we use the notation $\text{Reach}(\Lambda, S) = \bigcup_{s \in S} \text{Reach}(\Lambda, s)$.

In Claims 37, 38 and 39, we focus on source/target vectors in $[0, B] \times \mathbb{N}$ and, intuitively speaking, on arithmetic subsets of each 'line' $\{b\} \times \mathbb{N}$. First, Claim 37 states that the reachability set of a short 2-SLPS, intersected with each line, is a finite union of arithmetic sets. Second, Claim 38 states that the reachability set of a one-turn 2-SLPS from an arithmetic set inside a line, intersected with another line, is a finite union of arithmetic sets. Importantly, the starting point grows only additively, by a polynomially bounded amount, as we will apply Claim 38 $\mathcal{O}(k)$ times. Finally, Claim 39 states that the reachability set of a short 2-SLPS, from an arithmetic set inside a line, is a finite union of hybrid sets. All quantities in the claims are bounded polynomially.

▷ Claim 37. For every short 2-SLPS (Λ, s) and $u_1 \in [0, B]$, the set $\{u_2 \mid (u_1, u_2) \in \text{Reach}(\Lambda, s)\}$ is a finite union of (a, r, T)-arithmetic sets, where $a \leq \text{poly}(B, M)$, $r \leq \text{poly}(M)$, and $M = \text{SIZE}(\Lambda, s)$.

▷ Claim 38. Let Λ be a one-turn 2-SLPS, and $S_1 = a + r^{\leq K}$ for some $a, r, K \in \mathbb{N}$. Let $u_1, v_1 \in [0, B]$. The set $R(S_1) := \{v_2 \mid \exists_{u_2 \in S_1} (u_1, u_2) \xrightarrow{*} (v_1, v_2) \text{ in } \Lambda\}$ is a finite union of (a', r', T')-arithmetic sets with $a' \leq a + \text{poly}(B, M, r)$ and $r' \leq \max(\text{poly}(M), r)$, where $M = \text{SIZE}(\Lambda)$.

▷ Claim 39. For every short 2-SLPS Λ and $u = (u_1, u_2), p = (p_1, p_2) \in \mathbb{N}^2$, the set $\text{Reach}(\Lambda, u + \{p\}^{\leq T})$ is a finite union of hybrid sets (9), where $\text{NORM}(a), \text{NORM}(c), \text{NORM}(P), \text{NORM}(Q) \leq \text{poly}(\text{SIZE}(\Lambda), \text{NORM}(u), \text{NORM}(p))$.

We use the three claims to derive Lemma 35. As said above, we consider a fixed SLPS $\Lambda' = \Lambda_1 \Lambda_2 \Lambda_3$ and source $s \in \mathbb{N}^2$, and focus on the set $\text{Reach}(\Lambda', s)$ of vectors $t \in \mathbb{N}^2$ such that there are paths

$$s \xrightarrow{*} u \text{ in } \Lambda_1, \quad \text{a } B\text{-vertically close path } u \xrightarrow{*} u' \text{ in } \Lambda_2, \quad \text{and } u' \xrightarrow{*} t \text{ in } \Lambda_3, \tag{10}$$

for some $u, u' \in [0, B] \times \mathbb{N}$, where $u_1 = b$ and $u'_1 = b'$ are fixed. Let $M' := \text{SIZE}(\Lambda', s) \leq \text{poly}(M)$. First, by Claim 37, the set $\{u_2 \mid (u_1, u_2) \in \text{Reach}(\Lambda_1, s)\}$ is a finite union of (a, r, T)-arithmetic sets, where a, r are bounded polynomially in M' and B. Second, a path $u \xrightarrow{*} u'$ is a concatenation of $\ell \leq \text{SIZE}(\Lambda_2) \leq \text{poly}(M')$ paths of one-turn SLPS. By ℓ-fold application of Claim 38, the set $\{u'_2 \mid (u'_1, u'_2) \in \text{Reach}(\Lambda_1 \Lambda_2, s)\}$ is a finite union of arithmetic sets $a' + (r')^{\leq T'}$, where $a', r' \leq \text{poly}(\text{SIZE}(\Lambda_2)) \leq \text{poly}(M')$. Indeed, the bound on a' comes from ℓ-fold addition of values bounded by $\text{poly}(B, \text{poly}(M'), r) \leq \text{poly}(B, \text{poly}(M'), \text{poly}(M'))$, itself bounded by $\text{poly}(M')$:

$$a' \leq a + \ell \cdot \text{poly}(M') \leq a + \text{poly}(M') \leq \text{poly}(M'). \tag{11}$$

Finally, by Claim 39 the set $\text{Reach}(\Lambda', s)$ is a finite union of hybrid sets (9), where $\text{NORM}(a)$, $\text{NORM}(c)$, $\text{NORM}(P \cup Q) \leq \text{poly}(M', B) \leq \text{poly}(M)$. We conclude the proof of Lemma 35, keeping in mind that it still remains to demonstrate Claims 37, 38, and 39.

Here is a corollary of Lemma 4, useful in the proofs of the three claims:

► **Lemma 40.** *Consider a system $A \cdot x = b$ of m Diophantine linear equations with n unknowns, where absolute values of coefficients are bounded by N. Then, the set of solutions is of a form $U + P^*$, where $\text{NORM}(U \cup P) \leq \text{poly}(nN)^{\text{poly}(n,m)}$.*

Proof. The solution set is of the form $U + P^*$, where U is the set of pointwise minimal nonnegative integer solutions of the system, and P is the set of pointwise minimal nonnegative integer solutions of its homogeneous version $A \cdot x = \overrightarrow{0}$. By Lemma 4 each element of $U \cup P$ has norm at most $M = \mathcal{O}(nN)^m$. Therefore, the number of different solutions is at most $(M+1)^n$, which implies $\text{NORM}(U \cup P) \leq \text{poly}(nN)^{\text{poly}(n,m)}$. ◄

In the sequel we apply Lemma 40 in case when n and m are constants, in which case Lemma 40 yields the bound $\textsc{norm}(U+P) \leq \text{poly}(N)$.

Proof of Claim 37. For $d \in [1,2]$ and a path γ, let $\textsc{eff}_j(\gamma)$ denote the j-th coordinate of $\textsc{eff}(\gamma)$, and let $\textsc{drop}_j(\gamma) \in \mathbb{N}$ be the maximal value of $-\textsc{eff}_j(\delta)$, where δ ranges over prefixes of γ, that is the maximal amount by which the j-th coordinate can be decreased along γ.

W.l.o.g. we assume that Λ has exactly three loops, $\Lambda = \alpha_1\beta_1^*\alpha_2\beta_2^*\alpha_3\beta_3^*\alpha_4$. We describe paths $s = (s_1, s_2) \xrightarrow{*} (u_1, u_2)$ in Λ,

$$
\begin{aligned}
(s_1, s_2) = (a_1^1, a_2^1) \xrightarrow{\alpha_1} (b_1^1, b_2^1) \xrightarrow{\beta_1^{n_1}} (a_1^2, a_2^2) \xrightarrow{\alpha_2} (b_1^2, b_2^2) \xrightarrow{\beta_2^{n_2}} \\
(a_1^3, a_2^3) \xrightarrow{\alpha_3} (b_1^3, b_2^3) \xrightarrow{\beta_3^{n_3}} (a_1^4, a_2^4) \xrightarrow{\alpha_4} (b_1^4, b_2^4) = (u_1, u_2),
\end{aligned}
$$

by the following system of linear Diophantine inequalities, with unknowns $a_1^i, a_2^i, b_1^i, b_2^i$, for $i \in [1,4]$, and n_i, for $i \in [1,3]$, ensuring that the effects of α_i and $\beta_i^{n_i}$ are respected, and that all points along α_i remain nonnegative :

$$
\begin{array}{rcl}
a_j^i + \textsc{eff}_j(\alpha_i) & = & b_j^i \\
b_j^i + n_i \cdot \textsc{eff}_j(\beta_i) & = & a_j^{i+1} \\
a_j^i - \textsc{drop}_j(\alpha_i) & \geq & 0
\end{array}
\qquad\qquad
\begin{array}{rcl}
a_j^1 & = & s_j \\
b_1^4 & = & u_1
\end{array}
$$

Notice that each β_i is a single transition, so nonnegativity of (b_i^1, b_i^2) and (a_{i+1}^1, a_{i+1}^2) implies that all the vectors along $\beta_i^{n_i}$ are also nonnegative. Therefore, we do not need to add an analog of $a_j^i - \textsc{drop}_j(\alpha_i) \geq 0$ for β_i to the above system. By adding dummy variables we change inequalities into equations, thus obtaining a system $\mathcal{U}$ of linear Diophantine equations. All the coefficients in $\mathcal{U}$ are bounded by $\max(B, M)$, and all coefficients in its homogeneous version are bounded by M. Therefore, by Lemma 40 the solution set of $\mathcal{U}$ is $U + P^*$, where $\textsc{norm}(U) \leq \text{poly}(B, M)$ and $\textsc{norm}(P) \leq \text{poly}(M)$. By projecting the solution set to the variable b_2^4, we deduce that the set $S := \{u_2 \mid (u_1, u_2) \in \textsc{Reach}(\Lambda, s)\}$ is a finite union of linear sets $a + X^*$, where $a \leq P_1(B, M)$ and $X \subseteq [0, P_2(M)]$, for some nondecreasing polynomials P_1, P_2.

▷ Claim 41. For every $a, b \in \mathbb{N}$ and $B \subseteq [1, b]$, the linear set $a + B^*$ is a finite union of arithmetic sets $c + d^*$, where $c \leq a + b^3$ and $d \leq b$.

Proof. Let $B = \{b_1, \ldots, b_m\}$. Let $n \in a + B^*$, and let $(k_1, \ldots, k_m) \in \mathbb{N}^m$ be the lexicographically smallest vector such that $n = a + \sum_{i=1}^m k_i \cdot b_i$. We observe that $k_i < b$ for all $i \in [1, m-1]$ since, supposing $k_i \geq b$ for $i < m$, we would get a lexicographically smaller vector

$$(k_1, \ldots, k_{i-1}, k_i - b_m, k_{i+1}, \ldots, k_m + b_i) \in \mathbb{N}^m$$

with the same property. Thus $n \in a + r + b_m^*$, where $r = \sum_{i<m} k_i \cdot b_i \leq b^3$. As $n \in a + B^*$ was chosen arbitrarily, we deduce

$$a + B^* \;=\; \bigcup_{c \leq a + b^3,\, c \in a + B^*} c + b_m^*,$$

which concludes the proof of Claim 41. ◀

By Claim 41, the set S is a finite union of arithmetic sets $a + x^*$, where $a \leq P_1(B, M) + P_2(M)^3$ and $x \leq P_2(M)$, which concludes the proof of Claim 37. Notice that we actually get $T = \infty$ or $T = 0$ in all the arithmetic sets, but that is not needed for our considerations. ◀

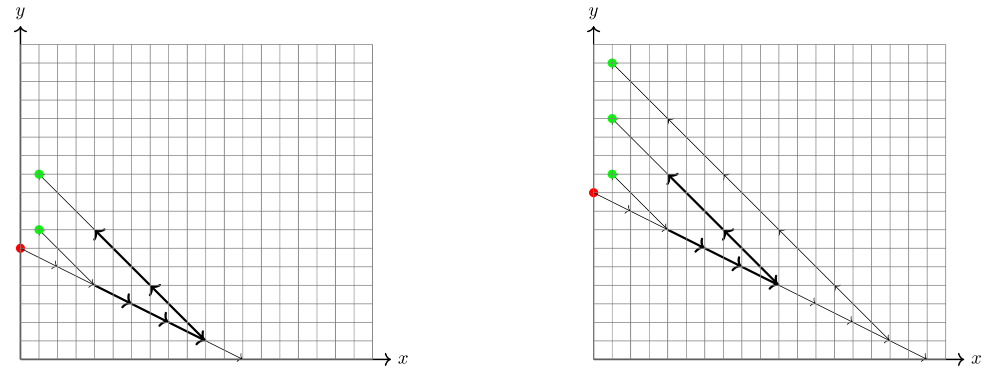

Figure 3 Left: $u_2 = 6$, $S_2 = \{7, 10\}$. Right: $u_2 = 9$, $S_2 = \{10, 13, 16\}$. Thick vectors add up to p.

Proof of Claim 38. Fix a one-turn SLPS $\Lambda = \alpha_1\beta_1^*\alpha_2\beta_2^*$, and let $M := \text{SIZE}(\Lambda)$ and $S_2 := R(S_1)$. Our goal is to show that the set $S_2 = \bigcup_{c\in S_1} R(\{c\})$ is a finite union of (a', r', T')-arithmetic sequences for $a' \leq a + \text{poly}(B, M, r)$ and $r' \leq \text{poly}(M)$. We write $R(c)$ instead of $R(\{c\})$.

Let $\text{EFF}(\beta_1) = (x_1, -y_1)$ and $\text{EFF}(\beta_2) = (-x_2, y_2)$, for some $x_1, x_2, y_1, y_2 \in \mathbb{N}_{>0}$. Every path $\rho = \alpha_1\beta_1^{n_1}\alpha_2\beta_2^{n_2}$ is determined by a pair $(n_1, n_2) \in \mathbb{N}^2$, and hence when $(u_1, u_2) \xrightarrow{*} (v_1, v_2)$, we may also write $(u_1, u_2) \xrightarrow{(n_1,n_2)} (v_1, v_2)$ for $(n_1, n_2) \in \mathbb{N}^2$, or $(u_1, u_2) \xrightarrow{(n_1,n_2)}$ when the target vector is not relevant. Whenever this happens, we necessarily have the equality $u_1 + \text{EFF}_1(\alpha_1\alpha_2) + n_1x_1 - n_2x_2 = v_1$ of effects on the first coordinate, which we transform to an equation with two unknowns n_1, n_2:

$$n_1x_1 - n_2x_2 = v_1 - u_1 - \text{EFF}_1(\alpha_1\alpha_2). \tag{12}$$

The set of solutions (n_1, n_2) of (12) is of the form $w + p^*$, where $w = (w_1, w_2) \in \mathbb{N}^2$ is the minimal solution of (12) and $p = (p_1, p_2) \in \mathbb{N}^2$ is the minimal solution of its uniform version, $n_1x_1 - n_2x_2 = 0$. By Lemma 40 we get $\text{NORM}(w) \leq \text{poly}(B, M)$. One can easily observe that $\text{NORM}(p) \leq 2M$ ($(x_1, x_2) = (n_2, n_1)$ is a solution). Let $\text{EFF}_2(w) = -w_1y_1 + w_2y_2$ and $\text{EFF}_2(p) = -p_1y_1 + p_2y_2 \in \mathbb{N}$ be the effects induced by w and p, respectively, on the second coordinate.

► **Example 42.** Let α_1 and α_2 be empty sequences, $\text{EFF}(\beta_1) = (2, -1)$ and $\text{EFF}(\beta_2) = (-3, 3)$. Let $u_1 = 0$, $v_1 = 1$. The set of solutions (n_1, n_2) of (12) is of the form $w + p^*$, where $w = (2, 1)$ and $p = (3, 2)$, and $\text{EFF}_2(p) = 3 \cdot (-1) + 2 \cdot 3 = 3$. Figure 3 shows $R(u_2) = \{7, 10\}$ when $u_2 = 6$, and $R(u_2) = \{10, 13, 16\}$ when $u_2 = 9$. In the latter case, elements of $R(u_2)$ correspond to the solutions w, $w + p$ and $w + 2p$ of (12), i.e., to $w + kp$ for k in an interval $[0, 2]$. According to Claim 43, this is true in general.

▷ Claim 43. For each $c \in S_1$ there exists an interval $I_c = [k_1, k_2]$, where $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}_\infty$, such that $R(c) = \{\text{EFF}_2(w) + k \cdot \text{EFF}_2(p) \mid k \in I_c\}$.

Proof. It is sufficient to show that whenever $(u_1, c) \xrightarrow{w+k_1\cdot p}$ and $(u_1, c) \xrightarrow{w+k_2\cdot p}$ for some $k_1 < k_2 \in \mathbb{N}$, then $(u_1, c) \xrightarrow{w+k\cdot p}$ for all $k \in [k_1, k_2]$. Fix $k \in [k_1, k_2]$. Every point x on the (possibly $\mathbb{Z}$-)path $(u_1, c) \xrightarrow{w+k\cdot p}$ is actually on a straight line between some two points, one on the path $(u_1, c) \xrightarrow{w+k_1\cdot p}$, and the other on the path $(u_1, c) \xrightarrow{w+k_2\cdot p}$. In consequence x, being a weighted average of the two points in $\mathbb{N}^2$, necessarily belongs to $\mathbb{N}^2$. Therefore $(u_1, c) \xrightarrow{w+k\cdot p}$ is a path. ◀

We notice that $\text{EFF}_2(p)$ can be negative, but this is irrelevant for our arguments. By Claim 43, for each $c \in S_1$ the set $R(c)$ is is an arithmetic sequence of difference $\overline{r} := |\text{EFF}_2(p)|$ and length equal to the cardinality of the interval I_c. Let $\text{SPAN}(c) \in \mathbb{N}_\infty$ be the difference between the supremum of $R(c)$ and the minimal element in $R(c)$. We say that $\text{SPAN}(c) = \infty$ if $R(c)$ is infinite. We split the proof into cases, depending on whether $\overline{r}$ divides the difference r of the sequence S_1, or not. Additionally we have a case when $\overline{r} = 0$, in other cases we silently assume that $\overline{r} \neq 0$.

Case I: r is divisible by $\overline{r}$. Therefore, if $\textsc{span}(c) \geq r$ then the sequence $R(c)$ actually touches the sequence $R(c+r)$, i.e., their union is a larger arithmetic sequence of difference $\overline{r}$:

▷ Claim 44. If $R(c+r^*) \neq \emptyset$ and $c \geq D := 3(M+r) \cdot M^2 + M \cdot \textsc{norm}(w)$, then $R(c+r^{\leq T}) = b + (\overline{r})^{\leq T'}$ for some $b \leq c + \textsc{eff}_2(w) + M \cdot \textsc{eff}_2(p)$ and $T' \in \mathbb{N}_\infty$.

Proof. We first show that $\textsc{span}(c) \geq r$. Suppose $R(c+r^*) \neq \emptyset$ and $c \geq D$. Due to the first assumption, for some $n \in \mathbb{N}$ there is a path $(u_1, c+nr) \xrightarrow{(n_1,n_2)}$ of the form

$$(u_1, c+nr) \xrightarrow{\alpha_1} (\overline{x}_1, \overline{y}_1) \xrightarrow{\beta_1^{n_1}} (\overline{x}_2, \overline{y}_2) \xrightarrow{\alpha_2} (\overline{x}_3, \overline{y}_3) \xrightarrow{\beta_2^{n_2}} (v_1, v_2). \tag{13}$$

In particular, $u_1 \geq \textsc{drop}_1(\alpha_1)$ and therefore $\overline{x}_1 \geq 0$. It is enough to take $(n_1, n_2) := w + Mp$ in (13). Then $n_1 \geq M$, so $\overline{x}_2 \geq \overline{x}_1 + M \geq M \geq \textsc{drop}_1(\alpha_2)$. Therefore $\overline{x}_3 \geq 0$. Now we show that c is large enough such that $(u_1, c) \xrightarrow{w+Mp}$ is nonnegative on the second coordinate as well. As $\textsc{norm}(p) \leq 2M$ then $n_1 + n_2 \leq M \cdot \textsc{norm}(p) + \textsc{norm}(w) \leq 2M^2 + \textsc{norm}(w)$. Therefore $\beta_1^{n_1}$ and $\beta_2^{n_2}$ can in total decrease the second coordinate by at most $M \cdot (2M^2 + \textsc{norm}(w))$. As $\alpha_1 + \alpha_2$ can in total decrease the second coordinate by at most M and $D \geq M + M \cdot (2M^2 + \textsc{norm}(w))$ we conclude that indeed the path $(u_1, c) \xrightarrow{w+Mp}$ is valid. For the same reasons, for every $m \in [M, M+r]$ there is a path $(u_1, c) \xrightarrow{w+mp}$, which guarantees $\textsc{span}(c) \geq r$.

Now we use that fact that $\textsc{span}(c) \geq r$. By monotonicity of $\textsc{vass}$, if $R(c) = b + (\overline{r})^{\leq T'}$ then $R(c+r)$ necessarily includes $R(c) + r = b + r + (\overline{r})^{\leq T'}$. Since $\textsc{span}(c) \geq r$, we have $b + r \in R(c)$, but also $b + r \in R(c+r)$, and therefore the union $R(c) \cup R(c+r)$ forms one arithmetic sequence $b + (\overline{r})^{\leq T'}$, for some $b \in \mathbb{N}$ and T'. The similar reasoning applies to any finite union, namely to $R(c + r^{\leq m})$ for $m \in \mathbb{N}$. In consequence, for every $T \in \mathbb{N}_\infty$ we have $R(c + r^{\leq T}) = b + (\overline{r})^{\leq T'}$, for some $b \in \mathbb{N}$ and $T' \in \mathbb{N}_\infty$, and since $c + \textsc{eff}_2(w) + M \cdot \textsc{eff}_2(p) \in R(c)$ we get the inequality $b \leq c + \textsc{eff}_2(w) + M \cdot \textsc{eff}_2(p)$, as required. ◀

We are ready for concluding Case I. As $\textsc{norm}(w) \leq \text{poly}(B, M)$ we have $D \leq \text{poly}(B, M, r)$. We partition $S_1 = a + r^{\leq T}$ into two subsets: $S_1' = S_1 \cap [0, D)$ and $S_1'' = S_1 \cap [D, \infty)$, both being arithmetic sequences of difference r, and consider S_1' and S_1'' separately.

Concerning $S_2' := R(S_1')$, as $\max(S_1') \leq D$, all elements of S_2' are upper-bounded by a polynomial in M and r, namely $\max(S_2') \leq M^2 \cdot (2M + D)$. Thus S_2' can be seen as a finite sum of singletons, each of which being an (a', r', T')-arithmetic sequence with $a' \leq M^2 \cdot (2M + D) \leq \text{poly}(B, M, r)$, $r' = 1$ and $T' = 0$. Clearly $r' = 1 \leq \text{poly}(M)$, and hence S_2' is of the required form.

Now we consider $S_2'' := R(S_1'')$. If $S_2'' = \emptyset$ we are done. Otherwise, let $c := \min(S_1'')$. Thus $S_1'' = c + r^{\leq T}$ for some $T \in \mathbb{N}_\infty$, and $D \leq c \leq \max(a, D + r)$. By Claim 44 we deduce that $S_2'' = b + (\overline{r})^{\leq T'}$ for some $b \leq c + \textsc{eff}_2(w) + M \cdot \textsc{eff}_2(p)$ and $T' \in \mathbb{N}_\infty'$. As $\textsc{eff}_2(w) \leq \text{poly}(B, M)$, $\textsc{eff}_2(p) \leq \text{poly}(M)$ and $c \leq a + \text{poly}(B, M, r)$ we get $b \leq a + \text{poly}(B, M, r)$. We also have $\overline{r} \leq \text{poly}(M)$, and hence S_2'' is of the required form.

Case II: r is not divisible by $\overline{r}$. In that case we split $S_1 = a + r^{\leq T}$ into several arithmetic sequences of difference $r \cdot \overline{r}$, namely into sequences of a form $(a + m \cdot r) + (r \cdot \overline{r})^{\leq T'}$, where $m < \overline{r}$, and apply the above reasoning to each of this sequences separately. As $\overline{r} \leq \text{poly}(M)$ we get also a finite set of arithmetic sequences with the base bounded by $a + \text{poly}(B, M, r)$ and difference bounded by $\text{poly}(M)$, as required.

Case III: $\overline{r} = 0$. W.l.o.g. we assume $R(S_1) \neq \emptyset$. For every $c \in S_1$ we have that either $R(c) = c + \textsc{eff}_2(w)$ or $R(c) = \emptyset$. By monotonicity of VASS we have that if $R(c) \neq \emptyset$ then $R(c+r) \neq \emptyset$. Let $D := 3(M+r) \cdot M^2 + M \cdot \textsc{norm}(w)$. Similarly as in the proof of Claim 44 we observe that if $c \geq D$ then for some $k \in N$ there is a run $(u_1, c) \xrightarrow{w+kp}$. Hence we have $R(S_1) = c + \text{EFF}_2(w) + r^{\leq T'}$ for some $T' \in \mathbb{N}_\infty$ and some $c \in S_1$ such that $c \leq \max(a, D + r)$. Therefore $R(S_1) = b + r^{\leq T'}$ for some $b \leq a + \text{poly}(B, M, r)$ as required.

◀

Proof of Claim 39. W.l.o.g. we assume $\Lambda = \alpha_1 \beta_1^* \alpha_2 \beta_2^* \alpha_3 \beta_3^* \alpha_4$. Let $S_1 = b + \{p\}^{\leq T}$ and $S_2 = \text{REACH}_\Lambda(S_1)$. Recall, that $\text{EFF}_j(\gamma)$ denotes the j-th coordinate of $\text{EFF}(\gamma)$, and $\text{DROP}_j(\gamma) \in \mathbb{N}$ is the maximal value of $-\text{EFF}_j(\delta)$, where δ ranges over prefixes of γ, that is the maximal amount by which the j-th coordinate can be decreased along γ.

Similarly as in the proof of Claim 37 we describe paths $s = (s_1, s_2) \xrightarrow{*} (t_1, t_2) = t$ in Λ,

$$\begin{aligned}(s_1, s_2) =& (a_1^1, a_2^1) \xrightarrow{\alpha_1} (b_1^1, b_2^1) \xrightarrow{\beta_1^{n_1}} (a_1^2, a_2^2) \xrightarrow{\alpha_2} (b_1^2, b_2^2) \xrightarrow{\beta_2^{n_2}} \\ & (a_1^3, a_2^3) \xrightarrow{\alpha_3} (b_1^3, b_2^3) \xrightarrow{\beta_3^{n_3}} (a_1^4, a_2^4) \xrightarrow{\alpha_4} (b_1^4, b_2^4) = (t_1, t_2).\end{aligned}$$

Notice that $(s_1, s_2) = (a_1^1, a_2^1) \in S_1$, so $(a_1^1, a_2^1) = u + p^n$ for some $n \in \mathbb{N}$. The following system of linear Diophantine inequalities $\mathcal{U}$ with unknowns $a_1^i, a_2^i, b_1^i, b_2^i$ for $i \in [1, 4]$, and n_i, for $i \in [1, 3]$, and unknown n, ensures that the effects of α_i and $\beta_i^{n_i}$ are respected, and that all points along α_i remain nonnegative and that $s \in S_1$:

$$\begin{aligned} a_j^i + \text{EFF}_j(\alpha_i) &= b_j^i \qquad & a_j^1 &= u_j + n \cdot p_j \\ b_j^i + n_i \cdot \text{EFF}_j(\beta_i) &= a_j^{i+1} & n &\leq T \\ a_j^i - \text{DROP}_j(\alpha_i) &\geq 0 \end{aligned}$$

Notice that each β_i is a single transition, so nonnegativity of (b_i^1, b_i^2) and (a_{i+1}^1, a_{i+1}^2) implies that all the vectors along $\beta_i^{n_i}$ are also nonnegative. Therefore, we do not need to add an analog of $a_j^i - \text{DROP}_j(\alpha_i) \geq 0$ for β_i to the above system. We first focus on the solutions of $\mathcal{U}$ without the equation $n \leq T$, and inequalities $a_j^i - \text{DROP}_j(\alpha_i) \geq 0$ transformed into equations with dummy variables on the right, similarly as in the proof of Claim 37. Let us call such system $\mathcal{U}'$. All the coefficients of $\mathcal{U}'$ are bounded by $\max(M, \text{NORM}(u), \text{NORM}(p))$. By Lemma 40 set of solutions of $\mathcal{U}'$ can be described as $L(U, V)$ for $\text{NORM}(U \cup V) \leq \text{poly}(M, \text{NORM}(u), \text{NORM}(p))$.

Now we have to care about the last inequality, namely $n \leq T$. If for some $a \in U$ we have $n > T$ then we can remove it from U. Let U' be the set U without the removed elements. As the set U' is finite it is enough to prove the conclusion of Claim 39 separately for each $a \in U'$. Fix $a \in U'$. We have $\text{NORM}(a), \text{NORM}(V) \leq \text{poly}(M, \text{NORM}(u), \text{NORM}(p))$. It is enough to prove that elements of the set $L(a, V)$ that additionally satisfy $n \leq T$, projected to the variables b_1^4 and b_2^4 can be described as a finite union of the sets of the form we need.

Let us consider all the elements of set V. Let $Q \subseteq V$ be the set of these elements $v \in V$, for which $v[n] = 0$ (that means that unknown n is equal to 0 in elements v), while $P = V \setminus Q$ be the set of the other elements $v \in V$, so that for which $v[n] > 0$. Notice that using elements in Q does not influence satisfying $n \leq T$, therefore they can be used unbounded number of times. Let $P = \{p_1, \ldots, p_\ell\}$ and let $p_i[n] = c_i$. For each $x \in L(a, V)$ we have $x = a + q + \Sigma_{i=1}^{\ell} k_i \cdot p_i$ where $q \in Q^*$. Hence, in order to satisfy $n \leq T$ we have to satisfy $c \diamond k \leq (T - a[n])$, where $c = (c_1, \ldots, c_\ell) \in \mathbb{N}_{>0}^\ell$ and $k = (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell$. As $c \in \mathbb{N}_{>0}^\ell$, if $T - a[n] < 0$ then the set of solutions is empty. Otherwise, the set of solutions can be represented as $a + P^{c \cdot x \leq T - a[n]} + Q^*$. Recall that $\text{NORM}(a), \text{NORM}(P \cup Q) \leq \text{poly}(M, \text{NORM}(u), \text{NORM}(p))$, and additionally $c \in \mathbb{N}_{>0}^\ell$. Additionally $\text{NORM}(c) \leq \ell \cdot \text{NORM}(P)$, where ℓ is the number of elements in P, thus $\ell \leq (\text{NORM}(P) + 1)^k$, where k is the number of unknowns in $\mathcal{U}$ (so k is a constant). In consequence $\text{NORM}(c) \leq \text{poly}(M, \text{NORM}(u), \text{NORM}(p))$. Summarising, the projection of $a + P^{c \cdot x \leq T - a[n]} + Q^*$ into variables b_1^4 and b_2^4 is of the required form.

◀

Claims 37, 38 and 39 are thus shown, and hence so is Lemma 35. ◀

Proof of Lemma 21. Fix an arbitrary geometrically 2-dimensional 3-VASS (V, s) and let $M = \text{SIZE}(V, s)$. Norms of vectors generating $\text{CONE}(V)$ — i.e., effects of simple cycles — are at most M, as no transition repeats along a simple cycle. The effect $\delta \in \mathbb{Z}^3$ of each simple cycle satisfies $a \diamond \delta = 0$, where $a \in \mathbb{Z}^3$ is a vector orthogonal to $\text{LIN}(V)$, or equivalently, orthogonal to some two effects of simple cycles. The vector a is thus an integer solution of a system of 2 linear equations with 3 unknowns, where absolute values of coefficients are bounded by M. By Lemma 4, there is such an integer solution $a = (a_1, a_2, a_3)$ with $\text{NORM}(a) \le D = \mathcal{O}(M^2)$.

In consequence, on every path $s \xrightarrow{*} t$ the value of inner product with a is bounded polynomially with respect to M:

▷ Claim 45. Every configuration $q(x) \in \text{REACH}(V, s)$ satisfies $-C \le a \diamond x \le C$, where $C = \mathcal{O}(M \cdot D)$.

We rely on the construction of [27, Lemma 5.1], which transforms a geometrically 2-dimensional 3-VASS (V, s) into a 2-VASS $(\overline{V}, \overline{s})$ of size at most $R(M)$ for some polynomial R, by dropping on of dimensions of V.

Case I: a contains both positive and negative numbers. W.l.o.g. assume that $a_1, a_2 \ge 0$ and $a_3 < 0$, in which case it is the third coordinate which is dropped by the construction of [27, Lemma 5.1]. States of $\overline{V}$ are of the form q_c, where $q \in Q$ and $c \in [-C, C]$, plus some further auxiliary states, omitted here. Due to Claim 45, there is a one-to-one correspondence between reachable configurations in V and reachable configurations in $\overline{V}$:

$$e = q(x_1, x_2, x_3) \quad \longmapsto \quad \overline{e} = q_c(x_1, x_2), \qquad \text{where } c = a \diamond x.$$

The tight correspondence between paths of V and $\overline{V}$, given in Claim 46 below, is essentially Lemma 5.1 of [27]:

▷ Claim 46. For every configurations s, u, here is a path $s \xrightarrow{*} u$ in $\overline{V}$ if, and only if, there is a path $\overline{s} \xrightarrow{*} \overline{u}$ in $\overline{V}$.

By Lemma 20, there is a polynomial F such that for every $B \in \mathbb{N}$, in the 2-VASS $(\overline{V}, \overline{s})$ obtained by the above construction, for every its state q_c, the set $\text{REACH}_{q_c}(\overline{V}, \overline{s})$ is $(F(M'), B)$-approximately semi-linear, where $M' = \text{SIZE}(\overline{V}, \overline{s}) \le R(M)$, and hence also $(F(R(M)), B)$-approximately semi-linear. We claim that for every state $q \in Q$, for every $B \in \mathbb{N}$, the set $\text{REACH}_q(V, s)$ is $(G(F(R(M))),$ $B)$-approximately semi-linear, for some nondecreasing polynomial G. Indeed, for any $B \in \mathbb{N}$, any (B-approximation of) a linear set $L = w + P^* \subseteq \mathbb{N}^2$, where $w = (w_1, w_2)$, witnessing that $\text{REACH}_{q_c}(\overline{V}, \overline{s})$ is $(F(R(M)), B)$-approximately semi-linear is transformed to a (B-approximation of) linear set L' witnessing that $\text{REACH}_q(V, s)$ is $(G(F(R(M))), B)$-approximately semi-linear, as follows. Take as base the unique vector $w' = (w_1, w_2, w_3)$ such that $a_1 w_1 + a_2 w_2 + a_3 w_3 = c$. For every period $p = (p_1, p_2) \in P$, take into the set P' the unique vector $p' = (p_1, p_2, p_3)$ such that $a_1 p_1 + a_2 p_2 + a_3 p_3 = 0$. Since $a_3 > 0$, it is guaranteed that $p_3 \ge 0$, and therefore $p' \in \mathbb{N}^3$. Let the polynomial G bound the blowup of $\text{NORM}(b')$ with respect to $\text{NORM}(b)$, and $\text{NORM}(p')$ with respect to $\text{NORM}(p)$, for instance $G(x) = M \cdot x + B$. The union of all (B-approximations of) so described sets $L' = w' + (P')^*$, for all $c \in [-C, C]$, provides the witness that $\text{REACH}_q(V, s)$ is $(G(F(R(M))),$ $B)$-approximately semi-linear.

Case II: a is non-negative or non-positive. W.l.o.g. assume $a \ge \overrightarrow{0}$ and $a_3 > 0$. By Claim 45, for each $q(x) \in \text{REACH}(V, s)$ we thus have $x_3 \le C$. We transform (V, s) into $(\overline{V}, \overline{s})$ with states of the form q_c, where $q \in Q$ and $c \in [0, C]$, by storing the third coordinate in state:

$$e = q(x_1, x_2, x_3) \quad \longmapsto \quad \overline{e} = q_c(x_1, x_2), \qquad \text{where } c = x_3.$$

As above, $\text{SIZE}(\overline{V}, \overline{s}) \le R(M)$, for a polynomial R. The argument that for every $B \in \mathbb{N}$, the set $\text{REACH}_q(V, s)$ is $(G(F(R(M))), B)$-approximately semi-linear, is similar to Case I (but simpler). ◀

C Proofs for Section 6 (Proof of Lemma 2)

Proof of Claim 22. $\textsc{SeqCone}(V)$ is equivalently definable as the last element C_k of the sequence of (rational) open cones $C_1, \ldots, C_k$, defined as follows. We put $C_1 := \textsc{Cone}(V_1) \cap (\mathbb{Q}_{>0})^3$, and for $i > 1$ we define inductively:

$$C_i \ := \ \big(C_{i-1} \ + \ \textsc{Cone}(V_i)\big) \cap (\mathbb{Q}_{>0})^3,$$

where the addition is Minkowski sum $X + Y = \{x + y \mid x \in X, y \in Y\}$. Then all $C_1, \ldots, C_k$ are finitely generated open cones, as $(\mathbb{Q}_{>0})^3$ is such a cone, and Minkowski sum and intersection preserve finitely generated open cones. ◀

Proof of Claim 27. The inequality is shown easily by induction on k. When $k = 1$, by definition of c we have $h_1(m) \leq m^c \leq m^C$, as required. Furthermore, assuming $h_{k-1}(m) \leq m^{C^{2^{k-1}-1}}$ for all $m > 1$, we get

$$h_k(m) \ \leq \ m^{C \cdot (C^{2^{k-1}-1})^2} \ = \ m^{C^{2^k-1}},$$

as required. ◀

Proof of Lemma 24. Consider a k-component 3-VASS $V = (V_1)u_1(V_2)u_2 \ldots u_{k-1}(V_k)$, where $V_i = (Q_i, T_i)$ and $u_i = (p'_i, \delta_i, p_{i+1})$, together with source and target configurations: $s = p_1(w)$ in V_1 and $t = p'_k(w')$ in V_k. Let $M = \textsc{size}(V, s, t) = \textsc{size}(V) + \textsc{norm}(s) + \textsc{norm}(t)$.

Suppose (V, s, t) is diagonal and wide, say $(\mathbb{Q}_{>0})^3 \subseteq \textsc{SeqCone}(V)$. We have $p_1(w) \xrightarrow{\pi} p_1(w+\Delta)$ and $p'_k(w' + \Delta') \xrightarrow{\pi'} p'_k(w')$ for some $\Delta, \Delta' \in (\mathbb{N}_{>0})^3$, and $(\mathbb{Q}_{>0})^3 \subseteq \textsc{SeqCone}(V)$.

Let P be a nondecreasing polynomial witnessing Lemma 8, i.e., 3-$\mathbb{Z}$-VASS are length-bounded by P. As V has a path $s \xrightarrow{*} t$, it also has a $\mathbb{Z}$-path $s \xrightarrow{*} t$. By Lemma 8, V has a $\mathbb{Z}$-path $s \xrightarrow{\sigma} t$ of length at most $P(M)$. The $\mathbb{Z}$-path factorises into components:

$$\sigma \ = \ \sigma_1;\, u_1;\, \ldots;\, \sigma_{k-1};\, u_{k-1};\, \sigma_k. \tag{14}$$

As in Case 1 of the proof of Lemma 9, let R be a nondecreasing polynomial such that in every 3-VASS of size m, the length of a covering path is at most $R(m)$ [24, Lemma 3.4]. We generalise Lemma 10 and prove that certain multiplicity of Δ' may be obtained by executing first a cycle in V_1, then a cycle in V_2, and so on, and finally a cycle in V_k, so that the total effect of the first j cycles is in $(\mathbb{N}_{>0})^3$, for every $j \in [1, k]$, and the lengths of all the cycles are bounded by a polynomial of degree $\mathcal{O}(k)$:

▶ **Lemma 47.** *There is an integer cascade $(\Delta'_1, \ldots, \Delta'_k)$ and $\ell \in \mathbb{N}_{>0}$ such that $\Delta'_1 + \ldots + \Delta'_k = \ell \cdot \Delta'$, and for $j \in [1, k]$ there are paths*

$$p_j(w + \Delta'_1 + \ldots + \Delta'_{j-1}) \xrightarrow{\pi_j} p_j(w + \Delta'_1 + \ldots + \Delta'_j)$$

in V_j of length $R(M)^{\mathcal{O}(k)}$.

Proof. Let ρ_j be a cycle in V_j that visits all states of V_j, and let $\Delta_j \in \mathbb{Z}^3$ be its effect, for $j \in [1, k]$. We have thus $\mathbb{Z}$-paths:

$$p_j(\Delta_1 + \ldots + \Delta_{j-1}) \xrightarrow{\rho_j} p_j(\Delta_1 + \ldots + \Delta_j).$$

Relying on $\Delta \in (\mathbb{N}_{>0})^3$, take a sufficiently large multiplicity $m \in \mathbb{N}_{>0}$ so that the $\mathbb{Z}$-paths become paths:

$$p_j(m \cdot \Delta + \Delta_1 + \ldots + \Delta_{j-1}) \xrightarrow{\rho_j} p_j(m \cdot \Delta + \Delta_1 + \ldots + \Delta_j). \tag{15}$$

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In particular, the tuple $(m \cdot \Delta + \Delta_1, \Delta_2, \ldots, \Delta_k)$ becomes a cascade. Let $\widetilde{\Delta} = m \cdot \Delta + \Delta_1 + \Delta_2 + \ldots + \Delta_k \in \mathbb{N}^3$ be the sum of the cascade. As $\Delta' \in (\mathbb{N}_{>0})^3$, there is $\ell' \in \mathbb{N}_{>0}$ such that $\ell' \cdot \Delta' - \widetilde{\Delta} \in (\mathbb{Q}_{>0})^3$, and hence, by wideness of (V, s), we have $\ell' \cdot \Delta' - \widetilde{\Delta} \in \textsc{SeqCone}(V)$, namely

$$\ell' \cdot \Delta' - \widetilde{\Delta} \; = \; s_1 + \ldots + s_k$$

is the sum of a cascade $(s_1, \ldots, s_k)$, where

$$\begin{aligned} s_1 \; &= \; r_{1,1} \cdot e_{1,1} + \ldots + r_{1,n_1} \cdot e_{1,n_1} \\ &\ldots \\ s_k \; &= \; r_{k,1} \cdot e_{k,1} + \ldots + r_{k,n_k} \cdot e_{k,n_k}. \end{aligned} \tag{16}$$

for some positive rational coefficients $r_{1,1}, \ldots, r_{1,n_1}, \ldots, r_{k,1}, \ldots, r_{k,n_k} \in \mathbb{Q}_{>0}$, where vectors $e_{j,1}, \ldots, e_{j,n_j} \in \mathbb{Z}^3$ are effects of simple cycles in V_j, for $j \in [1, k]$. Denote by RHS_j the jth right-hand side expression in (16). Therefore, the system $\mathcal{S}$ consisting of the equation

$$\ell \cdot (\ell' \cdot \Delta' - \widetilde{\Delta}) \; = \; \text{RHS}_1 + \ldots + \text{RHS}_k \tag{17}$$

together with the k equations

$$\begin{aligned} \ell_1 \; &= \; \text{RHS}_1 \\ \ell_2 \; &= \; \text{RHS}_1 + \text{RHS}_2 \\ &\ldots \\ \ell_k \; &= \; \text{RHS}_1 + \ldots + \text{RHS}_k, \end{aligned} \tag{18}$$

with unknowns $\ell, \ell_1, \ldots, \ell_k, r_{1,1}, \ldots, r_{k,n_k}$, has a positive integer solution. We rewrite the equation (17) to:

$$\ell\ell' \cdot \Delta' \; = \; \ell m \cdot \Delta + (\ell \cdot \Delta_1 + \text{RHS}_1) + \ldots + (\ell \cdot \Delta_k + \text{RHS}_k). \tag{19}$$

Irrespectively of the value of ℓ, the tuple $(\ell m \cdot \Delta + \ell \cdot \Delta_1, \ell \cdot \Delta_2, \ldots, \ell \cdot \Delta_k)$ is still a cascade, and due to $\ell_j > 0$ in (18) the tuple

$$(\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_k) \; := \; (\ell m \cdot \Delta + \ell \cdot \Delta_1 + \text{RHS}_1, \ell \cdot \Delta_2 + \text{RHS}_2, \ldots, \ell \cdot \Delta_k + \text{RHS}_k)$$

is a cascade as well. Let $r_j = r_0 + \ell \cdot (\Delta_1 + \ldots + \Delta_j) + \text{RHS}_1 + \ldots + \text{RHS}_j$ denote its jth partial sum, for $j \in [1, k]$, where $r_0 = \ell m \cdot \Delta$. Let $\sigma_{j,i}$ be a simple cycle of effect $e_{j,i}$ in V_j. Let σ_j be a $\mathbb{Z}$-path in V_j that starts (and ends) in state p_j and consists of the ℓ-fold concatenation of the cycle ρ_j, with attached $(r_{j,i})$-fold concatenation of each $\sigma_{j,i}$, for $i \in [1, n_j]$ (since ρ_j visits all states, this is possible):

$$p_j(r_{j-1}) \xrightarrow{\sigma_j} p_j(r_j). \tag{20}$$

The sum of effect of $\sigma_1, \ldots, \sigma_k$, plus r_0, yields the right-hand side of (19). Each of the paths starts and ends in $(\mathbb{N}_{>0})^3$ but may pass through non-positive points, and therefore it needs not be a path. Let $k \in \mathbb{N}_{>0}$ be a multiplicity large enough so that for every $j \in [1, k]$, the $\mathbb{Z}$-path (20) becomes a path when lifted by $(k-1) \cdot r_{j-1}$, i.e., when starting in $p_j(k \cdot r_{j-1})$, and also becomes a path when lifted by $(k-1) \cdot r_j$, i.e., when ending in $p_j(k \cdot r_j)$. In this case, the k-fold concatenation of each σ_j is also a path:

$$p_j(k \cdot r_{j-1}) \xrightarrow{(\sigma_j)^k} p_j(k \cdot r_j),$$

since all points visited in the inner iterations of σ_j are bounded from both sides by corresponding points visited in the first and the last iteration of σ_j. Therefore, the lifting of $(\sigma_j)^k$ by the source vector w is also a path:

$$p_j(w + k \cdot r_{j-1}) \xrightarrow{(\sigma_j)^k} p_j(w + k \cdot r_j). \tag{21}$$

Relying on (21), we define the cascade

$$(\Delta'_1, \ldots, \Delta'_k) \ := \ (k \cdot \widetilde{\Delta}_1, \ldots, k \cdot \widetilde{\Delta}_k),$$

and cycles $\pi_1, \ldots, \pi_k$: the cycle π_1 is $(\sigma_1)^k$ precomposed with the $(k\ell m)$-fold iteration of π,

$$p_1(w) \xrightarrow{\pi^{k\ell m}} p_1(w + k \cdot r_0) \xrightarrow{(\sigma_1)^k} p_1(w + \Delta'_1),$$

and for $j \in [2, k]$, the cycle π_j is $(\sigma_j)^k$,

$$p_j(w + \Delta'_1 + \ldots + \Delta'_{j-1}) \xrightarrow{(\sigma_j)^k} p_j(w + \Delta'_1 + \ldots + \Delta'_j),$$

as required.

The estimations of the length of the paths π_j are similar as in the proof of Lemma 10, so we focus only on new aspects. The number of different effects of cycles in each component is at most $(2M+1)^3 \leq \mathcal{O}(M^3)$, and therefore the number of unknowns $r_{j,i}$ in $\mathcal{S}$ is at most $k \cdot (2M+1)^3 \leq \mathcal{O}(M^4)$, and consequently so is the total number of unknowns in $\mathcal{S}$. The norm of a solution of the system $\mathcal{S}$ can now be bounded, due to Lemma 4, by $D = R(M)^{\mathcal{O}(k)}$. In consequence, we get the same bound $R(M)^{\mathcal{O}(k)}$ on lengths of paths π_j. This completes the proof of Lemma 47. ◀

We now use Lemma 47 to complete the proof of Lemma 24. Note that ℓ in Lemma 47 is necessarily also bounded by $R(M)^{\mathcal{O}(k)}$, and that for every $m \in \mathbb{N}_{>0}$ the m-fold iteration of the cycle π_j is also a path:

$$p_j(w + m \cdot (\Delta'_1 + \ldots + \Delta'_{j-1})) \xrightarrow{(\pi_j)^m} p_j(w + m \cdot (\Delta'_1 + \ldots + \Delta'_j)). \tag{22}$$

We pick an $m \in \mathbb{N}_{>0}$ and build a path ρ by interleaving the $\mathbb{Z}$-path (14) with m-fold iterations of the cycles $\pi_1, \ldots, \pi_k$ of Lemma 47:

$$\rho \ = \ (\pi_1)^m; \ \sigma_1; \ u_1; \ \ldots; \ \ (\pi_{k-1})^m; \ \sigma_{k-1}; \ u_{k-1}; \ (\pi_k)^m; \ \sigma_k;$$

As the effect of $(\pi_j)^m$ is $m \cdot \Delta'_j$, and the effect of σ is $w' - w$, we have:

$$p_1(w) \xrightarrow{\rho} p'_k(w' + m\ell \cdot \Delta').$$

We choose m sufficiently large to enforce that each of $\mathbb{Z}$-paths σ_j becomes a path, and hence the whole ρ is a path as well. It is enough to take $m = M \cdot P(M)$, which makes the length of ρ bounded by $P(M) \cdot R(M)^{\mathcal{O}(k)}$. Finally, we concatenate ρ with the $m\ell$-fold iteration of the path π',

$$p'_k(w' + m\ell \cdot \Delta') \xrightarrow{(\pi')^{m\ell}} p'_k(w'),$$

to get the required path ρ; $(\pi')^{m\ell}$ from $p_1(w)$ to $p'_k(w')$ of length bounded by $M \cdot P(M) \cdot R(M)^{\mathcal{O}(k)} \leq (M \cdot P(M) \cdot R(M)^{\mathcal{O}(1)})^k \leq M^{\mathcal{O}(k)}$. ◀

Proof of Lemma 28. Consider a non-easy k-component 3-VASS $V = (V_1)u_1(V_2)u_2 \ldots u_{k-1}(V_k)$, together with source and target configurations $s = p_1(w)$ and $t = p'_k(w')$. If V_1 is a geometrically 2-dimensional 3-VASS, there is nothing to prove as V is good-for-induction. If $(V_k)^{\mathrm{rev}}$ is a geometrically 2-dimensional 3-VASS then we are done too, as V^{rev} is good for induction and $\textsc{Len}(V, s, t) = \textsc{Len}(V^{\mathrm{rev}}, t, s)$. Therefore we assume from now on that V_1 and $(V_k)^{\mathrm{rev}}$ are of geometric dimension 3. In consequence, by Claim 22, all of $\textsc{Cone}(V_1), \textsc{SeqCone}(V)$, $\textsc{Cone}((V_k)^{\mathrm{rev}})$, $\textsc{SeqCone}(V^{\mathrm{rev}})$ are 3-dimensional open cones. We distinguish two cases, and hence the polynomial R is the sum of polynomials claimed in the respective cases.

Case I: (V, s, t) is non-diagonal. We may assume w.l.o.g. that V is non-forward-diagonal (otherwise replace V by V^{rev}), and therefore V_1 is so. Exactly as in Case 3 of the proof of Lemma 9, we transform (V_1, s) into three geometrically 2-dimensional 3-VASS $(\overline{V}_1, s_1), (\overline{V}_2, s_2), (\overline{V}_3, s_3)$. In each $(\overline{V}_i, s_i)$, we replace the target state p'_1 by $(p'_1)_b$, for an arbitrarily chosen value $b \in [0, B]$ of the bounded coordinate, and modify accordingly the first bridge transition $u_1 = (p'_1, \delta_1, p_2)$ to $\overline{u}_{i,b} = ((p'_1)_b, \delta_1, p_2)$. This yields a set S of $3(B + 1)$ good-for-induction k-component 3-VASS

$$S \;=\; \{\big((\overline{V}_i)\overline{u}_{i,b}(V_2)u_2 \ldots u_{k-1}(V_k), s_i, t\big) \mid i \in [1, 3],\; b \in [0, B]\},$$

which is length-equivalent to (V, s, t), as required. The size of each of these 3-VASS is at most $R(M)$, as in Claim 18 in Case 3 of the proof of Lemma 9.

Case II: (V, s, t) is non-wide. We proceed similarly to Case 2 of the proof of Lemma 9, and transform (V_1, s) into a geometrically 2-dimensional 3-VASS $(\overline{V}, \overline{s})$, defined as a (a, B)-trim of V_1 for some vector $a \in \mathbb{Z}^3$ and $B \in \mathbb{N}$. To this aim we need an analog of Claim 13 (Claim 50 below).

▷ Claim 48. $C_1 := \textsc{Cone}(V_1)$ and $S := \textsc{SeqCone}(V^{\text{rev}})$ are disjoint.

Proof. Let $S' = \textsc{SeqCone}((V')^{\text{rev}})$, where $V' = (V_2)u_2 \ldots u_{k-1}(V_k)$ is V without the first component and the first bridge. By definition, $S = (S' + \textsc{Cone}((V_1)^{\text{rev}})) \cap (\mathbb{Q}_{>0})^3$, but since $\textsc{Cone}((V_1)^{\text{rev}}) = -C_1$, we get

$$S = (S' - C_1) \cap (\mathbb{Q}_{>0})^3.$$

By Claim 22 the set S' is an open cone, and hence $S' - C_1$ is also so, as Minkowski sum preserves such cones. By definition $S' - C_1$ contains, for every vector $v \in C_1$, a vector $\varepsilon - v$ for some vector $\varepsilon \in S'$ of arbitrarily small norm $(*)$.

Towards a contradiction, suppose $C_1 \cap S$ is nonempty. Therefore, S, and hence also $S' - C_1$ contains some vector $v \in C_1$. Being an open cone, it also contains $v - \varepsilon$ for every vector ε of sufficiently small norm $(**)$. The two properties $(*)$ and $(**)$,

$$(*)\quad \forall_{v \in C_1}\, \forall_{N \in \mathbb{Q}}\, \exists_{\varepsilon, \textsc{norm}(\varepsilon) < N}\, (\varepsilon - v \in S' - C_1) \qquad (**)\quad \exists_{v \in C_1}\, \exists_{N \in \mathbb{Q}}\, \forall_{\varepsilon, \textsc{norm}(\varepsilon) < N}\, (v - \varepsilon \in S' - C_1),$$

imply that $S' - C_1$ contains both $v - \varepsilon$ and $\varepsilon - v$, for some $v \in C_1$ and $\varepsilon \in \mathbb{Q}^3$, and hence $S' - C_1$ includes a line. As $S - C_1$ is an open cone, we deduce $S' - C_1 = \mathbb{Q}^3$ and $S = (\mathbb{Q}_{>0})^3$, which means that V is wide, a contradiction. ◀

▷ Claim 49. Let $C \subseteq \mathbb{Q}^3$, $C' \subseteq (\mathbb{Q}_{>0})^3$ be 3-dimensional disjoint open cones, and $D \in \mathbb{Q}_{>0}$. All points whose distance to both cones is at most D, are at distance at most $3D$ to one of facet planes of C.

Proof. We give a geometric argument. Let S be any plane *separating* C and C', namely the two cones are on the opposite sides of S. Since $C' \subseteq (\mathbb{Q}_{>0})^3$ is included the positive quadrant, the plane S may be chosen to be adjacent to C, namely to satisfy one of the following conditions:

(1) S includes a facet F of C, or
(2) S includes an edge of C (adjacent to two facets F_1 and F_2).

Consider an arbitrary point $x \in \mathbb{Q}^3$ such that $d(x, C) \leq D$ and $d(x, C') \leq D$. Therefore $d(x, S) \leq D$, as C and C' are on opposite sides of S. Let $x' \in S$ be the point in S which is the closest to x. In case (1), we have $d(x, S) \leq D \leq 3D$, i..e, x is at distance at most D to the facet plane S. In case (2), let H_1, H_2 be the planes including F_1, F_2, respectively. Since $d(x, C) \leq D$ and $d(x, S) \leq D$, we deduce that $d(x, H_1) \leq D$ or $d(x, H_2) \leq D$. W.l.o.g. we assume that the angle between H_1 and S is at most as large as the angle between H_2 and S, and aim at showing

$d(x, H_1) \le 3D$. If $d(x, H_1) \le D$, we are done. Otherwise $d(x, H_2) \le D$, and hence by the triangle inequality we get $d(x', H_2) \le d(x', x) + d(x, H_2) \le 2D$. Since the angle between H_1 and S is not larger than the angle between H_2 and S, we deduce $d(x', H_1) \le d(x', H_2) \le 2D$, which implies $d(x, H_1) \le d(x, x') + d(x', H_1) \le 3D$, as required. As x was chosen arbitrarily, this completes the proof. ◀

Let $B := 9 \cdot D^2 \cdot P(M)^2$, where P comes from Lemma 23 and $D \le \mathcal{O}(M^2)$ from Claim 11.

▷ Claim 50. There is a vector $a \in \mathbb{Z}^3$ of $\textsc{norm}(a) \le D$ such that all configurations $q(x)$ in V_1 appearing on a path $s \xrightarrow{*} t$ in V satisfy $-B \le a \diamond x \le B$.

Proof. Consider a path $s \xrightarrow{\pi} t$ and let π_1 be its prefix in V_1. By Lemma 23, all the vectors appearing in π_1 are not further than $P(M)$ from $S = \textsc{SeqCone}(V^{\mathrm{rev}})$, but also not further than $P(M)$ from $C_1 = \textsc{Cone}(V_1)$. By disjointness of C_1 and S, due to Claim 48, and by Claim 49, there is a facet F of C such that all the vectors x appearing in π_1 are at distance at most $3 \cdot P(M)$ to the hyperplane H including F. Due to Claim 11, $H = \{y \mid a \diamond y = 0\}$ for some vector $a \in \mathbb{Z}^3$ of $\textsc{norm}(a) \le D = \mathcal{O}(M^2)$.

In order to bound the value of $a \diamond x$ for an arbitrarily vector x appearing in π_1, split x into $x = x_1 + x_2$, where x_1 is orthogonal to H_i, i.e., $x_1 = \ell \cdot a$ for some $\ell \in \mathbb{Q}$. W.l.o.g. assume that the length of a is at least 1. The length of x is $x_1 \diamond x_1 \le 9 \cdot P(M)^2$, and hence $x_1 = \ell \cdot a$ for some ℓ satisfying $|\ell| \le 9 \cdot P(M)^2$. In consequence, $a \diamond x = a \diamond x_1 = \ell \cdot a \diamond a$, and hence $|a \diamond x| \le \ell \cdot \textsc{norm}(a)^2 \le B = 9 \cdot D^2 \cdot P(M)^2$, as required. ◀

We complete the proof of Case II as in Case 2 of the proof of Lemma 9. We replace the first component V_1 by the geometrically 2-dimensional 3-VASS $\overline{V}$, as defined there, of size $E = \mathcal{O}(M \cdot B)$ (as stated in Claim 16), and the source configuration s by $\overline{s}$. We also replace the first bridge transition $u_1 = (p'_1, \delta_1, p_2)$ by $\overline{u}_b = ((p'_1)_b, \delta_1, p_2)$, for any $b \in [-B, B]$. This yields a set S of $2B+1$ good-for-induction k-component 3-VASS

$$S \;=\; \{\big((\overline{V})\overline{u}_b(V_2)u_2 \ldots u_{k-1}(V_k), \overline{s}_a, t\big) \mid a \in A,\; b \in [-B, B]\},$$

which is length-equivalent to (V, s, t), as required. The size of each of these 3-VASS is at most $R(M) = E + M \le \mathcal{O}(M \cdot B)$. ◀