A hypergraph bipartite Turán problem with odd uniformity

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Abstract

In this paper, we investigate the hypergraph Turán number $\mathrm{ex}(n, K_{s,t}^{(r)})$. Here, $K_{s,t}^{(r)}$ denotes the r-uniform hypergraph with vertex set $\left(\cup_{i\in[t]}X_i\right)\cup Y$ and edge set $\{X_i \cup \{y\} : i \in [t], y \in Y\}$, where $X_1, X_2, \cdots, X_t$ are t pairwise disjoint sets of size $r-1$ and Y is a set of size s disjoint from each X_i. This study was initially explored by Erdős and has since received substantial attention in research. Recent advancements by Bradač, Gishboliner, Janzer and Sudakov have greatly contributed to a better understanding of this problem. They proved that $\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{s,t}(n^{r-\frac{1}{s-1}})$ holds for any $r \geq 3$ and $s, t \geq 2$. They also provided constructions illustrating the tightness of this bound if $r \geq 4$ is *even* and $t \gg s \geq 2$. Furthermore, they proved that $\mathrm{ex}(n, K_{s,t}^{(3)}) = O_{s,t}(n^{3-\frac{1}{s-1}-\varepsilon_s})$ holds for $s \geq 3$ and some $\epsilon_s > 0$. Addressing this intriguing discrepancy between the behavior of this number for $r = 3$ and the even cases, Bradač et al. post a question of whether

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{r,s,t}(n^{r-\frac{1}{s-1}-\varepsilon}) \text{ holds for } \textit{odd } r \geq 5 \text{ and any } s \geq 3.$$

In this paper, we provide an affirmative answer to this question, utilizing novel techniques to identify regular and dense substructures. This result highlights a rare instance in hypergraph Turán problems where the solution depends on the parity of the uniformity.

1 Introduction

For a given r-uniform hypergraph H, we say an r-uniform hypergraph is *H-free* if it does not contain a copy of H as its subgraph. The *Turán number* $\mathrm{ex}(n, H)$ denotes the maximum number of edges in an H-free r-uniform hypergraph on n vertices. The study of Turán number is a central problem in extremal combinatorics. We refer interested readers to the survey by Füredi and Simonovits [8] for ordinary graphs and the survey by Keevash [9] for non-r-partite r-uniform hypergraphs. Here, our focus lies on the Turán numbers of r-partite r-uniform hypergraphs H for $r \geq 3$. A fundamental result proved by Erdős states that for every such H, $\mathrm{ex}(n, H) = O(n^{r-\varepsilon_H})$ holds for some $\varepsilon_H > 0$. The primary objective of this aspect is to determine the optimal constant ε_H. However, this problem is notoriously difficult and to date, there are very few cases that have been fully understood.

In this paper, we consider the Turán number of the following r-partite r-uniform hypergraphs, which were initially defined by Mubayi and Verstraëte [12]: for positive integers r, s, t, let $K_{s,t}^{(r)}$ denote the r-uniform hypergraph with vertex set $\left(\cup_{i\in[t]}X_i\right) \cup Y$ and edge set $\{X_i \cup \{y\} : i \in [t], y \in Y\}$,

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where $X_1, X_2, \cdots, X_t$ are t pairwise disjoint sets of size $r-1$ and Y is a set of size s disjoint from each X_i. This study can be traced back to an old problem posted by Erdős [5], who asked to determine the maximum number $f_r(n)$ of edges in an r-uniform hypergraph on n vertices that does not contain four distinct edges A, B, C, D satisfying $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$. This generalizes the Turán number of the four-cycle, and it is evident to see that $f_3(n) = \mathrm{ex}(n, K_{2,2}^{(3)})$ and $f_r(n) \leq \mathrm{ex}(n, K_{2,2}^{(r)})$ for any $r \geq 4$. Füredi [7] resolved a conjecture of Erdős made in [5], by showing that $f_r(n) \leq 3.5\binom{n}{r-1}$ for any $r \geq 3$. This was first improved by Mubayi and Verstraëte [12], and later on, further improvements were made by Pikhurko and Verstraëte [13].

Returning to the Turán number $\mathrm{ex}(n, K_{s,t}^{(r)})$, Mubayi and Verstraëte [12] primarily focus on the case $r = 3$. They proved that $\mathrm{ex}(n, K_{s,t}^{(3)}) = O_{s,t}(n^{3-1/s})$ for $t \geq s \geq 3$ and $\mathrm{ex}(n, K_{s,t}^{(3)}) = \Omega_t(n^{3-2/s})$ for $t > (s-1)!$. For the particular case $s = 2$, Mubayi and Verstraëte [12] also provided that $\mathrm{ex}(n, K_{2,t}^{(3)}) \leq t^4\binom{n}{2}$ for $t \geq 3$, and they further posed the question of determining the order of the magnitude of the leading coefficient in terms of t. Among other results, Ergemlidze, Jiang and Methuku [6] obtained an improvement by showing $\mathrm{ex}(n, K_{2,t}^{(3)}) \leq (15t\log t + 40t)\binom{n}{2}$, which can be extended to all $r \geq 3$. Using the random algebraic method (see [3, 4]), Xu, Zhang and Ge [14, 15] proved that $\mathrm{ex}(n, K_{s,t}^{(r)}) = \Theta(n^{r-1/t})$, assuming that s is sufficiently large than r, t.

Very recently, Bradač, Gishboliner, Janzer and Sudakov [2] made significant contributions towards a better understanding of the behavior of the Turán number $\mathrm{ex}(n, K_{s,t}^{(r)})$. Using a novel variant of the dependent random choice, they proved a general upper bound that $\mathrm{ex}(n, K_{s,t}^{(r)}) = O_s\left(t^{\frac{1}{s-1}} n^{r-\frac{1}{s-1}}\right)$ holds for any $r \geq 3$ and $s, t \geq 2$. Moreover, they built upon norm graphs ([1, 10]) and provided matching constructions, which led to

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = \Theta_{r,s}\left(t^{\frac{1}{s-1}} n^{r-\frac{1}{s-1}}\right) \text{ for any } \textit{even } r \geq 4,\ s \geq 2, \text{ and } t > (s-1)!.$$

Furthermore, they derived a different order of magnitude for n in the case $r = 3$ by proving that

$$\mathrm{ex}(n, K_{s,t}^{(3)}) = O_{s,t}\left(n^{3-\frac{1}{s-1}-\varepsilon_s}\right) \text{ holds for any } s \geq 3,\ t, \text{ and some positive constant } \varepsilon_s = O(s^{-5}).$$

Bradač et al. posed the question of whether the above upper bound can be extended to all odd uniformities. They noted that if the question is affirmative, then "this would be a rare example of an extremal problem where the answer depends on the parity of the uniformity", quoted from [2].

In this work, we provide a positive answer to the aforementioned question posed by Bradač et al. Our main result can be stated as follows.

Theorem 1.1. *For any odd $r \geq 3$ and any $s \geq 3$, there exists some $\varepsilon = \varepsilon(s) > 0$ depending only on s such that for any positive integer t,*

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{r,s,t}\left(n^{r-\frac{1}{s-1}-\varepsilon}\right).$$

We use a different proof approach from [2] (see Section 2.2 for an outline of the proof). The core ideas are to find some regular and dense substructures in hypergraphs. Our proof works for $\varepsilon(s) = \frac{1}{6(s+2)^2}$, although we have made no serious attempt to optimize the leading coefficient. This improves the choice of $\varepsilon_s = O(s^{-5})$ in [2] for the case $r = 3$.

The remainder of the paper is organized as follows. In Section 2, we introduce the necessary notation and provide an outline of the proof for Theorem 1.1. In Section 3, we break down the proof of Theorem 1.1 into three lemmas. The full proofs of these lemmas are presented in Section 4. Finally, in Section 5, we offer some concluding remarks.

2 Preliminaries

In this section, we begin by introducing the necessary notation, followed by providing a preliminary outline of the proof for our main result, namely Theorem 1.1.

2.1 Notation

Let $r \geq 3$, $s \geq 3$, t and n be positive integers throughout the rest of this paper. Let $[n] = \{1, 2, \cdots, n\}$.

Assume that $\mathcal{G}$ is an r-partite r-uniform hypergraph with parts $V_1, \cdots, V_r$, each of size n, throughout this section. For a given vertex v, the *link hypergraph* of v, denote as $N_{\mathcal{G}}(v)$, comprises all $(r-1)$-sets that, when combined with v, form an edge in $\mathcal{G}$. Similarly, for $k < r$ and a set T with k vertices, we write $N_{\mathcal{G}}(T)$ for the $(r-k)$-uniform hypergraph containing all $(r-k)$-sets which together with T form an edge in $\mathcal{G}$. We denote its cardinality as $d_{\mathcal{G}}(T) = |N_{\mathcal{G}}(T)|$. Specially if $d_{\mathcal{G}}(T) \neq 0$, we say the k-set T is a *k-tuple* of $\mathcal{G}$. We define the set of all $(r-1)$-tuples of $\mathcal{G}$ as $\mathcal{T}(\mathcal{G})$. For $i \in [r]$, let $\mathcal{T}_i(\mathcal{G})$ be the set of $(r-1)$-tuples of $\mathcal{G}$ contained in $V(\mathcal{G}) \backslash V_i$.

For an s-set $S = \{v_1, v_2 \cdots, v_s\} \subseteq V(\mathcal{G})$, we define $\mathrm{CN}_{\mathcal{G}}(S)$ as the set of common edges in all link hypergraphs of vertices $v_i \in S$. That is, $\mathrm{CN}_{\mathcal{G}}(S) = \bigcap_{i\in[s]} N_{\mathcal{G}}(v_i)$. A *vertex cover* of a set $\mathcal{E}$ of edges is a set of vertices that intersects with every edge in $\mathcal{E}$. For each s-set S, we choose and fix a minimum vertex-cover of $\mathrm{CN}_{\mathcal{G}}(S)$, and call these vertices the *roots* of S. For a root v of S, we also say S is rooted on v. The following simple yet crucial property will be repeatedly used in the proofs:

$$\text{If } \mathcal{G} \text{ is } K_{s,t}^{(r)}\text{-free, then any } s\text{-set } S \subseteq V(\mathcal{G}) \text{ has less than } rt \text{ roots.} \tag{1}$$

To see this, consider a maximum set of disjoint edges in $\mathrm{CN}_{\mathcal{G}}(S)$, and let A be the vertex set of these edges. Due to the maximality, every edge in $\mathrm{CN}_{\mathcal{G}}(S)$ contains at least one vertex in A. So A is a vertex-cover of $\mathrm{CN}_{\mathcal{G}}(S)$. Since $\mathcal{G}$ is $K_{s,t}^{(r)}$-free, $\mathrm{CN}_{\mathcal{G}}(S)$ has at most $t-1$ disjoint edges. Therefore, $|A| \leq (t-1)(r-1) < tr$, as desired.

Let S be an s-set in $V(\mathcal{G})$. We denote $cd_{\mathcal{G}}(S) = |\mathrm{CN}_{\mathcal{G}}(S)|$ to be the *codegree* of S in $\mathcal{G}$. For a vertex $u \notin S$, we write $cd_{\mathcal{G}}(S|u)$ for the number of edges in $\mathrm{CN}_{\mathcal{G}}(S)$ containing u. It is clear that $cd_{\mathcal{G}}(S) \leq \sum_u cd_{\mathcal{G}}(S|u)$, where the summation is over all roots u of S.

2.2 Proof sketch

In this concise overview of the proof for Theorem 1.1, we outline crucial intermediate properties and emphasize the differences between the cases when r is odd or even.

Consider $\mathcal{G}$ as a $K_{s,t}^{(r)}$-free r-partite r-uniform hypergraph with parts $V_1, \cdots, V_r$, each of size n, and possessing at least $n^{r-1/(s-1)-\epsilon}$ edges, where $\epsilon > 0$ is a small constant. First, we demonstrate that $\mathcal{G}$ can be assumed to be "regular" in the sense that every $(r-1)$-tuple has bounded degree. This regularity property is proven in Lemma 3.2 and simplifies the subsequent analysis.

The key ideas of the proof culminate in an auxiliary digraph $D(\mathcal{G})$, where the vertex set is $\{V_1, \cdots, V_r\}$, and directed edges $V_i \to V_j$ are formed for distinct i, j if there is a significant number of s-sets in V_j rooted on vertices in V_i within a relatively dense subgraph of $\mathcal{G}$ (refer to Definition 3.3 for a precise description). The main body of the proof is then divided into the following two properties, which are established in Lemmas 3.4 and 3.5, respectively:

(I). Every vertex in $D(\mathcal{G})$ has non-zero in-degree, and

(II). There exist no three distinct vertices forming a directed path $V_i \to V_j \to V_k$ in $D(\mathcal{G})$.

The proof of Property (II) is the most involved. In essence, if $V_i \to V_j$ holds, it can be shown that there exist large subsets $Y \subseteq V_j$ and $Z \subseteq V_k$ for $k \notin \{i,j\}$ such that for any $y \in Y$, the projection of $N_{\mathcal{G}}(y)$ onto Z is nearly complete (see Lemma 4.4 in more details). If $V_j \to V_k$ also holds for some $k \notin \{i,j\}$, then the number of pairs (S,y) where $y \in Y$ is a root of an s-set $S \subseteq Z$ can be shown to be at least $(|Y||Z|^s)^{1-O(\epsilon)}$. However, due to (1), the number of such pairs (S,y) is at most $O_{r,t}(|Z|^s)$. This would lead to a contradiction and establish Property (II).

Now we can distinguish between the cases when r is odd or even. If r is even, using Properties (I) and (II), one can conclude that $D(\mathcal{G})$ must be isomorphic to the union of 2-cycles, say $V_{2i-1} \leftrightarrows V_{2i}$ for $1 \le i \le r/2$. This configuration is feasible, as justified in the construction in [2]. However, if r is odd, Property (I) would force the existence of a directed path of length two say $V_i \to V_j \to V_k$. This clearly contradicts Property (II) and thus completes the proof of Theorem 1.1.

3 Proof of Theorem 1.1

In this section, we establish the proof of Theorem 1.1 by reducing it to Lemmas 3.2, 3.4, and 3.5.

Let us proceed to present the statements of these lemmas. The first lemma demonstrates that for any $K_{s,t}^{(r)}$-free r-uniform hypergraph $\mathcal{G}$, one can find a subgraph of $\mathcal{G}$ with nearly the same edge density and possessing the following useful property of being "almost-regular".

Definition 3.1. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraph with parts $V_1, \cdots, V_r$, each of size n. Let $\varepsilon \in (0,1)$ and $\alpha > 0$ be constants. We say $\mathcal{G}$ is (ε,α)-regular, if $e(\mathcal{G}) \ge n^{r-\frac{1}{s-1}-\varepsilon}$ and for each $i \in [r]$, there is a constant Δ_i such that every $(r-1)$-tuple $T \in \mathcal{T}_i(\mathcal{G})$ has bounded degree:*

$$\Delta_i/\alpha \le d_{\mathcal{G}}(T) \le \Delta_i, \text{ where } n^{1-\frac{1}{s-1}-\varepsilon} \le \Delta_i \le n^{1-\frac{1}{s-1}+\varepsilon}.$$

Note that if $\varepsilon' \ge \varepsilon$, $\alpha' \ge \alpha$ and $\mathcal{G}$ is (ε,α)-regular, then $\mathcal{G}$ is also (ε',α')-regular.

Lemma 3.2. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform hypergraph on rn vertices and with at least $n^{r-\frac{1}{s-1}-\varepsilon}$ edges, where $\varepsilon > 0$. Then $\mathcal{G}$ has an $\big(\varepsilon + (\log_2 n)^{-1/2}, 4r\log_2^r n\big)$-regular subgraph $\mathcal{H}$.*

The following definition plays a crucial role in the approach outlined in the previous section.

Definition 3.3. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraph with parts $V_1, \cdots, V_r$, each of size n. Let $\delta > 0$ be a constant.*

- *Fix an $(r-1)$-tuple T, a vertex $u \in T$ and a vertex $v \in N_{\mathcal{G}}(T)$. If there are at least $d_{\mathcal{G}}(T)^{s-1}/r$ many s-sets S satisfying that $v \in S \subseteq N_{\mathcal{G}}(T)$ and $cd_{\mathcal{G}}(S|u) \ge n^{r-2-\frac{1}{s-1}-\delta}$, then we say the pair $(T;v)$ is δ-dense on u in $\mathcal{G}$.*
- *Let $\mathcal{H}$ be a subgraph of $\mathcal{G}$ and $i,j \in [r]$ be two distinct integers. If for any $(r-1)$-tuple $T \in \mathcal{T}_j(\mathcal{H})$ and any $v \in N_{\mathcal{H}}(T)$, $(T;v)$ is δ-dense on the vertex $T \cap V_i$ in $\mathcal{G}$, then we write as $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$.*

Before turning to the statements of remaining lemmas, we would like to make several technical remarks about Definition 3.3. Firstly, with appropriate choices of δ and ε, the condition $cd_{\mathcal{G}}(S|u) \ge n^{r-2-\frac{1}{s-1}-\delta}$ would imply that u is a root of S.[1] Secondly, the notation $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$ can be equivalently

[1]This fact will be explicitly demonstrated in the proof of the first conclusion of Lemma 4.4.

expressed as follows: for any $e \in E(\mathcal{H})$, the pair $(e\backslash V_j; e \cap V_j)$ is δ-dense on the vertex $e \cap V_i$ in $\mathcal{G}$. Lastly, if $\delta' \geq \delta$ and $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$, then we also have $V_i \xrightarrow[\delta']{\mathcal{H},\mathcal{G}} V_j$.

The following two lemmas will be utilized to establish Property (I) and Property (II), respectively.

Lemma 3.4. *Let $\varepsilon \in (0,1)$ and $\alpha > 0$ be constants satisfying that $\alpha = o\left(n^{\varepsilon/s}\right)$. Suppose $\mathcal{G}$ is an (ε, α)-regular $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraph with parts $V_1, \cdots, V_r$, each of size n. For any part V_j, there exists an $(\varepsilon + \log_n 4r, 4r^2\alpha)$-regular subgraph $\mathcal{H} \subseteq \mathcal{G}$ and a distinct part V_i such that $V_i \xrightarrow[\delta]{\mathcal{H},\mathcal{G}} V_j$, where $\delta := (s+1)\varepsilon$.*

Lemma 3.5. *Let $\varepsilon, \delta, \alpha > 0$ satisfy $6(s+1)(\varepsilon + \delta) \leq 1$ and $\alpha = o(n^{\varepsilon})$. Let n be sufficiently large and $\mathcal{H}_1 \subseteq \mathcal{H} \subseteq \mathcal{G}_1 \subseteq \mathcal{G}$ be a sequence of (ε, α)-regular $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraphs with parts $V_1, \cdots, V_r$, each of size n. If $V_i \xrightarrow[\delta]{\mathcal{H}_1,\mathcal{H}} V_j \xrightarrow[\delta]{\mathcal{G}_1,\mathcal{G}} V_k$ holds for $j \notin \{i,k\}$, then $k = i$.*

Finally, we are prepared to prove Theorem 1.1, assuming Lemmas 3.2, 3.4, and 3.5.

Proof of Theorem 1.1. Let $\varepsilon = \frac{1}{6(s+2)^2}$ throughout this proof. It suffices to show that for any odd $r \geq 3$ and sufficiently large integer n, every $K_{s,t}^{(r)}$-free r-uniform hypergraph $\mathcal{G}$ on rn vertices has at most $n^{r-\frac{1}{s-1}-\varepsilon}$ edges. Suppose for a contradiction that there exists such an r-uniform hypergraph $\mathcal{G}$ with more than $n^{r-\frac{1}{s-1}-\varepsilon}$ edges.

By Lemma 3.2, $\mathcal{G}$ contains an $(\varepsilon_1, \alpha_1)$-regular subgraph $\mathcal{G}_1$, where $\varepsilon_1 = \varepsilon + (\log_2 n)^{-1/2}$ and $\alpha_1 = 4r \log_2^r n$. Then $\mathcal{G}_1$ is balanced r-partite, say with parts $V_1, \cdots, V_r$, each of size n. Let

$$\varepsilon_i = \varepsilon_1 + (i-1)\log_n 4r \quad \text{and} \quad \alpha_i = (4r^2)^{i-1}\alpha_1 \text{ for all } i \geq 1.$$

Also let

$$\varepsilon^* = \varepsilon + 2r(\log_2 n)^{-1/2} \quad \text{and} \quad \alpha^* = (4r^2)^{r+2}\log_2^r n.$$

We note that for each $1 \leq i \leq r+2$,

$$\varepsilon^* \geq \varepsilon_i, \ \ \alpha^* \geq \alpha_i, \ \text{ and } \alpha_i \text{ satisfies the condition of Lemma 3.4.} \tag{2}$$

We will iteratively apply Lemma 3.4 to obtain a sequence of $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraphs $\mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \cdots \supseteq \mathcal{G}_{r+2}$ on rn vertices as follows. Initially, let $a_1 = 1$; applying Lemma 3.4 with respect to $\mathcal{G}_1$ (which is $(\varepsilon_1, \alpha_1)$-regular) and the part V_{a_1}, there exist an $(\varepsilon_2, \alpha_2)$-regular subgraph $\mathcal{G}_2 \subseteq \mathcal{G}_1$ and an index $b_1 \neq a_1$ such that $V_{b_1} \xrightarrow[(s+1)\varepsilon_1]{\mathcal{G}_2,\mathcal{G}_1} V_{a_1}$; then applying Lemma 3.4 with respect to $\mathcal{G}_2$ and the part V_{b_1}, there exist an $(\varepsilon_3, \alpha_3)$-regular subgraph $\mathcal{G}_3 \subseteq \mathcal{G}_2$ and an index $c_1 \neq b_1$ such that $V_{c_1} \xrightarrow[(s+1)\varepsilon_2]{\mathcal{G}_3,\mathcal{G}_2} V_{b_1}$. Now assume that the sequence has been defined for $\mathcal{G}_1 \supseteq \cdots \supseteq \mathcal{G}_{2i-1}$ for some $2 \leq i \leq (r+1)/2$. We choose an index $a_i \in [r]\backslash\left(\cup_{1\leq j\leq i-1}\{a_j, b_j, c_j\}\right)$,[2] and then apply Lemma 3.4 twice to get subgraphs $\mathcal{G}_{2i+1} \subseteq \mathcal{G}_{2i} \subseteq \mathcal{G}_{2i-1}$ and indices $b_i, c_i \in [r]$ such that

$$V_{c_i} \xrightarrow[(s+1)\varepsilon_{2i}]{\mathcal{G}_{2i+1},\mathcal{G}_{2i}} V_{b_i} \xrightarrow[(s+1)\varepsilon_{2i-1}]{\mathcal{G}_{2i},\mathcal{G}_{2i-1}} V_{a_i}, \text{where } \mathcal{G}_j \text{ is } (\varepsilon_j, \alpha_j)\text{-regular for } j \in \{2i, 2i+1\}.$$

[2]We will see later that such an index is always valid as long as $i \leq (r+1)/2$.

Let $\delta^* = (s+1)\varepsilon^*$. Then as n is sufficiently large, it follows that

$$6(s+1)(\varepsilon^* + \delta^*) = 6(s+1)(s+2)\left(\frac{1}{6(s+2)^2} + 2r(\log_2 n)^{-1/2}\right) < 1. \tag{3}$$

In view of (2) and the remarks after Definitions 3.1 and 3.3, we see that $\mathcal{G}_\ell$ is $(\varepsilon^*, \alpha^*)$-regular for each $1 \le \ell \le r+2$, and

$$V_{c_i} \xrightarrow[\delta^*]{\mathcal{G}_{2i+1},\mathcal{G}_{2i}} V_{b_i} \xrightarrow[\delta^*]{\mathcal{G}_{2i},\mathcal{G}_{2i-1}} V_{a_i} \text{ holds for each } 1 \le i \le (r+1)/2. \tag{4}$$

By (3) and the fact $\alpha^* = o(n^\varepsilon)$, using Lemma 3.5, we can easily conclude that $c_i = a_i$ for all $1 \le i \le (r+1)/2$. By the choice of a_i, evidently a_i is distinct from the indices in $\cup_{1\le j\le i-1}\{a_j, b_j\}$. We claim that b_i is also distinct from the indices in $\cup_{1\le j\le i-1}\{a_j, b_j\}$. Otherwise, $b_i \in \{a_j, b_j\}$ for some $1 \le j \le i-1$, which, combining (4) for the index j (also using $c_j = a_j$), would yield that

$$\text{either } V_{a_i} \xrightarrow[\delta^*]{\mathcal{G}_{2i+1},\mathcal{G}_{2i}} V_{b_i} \xrightarrow[\delta^*]{\mathcal{G}_{2j+1},\mathcal{G}_{2j}} V_{b_j} \text{ (if } b_i = a_j\text{)}, \text{ or } V_{a_i} \xrightarrow[\delta^*]{\mathcal{G}_{2i+1},\mathcal{G}_{2i}} V_{b_i} \xrightarrow[\delta^*]{\mathcal{G}_{2j},\mathcal{G}_{2j-1}} V_{a_j} \text{ (if } b_i = b_j\text{)}.$$

Using Lemma 3.5 again, we then deduce that either $a_i = b_j$ or $a_i = a_j$, a contradiction to the choice of a_i, proving the claim. This shows that $\{a_i, b_i\}$ is disjoint from $\{a_j, b_j\}$ whenever $1 \le i \ne j \le (r+1)/2$. Consequently, $|\bigcup_{1\le i\le (r+1)/2}\{a_j, b_j\}| = 2\cdot(r+1)/2 = r+1$. However, this contradicts the fact that $\bigcup_{1\le i\le (r+1)/2}\{a_j, b_j\} \subseteq [r]$. This final contradiction completes the proof of Theorem 1.1. □

4 Proof of lemmas

This section is devoted to the proofs of Lemmas 3.2, 3.4 and 3.5.

4.1 Finding (ε, α)-regular subgraphs

In this subsection, we establish Lemma 3.2 along with several related properties regarding (ε, α)-regularity. The first lemma will be frequently used later to provide upper bounds on the (co-)degrees of subsets.

Lemma 4.1. *Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform balanced r-partite hypergraph on rn vertices. Suppose there is a constant Δ such that $d_{\mathcal{G}}(T) \le \Delta$ holds for every $(r-1)$-tuple T. Let A be a k-tuple and S be an s-set of $\mathcal{G}$. Then the following hold that*

$$d_{\mathcal{G}}(A) \le \Delta n^{r-k-1} \quad \text{and} \quad cd_{\mathcal{G}}(S) \le rt\Delta n^{r-3}.$$

Moreover, if $\mathcal{G}$ is (ε, α)-regular, then

$$d_{\mathcal{G}}(A) \le n^{r-k-\frac{1}{s-1}+\varepsilon} \quad \text{and} \quad cd_{\mathcal{G}}(S) \le rtn^{r-2-\frac{1}{s-1}+\varepsilon}.$$

Proof. By definition, $N_{\mathcal{G}}(A)$ is an $(r-k)$-uniform $(r-k)$-paritite hypergraph with each part of size n. On average, there is an $(r-k-1)$-set B such that at least $d_{\mathcal{G}}(A)/n^{r-k-1}$ edges in $N_{\mathcal{G}}(A)$ containing B. It also means that $\mathcal{G}$ has at least $d_{\mathcal{G}}(A)/n^{r-k-1}$ edges containing the $(r-1)$-set $A \cup B$. Consequently, one can get $\Delta \ge d_{\mathcal{G}}(A\cup B) \ge d_{\mathcal{G}}(A)/n^{r-k-1}$.

Let R be the set of roots of the s-set S. Since $\mathcal{G}$ is $K_{s,t}^{(r)}$-free, by (1) we have $|R| < rt$. Fix a vertex $v \in S$. For any $u \in R$, by the previous paragraph we have $d_{\mathcal{G}}(\{u, v\}) \le \Delta n^{r-3}$. Thus we can derive that $cd_{\mathcal{G}}(S) \le \sum_{u\in R} d_{\mathcal{G}}(\{u, v\}) \le rt\Delta n^{r-3}$.

If $\mathcal{G}$ is (ε, α)-regular, then we can substitute Δ with $n^{1-\frac{1}{s-1}+\varepsilon}$ to get the desired inequalities. □

We now present the proof of Lemma 3.2 using standard deletion arguments.

Proof of Lemma 3.2. Let $\mathcal{G}$ be a $K_{s,t}^{(r)}$-free r-uniform hypergraph on rn vertices and with at least $n^{r-\frac{1}{s-1}-\varepsilon}$ edges. It is well known that $\mathcal{G}$ has an r-partite subgraph $\mathcal{G}'$ with parts $V_1, V_2, \cdots, V_r$ of size n such that $e(\mathcal{G}') \geq \frac{r!}{r^r} e(\mathcal{G}) \geq \frac{r!}{r^r} n^{r-\frac{1}{s-1}-\varepsilon}$.

We first consider $(r-1)$-tuples in $\mathcal{T}_1(\mathcal{G}')$ and partition them into sub-families based on their degrees. We define $\mathcal{T}_{1,j} \subseteq \mathcal{T}_1(\mathcal{G}')$ as the set consisting of all $T \in \mathcal{T}_1(\mathcal{G}')$ with $2^{j-1} \leq d_{\mathcal{G}'}(T) \leq 2^j$ for $j \in [\log_2 n]$. By averaging, we can find a set $\mathcal{T}_{1,k}$ such that the number of the corresponding edges of $\mathcal{G}'$ is at least $e(\mathcal{G}')/\log_2 n$. Let $\mathcal{G}_1$ be the union of these edges, and let $\Delta_1 = 2^k$. Then we have $e(\mathcal{G}_1) \geq e(\mathcal{G}')/\log_2 n$ and $\Delta_1/2 \leq d_{\mathcal{G}_1}(T) \leq \Delta_1$ for any $T \in \mathcal{T}_1(\mathcal{G}_1)$. Consequently, we obtain $|\mathcal{T}_1(\mathcal{G}_1)| \leq e(\mathcal{G}_1)/(\Delta_1/2) \leq 2e(\mathcal{G}')/\Delta_1$. Similarly, we can get $\mathcal{G}_2 \subseteq \mathcal{G}_1$ and a constant Δ_2 such that $e(\mathcal{G}_2) \geq e(\mathcal{G}_1)/\log_2 n$ and $\Delta_2/2 \leq d_{\mathcal{G}_2}(T) \leq \Delta_2$ for every $T \in \mathcal{T}_2(\mathcal{G}_2)$. Repeating this process, we can obtain subgraphs $\mathcal{G}_i \subseteq \mathcal{G}_{i-1}$ and constants Δ_i for all $2 \leq i \leq r$ such that $e(\mathcal{G}_i) \geq e(\mathcal{G}_{i-1})/\log_2 n$ and $\Delta_i/2 \leq d_{\mathcal{G}_i}(T) \leq \Delta_i$ for every $T \in \mathcal{T}_i(\mathcal{G}_i)$. Then $|\mathcal{T}_i(\mathcal{G}_i)| \leq e(\mathcal{G}_i)/(\Delta_i/2) \leq 2e(\mathcal{G}')/\Delta_i$ for any $i \in [r]$.

Collecting the above estimations in $\mathcal{G}_r$, we have $e(\mathcal{G}_r) \geq e(\mathcal{G}')/\log_2^r n$, $|\mathcal{T}_i(\mathcal{G}_r)| \leq |\mathcal{T}_i(\mathcal{G}_i)| \leq 2e(\mathcal{G}')/\Delta_i$ for all $i \in [r]$, and for each $T \in \mathcal{T}_i(\mathcal{G}_r)$ (which is also in $\mathcal{T}_i(\mathcal{G}_i)$), $d_{\mathcal{G}_r}(T) \leq d_{\mathcal{G}_i}(T) \leq \Delta_i$.

Next, we want to identify a subgraph $\mathcal{H} \subseteq \mathcal{G}_r$ with a proper lower bound on $d_{\mathcal{H}}(T)$ for any $(r-1)$-tuple T through the following deleting process. Initially, let $\mathcal{H} = \mathcal{G}_r$. If there is an $(r-1)$-tuple $T \in \mathcal{T}_i(H)$ for some $i \in [r]$ with $d_{\mathcal{H}}(T) < \Delta_i/4r\log_2^r n$, then we delete all edges containing T, and denote the resulting hypergraph as $\mathcal{H}$. Repeat this process until either $\mathcal{H}$ is empty or every $T \in \mathcal{T}_i(H)$ satisfies $d_{\mathcal{H}}(T) \geq \Delta_i/4r\log_2^r n$. The number of edges we deleted is

$$e(\mathcal{G}_r) - e(\mathcal{H}) \leq \sum_{i\in[r]} |\mathcal{T}_i(\mathcal{G}_r)| \cdot \frac{\Delta_i}{4r\log_2^r n} \leq \sum_{i\in[r]} \frac{2e(\mathcal{G}')}{\Delta_i} \cdot \frac{\Delta_i}{4r\log_2^r n} = \frac{e(\mathcal{G}')}{2\log_2^r n} \leq \frac{e(\mathcal{G}_r)}{2}.$$

Therefore, $\mathcal{H}$ is a non-empty subgraph of $\mathcal{G}_r \subseteq \mathcal{G}$ with

$$e(\mathcal{H}) \geq \frac{e(\mathcal{G}_r)}{2} \geq \frac{e(\mathcal{G}')}{2\log_2^r n} \geq \frac{n^{r-\frac{1}{s-1}-\varepsilon}}{(r\log_2 n)^r} \geq n^{r-\frac{1}{s-1}-\varepsilon'}, \text{ where } \varepsilon' = \varepsilon + (\log_2 n)^{-1/2},$$

and $\Delta_i/4r\log_2^r n \leq d_{\mathcal{H}}(T) \leq d_{\mathcal{G}_r}(T) \leq \Delta_i$ for every $T \in \mathcal{T}_i(H)$.

To show $\mathcal{H}$ is the desired $(\varepsilon', 4r\log_2^r n)$-regular subgraph of $\mathcal{G}$, it remains to bound every Δ_i for $i \in [r]$. For any $i \in [r]$, since Δ_i is the upper bound of $d_{\mathcal{H}}(T)$ for every $T \in \mathcal{T}_i(H)$, it is clear that

$$\Delta_i \geq e(\mathcal{H})/n^{r-1} \geq n^{1-\frac{1}{s-1}-\varepsilon'}.$$

So it will suffice to show $\Delta_i \leq n^{1-\frac{1}{s-1}+\varepsilon'}$ for every $i \in [r]$. By symmetry, we may assume that Δ_1 is the maximum one among all Δ_i. Let us count the number m of pairs (S,T), where $T \in \mathcal{T}_1(\mathcal{H})$ and S is an s-set in $N_{\mathcal{H}}(T)$. Since $\sum_{T\in\mathcal{T}_1(\mathcal{H})} d_{\mathcal{H}}(T) = e(\mathcal{H})$ and for each such T, $d_{\mathcal{H}}(T) \geq \Delta_1/4r\log_2^r n \gg s$, we can obtain that

$$m = \sum_{T\in\mathcal{T}^1(\mathcal{H})} \binom{d_{\mathcal{H}}(T)}{s} \geq e(\mathcal{H}) \cdot \left(\frac{d_{\mathcal{H}}(T)}{s}\right)^{s-1} \geq \frac{n^{r-\frac{1}{s-1}-\varepsilon}}{(r\log_2 n)^r} \cdot \left(\frac{\Delta_1}{4rs\log_2^r n}\right)^{s-1}.$$

By averaging, there exists an s-set S which is contained in $N_{\mathcal{H}}(T)$ for at least m/n^s many $(r-1)$-tuples T. Using Lemma 4.1, we have

$$rt\Delta_1 n^{r-3} \geq cd_{\mathcal{H}}(S) \geq \frac{m}{n^s} \geq \frac{n^{r-s-\frac{1}{s-1}-\varepsilon}\Delta_1^{s-1}}{r^r \cdot (4rs\log_2^r n)^s}.$$

As n is sufficiently large, this gives $\Delta_1^{s-2} \le n^{s-3+\frac{1}{s-1}+\varepsilon} \log_2^{rs+1} n$. As $s \ge 3$, it further implies that

$$\max_{i\in[r]} \Delta_i = \Delta_1 \le n^{1-\frac{1}{s-1}+\frac{\varepsilon}{s-2}} (\log_2 n)^{\frac{rs+1}{s-2}} \le n^{1-\frac{1}{s-1}+\varepsilon'}.$$

Here, $\varepsilon' = \varepsilon + (\log_2 n)^{-1/2}$. Hence $\mathcal{G}$ has an $(\varepsilon', 4r\log_2^r n)$-regular subgraph $\mathcal{H}$. $\square$

We would like to point out that the proof of Lemma 3.2 can be slightly modified to show that for any $r \ge 3$, $s, t \ge 2$ and sufficiently large n, $\mathrm{ex}(n, K_{s,t}^{(r)}) \le n^{r-\frac{1}{s-1}} \log^{2r} n$ holds.

Applying a similar deletion argument given in the above proof, we can also derive the following.

Lemma 4.2. *Let $\mathcal{G}$ be an (ε,α)-regular $K_{s,t}^{(r)}$-free r-uniform balanced r-partite hypergraph on rn vertices. Let $c > 0$ be a constant. If $\mathcal{G}'$ is a subgraph of $\mathcal{G}$ with at least $e(\mathcal{G})/c$ edges, then $\mathcal{G}'$ has an $(\varepsilon + \log_n 2c, 2\alpha cr)$-regular subgraph $\mathcal{H}$.*

Proof. We apply the deletion argument to get a subgraph $\mathcal{H} \subseteq \mathcal{G}'$ as follows. Initially set $\mathcal{H} = \mathcal{G}'$. If $\mathcal{H}$ has an $(r-1)$-tuple T with $d_{\mathcal{H}}(T) < d_{\mathcal{G}}(T)/2cr$, then we delete all edges containing T, and still denote the resulting hypergraph as $\mathcal{H}$. Repeat this process until $\mathcal{H}$ is empty or every $(r-1)$-tuple T of $\mathcal{H}$ satisfies that $d_{\mathcal{H}}(T) \ge d_{\mathcal{G}}(T)/2cr$. The number of edges we deleted is

$$e(\mathcal{G}') - e(\mathcal{H}) \le \sum_{T\in\mathcal{T}(\mathcal{G})} \frac{d_{\mathcal{G}}(T)}{2cr} = \frac{r\cdot e(\mathcal{G})}{2cr} = \frac{e(\mathcal{G}')}{2}.$$

So we have $e(\mathcal{H}) \ge \frac{e(\mathcal{G}')}{2} \ge \frac{e(\mathcal{G})}{2c} \ge n^{r-\frac{1}{s-1}-\varepsilon-\log_n 2c}$ and $d_{\mathcal{G}}(T)/2cr \le d_{\mathcal{H}}(T) \le d_{\mathcal{G}}(T)$ for each $T \in \mathcal{T}(\mathcal{H})$. Since $\mathcal{G}$ is (ε,α)-regular, there exists $\Delta_i \in [n^{1-\frac{1}{s-1}-\varepsilon}, n^{1-\frac{1}{s-1}+\varepsilon}]$ such that for each $T \in \mathcal{T}_i(\mathcal{G})$, $\Delta_i/\alpha \le d_{\mathcal{G}}(T) \le \Delta_i$. This implies that for each $T \in \mathcal{T}_i(\mathcal{H})$, we have $\Delta_i/2\alpha cr \le d_{\mathcal{G}}(T)/2cr \le d_{\mathcal{H}}(T) \le d_{\mathcal{G}}(T) \le \Delta_i$. Therefore, $\mathcal{H}$ is an $(\varepsilon + \log_n 2c, 2\alpha cr)$-regular subgraph of $\mathcal{G}'$. $\square$

4.2 Finding δ-dense structures: Property (I)

In this subsection, we prove Lemma 3.4.

Proof of Lemma 3.4. Let $\varepsilon \in (0,1)$ and $\alpha > 0$ satisfy that $\alpha = o\left(n^{\varepsilon/s}\right)$. Let $\mathcal{G}$ be an (ε,α)-regular $K_{s,t}^{(r)}$-free r-uniform r-partite hypergraph with parts $V_1, \cdots, V_r$, each of size n. Without loss of generality, we may assume that $j = r$. Our goal is to show that there exists an $(\varepsilon + \log_n 4r, 4r^2\alpha)$-regular subgraph $\mathcal{H} \subseteq \mathcal{G}$ and an integer $i \in [r]\backslash\{r\}$ such that $V_i \xrightarrow[(s+1)\varepsilon]{\mathcal{H},\mathcal{G}} V_r$.

Let us first introduce some notation needed in this proof. For an s-set S and an $(r-1)$-tuple T in $\mathcal{G}$ with $S \subseteq N_{\mathcal{G}}(T)$, we define the *co-degree of S under T* as follows:

$$cd_{\mathcal{G}}(S|T) = \max\{cd_{\mathcal{G}}(S|u) : u \in T \text{ and } u \text{ is a root of } S\}.^3$$

We say the pair (S,T) is *small* if $cd_{\mathcal{G}}(S|T) < n^{r-2-\frac{1}{s-1}-(s+1)\varepsilon}$ and *large* otherwise.

We define an auxiliary function $f : E(\mathcal{G}) \to \{0, 1, \cdots, r-1\}$ on the edges of $\mathcal{G}$ as follows. For an edge $e = \{v_1, \cdots, v_r\} \in E(\mathcal{G})$ with $v_i \in V_i$, let $T_e = \{v_1, \cdots, v_{r-1}\}$ and let $\mathcal{S}(e)$ be the family consisting of all s-sets S satisfying that $v_r \in S \subseteq N_{\mathcal{G}}(T_e)$. We define $f(e) = 0$ if there are at least $(n^{1-\frac{1}{s-1}-\varepsilon}/(\alpha\log_2 n))^{s-1} = n^{s-2-(s-1)\varepsilon}/(\alpha\log_2 n)^{s-1}$ many s-sets $S \in \mathcal{S}(e)$ such that the

[3] Note that this is well-defined as there exists at least one vertex $u \in T$ which is a root of S.

pair (S, T_e) is small. Subsequently, if $f(e) \neq 0$, considering that $d_{\mathcal{G}}(T_e) \geq n^{1-\frac{1}{s-1}-\varepsilon}/\alpha$, there are $(1-o(1))d_{\mathcal{G}}(T_e)^{s-1}$ many s-sets $S \in \mathcal{S}(e)$ such that (S, T_e) is large. Note that in this case, for every s-set $S \in \mathcal{S}(e)$ with large (S, T_e), there exists a root of S, denoted as $v_k \in T_e$, satisfying $cd_{\mathcal{G}}(S|v_k) \geq n^{r-2-\frac{1}{s-1}-(s+1)\varepsilon}$. We will refer to such $S \in \mathcal{S}(e)$ as having *index* k, where $k \in \{1, \cdots, r-1\}$.[4] Let ℓ be the index such that the number of s-sets $S \in \mathcal{S}(e)$ with index ℓ is maximum among all indices in $\{1, \cdots, r-1\}$. By averaging, this number is at least $(1-o(1))d_{\mathcal{G}}(T_e)^{s-1}/(r-1) \geq d_{\mathcal{G}}(T_e)^{s-1}/r$. In this case, we define $f(e) = \ell$ and according to Definition 3.3, $(T_e; v_r)$ is $(s+1)\varepsilon$-dense on v_ℓ in $\mathcal{G}$.

Let m be the number of edges e in $\mathcal{G}$ with $f(e) = 0$. Now we demonstrate that to complete this proof, it is sufficient to show that $m = o(e(\mathcal{G}))$. Suppose indeed $m = o(e(\mathcal{G}))$. Then by averaging, there exists an integer $i \in \{1, \cdots, r-1\}$ such that there are at least $(1-o(1))e(\mathcal{G})/(r-1) \geq e(\mathcal{G})/r$ many edges $e \in E(\mathcal{G})$ with $f(e) = i$. By applying Lemma 4.2 (with the constant $c = r$), one can get an $(\varepsilon + \log_n 4r, 4r^2\alpha)$-regular subgraph $\mathcal{H}$ from these edges. Every edge e in $\mathcal{H}$ has $f(e) = i$, which also means that $(e \backslash V_r; e \cap V_r)$ is $(s+1)\varepsilon$-dense on the vertex $e \cap V_i$ in $\mathcal{G}$. Therefore, $V_i \xrightarrow[(s+1)\varepsilon]{\mathcal{H},\mathcal{G}} V_r$ holds and $\mathcal{H}$ is the desired subgraph of $\mathcal{G}$.

It remains to show that $m = o(e(\mathcal{G}))$. We count the number of pairs (S, e), where S is an s-set in V_r and $e \in E(\mathcal{G})$ such that (S, T_e) is small. By definition of $f(e) = 0$, we see that this number, denoted by M, is at least $m \cdot n^{s-2-(s-1)\varepsilon}/(\alpha \log_2 n)^{s-1}$. Now fix an s-set S_0. Recall the definition of smallness. Any $(r-1)$-tuple T such that the pair (S_0, T) is small must contain a root u of S with $cd_{\mathcal{G}}(S|u) < n^{r-2-\frac{1}{s-1}-(s+1)\varepsilon}$; call such root u *small*. Thus by (1), the number of such $(r-1)$-tuples T is at most $\sum_u cd_{\mathcal{G}}(S|u) < rtn^{r-2-\frac{1}{s-1}-(s+1)\varepsilon}$, where the summation is over all small roots u of S. There are at most n^s choices of s-sets S_0, and each small pair (S_0, T) can contribute s pairs (S_0, e) with $T \subseteq e \subseteq S_0 \cup T$ to the counting M. Thus we have

$$m \cdot n^{s-2-(s-1)\varepsilon}/(\alpha \log_2 n)^{s-1} \leq M < sn^s \cdot rtn^{r-2-\frac{1}{s-1}-(s+1)\varepsilon}.$$

Note that $(\alpha \log_2 n)^{s-1} = o\left((n^{\varepsilon/s} \log_2 n)^{s-1}\right) = o\left(n^{\varepsilon}\right)$ and $e(\mathcal{G}) \geq n^{r-\frac{1}{s-1}-\varepsilon}$. So the above inequality implies that $m \leq srt \cdot (\alpha \log_2 n)^{s-1} \cdot n^{r-\frac{1}{s-1}-2\varepsilon} = o(e(\mathcal{G}))$. We have proved Lemma 3.4. $\square$

4.3 Finding δ-dense structures: Property (II)

In this subsection, we prove Lemma 3.5. Before presenting the proof, we need to establish two technical lemmas. The first one involves some averaging statements for bipartite graphs.

Lemma 4.3. *Let $G = (A, B)$ be a bipartite graph with $e(G) \geq \rho|A||B|$ for some $\rho \in (0, 1)$.*

(1). There are at least $\rho|A|/2$ vertices $a \in A$ with $|N_G(a) \cap B| \geq \rho|B|/2$.

(2). Let s be a positive integer. If $\rho|A| \gg s$, then there are at least $(\rho|A|)^s/(3s!)$ many s-sets in A that have at least $\rho^s B/3$ common neighbors in G.

Proof. For (1), let A' be the set of vertices $a \subseteq A$ with $|N(a) \cap B| \geq \rho|B|/2$. Then we have $\rho|A||B| \leq e(G) \leq |A'||B| + (|A| - |A'|)\rho|B|/2$. This gives that $\rho|A||B|/2 \leq (1-\rho/2)|A'||B|$ and thus $|A'| \geq \rho|A|/(2-\rho) \geq \rho|A|/2$, as desired.

For (2), we construct an auxiliary bipartite graph H from G as follows. The two parts of H are B and $\mathcal{C}$, where $\mathcal{C}$ denotes the family of all s-sets in A. Let $bc \in E(H)$ if and only if $b \in B$ and $c \in \mathcal{C}$

[4] If there are multiple choices of k, we arbitrarily select one of them.

form a $K_{1,s}$ in G. In view of $e(G)=\rho|A||B|$ and $\rho|A|\gg s$, by Jensen's inequality, we have

$$e(H)=\sum_{b\in B}\binom{d_G(b)}{s}\geq |B|\binom{\rho|A|}{s}=(1-o(1))\rho^s|\mathcal{C}||B|,\ \text{where } |\mathcal{C}|=\binom{|A|}{s}.$$

By applying the conclusion (1) for H, there are at least $(1-o(1))\rho^s|\mathcal{C}|/2\geq(\rho|A|)^s/(3s!)$ many vertices $c\in\mathcal{C}$ with $|N_H(c)\cap B|\geq(1-o(1))\rho^s|B|/2\geq\rho^s|B|/3$. This completes the proof. □

The following lemma provides the crucial techniques for proving Lemma 3.5. Roughly speaking, given the assumption $V_1\xrightarrow[\delta]{\mathcal{H},\mathcal{G}}V_2$, it reveals some dense structures concerning the "adjacency" between V_2 and V_1, as well as between V_2 and any predetermined part V_j.

Lemma 4.4. *Let $\varepsilon,\delta,\alpha>0$ satisfy $6(s+1)(\varepsilon+\delta)\leq 1$ and $\alpha=o(n^{\varepsilon})$. Let $\mathcal{G}$ be an (ε,α)-regular $K_{s,t}^{(r)}$-free r-uniform balanced r-partite hypergraph on rn vertices with parts $V_1,\cdots,V_r$.*

Fix $T=\{v_1,v_3,\cdots,v_r\}\in\mathcal{T}_2(\mathcal{G})$, where $v_i\in V_i$ for $i\in[r]\backslash\{2\}$. Let X be a subset of $N_{\mathcal{G}}(T)\subseteq V_2$ such that for every vertex $v\in X$, $(T;v)$ is δ-dense on v_1 in $\mathcal{G}$. Then the following hold.

- *(1). If $|X|\gg n^{\varepsilon+\delta}$, then X contains at least $n^{-s\varepsilon-s\delta}|X|^s/(3s!r^s)$ different s-sets rooted on v_1.*
- *(2). Suppose $X\neq\emptyset$. Then for any given $j\in[r]\backslash\{1,2\}$, there exist subsets $Y\subseteq N_{\mathcal{G}}(T)$, $Z\subseteq V_j$, and an $(r-3)$-tuple $R\subseteq V(\mathcal{G})\backslash(V_1\cup V_2\cup V_j)$ such that $|Y|\geq n^{1-\frac{1}{s-1}-2\varepsilon-\delta}/2r\alpha$, $n^{1-\frac{1}{s-1}-\delta}\leq|Z|\leq n^{1-\frac{1}{s-1}+\varepsilon}$, and for any $y\in Y$, $|N_{\mathcal{G}}(\{v_1,y\}\cup R)\cap Z|\geq n^{1-\frac{1}{s-1}-\varepsilon-2\delta}/2$.*

Proof. We fix such an $(r-1)$-tuple T. For each $v\in N_{\mathcal{G}}(T)$, let $\mathcal{A}(v)$ be the set of s-sets S such that $v\in S\subseteq N_{\mathcal{G}}(T)$ and $cd_{\mathcal{G}}(S|v_1)\geq n^{r-2-\frac{1}{s-1}-\delta}$. If $v\in X$, then $(T;v)$ is δ-dense on v_1 in $\mathcal{G}$ and thus $|\mathcal{A}(v)|\geq d_{\mathcal{G}}(T)^{s-1}/r$. Let $A(v)=\bigcup_{S\in\mathcal{A}(v)}S$. Then $A(v)\subseteq N_{\mathcal{G}}(T)$, and as clearly $\binom{|A(v)|}{s-1}\geq|\mathcal{A}(v)|$, we have $|A(v)|\geq|\mathcal{A}(v)|^{1/(s-1)}\geq d_{\mathcal{G}}(T)/r$. Note that for each $u\in A(v)$, there exists some $S\in\mathcal{A}(v)$ with $\{u,v\}\subseteq S$. This implies that for each $u\in A(v)$,

$$|N_{\mathcal{G}}(\{u,v_1\})\cap N_{\mathcal{G}}(\{v,v_1\})|\geq cd_{\mathcal{G}}(S|v_1)\geq n^{r-2-\frac{1}{s-1}-\delta}. \tag{5}$$

We first consider the conclusion (1). We have seen that for all $v\in X$, $A(v)\subseteq N_{\mathcal{G}}(T)$ has size at least $d_{\mathcal{G}}(T)/r$. By averaging, there exists $u_0\in N_{\mathcal{G}}(T)$ and a subset $X'\subseteq X$ with $|X'|\geq|X|/r$ such that $u_0\in A(v)$ for every $v\in X'$. Let $\mathcal{B}=N_{\mathcal{G}}(\{u_0,v_1\})$ be a subset of $(r-2)$-tuples. Then we have

$$n^{r-2-\frac{1}{s-1}-\delta}\leq|\mathcal{B}|\leq n^{r-2-\frac{1}{s-1}+\varepsilon},$$

where the first inequality holds by (5) and the second inequality is given by Lemma 4.1. We now define a bipartite graph $H=(X',\mathcal{B})$ as follows. For $v\in X'$ and $B\in\mathcal{B}$, we define $vB\in E(H)$ if and only if $\{v,v_1\}\cup B\in E(\mathcal{G})$. Note that it means $N_H(v)=\mathcal{B}\cap N_{\mathcal{G}}(\{v,v_1\})$ for $v\in X'$. By (5), we see that each $v\in X'$ has degree at least $n^{r-2-\frac{1}{s-1}-\delta}$ in H. Consequently, $e(H)\geq|X'|n^{r-2-\frac{1}{s-1}-\delta}\geq n^{-\varepsilon-\delta}|X'||\mathcal{B}|$. Since $|X'|\geq|X|/r\gg n^{\varepsilon+\delta}$, applying Lemma 4.3 (2), there are at least $(n^{-\varepsilon-\delta}|X'|)^s/(3s!)\geq n^{-s\varepsilon-s\delta}|X|^s/(3s!r^s)$ many s-sets $S\subseteq X'\subseteq X$ such that the common neighbor of S in H is at least $n^{-s\varepsilon-s\delta}|\mathcal{B}|/3$. In other words,

$$cd_{\mathcal{G}}(S|v_1)\geq n^{-s\varepsilon-s\delta}|\mathcal{B}|/3\geq n^{r-2-\frac{1}{s-1}-s\varepsilon-(s+1)\delta}/3. \tag{6}$$

It suffices to show that v_1 indeed is a root of such S if $d_{\mathcal{G}}(S|v_1) \geq n^{r-2-\frac{1}{s-1}-s\varepsilon-(s+1)\delta}/3$. Write $S = \{w_1, \cdots, w_s\}$ and let R be the set of all roots of S. We know $|R| \leq rt$ from (1). If $v_1 \notin R$, then we can obtain

$$cd_{\mathcal{G}}(S|v_1) \leq \sum_{x \in R} d_{\mathcal{G}}(\{w_1, v_1, x\}) \leq |R| n^{r-3-\frac{1}{s-1}+\varepsilon} \leq rt \cdot n^{r-3-\frac{1}{s-1}+\varepsilon} < n^{r-2-\frac{1}{s-1}-s\varepsilon-(s+1)\delta}/3,$$

where the first inequality holds because every set in $\mathrm{CN}_{\mathcal{G}}(S)$ containing v_1 must also contain w_1 and a root in R, the second inequality follows from Lemma 4.1, and the last inequality holds because $6(s+1)(\varepsilon+\delta) \leq 1$. This contradicts (6). We have finished the proof for the first conclusion.

Next we prove the second conclusion. Without loss of generality, we assume $j = 3$. Fix a vertex $v \in X \subseteq V_2$. We define a bipartite graph $H = (A, \mathcal{B})$ similarly as above, where $A := A(v)$, and $\mathcal{B} := N_{\mathcal{G}}(\{v, v_1\})$. Let $a \in A$ and $B \in \mathcal{B}$ form an edge in H if and only if $\{v_1, a\} \cup B \in E(\mathcal{G})$. For each $a \in A$, we have $N_H(a) = \mathcal{B} \cap N_{\mathcal{G}}(\{v_1, a\})$, and by (5), each a has at least $n^{r-2-\frac{1}{s-1}-\delta}$ neighbors in $\mathcal{B}$. So $e(H) \geq |A| n^{r-2-\frac{1}{s-1}-\delta}$. Let $\mathcal{R}$ be the set of all $(r-3)$-tuples $R \subseteq V_4 \cup \cdots \cup V_r$ of $\mathcal{G}$. For $R \in \mathcal{R}$, define $\mathcal{B}_R$ as the set of $(r-2)$-tuples in $\mathcal{B}$ containing R. So $\mathcal{B}$ has a partition $\cup_{R \in \mathcal{R}} \mathcal{B}_R$. As $|\mathcal{R}| \leq n^{r-3}$, by averaging there exists some $R^* \in \mathcal{R}$ such that $e_H(A, \mathcal{B}_{R^*}) \geq e(H)/n^{r-3} \geq |A| n^{1-\frac{1}{s-1}-\delta}$. Let $Z := N_{\mathcal{G}}(\{v, v_1\} \cup R^*) \subseteq V_3$. Clearly there is a bijection between Z and $\mathcal{B}_{R^*} = \{\{z\} \cup R^* | z \in Z\}$. So we can identify the bipartite subgraph $(A, \mathcal{B}_{R^*})$ of H as $H' = (A, Z)$, where H' is defined such that $az \in E(H')$ if and only if $\{v_1, a, z\} \cup R^* \in E(\mathcal{G})$ for $a \in A$ and $z \in Z$. Then we have

$$n^{1-\frac{1}{s-1}+\varepsilon} \geq |Z| \geq e(H')/|A| = e_H(A, \mathcal{B}_{R^*})/|A| \geq n^{1-\frac{1}{s-1}-\delta},$$

where the first inequality holds due to Lemma 4.1. This further shows that $e(H') \geq |A| n^{1-\frac{1}{s-1}-\delta} \geq n^{-\varepsilon-\delta}|A||Z|$. Recalling that $|A| = |A(v)| \geq d_{\mathcal{G}}(T)/r$, and combining it with $d_{\mathcal{G}}(T) \geq n^{1-\frac{1}{s-1}-\varepsilon}/\alpha$ and the given restrictions on $\varepsilon, \delta, \alpha$, we have $n^{-\varepsilon-\delta}|A| \geq n^{1-\frac{1}{s-1}-2\varepsilon-\delta}/(\alpha r) \gg s$. Applying Lemma 4.3 (1) to $H' = (A, Z)$, there exists a subset $Y \subseteq A \subseteq N_{\mathcal{G}}(T)$ of size

$$|Y| \geq n^{-\varepsilon-\delta}|A|/2 \geq n^{-\varepsilon-\delta} d_{\mathcal{G}}(T)/2r \geq n^{1-\frac{1}{s-1}-2\varepsilon-\delta}/2r\alpha$$

such that every $y \in Y$ has $|N_{\mathcal{G}}(\{v_1, y\} \cup R^*) \cap Z| = |N_{H'}(y) \cap Z| \geq n^{-\varepsilon-\delta}|Z|/2 \geq n^{1-\frac{1}{s-1}-\varepsilon-2\delta}/2$. $\square$

Finally, we are ready to show Lemma 3.5.

Proof of Lemma 3.5. Let $\varepsilon, \delta, \alpha$ be constants and $\mathcal{H}_1 \subseteq \mathcal{H} \subseteq \mathcal{G}_1 \subseteq \mathcal{G}$ be the sequence of (ε, α)-regular $K_{s,t}^{(r)}$-free hypergraphs given by the statement. Without loss of generality, we may assume that $V_1 \xrightarrow[\delta]{\mathcal{H}_1, \mathcal{H}} V_2 \xrightarrow[\delta]{\mathcal{G}_1, \mathcal{G}} V_k$ holds for some $k \neq 2$. We aim to show that $k = 1$.

Suppose for a contradiction that $k \notin \{1, 2\}$. Let $T \in \mathcal{T}_2(\mathcal{H}_1)$ be an $(r-1)$-tuple with $T \cap V_1 = \{v_1\}$. Since $V_1 \xrightarrow[\delta]{\mathcal{H}_1, \mathcal{H}} V_2$, for any $x \in N_{\mathcal{H}_1}(T)$, $(T; x)$ is δ-dense on v_1 in $\mathcal{H}$. Using Lemma 4.4 (2), there exist subsets $Y \subseteq N_{\mathcal{H}}(T) \subseteq V_2$, $Z \subseteq V_k$ and an $(r-3)$-tuple $R \subseteq V(\mathcal{G}) \backslash (V_1 \cup V_2 \cup V_k)$ such that

- $|Y| \geq n^{1-\frac{1}{s-1}-2\varepsilon-\delta}/2r\alpha$,
- $n^{1-\frac{1}{s-1}-\delta} \leq |Z| \leq n^{1-\frac{1}{s-1}+\varepsilon}$, and
- for each $y_j \in Y$, if we let $T_j = \{v_1, y_j\} \cup R$ and $X_j = N_{\mathcal{H}}(T_j) \cap Z$, then $|X_j| \geq n^{1-\frac{1}{s-1}-\varepsilon-2\delta}/2$.

Let us count the number m of pairs (S,y) such that $y \in Y$ is a root of an s-set $S \subseteq Z$ in $\mathcal{G}$. Since Z has at most $|Z|^s$ different s-sets, and each s-set has at most rt root in $\mathcal{G}$, we have $m \le rt \cdot |Z|^s$. Now consider a fixed vertex $y_j \in Y$ (so $y_j \in V_2$). As $\mathcal{H} \subseteq \mathcal{G}_1$, we see $T_j \in \mathcal{T}_k(\mathcal{H}) \subseteq \mathcal{T}_k(\mathcal{G}_1)$. Since $V_2 \xrightarrow[\delta]{\mathcal{G}_1,\mathcal{G}} V_k$, by Definition 3.3, every $x \in X_j \subseteq N_{\mathcal{H}}(T_j) \subseteq N_{\mathcal{G}_1}(T_j)$ satisfies that $(T_j;x)$ is δ-dense on y_j in $\mathcal{G}$. Since $|X_j| \ge n^{1-\frac{1}{s-1}-\varepsilon-2\delta}/2 \gg n^{\varepsilon+\delta}$, using Lemma 4.4 (1), we can derive that $X_j \subseteq Z$ contains at least $n^{-s\varepsilon-s\delta}|X_j|^s/(3s!r^s)$ different s-sets rooted on y_j in $\mathcal{G}$. This shows that $m \ge \sum_{y_j\in Y} n^{-s\varepsilon-s\delta}|X_j|^s/(3s!r^s)$. Putting everything together, we get

$$rt \cdot \left(n^{1-\frac{1}{s-1}+\varepsilon}\right)^s \ge rt\cdot|Z|^s \ge m \ge \frac{1}{2r\alpha\cdot 2^s\cdot 3s!r^s}\cdot n^{1-\frac{1}{s-1}-2\varepsilon-\delta}\cdot n^{-s\varepsilon-s\delta}\cdot\left(n^{1-\frac{1}{s-1}-\varepsilon-2\delta}\right)^s.$$

Since $\alpha = o(n^\varepsilon)$, this implies

$$(3s+3)\varepsilon + (3s+1)\delta \ge 1 - 1/(s-1) + o(1), \text{ where } o(1)\to 0 \text{ as } n\to\infty. \tag{7}$$

Since $s \ge 3$ and $6(s+1)(\varepsilon+\delta) \le 1$, we have $(3s+3)\varepsilon+(3s+1)\delta < 3(s+1)(\varepsilon+\delta) \le 1/2 \le 1-1/(s-1)$, which contradicts (7) as n is sufficiently large. The proof of this lemma is now complete. $\square$

5 Concluding remarks

In this paper, we prove that for any odd $r \ge 3$ and any $s \ge 3$, there exists an $\varepsilon_s > 0$ such that

$$\mathrm{ex}(n, K_{s,t}^{(r)}) = O_{r,s,t}\left(n^{r-\frac{1}{s-1}-\varepsilon}\right).$$

It would be interesting to determine the optimal constant ε_s for any odd $r \ge 3$. It is also worth noting that Mubayi and Verstraëte [12] conjectured that the Turán number for 3-uniform hypergraphs satisfies $\mathrm{ex}(n, K_{s,t}^{(3)}) = \Theta_{s,t}\left(n^{3-\frac{2}{s}}\right)$ for any $t \ge s \ge 2$, which is still open for $s \ge 3$.

As briefly discussed in Subsection 2.2, the proofs of Lemmas 3.4 and 3.5 yield certain rich adjacency structures in dense $K_{s,t}^{(r)}$-free r-uniform hypergraphs for even $r \ge 4$. There structures also align with the construction provided in Section 3 of [2]. These observations suggest that perhaps there exists a stability result for $K_{s,t}^{(r)}$ for even $r \ge 4$.

The asymptotics of $f_r(n)$ and $\mathrm{ex}(n, K_{2,2}^{(r)})$ remain intriguing open problems. We conclude this paper by mentioning two related conjectures. The first conjecture due to Füredi [7] states that $\mathrm{ex}(n, K_{2,2}^{(r)}) = (1+o(1))\binom{n-1}{r-1}$ for any $r \ge 3$. Note that the lower bound $\binom{n-1}{r-1}$ can be achieved by the hypergraph star. Despise significant progress made in [12, 13], this conjecture remains open for any $r \ge 3$. The second conjecture, posed by Mubayi (see Conjecture 6.2 in [11]), suggests that the $f_r(n)$-problem is *stable* for $r \ge 4$. This says that for any $r \ge 4$ and $\delta > 0$, there exist $\epsilon > 0$ and n_0 such that any n-vertex r-uniform hypergraph with at least $(1-\epsilon)\binom{n}{r-1}$ edges, which does not contain four distinct edges A, B, C, D satisfying $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$, must contain a vertex v belonging to at least $(1-\delta)\binom{n}{r-1}$ edges.

Acknowledgment. The authors would like to thank Prof. Hao Huang for helpful discussions.

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