Towards the Proximity Conjecture on Group-Labeled Matroids

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Abstract

Consider a matroid M whose ground set is equipped with a labeling to an abelian group. A basis of M is called F-avoiding if the sum of the labels of its elements is not in a forbidden label set F. Hörsch, Imolay, Mizutani, Oki, and Schwarcz (2024) conjectured that if an F-avoiding basis exists, then any basis can be transformed into an F-avoiding basis by exchanging at most $|F|$ elements. This *proximity conjecture* is known to hold for certain specific groups; in the case where $|F| \leq 2$; or when the matroid is subsequence-interchangeably base orderable (SIBO), which is a weakening of the so-called strongly base orderable (SBO) property.

In this paper, we settle the proximity conjecture for sparse paving matroids or in the case where $|F| \leq 4$. Related to the latter result, we present the first known example of a non-SIBO matroid. We further address the setting of multiple group-label constraints, showing proximity results for the cases of two labelings, SIBO matroids, matroids representable over a fixed, finite field, and sparse paving matroids.

Keywords: sparse paving matroid, subsequence-interchangeable base orderability, congruency constraint, multiple labelings

1 Introduction

Let E be a finite ground set and let $\psi\colon E \to \Gamma$ be a labeling from E to an abelian group Γ. A *group-label constraint* requires for a solution $X \subseteq E$ to satisfy $\psi(X) := \sum_{e \in X} \psi(e) \notin F$, where $F \subseteq \Gamma$ is a prescribed set of forbidden labels. Such a solution X is called *F-avoiding*. An F-avoiding X is also called *zero* in the case when $F = \Gamma \setminus \{0\}$ (i.e., $\psi(X) = 0$), and *non-zero* in the case when $F = \{0\}$ (i.e., $\psi(X) \neq 0$). Several constraints in combinatorial optimization, such as parity, congruency, and exact-weight constraints, are representable as group-label constraints by letting Γ be a cyclic group $\mathbb{Z}_m$ or the integers $\mathbb{Z}$, and F be the complement of a singleton. These constraints have been studied for many classical combinatorial optimization problems, including matching [1, 14, 25, 36, 40], arborescence [2], submodular function minimization [17, 37], minimum cut [38], and independent sets or bases in a matroid [6, 12, 43]. Also, the non-zero and F-avoiding constraints have been particularly well-studied for path and cycle problems on graphs [8, 9, 23, 27, 29, 42, 47, 49, 50].

In this paper, we study group-label constraints on matroid bases. This line of research was initiated by Liu and Xu [32], who addressed the problem of finding a zero basis. Hörsch, Imolay, Mizutani, Oki, and Schwarcz [20] considered non-zero bases, and more generally, F-avoiding bases, posing the following conjecture.

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Conjecture 1.1 (Proximity Conjecture [20]). *Let M be a matroid, $\psi\colon E \to \Gamma$ a labeling from the ground set E of M to an abelian group Γ, and $F \subseteq \Gamma$ a finite collection of forbidden labels. Then, for any basis A of M, there exists an F-avoiding basis B of M such that $|A \setminus B| \le |F|$, provided that at least one F-avoiding basis exists.*

It is clear that Conjecture 1.1 implies an algorithm for finding an F-avoiding basis using $O((rn)^{|F|})$ independence oracle queries, where r is the rank of M and $n \coloneqq |E|$. Another consequence of Conjecture 1.1 is that the number of F-avoiding bases is at least $|\mathcal{B}|/\sum_{i=0}^{|F|}\binom{r}{i}\binom{n-r}{i}$, where $\mathcal{B}$ is the basis family of M, provided that at least one F-avoiding basis exists.

Conjecture 1.1 is known to hold if (i) $|F| \le 2$, (ii) Γ is an ordered group, (iii) Γ has prime order, (iv) $|\Gamma \setminus F| = 1$ and Γ is a cyclic group with the order being a prime power or the product of two primes, or (v) M is strongly base orderable (SBO). The claim for $|F| = 1$ essentially follows from Rieder's characterization [43] of basis lattices, and a simpler proof can be found in [20]. The proof for $|F| = 2$ in [20] first reduces the problem to 6-element, rank-3 matroids and then shows the claim for them, treating one special matroid $M(K_4)$ separately. Case (ii) was also proven in [20], where an ordered group is a group equipped with a total order consistent with the group operation, such as the integers $\mathbb{Z}$ and the reals $\mathbb{R}$.

Conjecture 1.1 for (iii) and (iv) was shown by Liu and Xu [32]. They showed (iii) with $|\Gamma \setminus F| = 1$ using an additive combinatorics result by Schrijver and Seymour [45] (which is Conjecture 1.2 below for prime-order cyclic groups), and it immediately extends to general F. For (iv), Liu and Xu observed more generally that Conjecture 1.1 for finite Γ with $|\Gamma \setminus F| = 1$ holds if the following long-standing conjecture by Schrijver and Seymour [45][1] is met for every subgroup of Γ.

Conjecture 1.2 (Schrijver and Seymour [45]; see also [11]). *Let M be a matroid with ground set E, basis family $\mathcal{B}$, and rank function ρ. Let $\psi\colon E \to \Gamma$ be a labeling to an abelian group Γ and $H \coloneqq \{\, g \in \Gamma \mid g + \psi(\mathcal{B}) = \psi(\mathcal{B}) \,\}$ the stabilizer subgroup of $\psi(\mathcal{B}) \coloneqq \{\, \psi(B) \mid B \in \mathcal{B} \,\}$. Then,*

$$|\psi(\mathcal{B})| \ge |H| \left(\sum_{Q \in \Gamma/H} \rho(\psi^{-1}(Q)) - \rho(E) + 1 \right).$$

Schrijver and Seymour [45] showed Conjecture 1.2 for prime-order cyclic groups, and DeVos, Goddyn, and Mohar [11] proved it for the cases when M is obtained from a uniform matroid by adding parallel elements and when Γ is one of the groups in (iv).

Case (v) was shown in [32] for $|\Gamma \setminus F| = 1$ and in [20] for general F. SBO matroids are a class of matroids that admit a certain basis exchange property and includes gammoids (so in particular, uniform, partition, laminar, and transversal matroids); see, e.g., [44, Section 42.6c]. As mentioned in the full version [21] of [20], the same proof works if we only assume a weaker property called *subsequence-interchangeable base orderability* (cf. Lemma 2.2). Following Baumgart [3], we say that a rank-r matroid is *subsequence-interchangeably base orderable* (SIBO) if every pair of bases A and B admits orderings $a_1, \dots, a_r$ of A and $b_1, \dots, b_r$ of B such that $(B \setminus \{b_i, \dots, b_j\}) \cup \{a_i, \dots, a_j\} = \{b_1, \dots, b_{i-1}, a_i, \dots, a_j, b_{j+1}, \dots, b_r\}$ is a basis for any pair (i, j) with $1 \le i \le j \le r$. For each pair of bases, we call such a pair of orderings an *SI-ordering*. Baumgart [3] posed the following conjecture.

Conjecture 1.3 (Baumgart [3]). *Every graphic matroid is SIBO.*

By case (v) above, Conjecture 1.3 would imply Conjecture 1.1 for graphic matroids. Let us note that Conjecture 1.3 is a strengthening of the graphic matroid case of the following celebrated conjecture.

Conjecture 1.4 (Gabow [16], see also [10,48]). *Let A and B be bases of a rank-r matroid M. Then, there are orderings $a_1, \dots, a_r$ of A and $b_1, \dots, b_r$ of B such that $\{a_1, \dots, a_i, b_{i+1}, \dots, b_r\}$ and $\{b_1, \dots, b_i, a_{i+1}, \dots, a_r\}$ are bases for any $i = 1, \dots, r$.*

In contrast to Conjecture 1.3, Conjecture 1.4 is known to hold for graphic [10,26,48], and more generally for regular matroids [4].

[1] The original conjecture by Schrijver and Seymour [45] is stated only for the case when the stabilizer subgroup H is trivial. Conjecture 1.2 is the form given in [11] and is obtained by applying the original one to a labeling $\psi'\colon E \to \Gamma/H$ defined by $\psi'(e) \coloneqq \psi(e) + H$ for each $e \in E$.

Our contributions. Our first main result is a proof of Conjecture 1.1 for sparse paving matroids. A rank-r matroid is *paving* if every circuit is of size either r or $r+1$, and is *sparse paving* if it and its dual are both paving. Sparse paving matroids are significant in matroid theory as they have been used in hardness proofs for several algorithmic problems [12, 21, 24, 33] as well as counterexamples of conjectures. In fact, sparse paving matroids were used in [20] to disprove a strengthening of Conjecture 1.1 for $|\Gamma \setminus F| = 1$ posed in the initial preprint version [31] of [32]. Given this context, our positive result for sparse paving matroids provides additional evidence for Conjecture 1.1. Moreover, since it is believed that asymptotically almost all matroids are sparse paving [34], our result would imply that Conjecture 1.1 holds in an asymptotic sense.

We also show Conjecture 1.1 for $|F| \leq 4$ using a computer-aided proof. First, we use an observation from [21] (see Lemma 2.1) to reduce to the case of matroids on at most 10 elements having rank at most 5. By checking all such matroids using a SAT solver, it turns out that all of them are SIBO, except for a single matroid called R_{10}. We complete the proof by showing Conjecture 1.1 for R_{10} separately. For completeness, we also provide an elementary proof that R_{10} is not SIBO. This is noteworthy, as it serves as the first example of a non-SIBO matroid and shows that Conjecture 1.3 does not extend to regular matroids.

As the second main thread of the paper, we consider an analog of Conjecture 1.1 for multiple group labelings. In this setting, given k labelings $\psi_1, \dots, \psi_k$, where each ψ_i is a map from E to an abelian group Γ_i, and k forbidden labels $f_1 \in \Gamma_1, \dots, f_k \in \Gamma_k$, we are to find a basis B such that $\psi_i(B) \neq f_i$ for all $i \in \{1, \dots, k\}$. Questions with similar constraints have also been studied for paths and cycles in graphs [7, 18, 19, 22]. Note that the $\psi_1 = \cdots = \psi_k$ case corresponds to the single F-avoiding constraint with $F = \{f_1, \dots, f_k\}$. We pose the following conjecture.

Conjecture 1.5 (Multi-Labeled Proximity Conjecture)**.** *There is a computable function $d\colon \mathbb{N} \to \mathbb{N}$ such that for each $k \in \mathbb{N}$, matroid M with ground set E, group labelings $\psi_i\colon E \to \Gamma_i$, group elements $f_i \in \Gamma_i$ for $i = 1, \dots, k$, and basis A of M, there exists a basis B of M with $\psi_i(B) \neq f_i$ for $i = 1, \dots, k$ and $|A \setminus B| \leq d(k)$, provided that at least one such basis exists.*

As a lower bound, we show using uniform matroids that $d(k)$, if exists, must be at least $2^k - 1$. Conjecture 1.1 for $|F| = 1$ implies that this is tight if $k = 1$. For $k = 2$, we show that $2^2 - 1 = 3$ is tight. We further show that $d(k) = \lfloor (e - 1/2)k! \rfloor - 1$ suffices for SIBO matroids. We combine an extension of this result with a result of [20] (Theorem 5.4) to show the existence of such a function $d(k)$ for matroids representable over a fixed, finite field. Finally, we prove an analogous result for sparse paving matroids using a similar method, but relying on a new structural observation on sparse paving matroids instead of Theorem 5.4.

Related work. Eisenbrand, Rohwedder, and Węgrzycki [13] showed a proximity result on basis pairs of integer-labeled matroids. For a matroid labeled with $\mathbb{Z}_m$ for a positive integer $m \geq 2$, this result implies that if there exists a zero basis, then for any basis A, there exists a zero basis B such that $|A \setminus B| = O(m^5)$. This bound is weaker than the bound $|A \setminus B| \leq m - 1$ implied by Conjecture 1.1. Their result also implies a bound $|A \setminus B| = |\Gamma|^{O(\log |\Gamma|)}$ for a matroid labeled with a finite abelian group Γ. The paper [13] additionally provided an FPT algorithm for finding a zero basis of a matroid labeled with a finite abelian group when parameterized by group size.

Hörsch, Imolay, Mizutani, Oki, and Schwarcz [20] also posed a weighted variant of Conjecture 1.1, which was settled for SBO matroids as well as the case when $|F| = 1$ [20]. Extending the proofs of Conjecture 1.1 in other cases to the weighted conjecture is left for future work.

Organization. The rest of this paper is organized as follows. Section 2 describes preliminaries. Sections 3 and 4 prove Conjecture 1.1 for sparse paving matroids and the case when $|F| \leq 4$, respectively. Section 5 gives proximity results in the setting of multiple labelings. Finally, in Section 6, we conclude the paper with several open questions.

2 Preliminaries

For a nonnegative integer k and a set S, let $[k] \coloneqq \{1, \dots, k\}$ and $\binom{S}{k} \coloneqq \{\, X \subseteq S \mid |X| = k \,\}$. For a set S, $x \notin S$, and $y \in S$, we abbreviate $S \cup \{x\}$ as $S + x$ and $S \setminus \{y\}$ as $S - y$. All groups are implicitly assumed to be abelian. We use the additive notation for the group operation. Let $\mathbb{Z}_m$ be the cyclic group of order m. For a prime power q, let $\mathrm{GF}(q)$ be the finite field of size q.

We refer the readers to [39] for basic concepts and terminology in matroid theory. A matroid M consists of a finite ground set $E(M)$ and a nonempty set family $\mathcal{B}(M)$ such that for any $B, B' \in \mathcal{B}(M)$ and $e \in B \setminus B'$, there exists $f \in B' \setminus B$ such that $B - e + f \in \mathcal{B}(M)$. Every element in $\mathcal{B}(M)$ is called a *basis* (or a *base*). The *rank* of M is the size of any basis of M. The *dual* M^* of M is a matroid on the same ground set defined by $\mathcal{B}(M^*) = \{\, E(M) \setminus B \mid B \in \mathcal{B}(M) \,\}$. For $X \subseteq E(M)$, the *restriction* of M to X, denoted by $M|X$, is a matroid on X with $\mathcal{B}(M|X) = \{\, B' \in \binom{X}{r'} \mid B' \subseteq B \ (\exists B \in \mathcal{B}(M)) \,\}$, where $r' \coloneqq \max_{B \in \mathcal{B}(M)} |B \cap X|$. Also, the *contraction* of M by X is a matroid $M/X \coloneqq (M^*|(E(M) \setminus X))^*$. A matroid M' is a *minor* of M if $M' = (M|X)/Y$ for some X, Y with $Y \subseteq X \subseteq E(M)$. Two matroids M_1 and M_2 are *isomorphic* if there exists a bijection $\sigma \colon E(M_1) \to E(M_2)$ such that $\mathcal{B}(M_2) = \{\, \{\, \sigma(e) \mid e \in B \,\} \mid B \in \mathcal{B}(M_1) \,\}$.

A matroid M of rank r is called *uniform* if $\mathcal{B}(M) = \binom{E(M)}{r}$; it is denoted by $U_{r,n}$ up to isomorphism, where $n = |E(M)|$. A matroid M is called $\mathbb{F}$-*representable* if for some matrix A over a field $\mathbb{F}$, $E(M)$ corresponds to the set of columns of A and $\mathcal{B}(M)$ consists of the subsets of columns of A each of which forms a basis of the vector space spanned by the columns of A. We will use the following characterizations of paving and sparse paving matroids. A matroid M of rank r is paving if and only if there exists a collection $\mathcal{H} = \{H_1, \dots, H_k\}$ of subsets of $E(M)$ such that $|H_i| \geq r$ for each $i \in [k]$, $|H_i \cap H_j| \leq r - 2$ if $i \neq j$, and $\mathcal{B}(M) = \{\, B \in \binom{E(M)}{r} \mid B \not\subseteq H_i \ (\forall i \in [k]) \,\}$ (cf. [15, Theorem 5.3.5]). Also, M is sparse paving if and only if there is such a representation with $|H_i| = r$ for each $i \in [k]$; in this case, $\mathcal{B}(M) = \binom{E(M)}{r} \setminus \mathcal{H}$ (cf. [5, Lemma 2.1]). Note that the class of sparse paving matroids is minor-closed, since it is closed under taking restrictions and duals.

In an indirect approach to Conjecture 1.1, the following lemma is useful. It is similar to [32, Corollary 4.4] and is obtained from [21, Lemmas 5.35 and 5.36].

Lemma 2.1 (see [21])**.** *Let M be a matroid, $\psi \colon E(M) \to \Gamma$ a group labeling, and $F \subseteq \Gamma$ a finite set of forbidden labels. Assume that (M, ψ, F) is a counterexample to Conjecture 1.1, i.e., M has an F-avoiding basis and it has a basis A with $|A \setminus B| \geq |F| + 1$ for any F-avoiding basis B. Then, there exists a minor M' of M having rank $|F| + 1$, a labeling $\psi' \colon E(M) \to \Gamma$, a set of labels $F' \subseteq \Gamma$ with $|F'| = |F|$, and a basis B' of M such that B' is the only F'-avoiding basis of M' and $E(M') \setminus B'$ is a basis.*

The following observation was essentially noted in [21, Remark 5.16]; we include a proof for completeness.

Lemma 2.2 (see [21])**.** *Let E be a finite set, $\psi \colon E \to \Gamma$ a group labeling, $F \subseteq \Gamma$ a finite collection of labels, and $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_r\}$ disjoint subsets of E, where $r = |F| + 1$. If B is F-avoiding, then there exists a pair (i, j) with $1 \leq i \leq j \leq r$ and $(i, j) \neq (1, r)$ such that $\hat{B}_{i,j} \coloneqq (B \setminus \{b_i, \dots, b_j\}) \cup \{a_i, \dots, a_j\} = \{b_1, \dots, b_{i-1}, a_i, \dots, a_j, b_{j+1}, \dots, b_r\}$ is F-avoiding. Furthermore, if $\psi(\hat{B}_{1,j}) \in F$ for every $j \geq 1$, then there exists a pair with $i > 1$ such that $\psi(\hat{B}_{i,j}) = \psi(B)$.*

Proof. TOPROVE 0 □

3 Proximity Theorem for Sparse Paving Matroids

In this section, we prove the following theorem.

Theorem 3.1. *Conjecture 1.1 is true when M is a sparse paving matroid.*

Proof. TOPROVE 1 □

4 Proximity Theorem for at Most 4 Forbidden Labels

We first observe that Conjecture 1.3 does not hold for regular matroids: a pair of disjoint bases of the matroid R_{10} does not have an SI-ordering. R_{10} is the matroid appearing in Seymour's fundamental decomposition theorem of regular matroids [46], and it can be defined as the even-cycle matroid of the complete graph K_5. The ground set of this matroid is the edge set of K_5 and its bases are the sets of five edges forming a subgraph containing exactly one odd cycle and no even cycle. It is not difficult to check the following statement; see also the proof of [4, Proposition 5.5].

Lemma 4.1. *Consider R_{10} as the even-cycle matroid of the complete graph K_5 on vertex set $\{v_1, \ldots, v_5\}$. Then, for any two disjoint bases A and B of R_{10}, there exists an automorphism of R_{10} mapping A and B to the 5-cycles $\{\, v_i v_{i+1} \mid i \in [5] \,\}$ and $\{\, v_i v_{i+2} \mid i \in [5] \,\}$ of K_5, respectively, where indices are meant in a cyclic order (e.g., $v_6 = v_1$).*

We show the following using Lemma 4.1.

Theorem 4.2. *R_{10} is not SIBO.*

Proof. TOPROVE 2 □

Using a computer program, we verified that this is the only example of a basis pair not having an SI-ordering up to rank 5.

Proposition 4.3. *Let M be a matroid of rank at most 5, and (A, B) a basis pair of M not having an SI-ordering. Then, A and B are disjoint and the restriction $M|(A \cup B)$ is isomorphic to R_{10}.*

Giving a human-readable proof of Proposition 4.3 seems difficult, as even Conjecture 1.4 was verified only up to rank 4 [30]. (Note that R_{10} is known to satisfy Conjecture 1.4 [4], thus Proposition 4.3 implies that the conjecture holds up to rank 5.) Up to rank 4, one can check the validity of Proposition 4.3 by using one of the existing databases of small matroids [35]. As the list (or number) of rank-5 matroids on 10 elements is unknown and expected to be very large [35], we used a different approach: we encoded a basis pair of a matroid of given rank not having an SI-ordering as a Boolean formula, with variables encoding which subsets are bases, and decided the satisfiability with a SAT solver; see Appendix A for details. We note that a similar but much more sophisticated approach has been used to study Rota's basis conjecture [28].

We are ready to prove the following theorem.

Theorem 4.4. *Conjecture 1.1 is true when $|F| \leq 4$.*

Proof. TOPROVE 3 □

5 Proximity for Multiple Labelings

In this section, we verify Conjecture 1.5 for various classes of matroids. We begin with an example showing that the function d in Conjecture 1.5 must satisfy $d(k) \geq 2^k - 1$ for each $k \geq 1$, even for uniform matroids.

Example 5.1. Let k be a positive integer. Let $r = 2^k - 1$, and let $U_{r,2r}$ denote the uniform matroid of rank r on some ground set E of size $2r$. We show that there exist group labelings $\psi_i \colon E \to \Gamma_i$ for $i \in [k]$, and a basis A of $U_{r,2r}$ such that $B \coloneqq E \setminus A$ is the only basis of M satisfying $\psi_i(B) \neq 0$ for all $i \in [k]$.

Let us fix a set $A \subseteq E$ of size r. We set $\Gamma_k = \mathbb{Z}$, and $\Gamma_i = \mathbb{Z}_{2^i}$ for $i \in [k-1]$. Finally, we define the group labelings by

$$\psi_i(e) \coloneqq \begin{cases} 2^{i-1} - 1 & \text{if } e \in A, \\ 2^{i-1} & \text{if } e \notin A \text{ and } i \neq k, \\ -2^{k-1} & \text{if } e \notin A \text{ and } i = k. \end{cases}$$

Note that $r \equiv -1 \pmod{2^i}$ for all $i \in [k-1]$. It is easy to see that $\psi_i(B) \neq 0$ for all $i \in [k]$, where $B = E \setminus A$. Hence, we only need to show that for each $\ell \in [r-1]$, any set A' obtained by exchanging ℓ elements of A into B is zero in at least one of the labelings.

Let $i \in \mathbb{N}$ be the largest for which 2^{i-1} divides $\ell + 1$. Note that $\ell \equiv 2^{i-1} - 1 \pmod{2^i}$. Moreover, as $\ell < r = 2^k - 1$, we have $i \leq k$. Now if $i < k$, then

$$\begin{aligned} \psi_i(A') &= (2^{i-1} - 1)(r - \ell) + 2^{i-1}\ell \\ &= (2^{i-1} - 1)r + \ell \\ &\equiv (2^{i-1} - 1) \cdot (-1) + (2^{i-1} - 1) && \pmod{2^i} \\ &\equiv 0 && \pmod{2^i}. \end{aligned}$$

On the other hand, if $i = k$, then $\ell = 2^{k-1} - 1$, and in this case

$$\psi_k(A') = (2^{k-1} - 1)(r - \ell) - 2^{k-1}\ell = (2^{k-1} - 1)2^{k-1} - 2^{k-1}(2^{k-1} - 1) = 0.$$

Next, we show the existence of $d(k)$ as in Conjecture 1.5 for several matroid classes. The proofs will be based on the following lemma.

Lemma 5.2. *Let $\psi_i \colon E \to \Gamma_i$ be group labelings on a finite set E and $f_i \in \Gamma_i$ group elements for $i \in [k]$. Let $B \subseteq E$ be a subset with $\psi_t(B) \neq f_t$ for all $t \in [k]$. Let $X_1, \dots, X_\ell \subseteq B$ and $Y_1, \dots, Y_\ell \subseteq E \setminus B$ be pairwise disjoint nonempty subsets. For $1 \leq i \leq j \leq \ell$, let*

$$B_{i,j} \coloneqq (B \setminus (X_i \cup \dots \cup X_j)) \cup (Y_i \cup \dots \cup Y_j).$$

If $\ell \geq \lfloor (e - 1/2)k! \rfloor$, there are indices i, j such that $\psi_t(B_{i,j}) \neq f_t$ for all $t \in [k]$.

Proof. TOPROVE 4 □

Theorem 5.3. *Let A be a basis of a matroid M on ground set E, and $\psi_i \colon E \to \Gamma_i$ group labelings and $f_i \in \Gamma_i$ group elements for $i \in [k]$. Assume that M has at least one basis B with $\psi_t(B) \neq f_i$ for all $i \in [k]$.*

(i) If M is SIBO, then it has a basis B with $\psi_t(B) \neq f_t$ for all $t \in [k]$ and $|A \setminus B| \leq \lfloor (e - 1/2)k! \rfloor - 1$.

(ii) If $k = 2$, then M has a basis B with $\psi_1(A) \neq f_1$, $\psi_2(A) \neq f_2$, and $|A \setminus B| \leq 3$.

Proof. TOPROVE 5 □

Example 5.1 shows that for $k = 2$, the bound 3 in Theorem 5.3(ii) is tight. Observe that this differs from the case $\psi_1 = \psi_2$ where the tight bound is 2 [20].

Finally, we derive the validity of Conjecture 1.5 for matroids representable over a fixed, finite field and sparse paving matroids. For positive integers α and k, we define a matroid M to be *weakly (α, k)-base orderable* if for every ordered basis pair (A, B) of M with $|A \setminus B| \geq \alpha$, there exist pairwise disjoint nonempty subsets $X_1, \dots, X_k \subseteq B \setminus A$ and $Y_1, \dots, Y_k \subseteq A \setminus B$ such that $\left(B \setminus \bigcup_{i \in Z} X_i\right) \cup \bigcup_{i \in Z} Y_i$ is a basis for each $Z \subseteq [k]$. The following was shown in [20].

Theorem 5.4 (Hörsch, Imolay, Mizutani, Oki, Schwarcz [20])**.** *There is a computable function $h \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every prime power q, every $\mathrm{GF}(q)$-representable matroid is weakly $(h(q, k), k)$-orderable for any $k \in \mathbb{N}$.*

Following the terminology of [20], for a basis B of a matroid M, we say that a minor M' of M is a *B-minor* if $M' = (M|X)/Y$ for some X, Y with $Y \subseteq B \subseteq X \subseteq E(M)$. In this case, $\mathcal{B}(M') = \{\, B' \subseteq X \setminus Y \mid B' \cup Y \in \mathcal{B}(M) \,\}$. We show the following result, which is related to results of [41].

Theorem 5.5. *Let $k \geq 0$ be an integer and M a sparse paving matroid of rank r. If $\min\{r, |E(M)| - r\} \geq \binom{2k}{k}$, then for each basis B, M has a B-minor isomorphic to $U_{k,2k}$.*

Proof. TOPROVE 6 □

Remark 5.6. We note that Pendavingh and van der Pol [41] showed that for a fixed k, asymptotically almost all matroids contain $U_{k,2k}$ as a minor. If M is a sparse paving matroid, then a counting argument found in [41, Lemma 4.7] combined with the observation $|\mathcal{B}(M)| \geq \frac{r}{r+1}\binom{|E|}{r}$ implies that if $r \geq \binom{2k}{k}$ and $|E(M)| - r \geq k$ hold, then M contains $U_{k,2k}$ as a minor. It is not clear whether a similar argument can be used to give a simple proof of the existence of such a B-minor for any basis B as in Theorem 5.5.

Theorems 5.4 and 5.5 and Lemma 5.2 immediately verify Conjecture 1.5 for matroids representable over a fixed, finite field and sparse paving matroids.

Corollary 5.7. *Let $\mathcal{M}$ be the class of (1) $\mathrm{GF}(q)$-representable matroids for a fixed prime power q or (2) sparse paving matroids. Then, there is a computable function $d \colon \mathbb{N} \to \mathbb{N}$ such that if $M \in \mathcal{M}$, $\psi \colon E(M) \to \Gamma_i$ are group labelings, $f_i \in \Gamma_i$ are group elements for $i \in [k]$, and A is a basis of M, then M has a basis B with $\psi_i(B) \neq f_i$ for all $i \in [k]$ and $|A \setminus B| \leq d(k)$, provided that M has at least one basis B' with $\psi_i(B') \neq f_i$ for all $i \in [k]$.*

Proof. TOPROVE 7 □

6 Conclusion

In this paper, we have proven Conjecture 1.1 for the case when the matroid is sparse paving or $|F| \leq 4$, and settled Conjecture 1.5 for $k = 2$ and some classes of matroids. We conclude this paper by posing new conjectures.

Conjecture 6.1. *Every sparse paving matroid is SIBO.*

We have checked the validity of the conjecture up to rank 6 using a SAT solver. If true, Conjecture 6.1 would give another proof of Theorem 3.1. Unfortunately, the proof [5] of Conjecture 1.4 for sparse paving matroids does not seem to generalize to this conjecture.

We also pose the following refinement of Conjecture 1.5 in light of our lower bound on $d(k)$. We state it in the form of a question rather than a conjecture as we do not expect it to hold for general matroids, whereas it is more likely to hold for uniform, SBO, and SIBO matroids.

Question 6.2. *Let M be a matroid with the ground set E, $\psi_i \colon E \to \Gamma_i$ a group labeling, and $f_i \in \Gamma_i$ a group element for $i \in [k]$. Then, if at least one such basis exists, for any basis A of M, is there a basis B of M with $\psi_i(B) \neq f_i$ for $i \in [k]$ and $|A \setminus B| \leq 2^k - 1$?*

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References

[1] S. Artmann, R. Weismantel, and R. Zenklusen. A strongly polynomial algorithm for bimodular integer linear programming. In *Proceedings of the 49th Annual ACM Symposium on Theory of Computing (STOC '17)*, New York, 2017. ACM.

[2] F. Barahona and W. R. Pulleyblank. Exact arborescences, matchings and cycles. *Discrete Applied Mathematics*, 16(2):91–99, 1987.

[3] M. Baumgart. *Ranking and ordering problems of spanning trees.* PhD thesis, Technische Universität München, Munich, 2009.

[4] K. Bérczi, B. Mátravölgyi, and T. Schwarcz. Reconfiguration of basis pairs in regular matroids. *arXiv preprint* `arXiv:2311.07130`, 2023.

[5] J. E. Bonin. Basis-exchange properties of sparse paving matroids. *Advances in Applied Mathematics*, 50(1):6–15, 2013.

[6] P. M. Camerini, G. Galbiati, and F. Maffioli. Random pseudo-polynomial algorithms for exact matroid problems. *Journal of Algorithms*, 13(2):258–273, 1992.

[7] V. Chekan, C. Geniet, M. Hatzel, M. Pilipczuk, M. Sokołowski, M. T. Seweryn, and M. Witkowski. Half-integral Erdős–Pósa property for non-null S-T paths. *arXiv preprint* `arXiv:2408.16344`, 2024.

[8] M. Chudnovsky, W. H. Cunningham, and J. Geelen. An algorithm for packing non-zero A-paths in group-labelled graphs. *Combinatorica*, 28:145–161, 2008.

[9] M. Chudnovsky, J. Geelen, B. Gerards, L. Goddyn, M. Lohman, and P. Seymour. Packing non-zero A-paths in group-labelled graphs. *Combinatorica*, 26(5):521–532, 2006.

[10] R. Cordovil and M. L. Moreira. Bases-cobases graphs and polytopes of matroids. *Combinatorica*, 13(2):157–165, 1993.

[11] M. DeVos, L. Goddyn, and B. Mohar. A generalization of Kneser's Addition Theorem. *Advances in Mathematics*, 220(5):1531–1548, 2009.

[12] I. Doron-Arad, A. Kulik, and H. Shachnai. Lower bounds for matroid optimization problems with a linear constraint. In K. Bringmann, M. Grohe, G. Puppis, and O. Svensson, editors, *Proceedings of the 51th International Colloquium on Automata, Languages, and Programming (ICALP '24)*, volume 297 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 56:1–56:20, Dagstuhl, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[13] F. Eisenbrand, L. Rohwedder, and K. Węgrzycki. Sensitivity, proximity and FPT algorithms for exact matroid problems. In *Proceedings of the 65th Annual Symposium on Foundations of Computer Science (FOCS '24)*, Berkeley, 2024. IEEE. To appear.

[14] N. El Maalouly, R. Steiner, and L. Wulf. Exact matching: Correct parity and FPT parameterized by independence number. In S. Iwata and N. Kakimura, editors, *34th International Symposium on Algorithms and Computation (ISAAC '23)*, volume 283 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 28:1–28:18, Dagstuhl, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[15] A. Frank. *Connections in Combinatorial Optimization*, volume 38 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2011.

[16] H. Gabow. Decomposing symmetric exchanges in matroid bases. *Mathematical Programming*, 10(1):271–276, 1976.

[17] M. X. Goemans and V. Ramakrishnan. Minimizing submodular functions over families of sets. *Combinatorica*, 15(4):499–513, 1995.

[18] J. P. Gollin, K. Hendrey, K.-i. Kawarabayashi, O.-j. Kwon, and S.-i. Oum. A unified half-integral Erdős–Pósa theorem for cycles in graphs labelled by multiple abelian groups. *Journal of the London Mathematical Society*, 109(1):e12858, 2024.

[19] J. P. Gollin, K. Hendrey, O.-j. Kwon, S.-i. Oum, and Y. Yoo. A unified Erdős–Pósa theorem for cycles in graphs labelled by multiple abelian groups. *arXiv preprint* *arXiv:2209.09488*, 2022.

[20] F. Hörsch, A. Imolay, R. Mizutani, T. Oki, and T. Schwarcz. Problems on group-labeled matroid bases. In K. Bringmann, M. Grohe, G. Puppis, and O. Svensson, editors, *Proceedings of the 51st International Colloquium on Automata, Languages, and Programming (ICALP '24)*, volume 297 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 86:1–86:20, Dagstuhl, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[21] F. Hörsch, A. Imolay, R. Mizutani, T. Oki, and T. Schwarcz. Problems on group-labeled matroid bases. *arXiv preprint* *arXiv:2402.16259*, 2024.

[22] T. Huynh, F. Joos, and P. Wollan. A unified Erdős–Pósa theorem for constrained cycles. *Combinatorica*, 39(1):91–133, 2019.

[23] Y. Iwata and Y. Yamaguchi. Finding a shortest non-zero path in group-labeled graphs. *Combinatorica*, 42(S2):1253–1282, 2022.

[24] P. Jensen and B. Korte. Complexity of matroid property algorithms. *SIAM Journal on Computing*, 11(1):184–190, 1982.

[25] X. Jia, O. Svensson, and W. Yuan. The exact bipartite matching polytope has exponential extension complexity. In *Proceedings of the 34th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '23)*, pages 1635–1654. SIAM, 2023.

[26] Y. Kajitani, S. Ueno, and H. Miyano. Ordering of the elements of a matroid such that its consecutive w elements are independent. *Discrete Mathematics*, 72(1–3):187–194, 1988.

[27] Y. Kawase, Y. Kobayashi, and Y. Yamaguchi. Finding a path with two labels forbidden in group-labeled graphs. *Journal of Combinatorial Theory, Series B*, 143:65–122, 2020.

[28] M. Kirchweger, M. Scheucher, and S. Szeider. A SAT attack on Rota's Basis Conjecture. In *25th International Conference on Theory and Applications of Satisfiability Testing (SAT 2022)*, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[29] Y. Kobayashi and S. Toyooka. Finding a shortest non-zero path in group-labeled graphs via permanent computation. *Algorithmica*, 77(4):1128–1142, 2017.

[30] D. Kotlar and R. Ziv. On serial symmetric exchanges of matroid bases. *Journal of Graph Theory*, 73(3):296–304, 2013.

[31] S. Liu and C. Xu. On the congruency-constrained matroid base. *arXiv preprint* `arXiv:2311.11737v1`, 2023.

[32] S. Liu and C. Xu. On the congruency-constrained matroid base. In J. Vygen and J. Byrka, editors, *Proceedings of the 25th Integer Programming and Combinatorial Optimization (IPCO '24)*, volume 14679 of *Lecture Notes in Computer Science (LNCS)*, pages 280–293, Cham, 2024. Springer.

[33] L. Lovász. The matroid matching problem. In L. Lovász and V. T. Sós, editors, *Algebraic methods in graph theory*, volume 25 of *Colloquia Mathematica Societatis János Bolyai*, pages 495–517. North-Holland, Amsterdam, 1981.

[34] D. Mayhew, M. Newman, D. Welsh, and G. Whittle. On the asymptotic proportion of connected matroids. *European Journal of Combinatorics*, 32(6):882–890, 2011.

[35] D. Mayhew and G. F. Royle. Matroids with nine elements. *Journal of Combinatorial Theory, Series B*, 98(2):415–431, 2008.

[36] K. Mulmuley, U. V. Vazirani, and V. V. Vazirani. Matching is as easy as matrix inversion. *Combinatorica*, 7(1):105–113, 1987.

[37] M. Nägele, B. Sudakov, and R. Zenklusen. Submodular minimization under congruency constraints. *Combinatorica*, 39(6):1351–1386, 2019.

[38] M. Nägele and R. Zenklusen. A new contraction technique with applications to congruency-constrained cuts. *Mathematical Programming*, 183(1):455–481, 2020.

[39] J. Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[40] C. H. Papadimitriou and M. Yannakakis. The complexity of restricted spanning tree problems. *Journal of the ACM*, 29(2):285–309, 1982.

[41] R. Pendavingh and J. van der Pol. On the number of bases of almost all matroids. *Combinatorica*, 38(4):955–985, 2018.

[42] B. Reed. Mangoes and blueberries. *Combinatorica*, 2(19):267–296, 1999.

[43] J. Rieder. The lattices of matroid bases and exact matroid bases. *Archiv der Mathematik*, 56(6):616–623, 1991.

[44] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, Berlin, 2003.

[45] A. Schrijver and P. D. Seymour. Spanning trees of different weights. In W. J. Cook and P. D. Seymour, editors, *Polyhedral Combinatorics*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, pages 281–288. DIMACS/AMS, 1990.

[46] P. D. Seymour. Decomposition of regular matroids. *Journal of Combinatorial Theory, Series B*, 28(3):305–359, 1980.

[47] R. Thomas and Y. Yoo. Packing cycles in undirected group-labelled graphs. *Journal of Combinatorial Theory, Series B*, 161:228–267, 2023.

[48] D. Wiedemann. Cyclic base orders of matroids. Manuscript, 1984.

[49] P. Wollan. Packing non-zero A-paths in an undirected model of group labeled graphs. *Journal of Combinatorial Theory, Series B*, 100(2):141–150, 2010.

[50] P. Wollan. Packing cycles with modularity constraints. *Combinatorica*, 31(1):95–126, 2011.

Appendix

A CNF formulation of finding a non-SIBO matroid

In this section, we describe how we can reduce the problem of finding a $2r$-elements, rank-r, non-SIBO matroid to SAT by describing a CNF (conjunctive normal form) formulation.

Let $E = [2r]$ be the ground set. We prepare $\binom{2r}{r}$ Boolean variables x_B indexed by $B \in \binom{E}{r}$. We build a CNF such that $\left\{ B \in \binom{E}{r} \;\middle|\; x_B \text{ is true} \right\}$ forms the basis family of a matroid, $[r]$ and $E \setminus [r]$ are bases, and $([r], E \setminus [r])$ has no SI-ordering by collecting the following clauses.

Basis exchange property: for every $A, B \in \binom{E}{r}$ and $e \in A \setminus B$,

$$\neg x_A \vee \neg x_B \vee \bigvee_{f \in B \setminus A} x_{A-e+f}.$$

Fixed basis:

$$x_{[r]}.$$

Fixed basis:

$$x_{E \setminus [r]}.$$

No SI-ordering: for every permutation $a_1, \ldots, a_r$ of $[r]$ and $b_1, \ldots, b_r$ of $E \setminus [r]$,

$$\bigvee_{0 \le i < j \le r} \neg x_{\{a_1, \ldots, a_i, b_{i+1}, \ldots, b_j, a_{j+1}, \ldots, a_r\}}.$$

Note that if a non-disjoint basis pair (A, B) of a matroid M has no SI-ordering, then $(A \setminus B, B \setminus A)$ has no SI-ordering as well in $M/(A \cap B)$. Thus, we can restrict our attention to the disjoint basis pair $([r], E \setminus [r])$ by verifying the unsatisfiability of the CNF from small r.

Our Python script to solve the above SAT instance is available at https://github.com/taiheioki/sibo.