Counting Permutation Patterns with Multidimensional Trees

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Abstract

We consider the well-studied *pattern counting problem*: given a permutation $\pi \in \mathbb{S}_n$ and an integer $k > 1$, count the number of order-isomorphic occurrences of every pattern $\tau \in \mathbb{S}_k$ in π.

Our first result is an $\widetilde{\mathcal{O}}\left(n^2\right)$-time algorithm for $k = 6$ and $k = 7$. The proof relies heavily on a new family of graphs that we introduce, called *pattern-trees*. Every such tree corresponds to an integer linear combination of permutations in $\mathbb{S}_k$, and is associated with linear extensions of partially ordered sets. We design an evaluation algorithm for these combinations, and apply it to a family of linearly-independent trees. For $k = 8$, we show a barrier: the subspace spanned by trees in the previous family has dimension exactly $|\mathbb{S}_8| - 1$, one less than required.

Our second result is an $\widetilde{\mathcal{O}}(n^{7/4})$-time algorithm for $k = 5$. This algorithm extends the framework of pattern-trees by speeding-up their evaluation in certain cases. A key component of the proof is the introduction of pair-rectangle-trees, a data structure for dominance counting.

1 Introduction

A permutation $\tau \in \mathbb{S}_k$ occurs in a permutation $\pi \in \mathbb{S}_n$ if there exist k points in π that are order-isomorphic to τ. By way of example, in $\overline{1}3\overline{42} \in \mathbb{S}_4$,[1] the overlined points form an occurrence of $132 \in \mathbb{S}_3$. The number of occurrences $\#\tau\,(\pi)$ of a permutation $\tau \in \mathbb{S}_k$ (a *pattern*) within a larger permutation $\pi \in \mathbb{S}_n$ has been the basis of many interesting questions, both combinatorial and algorithmic.

In a classical result, MacMahon [Mac15] proved that the number of permutations $\pi \in \mathbb{S}_n$ that *avoid* the pattern 123 (i.e., $\#123\,(\pi) = 0$) is counted by the Catalan numbers. Another classical result is the well-known Erdős-Szekeres theorem [ES35], which states that any permutation of size $(s-1)(l-1)+1$ cannot simultaneously avoid both $(1,\ldots,s)$ *and* $(1,\ldots,1)$. These early results gave rise to an entire field of study regarding pattern avoidance, c.f. [Pra73, Knu97, SS85]. One particularly noteworthy result is Marcus and Tardos' resolution of the Stanley-Wilf conjecture [MT04]: for any fixed pattern $\tau \in \mathbb{S}_k$, the growth rate of the number of permutations $\pi \in \mathbb{S}_n$ avoiding τ is $c(\tau)^n$, where $c(\tau)$ is a constant depending only on τ.

Pattern avoidance can also be cast as an algorithmic problem. The *permutation pattern matching* problem is the task of determining, given a pattern $\tau \in \mathbb{S}_k$ and a permutation $\pi \in \mathbb{S}_n$, whether π avoids τ. What is the computational complexity of this task? Trivial enumeration over all k-tuples of points yields an $\mathcal{O}(k \cdot n^k)$-time algorithm. This bound has been improved upon by a long line of works: Albert et al. [AAAH01] lowered the bound to $\mathcal{O}(n^{2k/3+1})$, Ahal and Rabinovich [AR08] to $\mathcal{O}(n^{(0.47+o(1))k})$, and finally Guillemot and Marx [GM14] established the fixed-parameter tractability of the problem, i.e., whenever k is *fixed*, the problem can be solved in time linear in n

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[1] Throughout this paper, permutations are written in one-line notation. If they are short, we omit the parenthesis.

(see also [Fox13] for an improvement on this result). In stark contrast, when the pattern τ is *not fixed* (i.e., when $k = k(n) \to \infty$), permutation pattern matching is known to be NP-complete, as shown by Bose, Buss and Lubiw [BBL98].

A closely related algorithmic question is the *counting version* of permutation pattern matching. The *permutation pattern counting* problem is the task of counting, given a pattern $\tau \in \mathbb{S}_k$ and permutation $\pi \in \mathbb{S}_n$, the number of occurrences $\#\tau(\pi)$. Once again, there is a straightforward $\mathcal{O}(k \cdot n^k)$-time algorithm – how far is it from optimal? Albert et al. lowered the bound to $\mathcal{O}(n^{2k/3+1})$ [AAAH01][2] and the current best known bound is $\mathcal{O}(n^{(1/4+o(1))k})$, due to Berendsohn et al. [BKM21]. Berendsohn et al. also showed a barrier: assuming the exponential time hypothesis, there is no algorithm for pattern counting with running time $f(k) \cdot n^{o(k/\log k)}$, for *any* function f.

Another intriguing line of work focuses on the pattern counting problem, for *constant small* k. As the number of patterns $\tau \in \mathbb{S}_k$ is fixed in this regime, one can equivalently, up to a constant multiplicative factor, compute the entire $k!$-dimensional vector of *all* occurrences, $(\#\tau(\pi))_{\tau \in \mathbb{S}_k}$. This vector, which characterises the local structure of a permutation over size-k pointsets, is known as the k*-profile*. The k-profile has also featured in works aiming to understand the local structure of permutations, c.f. [BLL23, EZ20, CP08].

Even-Zohar and Leng [EZL21] designed a class of algorithms capable of computing the 3-profile in $\widetilde{\mathcal{O}}(n)$-time,[3] and the 4-profile in $\widetilde{\mathcal{O}}(n^{3/2})$-time. Improving on their result for $k = 4$, Dudek and Gawrychowski [DG20] gave a bidirectional reduction between the task of computing the 4-profile, and that of counting 4-cycles in a sparse graph. The best known algorithm for the latter problem has running time $\mathcal{O}(n^{2-3/(2\omega+1)})$ [WWWY14], where $\omega < 2.372$ [DWZ23] is the exponent of matrix multiplication. Consequently, Dudek and Gawrychowski obtain an $\mathcal{O}(n^{1.478})$-time algorithm for the 4-profile. Our paper continues this line of work: we design algorithms computing the 5, 6 and 7-profiles, and highlight a barrier in the way of computing the 8-profile.

1.1 Our Contribution

We introduce *pattern-trees*: a family of graphs that generalise the corner-trees of Even-Zohar and Leng [EZL21]. Pattern-trees are rooted labeled trees, in which every vertex is associated with a set of *point variables*, along with constraints that fix their relative ordering in the plane, and every edge is labeled by a list of constraints over the ordering of points associated with its incident vertices.

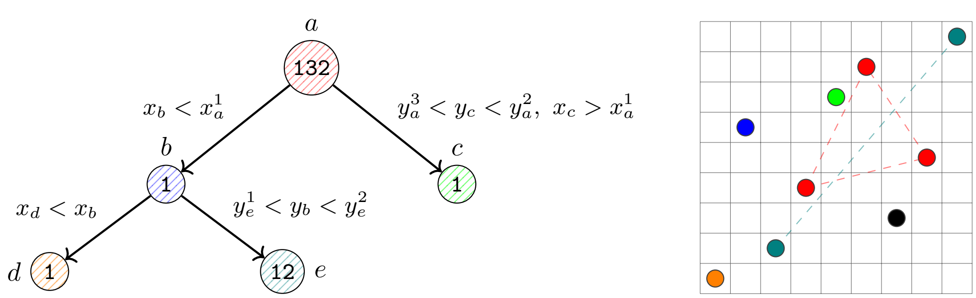


Figure 1: An embedding of a pattern-tree (left) into the permutation `162478359` $\in \mathbb{S}_9$ (right).

[2]Their algorithm also works for the counting version.

[3]As usual, the notation $\widetilde{\mathcal{O}}(\cdot)$ hides poly-logarithmic factors.

Using an algorithm derived from pattern-trees, we obtain our first result.

Theorem 1. *For every $1 \leq k \leq 7$, the k-profile of an n-element permutation can be computed in $\widetilde{\mathcal{O}}\left(n^2\right)$ time and space.*

Our proof of Theorem 1 relies on embeddings of trees into permutations. Consider the *number of distinct embeddings* of the points of a pattern-tree T into the points in the plane associated with a permutation $\pi \in \mathbb{S}_n$, in which the embedding satisfies all constraints defined by the tree. We show that this quantity can be expressed as a *fixed* integer linear combination of permutation pattern counts, irrespective of π. Interpreted as a formal sum of patterns, this is simply a vector in $\mathbb{Z}^{\mathbb{S}_{\leq k}}$, where k is the number of point variables in the tree. These vectors are associated with pairwise compositions of linear extensions of partially ordered sets, whose Hasse diagrams can be partitioned in a particular way.

The subspaces spanned by the vectors of trees, over the rationals, are central to our proof. It is not hard to show that when the subspace of a set of trees is full-dimensional, one can derive from those trees an algorithm for the k-profile. To this end, we design an *evaluation algorithm*: given a pattern-tree T and an input permutation $\pi \in \mathbb{S}_n$[4], the algorithm computes the number of occurrences of T in π, denoted $\#T(\pi)$. The complexity of this algorithm depends on properties of the tree. In our proof of Theorem 1, we construct a family of trees evaluable in $\widetilde{\mathcal{O}}\left(n^2\right)$-time, which are of full dimension for $\mathbb{S}_{\leq 7}$.

Compared to previous results, Theorem 1 offers an improvement whenever $k \in \{5, 6, 7\}$. The best known bound for the k-profile problem is $\mathcal{O}(n^{k/4+o(k)})$, due to Berendsohn et al. [BKM21]. Their approach relies on formulating a binary CSP, and bounding its tree-width. It is well known that binary CSPs can be solved in time $\mathcal{O}(n^{t+1})$ [DP89, Fre90], where n is the domain size, and t is the tree-width of the constraint graph. In the algorithm of [BKM21], the tree-width is bounded by $k/4 + o(k)$, where the $o(k)$-term is greater than one. Therefore, their algorithm has at least cubic running time when $k \geq 4$.

The relationship between properties of pattern-trees and the dimensions of the subspaces spanned by them is still far from understood (see Section 5). Corner-trees, which are exactly the pattern-trees whose evaluation is quasi-linear, were shown in [EZL21] to have full rank for $\mathbb{S}_{\leq 3}$, and rank only $|\mathbb{S}_4| - 1 = 23$, restricted to $\mathbb{S}_4$. Intriguingly, we show that the family of pattern-trees with which our proof of Theorem 1 is obtained, whose evaluation complexity is quadratic, have full rank for $\mathbb{S}_{\leq 7}$, and rank only $|\mathbb{S}_8| - 1 = 40319$ restricted to $\mathbb{S}_8$. We observe several striking resemblances between the two vectors spanning the orthogonal complements, for $\mathbb{S}_4$ and $\mathbb{S}_8$ respectively, in terms of their symmetries. In fact, we extend a characterisation of [DG20] regarding the symmetries for $\mathbb{S}_4$ to the case of $\mathbb{S}_8$ (see Section 3.4).

Our second result is a sub-quadratic algorithm for the 5-profile.

Theorem 2. *The 5-profile of an n-element permutation can be computed in time $\widetilde{\mathcal{O}}\left(n^{7/4}\right)$.*

The proof of Theorem 2 is obtained by speeding-up the evaluation algorithm of pattern-trees. The original algorithm for pattern-trees has an integral exponent in its complexity, which is determined by properties of the tree. We show that trees with certain topological properties, i.e.,

[4]Throughout this paper we operate on n-element permutations as input. Such inputs are assumed to be presented to the algorithm *sparsely*, e.g., as a length-n vector representing the permutation in one-line notation.

containing a particular set of "gadgets", can be evaluated faster. The family of trees constituting all corner-trees, and their augmentation by our gadgets, span a full-dimensional subspace over $\mathbb{S}_5$. This allows us to break the quadratic barrier for the 5-profile.

One of the key ingredients, both in the original evaluation algorithm and in its extended version, is a data structure known as a multidimensional segment-tree, or rectangle-tree [JMS05, Cha88].[5] A d-dimensional rectangle-tree holds (possibly weighted) points in $[n]^d$, and answers sum-queries over rectangles $\mathcal{R} \subseteq [n]^d$ (i.e., Cartesian products of segments) in poly-logarithmic time.

The gadgets appearing in the proof of Theorem 2 are sub-structures related to the patterns `3214` and `43215`. For the former, we extend an algorithm of [EZL21] into a weighted variant, and provide an evaluation algorithm of complexity $\widetilde{\mathcal{O}}(n^{5/3})$. We then further extend this into an algorithm for the latter gadget, of complexity $\widetilde{\mathcal{O}}(n^{7/4})$. The latter proof is involved, and requires the introduction of a new data structure, which we call a *pair-rectangle-tree*. A pair-rectangle-tree is an extension of rectangle-trees that can facilitate more complex queries, in particular, regarding the dominance counting (see [JMS05, CE87]) of a set of points in a rectangle. We remark that the original pattern-tree evaluation algorithm can only compute *equivalent* gadgets in quadratic time. That is, the evaluation algorithm is not always optimal.

1.2 Paper Organization

In Section 3 we introduce pattern-trees. Our construction for $3 \leq k \leq 7$ can be found in Section 3.3, and the case $k = 8$ is dealt with in Section 3.4. A straightforward application of pattern-trees for general k is given in Section 3.5. Section 4 revolves around our construction of an $\widetilde{\mathcal{O}}\left(n^{7/4}\right)$-time algorithm for the 5-profile. The augmentation of the pattern-trees evaluation algorithm can be found in Section 4.2, and the particular gadgets used in the 5-profile are obtained in Section 4.3 and Section 4.4. The data structure we introduce for dominance counting in rectangles, pair-rectangle-tree, is given in Section 4.5. Finally, in Section 5 we discuss open questions and possible extensions of this work.

2 Preliminaries

2.1 Permutations

A permutation $\pi \in \mathbb{S}_n$ over n elements is a bijection from $[n]$ to itself, where $[n] := \{1, 2, \ldots, n\}$. Throughout this paper, we express permutations using one-line notation, and if the permutation range is sufficiently small, we omit the parentheses. For instance, `123` is the identity permutation over 3 elements. Associated with any permutation $\pi \in \mathbb{S}_n$ is a set of n points in the plane, $p(\pi) := \{(i, \pi(i)) : i \in [n]\}$, which we refer to as the *points of* π. In the other direction, any set of n points in the plane defines a permutation $\pi \in \mathbb{S}_n$, provided that no two points lie on an axis-parallel line. Given such a set $S \subset \mathbb{R}^2$, we use the notation $S \cong \pi$ to indicate that the points are *order-isomorphic* to π.

An *occurrence* of a *pattern* $\tau \in \mathbb{S}_k$ in a permutation $\pi \in \mathbb{S}_n$ is a k-tuple $1 \leq i_1 < \cdots < i_k \leq n$ such that the set of points $(i_j, \pi(i_j))$ is order-isomorphic to τ. That is, $\pi(i_j) < \pi(i_l)$ if and only if $\tau(j) < \tau(l)$ for all $j, l \in [k]$. The number of occurrences of τ in π is denoted by $\#\tau\,(\pi)$.

[5] A 2-dimensional version of this data structure features in both [EZL21] and [DG20].

The dihedral group D_4 naturally acts on the symmetric group $\mathbb{S}_n$, by acting on $[1,n]^2$. Formally, for any element $g \in D_4$ and permutation $\pi \in \mathbb{S}_n$, we have that $(g.\pi) \in \mathbb{S}_n$ is the permutation for which $g.(p(\pi)) \cong g.\pi$. Our algorithms usually receive permutations as input, and compute some combination of pattern-counts. To this end, it is sometimes helpful to first act on the input with an element $g \in D_4$ (as a preprocessing step), and only then invoke the algorithm as usual. In this way, if an algorithm computes the count $\#\tau\,(\pi)$, then after the action we obtain $\#\tau\,(g.\pi) = \#(g^{-1}.\tau)\,(\pi)$.

Our main focus in this paper is the computation of $\#\tau\,(\pi)$ for all $\tau \in \mathbb{S}_k$, where $\pi \in \mathbb{S}_n$ is given as input and and k is fixed. This collection of counts is defined as follows.

Definition 2.1. *The k-*profile *of a permutation $\pi \in \mathbb{S}_n$ is the vector $(\#\tau\,(\pi))_{\tau \in \mathbb{S}_k} \in \mathbb{Z}^{\mathbb{S}_k}$.*

2.2 Partially Ordered Sets

A partially ordered set (*poset*) $\mathcal{P}(X, \leq)$ over a ground set X is a partial arrangement of the elements in X according to the order relation $\leq$. If $\leq$ is not reflexive, we say that $\mathcal{P}$ is *strict*. A partial order $\leq^\star$ is said to be an extension of $\leq$ if $x \leq y$ implies $x \leq^\star y$ for all $x, y \in X$. If an extension $\leq^\star$ is a total order, it is called a *linear extension* of $\leq$. As usual, the set of all linear extensions of a poset $\mathcal{P}$ is denoted by $\mathcal{L}(\mathcal{P})$.

2.3 Computational Model

Throughout this paper we disregard all polylog(n)-factors, so our results hold for any choice of standard computational model (say, word-RAM). The notation $\widetilde{\mathcal{O}}\left(n^k\right)$ (adding the tilde) is used to hide poly-logarithmic factors. The algorithms presented in this paper operate on n-element permutations as input, and we remark that such inputs are assumed to be presented to the algorithm *sparsely*, e.g., as a length-n vector representing the permutation in one-line notation.

2.4 Rectangle-Trees

Our algorithms for efficiently computing profiles rely heavily on a simple and powerful data structure, which we refer to as a *rectangle-tree*[6] or a *multidimensional segment-tree*. Concretely, we require the following folklore fact.

Proposition 2.2 ([Cha88, JMS05], see also [DG20])**.** *For any fixed dimension $d \geq 1$, there exists a deterministic data structure $\mathcal{T}$ that supports each of the following actions in $\widetilde{\mathcal{O}}\left(1\right)$ time:*

1. *Initialisation: Given $n \in \mathbb{N}$, construct an empty tree over $[n]^d$.*
2. *Insertion: Given $x \in [n]^d$ and $w = \mathcal{O}\left(\text{poly}(n)\right)$, add weight w to point x.*
3. *Query: Given a rectangle $\mathcal{R} \subseteq [n]^d$, the query $\mathcal{T}(\mathcal{R})$ returns the sum of weights over all points in $\mathcal{R}$.*

Let us illustrate the application of rectangle-trees to pattern counting, through the simple (and again, folklore) case of *monotone pattern counting.*

Proposition 2.3. *Let $k \geq 1$ be a fixed integer and let $\pi \in \mathbb{S}_n$ be an input permutation. The pattern counts $\#(\mathtt{1},\ldots,\mathtt{k})\,(\pi)$ and $\#(\mathtt{k},\ldots,\mathtt{1})\,(\pi)$ can be computed in $\widetilde{\mathcal{O}}\left(n\right)$ time.*

[6] A rectangle $\mathcal{R} \subseteq [n]^d$ is a Cartesian product of *segments*, i.e., invervals of the form $\{a, a+1, \ldots, b\} \subseteq [n]$.

Proof. TOPROVE 0 □

Remark 2.4. It is also possible to count monotone patterns using 1-dimensional segment-trees, somewhat more efficiently. However, the difference is only in logarithmic factors. The multidimensional structure highlighted above will serve us in more complicated cases.

3 Pattern-Trees

In this section we introduce a family of graphs, called *pattern-trees*. Using pattern-trees we derive algorithms for computing the k-profile of a permutation. Our main result for this section (see Section 3.3) is a quadratic-time algorithm for the k-profile of a permutation, for every $k \leq 7$:

Theorem 1. *For $1 \leq k \leq 7$, the k-profile of an n-element permutation is computable in $\widetilde{\mathcal{O}}\left(n^2\right)$ time and space.*

In Section 3.4 we consider the subspaces spanned by the same family of pattern-trees, restricted to $\mathbb{S}_8$. We show that this subspace is of dimension $|\mathbb{S}_8| - 1$, one less than required. In Section 3.5 we consider the case of general (constant) k, and show a straightforward application of pattern-trees yielding an $\widetilde{\mathcal{O}}(n^{\lceil k/2 \rceil})$-time algorithm for the k-profile.

Before we present pattern-trees, let us begin by recalling corner-trees.

3.1 Warmup: Corner-Trees

One of the main components in the work of [EZL21] is the introduction of *corner-trees*. Corner-trees are a family of rooted edge-labeled trees. Every corner-tree of k vertices is associated with a particular vector in $\mathbb{Z}^{\mathbb{S}_{\leq k}}$; i.e., a formal integer linear combination of permutations, each of size at most k. Furthermore, there exists an efficient evaluation algorithm for corner-trees: given any input permutation $\pi \in \mathbb{S}_n$ and corner-tree T, the integer sum of permutation *pattern counts* in π, called the vector of T, can be computed in time $\widetilde{\mathcal{O}}\left(n\right)$. We refer to this operation as *evaluating the vector of T over π*.

Definition 3.1 (corner-tree [EZL21][7]). *A corner-tree is a rooted*[8] *edge-labeled tree, with edge labels in the set* $\{\mathrm{NE}, \mathrm{NW}, \mathrm{SE}, \mathrm{SW}\}$.

An *occurrence* of a corner-tree T in a permutation π is a map $\varphi : V(T) \to p(\pi)$, in which the image *agrees* with the edge-labels of the tree. That is, for every edge $(u \to v) \in E(T)$, $\varphi(v)$ is to the *left* of $\varphi(u)$ if the edge is labeled NW or SW, and to its right otherwise. Similar rules apply for their vertical ordering. As in [EZL21], the *number of occurrences* of a corner-tree T in a permutation π is denoted by $\#T(\pi)$.

The *vector* of a corner-tree is a formal sum of permutation patterns with integer coefficients, representing the number of occurrences of the tree in *any* input permutation. For instance, the vector of $\bullet \xrightarrow{\text{SE}} \circ \xrightarrow{\text{NE}} \circ$ is $\#213 + \#312$. Clearly, the vector of a corner-tree over k vertices may involve patterns of size at most k, as the tree conditions on the relative ordering of at most $|V(T)|$ points (smaller patterns may appear as well, since occurrences are not necessarily injective).

[7]For convenience, we consider corner-trees to be edge-labeled, rather than vertex-labeled as in [EZL21].
[8]Hereafter, whenever we consider rooted trees, we orient their edges away from the root.

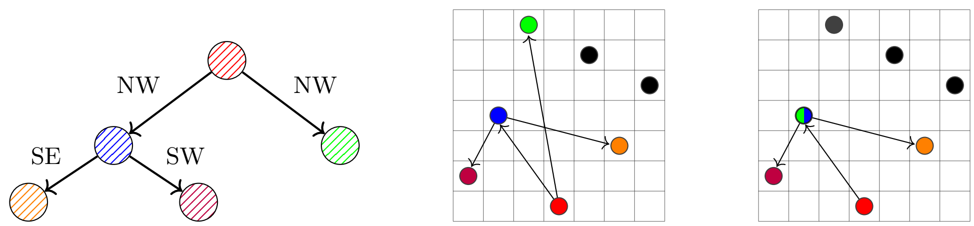


Figure 2: Two occurrences of a corner-tree (left) in $\pi = \texttt{2471635} \in \mathbb{S}_7$ (centre, right). Occurrences need not be injective; for instance, on the right, the blue and green points are identified.

Theorem 1.1 of [EZL21] presents an algorithm for evaluating the vector of a corner-tree over an input permutation $\pi \in \mathbb{S}_n$. For expositionary purposes, we sketch a simplified version of their algorithm, phrased in terms of rectangle-trees.

Proposition 3.2 (Theorem 1.1 of [EZL21])**.** *The vector of any corner-tree with a constant number of vertices can be evaluated over an input permutation $\pi \in \mathbb{S}_n$ in time $\widetilde{\mathcal{O}}(n)$.*

Proof Sketch. Let T be a corner-tree and let $\pi \in \mathbb{S}_n$ be a permutation. To start, construct a 2-dimensional rectangle-tree (see Section 2.4), and insert the points $p(\pi)$ with weight 1, in time $\widetilde{\mathcal{O}}(n)$. Associate this tree with the *leaves* of T. Next, traverse the vertices of T in post-order. At every internal vertex u, construct a new (empty) rectangle-tree $\mathcal{T}_u$, and associate it with u. Then, iterate over every point in π, and at each point perform one *rectangle query* to the rectangle-tree associated with each of u's children, querying the rectangle corresponding to the edge label in T written on the parent-child edge. For example, if $u \to v$ is labeled SW, the iteration over a point $(i, \pi(i)) \in p(\pi)$ queries the rectangle $[1, i-1] \times [1, \pi(i)-1]$. Store the product of all answers to these queries in $\mathcal{T}_u$, at the position of the current permutation point. It can be shown that the sum of all values at the root's tree (i.e., a full rectangle query) is the number of occurrences, $\#T(\pi)$. $\square$

3.2 Pattern-Trees

We introduce *pattern-trees*: a family of graphs that generalise the corner-trees of [EZL21]. In pattern-trees, every *vertex* is labeled by a permutation, and every *edge* is labeled by a list of constraints. The permutations written on the vertices fix the exact ordering of the points corresponding to them, and the edge-constraints are similarly imposed over the points corresponding to the two incident vertices. As with corner-trees, pattern-trees serve two purposes: firstly, every pattern-tree is associated with a set of constraints over permutation points, the number of satisfying assignments to which can be expressed as a formal integer linear combination of patterns (that is, a vector). Secondly, we present an algorithm for evaluating this vector over an input permutation. This allows us to efficiently compute certain pattern combinations not spanned by corner-trees.

Definition 3.3 (pattern-tree)**.** *A pattern-tree T is a rooted edge- and vertex-labeled tree, where:*

1. *Every vertex $v \in V(T)$ is:*
   - *Labeled by a permutation $\tau_v \in \mathbb{S}_r$, for some integer $r \geq 1$.*
   - *Associated with two sets of fresh variables,*

$$x_v := \{x_v^1, \ldots, x_v^r\}, \text{ and } y_v := \{y_v^1, \ldots, y_v^r\},$$

where we denote $p_v^i := (x_v^i, y_v^i)$ for every $i \in [r]$, and $p_v := \{p_v^i : i \in [r]\}$.

2. *Every edge $(u \to v) \in E(T)$ is labeled by:*
   - *Two strict posets, $\mathcal{P}^x_{uv} = (x_u \sqcup x_v, <)$ and $\mathcal{P}^y_{uv} = (y_u \sqcup y_v, <)$.*
   - *A set $E_{uv} \subseteq p_u \times p_v$ of equalities between the points of u and those of v.*

The *size* $s(v)$ of a vertex v is the size r of the permutation $\tau_v \in \mathbb{S}_r$ with which it is labeled. The *maximum size* of a pattern-tree, denoted $s(T)$, is the maximum over all vertex sizes. The *total size*, denoted $\Sigma(T)$, is the sum over all vertex sizes. Under this notation, a corner-tree is a pattern-tree of maximum size one. Lastly, $p(T) := \bigsqcup_{v \in V(T)} p_v$ is the set of all $\Sigma(T)$ points in the tree.

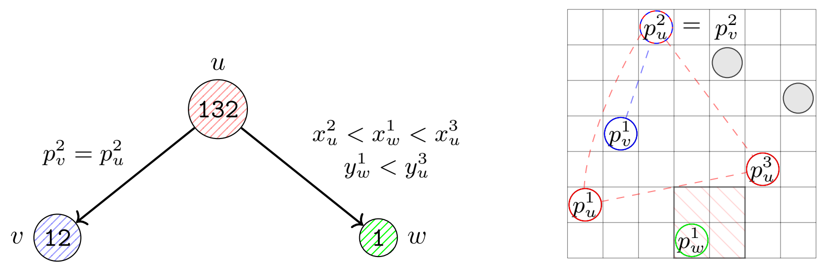


Figure 3: An occurrence of a pattern-tree T (left) in the permutation $\pi = \texttt{2471635} \in \mathbb{S}_7$ (right). Every set of coloured points on the right induces the permutation with which the similarly coloured vertex on the left is labeled ("vertex constraints"). All of the edge-constraints are also satisfied: points p_v^2 and p_u^2 are *identified*, and point p_w^1 (green) must reside within the red shaded square. This tree corresponds to a linear combination, $\#\texttt{1423} + \#\texttt{2413} + 2 \cdot \#\texttt{12534} + \cdots + \#\texttt{24513}$, of patterns in $\mathbb{S}_4$ and $\mathbb{S}_5$. The tree has total size $\Sigma(T) = 6$ and maximum size $s(T) = 3$.

Pattern-Tree Constraints. Any pattern-tree T defines constraints $\mathcal{C}(T)$ over *points* $p(T)$:

1. Every vertex v labeled by $\tau_v \in \mathbb{S}_r$ contributes the following inequalities,[9]

$$x_v^1 < x_v^2 < \cdots < x_v^r, \text{ and } y_v^i < y_v^j \text{ for all } i, j \in [r] \text{ such that } \tau_v(i) < \tau_v(j).$$

2. Every edge $u \to v$ contributes the inequalities in $\mathcal{P}^x_{uv}$ and $\mathcal{P}^y_{uv}$, and the equalities in E_{uv}.

Hereafter, we partition $\mathcal{C}(T)$ into two parts: its *equalities*, which define an equivalence relation $E^T := \bigsqcup_{u \to v} E_{uv}$ over the points $p(T)$, and its *inequalities*, which define *strict* posets,

$$\mathcal{P}_x^T = \Big(\bigsqcup_{v \in V(T)} x_v, < \Big), \text{ and } \mathcal{P}_y^T = \Big(\bigsqcup_{v \in V(T)} y_v, < \Big).$$

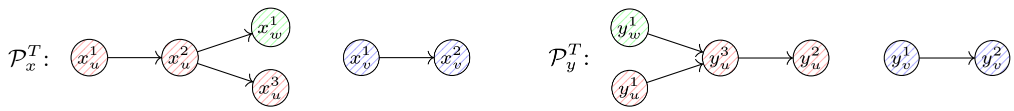
Given an equivalence relation $E \supseteq E^T$, the posets $\mathcal{P}_x^E$ and $\mathcal{P}_y^E$ are the strict posets obtained from $\mathcal{P}_x^T$ and $\mathcal{P}_y^T$ by replacing every coordinate variable corresponding to a point $p \in p(T)$ by a single variable corresponding to the equivalence class of p in E.

[9]These vertex-constraints enforce the pattern τ_v over the points p_v.

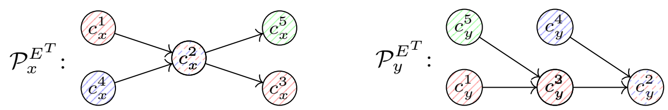
Example 3.1. *The pattern-tree T appearing in Figure 3 corresponds to the constraints*

$$\mathcal{C}(T) = \left\{x_u^1 < x_u^2 < x_u^3,\ y_u^1 < y_u^3 < y_u^2,\ x_u^2 < x_w^1 < x_u^3,\ y_w^1 < y_u^3,\ p_v^2 = p_u^2,\ x_v^1 < x_v^2,\ y_v^1 < y_v^2\right\},$$

whose posets are:



Applying $E^T = \left\{c_1 = \{p_u^1\},\ c_2 = \{p_v^2, p_u^2\},\ c_3 = \{p_u^3\},\ c_4 = \{p_v^1\},\ c_5 = \{p_w^1\}\right\}$ *yields the posets:*



Pattern-Tree Occurrences. As with corner-trees, we define *pattern-tree occurrences.*

Definition 3.4. *An occurrence $\varphi : p(T) \to p(\pi)$ of a pattern-tree T in a permutation $\pi \in \mathbb{S}_n$ is a map whose image $\varphi(p(T))$ conforms to the constraints $\mathcal{C}(T)$.*

An illustration of a pattern-tree occurrence is shown in Figure 3. Note that, as with corner-trees, occurrence maps need not be injective. We remark that some pattern-trees may have no occurrences, in any permutation $\pi \in \mathbb{S}_n$. For example, $u \bullet\!\xrightarrow[p_u = p_v,\, x_u < x_v]{}\!\circ\ v$ is infeasible.

Pattern-Tree Vectors. As with corner-trees, one can associate a vector with every pattern-tree T, which is a formal integer linear combination of pattern-counts representing the number of occurrences of T in any input permutation $\pi \in \mathbb{S}_n$.

Lemma 3.5. *Let T be a pattern-tree. The number of occurrences of T in an input permutation $\pi \in \mathbb{S}_n$ is given by the following sum of pattern-counts, each of size at most $\Sigma(T)$:*

$$\#T(\pi) = \sum_{E \supseteq E^T} \sum_{\substack{\sigma \in \mathcal{L}(\mathcal{P}_x^E) \\ \tau \in \mathcal{L}(\mathcal{P}_y^E)}} \#\left(\tau\sigma^{-1}\right)(\pi).$$

Proof. TOPROVE 1 □

Evaluating a Pattern-Tree. It remains to construct an evaluation algorithm for the vector of a pattern-tree. To present our algorithm, we require some notation.

1. Points: To every set of points $S := \{s_1, \ldots, s_r\} \subseteq p(\pi)$, where $\pi \in \mathbb{S}_n$ is a permutation and $(s_1)_x < \cdots < (s_r)_x$, we associate a $2r$-dimensional point,

$$p(S) := \left((s_1)_x, \ldots, (s_r)_x, (s_1)_y, \ldots, (s_r)_y\right) \in [n]^{2r}$$

2. Rectangles: To every combination of an edge $(u \to v) \in E(T)$ in a pattern-tree T, where v and u are of sizes d and r respectively, and set of points $S := \{s_1, \ldots, s_r\} \subseteq p(\pi)$, we associate a $2d$-dimensional rectangle,

   $$\mathcal{R}_{uv}^{S} := \mathcal{R}_{uv}^{S,x} \times \mathcal{R}_{uv}^{S,y} \subseteq [n]^{2d}, \text{ where } \mathcal{R}_{uv}^{S,x}, \mathcal{R}_{uv}^{S,y} \subseteq [n]^{d}.$$

   The i-th segment of $\mathcal{R}_{uv}^{S,x}$ contains the x coordinates that x_v^i can take under the constraints of $u \to v$, when x_u^j is assigned $(s_j)_x$. Namely, the intersection of the following segments:

   $$\underbrace{\bigcap_{j:(p_u^j, p_v^i) \in E_{uv}} \{(s_j)_x\}}_{\text{equals}}, \quad \underbrace{\bigcap_{j:(x_v^i < x_u^j) \in \mathcal{P}_{uv}^x} \{1, \ldots, (s_j)_x - 1\}}_{\text{less-than}}, \quad \underbrace{\bigcap_{j:(x_v^i > x_u^j) \in \mathcal{P}_{uv}^x} \{(s_j)_x + 1, \ldots, n\}}_{\text{greater-than}}$$

   The y-segments are similarly defined.

Observe that the rectangle $\mathcal{R}_{uv}^{S}$ is the set of permissible locations for the points p_v, subject to the edge-constraints on the edge $u \to v$, when the points p_u are mapped to $p(S)$. That is, it enforces both the equalities (left) and inequalities (centre, right) written on the edge $u \to v$.

The evaluation algorithm now follows.

Algorithm 1 Bottom-Up Evaluation of Pattern-Tree Vector

Input: A pattern-tree T, and a permutation $\pi \in \mathbb{S}_n$.

1. Traverse the vertices of T in post-order. For every vertex u labeled by $\tau_u \in \mathbb{S}_r$:
   (a) Construct a new (empty) rectangle-tree $\mathcal{T}_u$ of dimension $2r$.
   (b) Iterate over all sets $S := \{s_1, \ldots, s_r\} \subseteq p(\pi)$. If $S \cong \tau_u$, then:
      i. For every child v of u, issue the query $\mathcal{T}_v(\mathcal{R}_{uv}^{S})$.
      ii. Add the weight $\prod_{u \to v} \mathcal{T}_v(\mathcal{R}_{uv}^{S})$ (or 1, if u is a leaf) to point $p(S)$ in $\mathcal{T}_u$.
2. Return the answer to the query $\mathcal{T}_z(\mathcal{R})$, where z is the root of T, and $\mathcal{R} = [n]^{2|\tau_z|}$.

Theorem 3.6. *Let T be a pattern-tree of constant total size, and let $\pi \in \mathbb{S}_n$ be a permutation. The vector of T can be evaluated over π in $\widetilde{\mathcal{O}}\left(n^{s(T)}\right)$ time, where $s(T)$ is the maximum size.*[10]

Proof. TOPROVE 2 □

Remark 3.7. The algorithm presented in Theorem 3.6 is not necessarily the most efficient way to compute the vector of a pattern-tree, for several reasons. Firstly, many trees may correspond to the same vector, and these trees need not have the same maximum size. For example, both

$$u \bullet \xrightarrow{x_u < x_v,\, y_u < y_v} \circ\, v \quad \text{and} \quad 12$$

correspond to the vector #12. Secondly, as we will see in Section 4, there exist vectors for which bespoke *efficient* algorithms can be constructed, whose running time is strictly smaller than the maximum size of *any* pattern-tree with the same vector.

[10] The space-complexity is also $\widetilde{\mathcal{O}}\left(n^{s(T)}\right)$, since at every vertex of size r, we insert $\leq \binom{n}{r}$ points to a rectangle-tree.

3.3 $\widetilde{\mathcal{O}}(n^2)$ Algorithm for the k-Profile, for $1 \le k \le 7$

The corner-trees of [EZL21] are very efficiently computable. However, asymptotically, there are quite few of them: the number of rooted unlabeled trees over k vertices is only exponential in k (see, e.g., [Knu97] for a more accurate estimate), and clearly so is the number of corner-tree edge labels. Therefore, as $k \to \infty$, even if asymptotically almost all corner-trees vectors were linearly independent over $\mathbb{S}_{\le k}$, they would nevertheless contribute only a *negligible* proportion with respect to the full dimension, $|\mathbb{S}_{\le k}| = \sum_{r=1}^{k} r!$.

In contrast, it is not hard to see that pattern-trees are *fully expressive*: for every pattern $\tau \in \mathbb{S}_k$, there exists a pattern-tree T with $s(T) = k$, whose vector is precisely that pattern (in fact, $s(T) = \lceil k/2 \rceil$ suffices, see Section 3.5). To design *efficient* algorithms for the k-profile, we are interested in finding families of pattern-trees of *least maximum size*, whose corresponding vectors are linearly independent.

In [EZL21], corner-trees (i.e., pattern-tree of maximum size 1) over k vertices were shown to have full rank over $\mathbb{Q}^{\mathbb{S}_{\le k}}$ for $k = 3$, and in the cases $k = 4$ and $k = 5$, the subspaces spanned by them, restricted to $\mathbb{S}_4$ and $\mathbb{S}_5$, were found to be of dimensions only 23 and 100, respectively. Here, we show that for $k \le 7$, pattern-trees of maximum size ≤ 2 suffice.

Proof. TOPROVE 3 □

3.4 The case $k = 8$

Do pattern-trees over at most 8 points, and with $s(T) \le 2$, have full dimension for $\mathbb{S}_{\le 8}$? Using a computer program, we exhaustively enumerate all pattern-trees with the following properties,[11]

1. Every tree has $|p(T)| = 8$ points, and maximum size $s(T) \le 2$.
2. No edge is labeled with an equality.

In [EZL21] it was shown that pattern-trees with 4 vertices and maximum size 1 (corner-trees) span a subspace of dimension only $|\mathbb{S}_4| - 1 = 23$, when restricted to $\mathbb{S}_4$. Our pattern-trees extend this result: the subspace spanned by the above family of pattern-trees, with 8 points and maximum size ≤ 2, is of dimension exactly $|\mathbb{S}_8| - 1 = 40319$, when restricted to $\mathbb{S}_8$. The two vectors spanning the orthogonal complements of the subspaces for $\mathbb{S}_4$ and $\mathbb{S}_8$, $v_4 \in \mathbb{Q}^{\mathbb{S}_4}$ and $v_8 \in \mathbb{Q}^{\mathbb{S}_8}$ respectively, bear striking resemblance, as we detail below.

One of the central components in the 4-profile algorithm of [DG20] is the classification of patterns in $\mathbb{S}_4$ into two sets: *trivial* and *non-trivial*. A pattern $\tau \in \mathbb{S}_4$ is called non-trivial if its four points appear each in a different quadrant of the square $[4]^2$. A pattern is called trivial otherwise. An occurrence of a non-trivial permutation τ in a permutation $\pi \in \mathbb{S}_n$, in which the points of τ appear in the four quadrants of $[n]^2$, is called 4-partite.

There are 16 non-trivial patterns in $\mathbb{S}_4$, and they exactly form the support of the vector v_4. Half appear with magnitude 1, and half with magnitude -1. Clearly this implies that all trivial patterns can be counted in quasi-linear time (see Proposition 3.2). In fact, Dudek and Gawrychowski [DG20] observe that the only "hard case" in computing the 4-profile is counting the 4-partite occurrences of

[11] See Appendix A for a description of the enumeration process. For $k = 8$, this yields a matrix with $|\mathbb{S}_8| = 8!$ columns, and $|\mathbb{S}_8 \times \mathbb{S}_8 \times \{T_\lambda\}| \approx 2^{37}$ rows. We remark that we explicitly do not consider pattern-trees over *more* than 8 points, and trees whose edges are labeled by equalities. Whether this is without loss of generality, i.e., could their inclusion increase the rank, is unknown to us.

the non-trivial patterns, and prove a bidirectional reduction between enumerating such occurrences, and counting 4-cycles in sparse graphs.

At the heart of their algorithm for 4-partite occurrences lies an observation regarding the symmetries of the non-trivial permutations: they are closed both under the action of $D_4 \curvearrowright \mathbb{S}_4$, *and* the action of swapping the first two points (i.e., reflecting the *left half* of the square horizontally).

We extend all of the above characterisations to $\mathbb{S}_8$, as follows. Say that a pattern $\tau \in \mathbb{S}_8$ is *non-trivial* if it satisfies the following:

1. Each quadrant contains exactly two points.

2. The number of ascending (resp. descending) pairs in the four quadrants is odd.

3. Every *half* (top, bottom, left and right) of τ is a non-trivial permutation in $\mathbb{S}_4$.

We call a pattern trivial otherwise.

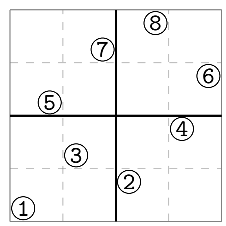


Figure 4: A non-trivial pattern $\tau = \texttt{15372846} \in \mathbb{S}_8$. There are three ascending pairs in its quadrants, and one descending pair. Its halves are order-isomorphic to the non-trivial permutations, `1342` (top), `1324` (bottom, left) and `1423` (right).

There are 2048 non-trivial permutations in $\mathbb{S}_8$. The support of the vector v_8 consists exactly of the non-trivial patterns of $\mathbb{S}_8$. Again, half appear with magnitude 1, and the other half (which are the vertical or horizontal reflections of the first set) appear with magnitude -1. This of course implies that all trivial patterns can be counted in $\widetilde{\mathcal{O}}\left(n^2\right)$-time. One can further extend the analogy to [DG20] by noting that all non-trivial patterns are closed under the action of $D_4 \curvearrowright \mathbb{S}_8$, *and* the actions of swapping the first two elements, or the first two pairs (i.e., reflecting the left quarter-strip, or the left half of the square horizontally).

We find the emergence of this "pattern" of non-trivial permutations and their relation to pattern-trees to be highly interesting. In fact, in direct analogy to [DG20], we conjecture that, as with $\mathbb{S}_4$, the occurrences of non-trivial patterns $\tau \in \mathbb{S}_8$ in a permutation $\pi \in \mathbb{S}_n$, in which the points of τ appear in the above configuration within the square $[n]^2$, constitute the "hard case" for computing the 8-profile. Settling this question, as well as understanding the (possibly algebraic) relation between pattern-trees of maximum size $\leq s$, and non-trivial permutations, are left as open questions.[12]

[12] Another possible extension of the analogy with regards to [DG20] is the following: it is known that for $3 \leq k \leq 7$, the number of length-k cycles in an n-vertex graph can be counted in time $\widetilde{\mathcal{O}}\left(n^{\omega}\right)$ [AYZ97], where ω is the exponent of matrix multiplication. Whether this cutoff at $k = 8$ relates to the 8-profile problem is unknown to us.

3.5 $\widetilde{\mathcal{O}}\left(n^{\lceil k/2 \rceil}\right)$ Algorithm for the k-Profile

We end this section by considering the problem of computing the k-profile via pattern-trees, for arbitrary (fixed) k. In the following proposition, we show that families of pattern-trees of maximal size $s(T) = \lceil k/2 \rceil$ suffice for computing the k-profile, through Algorithm 1. See Section 5 for a discussion on the relationship between $s(T)$ and k.

Proposition 3.8. *Let $\pi \in \mathbb{S}_n$ be an input permutation, and let $k \geq 2$ be a fixed integer. The k-profile of π can be computed in $\widetilde{\mathcal{O}}\left(n^{\lceil k/2 \rceil}\right)$ time.*

Proof. TOPROVE 4 □

4 $\widetilde{\mathcal{O}}\left(n^{7/4}\right)$ Algorithm for the 5-Profile

In Section 3, we recalled that pattern-trees of maximum size 1 (i.e., corner-trees) have full rational rank for $\mathbb{S}_{\leq 3}$ [EZL21], and proved that trees of maximal size at most 2 have full rank for $\mathbb{S}_{\leq 7}$ (see Theorem 1). Therefore, up to $k = 3$, the k-profile of an n-element permutation can be computed in $\widetilde{\mathcal{O}}(n)$ time, and up to $k = 7$, it is computable in $\widetilde{\mathcal{O}}\left(n^2\right)$ time. This naturally raises the question: is there a sub-quadratic time algorithm for these cases, where $k \geq 4$? We prove the following.

Theorem 2. *The 5-profile of any n-element permutation can be computed in time $\widetilde{\mathcal{O}}\left(n^{7/4}\right)$.*

We remark that the case $k = 4$ has been extensively studied in [DG20] and [EZL21]. There, they construct sub-quadratic algorithms of complexities $\mathcal{O}\left(n^{1.478}\right)$ and $\widetilde{\mathcal{O}}\left(n^{3/2}\right)$, respectively.

4.1 Marked and Weighted Patterns

For the proof of Theorem 2, we introduce the following notation.

Marked Patterns. A *marked pattern* is a pattern $\tau \in \mathbb{S}_k$ associated with an index $1 \leq j \leq k$. We say that a marked pattern τ occurs at index $1 \leq i \leq n$ in $\pi \in \mathbb{S}_n$, if there exists an occurrence of τ in π, in which the j-th x-coordinate is i. When the marked pattern τ is short, we underline the j-th index to indicate that marked index. For instance, $\underline{2}1$ occurs in 132 at index 2.

The *marked pattern count* is a 2-dimensional rectangle-tree containing the points $p(\pi)$, in which the weight of every point $(i, \pi(i))$ is the number of marked pattern occurrences at position i. For example, the tree $\mathcal{T}_2$ appearing in Proposition 2.3 is precisely the marked pattern count $\#1\underline{2}\,(\pi)$.

Weighted Pattern Counts. Let $\pi \in \mathbb{S}_n$ and let $w_1, \ldots, w_k : [n] \to \mathbb{Z}$ be *weight functions*, where $k \geq 1$ is a fixed integer. The *weighted pattern count* of $\tau \in \mathbb{S}_k$ in π, denoted $\#_w\tau\,(\pi)$, is the sum of $\prod_{j=1}^{k} w_j(i_j)$ over all occurrences $1 \leq i_1 < \cdots < i_k \leq n$ of τ in π. In other words, we count occurrences where every point has weight depending on its position, rather than 1 as usual.

The two concepts of marked patterns and weighted patterns can be combined in a straightforward way: the *weighted marked pattern count* is once again defined as a 2-dimensional rectangle-tree, as with marked pattern counts, but where now the number of occurrences for each point $(i, \pi(i))$ is appropriately weighted.

4.2 An Improvement to the Bottom-Up Algorithm

Recall that Algorithm 1 has time complexity $\widetilde{\mathcal{O}}\left(n^{s(T)}\right)$, where $s(T)$ is an integer. As we seek sub-quadratic algorithms, and since trees of $s(T) = 1$ (i.e., corner-trees) do not have full rank for $\mathbb{S}_{\leq 5}$, we take an alternative approach.

Let u be vertex of a pattern-tree T, labeled by some permutation $\tau_u \in \mathbb{S}_r$, such that:

1. The incoming edge to u (if any) conditions on a single point of u, say p_u^l.
2. Each outgoing edge of u (if any) is labeled by a single equality to a point of u.

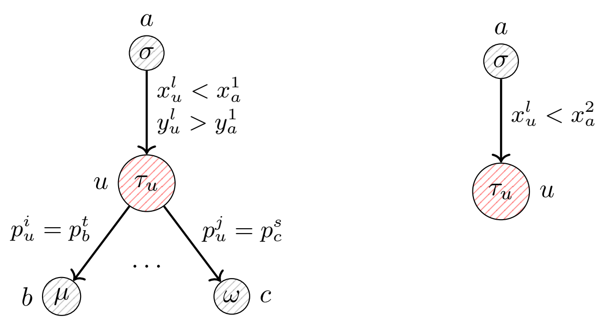


Figure 5: Two "gadgets" in a pattern-tree. The left corresponds to a weighted marked pattern count of τ_u, marked at l. The right corresponds to a (unweighted) marked pattern count of τ_u.

Suppose that, for the permutation $\tau_u \in \mathbb{S}_r$ and index $l \in [r]$, and given a set of weight functions $\{w_j\}_j$,[13] we are able to construct a 2-dimensional rectangle-tree representing the *weighted marked pattern-count*, $\#_w \tau_u(\pi)$ marked at l. Then, we claim that one can *modify* Algorithm 1 by replacing the rectangle-tree $\mathcal{T}_u$ associated with u, with the weighted marked pattern count of τ, marked at l, for a particular choice of weight functions. Concretely, we make the following modifications in Algorithm 1:

Traversing u. Instead of the routine operation of Algorithm 1, when u is visited we compute a weighted l-marked pattern count $\#_w \tau_u(\pi)$, abbreviated as $\mathcal{T}'_u$, with the following weights: for every point p_u^j, define a weight function $w_j : [n] \to \mathbb{Z}$ by

$$w_j(a) := \prod_{\substack{u \to v \\ p_u^j \text{ constrained}}} \mathcal{T}_v(\mathcal{R}_v^i)$$

where for an edge $u \to v$ labeled $p_u^j = p_v^i$, we define $\mathcal{R}_v^i$ as the rectangle in which the i-th x-segment is $\{a\}$ and all other segments are unconstrained (if p_u^j is not constrained by any outgoing edges, set its weight function to 1). By the invariant of Algorithm 1, the query $\mathcal{T}_v(\mathcal{R}_v^i)$ counts the number of occurrences of $T_{\leq v}$ in π such that $x_v^i = a$. Therefore, the resulting tree $\mathcal{T}'_u$ contains, at every point $(i, \pi(i))$, the number of occurrences of $T_{\leq u}$ in π such that $x_u^l = i$.

[13]We assume that for every weight function $w_j : [n] \to \mathbb{Z}$, the value $w_j(a)$ can be computed in $\widetilde{\mathcal{O}}(1)$-time.

Querying u. In Algorithm 1 we query the rectangle-trees of vertices in two scenarios:

1. If u is an internal vertex: In the original formulation of Algorithm 1, when the parent z of u is visited, we issue queries of the form $\mathcal{T}_u(\mathcal{R}^S_{zu})$, for pointsets $S \subseteq p(\pi)$. As the edge $z \to u$ only constrains p^l_u, the rectangles $\mathcal{R}^S_{zu}$ are *degenerate*, i.e., all of their segments are complete, except the two segments corresponding to p^l_u. These queries can be answered by $\mathcal{T}'_u(\mathcal{R}^l_u)$, where $\mathcal{R}^l_u \subseteq [n]^2$ is the 2-dimensional projection of $\mathcal{R}^S_{zu}$ onto those two segments.

2. If u is the root: The final step of the algorithm performs the full rectangle query $\mathcal{T}_u([n]^{2|\tau_u|})$, which counts the occurrences of $T_{\leq u} = T$ in all of π. This can be answered by the full rectangle query $\mathcal{T}'_u([n]^2)$.

As for the correctness of this modification to Algorithm 1, it remains to show that the new queries return the same values as the original ones. Let $\mathcal{R}$ be some rectangle query to $\mathcal{T}_u$. The value of $\mathcal{T}_u(\mathcal{R})$ is the number of occurrences of $T_{\leq u}$ in π, constrained to the coordinates allowed by $\mathcal{R}$. Since in both cases, all segments in $\mathcal{R}$ are complete except possibly those corresponding to p^l_u, this counts the occurrences of $T_{\leq u}$ in π constrained only to $p^l_u \in \mathcal{R}'$, for a 2-dimensional projection $\mathcal{R}'$ of $\mathcal{R}$ to the corresponding segments. By definition of a weighted marked pattern count, this is exactly the value of $\mathcal{T}'_u(\mathcal{R}')$.

In the remainder of this section, we design algorithms computing the pattern counts $\#_w 321\underline{4}$ and $\#4321\underline{5}$ in sub-quadratic time. Consequently, we can insert vertices labeled 3214 and 43215 into pattern-trees of maximum size 1 and with at most 5 points. Using the above modification to Algorithm 1, the overall time complexity for the evaluation of such trees remains sub-quadratic.

4.3 Computing $\#_w 321\underline{4}$ in $\widetilde{\mathcal{O}}\left(n^{5/3}\right)$ time

Theorem 1.2 of [EZL21] describes an algorithm for counting #3214. We require a slight alteration of their algorithm, and in particular, a weighted variant.

Lemma 4.1 (weighted version of Theorem 1.2 in [EZL21])**.** *Given an input permutation $\pi \in \mathbb{S}_n$ and weight functions $w_1, \ldots, w_4 : [n] \to \mathbb{Z}$, the tree $\#_w 321\underline{4}\,(\pi)$ can be computed in $\widetilde{\mathcal{O}}\left(n^{5/3}\right)$ time.*

Proof. TOPROVE 5 □

4.4 Computing $\#4321\underline{5}$ in $\widetilde{\mathcal{O}}\left(n^{7/4}\right)$ time

Lemma 4.2. *Let $\pi \in \mathbb{S}_n$ be a permutation. Then, $\#4321\underline{5}\,(\pi)$ can be computed in $\widetilde{\mathcal{O}}\left(n^{7/4}\right)$ time.*

Proof. TOPROVE 6 □

4.5 Pair-Rectangle-Trees

Theorem 4.3. *(pair-rectangle-tree) There exists a data structure with the following properties:*

1. *Preprocessing: Given an input permutation $\pi \in \mathbb{S}_n$, the tree is initialised in time $\widetilde{\mathcal{O}}\left(n^2/q\right)$.*

2. *Query: Given any rectilinear rectangle $\mathcal{R} \subseteq [n] \times [n]$, return the number of descending (resp. ascending) pairs of permutations points in $\mathcal{R}$, in time $\widetilde{\mathcal{O}}\left(q\right)$.*

where $q = q(n) \in [n]$ is a parameter that can be chosen arbitrarily.

Proof. TOPROVE 7 $\square$

4.6 Algorithm for the 5-Profile

Proof. TOPROVE 8 $\square$

5 Discussion

Some immediate extensions of this work, such as the application of our methods to the 9-profile, or the use of pattern-trees with maximum size 3, are computationally difficult and likely require further analysis or a different approach (say, algebraic). Several interesting open questions remain:

1. **Maximum size versus rank.** For a given integer s, denote by $f(s)$ the largest integer k such that the subspace spanned by the vectors of pattern-trees over at most k points, of maximum size s and with no equalities, is of full dimension, $|\mathbb{S}_{\leq k}|$. The results of [EZL21] imply that $f(1) = 3$. In Section 3.3 and Section 3.4 we prove $f(2) = 7$, and in Section 3.5 we show that $f(s) \geq 2s$, for every $s \geq 1$. What is the behavior of $f(s)$? For example, do we have $f(s) \geq 4s \pm o(s)$, as attained by the technique of [BKM21]?

2. **A fine-grained variant of $f(s)$.** For integers s and k, let $g(s, k)$ be the number of linearly independent vectors in $\mathbb{Q}^{\mathbb{S}_k}$ generated by Algorithm 1 when applied to trees of maximum size s over k permutation points with no equalities. What is the general behavior of $g(s, k)$? This generalises a question of [EZL21] about corner-trees. The following values are presently known.

$s \backslash k$	1	2	3	4	5	6	7	8
1	1	2	6	23	100	463	2323	**12173**
2	**1**	**2**	**6**	**24**	**120**	**720**	**5040**	**40319**

Table 1: Bolded values in this table are computed in this paper (new).

3. **Complexity of k-profile, for $5 \leq k \leq 7$.** Can the time complexity for finding the $5, 6, 7$-profiles be improved further, perhaps by utilising techniques along the lines of Section 4? In particular, we ask whether the 6-profile can be computed in sub-quadratic time.

4. **Study of the 8-profile.** [DG20] shows the equivalence between the computation of the 4-profile and counting 4-cycles in sparse graphs. In Section 3.4 we show that many of the observations of [DG20] can be extended to $\mathbb{S}_8$. In fact, we conjecture that there exists an analogous hardness result for $k = 8$, and we refer the reader to Section 3.4 where the details are discussed.

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A Enumeration of Pattern-Tree Vectors

Let s and k be two positive integers, where $s \leq k$. Consider the following enumeration process, which computes the matrix whose rows are the vectors of all pattern-trees of maximum size $\leq s$, with exactly k points, restricted to $\mathbb{S}_k$.

For every ordered partition $\lambda \vdash k$ with no part larger than s, and for every vertex-labeled tree $T \in \mathbb{T}_{|\lambda|}$,[14] let T_λ be the tree in which vertex i has size $\lambda(i)$, and is assigned point variables

$$p(v_i) = \left\{p_{r_{i-1}+1}, \ldots, p_{r_i}\right\}, \text{ where } r_i := \sum_{j \leq i} \lambda(j), \text{ and } r_0 := 0.$$

Think of T_λ as a "template" for a pattern-tree, where the topology, the sizes of vertices, and the names of their variables have been determined, but the edge-constraints have not. Next, iterate over all pairs of permutations, $\sigma, \tau \in \mathbb{S}_k$, and over all trees T_λ.

Any such combination maps to a pattern-tree in the above family. For every i and j such that p_i and p_j are associated with the same vertex v, write the constraint $x_i < x_j$ in the vertex v if $\sigma(i) < \sigma(j)$, and write $x_i > x_j$ otherwise. Do likewise for the y constraints and τ, and repeat the same operation for every pair i, j such that the points p_i and p_j are associated with adjacent

[14] Here $\mathbb{T}_r$ is the set of all vertex-labeled trees over r vertices.

vertices in T_λ – in this case, we write the inequality on the edge. Observe that the ordering of points in each vertex is fully determined, i.e., defines a *permutation*.

Therefore, for any combination of T_λ, σ and τ, we obtain a pattern-tree T for which σ is a linear extension of the x-poset, and τ is a linear extension of the y-poset. In this case, we add 1 to the vector of T, at the index of $\tau\sigma^{-1}$ (see Lemma 3.5). Once the process is completed, we obtain a matrix with $|\mathbb{S}_k| = k!$ columns, and no more than $|\mathbb{S}_k \times \mathbb{S}_k \times \{T_\lambda\}|$ rows. The row-space of this matrix over the rationals is the subspace spanned by the above family of trees, restricted to $\mathbb{S}_k$.

We remark that if for every $k' \leq k$ this process produces a matrix of full rank, then by induction, the vectors of the union of all trees in these families spans the entire subspace, for $\mathbb{S}_{\leq k}$ (no tree over $k' < k$ points has a component in $\mathbb{S}_k$).