# ON THE INVERSE PROBLEM OF THE $k$-TH DAVENPORT CONSTANTS FOR GROUPS OF RANK 2

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Abstract. For a finite abelian group $G$ and a positive integer $k$, let $\mathsf{D}_k(G)$ denote the smallest integer $\ell$ such that each sequence over $G$ of length at least $\ell$ has $k$ disjoint nontrivial zero-sum subsequences. It is known that $\mathsf{D}_k(G) = n_1 + kn_2 - 1$ if $G \cong C_{n_1} \oplus C_{n_2}$ is a rank 2 group, where $1 < n_1 \mid n_2$. We investigate the associated inverse problem for rank 2 groups, that is, characterizing the structure of zero-sum sequences of length $\mathsf{D}_k(G)$ that can not be partitioned into $k+1$ nontrivial zero-sum subsequences.

## 1. Introduction

Let $(G, +, 0)$ be a finite abelian group. By a sequence $S$ over $G$, we mean a finite sequence of terms from $G$ which is unordered, repetition of terms allowed. We say that $S$ is a zero-sum sequence if the sum of its terms equals zero and denote by $|S|$ the length of the sequence.

Let $k$ be a positive integer. We denote by $\mathsf{D}_k(G)$ the smallest integer $\ell$ such that every sequence over $G$ of length at least $\ell$ has $k$ disjoint nontrivial zero-sum subsequences. We call $\mathsf{D}_k(G)$ the $k$-th Davenport constant of $G$, while the Davenport constant $\mathsf{D}(G) = \mathsf{D}_1(G)$ is one of the most important zero-sum invariants in Combinatorial Number Theory and, together with Erdős-Ginzburg-Ziv constant, $\eta$-constant, etc., has been studied a lot (see [39, 40, 1, 29, 49, 50, 30, 16, 21, 41, 5, 43, 6, 14, 7, 22, 38]). This variant $\mathsf{D}_k(G)$ of the Davenport constant was introduced and investigated by F. Halter-Koch [37], in the context of investigations on the asymptotic behavior of certain counting functions of algebraic integers defined via factorization properties ( see the monograph [27, Section 6.1], and the survey article [19, Section 5]). In 2014, K. Cziszter and M. Domokos ([9, 8]) introduced the generalized Noether Number $\beta_k(G)$ for general groups, which equals $\mathsf{D}_k(G)$ when $G$ is abelian (see [11, 12, 10] for more about this direction). Knowledge of those constants is highly relevant when applying the inductive method to determine or estimate the Davenport constant of certain finite abelian groups (see [13, 4, 3, 42]).

In 2010, M. Freeze and W. Schmid ([17]) showed that for each finite abelian group $G$ we have $\mathsf{D}_k(G) = \mathsf{D}_0(G) + k\exp(G)$ for some $\mathsf{D}_0(G) \in \mathbb{N}_0$ and all sufficiently large $k$. In fact, it is known that for groups of rank at most two, and for some other types of groups, an equality of the form

2010 *Mathematics Subject Classification.* 11B30, 11P70.

*Key words and phrases.* zero-sum sequences, Davenport constant, $k$-th Davenport constant.

This work was supported by the Austrian Science Fund FWF [grant DOI:10.55776/P36852].

$\mathsf{D}_k(G) = \mathsf{D}_0(G) + k\exp(G)$ for some $\mathsf{D}_0(G) \in \mathbb{N}_0$ holds for all $k$. In particular, for a rank two abelian group $G = C_m \oplus C_n$, where $m \mid n$, we have $\mathsf{D}_k(G) = m + kn - 1$ ([27, Theorem 6.1.5]). Yet, it fails for elementary 2 and 3-groups of rank at least 3 (see [13, 4]). In general, computing (even bounding) $\mathsf{D}_k(G)$ is quite more complicated than for $\mathsf{D}(G)$, in particular for (elementary) $p$-groups, while $\mathsf{D}(G)$ is know for $p$-groups.

In zero-sum theory, the associated inverse problems of zero-sum invariants study the structure of extremal sequences that do not have enough zero-sum subsequences with the prescribed properties. The inverse problems of the Davenport constant, the $\eta$-constant, and the Erdős-Ginzburg-Ziv constant are central topics (see [51, 52, 45, 46, 23, 24, 15, 34, 35, 47, 48, 31, 36]). The associated inverse problem of $\mathsf{D}_k(G)$ is to characterize the maximal length zero-sum sequences that can not be partitioned into $k+1$ nontrivial zero-sum subsequences. In particular, the inverse problem of $\mathsf{D}(G)$ is to characterize the structure of minimal zero-sum subsequences of length $\mathsf{D}(G)$, which was accomplished for groups of rank 2 in a series of papers [44] [18] [20] [47] [2], where a minimal zero-sum sequence is a zero-sum sequence that can not be partitioned into two nontrivial zero-sum subsequences. Those inverse results can be used to construct minimal generating subsets in Invariant Theory (see [11, Proposition 4.7]).

Let $\mathcal{B}(G)$ be the set of all zero-sum sequences over $G$. We define

$$\mathcal{M}_k(G) = \{S \in \mathcal{B}(G) \colon S \text{ can not be partitioned into } k+1 \text{ nontrivial zero-sum subsequences}\}\,.$$

Then it is easy to see that $\mathsf{D}_k(G) = \max\{|S| \colon S \in \mathcal{M}_k(G)\}$. In this paper, we investigate the inverse problem of general Davenport constant $\mathsf{D}_k(G)$ for all rank 2 groups, that is, to study the structure of sequences of $\mathcal{M}_k(G)$ of length $\mathsf{D}_k(G)$. In 2003, Gao and Geroldinger ([18, Theorem 7.1]) studied the inverse problem of $\mathsf{D}_k(G)$ for $G = C_n \oplus C_n$ under some assumptions of $G$, which had been confirmed later. We reformulate this result in the following and a proof will be given in Section 3.

**Theorem 1.1.** *Let $G = C_n \oplus C_n$ with $n \ge 2$, let $k \ge 1$, and let $U \in \mathcal{B}(G)$ with $|U| = \mathsf{D}_k(G)$. Then $U \in \mathcal{M}_k(G)$ if and only if there exists a basis $(e_1, e_2)$ of $G$ such that it has one of the following two forms.*

(I)
$$U = e_1^{k_1 n - 1} \prod_{i \in [1, k_2 n]} (x_i e_1 + e_2), \quad \textit{where}$$

(a) $k_1, k_2 \in \mathbb{N}$ *with* $k_1 + k_2 = k + 1$,

(b) $x_1, \ldots, x_{k_2 n} \in [0, n-1]$ *and* $x_1 + \ldots + x_{k_2 n} \equiv 1 \mod n$.

(II)
$$U = e_1^{an} e_2^{bn-1} (x e_1 + e_2)^{cn-1} (x e_1 + 2e_2)\,, \quad \textit{where}$$

(a) $x \in [2, n-2]$ *with* $\gcd(x, n) = 1$,

(b) $a, b, c \ge 1$ *with* $a + b + c = k + 1$.

*Note that in this case, we have $k \ge 2$.*

For general groups, we have the following main theorem.

**Theorem 1.2.** *Let $G = C_{n_1} \oplus C_{n_2}$ with $1 < n_1 \mid n_2$ and $n_1 < n_2$, let $k \geq 1$, and let $U \in \mathcal{B}(G)$ with $|U| = \mathsf{D}_k(G)$. Then $U \in \mathcal{M}_k(G)$ if and only if it has one of the following four forms.*

(I)
$$U = e_1^{\operatorname{ord}(e_1)-1} \prod_{i \in [1, k \operatorname{ord}(e_2)]} (x_i e_1 + e_2), \quad \textit{where}$$

(a) $(e_1, e_2)$ *is a basis for* $G$ *with* $\operatorname{ord}(e_1) = n_1$ *and* $\operatorname{ord}(e_2) = n_2$,
(b) $x_1, \ldots, x_{k \operatorname{ord}(e_2)} \in [0, \operatorname{ord}(e_1) - 1]$ *and* $x_1 + \ldots + x_{k \operatorname{ord}(e_2)} \equiv 1 \mod \operatorname{ord}(e_1)$.

(II)
$$U = e_1^{k \operatorname{ord}(e_1)-1} \prod_{i \in [1, \operatorname{ord}(e_2)]} (x_i e_1 + e_2), \quad \textit{where}$$

(a) $(e_1, e_2)$ *is a basis for* $G$ *with* $\operatorname{ord}(e_1) = n_2$ *and* $\operatorname{ord}(e_2) = n_1$,
(b) $x_1, \ldots, x_{\operatorname{ord}(e_2)} \in [0, \operatorname{ord}(e_1) - 1]$ *and* $x_1 + \ldots + x_{\operatorname{ord}(e_2)} \equiv 1 \mod \operatorname{ord}(e_1)$.

(III)
$$U = g_1^{n_1 - 1} \prod_{i \in [1, kn_2]} (-x_i g_1 + g_2), \quad \textit{where}$$

(a) $(g_1, g_2)$ *is a generating set of* $G$ *with* $\operatorname{ord}(g_1) > n_1$ *and* $\operatorname{ord}(g_2) = n_2$,
(b) $x_1, \ldots, x_{kn_2} \in [0, n_1 - 1]$ *with* $x_1 + \ldots + x_{kn_2} = n_1 - 1$.

(IV)
$$U = e_1^{sn_1 - 1} \prod_{i \in [1, kn_2 - (s-1)n_1]} ((1 - x_i) e_1 + e_2), \quad \textit{where}$$

(a) $(e_1, e_2)$ *is a basis of* $G$ *with* $\operatorname{ord}(e_1) = n_2$ *and* $\operatorname{ord}(e_2) = n_1$,
(b) $s \in [2, kn_2/n_1 - 1]$,
(c) $x_1, \ldots, x_{kn_2 - (s-1)n_1} \in [0, n_1 - 1]$ *with* $x_1 + \ldots + x_{kn_2 - (s-1)n_1} = n_1 - 1$.

## 2. Notation and Preliminaries

Our notations and terminology are consistent with [25] and [32]. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set the discrete interval $[a, b] = \{x \in \mathbb{Z} \colon a \leq x \leq b\}$. Throughout this paper, all abelian groups will be written additively, and for $n \in \mathbb{N}$, we denote by $C_n$ a cyclic group with $n$ elements.

Let $G$ be a finite abelian group. It is well-known that $|G| = 1$ or $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$, where $r = \mathsf{r}(G) \in \mathbb{N}$ is the *rank* of $G$, and $n_r = \exp(G)$ is the *exponent* of $G$. We denote by $|G|$ the *order* of $G$, and $\operatorname{ord}(g)$ the *order* of $g \in G$.

Let $\mathcal{F}(G)$ be the free abelian (multiplicatively written) monoid with basis $G$. Then sequences over $G$ could be viewed as elements of $\mathcal{F}(G)$. A sequence $S \in \mathcal{F}(G)$ could be written as

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \in \mathbb{N}_0$ is the multiplicity of $g$ in $S$. We call

- $\mathrm{supp}(S) = \{g \in G \colon \mathsf{v}_g(S) > 0\} \subset G$ the *support* of $S$, and
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of $S$.

Let $t \in \mathbb{N}$. We denote

$$\Sigma_{\le t}(S) = \left\{ \sum_{i \in I} g_i \colon I \subseteq [1, l] \text{ with } 1 \le |I| \le t \right\} .$$

A sequence $T \in \mathcal{F}(G)$ is called a subsequence of $S$ if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G$, and denoted by $T \mid S$. If $T \mid S$, then we denote

$$T^{-1}S = \prod_{g \in G} g^{\mathsf{v}_g(S) - \mathsf{v}_g(T)} \in \mathcal{F}(G) .$$

Let $T_1, T_2 \in \mathcal{F}(G)$. We set

$$T_1 T_2 = \prod_{g \in G} g^{\mathsf{v}_g(T_1) + \mathsf{v}_g(T_2)} \in \mathcal{F}(G) .$$

If $T_1, \ldots, T_t \in \mathcal{F}(G)$ such that $T_1 \cdot \ldots \cdot T_t \,|\, S$, where $t \ge 2$, then we say $T_1, \ldots, T_t$ are disjoint subsequences of $S$.

Every map of abelian groups $\phi : G \to H$ extends to a map from the sequences over $G$ to the sequences over $H$ by setting $\phi(S) = \phi(g_1) \cdot \ldots \cdot \phi(g_l)$. If $\phi$ is a homomorphism, then $\phi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \mathsf{ker}(\phi)$.

We denote by $\mathsf{E}(G)$ the Gao's constant which is the smallest integer $\ell$ such that every sequence over $G$ of length $\ell$ has a zero-sum subsequence of length $|G|$ and by $\eta(G)$ the smallest integer $\ell$ such that every sequence over $G$ of length $\ell$ has a zero-sum subsequence $T$ of length $1 \le |T| \le \exp(G)$. Let $\mathsf{d}(G)$ be the maximal length of a sequence over $G$ that has no zero-sum subsequence. Then it is easy to see that $\mathsf{d}(G) = \mathsf{D}(G) - 1$. The following result is well-known and we may use it without further mention.

**Lemma 2.1.** *Let $G$ be a finite abelian group. Then $\mathsf{E}(G) = |G| + \mathsf{d}(G) \le 2|G| - 1$.*

*Proof.* TOPROVE 0 □

We also need the following lemmas.

**Lemma 2.2.** *Let $G$ be a finite abelian group. If $\mathsf{D}(G) = |G|$, then $G$ is cyclic and for every minimal zero-sum sequence $S$ over $G$ of length $|G|$, there exists $g \in G$ with $\mathrm{ord}(g) = |G|$ such that $S = g^{|G|}$.*

*Proof.* TOPROVE 1 □

**Lemma 2.3.** *Let $G$ be a finite abelian group and let $H \subset G$ be a proper subgroup. Then $\mathsf{D}_k(H) < \mathsf{D}_k(G)$ for all $k \in \mathbb{N}$.*

*Proof.* TOPROVE 2 □

**Theorem 2.4.** *Let $G = C_{n_1} \oplus C_{n_2}$ with $n_1 \mid n_2$, where $n_1, n_2 \in \mathbb{N}$, and let $k \in \mathbb{N}$. Then $\eta(G) = 2n_1 + n_2 - 2$ and $\mathsf{D}_k(G) = n_1 + kn_2 - 1$. In particular, $\mathsf{D}(G) = n_1 + n_2 - 1$.*

*Proof.* TOPROVE 3 □

**Theorem 2.5.** *Let $G = C_n \oplus C_{mn}$ with $n \geq 2$ and $m \geq 1$. A sequence $S$ over $G$ of length $\mathsf{D}(G) = n + mn - 1$ is a minimal zero-sum sequence if and only if it has one of the following two forms:*

(I)

$$S = e_1^{\operatorname{ord}(e_1)-1} \prod_{i=1}^{\operatorname{ord}(e_2)} (x_i e_1 + e_2),$$

*where*

(a) $\{e_1, e_2\}$ *is a basis of* $G$,

(b) $x_1, \ldots, x_{\operatorname{ord}(e_2)} \in [0, \operatorname{ord}(e_1) - 1]$ *and* $x_1 + \ldots + x_{\operatorname{ord}(e_2)} \equiv 1 \mod \operatorname{ord}(e_1)$.

*In this case, we say that $S$ is of type I(a) or I(b) according to whether $\operatorname{ord}(e_2) = n$ or $\operatorname{ord}(e_2) = mn > n$.*

(II)

$$S = f_1^{sn-1} f_2^{(m-s)n+\epsilon} \prod_{i=1}^{n-\epsilon} (-x_i f_1 + f_2),$$

*where*

(a) $\{f_1, f_2\}$ *is a generating set for $G$ with $\operatorname{ord}(f_2) = mn$ and $\operatorname{ord}(f_1) > n$,*

(b) $\epsilon \in [1, n-1]$ *and* $s \in [1, m-1]$,

(c) $x_1, \ldots, x_{n-\epsilon} \in [1, n-1]$ *with* $x_1 + \ldots + x_{n-\epsilon} = n - 1$,

(d) *either $s = 1$ or $nf_1 = nf_2$, with both holding when $m = 2$, and*

(e) *either $\epsilon \geq 2$ or $nf_1 \neq nf_2$.*

*In this case, we say that $S$ is of type II.*

*Proof.* TOPROVE 4 □

**Lemma 2.6.** *Let $G$ be a finite abelian group, let $H$ be a cyclic subgroup of $G$, and let $\varphi\colon G \to G/H$ be the canonical epimorphism. If $S \in \mathcal{M}_k(G)$, then $\varphi(S) \in \mathcal{M}_{k|H|}(G/H)$.*

*Proof.* TOPROVE 5 □

## 3. Proof of main theorems

**Proposition 3.1.** *Let $G$ be a finite abelian group of rank at most 2, let $k \in \mathbb{N}$, and let $S$ be a zero-sum sequence over $G$ of length $\mathsf{D}_k(G)$. Then $S \in \mathcal{M}_k(G)$ if and only if $0 \notin \Sigma_{\leq \exp(G)-1}(S)$.*

*Proof.* TOPROVE 6 □

We first investigate the associated inverse problem for cyclic groups.

**Theorem 3.2.** *Let $G$ be cyclic, let $k \in \mathbb{N}$, and let $S$ be a zero-sum sequence over $G$ of length $\mathsf{D}_k(G)$. Then $S \in \mathcal{M}_k(G)$ if and only if there exists $g \in G$ with $\operatorname{ord}(g) = |G|$ such that $S = g^{k|G|}$.*

*Proof.* TOPROVE 7 □

Next, we prove Theorem 1.1 which could be handled easily by Proposition 3.1 and [18, Theorem 7.1].

*Proof.* TOPROVE 8 □

**Lemma 3.3.** *Let $G = C_n \oplus C_n$ with $n \ge 2$ and let $k \ge 2$. If $S \in \mathcal{F}(G)$ is a zero-sum sequence with $|S| = (k+1)n - 1$ and $0 \notin \Sigma_{\le n-1}(S)$, then there is a basis $(e_1, e_2)$ for $G$ such that either*

1. $\operatorname{supp}(S) \subseteq \{e_1\} \cup \big(\langle e_1\rangle + e_2\big)$ *and* $\mathsf{v}_{e_1}(S) \equiv -1 \mod n$, *or*
2. $S = e_1^{an} e_2^{bn-1} (xe_1 + e_2)^{cn-1} (xe_1 + 2e_2)$ *for some* $x \in [2, n-2]$ *with* $\gcd(x, n) = 1$, *and some* $a, b, c \ge 1$ *with* $k + 1 = a + b + c$.

*Proof.* TOPROVE 9 □

The following lemma shows two special cases of Theorem 1.2.

**Lemma 3.4.** *Let $G = C_{n_1} \oplus C_{n_2}$ with $1 < n_1 \mid n_2$ and $n_1 < n_2$, let $k \ge 2$, and let $U \in \mathcal{M}_k(G)$ with $|U| = \mathsf{D}_k(G)$.*

1. *If there is some $e_1 \in \operatorname{supp}(U)$ such that $\operatorname{ord}(e_1) = n_1$ and $\mathsf{v}_{e_1}(U) \ge n_1 - 1$, then there exists $e_2 \in G$ with $\operatorname{ord}(e_2) = n_2$ such that $(e_1, e_2)$ is a basis of $G$ and*
$$U = e_1^{n_1 - 1} \prod_{i \in [1, kn_2]} (x_i e_1 + e_2)\,,$$
*where $x_1, \ldots, x_{kn_2} \in [0, n_1 - 1]$ and $x_1 + \ldots + x_{kn_2} \equiv 1 \mod n_1$.*
2. *If there is some $e_2 \in \operatorname{supp}(U)$ such that $\operatorname{ord}(e_2) = n_2$ and $\mathsf{v}_{e_2}(U) \ge kn_2 - 1$, then there exists $e_1 \in G$ with $\operatorname{ord}(e_1) = n_1$ such that $(e_1, e_2)$ is a basis of $G$ and*
$$U = e_2^{kn_2 - 1} \prod_{i \in [1, n_1]} (e_1 + x_i e_2)\,,$$
*where $x_1, \ldots, x_{n_1} \in [0, n_2 - 1]$ and $x_1 + \ldots + x_{n_1} \equiv 1 \mod n_2$.*

*Proof.* TOPROVE 10 □

Now we are ready to prove our main Theorem 1.2.

*Proof.* TOPROVE 11 □

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