A hypergraph bipartite Turán problem with odd uniformity

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Abstract

In this paper, we investigate the hypergraph Turán number $\mathrm{ex}(n, K_{s,t}^{(r)})$. Here, $K_{s,t}^{(r)}$ denotes the r-uniform hypergraph with vertex set $\left(\cup_{i\in[t]}X_i\right)\cup Y$ and edge set $\{X_i\cup\{y\} : i \in [t], y \in Y\}$, where $X_1, X_2, \cdots, X_t$ are t pairwise disjoint sets of size $r-1$ and Y is a set of size s disjoint from each X_i. This study was initially explored by Erdős and has since received substantial attention in research. Recent advancements by Bradač, Gishboliner, Janzer and Sudakov have greatly contributed to a better understanding of this problem. They proved that $\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{s,t}(n^{r-\frac{1}{s-1}})$ holds for any $r \geq 3$ and $s, t \geq 2$. They also provided constructions illustrating the tightness of this bound if $r \geq 4$ is *even* and $t \gg s \geq 2$. Furthermore, they proved that $\mathrm{ex}(n, K_{s,t}^{(3)}) = O_{s,t}(n^{3-\frac{1}{s-1}-\varepsilon_s})$ holds for $s \geq 3$ and some $\epsilon_s > 0$. Addressing this intriguing discrepancy between the behavior of this number for $r = 3$ and the even cases, Bradač et al. post a question of whether

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{r,s,t}(n^{r-\frac{1}{s-1}-\varepsilon}) \text{ holds for } odd\ r \geq 5 \text{ and any } s \geq 3.$$

In this paper, we provide an affirmative answer to this question, utilizing novel techniques to identify regular and dense substructures. This result highlights a rare instance in hypergraph Turán problems where the solution depends on the parity of the uniformity.

1 Introduction

For a given r-uniform hypergraph H, we say an r-uniform hypergraph is *H-free* if it does not contain a copy of H as its subgraph. The *Turán number* $\mathrm{ex}(n, H)$ denotes the maximum number of edges in an H-free r-uniform hypergraph on n vertices. The study of Turán number is a central problem in extremal combinatorics. We refer interested readers to the survey by Füredi and Simonovits [8] for ordinary graphs and the survey by Keevash [9] for non-r-partite r-uniform hypergraphs. Here, our focus lies on the Turán numbers of r-partite r-uniform hypergraphs H for $r \geq 3$. A fundamental result proved by Erdős states that for every such H, $\mathrm{ex}(n, H) = O(n^{r-\varepsilon_H})$ holds for some $\varepsilon_H > 0$. The primary objective of this aspect is to determine the optimal constant ε_H. However, this problem is notoriously difficult and to date, there are very few cases that have been fully understood.

In this paper, we consider the Turán number of the following r-partite r-uniform hypergraphs, which were initially defined by Mubayi and Verstraëte [12]: for positive integers r, s, t, let $K_{s,t}^{(r)}$ denote the r-uniform hypergraph with vertex set $\left(\cup_{i\in[t]}X_i\right) \cup Y$ and edge set $\{X_i \cup \{y\} : i \in [t], y \in Y\}$,

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where $X_1, X_2, \cdots, X_t$ are t pairwise disjoint sets of size $r-1$ and Y is a set of size s disjoint from each X_i. This study can be traced back to an old problem posted by Erdős [5], who asked to determine the maximum number $f_r(n)$ of edges in an r-uniform hypergraph on n vertices that does not contain four distinct edges A, B, C, D satisfying $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$. This generalizes the Turán number of the four-cycle, and it is evident to see that $f_3(n) = \mathrm{ex}(n, K_{2,2}^{(3)})$ and $f_r(n) \leq \mathrm{ex}(n, K_{2,2}^{(r)})$ for any $r \geq 4$. Füredi [7] resolved a conjecture of Erdős made in [5], by showing that $f_r(n) \leq 3.5\binom{n}{r-1}$ for any $r \geq 3$. This was first improved by Mubayi and Verstraëte [12], and later on, further improvements were made by Pikhurko and Verstraëte [13].

Returning to the Turán number $\mathrm{ex}(n, K_{s,t}^{(r)})$, Mubayi and Verstraëte [12] primarily focus on the case $r = 3$. They proved that $\mathrm{ex}(n, K_{s,t}^{(3)}) = O_{s,t}(n^{3-1/s})$ for $t \geq s \geq 3$ and $\mathrm{ex}(n, K_{s,t}^{(3)}) = \Omega_t(n^{3-2/s})$ for $t > (s-1)!$. For the particular case $s = 2$, Mubayi and Verstraëte [12] also provided that $\mathrm{ex}(n, K_{2,t}^{(3)}) \leq t^4\binom{n}{2}$ for $t \geq 3$, and they further posed the question of determining the order of the magnitude of the leading coefficient in terms of t. Among other results, Ergemlidze, Jiang and Methuku [6] obtained an improvement by showing $\mathrm{ex}(n, K_{2,t}^{(3)}) \leq (15t\log t + 40t)\binom{n}{2}$, which can be extended to all $r \geq 3$. Using the random algebraic method (see [3, 4]), Xu, Zhang and Ge [14, 15] proved that $\mathrm{ex}(n, K_{s,t}^{(r)}) = \Theta(n^{r-1/t})$, assuming that s is sufficiently large than r, t.

Very recently, Bradač, Gishboliner, Janzer and Sudakov [2] made significant contributions towards a better understanding of the behavior of the Turán number $\mathrm{ex}(n, K_{s,t}^{(r)})$. Using a novel variant of the dependent random choice, they proved a general upper bound that $\mathrm{ex}(n, K_{s,t}^{(r)}) = O_s\left(t^{\frac{1}{s-1}} n^{r-\frac{1}{s-1}}\right)$ holds for any $r \geq 3$ and $s, t \geq 2$. Moreover, they built upon norm graphs ([1, 10]) and provided matching constructions, which led to

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = \Theta_{r,s}\left(t^{\frac{1}{s-1}} n^{r-\frac{1}{s-1}}\right) \text{ for any } \textit{even } r \geq 4,\ s \geq 2, \text{ and } t > (s-1)!.$$

Furthermore, they derived a different order of magnitude for n in the case $r = 3$ by proving that

$$\mathrm{ex}(n, K_{s,t}^{(3)}) = O_{s,t}\left(n^{3-\frac{1}{s-1}-\varepsilon_s}\right) \text{ holds for any } s \geq 3,\ t, \text{ and some positive constant } \varepsilon_s = O(s^{-5}).$$

Bradač et al. posed the question of whether the above upper bound can be extended to all odd uniformities. They noted that if the question is affirmative, then "this would be a rare example of an extremal problem where the answer depends on the parity of the uniformity", quoted from [2].

In this work, we provide a positive answer to the aforementioned question posed by Bradač et al. Our main result can be stated as follows.

Theorem 1.1. *For any odd $r \geq 3$ and any $s \geq 3$, there exists some $\varepsilon = \varepsilon(s) > 0$ depending only on s such that for any positive integer t,*

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{r,s,t}\left(n^{r-\frac{1}{s-1}-\varepsilon}\right).$$

We use a different proof approach from [2] (see Section 2.2 for an outline of the proof). The core ideas are to find some regular and dense substructures in hypergraphs. Our proof works for $\varepsilon(s) = \frac{1}{6(s+2)^2}$, although we have made no serious attempt to optimize the leading coefficient. This improves the choice of $\varepsilon_s = O(s^{-5})$ in [2] for the case $r = 3$.

The remainder of the paper is organized as follows. In Section 2, we introduce the necessary notation and provide an outline of the proof for Theorem 1.1. In Section 3, we break down the proof of Theorem 1.1 into three lemmas. The full proofs of these lemmas are presented in Section 4. Finally, in Section 5, we offer some concluding remarks.

2 Preliminaries

In this section, we begin by introducing the necessary notation, followed by providing a preliminary outline of the proof for our main result, namely Theorem 1.1.

2.1 Notation

Let $r \geq 3$, $s \geq 3$, t and n be positive integers throughout the rest of this paper. Let $[n] = \{1, 2, \cdots, n\}$.

Assume that $\mathcal{G}$ is an r-partite r-uniform hypergraph with parts $V_1, \cdots, V_r$, each of size n, throughout this section. For a given vertex v, the *link hypergraph* of v, denote as $N_{\mathcal{G}}(v)$, comprises all $(r-1)$-sets that, when combined with v, form an edge in $\mathcal{G}$. Similarly, for $k < r$ and a set T with k vertices, we write $N_{\mathcal{G}}(T)$ for the $(r-k)$-uniform hypergraph containing all $(r-k)$-sets which together with T form an edge in $\mathcal{G}$. We denote its cardinality as $d_{\mathcal{G}}(T) = |N_{\mathcal{G}}(T)|$. Specially if $d_{\mathcal{G}}(T) \neq 0$, we say the k-set T is a *k-tuple* of $\mathcal{G}$. We define the set of all $(r-1)$-tuples of $\mathcal{G}$ as $\mathcal{T}(\mathcal{G})$. For $i \in [r]$, let $\mathcal{T}_i(\mathcal{G})$ be the set of $(r-1)$-tuples of $\mathcal{G}$ contained in $V(\mathcal{G}) \backslash V_i$.

For an s-set $S = \{v_1, v_2 \cdots, v_s\} \subseteq V(\mathcal{G})$, we define $\mathrm{CN}_{\mathcal{G}}(S)$ as the set of common edges in all link hypergraphs of vertices $v_i \in S$. That is, $\mathrm{CN}_{\mathcal{G}}(S) = \bigcap_{i \in [s]} N_{\mathcal{G}}(v_i)$. A *vertex cover* of a set $\mathcal{E}$ of edges is a set of vertices that intersects with every edge in $\mathcal{E}$. For each s-set S, we choose and fix a minimum vertex-cover of $\mathrm{CN}_{\mathcal{G}}(S)$, and call these vertices the *roots* of S. For a root v of S, we also say S is rooted on v. The following simple yet crucial property will be repeatedly used in the proofs:

$$\text{If } \mathcal{G} \text{ is } K_{s,t}^{(r)}\text{-free, then any } s\text{-set } S \subseteq V(\mathcal{G}) \text{ has less than } rt \text{ roots.} \tag{1}$$

To see this, consider a maximum set of disjoint edges in $\mathrm{CN}_{\mathcal{G}}(S)$, and let A be the vertex set of these edges. Due to the maximality, every edge in $\mathrm{CN}_{\mathcal{G}}(S)$ contains at least one vertex in A. So A is a vertex-cover of $\mathrm{CN}_{\mathcal{G}}(S)$. Since $\mathcal{G}$ is $K_{s,t}^{(r)}$-free, $\mathrm{CN}_{\mathcal{G}}(S)$ has at most $t-1$ disjoint edges. Therefore, $|A| \leq (t-1)(r-1) < tr$, as desired.

Let S be an s-set in $V(\mathcal{G})$. We denote $cd_{\mathcal{G}}(S) = |\mathrm{CN}_{\mathcal{G}}(S)|$ to be the *codegree* of S in $\mathcal{G}$. For a vertex $u \notin S$, we write $cd_{\mathcal{G}}(S|u)$ for the number of edges in $\mathrm{CN}_{\mathcal{G}}(S)$ containing u. It is clear that $cd_{\mathcal{G}}(S) \leq \sum_u cd_{\mathcal{G}}(S|u)$, where the summation is over all roots u of S.

2.2 Proof sketch

In this concise overview of the proof for Theorem 1.1, we outline crucial intermediate properties and emphasize the differences between the cases when r is odd or even.

Consider $\mathcal{G}$ as a $K_{s,t}^{(r)}$-free r-partite r-uniform hypergraph with parts $V_1, \cdots, V_r$, each of size n, and possessing at least $n^{r-1/(s-1)-\epsilon}$ edges, where $\epsilon > 0$ is a small constant. First, we demonstrate that $\mathcal{G}$ can be assumed to be "regular" in the sense that every $(r-1)$-tuple has bounded degree. This regularity property is proven in Lemma 3.2 and simplifies the subsequent analysis.

The key ideas of the proof culminate in an auxiliary digraph $D(\mathcal{G})$, where the vertex set is $\{V_1, \cdots, V_r\}$, and directed edges $V_i \to V_j$ are formed for distinct i, j if there is a significant number of s-sets in V_j rooted on vertices in V_i within a relatively dense subgraph of $\mathcal{G}$ (refer to Definition 3.3 for a precise description). The main body of the proof is then divided into the following two properties, which are established in Lemmas 3.4 and 3.5, respectively:

(I). Every vertex in $D(\mathcal{G})$ has non-zero in-degree, and

(II). There exist no three distinct vertices forming a directed path $V_i \to V_j \to V_k$ in $D(\mathcal{G})$.

The proof of Property (II) is the most involved. In essence, if $V_i \to V_j$ holds, it can be shown that there exist large subsets $Y \subseteq V_j$ and $Z \subseteq V_k$ for $k \notin \{i,j\}$ such that for any $y \in Y$, the projection of $N_{\mathcal{G}}(y)$ onto Z is nearly complete (see Lemma 4.4 in more details). If $V_j \to V_k$ also holds for some $k \notin \{i,j\}$, then the number of pairs (S, y) where $y \in Y$ is a root of an s-set $S \subseteq Z$ can be shown to be at least $(|Y||Z|^s)^{1-O(\epsilon)}$. However, due to (1), the number of such pairs (S, y) is at most $O_{r,t}(|Z|^s)$. This would lead to a contradiction and establish Property (II).

Now we can distinguish between the cases when r is odd or even. If r is even, using Properties (I) and (II), one can conclude that $D(\mathcal{G})$ must be isomorphic to the union of 2-cycles, say $V_{2i-1} \leftrightarrows V_{2i}$ for $1 \le i \le r/2$. This configuration is feasible, as justified in the construction in [2]. However, if r is odd, Property (I) would force the existence of a directed path of length two say $V_i \to V_j \to V_k$. This clearly contradicts Property (II) and thus completes the proof of Theorem 1.1.

3 Proof of Theorem 1.1

In this section, we establish the proof of Theorem 1.1 by reducing it to Lemmas 3.2, 3.4, and 3.5.

Let us proceed to present the statements of these lemmas. The first lemma demonstrates that for any $K_{s,t}^{(r)}$-free r-uniform hypergraph $\mathcal{G}$, one can find a subgraph of $\mathcal{G}$ with nearly the same edge density and possessing the following useful property of being "almost-regular".

Definition 3.1. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraph with parts $V_1, \cdots, V_r$, each of size n. Let $\varepsilon \in (0,1)$ and $\alpha > 0$ be constants. We say $\mathcal{G}$ is (ε, α)-regular, if $e(\mathcal{G}) \ge n^{r-\frac{1}{s-1}-\varepsilon}$ and for each $i \in [r]$, there is a constant Δ_i such that every $(r-1)$-tuple $T \in \mathcal{T}_i(\mathcal{G})$ has bounded degree:*

$$\Delta_i/\alpha \le d_{\mathcal{G}}(T) \le \Delta_i, \text{ where } n^{1-\frac{1}{s-1}-\varepsilon} \le \Delta_i \le n^{1-\frac{1}{s-1}+\varepsilon}.$$

Note that if $\varepsilon' \ge \varepsilon$, $\alpha' \ge \alpha$ and $\mathcal{G}$ is (ε, α)-regular, then $\mathcal{G}$ is also (ε', α')-regular.

Lemma 3.2. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform hypergraph on rn vertices and with at least $n^{r-\frac{1}{s-1}-\varepsilon}$ edges, where $\varepsilon > 0$. Then $\mathcal{G}$ has an $\big(\varepsilon + (\log_2 n)^{-1/2}, 4r\log_2^r n\big)$-regular subgraph $\mathcal{H}$.*

The following definition plays a crucial role in the approach outlined in the previous section.

Definition 3.3. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraph with parts $V_1, \cdots, V_r$, each of size n. Let $\delta > 0$ be a constant.*

- *Fix an $(r-1)$-tuple T, a vertex $u \in T$ and a vertex $v \in N_{\mathcal{G}}(T)$. If there are at least $d_{\mathcal{G}}(T)^{s-1}/r$ many s-sets S satisfying that $v \in S \subseteq N_{\mathcal{G}}(T)$ and $cd_{\mathcal{G}}(S|u) \ge n^{r-2-\frac{1}{s-1}-\delta}$, then we say the pair $(T; v)$ is δ-dense on u in $\mathcal{G}$.*
- *Let $\mathcal{H}$ be a subgraph of $\mathcal{G}$ and $i, j \in [r]$ be two distinct integers. If for any $(r-1)$-tuple $T \in \mathcal{T}_j(\mathcal{H})$ and any $v \in N_{\mathcal{H}}(T)$, $(T; v)$ is δ-dense on the vertex $T \cap V_i$ in $\mathcal{G}$, then we write as $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$.*

Before turning to the statements of remaining lemmas, we would like to make several technical remarks about Definition 3.3. Firstly, with appropriate choices of δ and ε, the condition $cd_{\mathcal{G}}(S|u) \ge n^{r-2-\frac{1}{s-1}-\delta}$ would imply that u is a root of S.[1] Secondly, the notation $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$ can be equivalently

[1]This fact will be explicitly demonstrated in the proof of the first conclusion of Lemma 4.4.

expressed as follows: for any $e \in E(\mathcal{H})$, the pair $(e\backslash V_j; e \cap V_j)$ is δ-dense on the vertex $e \cap V_i$ in $\mathcal{G}$. Lastly, if $\delta' \geq \delta$ and $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$, then we also have $V_i \xrightarrow[\delta']{\mathcal{H},\mathcal{G}} V_j$.

The following two lemmas will be utilized to establish Property (I) and Property (II), respectively.

Lemma 3.4. *Let $\varepsilon \in (0,1)$ and $\alpha > 0$ be constants satisfying that $\alpha = o\left(n^{\varepsilon/s}\right)$. Suppose $\mathcal{G}$ is an (ε,α)-regular $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraph with parts $V_1, \cdots, V_r$, each of size n. For any part V_j, there exists an $(\varepsilon + \log_n 4r, 4r^2\alpha)$-regular subgraph $\mathcal{H} \subseteq \mathcal{G}$ and a distinct part V_i such that $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$, where $\delta := (s+1)\varepsilon$.*

Lemma 3.5. *Let $\varepsilon, \delta, \alpha > 0$ satisfy $6(s+1)(\varepsilon+\delta) \leq 1$ and $\alpha = o(n^{\varepsilon})$. Let n be sufficiently large and $\mathcal{H}_1 \subseteq \mathcal{H} \subseteq \mathcal{G}_1 \subseteq \mathcal{G}$ be a sequence of (ε,α)-regular $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraphs with parts $V_1, \cdots, V_r$, each of size n. If $V_i \xrightarrow[\delta]{\mathcal{H}_1,\mathcal{H}} V_j \xrightarrow[\delta]{\mathcal{G}_1,\mathcal{G}} V_k$ holds for $j \notin \{i,k\}$, then $k = i$.*

Finally, we are prepared to prove Theorem 1.1, assuming Lemmas 3.2, 3.4, and 3.5.

Proof. TOPROVE 0 $\square$

4 Proof of lemmas

This section is devoted to the proofs of Lemmas 3.2, 3.4 and 3.5.

4.1 Finding (ε,α)-regular subgraphs

In this subsection, we establish Lemma 3.2 along with several related properties regarding (ε,α)-regularity. The first lemma will be frequently used later to provide upper bounds on the (co-)degrees of subsets.

Lemma 4.1. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform balanced r-partite hypergraph on rn vertices. Suppose there is a constant Δ such that $d_{\mathcal{G}}(T) \leq \Delta$ holds for every $(r-1)$-tuple T. Let A be a k-tuple and S be an s-set of $\mathcal{G}$. Then the following hold that*

$$d_{\mathcal{G}}(A) \leq \Delta n^{r-k-1} \quad \text{and} \quad cd_{\mathcal{G}}(S) \leq rt\Delta n^{r-3}.$$

Moreover, if $\mathcal{G}$ is (ε,α)-regular, then

$$d_{\mathcal{G}}(A) \leq n^{r-k-\frac{1}{s-1}+\varepsilon} \quad \text{and} \quad cd_{\mathcal{G}}(S) \leq rtn^{r-2-\frac{1}{s-1}+\varepsilon}.$$

Proof. TOPROVE 1 $\square$

We now present the proof of Lemma 3.2 using standard deletion arguments.

Proof. TOPROVE 2 $\square$

We would like to point out that the proof of Lemma 3.2 can be slightly modified to show that for any $r \geq 3$, $s,t \geq 2$ and sufficiently large n, $\mathrm{ex}(n, K_{s,t}^{(r)}) \leq n^{r-\frac{1}{s-1}} \log^{2r} n$ holds.

Applying a similar deletion argument given in the above proof, we can also derive the following.

Lemma 4.2. *Let $\mathcal{G}$ be an (ε,α)-regular $K_{s,t}^{(r)}$-free r-uniform balanced r-partite hypergraph on rn vertices. Let $c > 0$ be a constant. If $\mathcal{G}'$ is a subgraph of $\mathcal{G}$ with at least $e(\mathcal{G})/c$ edges, then $\mathcal{G}'$ has an $(\varepsilon + \log_n 2c, 2\alpha cr)$-regular subgraph $\mathcal{H}$.*

Proof. TOPROVE 3 $\square$

4.2 Finding δ-dense structures: Property (I)

In this subsection, we prove Lemma 3.4.

Proof. TOPROVE 4 $\square$

4.3 Finding δ-dense structures: Property (II)

In this subsection, we prove Lemma 3.5. Before presenting the proof, we need to establish two technical lemmas. The first one involves some averaging statements for bipartite graphs.

Lemma 4.3. *Let $G = (A, B)$ be a bipartite graph with $e(G) \geq \rho|A||B|$ for some $\rho \in (0, 1)$.*

(1). There are at least $\rho|A|/2$ vertices $a \in A$ with $|N_G(a) \cap B| \geq \rho|B|/2$.

(2). Let s be a positive integer. If $\rho|A| \gg s$, then there are at least $(\rho|A|)^s/(3s!)$ many s-sets in A that have at least $\rho^s B/3$ common neighbors in G.

Proof. TOPROVE 5 $\square$

The following lemma provides the crucial techniques for proving Lemma 3.5. Roughly speaking, given the assumption $V_1 \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_2$, it reveals some dense structures concerning the "adjacency" between V_2 and V_1, as well as between V_2 and any predetermined part V_j.

Lemma 4.4. *Let $\varepsilon, \delta, \alpha > 0$ satisfy $6(s+1)(\varepsilon+\delta) \leq 1$ and $\alpha = o(n^\varepsilon)$. Let $\mathcal{G}$ be an (ε, α)-regular $K_{s,t}^{(r)}$-free r-uniform balanced r-partite hypergraph on rn vertices with parts $V_1, \cdots, V_r$.*

Fix $T = \{v_1, v_3, \cdots, v_r\} \in \mathcal{T}_2(\mathcal{G})$, where $v_i \in V_i$ for $i \in [r]\backslash\{2\}$. Let X be a subset of $N_{\mathcal{G}}(T) \subseteq V_2$ such that for every vertex $v \in X$, $(T; v)$ is δ-dense on v_1 in $\mathcal{G}$. Then the following hold.

(1). If $|X| \gg n^{\varepsilon+\delta}$, then X contains at least $n^{-s\varepsilon-s\delta}|X|^s/(3s!r^s)$ different s-sets rooted on v_1.

(2). Suppose $X \neq \emptyset$. Then for any given $j \in [r]\backslash\{1, 2\}$, there exist subsets $Y \subseteq N_{\mathcal{G}}(T)$, $Z \subseteq V_j$, and an $(r-3)$-tuple $R \subseteq V(\mathcal{G})\backslash(V_1 \cup V_2 \cup V_j)$ such that $|Y| \geq n^{1-\frac{1}{s-1}-2\varepsilon-\delta}/2r\alpha$, $n^{1-\frac{1}{s-1}-\delta} \leq |Z| \leq n^{1-\frac{1}{s-1}+\varepsilon}$, and for any $y \in Y$, $|N_{\mathcal{G}}(\{v_1, y\} \cup R) \cap Z| \geq n^{1-\frac{1}{s-1}-\varepsilon-2\delta}/2$.

Proof. TOPROVE 6 $\square$

Finally, we are ready to show Lemma 3.5.

Proof. TOPROVE 7 $\square$

5 Concluding remarks

In this paper, we prove that for any odd $r \geq 3$ and any $s \geq 3$, there exists an $\varepsilon_s > 0$ such that

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{r,s,t}\left(n^{r-\frac{1}{s-1}-\varepsilon}\right).$$

It would be interesting to determine the optimal constant ε_s for any odd $r \geq 3$. It is also worth noting that Mubayi and Verstraëte [12] conjectured that the Turán number for 3-uniform hypergraphs satisfies $\mathrm{ex}(n, K_{s,t}^{(3)}) = \Theta_{s,t}\left(n^{3-\frac{2}{s}}\right)$ for any $t \geq s \geq 2$, which is still open for $s \geq 3$.

As briefly discussed in Subsection 2.2, the proofs of Lemmas 3.4 and 3.5 yield certain rich adjacency structures in dense $K_{s,t}^{(r)}$-free r-uniform hypergraphs for even $r \geq 4$. There structures also align with the construction provided in Section 3 of [2]. These observations suggest that perhaps there exists a stability result for $K_{s,t}^{(r)}$ for even $r \geq 4$.

The asymptotics of $f_r(n)$ and $\mathrm{ex}(n, K_{2,2}^{(r)})$ remain intriguing open problems. We conclude this paper by mentioning two related conjectures. The first conjecture due to Füredi [7] states that $\mathrm{ex}(n, K_{2,2}^{(r)}) = (1 + o(1))\binom{n-1}{r-1}$ for any $r \geq 3$. Note that the lower bound $\binom{n-1}{r-1}$ can be achieved by the hypergraph star. Despise significant progress made in [12, 13], this conjecture remains open for any $r \geq 3$. The second conjecture, posed by Mubayi (see Conjecture 6.2 in [11]), suggests that the $f_r(n)$-problem is *stable* for $r \geq 4$. This says that for any $r \geq 4$ and $\delta > 0$, there exist $\epsilon > 0$ and n_0 such that any n-vertex r-uniform hypergraph with at least $(1-\epsilon)\binom{n}{r-1}$ edges, which does not contain four distinct edges A, B, C, D satisfying $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$, must contain a vertex v belonging to at least $(1 - \delta)\binom{n}{r-1}$ edges.

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