On r-wise t-intersecting uniform families

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Abstract

We consider families, $\mathcal{F}$ of k-subsets of an n-set. For integers $r \geq 2$, $t \geq 1$, $\mathcal{F}$ is called r-wise t-intersecting if any r of its members have at least t elements in common. The most natural construction of such a family is the full t-star, consisting of all k-sets containing a fixed t-set. In the case $r = 2$ the Exact Erdős-Ko-Rado Theorem shows that the full t-star is largest if $n \geq (t+1)(k-t+1)$. In the present paper, we prove that for $n \geq (2.5t)^{1/(r-1)}(k-t)+k$, the full t-star is largest in case of $r \geq 3$. Examples show that the exponent $\frac{1}{r-1}$ is best possible. This represents a considerable improvement on a recent result of Balogh and Linz.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set. Let $2^{[n]}$ denote the power set of $[n]$ and let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. A subset $\mathcal{F} \subset \binom{[n]}{k}$ is called a *k-uniform family.*

The central notion of this paper is that of r-wise t-intersecting.

Definition 1.1. For positive integers r, t, $r \geq 2$, a family $\mathcal{F} \subset 2^{[n]}$ is called r-wise t-intersecting if $|F_1 \cap F_2 \cap \ldots \cap F_r| \geq t$ for all $F_1, F_2, \ldots, F_r \in \mathcal{F}$.

Let us define

$$m(n,r,t) = \max\left\{|\mathcal{F}|\colon \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-intersecting}\right\},$$
$$m(n,k,r,t) = \max\left\{|\mathcal{F}|\colon \mathcal{F} \subset \binom{[n]}{k} \text{ is } r\text{-wise } t\text{-intersecting}\right\}.$$

Let us define the so-called Frankl families (cf. [7])

$$\mathcal{A}_i(n,r,t) = \{A \subset [n]\colon A \cap [t+ri] \geq t + (r-1)i\},\ 0 \leq i \leq \frac{k-t}{r},$$
$$\mathcal{A}_i(n,k,r,t) = \mathcal{A}_i(n,t) \cap \binom{[n]}{k}.$$

Since $\mathcal{A}_i(n,r,t)$ consists of the sets A satisfying $|[t+ri] \setminus A| \leq i$, that is, sets that leave out at most i elements out of the first $t+ri$, $|A_1 \cap \ldots \cap A_r \cap [t+ri]| \geq t + ri - ri \geq t$ for all $A_1, \ldots, A_r \in \mathcal{A}_i(n,r,t)$.

Conjecture 1.2 ([7]).

$$m(n,r,t) = \max_i |\mathcal{A}_i(n,r,t)|; \tag{1.1}$$

$$m(n,k,r,t) = \max_i |\mathcal{A}_i(n,k,r,t)|. \tag{1.2}$$

Let us note that for $r=2$ the statement (1.1) is a consequence of the classical Katona Theorem [21].

Theorem 1.3 (The Katona Theorem [21]).

$$m(n,2,t) = |\mathcal{A}_{\lfloor \frac{n-t}{2} \rfloor}(n,2,t)|.$$

The case $r=2$ of (1.2) was a longstanding conjecture. It was proved in [15] for a wide range and it was completely established by the celebrated Complete Intersection Theorem of Ahlswede and Khachatrain [2].

A family $\mathcal{F} \subset \binom{[n]}{k}$ is called a *t-star* if there exists $T \subset [n]$ with $|T| = t$ such that $T \subset F$ for all $F \in \mathcal{F}$. The family $\{F \in \binom{[n]}{k} \colon T \subset F\}$ with some $T \in \binom{[n]}{t}$ is called a *full t-star.*

Let us recall a part of it that was proved earlier.

Theorem 1.4 (Exact Erdős-Ko-Rado Theorem [5], [9], [25]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 2-wise t-intersecting family. Then for $n \geq (t+1)(k-t+1)$,*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Moreover, for $n > (t+1)(k-t+1)$ equality holds if and only if $\mathcal{F}$ is the full t-star.

Theorem 1.4 motivates the following question that is the central problem of the present paper: determine or estimate $n_0(k,r,t)$, the minimal integer n_0 such that for all $n \geq n_0$ and all r-wise t-intersecting families $\mathcal{F} \subset \binom{[n]}{k}$, $|\mathcal{F}| \leq |\mathcal{A}_0(n,k,r,t)| = \binom{n-t}{k-t}$. Theorem 1.4 shows $n_0(k,2,t) = (t+1)(k-t+1)$.

Since the value $\binom{n-t}{k-t}$ is independent of r, it should be clear that $n_0(k,r,t)$ is a monotone decreasing function of r. Thus $n_0(k,r,t) \leq n_0(k,2,t) = (t+1)(k-t+1)$. For $t=1$ the exact value of $m(n,k,r,t)$ and thereby $n_0(k,r,t)$ is known (cf. [6]):

$$m(n,k,r,1) = \begin{cases} \binom{n-1}{k-1}, & \text{if } n \geq \frac{r}{r-1}k \\ \binom{n}{k}, & \text{if } n < \frac{r}{r-1}k. \end{cases} \tag{1.3}$$

Recently, Balogh and Linz [3] showed that

$$n_0(k,r,t) < (t+r-1)(k-t-r+3).$$

The main result of the present paper is

Theorem 1.5. *For $r = 3, 4$,*

$$n_0(k,r,t) \leq (2.5t)^{\frac{1}{r-1}}(k-t) + k. \tag{1.4}$$

For $r \geq 5$,

$$n_0(k,r,t) \leq (2t)^{\frac{1}{r-1}}(k-t) + k. \tag{1.5}$$

Let us show that (1.5) is essentially best possible for $t \geq 2^r - r$ and r sufficiently large. Precisely, for $t \geq 2^r - r$ we have

$$\left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}(k-t) < n_0(k,r,t) \leq (2t)^{\frac{1}{r-1}}(k-t)+k.$$

Let us prove the lower bound by showing that $|\mathcal{A}_1(n,k,r,t)| > \binom{n-t}{k-t}$ for $n = \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}(k-t-r+2)+t+r-2$. Note that

$$|\mathcal{A}_1(n,k,r,t)| = \binom{n-t-r}{k-t-r}+(t+r)\binom{n-t-r}{k-t-r+1} = \binom{n-t-r}{k-t-r}\left(1+\frac{(t+r)(n-k)}{k-t-r+1}\right)$$

and

$$\begin{aligned}
\frac{|\mathcal{A}_1(n,k,r,t)|}{\binom{n-t}{k-t}} &= \frac{(k-t)(k-t-1)\dots(k-t-r+1)}{(n-t)(n-t-1)\dots(n-t-r+1)}\left(1+\frac{(t+r)(n-k)}{k-t-r+1}\right)\\
&= \frac{(k-t)(k-t-1)\dots(k-t-r+2)}{(n-t)(n-t-1)\dots(n-t-r+2)}\frac{(t+r)n-(k+1)(t+r-1)}{n-t-r+1}\\
&> \left(\frac{k-t-r+2}{n-t-r+2}\right)^{r-1}\frac{(t+r)n-(k+1)(t+r-1)}{n-t-r+1}.
\end{aligned}$$

If $t \geq 2^r - r$ then $n = \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}(k-t-r+2)+t+r-2 \geq 2k-t-r+2$. Let us assume $k \geq t+r$ (this is no real restriction, cf. Proposition 1.9 below). It follows that

$$\frac{(t+r)n-(k+1)(t+r-1)}{n-t-r+1} \geq (t+r)\frac{n-k-1+\frac{k+1}{t+r}}{n-t-r+1} > \frac{(t+r)(n-k)}{n-t-r+1} > \frac{t+r}{2}.$$

Thus,

$$\frac{|\mathcal{A}_1(n,k,r,t)|}{\binom{n-t}{k-t}} > \left(\frac{k-t-r+2}{n-t-r+2}\right)^{r-1}\frac{t+r}{2} = 1.$$

Therefore for $t \geq 2^r - r$ we obtain that

$$\begin{aligned}
n_0(k,r,t) &> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}(k-t-r+2)+t+r-2\\
&> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}(k-t)+\left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}\left(2\left(\frac{t+r}{2}\right)^{\frac{r-2}{r-1}}-r\right)\\
&> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}(k-t)+\left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}\left(2^{r-1}-r\right)\\
&> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}}(k-t).
\end{aligned}$$

Our next result determines $m(n,k,3,2)$ for $n > 2k \geq 4$.

Theorem 1.6. *For $n > 2k \geq 4$,*

$$m(n,k,3,2) = \binom{n-2}{k-2}. \tag{1.6}$$

Moreover, in case of equality $\mathcal{F}$ is the full 2-star.

Let us note that Balogh and Linz [3] proved this for $n \geq 4(k-2)$ and in the much older paper [16] the weaker result $m(n,k,3,2) = (1+o(1))\binom{n-2}{k-2}$ was established for $k < 0.501n$.

Let us give two more numerical examples.

Proposition 1.7. *For $n \geq 2k$,*

$$m(n,k,4,3) = \binom{n-3}{k-3} \text{ and } m(n,k,4,4) = \binom{n-4}{k-4}.$$

The next result establishes the analogue of (1.6) for a wide range of the pair (r,t).

Theorem 1.8. *Let $n \geq \max\left\{2k, \frac{t(t-1)}{2\log 2} + 2t - 1\right\}$ and $t \leq 2^{r-2}\log 2 - 2$. Then*

$$m(n,k,r,t) = \binom{n-t}{k-t}. \tag{1.7}$$

Moreover, in case of equality $\mathcal{F}$ is the full t-star.

Let us show that for $k \leq t+r-2$ the only r-wise t-intersecting family is the t-star.

Proposition 1.9. *Suppose that $\mathcal{G}$ is an r-wise t-intersecting k-graph that is not a t-star ($|\cap \mathcal{G}| < t$). Then $k \geq t+r$ or $k = t+r-1$ and $\mathcal{G} \subset \binom{Y}{k}$ for some $(k+1)$-element set Y.*

Proof. We distinguish two cases.

(i) There exist $G_1, G_2 \in \mathcal{G}$ with $|G_1 \cap G_2| \leq k-2$.

Since $\mathcal{G}$ is 2-wise t-intersecting, we infer that $|G_1 \cap G_2| \geq t$. Choose a t-subset T of $G_1 \cap G_2$. Since $\mathcal{G}$ is not a t-star, there exist $G_3 \in \mathcal{G}$ and $x \in T$ such that $x \notin G_3$. Then $|G_1 \cap G_2 \cap G_3| \leq |(G_1 \cap G_2) \setminus \{x\}| = k-3$. Similarly, we can choose successively $G_4, \ldots, G_r$ to satisfy $|G_1 \cap \ldots \cap G_r| \leq k-r$. This proves $k-r \geq t$, i.e., $k \geq r+t$.

(ii) $\mathcal{G}$ is 2-wise $(k-1)$-intersecting.

Pick arbitrary $G_1, G_2 \in \mathcal{G}$ and set $Y = G_1 \cup G_2$, $Z = G_1 \cap G_2$. Then $|Y| = k+1$ and $|Z| = k-1$. Since $\mathcal{G}$ is 2-wise t-intersecting and $|Z| = k-1 > t$, there exists $G_3 \in \mathcal{G}$ with $Z \not\subset G_3$. Since $\mathcal{G}$ is 2-wise $(k-1)$-intersecting, $|G_i \cap G_3| \geq k-1$, $i = 1,2$. It follows that $G_3 \subset Y$. Without loss of generality, assume that $Y = [k+1]$ and $G_i = [k+1] \setminus \{i\}$, $i = 1,2,3$. If there exists $G \in \mathcal{G}$ with $|G \cap [k+1]| \leq k-1$. Then there exist $x, y \in [k+1]$ such that $G \subset [k+1] \setminus \{x,y\}$. Let $i \in [3] \setminus \{x,y\}$. Then $|G \cap G_i| \leq k+1-3 = k-2$, contradicting the assumption that $\mathcal{G}$ is 2-wise $(k-1)$-intersecting. Thus $\mathcal{G} \subset \binom{Y}{k}$. □

Based on Proposition 1.9 in the sequel we always assume that $n \geq k \geq t+r$.

As to the corresponding problem for the non-uniform case, Erdős-Ko-Rado [5] proved $m(n,2,1) = 2^{n-1}$. Then the first author [8] established $m(n,3,2) = 2^{n-2}$. After several partial results the proof of the following result was concluded in [14]:

$$m(n,r,t) = 2^{n-t} \text{ if and only if } t \leq 2^r - r - 1. \tag{1.8}$$

We call a family $\mathcal{F} \subset \binom{[n]}{k}$ *non-trivial* if $\cap\{F : F \in \mathcal{F}\} = \emptyset$. Define

$$m^*(n,r,t) = \max\left\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\right\},$$

$$m^*(n,k,r,t) = \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\right\}.$$

Theorem 1.10 (Brace-Daykin-Frankl Theorem (cf. [4] for $t=1$ and [12] for $t \geq 2$)). *For $t+r \leq n$ and $t < 2^r - r - 1$,*

$$m^*(n,r,t) = |\mathcal{A}_1(n,r,t)| = (t+r+1)2^{n-t-r}. \tag{1.9}$$

Let us recall some notations and useful results. For $i \in [n]$, define

$$\mathcal{F}(i) = \{F \setminus \{i\} \colon i \in F \in \mathcal{F}\}, \ \mathcal{F}(\bar{i}) = \{F \colon i \notin F \in \mathcal{F}\}.$$

For $P \subset Q \subset [n]$, define

$$\mathcal{F}(Q) = \{F \setminus Q \colon Q \subset F\}, \ \mathcal{F}(P,Q) = \{F \setminus Q \colon F \cap Q = P\}.$$

Let X be a finite set. For any $\mathcal{F} \subset \binom{X}{k}$ and $1 \leq b < k$, define the *bth shadow* $\partial^{(b)}\mathcal{F}$ as

$$\partial^{(b)}\mathcal{F} = \left\{E \in \binom{X}{k-b} \colon \text{there exists } F \in \mathcal{F} \text{ such that } E \subset F\right\}.$$

If $b=1$ then we simply write $\partial\mathcal{F}$ and call it *the shadow* of $\mathcal{F}$. Define the *up shadow* $\partial^+\mathcal{F}$ as

$$\partial^+\mathcal{F} = \left\{G \in \binom{X}{k+1} \colon \text{ there exists } F \in \mathcal{F} \text{ such that } F \subset G\right\}.$$

Sperner [24] proved the following result.

Theorem 1.11 ([24]). *For $\mathcal{F} \subset \binom{[n]}{k}$,*

$$\frac{|\partial^+\mathcal{F}|}{\binom{n}{k+1}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}}. \tag{1.10}$$

For $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$, we say that $\mathcal{A}, \mathcal{B}$ are *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Theorem 1.12 ([18]). *Let $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ be cross-intersecting. Then for $n \geq 2k$,*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k}. \tag{1.11}$$

We need the following version of the Kruskal-Katona Theorem.

Theorem 1.13 ([23, 22]). *Let n, k, m be positive integers with $k \leq m \leq n$ and let $\mathcal{F} \subset \binom{[n]}{k}$ and. If $|\mathcal{F}| > \binom{m}{k}$ then*

$$|\partial\mathcal{F}| > \binom{m}{k-1}.$$

We also need an inequality concerning the bth shadow of an r-wise t-intersecting family.

Theorem 1.14 ([13]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family. Then for $0 < b \leq t$ we have*

$$|\partial^{(b)}\mathcal{F}| \geq |\mathcal{F}| \min_{0 \leq i \leq \frac{k-t}{r-1}} \frac{\binom{ri+t}{i+b}}{\binom{ri+t}{i}}. \tag{1.12}$$

2 Shifting and lattice paths

In [5], Erdős, Ko and Rado introduced a very powerful tool in extremal set theory, called shifting. For $\mathcal{F} \subset \binom{[n]}{k}$ and $1 \leq i < j \leq n$, define the shifting operator

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) \colon F \in \mathcal{F}\},$$

where

$$S_{ij}(F) = \begin{cases} F' := (F \setminus \{j\}) \cup \{i\}, & \text{if } j \in F, i \notin F \text{ and } F' \notin \mathcal{F}; \\ F, & \text{otherwise.} \end{cases}$$

It is well known (cf. [11]) that the shifting operator preserves the size of $\mathcal{F}$ and the r-wise t-intersecting property. Thus one can apply the shifting operator to $\mathcal{F}$ when considering $m(n,k,r,t)$.

A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *shifted* if $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$. It is easy to show (cf. [11]) that every family can be transformed into a shifted family by applying the shifting operator repeatedly. Thus we can always assume that the family $\mathcal{F}$ is shifted when detemining $m(n,k,r,t)$.

Let us define the shifting partial order. Let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$ be two distinct k-sets with $a_1 < a_2 < \dots < a_k$ and $b_1 < b_2 < \dots < b_k$. We say that A *precedes* B in shifting partial order, denoted by $A \prec B$ if $a_i \leq b_i$ for $i = 1, 2, \dots, k$.

Let us recall two properties of shifted families:

Lemma 2.1 (cf. [11]). *If $\mathcal{F} \subset \binom{[n]}{k}$ is a shifted family, then $A \prec B$ and $B \in \mathcal{F}$ always imply $A \in \mathcal{F}$.*

Lemma 2.2 ([11]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted family. Then $\mathcal{F}$ is r-wise t-intersecting if and only if for every $F_1, \dots, F_r \in \mathcal{F}$ there exists s such that*

$$\sum_{1 \leq i \leq r} |F_i \cap [s]| \geq (r-1)s + t. \tag{2.1}$$

Note that $\sum\limits_{1 \leq i \leq r} |F_i \cap [s]| \leq rs$ implies $s \geq t$ if such an s exists. For completeness let us include the proof.

Proof. First we show that if $\mathcal{F}$ is r-wise t-intersecting then for every $F_1, \dots, F_r \in \mathcal{F}$ there exists s such that (2.1) holds. Argue indirectly and suppose $F_1, F_2, \dots, F_r$ is counter-example with $\sum_{1 \leq i \leq r} \sum_{j \in F_i} j$ minimal.

Let x be the t-th common vertex of $F_1, \dots, F_r$. By our assumption,

$$\sum_{1 \leq i \leq r} |F_i \cap [x]| < (r-1)x + t = rt + (r-1)(x-t). \tag{2.2}$$

Note that

$$|(F_1 \cap [x]) \cap (F_2 \cap [x]) \cap \dots \cap (F_r \cap [x])| = t.$$

By (2.2), there exists $y < x$ such that y is contained in at most $r-2$ of $F_1 \cap [x], F_2 \cap [x], \dots, F_r \cap [x]$. Since $\mathcal{F}$ is shifted, $F_1' := (F_1 \setminus \{x\}) \cup \{y\} \in \mathcal{F}$. Then $F_1', F_2, \dots, F_r$ is also a counter-example, contradicting the minimality of $\sum_{1 \leq i \leq r} \sum_{j \in F_i} j$.

Next we show that if (2.2) holds for every $F_1, \ldots, F_r \in \mathcal{F}$ then $\mathcal{F}$ is r-wise t-intersecting. Indeed, suppose that there exist $F_1, \ldots, F_r \in \mathcal{F}$ with $|F_1 \cap F_2 \cap \ldots \cap F_r| < t$. Then for any $s \geq t$ at most $t-1$ elements in $[s]$ are contained in r of $F_1, F_2, \ldots, F_r$. It follows that

$$\sum_{1 \leq i \leq r} |F_i \cap [s]| \leq r(t-1) + (r-1)(s-t+1) \leq (r-1)s + t - 1,$$

a contradiction. Thus the lemma holds. $\square$

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. For any $F_1, \ldots, F_r \in \mathcal{F}$, define $s(F_1, \ldots, F_r)$ to be the minimum s such that

$$\sum_{1 \leq i \leq r} |F_i \cap [s]| \geq (r-1)s + t.$$

Set $s := s(F_1, \ldots, F_r)$. Then we must have

$$\sum_{1 \leq i \leq r} |F_i \cap [s]| = (r-1)s + t.$$

Indeed, if $\sum\limits_{1 \leq i \leq r} |F_i \cap [s]| \geq (r-1)s + t + 1$ then

$$\sum_{1 \leq i \leq r} |F_i \cap [s-1]| \geq (r-1)s + t + 1 - r \geq (r-1)(s-1) + t,$$

contradicting the minimality of s. Set $F_1 = F_2 = \ldots = F_r = F$ for $F \in \mathcal{F}$, we obtain $r|F \cap [s]| = (r-1)s + t$. It follows that $\frac{s-t}{r} =: i$ is an integer. Then $s = t + ri$ and

$$\frac{(r-1)s + t}{r} = t + \frac{(r-1)(s-t)}{r} = t + (r-1)i.$$

Thus $|F \cap [t + ri]| \geq t + (r-1)i$ holds and we get the following corollary.

Corollary 2.3 ([11]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. Then for every $F \in \mathcal{F}$, there exists $i \geq 0$ so that $|F \cap [t + ri]| \geq t + (r-1)i$.*

In [9] a bijection between subsets and certain lattice paths was established. For $F \in \binom{[n]}{k}$, define $P(F)$ to be the lattice path in the two-dimensional integer grid $\mathbb{Z}^2$ starting at origin as follows. In the ith step for $i = 1, 2, \ldots, n$, from the current point (x, y) the path $P(F)$ goes to $(x, y+1)$ if $i \in F$ and goes to $(x+1, y)$ if $i \notin F$. Since $|F| = k$, there are exactly k vertical steps. Thus the end point of $P(F)$ is $(n-k, k)$.

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. By Corollary 2.3 we infer that $P(F)$ hits $y = (r-1)x + t$ for every $F \in \mathcal{F}$. For $F \in \mathcal{F}$, define $i(F)$ to be the minimum integer i such that $|F \cap [t + ri]| = t + (r-1)i$. Define

$$\mathcal{F}_i = \{F \in \mathcal{F} \colon i(F) = i\}, i = 0, 1, 2, \ldots, \left\lfloor \frac{k-t}{r-1} \right\rfloor.$$

By Corollary 2.3, $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{\lfloor \frac{k-t}{r-1} \rfloor}$ form a partition of $\mathcal{F}$.

The next lemma gives a universal bound ont the size of an r-wise t-intersecting family for $n \geq 2k - t$.

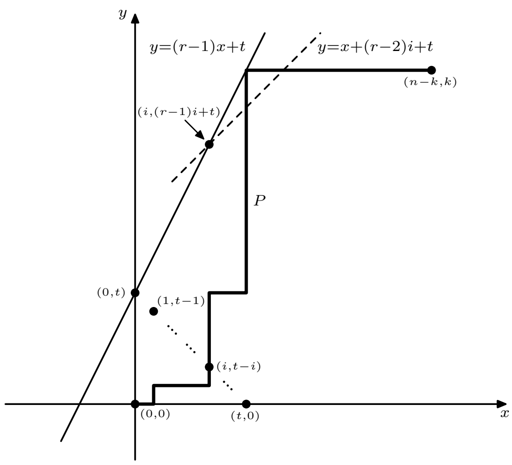

Figure 1: The lattice path P goes through $(i,t-i)$ and hits the line $y=(r-1)x+t$.

Lemma 2.4. *Let $\mathcal{F}\subset\binom{[n]}{k}$ be an r-wise t-intersecting family with $r\geq 3$ and $n\geq 2k-t$. Then*

$$|\mathcal{F}|\leq \sum_{0\leq i\leq t}\binom{t}{i}\binom{n-t}{k-t-(r-1)i}. \tag{2.3}$$

Moreover,

$$\sum_{i\geq 1}|\mathcal{F}_i|\leq \sum_{1\leq i\leq t}\binom{t}{i}\binom{n-t}{k-t-(r-1)i}. \tag{2.4}$$

Proof. Without loss of generality, we may assume that $\mathcal{F}$ is shifted. For each $F\in\mathcal{F}$, by Corollary 2.3 we infer that $P(F)$ hits the line $y=(r-1)x+t$. Note that the number of lattice paths that go through $(0,t)$ is exactly $\binom{n-t}{k-t}$. Let us count the number of lattice paths P that do not pass $(0,t)$. Then P has to go through exactly one of $(1,t-1),(2,t-2),\ldots,(t,0)$. Since $r\geq 3$, the paths that start at $(i,t-i)$ and hit the line $y=(r-1)x+t$ have to hit the line $y=x+(r-2)i+t$ (as shown in Figure 1). Note that the number of lattice paths from $(0,0)$ to $(i,t-i)$ is $\binom{t}{i}$. By the reflection principle (cf. e.g. [9]), the number of paths from $(i,t-i)$ to $(n-k,k)$ hitting $y=x+(r-2)i+t$ equals the the number of paths from $(-(r-1)i,(r-1)i+t)$ to $(n-k,k)$, which is $\binom{n-t}{k-t-(r-1)i}$. Thus,

$$\sum_{i\geq 1}|\mathcal{F}_i|\leq \sum_{1\leq i\leq t}\binom{t}{i}\binom{n-t}{k-t-(r-1)i}.$$

Since $|\mathcal{F}_0|\leq\binom{n-t}{k-t}$, (2.3) follows. $\square$

Fact 2.5. *Suppose $\mathcal{F}\subset 2^{[n]}$ is r-wise t-intersecting but $\mathcal{F}$ is not a t-star. Then for $2\leq s<r$, $\mathcal{F}$ is s-wise $(t+r-s)$-intersecting.*

Proof. Set $Y=\cap\{F\colon F\in\mathcal{F}\}$. Then $|Y|<t$ and by definition $\mathcal{F}(Y)$ is r-wise $(t-|Y|)$-intersecting and non-trivial. We need to show that $\mathcal{F}(Y)$ is s-wise $(t-|Y|+r-s)$-intersecting. Suppose the contrary and fix $G_1,G_2,\ldots,G_s\in\mathcal{F}(Y)$ satisfy $|G_1\cap\ldots\cap G_s|<t-|Y|+r-s$.

Using non-triviality we may choose successively $G_{s+1},\ldots,G_r$ to satisfy $|G_1\cap\ldots\cap G_s\cap G_{s+1}\cap\ldots\cap G_r|<t-|Y|$, i.e., $|(G_1\cup Y)\cap\ldots\cap(G_r\cap Y)|<t$, a contradiction. $\square$

Corollary 2.6. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family with $r \geq 3$. If $\mathcal{F}$ is not a t-star, then*

$$|\mathcal{F}| \leq \sum_{0 \leq i \leq t} \binom{t}{i} \binom{n-t}{k-t-(r-1)i} - \binom{n-t-1}{k-t}. \tag{2.5}$$

Proof. In the proof of (2.3) we counted $\binom{n-t}{k-t}$ for the paths through $(0,t)$. Since $\mathcal{F}$ is not a t-star, by Fact 2.5 we infer that $\mathcal{F}$ is $(r-1)$-wise $(t+1)$-intersecting. It follows that $\mathcal{F}([t])$ is $(r-1)$-wise intersecting. By (1.3) we have $|\mathcal{F}([t])| \leq \binom{n-t-1}{k-t-1}$. Now $\binom{n-t}{k-t} - \binom{n-t-1}{k-t-1} = \binom{n-t-1}{k-t}$ proves (2.5). □

3 Proof of Theorem 1.5

Proof of Theorem 1.5. Let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family with $n \geq (ct)^{\frac{1}{r-1}}(k-t)+k$ ($c \geq 1$ to be specified later). Without loss of generality, assume that $\mathcal{F}$ is shifted and is not a t-star. By Fact 2.5, $\mathcal{F}$ is $(r-1)$-wise $(t+1)$-intersecting and $(r-2)$-wise $(t+2)$-intersecting. It follows that $\mathcal{F}([t])$ is $(r-1)$-wise intersecting and $(r-2)$-wise 2-intersecting.

Since $n \geq (ct)^{\frac{1}{r-1}}(k-t)+k > 2k-t$, $n-t > 2(k-t)$ follows. By (1.3) we have

$$|\mathcal{F}_0| = |\mathcal{F}([t])| \leq \binom{n-t-1}{k-t-1} = \frac{k-t}{n-t}\binom{n-t}{k-t} < \frac{1}{2}\binom{n-t}{k-t}. \tag{3.1}$$

If $r \geq 5$, then $\mathcal{F}([t])$ is 3-wise 2-intersecting. By Theorem 1.6,

$$|\mathcal{F}_0| = |\mathcal{F}([t])| \leq \binom{n-t-2}{k-t-2} = \frac{(k-t)(k-t-1)}{(n-t)(n-t-1)}\binom{n-t}{k-t} < \frac{1}{4}\binom{n-t}{k-t}. \tag{3.2}$$

Using (2.4) we have

$$|\mathcal{F}| \leq |\mathcal{F}_0| + \sum_{1 \leq i \leq t} \binom{t}{i}\binom{n-t}{k-t-(r-1)i}.$$

Note that if $k-t-(r-1)i < 0$ then $\binom{n-t}{k-t-(r-1)i} = 0$. Let

$$f(n,k,r,t,i) := \binom{t}{i}\binom{n-t}{k-t-(r-1)i}.$$

Then for $1 \leq i \leq t-1$ and $n-k \geq (ct)^{\frac{1}{r-1}}(k-t)$,

$$\begin{aligned}\frac{f(n,k,r,t,i+1)}{f(n,k,r,t,i)} &\leq \frac{t-i}{i+1} \cdot \left(\frac{k-t-(r-1)i}{n-k+(r-1)(i+1)}\right)^{r-1} \\ &< \frac{t}{i+1} \cdot \left(\frac{k-t}{n-k}\right)^{r-1} \\ &\leq \frac{1}{c(i+1)}.\end{aligned}$$

It follows that for $i > 1$,

$$f(n,k,r,t,i) < \frac{1}{ci} f(n,k,r,t,i-1) < \frac{1}{c^{i-1} i!} f(n,k,r,t,1).$$

By $\sum\limits_{1\le i\le t}\frac{1}{c^i i!}<e^{1/c}-1$,

$$\sum_{1\le i\le t} f(n,k,r,t,i)\le f(n,k,r,t,1)\sum_{1\le i\le t}\frac{1}{c^{i-1}i!}<t\binom{n-t}{k-t-r+1}c(e^{1/c}-1).$$

Note that $n-k\ge (ct)^{\frac{1}{r-1}}(k-t)$ implies

$$\frac{\binom{n-t}{k-t-r+1}}{\binom{n-t}{k-t}}<\left(\frac{k-t}{n-k}\right)^{r-1}\le\frac{1}{ct}.$$

It follows that

$$\sum_{1\le i\le t}\binom{t}{i}\binom{n-t}{k-t-(r-1)i}=\sum_{1\le i\le t} f(n,k,r,t,i)<(e^{1/c}-1)\binom{n-t}{k-t}. \tag{3.3}$$

Note that $e^{1/2.5}-1<\frac{1}{2}$ and $e^{1/2}-1<\frac{3}{4}$. For $r=3,4$ and $c=2.5$, adding (3.1) and (3.3) we get

$$|\mathcal{F}|<\frac{1}{2}\binom{n-t}{k-t}+\sum_{1\le i\le t}\binom{t}{i}\binom{n-t}{k-t-(r-1)i}<\frac{1}{2}\binom{n-t}{k-t}+\frac{1}{2}\binom{n-t}{k-t}=\binom{n-t}{k-t}.$$

For $r\ge 5$ and $c=2$, adding (3.2) and (3.3) we conclude that

$$|\mathcal{F}|<\frac{1}{4}\binom{n-t}{k-t}+\sum_{1\le i\le t}\binom{t}{i}\binom{n-t}{k-t-(r-1)i}<\frac{1}{4}\binom{n-t}{k-t}+\frac{3}{4}\binom{n-t}{k-t}=\binom{n-t}{k-t}.$$

$\square$

4 The probability of hitting the line, uniform vs non-uniform

We need the following version of the Chernoff bound for the binomial distribution.

Theorem 4.1 ([20]). *Let $X\in Bi(n,p)$ and $\lambda=np$. Then*

$$Pr(X<\lambda-a)\le e^{-\frac{a^2}{2\lambda}}. \tag{4.1}$$

We call $P(n)$ a *p-random walk of length n* if it starts at origin and goes up a unit with probability p and goes right a unit with probability $1-p$ at each step. Let $f(n,r,t,p)$ be the probability that a p-random walk $P(n)$ hits the line $y=(r-1)x+t$. Set $f(r,t,p)=\lim\limits_{n\to\infty} f(n,r,t,p)$. That is, $f(r,t,p)$ is the probability that an infinite p-random walk hits the line $y=(r-1)x+t$.

Lemma 4.2 ([11],[12]). *(i) $f(n,r,t,p)\le f(n+1,r,t,p)$.*

(ii) $f(n+1,r,t,p)=pf(n,r,t-1,p)+(1-p)f(n,r,t+r-1,p)$.

(iii)

$$f(r,t,p)=\gamma^t,$$

where γ is the unique root of $x=p+(1-p)x^r$ in the open interval $(0,1)$.

(iv) Let α_r be the unique root of $x = \frac{1}{2} + \frac{1}{2}x^r$. Then

$$\alpha_3 = \frac{\sqrt{5}-1}{2},\ \frac{1}{2} < \alpha_r < \frac{1}{2} + \frac{1}{2^r} \text{ for } r \geq 4.$$

Moreover,

$$\frac{1}{2^r - r} < \alpha_r^r \leq \frac{1}{2^r - r - 1} \text{ for } r \geq 3. \tag{4.2}$$

Let us define another type of random walk. We call $Q(n,i)$ a *uniform random walk* if it is chosen uniformly from all lattice paths from $(0,0)$ to $(n-i,i)$. Let $g(n,i,r,t)$ be the probability that a uniform random walk $Q(n,i)$ hits the line $y = (r-1)x + t$.

Proposition 4.3. *(i) $g(n,i,r,t) \leq g(n,i+1,r,t)$.*

(ii) $g(n+1,k,r,t) \leq g(n,k,r,t)$.

(iii) For $r \geq 3$ and $t \geq 2$, $g(2k,k,r,t) \leq g(2k+2,k+1,r,t)$.

(iv) $\lim_{k\to\infty} g(2k,k,r,t) \leq f(r,t,\frac{1}{2})$.

Proof. First we prove (i). Let $\mathcal{G}_i \subset \binom{[n]}{i}$ be the collection of all i-sets F such that $P(F)$ hits the line $y = (r-1)x + t$. Let $E \in \binom{[n]}{i}$. If $P(E)$ hits $y = (r-1)x + t$ then so does $P(F)$ for every F with $E \subset F$. Thus $\partial^+\mathcal{G}_i \subset \mathcal{G}_{i+1}$. Note that $g(n,i,r,t) = \frac{|\mathcal{G}_i|}{\binom{n}{i}}$. By (1.10), we conclude that

$$g(n,i+1,r,t) = \frac{|\mathcal{G}_{i+1}|}{\binom{n}{i+1}} \geq \frac{|\partial^+\mathcal{G}_i|}{\binom{n}{i+1}} \geq \frac{|\mathcal{G}_i|}{\binom{n}{i}} = g(n,i,r,t).$$

Next we prove (ii). Note that a lattice path from $(0,0)$ to $(n+1-k,k)$ goes through either $(n-k,k)$ or $(n-(k-1),k-1)$. It follows that

$$g(n+1,k,r,t)\binom{n+1}{k} = g(n,k,r,t)\binom{n}{k} + g(n,k-1,r,t)\binom{n}{k-1}.$$

By (i) we have $g(n,k,r,t) \geq g(n,k-1,r,t)$. Thus,

$$g(n+1,k,r,t)\binom{n+1}{k} \leq g(n,k,r,t)\binom{n}{k} + g(n,k,r,t)\binom{n}{k-1} = g(n,k,r,t)\binom{n+1}{k}$$

and (ii) follows.

Thirdly we prove (iii). Let $\ell(t,i)$ be the number of lattice paths from $(0,0)$ to $(i,(r-1)i+t)$ that hit $y = (r-1)x+t$ first at $x = i$. Note that the number of lattice paths from $(i,(r-1)i+t)$ to (k,k) is $\binom{2k-ri-t}{k-(r-1)i-t}$. Thus,

$$g(2k,k,r,t) = \sum_{0\leq i\leq \frac{k-t}{r-1}} \ell(t,i)\frac{\binom{2k-ri-t}{k-(r-1)i-t}}{\binom{2k}{k}}.$$

Let $c_r(k,t,i) = \frac{\binom{2k-ri-t}{k-(r-1)i-t}}{\binom{2k}{k}}$. Then, using $\binom{2k}{k}/\binom{2k+2}{k+1} = \frac{k+1}{4k+2}$,

$$\begin{aligned}\frac{c_r(k+1,t,i)}{c_r(k,t,i)} &= \frac{\binom{2k+2-ri-t}{k+1-(r-1)i-t}}{\binom{2k-ri-t}{k-(r-1)i-t}} \cdot \frac{\binom{2k}{k}}{\binom{2k+2}{k+1}} \\ &= \frac{(2k+2-ri-t)(2k+1-ri-t)}{(k+1-(r-1)i-t)(k+1-i)} \cdot \frac{k+1}{4k+2}.\end{aligned}$$

Note that for $r \geq 3$ and $t \geq 2$ we have

$$\begin{aligned}
&(2k+2-ri-t)(2k+1-ri-t)(k+1)-(k+1-(r-1)i-t)(k+1-i)(4k+2)\\
&=\big(t(t-1)+2i(r-2)t+i(i(r-2)^2-r)\big)\,k+t(t-1)+2i(r-1)t+i(i(r-1)^2-r)+i^2>0.
\end{aligned}$$

It follows that $c_r(k+1,t,i) > c_r(k,t,i)$ for all $0 \leq i \leq \frac{k-t}{r-1}$. Thus,

$$\begin{aligned}
g(2k,k,r,t) &= \sum_{0\leq i\leq \frac{k-t}{r-1}} \ell(t,i)c_r(k,t,i)\\
&< \sum_{0\leq i\leq \frac{k-t}{r-1}} \ell(t,i)c_r(k+1,t,i) + \sum_{\frac{k-t}{r-1}<i\leq \frac{k+1-t}{r-1}} \ell(t,i)c_r(k+1,t,i)\\
&= g(2k+2,k+1,r,t).
\end{aligned}$$

Lastly we prove (iv). Let $k > 4\log k$ and let P be a p-random walk of length $2k$ with $p = \frac{1}{2} + \sqrt{\frac{\log k}{k}}$. Let X be the number of vertical steps on P. Then

$$\mathbb{E}X = (2k)p = k + 2\sqrt{k\log k}.$$

Since $k \geq 4\log k$ implies $2k \geq k + 2\sqrt{k\log k}$, by (4.1) we have

$$Pr(X<k) \leq e^{-\frac{2k\log k}{k+2\sqrt{k\log k}}} \leq e^{-\log k} = \frac{1}{k}. \tag{4.3}$$

Note that

$$\begin{aligned}
f(2k,r,t,p) &= \sum_{t\leq i\leq 2k} Pr(X=i)Pr[P \text{ hits } y=(r-1)x+t|X=i]\\
&= \sum_{t\leq i\leq 2k} Pr(X=i)g(2k,i,r,t)\\
&\geq \sum_{k\leq i\leq 2k} Pr(X=i)g(2k,i,r,t).
\end{aligned}$$

By Proposition 4.3 (i) we have $g(2k,i,r,t) \geq g(2k,k,r,t)$ for all $i \geq k$. It follows that

$$f(2k,r,t,p) \geq g(2k,k,r,t)\sum_{k\leq i\leq 2k} Pr(X=i) = g(2k,k,r,t)Pr(X\geq k).$$

By (4.3), we obtain that

$$f\left(2k,r,t,\frac{1}{2}+\sqrt{\frac{\log k}{k}}\right) \geq g(2k,k,r,t)\frac{k-1}{k}.$$

Letting k go to infinity on both sides, we obtain that

$$f\left(r,t,\frac{1}{2}\right) \geq \lim_{k\to\infty} g(2k,k,r,t).$$

□

Proposition 4.4. *For $n \geq 2k$,*

$$m(n,k,r,t) \leq \alpha_r^t\binom{n}{k}, \tag{4.4}$$

where α_r is the unique root of $x = \frac{1}{2}+\frac{1}{2}x^r$ in the interval $(0,1)$.

Proof. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family with $|\mathcal{F}| = m(n,k,r,t)$. By Corollary 2.3, we infer that $|\mathcal{F}|$ is at most the number of lattice paths from $(0,0)$ to $(n-k,k)$ hitting $y=(r-1)x+t$. By $n \geq 2k$ and Proposition 4.3 (ii) (iii) (iv), it follows that

$$|\mathcal{F}| = m(n,k,r,t) \leq g(n,k,r,t)\binom{n}{k} \leq g(2k,k,r,t)\binom{n}{k} \leq f\left(r,t,\frac{1}{2}\right)\binom{n}{k} = \alpha_r^t\binom{n}{k}.$$

□

5 Proof of Theorem 1.6

Let us prove a useful corollary of Theorem 1.14.

Corollary 5.1. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise t-intersecting family. If $t \geq 4$ then $|\partial^{(2)}\mathcal{F}| > 4|\mathcal{F}|$. If $t \geq 7$ then $|\partial^{(4)}\mathcal{F}| > 16|\mathcal{F}|$.*

Proof. For $t \geq 4$, we have

$$\frac{\binom{3i+t}{i+2}}{\binom{3i+t}{i}} = \frac{(2i+t-1)(2i+t)}{(i+1)(i+2)} > 2 \times 2 = 4$$

Applying Theorem 1.14 with $b=2$, we obtain that

$$|\partial^{(2)}\mathcal{F}| \geq |\mathcal{F}| \min_{0 \leq i \leq \frac{k-t}{2}} \frac{\binom{3i+t}{i+2}}{\binom{3i+t}{i}} > 4|\mathcal{F}|.$$

Similarly, if $t \geq 7$ then

$$\begin{aligned}\frac{\binom{3i+t}{i+4}}{\binom{3i+t}{i}} &= \frac{(2i+t)(2i+t-1)(2i+t-2)(2i+t-3)}{(i+4)(i+3)(i+2)(i+1)} \\ &\geq \frac{(2i+7)(2i+6)(2i+5)(2i+4)}{(i+4)(i+3)(i+2)(i+1)} \\ &= 4\frac{(2i+7)(2i+5)}{(i+4)(i+1)}.\end{aligned}$$

Since

$$(2i+7)(2i+5) = 4\left(i^2+6i+\frac{35}{4}\right) > 4(i^2+5i+4) = 4(i+4)(i+1),$$

we infer that $\frac{\binom{3i+t}{i+4}}{\binom{3i+t}{i}} > 4 \times 4 = 16$. Applying Theorem 1.14 with $b=4$, we obtain that

$$|\partial^{(4)}\mathcal{F}| \geq |\mathcal{F}| \min_{0 \leq i \leq \frac{k-t}{2}} \frac{\binom{3i+t}{i+4}}{\binom{3i+t}{i}} > 16|\mathcal{F}|.$$

□

Fact 5.2. *For $n \geq \frac{\sqrt{4t+9}-1}{2}k$, $|\mathcal{A}_1(n,k,3,t)| < \binom{n-t}{k-t}$. For $n = \left(\frac{\sqrt{4t+9}-1}{2} - \epsilon\right)k$ with some $0 < \epsilon < \frac{1}{10}$ and $k \geq \frac{t^2+2t}{2\epsilon}$, $|\mathcal{A}_1(n,k,3,t)| > \binom{n-t}{k-t}$.*

Proof. By Proposition 1.9 we assume $k \geq t+3$. Note that $|\mathcal{A}_1(n,k,3,t)| = (t+3)\binom{n-t-3}{k-t-2} + \binom{n-t-3}{k-t-3} = \frac{(t+3)n-(t+2)(k+1)}{k-t-2}\binom{n-t-3}{k-t-3}$. Then

$$\frac{|\mathcal{A}_1(n,k,3,t)|}{\binom{n-t}{k-t}} = \frac{(k-t)(k-t-1)((t+3)n-(t+2)(k+1))}{(n-t)(n-t-1)(n-t-2)}.$$

Let $n = xk$ and define

$$\begin{aligned} f(k,x) &:= (k-t)(k-t-1)((t+3)n-(t+2)(k+1)) - (n-t)(n-t-1)(n-t-2) \\ &= (k-t)(k-t-1)\left(((t+3)x-(t+2))k-(t+2)\right) - (xk-t)(xk-t-1)(xk-t-2). \end{aligned}$$

By simplification, we obtain that

$$f(k,x) = -(x-1)k\left((x^2+x-t-2)k^2 + (2t^2+4t-3(t+1)x)k - (t+1)(t^2-t-1)\right).$$

Let

$$g(k,x) = (x^2+x-t-2)k^2 + (2t^2+4t-3(t+1)x)k - (t+1)(t^2-t-1).$$

If $x \geq \frac{\sqrt{4t+9}-1}{2}$, then $x^2+x-t-2 \geq 0$. By $k \geq t+3$, it follows that

$$\begin{aligned} g(k,x) &\geq (x^2+x-t-2)k(t+3) + (2t^2+4t-3(t+1)x)k - (t+1)(t^2-t-1) \\ &= (t^2+3(x^2-2)+t(x^2-2x-1))k + 1 + 2t - t^3. \end{aligned}$$

Since $x \geq \frac{\sqrt{4t+9}-1}{2} > 1.56$ and $t \geq 2$ imply $3(x^2-2) > 0$ and $t(x^2-2x-1)+t^2 > 0$,

$$\begin{aligned} g(k,x) &\geq \left(t^2+3(x^2-2)+t(x^2-2x-1)\right)(t+3) + 1 + 2t - t^3 \\ &\geq (x^2-2x+2)t^2 + (6x^2-6x-7)t + 9x^2 - 17 \\ &\geq t^2 + (6x^2-6x-7)t + (9x^2-17) \\ &\geq (6x^2-6x-5)t + (9x^2-17) \\ &> 0. \end{aligned}$$

Thus $f(k,x) < 0$ for $x \geq \frac{\sqrt{4t+9}-1}{2}$. Therefore $|\mathcal{A}_1(n,k,3,t)| < \binom{n-t}{k-t}$ for $n \geq \frac{\sqrt{4t+9}-1}{2}k$.

If $x = \frac{\sqrt{4t+9}-1}{2} - \epsilon$, then by $\epsilon < \frac{1}{10}$ and $t \geq 2$,

$$x^2 - x + 2 = -\epsilon(\sqrt{4t+9}-\epsilon) \leq -\epsilon(\sqrt{17}-\epsilon) < -4\epsilon.$$

It follows that for $k \geq \frac{t^2+2t}{2\epsilon}$,

$$(x^2+x-t-2)k^2 + (2t^2+4t-3(t+1)x)k - (t+1)(t^2-t-1) < -4\epsilon k^2 + (2t^2+4t)k \leq 0$$

Thus $|\mathcal{A}_1(n,k,3,t)| > \binom{n-t}{k-t}$ for $k \geq \frac{t^2+2t}{2\epsilon}$ and $n = (\frac{\sqrt{4t+9}-1}{2} - \epsilon)k$. $\square$

Proof of Theorem 1.6. By Proposition 1.9, we assume $k \geq t+r = 5$. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted 3-wise 2-intersecting family that is not a 2-star. By Fact 5.2, $|\mathcal{A}_1(n,k,3,2)| < \binom{n-2}{k-2}$

for $n \geq 2k$. Thus we may assume as well $\mathcal{F} \not\subset \mathcal{A}_1(n,k,3,2)$. We partition $\mathcal{F}$ according $F \cap [5]$. Define

$$\begin{aligned}
\mathcal{F}_0 &= \{F \in \mathcal{F} \colon \{1,2\} \subset F\}\,, \\
\mathcal{F}_i &= \{F \in \mathcal{F} \colon F \cap [5] = [5] \setminus \{i\}\}\,,\ i = 1,2, \\
\mathcal{F}_3 &= \{F \in \mathcal{F} \colon \{1,2\} \not\subset F \text{ and } |F \cap [5]| = 3\}\,, \\
\mathcal{F}_4 &= \{F \in \mathcal{F} \colon \{1,2\} \not\subset F \text{ and } |F \cap [5]| = 2\}\,, \\
\mathcal{F}_5 &= \{F \in \mathcal{F} \colon |F \cap [5]| = 1\}\,, \\
\mathcal{F}_6 &= \{F \in \mathcal{F} \colon F \cap [5] = \emptyset\}\,.
\end{aligned}$$

Then

$$|\mathcal{F}| = \sum_{0 \leq i \leq 5} |\mathcal{F}_i|.$$

Since $\mathcal{F}$ is 3-wise 2-intersecting and it is not a 2-star, by Fact 2.5 $\mathcal{F}$ is 2-wise 3-intersecting. It follows that $\mathcal{F}(\{1,2\})$ is 2-wise intersecting. By (1.3) we have

$$|\mathcal{F}_0| = |\mathcal{F}(\{1,2\})| \leq \binom{n-3}{k-3}. \tag{5.1}$$

Set

$$\mathcal{A}_i = \{F \setminus [5] \colon F \cap [5] = [5] \setminus \{i\}\}\,, i = 1,2.$$

Claim 5.3. We may assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are cross-intersecting.

Proof. Indeed, otherwise there exist $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap F_2 = \{3,4,5\}$. Using shiftedness and the 3-wise 2-intersecting property, $|F \cap [5]| \geq 4$ for all $F \in \mathcal{F}$. Then $\mathcal{F} \subset \mathcal{A}_1(n,k,3,2)$, contradicting our assumption. $\square$

Since $\mathcal{A}_1, \mathcal{A}_2 \subset \binom{[6,n]}{k-4}$ are cross-intersecting, $n-5 > 2(k-4)$, by (1.11) we have

$$|\mathcal{F}_1| + |\mathcal{F}_2| = |\mathcal{A}_1| + |\mathcal{A}_2| \leq \binom{n-5}{k-4}. \tag{5.2}$$

Note that $P(F)$ goes through $(2,3)$ and hits $y = 2x+2$ for every $F \in \mathcal{F}_3$. Since $n-5 \geq 2(k-3)$, by Proposition 4.3 (ii) and (iv) the number of lattice paths from $(2,3)$ to $(n-k,k)$ hitting $y = 2x+2$ is at most $\left(\frac{\sqrt{5}-1}{2}\right)^3 \binom{n-5}{k-3}$. The number of 3-sets $B \subset [5]$ with $[2] \not\subset B$ is $\binom{5}{3} - 3 = 7$. Thus,

$$|\mathcal{F}_3| \leq 7 \left(\frac{\sqrt{5}-1}{2}\right)^3 \binom{n-5}{k-3} < 1.66 \binom{n-5}{k-3}. \tag{5.3}$$

Similarly, $P(F)$ goes through $(3,2)$ and hits $y = 2x+2$ for every $F \in \mathcal{F}_4$ and the number of 2-sets $B \subset [5]$ with $[2] \not\subset B$ is $\binom{5}{2} - 1 = 9$. Since $n > 2k$ implies $n-5 \geq 2(k-2)$, by Proposition 4.3 (ii) and (iv) we infer that

$$|\mathcal{F}_4| \leq 9 \left(\frac{\sqrt{5}-1}{2}\right)^6 \binom{n-5}{k-2} < 0.51 \binom{n-5}{k-2}. \tag{5.4}$$

Let $\mathcal{B}_i = \mathcal{F}(\{i\},[5]) \subset \binom{[6,n]}{k-1}$, $i=1,2,\dots,5$. By shiftedness $\mathcal{B}_i$ is 3-wise 9-intersecting. Let $\mathcal{D}_i = \partial^{(2)}\mathcal{B}_i$. Then it is easy to see that $\mathcal{D}_i$ is 3-wise $9-2\times 3=3$-intersecting. Since $n-5>2(k-3)$, by Proposition 4.4 we infer that

$$|\mathcal{D}_i| \leq \left(\frac{\sqrt{5}-1}{2}\right)^3 \binom{n-5}{k-3}.$$

Since $\mathcal{B}_i$ is 3-wise 9-intersecting, by Corollary 5.1 we get

$$|\mathcal{D}_i| > 4|\mathcal{B}_i|,\ \ i=1,2,3,4,5.$$

Thus,

$$|\mathcal{F}_5| = \sum_{1\leq i\leq 5} |\mathcal{B}_i| \leq \frac{1}{4}\sum_{1\leq i\leq 5}|\mathcal{D}_i| < \frac{5}{4}\left(\frac{\sqrt{5}-1}{2}\right)^3\binom{n-5}{k-3} < 0.3\binom{n-5}{k-3}. \tag{5.5}$$

Let $\mathcal{A}_6 = \partial^{(4)}\mathcal{F}_6 \subset \binom{[6,n]}{k-4}$. Since $\mathcal{F}_6$ is 3-wise 12-intersecting, by Corollary 5.1 we get $|\mathcal{A}_6| > 16|\mathcal{F}_6|$. By shiftedness, $\mathcal{A}_6 \subset \mathcal{A}_i$ for $i=1,2$. By Claim 5.3, we infer that $\mathcal{A}_6$ is intersecting. Thus, by $k-2<(n-5)-(k-5)$ we obtain that

$$|\mathcal{F}_6| < \frac{1}{16}|\mathcal{A}_6| \leq \frac{1}{16}\binom{n-6}{k-5} < \frac{1}{16}\binom{n-5}{k-5} < \frac{1}{16}\binom{n-5}{k-2}. \tag{5.6}$$

Adding (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6), we conclude that

$$\begin{aligned}
|\mathcal{F}| &= \sum_{0\leq i\leq 6}|\mathcal{F}_i| \\
&\leq \binom{n-3}{k-3} + \binom{n-5}{k-4} + 1.66\binom{n-5}{k-3} + 0.51\binom{n-5}{k-2} + 0.3\binom{n-5}{k-3} + \frac{1}{16}\binom{n-5}{k-2} \\
&< \binom{n-5}{k-5} + 3\binom{n-5}{k-4} + 3\binom{n-5}{k-3} + \binom{n-5}{k-2} \\
&= \binom{n-2}{k-2}.
\end{aligned}$$

□

6 Proof of Proposition 1.7 and Theorem 1.8

Let us prove a useful inequality.

Lemma 6.1. *For $n > \frac{rk-t}{r-1}$,*

$$m(n,k,r,t) \leq m(n-1,k,r,t) + m(n-1,k-1,r,t). \tag{6.1}$$

Proof. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family with $|\mathcal{F}| = m(n,k,r,t)$. Clearly $\mathcal{F}(\bar{n})$ is r-wise t-intersecting. It follows that $|\mathcal{F}(\bar{n})| \leq m(n-1,k,r,t)$. We claim that $\mathcal{F}(n)$ is also r-wise t-intersecting. Indeed, otherwise there exist $G_1,G_2,\dots,G_r \in \mathcal{F}(n)$ such that $|G_1\cap G_2\cap\dots\cap G_r| = t-1$. If each $i\in[n-1]$ is contained in at least $r-1$ of $G_1,G_2,\dots,G_r$, then

$$\sum_{1\leq i\leq r} G_i = rk \geq (r-1)\left((n-1)-(t-1)\right) + rt = (r-1)n + t,$$

contradicting $n > \frac{rk-t}{r-1}$. Thus there exists $x \in [n-1]$ such that x is contained in at most $r-2$ of $G_1, G_2, \ldots, G_r$. Note that $G_i \cup \{n\} \in \mathcal{F}$. Since $G_1 \cup \{x\} \prec G_1 \cup \{n\}$, by shiftedness we have $G_1 \cup \{x\} \in \mathcal{F}$. However,

$$|(G_1 \cup \{x\}) \cap (G_2 \cup \{n\}) \cap \ldots \cap (G_r \cup \{n\})| = |G_1 \cap G_2 \cap \ldots \cap G_r| = t-1,$$

contradicting the fact that $\mathcal{F}$ is r-wise t-intersecting. Thus $\mathcal{F}(n)$ is r-wise t-intersecting, implying that $|\mathcal{F}(n)| \le m(n-1, k-1, r, t)$. Therefore,

$$m(n,k,r,t) = |\mathcal{F}| = |\mathcal{F}(\bar{n})| + |\mathcal{F}(n)| \le m(n-1,k,r,t) + m(n-1,k-1,r,t). \qquad \square$$

Lemma 6.2. *Suppose that $m(n,k,r,t) = \binom{n-t}{k-t}$ then*

$$m(n, k-1, r, t) = \binom{n-t}{k-1-t}.$$

Proof. Assume that $\mathcal{G} \subset \binom{[n]}{k-1}$ is an r-wise t-intersecting family and $|\mathcal{G}| > \binom{n-t}{k-1-t} = \binom{n-t}{n-k+1}$. Set

$$\mathcal{G}^c = \{[n] \setminus G \colon G \in \mathcal{G}\}.$$

Note that $|\mathcal{G}^c| = |\mathcal{G}| > \binom{n-t}{n-k+1}$. By Theorem 1.13 we have $|\partial \mathcal{G}^c| > \binom{n-t}{n-k}$. Define

$$\mathcal{F} = \{[n] \setminus G \colon G \in \partial \mathcal{G}^c\}.$$

It is easy to see that $\mathcal{F} = \partial^+(\mathcal{G})$. It follows that $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise t-intersecting. Then

$$|\mathcal{F}| \le m(n,k,r,t) = \binom{n-t}{k-t},$$

contradicting $|\mathcal{F}| = |\partial \mathcal{G}^c| > \binom{n-t}{n-k} = \binom{n-t}{k-t}$. $\square$

Let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family. We say that $\mathcal{F}$ is *saturated* if any addition of an extra k-set to $\mathcal{F}$ would destroy the r-wise t-intersecting property. We say $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r \subset \binom{[n]}{k}$ are cross t-intersecting if $|F_1 \cap F_2 \cap \ldots \cap F_r| \ge t$ for all $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$, …, $F_r \in \mathcal{F}_r$.

Lemma 6.3. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted and saturated r-wise t-intersecting family. Let $\mathcal{G}_i = \mathcal{F}([t+1] \setminus \{i\}, [t+1])$, $i = 1, 2, 3, \ldots, t$. If $\mathcal{F}$ is not a t-star, then $\mathcal{G}_i = \mathcal{G}_j$ for all $1 \le i < j \le t$.*

Proof. Since $\mathcal{F}$ is not a t-star, there exists some $F_0 \in \mathcal{F}$ such that $|F_0 \cap [t]| \le t-1$. By shiftedness, we may assume that $F_0 \cap [t] = [t] \setminus \{t\}$. By shiftedness again,

$$\mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots \subset \mathcal{G}_t. \tag{6.2}$$

Since $F_0 \setminus [t+1] \in \mathcal{G}_t$, we have $\mathcal{G}_t \neq \emptyset$.

By (6.2) it suffices to show that $\mathcal{G}_1 = \mathcal{G}_t$. Suppose for contradiction that $\mathcal{G}_1 \subsetneq \mathcal{G}_t$. Then there exists some $G_t \in \mathcal{G}_t \setminus \mathcal{G}_1$. Then $F := G_t \cup ([t+r] \setminus \{t\}) \in \mathcal{F}$ and $F' := G_t \cup ([t+1] \setminus \{1\}) \notin \mathcal{F}$. By saturatedness and Lemma 2.2, we infer that there exist $F_1, F_2, \ldots, F_{r-1} \in \mathcal{F}$ such that for all $x \ge 0$,

$$|F' \cap [x]| + \sum_{1 \le i \le r-1} |F_i \cap [x]| \le (r-1)x + t - 1.$$

Since $F, F_1, F_2, \ldots, F_{r-1} \in \mathcal{F}$, by Lemma 2.2 there exists some $s \geq t$ such that

$$|F \cap [s]| + \sum_{1 \leq i \leq r-1} |F_i \cap [s]| \geq (r-1)s + t.$$

It follows that $|F' \cap [s]| < |F \cap [s]|$, contradicting the fact that $s \geq t$. Thus $\mathcal{G}_1 = \mathcal{G}_t$ and the lemma follows. $\square$

Lemma 6.4. *For $k \geq 3$,*

$$m(2k, k, 4, 3) = \binom{n-3}{k-3}.$$

Proof. Let $n = 2k$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted 4-wise 3-intersecting family. Without loss of generality, assume that $\mathcal{F}$ is saturated and is not a 3-star. We distinguish two cases.

Case A. There exist $F_1, F_2, F_3 \in \mathcal{F}$ such that $|F_1 \cap F_2 \cap F_3| = 4$.

By shiftedness, we may assume $F_1 \cap F_2 \cap F_3 = [4]$. Then the 4-wise 3-intersecting property implies $|F \cap [4]| \geq 3$ for all $F \in \mathcal{F}$. Define $\mathcal{H}_i = \mathcal{F}([4] \setminus \{i\}, [4])$ for $i = 1, 2, 3$. By Lemma 6.3 these three families are identical. Set $\mathcal{H} = \mathcal{H}_1$. Since $\mathcal{F}$ is 4-wise 3-intersecting, $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are cross 3-intersecting. Thus $\mathcal{H}$ is 3-wise 3-intersecting. As $n - 4 > 2(k-3)$, by Proposition 4.4,

$$|\mathcal{H}_1| + |\mathcal{H}_2| + |\mathcal{H}_3| = 3|\mathcal{H}| < 3\left(\frac{\sqrt{5}-1}{2}\right)^3 \binom{n-4}{k-3} < \binom{n-4}{k-3}. \tag{6.3}$$

Since $\mathcal{F}$ is not a 3-star, by Fact 2.5, $\mathcal{F}([3])$ is 3-wise intersecting. By (1.3),

$$|\mathcal{F}([3])| \leq \binom{n-4}{k-4}. \tag{6.4}$$

Adding (6.3) and (6.4), $|\mathcal{F}| < \binom{n-3}{k-3}$ follows.

Case B. $\mathcal{F}$ is 3-wise 5-intersecting.

By Proposition 4.4,

$$|\mathcal{F}| \leq \left(\frac{\sqrt{5}-1}{2}\right)^5 \binom{2k}{k} < 0.0902\binom{2k}{k}.$$

Note that

$$\frac{\binom{2k-3}{k-3}}{\binom{2k}{k}} = \frac{k(k-1)(k-2)}{2k(2k-1)(2k-2)} = \frac{1}{4} \times \frac{k-2}{2k-1}.$$

Since we may assume $k \geq 4 + 3 = 7$,

$$|\mathcal{F}| < 0.0902\binom{2k}{k} \leq 0.0902 \times 4 \times \frac{2k-1}{k-2}\binom{2k-3}{k-3} \leq 0.0902 \times 4 \times \frac{13}{5}\binom{2k-3}{k-3} < \binom{2k-3}{k-3}.$$

Thus $m(2k, k, 4, 3) = \binom{2k-3}{k-3}$. $\square$

Lemma 6.5. *For $k \geq 4$,*

$$m(2k, k, 4, 4) = \binom{n-4}{k-4}.$$

Proof. Let $n = 2k$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted 4-wise 4-intersecting family. Without loss of generality, assume that $\mathcal{F}$ is saturated and is not a 4-star. We distinguish three cases.

Case A. $\mathcal{F}$ is 3-wise 5-intersecting but not 3-wise 6-intersecting.

By shiftedness, we may assume $F_1 \cap F_2 \cap F_3 = [5]$ for some $F_1, F_2, F_3 \in \mathcal{F}$. Then $|F \cap [5]| \geq 4$ for all $F \in \mathcal{F}$. Define $\mathcal{H}_i = \mathcal{F}([5] \setminus \{i\}, [5])$ for $i = 1, 2, 3, 4$. By Lemma 6.3 these four families are identical. Since $\mathcal{F}$ is 4-wise 4-intersecting, $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ are cross 4-intersecting. Thus $\mathcal{H}$ is 4-wise 4-intersecting. As $n - 5 > 2(k-4)$, by Proposition 4.4,

$$\sum_{1 \leq i \leq 4} |\mathcal{H}_i| = 4|\mathcal{H}| < 4\alpha_4^4 \binom{n-5}{k-4} \overset{(4.2)}{<} \frac{4}{2^4 - 4 - 1} \binom{n-5}{k-4} < \binom{n-5}{k-4}. \tag{6.5}$$

Since $\mathcal{F}$ is not a 4-star, by Fact 2.5, $\mathcal{F}([4])$ is 3-wise intersecting. By (1.3),

$$|\mathcal{F}([4])| \leq \binom{n-5}{k-5}. \tag{6.6}$$

Adding (6.5) and (6.6), $|\mathcal{F}| < \binom{n-4}{k-4}$ follows.

Case B. $\mathcal{F}$ is 3-wise 6-intersecting but not 3-wise 7-intersecting.

Then $\mathcal{F}([4])$ is 3-wise 2-intersecting. Since $n - 4 > 2(k-4)$, by Theorem 1.6 we have $|\mathcal{F}[4]| \leq \binom{n-6}{k-6}$. Fix $H_1, H_2, H_3 \in \mathcal{F}$ with $H_1 \cap H_2 \cap H_3 = [6]$. Then the 4-wise 4-intersecting property implies $|F \cap [6]| \geq 4$ for all $F \in \mathcal{F}$. Let

$$\mathcal{F}_i = \{F \in \mathcal{F} \colon [4] \not\subset F,\ |F \cap [6]| = i\},\ i = 4, 5.$$

For $B \in \binom{[6]}{4}$ with $B \neq [4]$ and $F \in \mathcal{F}(B, [6])$, $P(F)$ is a lattice path from $(0,0)$ to $(n-k, k)$ that goes through $(2,4)$ and hits $y = 3x + 4$. By Proposition 4.3 (ii) and (iv) we infer that

$$|\mathcal{F}_4| = \sum_{B \in \binom{[6]}{4},\ B \neq [4]} |\mathcal{F}(B, [6])| < 14\alpha_4^6 \binom{n-6}{k-4} \overset{(4.2)}{<} \frac{14}{(2^4 - 4 - 1)^{3/2}} \binom{n-6}{k-4} < \binom{n-6}{k-4}.$$

Note that $\mathcal{F}_5 = \cup_{1 \leq i \leq 4} \mathcal{F}([6] \setminus \{i\}, [6])$. Let $\mathcal{G}_i = \mathcal{F}([6] \setminus \{i\}, [6])$, $i = 1, 2, 3, 4$. By Lemma 6.3, $\mathcal{G}_i = \mathcal{G}_j$ for $1 \leq i < j \leq 4$. Since $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ are cross 2-intersecting, $\mathcal{G}_1$ is 4-wise 2-intersecting. Since $n - 6 > 2(k-5)$, by Theorem 1.6

$$|\mathcal{F}_5| = \sum_{1 \leq i \leq 4} |\mathcal{G}_i| \leq 4\binom{n-8}{k-7} = 4\frac{(k-5)(k-6)}{(n-6)(n-7)} \binom{n-6}{k-5} < \binom{n-6}{k-5}.$$

Thus,

$$|\mathcal{F}| \leq |\mathcal{F}([4])| + |\mathcal{F}_4| + |\mathcal{F}_5| < \binom{n-6}{k-6} + \binom{n-6}{k-4} + 2\binom{n-6}{k-5} = \binom{n-4}{k-4}.$$

Case C. $\mathcal{F}$ is 3-wise 7-intersecting.

If $\mathcal{F}$ is 3-wise 8-intersecting, then we may assume $k \geq 3 + 8 = 11$ and by Proposition 4.4,

$$\begin{aligned} |\mathcal{F}| &\leq \left(\frac{\sqrt{5}-1}{2}\right)^8 \binom{n}{k} < \left(\frac{\sqrt{5}-1}{2}\right)^8 \left(\frac{2k-3}{k-3}\right)^4 \binom{n-4}{k-4} \\ &< \left(\frac{\sqrt{5}-1}{2}\right)^8 \left(\frac{19}{8}\right)^4 \binom{n-4}{k-4} < \binom{n-4}{k-4}. \end{aligned}$$

Thus there exist $F_1, F_2, F_3 \in \mathcal{F}$ such that $F_1 \cap F_2 \cap F_3 = [7]$. Then $|F \cap [7]| \geq 4$ for all $F \in \mathcal{F}$. Since $\mathcal{F}([4])$ is 3-wise 3-intersecting, by Proposition 4.4,

$$|\mathcal{F}([4])| < \left(\frac{\sqrt{5}-1}{2}\right)^3 \binom{n-4}{k-4} < 0.24\binom{n-4}{k-4}.$$

Let

$$\mathcal{F}_i = \{F \in \mathcal{F} \colon [4] \not\subset F,\ |F \cap [7]| = i\},\ i = 4, 5, 6$$

and let $\mathcal{G}_i = \mathcal{F}([7] \setminus \{i\}, [7])$. Then by Lemma 6.3, $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3 = \mathcal{G}_4 =: \mathcal{G}$. Moreover, $\mathcal{G}$ is 4-wise intersecting. Thus,

$$|\mathcal{F}_6| = 4|\mathcal{G}| \leq 4\binom{n-8}{k-7} < 3.4 \times \frac{k-6}{n-7}\binom{n-7}{k-6} + 0.6\binom{n-7}{k-7} < 1.8\binom{n-7}{k-6} + 0.6\binom{n-7}{k-7}.$$

Note that $P(F)$ goes though $(7-i, i)$ and hits $y = 3x + 4$ for each $F \in \mathcal{F}_i$, $i = 4, 5$. Using Proposition 4.3 (ii) and (iv), we have

$$|\mathcal{F}_5| = 18\alpha_4^5\binom{n-7}{k-5} < 18 \times \frac{1}{2^4-4-1}\alpha_4\binom{n-7}{k-5} = 2\alpha_4\binom{n-7}{k-5} < 1.8\binom{n-7}{k-5}$$

and

$$|\mathcal{F}_4| \leq 34 \times \alpha_4^9\binom{n-7}{k-4} < \frac{34}{(2^4-4-1)^2} \times \left(\frac{1}{2} + \frac{1}{2^4}\right)\binom{n-7}{k-4} = \frac{17}{72}\binom{n-7}{k-4} < 0.6\binom{n-7}{k-4}.$$

Thus,

$$\begin{aligned}
|\mathcal{F}| &= |\mathcal{F}([4])| + |\mathcal{F}_4| + |\mathcal{F}_5| + |\mathcal{F}_6| \\
&< 0.24\binom{n-4}{k-4} + 0.6\left(\binom{n-7}{k-4} + 3\binom{n-7}{k-5} + 3\binom{n-7}{k-6} + \binom{n-7}{k-7}\right) \\
&= 0.84\binom{n-4}{k-4} < \binom{n-4}{k-4}. \qquad \square
\end{aligned}$$

Proof of Proposition 1.7. Let $(r, t) \in \{(4, 3), (4, 4)\}$. By Lemmas 6.4 and 6.5, we infer $m(2k, k, r, t) = \binom{n-t}{k-t}$. For $n \geq 2k$, if n is even then by $m(n, n/2, r, t) = \binom{n-t}{n/2-t}$ and Lemma 6.6 we have $m(n, k, r, t) = \binom{n-t}{k-t}$. If n is odd, then $n \geq 2k$ implies $n - 1 \geq 2k$. Using (6.1) we conclude that

$$m(n, k, r, t) \leq m(n-1, k, r, t) + m(n-1, k-1, r, t) = \binom{n-t}{k-t}. \qquad \square$$

Lemma 6.6. *If $k \geq \frac{t(t-1)}{4\log 2} + t - 1$ and $t \leq 2^{r-2}\log 2 - 2$, then*

$$m(2k, k, r, t) = \binom{n-t}{k-t}. \tag{6.7}$$

Moreover, in case of equality $\mathcal{F}$ is the full t-star.

Proof. Let $n = 2k$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted and saturated r-wise t-intersecting family. If there exist $F_1, F_2, \ldots, F_{r-1} \in \mathcal{F}$ with $|F_1 \cap F_2 \cap \ldots \cap F_{r-1}| = t$, then $\mathcal{F}$ is a t-star and (6.7) follows. Thus we may assume that $\mathcal{F}$ is and $(r-1)$-wise $(t+1)$-intersecting. We distinguish two cases.

Case 1. $\mathcal{F}$ is $(r-1)$-wise $(t+2)$-intersecting.

Then by (4.4) we have

$$|\mathcal{F}| \leq \alpha_{r-1}^{t+2}\binom{n}{k} < \left(\frac{1}{2}+\frac{1}{2^{r-1}}\right)^{t+2}\binom{n}{k} \leq \left(\frac{1}{2}+\frac{1}{2^{r-1}}\right)^{t+2}\frac{n(n-1)\ldots(n-t+1)}{k(k-1)\ldots(k-t+1)}\binom{n-t}{k-t}.$$

Since $n = 2k$, by $t \leq 2^{r-2}\log 2 - 2$ and $k \geq \frac{t(t-1)}{4\log 2} + t - 1$ we infer that

$$\begin{aligned}
\frac{1}{2^{t+1}}\left(1+\frac{1}{2^{r-2}}\right)^{t+2}\frac{(2k-1)\ldots(2k-t+1)}{(k-1)\ldots(k-t+1)} &\leq \frac{1}{4}e^{\frac{t+2}{2^{r-2}}}\prod_{1\leq i\leq t-1}\left(1+\frac{i}{2(k-i)}\right)\\
&< \frac{1}{4}\exp\left(\frac{t+2}{2^{r-2}}+\sum_{1\leq i\leq t-1}\frac{i}{2(k-t+1)}\right)\\
&\leq \frac{1}{4}\exp\left(\log 2+\frac{t(t-1)}{4(k-t+1)}\right)\\
&\leq \frac{1}{4}\exp\left(\log 2+\log 2\right) = 1.
\end{aligned}$$

Thus $|\mathcal{F}| < \binom{n-t}{k-t}$.

Case 2. There exist $F_1, \ldots, F_{r-1} \in \mathcal{F}$ with $|F_1 \cap \ldots \cap F_{r-1}| = t+1$.

By shiftedness, we may assume $F_1 \cap \ldots \cap F_{r-1} = [t+1]$. Then the r-wise t-intersecting property implies $|F \cap [t+1]| \geq t$ for all $F \in \mathcal{F}$. Let $\mathcal{G}_i = \mathcal{F}([t+1]\setminus\{i\}, [t+1])$ for $i = 1, 2, \ldots, t+1$. By Lemma 6.3 we infer $\mathcal{G}_1 = \mathcal{G}_2 = \ldots = \mathcal{G}_t$. Let $\mathcal{G} = \mathcal{G}_i$, $i = 1, 2, \ldots, t$. Then

$$|\mathcal{F}| = |\mathcal{F}([t])| + t|\mathcal{G}|.$$

By Fact 2.5, $\mathcal{F}([t])$ is $(r-1)$-wise intersecting. Using (1.3), we obtain that

$$|\mathcal{F}([t])| \leq \binom{n-t-1}{k-t-1} < \frac{k-t}{2k-t}\binom{n-t}{k-t} < \frac{1}{2}\binom{n-t}{k-t}.$$

We are left to show $t|\mathcal{G}| \leq \frac{1}{2}\binom{n-t}{k-t}$.

If $t \geq r$, then $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_r$ is cross $(r-1)$-intersecting on $[t+2, n]$. Since $\mathcal{G}_1 = \mathcal{G}_2 = \ldots = \mathcal{G}_r$, $\mathcal{G}$ is r-wise $(r-1)$-intersecting. Note that $t \leq 2^{r-2}\log 2 - 2 \leq \frac{2^r-r-1}{4}$ holds for all $r \geq 3$. By Proposition 4.4,

$$|\mathcal{G}| \leq \alpha_r^{r-1}\binom{n-t-1}{k-t} \overset{(4.2)}{<} \frac{1}{\alpha_r(2^r-r-1)}\binom{n-t-1}{k-t} < \frac{1}{2t}\binom{n-t}{k-t}$$

and we are done.

By Fact 2.5, $\mathcal{F}$ is t-wise r-intersecting. If $r > t$ then $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_t$ is cross $(r-1)$-intersecting on $[t+2, n]$. Since $\mathcal{G}_1 = \mathcal{G}_2 = \ldots = \mathcal{G}_t$, $\mathcal{G}$ is t-wise $(r-1)$-intersecting. By Theorem 1.6 we may assume $t \geq 3$. Then

$$|\mathcal{G}| \leq \alpha_t^{r-1}\binom{n-t-1}{k-t} \leq \alpha_t^{t}\binom{n-t-1}{k-t} \overset{(4.2)}{<} \frac{1}{2^t-t-1}\binom{n-t-1}{k-t} \leq \frac{1}{2t}\binom{n-t}{k-t}.$$

Thus $t|\mathcal{G}| \leq \frac{1}{2}\binom{n-t}{k-t}$ and the lemma is proven. $\square$

Proof of Theorem 1.8. Note that $n \geq \frac{t(t-1)}{2\log 2} + 2t - 1$ implies

$$\frac{n}{2} > \frac{n-1}{2} \geq \frac{t(t-1)}{4\log 2} + t - 1. \tag{6.8}$$

If n is even, then by applying Lemma 6.6 we infer that

$$m\left(n, \frac{n}{2}, r, t\right) = \binom{n-t}{\frac{n}{2}-t}.$$

Since $\frac{n}{2} \geq k$, by Lemma 6.2 we have

$$m\left(n, k, r, t\right) = \binom{n-t}{k-t}.$$

If n is odd, then $n \geq 2k$ implies $n \geq 2k+1$. By (6.8) and applying Lemma 6.6,

$$m\left(n-1, \frac{n-1}{2}, r, t\right) = \binom{n-1-t}{\frac{n-1}{2}-t}.$$

Since $\frac{n-1}{2} \geq k > k-1$, by Lemma 6.2

$$m(n-1, k, r, t) = \binom{n-1-t}{k-t} \text{ and } m(n-1, k-1, r, t) = \binom{n-1-t}{k-1-t}.$$

Using (6.1) we conclude that

$$m(n, k, r, t) \leq m(n-1, k, r, t) + m(n-1, k-1, r, t) = \binom{n-t}{k-t}. \qquad \square$$

7 Concluding remarks

The area of research concerning r-wise t-intersecting non-uniform families is quite large and there are several results we could not even mention. The case of uniform families, that is, adding a new parameter k, increases this variety. In the present paper we stayed mostly in the range $k \leq \frac{1}{2}n$. However, it is completely legitimate to consider the range $k \sim cn$ for any fixed $c < 1$ as long as $c \leq \frac{r-1}{r}$.

If one wants to extend the results to such a range it seems to be essential to answer the following question.

Problem 7.1. *Let $c < \frac{r-1}{r}$ and denote by $p(n,k,r,t)$ the probability that a random lattice path from $(0,0)$ to $(n-k,k)$ hits the line $y = (r-1)x + t$. Let α be the unique root of $c - x + (1-c)x^r = 0$ in $(0,1)$. Does the inequality*

$$p(n,k,r,t) < \alpha^t \text{ holds always if } k \leq cn? \tag{7.1}$$

It seems to be rather difficult to determine the exact value of $n_0(k,r,t)$. Based on Fact 5.2, let us make the following:

Conjecture 7.2. For $n \geq \frac{\sqrt{4t+9}-1}{2}k$,

$$m(n,k,3,t) = \binom{n-t}{k-t}.$$

Another important problem would be to determine $m^*(n,k,r,1)$, the uniform version of the Brace-Daykin Theorem (the case $t=1$ of Theorem 1.10). In the case $r=2$ the solution is given by the Hilton-Milner Theorem [19].

Let us recall the Hilton-Milner-Frankl Theorem. Define

$$\mathcal{B}(n,k,r,t) = \left\{ B \in \binom{[n]}{k} : [t+r-2] \subset B,\ B \cap [t+r-1, k+1] \neq \emptyset \right\}$$
$$\cup \left\{ [k+1] \setminus \{j\} : 1 \le j \le t+r-2 \right\}.$$

Theorem 7.3 (Hilton-Milner-Frankl Theorem [19, 10, 1]). *For $n \ge (k-t+1)(t+1)$,*

$$m^*(n,k,2,t) = \max\left\{ |\mathcal{A}_1(n,k,2,t)|, |\mathcal{B}(n,k,2,t)| \right\}. \tag{7.2}$$

Note that both families $\mathcal{A}_1(n,k,2,t)$ and $\mathcal{B}(n,k,2,t)$ are r-wise $(t+2-r)$-intersecting, in particular, $(t+1)$-wise 1-intersecting. Thus in the range $(k-t+1)(t+1) < n$, i.e., $k < \frac{n}{t+1} + t - 1$,

$$m^*(n,k,r,t+2-r) = m^*(n,k,2,t).$$

However the case $k \sim cn$ with $\frac{1}{t+1} < c < \frac{r-1}{r}$ appears to be much harder. In [17] the following was proved.

Theorem 7.4 ([17]). *Let $0 < \varepsilon < \frac{1}{10}$. For $n \ge \frac{4}{\varepsilon^2} + 7$ and $\left(\frac{1}{2} + \varepsilon\right) n \le k \le \frac{3n}{5} - 3$,*

$$m^*(n,k,3,1) = |\mathcal{A}_1(n,k,3,1)|.$$

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