Minimum Cost Nowhere-zero Flows and Cut-balanced Orientations

Karthekeyan Chandrasekaran* Siyue Liu† R. Ravi‡

Abstract

Flows and colorings are disparate concepts in graph algorithms—the former is tractable while the latter is intractable. Tutte [43, 44] introduced the concept of nowhere-zero flows to unify these two concepts. Jaeger [23] showed that nowhere-zero flows are equivalent to cut-balanced orientations. Motivated by connections between nowhere-zero flows, cut-balanced orientations, Nash-Williams' well-balanced orientations, and postman problems, we study optimization versions of nowhere-zero flows and cut-balanced orientations. Given a bidirected graph with asymmetric costs on two orientations of each edge, we study the min cost nowhere-zero k-flow problem and min cost k-cut-balanced orientation problem. We show that both problems are NP-hard to approximate within any finite factor. Given the strong inapproximability result, we design bicriteria approximations for both problems: we obtain a $(6, 6)$-approximation to the min cost nowhere-zero k-flow and a $(k, 6)$-approximation to the min cost k-cut-balanced orientation. For the case of symmetric costs (where the costs of both orientations are the same for every edge), we show that the nowhere-zero k-flow problem remains NP-hard and admits a 3-approximation.

*Grainger College of Engineering, University of Illinois, Urbana-Champaign. Email: karthe@illinois.edu. Supported in part by NSF grant CCF-2402667.

†Tepper School of Business, Carnegie Mellon University. Email: siyueliu@andrew.cmu.edu. This material is based upon work supported in part by the Air Force Office of Scientific Research under award number FA9550-23-1-0031.

‡Tepper School of Business, Carnegie Mellon University. Email: ravi@andrew.cmu.edu. This material is based upon work supported in part by the Air Force Office of Scientific Research under award number FA9550-23-1-0031.

1 Introduction

Flows and colorings are disparate concepts in graph theory, especially from a computational perspective—flow problems are typically tractable while coloring problems are typically intractable. However, these two concepts are related in planar graphs via planar duality. Inspired by this connection, Tutte [43, 44] introduced nowhere-zero flows to unify flows and colorings in arbitrary graphs. This unified viewpoint has driven several fundamental results in connectivity and orientations (e.g., see [46] and [5]). Existence of nowhere zero flows is characterized by the existence of cut-balanced orientations [23]. Motivated by these connections, we investigate optimization problems associated with nowhere-zero flows and cut-balanced orientations.

We begin by defining nowhere-zero flows, cut-balanced orientations, and the optimization problems of interest. In an undirected graph $G = (V, E)$, for a subset of vertices $U \subseteq V$, we denote the set of edges with exactly one end-vertex in U by $\delta_G(U)$. In a directed graph $D = (V, A)$, for $U \subseteq V$, we denote the arcs leaving and entering U by $\delta^+_D(U)$ and $\delta^-_D(U)$ respectively. For a subset of edges $F \subseteq E$, we denote $\delta_F(U) := \delta_G(U) \cap F$. Similarly, for a subset of arcs $B \subseteq A$, we define $\delta^{\pm}_B(U) := \delta^{\pm}_D(U) \cap B$. Let $G = (V, E)$ be an undirected graph and $k \geq 2$ be an integer. A *nowhere-zero k-flow* in G is a tuple $(\vec{E}, f)$, where $\vec{E}$ is an orientation of E and $f : \vec{E} \to \{1, 2, ..., k-1\}$ is a function that satisfies flow conservation, i.e., $f(\delta^+_{\vec{E}}(v)) = f(\delta^-_{\vec{E}}(v))$ for every $v \in V$. A *nowhere-zero flow* is a nowhere-zero k-flow for some finite integer $k \geq 2$. An orientation $\vec{E}$ of E *induces* a nowhere-zero k-flow if there is a function $f : \vec{E} \to \{1, 2, ..., k-1\}$ such that $(\vec{E}, f)$ is a nowhere-zero k-flow. An orientation $\vec{E}$ of E is *k-cut-balanced* if $|\delta^+_{\vec{E}}(U)| \geq \frac{1}{k}|\delta_E(U)|\ \forall U \subseteq V$. A *cut-balanced orientation* is a k-cut-balanced orientation for some finite integer $k \geq 2$. The *bidirected graph* of $G = (V, E)$, denoted $\vec{G} = (V, E^+ \cup E^-)$, is obtained by making two copies $e^+ \in E^+$ and $e^- \in E^-$ of every edge $e \in E$ and orienting them in opposite directions. Let $c : E^+ \cup E^- \to \mathbb{Z}_{\geq 0}$ be costs on the arcs of the bidirected graph $\vec{G}$. The cost of an orientation $\vec{E}$ is denoted by $c(\vec{E}) := \sum_{e \in \vec{E}} c(e)$ and the cost of a nowhere-zero flow $(\vec{E}, f)$ is denoted by $c(f) := \sum_{e \in \vec{E}} c(e)f(e)$. We consider the following problems for a fixed integer $k \geq 2$:

Weighted nowhere-zero k-flow (WNZF(k)):
Given: Undirected 2-edge-connected graph $G = (V, E)$ and costs $c : E^+ \cup E^-$
Goal: Find a nowhere-zero k-flow $(\vec{E}, f)$ with minimum cost $c(f)$.

Weighted k-cut-balanced orientations (WCBO(k)):
Given: Undirected 2-edge-connected graph $G = (V, E)$ and costs $c : E^+ \cup E^-$
Goal: Find a k-cut-balanced orientation $\vec{E}$ with minimum cost $c(\vec{E})$.

The *feasibility versions* of WNZF(k) and WCBO(k) are the problems of determining whether a given undirected 2-edge-connected graph has a nowhere-zero k-flow and a k-cut-balanced orientation, respectively. An orientation induces a nowhere-zero k-flow if and only if it is k-cut-balanced [23, 16, 17, 41]. Hence, the feasibility versions of WNZF(k) and WCBO(k) are equivalent, and we will discuss their complexity shortly. We say that the cost function $c : E^+ \cup E^-$ is *symmetric* if $c(e^+) = c(e^-)$ for every $e \in E$ and *asymmetric* otherwise. If the costs c are symmetric, WCBO(k) is equivalent to the feasibility problem while WNZF(k) is non-trivial. We denote WNZF(k) for symmetric costs as SWNZF(k). We define WNZF(∞) as the problem of finding a min-cost nowhere-zero flow with no restriction on k, i.e.,

$$\text{WNZF}(\infty) := \min\{c(f) : \exists\, \vec{E} \text{ such that } (\vec{E}, f) \text{ is a nowhere-zero } k\text{-flow for some integer } k \geq 2\}.$$

We define WCBO(∞) and SWNZF(∞) analogously. In all optimization problems above, we assume that the input graph is 2-edge-connected since this is a necessary condition for the existence of a nowhere-zero flow/cut-balanced orientation.

1.1 Background, Motivations, and Connections

Nowhere-zero Flows. Nowhere-zero flow is a rich area of study in graph theory, with close connections to graph coloring and chromatic polynomials. We begin by observing that 2-edge-connectivity is necessary and sufficient for the existence of a nowhere-zero flow.[1] Moreover, for integers $k_2 \geq k_1 \geq 2$, a nowhere-zero k_1-flow is also a nowhere-zero k_2-flow. Given this, most works on nowhere-zero flows focused on the least integer k for which every 2-edge-connected graph admits a nowhere-zero k-flow. Tutte [43, 44] conjectured that every 2-edge-connected graph has a nowhere-zero 5-flow—we will call this Tutte's 5-flow conjecture. The Petersen graph does not have a nowhere-zero 4-flow, showing that 5 is best possible. Jaeger [22, 24] proved that every 2-edge-connected graph has a nowhere-zero 8-flow. Seymour [39] improved Jaeger's result by showing that every 2-edge-connected graph has a nowhere-zero 6-flow. Seymour's proof is non-constructive and a subsequent algorithmic proof was given by Younger [48]. DeVos and Nurse [7] gave a shorter proof of Seymour's result which can also be made constructive.

We now discuss the feasibility variants of WNZF(k) and WCBO(k). A consequence of Seymour's result is that for every integer $k \geq 6$, all instances of both WNZF(k) and WCBO(k) are feasible. We now discuss the status for integers $k \in \{2, 3, 4, 5\}$. An undirected graph admits a nowhere-zero 2-flow if and only if it is Eulerian; we recall that Eulerian property is verifiable in polynomial time and hence, the feasibility versions of WNZF(2) and WCBO(2) are solvable in polynomial time. A planar graph admits a nowhere-zero k-flow if and only if its dual is k-vertex-colorable [25]; deciding 3-vertex-colorability of a planar graph is NP-complete [15] and consequently, feasibility variants of WNZF(3) and WCBO(3) are NP-complete (even in planar graphs) [28]. Tutte [43] showed that a simple cubic graph has a nowhere-zero 4-flow if and only if it is 3-edge-colorable. Deciding 3-edge-colorability of a given simple cubic graph is NP-complete [20] and hence, feasibility variants of WNZF(4) and WCBO(4) are also NP-complete (see also [24] and Section 7.3 of [46]). Interestingly, Kochol [28] proved that if Tutte's 5-flow conjecture is false, then deciding whether a cubic graph has a nowhere-zero 5-flow is NP-complete. In fact, for $k = 3, 4, 5$, NP-completeness of the feasibility variants implies no finite approximations for corresponding optimization problems WNZF(k) and WCBO(k). This is because a finite approximation algorithm for the instance with all costs set to 0 would also certify feasibility. We summarize these facts in the first two columns of Table 1.

From the discussion above, we have that the feasibility variants of WNZF(2) and WCBO(2) are polynomial-time solvable and for $k \geq 6$, all instances of WNZF(k) and WCBO(k) are feasible. Given this status, it is natural to ask whether the optimization variants WNZF(k) and WCBO(k) are solvable in polynomial time for $k = 2$ and $k \geq 6$. We observe that WNZF(2) and WCBO(2) are equivalent since a nowhere-zero 2-flow $(\vec{E}, f)$ has $f(e) = 1\ \forall e \in \vec{E}$, and thus the cost of the flow is equal to the cost of the orientation. In particular, WNZF(2) and WCBO(2) are equivalent to the min cost Eulerian orientation problem, which can be solved in polynomial time [47]. We emphasize that these connections already highlight the discrete nature of WNZF(k) in contrast to the classic min cost flow problem which is inherently a continuous optimization problem.

Well-balanced Orientations. The notion of k-cut-balanced orientation is closely related to that of well-balanced orientation, a fundamental notion in connectivity-preserving orientation problems. Let $G = (V, E)$ be an undirected graph. An orientation $\vec{E}$ is *well-balanced* if $\lambda_{\vec{E}}(u, v) \geq \lfloor \lambda_E(u, v)/2 \rfloor$ for every $u, v \in V$, where $\lambda_E(u, v) := \min_{u \in U \subseteq V - v} |\delta_E(U)|$ and $\lambda_{\vec{E}}(u, v) := \min_{u \in U \subseteq V - v} |\delta^+_{\vec{E}}(U)|$. Nash-Williams [32] showed that every 2-edge-connected graph admits a well-balanced orientation—this result is known as the strong orientation theorem in the literature. The strong orientation theorem guarantees a single orientation that

[1] The necessity of 2-edge-connectivity for nowhere-zero flows follows from flow-conservation and nowhere-zero property. We sketch its sufficiency: we recall that every 2-edge-connected graph $G = (V, E)$ admits a strongly connected orientation $\vec{E}$, i.e., an orientation $\vec{E}$ such that $|\delta^+_{\vec{E}}(U)| \geq 1$ for every $\emptyset \neq U \subsetneq V$; such an orientation is a k-cut-balanced orientation for some sufficiently large integer k; now, recall that an orientation is a k-cut-balanced orientation if and only if it induces a nowhere-zero k-flow [23].

halves the pairwise connectivity of all pairs. In contrast, Nash-Williams' weak orientation theorem is about global connectivity: it states that every $2k$-edge-connected graph admits a k-arc-connected orientation. The weak orientation theorem can be derived from the strong orientation theorem. Nash-Williams' proof of the strong orientation theorem is rather involved. Subsequent works [30, 11] have given alternative proofs, but they are still complex. In fact, discovering a simpler proof of Nash-Williams' strong orientation theorem has been a long-standing question in graph theory and combinatorial optimization (see Section 9.8 in [12]).

As a motivating question for this quest for simplification, Frank [11] posed an optimization variant of Nash-Williams' strong orientation theorem, namely the min-cost well-balanced orientation problem: given an undirected graph with costs on both orientations of each edge, find a min cost well-balanced orientation. Bernáth, Iwata, Király, Király, and Szigeti [1] proved that this problem is NP-hard. Further, Bernáth and Joret [2] proved that deciding if a mixed graph has a well-balanced orientation is also NP-hard. These results rule out the possibility of approximating the min cost well-balanced orientation within any finite factor. On the other hand, the optimization variant of Nash-Williams' weak orientation theorem—namely, given an undirected $2k$-edge-connected graph with costs on both orientations of each edge, finding a min-cost k-arc-connected orientation—is solvable in polynomial time via Edmonds-Giles' theory of submodular flows [8].

We observe that k-cut-balanced orientations are closely related to well-balanced orientations: if $\vec{E}$ is a k-cut-balanced orientation, then for every pair $(u, v) \in V$, we have that

$$\lambda_{\vec{E}}(u, v) = \min_{u \in U \subseteq V - v} |\delta^+_{\vec{E}}(U)| \geq \min_{u \in U \subseteq V - v} \frac{1}{k}|\delta_E(U)| = \frac{1}{k}\lambda_E(u, v).$$

Thus, a k-cut-balanced orientation can be viewed as an "approximately" well-balanced orientation with the approximation factor being $k/2$. Since there exists an efficient algorithm to find a 6-cut-balanced orientation in every 2-edge-connected graph, it follows that there exists an efficient algorithm to find a "3-approximately" well-balanced orientation. More generally, a k-cut-balanced orientation with small cost is a "$k/2$-approximately" well-balanced orientation with small cost, thus relating WCBO(k) to a bicriteria version of Frank's problem. This connection was one of the motivations for us to study the optimization version of k-cut-balanced orientation problem, namely WCBO(k).

Orientation and Postman problems. We observe that WCBO(∞) is polynomial-time solvable. In fact, it is equivalent to the problem of finding a min cost strongly connected orientation: an orientation $\vec{E}$ is k-cut-balanced for some finite integer $k \geq 2$ if and only if for every $\emptyset \neq U \subsetneq V$, we have that $|\delta^+_{\vec{E}}(U)| \geq \frac{1}{k}|\delta_E(U)|$, i.e., $|\delta^+_{\vec{E}}(U)| \geq \left\lceil \frac{1}{k}|\delta_E(U)| \right\rceil$, i.e., $|\delta^+_{\vec{E}}(U)| \geq 1$. Min cost strongly connected orientation can be solved in polynomial time (via [29, 8]).

In WNZF(∞), we seek a nowhere-zero flow $(\vec{E}, f)$ which is equivalent to an integer-valued flow $f : E^+ \cup E^- \to \mathbb{Z}_{\geq 0}$ such that the flow is positive in *exactly* one of e^+ and e^- for every edge $e \in E$; instead, if we seek an integer-valued flow $f : E^+ \cup E^- \to \mathbb{Z}_{\geq 0}$ such that *at least* one of e^+ and e^- has positive flow value for every $e \in E$, then the resulting problem is the *asymmetric postman problem* [31]: find a min cost directed Eulerian tour of the graph G that traverses every edge $e \in E$ at least once. In contrast to the symmetric postman problem which can be solved in polynomial time [9], the asymmetric postman problem is already NP-hard but admits a 3/2-approximation [47, 35]. A well-studied special case of the asymmetric postman problem is the *mixed postman* problem, where we are given a mixed graph (consisting of undirected edges and directed arcs) with costs on the arcs and *symmetric* costs on the edges and the goal is to find a min cost Eulerian tour that traverses every arc and edge at least once. This is also NP-hard [33] and several works have focused on its approximability [9, 4, 14, 34]—the current best approximation ratio is $\frac{3}{2}$ [34]. The mixed postman problem with the restriction that each arc/edge can be traversed at most once has also been studied. The arc-restricted problem is NP-hard and admits a 4/3-approximation [49], while the edge-restricted problem is NP-hard to approximate within any finite factor [49, 45, 50].

WNZF(∞) can be viewed as the asymmetric postman problem with *orientation constraints*: i.e., flow is allowed to be positive on exactly one of the two orientations of each edge. We will subsequently take this viewpoint while designing LP-based approximation algorithms for WNZF(k). We remark that WNZF(∞), i.e., postman problem with orientation constraints, is different from the (mixed) postman problem with the restriction that each edge can be traversed at most once—the latter requires the flow value to be exactly 1 on one orientation of each edge and 0 on the other while WNZF(∞) requires the flow value to be positive on exactly one orientation of each edge. Orientation constraints requiring flow value to be positive on exactly one orientation of each edge arise naturally in network design problems since they model one-way roads. We mention that several prior works have considered orientation constraints in directed network design problems and most of them have concluded that orientation constraints tend to make the problem much harder (e.g., see [13, 27, 40, 6]).

1.2 Our Results

We first show hardness and inapproximability of WNZF(k) and WCBO(k).

Theorem 1. *For every finite integer $k \geq 3$,* WNZF(k) *and* WCBO(k) *are NP-hard to approximate within any finite factor.*

Theorem 1 rules out the possibility of approximation algorithms for WNZF(k) and WCBO(k). Given this status, we investigate bicriteria approximation algorithms for both these problems. We say that an algorithm is an (α, β)*-approximation* for WNZF(k) (resp. WCBO(k)) if the algorithm returns a nowhere-zero βk-flow (resp. βk-cut-balanced orientation) with cost at most $\alpha c(\mathrm{OPT})$ where $c(\mathrm{OPT})$ is the minimum cost of a nowhere-zero k-flow (resp. k-cut-balanced orientation). We show the following bicriteria approximation for WNZF(k).

Theorem 2. *For every finite integer $k \geq 6$ and for $k = \infty$,* WNZF(k) *admits a $(6, 6)$-approximation.*

We observe that Theorem 2 implies a 6-approximation for WNZF(∞) as stated below.

Corollary 3. WNZF(∞) *admits a 6-approximation.*

Next, we turn to bicriteria approximation for WCBO(k). We recall that WCBO(∞) is equivalent to the min cost strongly connected orientation problem which is solvable in polynomial time [48, 8]. Hence, we focus on WCBO(k) for finite integers $k \geq 3$. Moreover, Theorem 1 implies that WCBO(k) is inapproximable for every finite integer $k \geq 3$. We complement these results by showing the following bicriteria approximation.

Theorem 4. *For every integer $k \geq 6$,* WCBO(k) *admits a $(k, 6)$-approximation.*

In most applications of nowhere-zero k-flows and k-cut-balanced orientations, we note that k is a small constant. In particular, Theorems 2 and 4 imply $(6, 6)$-approximation algorithms for min cost nowhere-zero 6-flow and min cost 6-cut-balanced orientation respectively.

Next, we turn to the symmetric cost variants of the problems. As mentioned earlier, the symmetric cost variant of WCBO(k) is equivalent to the feasibility variant, so we focus only on the symmetric cost variant of WNZF(k). We recall that SWNZF(∞) is closely related to the symmetric postman problem: it can be viewed as an orientation constrained min-cost symmetric postman problem. While the min-cost symmetric postman problem is polynomial-time solvable, we show that SWNZF(∞) is NP-hard even for unit costs. We show the following hardness and inapproximability results for SWNZF(k).

Theorem 5. *For every finite integer $k \geq 3$ and for $k = \infty$,* SWNZF(k) *for unit costs is NP-hard. Moreover, for $k = 3$ and $k = 4$,* SWNZF(k) *is NP-hard to approximate within any finite factor and* SWNZF(5) *is NP-hard to approximate within any finite factor assuming Tutte's 5-flow conjecture is false.*

Motivated by the hardness results in Theorem 5, we turn to approximation algorithms for SWNZF(k). If the costs are symmetric, we observe that every nowhere-zero 6-flow f is a 5-approximation for SWNZF(k) for every finite integer $k \geq 6$ and for $k = \infty$: this is because the flow value of f on each edge is at most 5, while the min cost nowhere-zero k-flow has to send at least one unit flow on each edge. We improve on this naive 5-approximation for SWNZF(k) to achieve an approximation factor of 3. We refer the reader to Table 1 for a summary of our results.

Theorem 6. *For every finite integer $k \geq 6$ and for $k = \infty$,* SWNZF(k) *admits a* 3*-approximation.*

Integer k	**Feasibility** **NZF**(k) & **CBO**(k)	**WNZF**(k)	**WCBO**(k)	**SWNZF**(k)
2	Poly-time [folklore]	Poly-time [47]	Poly-time [47]	Poly-time [47]
3, 4	NP-complete [25, 43, 20]	No finite approx	No finite approx	No finite approx
5 (assuming Tutte's 5-flow conjecture is false)	NP-complete [28]	No finite approx	No finite approx	No finite approx
finite $k \geq 6$	Always feasible [39, 48]	No finite approx (Theorem 1) $(6, 6)$-approx (Theorem 2)	No finite approx (Theorem 1) $(k, 6)$-approx (Theorem 4)	NP-hard for unit costs (Theorem 5) 3-approx (Theorem 6)
∞	Always feasible [39, 48]	6-approx (Corollary 3)	Poly-time [29, 8]	NP-hard for unit costs (Theorem 5) 3-approx (Theorem 6)

Table 1: Hardness and approximation algorithms for WNZF(k), WCBO(k) and SWNZF(k). Feasibility NZF(k) and CBO(k) refer to the feasibility versions of WNZF(k) and WCBO(k) respectively. All inapproximability results are assuming $P \neq NP$.

1.3 Techniques

We observe that the equivalence of nowhere-zero k-flows and k-cut-balanced orientations [23] implies a reduction between WNZF(k) and WCBO(k) with a small loss in approximation factor–see Lemma 9. A consequence of this reduction is that NP-hardness of approximating WNZF(k) within any finite factor implies the same for WCBO(k) and vice-versa. With this consequence, we observe that proving Theorem 1 reduces to showing that WCBO(k) does not admit a finite approximation. We prove the latter by a reduction from the satisfiability problem. We prove the hardness result for SWNZF(k) mentioned in Theorem 5 by a reduction from Not-All-Equal-3-SAT. Our reductions are based on careful choice of gadgets.

We now discuss the techniques underlying our algorithmic results. For WNZF(k), we write an integer program and show extreme points properties of its LP relaxation. In particular, let $x^* \in \mathbb{R}^{E^+ \cup E^-}$ be an extreme point optimum solution to the LP relaxation. We show that x^* is half-integral. We prove that the edges of the undirected graph can be partitioned into integral and non-integral edges: An integral edge has x^* being integral in both directions; a non-integral edge has x^* equal to $\frac{1}{2}$ in both directions. We show that the integral arcs in the solution already form a 'partial' k-flow f (that is conserved at every node but may be zero on some edges). Since we are interested in a nowhere-zero flow, we need to send flow along non-integral edges. To send flow on the non-integral edges, we use an arbitrary nowhere-zero 6-flow g of the whole graph which can be found in polynomial time. We then use the observation that the composed flow $6f + g$

is nonzero in every edge and has all flow values at most $6k-1$ to conclude that $6f+g$ is a nowhere-zero $6k$-flow. For the same reason, $6f-g$ is also a nowhere-zero $6k$-flow. Our algorithm returns the smaller cost one among $6f+g$ and $6f-g$. Finally since the non-integral edges have LP values equal to $\frac{1}{2}$ in both directions in the extreme point x^*, we can bound the approximation factor of the returned nowhere-zero $6k$-flow relative to the optimum objective value of the LP-relaxation.

For WCBO(k), the reduction mentioned in the first paragraph above (and detailed in Lemma 9) and Theorem 2 imply a $(6(k-1),6)$-approximation. We improve on the factors to achieve a $(k,6)$-approximation in Theorem 4. For this, we write an integer program and study its LP-relaxation. In particular, we show that the LP-relaxation is $1/k$-integral via a connection to the theory of submodular flows. The integral arcs of this LP-relaxation no longer form a 'partial' k-flow. However, we show that we can always complete the orientation given by the integral arcs into a k-cut-balanced orientation, which induces a partial k-flow f. Thus, we can use a similar approach as for WNZF(k) to obtain a $6k$-cut-balanced orientation with cost at most k times the cost of the LP-relaxation.

For SWNZF(k), we achieve a unicriteria approximation via combinatorial techniques. Our algorithm finds a *locally optimal* nowhere-zero 6-flow $(\vec{E},f)$, that is, a nowhere-zero 6-flow whose cost cannot be reduced by pushing 6 units of flow along any directed cycle in $\vec{E}$. We show that a locally optimal nowhere-zero 6-flow gives a 3-approximation to SWNZF(k) for every finite integer $k\geq 6$ and for $k=\infty$. Next, we turn to the question of existence and an efficient algorithm to find a locally optimal nowhere-zero 6-flow. We show that for an arbitrary nowhere-zero 6-flow f, there is a locally optimal nowhere-zero 6-flow f' that can be obtained starting from f and repeatedly pushing 6 units of flow along the reverse directions of directed cycles; this proves the existence of a locally optimal nowhere-zero 6-flow. We phrase the problem of finding such a locally optimal nowhere-zero 6-flow via such cycle augmentations from an arbitrary f as a min-cost circulation problem. This can be solved in polynomial time, and its optimum can be used to recover a locally optimal nowhere-zero 6-flow.

Organization. In Section 2, we setup some notation, present basic properties of nowhere-zero k-flows, and show the equivalence of nowhere-zero k-flows and k-cut-balanced orientations. In Section 3, we give the reduction between approximation algorithms for WNZF(k) and WCBO(k). In Section 4, we prove the hardness of WNZF(k) and WCBO(k). In Section 5, we give LP relaxations and design bicriteria approximation algorithms for WNZF(k) and WCBO(k). In Section 6, we prove the hardness and give unicriteria approximation algorithms for SWNZF(k).

2 Preliminaries

Let $G=(V,E)$ be an undirected graph. Let $\vec{G}=(V,E^+\cup E^-)$ denote its bidirected graph. A *partial orientation* is an orientation $\vec{F}$ on a subset $F\subseteq E$ of edges. A *partial k-flow* (or *k-flow*) is a partial orientation $\vec{F}$ and a function $f:\vec{F}\to\{1,2,\ldots,k-1\}$ such that $f(\delta^+_{\vec{F}}(v))=f(\delta^-_{\vec{F}}(v))$. Denote the reverse orientation of an arc e by e^{-1}. For a partial orientation $\vec{F}$, denote the reverse orientation by $\overleftarrow{F}$. Given a k-flow $(\vec{F},f)$, we can extend the domain of the function f to the arcs $E^+\cup E^-$ of the bidirected graph $\vec{G}$ by defining $f(e):=-f(e^{-1})\ \forall e\in\overleftarrow{F}$ and $f(e):=0\ \forall e\notin\vec{F}\cup\overleftarrow{F}$. We call the resulting extended function the *extension* of the partial k-flow f to $E^+\cup E^-$. We note that the extension of a partial k-flow satisfies flow conservation. This gives us the following equivalent definition of a k-flow: a k-flow is a function $f:E^+\cup E^-\to\{0,\pm1,\ldots,\pm(k-1)\}$ such that $f(e^+)=-f(e^-)\ \forall e\in E$ and $f(\delta^+_{\vec{G}}(v))=f(\delta^-_{\vec{G}}(v))$. A k-flow f is a nowhere-zero k-flow if $f(e)\neq 0\ \forall e\in E^+\cup E^-$. For a k-flow $f:E^+\cup E^-\to\{0,\pm1,\ldots,\pm(k-1)\}$, we denote $\mathrm{supp}(f):=\{e\in E: f(e^+)\neq 0\}$ as the edges oriented by f and $\mathrm{supp}^+(f):=\{e\in E^+\cup E^-: f(e)>0\}$ as the partial orientation associated with f. One can verify the two definitions of (nowhere-zero) k-flows are equivalent by letting $\vec{F}=\mathrm{supp}^+(f)$. Due to the equivalence of the two definitions, we will sometimes refer to a k-flow $(\vec{F}, f)$ by the extension $f : E^+ \cup E^- \to \{0, \pm 1, ..., \pm(k-1)\}$, omitting the orientation $\vec{F}$.

Equipped with this alternative definition, we can talk about negation, scaling and summation of k-flows. Let $G = (V, E)$ be an undirected graph. For a k-flow $(\vec{F}, f)$, $-f$ is defined as the negation of the extension of f to $E^+ \cup E^-$. We observe that $-f$ is also a k-flow and moreover, $\mathrm{supp}^+(-f) = \overleftarrow{F}$. Let $(\vec{E}_1, f_1)$ be a k_1-flow and $(\vec{E}_2, f_2)$ be a k_2-flow of G for some subsets $E_1, E_2 \subseteq E$. The sum of the two flows $f = f_1 + f_2$ is defined as the sum of the extensions of f_1 and f_2 to $E^+ \cup E^-$. Let $\vec{E} := \mathrm{supp}^+(f)$ be the partial orientation associated with f. We note that f may not be a nowhere-zero flow even if $E_1 \cup E_2 = E$ because of flow cancellation between f_1 and f_2. However, given $E_1 \cup E_2 = E$, if we scale f_1 by a factor of k_2, the resulting flow $f = k_2 f_1 + f_2$ is nowhere-zero. This observation has been very useful in constructing nowhere-zero k-flows in general graphs (e.g. [22, 24, 39]). We summarize it in the following proposition and give its proof for completeness.

Proposition 7. *Let $G = (V, E)$ be an undirected graph and $E_1, E_2 \subseteq E$ be such that $E_1 \cup E_2 = E$. Let $(\vec{E}_1, f_1)$ be a k_1-flow and $(\vec{E}_2, f_2)$ be a k_2-flow. Then, $f = k_2 f_1 + f_2$ is a nowhere-zero $k_1 k_2$-flow. Moreover, $\vec{E} := \mathrm{supp}^+(f)$ has the same orientation as $\vec{E}_1$ on E_1 and the same orientation as $\vec{E}_2$ on $E_2 \setminus E_1$.*

Proof. For each $e \in \mathrm{supp}^+(f_1)$, since $f(e) = k_2 f_1(e) + f_2(e) \geq k_2 - (k_2 - 1) \geq 1$, f_2 does not cancel $k_2 f_1$ to value 0. Thus, $\vec{E}$ has the same orientation as $\vec{E}_1$ on E_1 and the same orientation as $\vec{E}_2$ on $E_2 \setminus E_1$. Moreover, for each $e \in E^+ \cup E^-$, the flow value is bounded by $f(e) = k_2 f_1(e) + f_2(e) \leq k_2(k_1 - 1) + (k_2 - 1) \leq k_1 k_2 - 1$. Therefore, f is a nowhere-zero $k_1 k_2$-flow. □

The feasibility of nowhere-zero k-flows and k-cut-balanced orientations are equivalent due to the following lemma by Jaeger [23]. Since we will use this fact repeatedly, we include a proof.

Lemma 8 (Jaeger [23])**.** *Let $G = (V, E)$ be an undirected graph and $k \geq 2$ be an integer. An orientation is k-cut balanced if and only if it induces a nowhere-zero k-flow. Moreover, given a k-cut balanced orientation, there exists a polynomial-time algorithm to construct a nowhere-zero k-flow.*

Proof. Reverse direction: Let $\vec{E}$ and $f : \vec{E} \to \{1, ..., k-1\}$ be a nowhere-zero k-flow. By flow conservation, $f(\delta^+_{\vec{E}}(U)) = f(\delta^-_{\vec{E}}(U))\ \forall U \subseteq V$. Thus, we have $1 \cdot |\delta^+_{\vec{E}}(U)| \leq f(\delta^+_{\vec{E}}(U)) = f(\delta^-_{\vec{E}}(U)) \leq (k-1) \cdot |\delta^-_{\vec{E}}(U)|$. Similarly, we also have $1 \cdot |\delta^-_{\vec{E}}(U)| \leq f(\delta^-_{\vec{E}}(U)) = f(\delta^+_{\vec{E}}(U)) \leq (k-1) \cdot |\delta^+_{\vec{E}}(U)|$. It follows from the equality $|\delta_E(U)| = |\delta^+_{\vec{E}}(U)| + |\delta^-_{\vec{E}}(U)|$ that $\frac{1}{k}|\delta_E(U)| \leq |\delta^+_{\vec{E}}(U)| \leq \frac{k-1}{k}|\delta_E(U)|$. Thus, $\vec{E}$ is a k-cut-balanced orientation.

Forward direction: Let $\vec{E}$ be a k-cut-balanced orientation. By Hoffman's circulation theorem [19], there exists a circulation $f \in \mathbb{R}^{\vec{E}}$ satisfying $1 \leq f(e) \leq k-1\ \forall e \in \vec{E}$, if and only if $|\delta^-_{\vec{E}}(U)| \leq (k-1)|\delta^+_{\vec{E}}(U)|\ \forall U \subsetneq V, U \neq \emptyset$, which is equivalent to $|\delta^+_{\vec{E}}(U)| \geq \frac{1}{k}|\delta_E(U)|\ \forall U \subsetneq V, U \neq \emptyset$, satisfied by the k-cut-balanced condition. Thus, f is a nowhere-zero k-flow. Moreover, we can use any circulation algorithm to construct f in polynomial time (see e.g. [10, 3]). □

3 Reductions between WNZF(k) and WCBO(k)

Lemma 8 implies that approximation algorithms for WNZF(k) and WCBO(k) can be translated to each other with a small loss in factors as shown in the following lemma. A consequence of the lemma below is that NP-hardness of approximating WCBO(k) within any finite factor implies the same for WNZF(k), and vice versa.

Lemma 9. *If* WNZF(k) *has an (α, β)-approximation algorithm, then* WCBO(k) *has a $((k-1)\alpha, \beta)$-approximation algorithm; if* WCBO(k) *has an (α, β)-approximation algorithm, then* WNZF(k) *has a $((\beta k - 1)\alpha, \beta)$-approximation algorithm.*

Proof. Let $(\vec{E}_1^*, f_1^*)$ be a nowhere-zero k-flow with minimum cost and let $\vec{E}_2^*$ be a k-cut-balanced orientation with minimum cost on the same instance. Let f_2^* be a nowhere-zero k-flow induced by $\vec{E}_2^*$, which can be constructed in polynomial time according to Lemma 8.

Suppose that WNZF(k) has an (α, β)-approximation algorithm. Then, applying the algorithm returns a nowhere-zero βk-flow $(\vec{E}, f)$. By Lemma 8, $\vec{E}$ is a βk-cut-balanced orientation. Moreover, the orientation satisfies

$$c(\vec{E}) \leq c(f) \leq \alpha \cdot c(f_1^*) \leq \alpha \cdot c(f_2^*) \leq (k-1)\alpha \cdot c(\vec{E}_2^*),$$

where the first inequality follows from the fact that f is nowhere-zero. The second inequality follows from the fact that f is an (α, β)-approximate solution to WNZF(k). The third inequality follows from the fact that f_1^* is a min cost nowhere-zero k-flow. The fourth inequality follows from the fact that f_2^* is a nowhere-zero k-flow.

Suppose that WCBO(k) has an (α, β)-approximation algorithm. Then, applying the algorithm returns a βk-cut-balanced orientation $\vec{E}$. By Lemma 8, the orientation $\vec{E}$ induces a nowhere-zero βk-flow f. Moreover, the flow f satisfies

$$c(f) \leq (\beta k - 1) \cdot c(\vec{E}) \leq (\beta k - 1)\alpha \cdot c(\vec{E}_2^*) \leq (\beta k - 1)\alpha \cdot c(\vec{E}_1^*) \leq (\beta k - 1)\alpha \cdot c(f_1^*),$$

where the first inequality follows from the fact that f is a nowhere-zero βk-flow. The second inequality follows from the fact that $\vec{E}$ is an (α, β)-approximate solution to WCBO(k). The third inequality follows from the fact that $\vec{E}_2^*$ is a min cost k-cut-balanced orientation. The fourth inequality follows from the fact that f_1^* is nowhere-zero. □

4 Hardness of WNZF(k) and WCBO(k)

We prove Theorem 1 in this section, i.e., we show that both WNZF(k) and WCBO(k) do not admit an algorithm with finite approximation factors for every finite $k \geq 3$. By Lemma 9, NP-hardness of approximating WCBO(k) within any finite factor implies the same for WNZF(k), and vice versa. Thus, it suffices to prove the NP-hardness of approximation within any finite factor for WCBO(k). To achieve this, we study the problem of determining whether an arbitrary partial orientation $\vec{F}$ of a 2-edge-connected graph $G = (V, E)$ can be completed into a k-cut-balanced orientation. This is the key step towards proving the hardness of WCBO(k). This question is known to be NP-hard for $k = 3, 4$, because if we take $\vec{F} = \emptyset$, this recovers the problems of deciding whether a 2-edge-connected graph admits a nowhere-zero 3-flow and a nowhere-zero 4-flow, which are both NP-hard. This question is also hard for $k = 5$ if Tutte's 5-flow conjecture is false. Thus, the problem of deciding whether a partial orientation can be completed into a k-cut-balanced orientation is interesting only for $k \geq 6$, for which we know that the graph has a nowhere-zero k-flow, and possibly for $k = 5$. However, we show in Theorem 10 that this problem is hard for every finite integer $k \geq 3$.

Satisfiability (SAT) is a well-known NP-complete problem: the input is a collection of n variables and m clauses, where each clause is a disjunction of a subset of variables or their negation. The goal is to determine if there is an assignment of Boolean values to variables such that all clauses are satisfied. We call a SAT instance as a *restricted SAT* if every variable appears in at most 3 clauses. Restricted SAT is also NP-complete [42].

Theorem 10. *The following problem is NP-hard for every finite integer $k \geq 3$: given an undirected graph $G = (V, E)$ and a partial orientation $\vec{F}$ of a subset $F \subseteq E$ of edges, decide whether $\vec{F}$ can be completed into a k-cut-balanced orientation.*

Proof. If $k = 3$, take $\vec{F} = \emptyset$. The theorem follows from the NP-completeness of determining whether a given graph has a 3-cut-balanced orientation (see Table 1). Thus, we assume $k \geq 4$ from now on.

We reduce from restricted SAT where every variable appears at most 3 times. Consider a restricted SAT instance with variables $x_1, ..., x_n$ and clauses $C_1, ..., C_m$. We may assume that every variable x_i appears at least once positively and at least once negatively: otherwise, x_i always appears positive (resp. negative), in which case we can simply assign $x_i = 1$ (resp. $x_i = 0$) and delete all clauses containing x_i.

Construct a graph $G = (V, E)$ and a partial orientation $\vec{F}$ as follows. Fix a root r. Introduce $2n$ nodes $\{u_1, u_1', ..., u_n, u_n'\}$ and m nodes $\{v_1, ..., v_m\}$ such that $(u_i, v_j) \in \vec{F}$ if and only if x_i appears positively in C_j; $(u_i', v_j) \in \vec{F}$ if and only if x_i appears negatively in C_j. Suppose x_i appears $a_i \in \{1, 2\}$ times positively and $a_i' \in \{1, 2\}$ times negatively. Add one arc (r, u_i) and $k - a_i - 2$ arcs (u_i, r) to $\vec{F}$. Similarly, add one arc (r, u_i') and $k - a_i' - 2$ arcs (u_i', r) to $\vec{F}$. Note that $k - a_i - 2 \geq 0$ since $k \geq 4$ and $a_i \leq 2$. For each clause C_j, add $|C_j| + 1$ arcs (v_j, r) to $\vec{F}$. Finally, let E be the union of the underlying undirected graph of $\vec{F}$ and $\cup_{i=1}^{n} \{(u_i, u_i')\}$ (see Figure 1 left for details and Figure 2 left for an example).

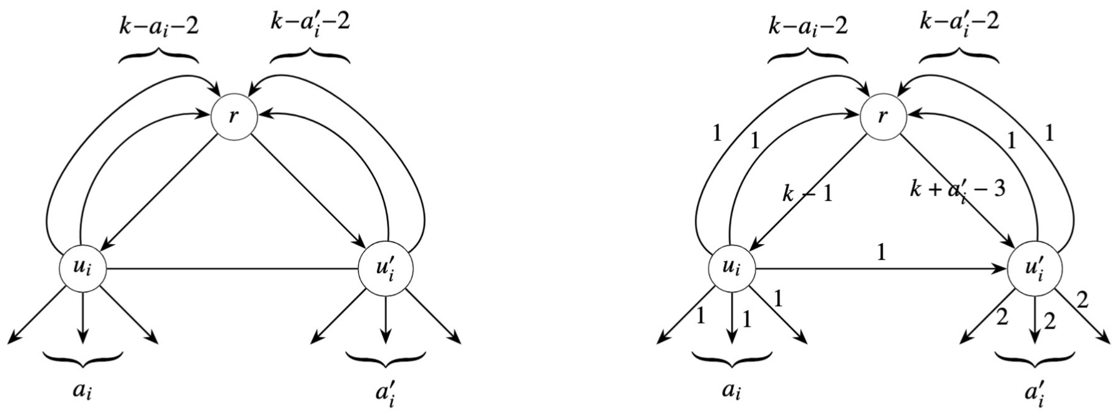

Figure 1: Left: part of graph $G = (V, E)$ and a partial orientation $\vec{F}$. Right: part of a nowhere-zero k-flow $(\vec{E}, f)$ of G such that $\vec{F} \subseteq \vec{E}$.

We claim that $\vec{F}$ can be completed into a k-cut-balanced orientation if and only if the instance of SAT is satisfiable. We first prove the reverse direction. Given an assignment to the variables $x_1, ..., x_n$ satisfying, complete $\vec{F}$ into an orientation $\vec{E}$ of all edges in the following way. Orient $(u_i, u_i') \in \vec{E}$ if $x_i = 0$; orient $(u_i', u_i) \in \vec{E}$ if $x_i = 1$. We claim that $\vec{E}$ is a k-cut-balanced orientation. To prove this, we construct a nowhere-zero k-flow f over $\vec{E}$. For an arbitrary $i \in [n]$, suppose $x_i = 0$, which implies $(u_i, u_i') \in \vec{E}$ (the case where $x_i = 1$ is symmetric). Let $f(r, u_i) = k - 1$ and $f(u_i, w) = 1\ \forall (u_i, w) \in \delta^+_{\vec{E}}(u_i)$. Let $f(r, u_i') = k + a_i' - 3 \in \{k - 1, k - 2\}$; $f(u_i', w) = 1\ \forall (u_i, w) \in \delta^+_{\vec{E}}(u_i)$ and $w = r$; and $f(u_i', w) = 2\ \forall (u_i, w) \in \delta^+_{\vec{E}}(u_i)$ and $w \neq r$ (see Figure 1 right). We note that f satisfies flow conservation at both u_i and u_i'. For each $j \in [m]$, since C_j is satisfied, w.l.o.g. there is some $i \in [n]$ such that x_i appears negatively in C_j and $x_i = 0$ (the case where there is some $i \in [n]$ such that x_i appears positively in C_j and $x_i = 1$ is symmetric), which implies $f(u_i', v_j) = 2$. Therefore, $|C_j| + 1 \leq f(\delta^-_{\vec{E}}(v_j)) \leq 2|C_j|$. Since $|\delta^+_{\vec{E}}(v_j)| = |C_j| + 1$, we can arbitrarily assign flow values 1 or 2 to arcs in $\delta^+_{\vec{E}}(v_j)$ such that $f(\delta^+_{\vec{E}}(v_j)) = f(\delta^-_{\vec{E}}(v_j))$. So far, we have assigned flow value for each arc in $\vec{E}$ and checked flow conservation for $u_i\ \forall i$ and $v_j\ \forall j$. Therefore, the flow conservation has to hold for r as well. This shows that $(\vec{E}, f)$ is a nowhere-zero k-flow. By Lemma 8, $\vec{E}$ is a k-cut-balanced orientation.

Next, we prove the forward direction. Let $\vec{E}$ be a k-cut-balanced orientation such that $\vec{F} \subseteq \vec{E}$. We assign $x_i = 0$ if $(u_i, u_i') \in \vec{E}$; $x_i = 1$ if $(u_i', u_i) \in \vec{E}$. We now show that all clauses are satisfied under this assignment. Suppose not, which means there is some clause $j \in [m]$ that is not satisfied. Consider the set $U := \{u_i : x_i \text{ appears positively in } C_j\}$ and $U' := \{u_i' : x_i \text{ appears negatively in } C_j\}$. Since C_j is

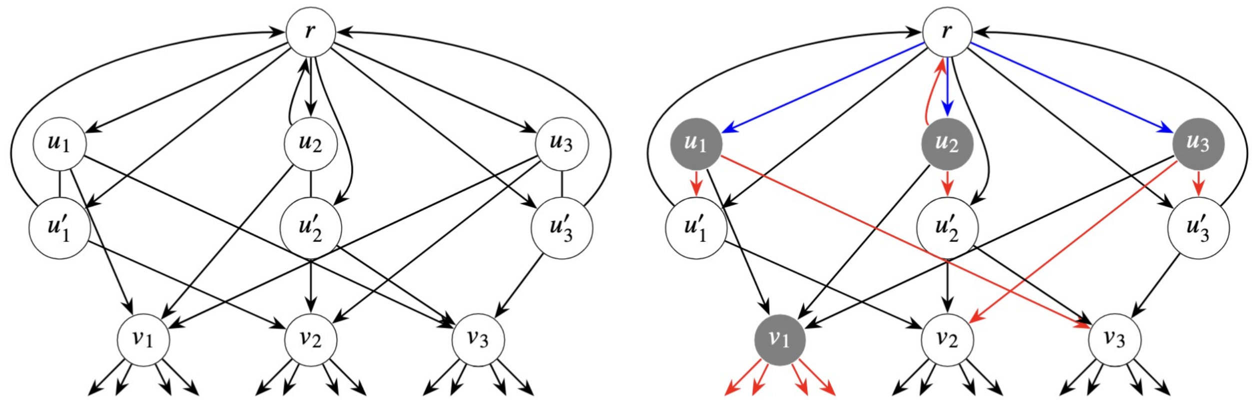

Figure 2: Left: the graph G for restricted SAT instance $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$ and $k = 4$. Right: an infeasible assignment $x_1 = x_2 = x_3 = 0$ yields a cut $X = \{u_1, u_2, u_3, v_1\}$ that violates the k-cut-balancedness condition. Here, arcs in $\delta^+(X)$ are colored red; arcs in $\delta^-(X)$ are colored blue.

unsatisfied, it follows that $(u_i, u_i') \in \vec{E}\ \forall u_i \in U$ and $(u_i', u_i) \in \vec{E}\ \forall u_i' \in U'$. Then, for every $u_i \in U$, $|\delta^+(u_i)| = (k - a_i - 2) + a_i + 1 = k - 1$, consisting of $k - a_i - 2$ arcs to r, a_i arcs to v_j's and one arc to u_i'. Similarly, for every $u_i' \in U'$, $|\delta^+(u_i')| = k - 1$. Let $X := U \cup U' \cup \{v_j\}$. Thus, $|\delta^+(X)| = \sum_{i:u_i \in U} |\delta^+(u_i)| + \sum_{i:u_i' \in U'} |\delta^+(u_i')| - |\delta^-(v_j)| + |\delta^+(v_j)| = (k-1)|C_j| - |C_j| + (|C_j|+1) = (k-1)|C_j|+1$. On the other hand, the only incoming arcs of X are from the root r to $U \cup U'$, and thus $|\delta^-(X)| = |U| + |U'| = |C_j|$. This violates the k-cut-balancedness condition $|\delta^+(X)| \leq \frac{k-1}{k}|\delta(X)|$, or equivalently $|\delta^+(X)| \leq (k-1)|\delta^-(X)|$, a contradiction. □

Theorem 1 follows as a corollary.

Proof of Theorem 1. We first prove that it is NP-hard to approximate WCBO(k) within any finite factor for every finite $k \geq 3$. We reduce from the decision problem of whether $\vec{F}$ can be completed into a k-cut-balanced orientation. Let $\vec{G} = (V, E^+ \cup E^-)$ be the bidirected graph of G. Set cost $c(e) = 0$ and $c(e^{-1}) = \infty$ for every arc $e \in \vec{F}$. Set cost $c(e^+) = c(e^-) = 0$ for every $e \in E \setminus F$. A finite approximation algorithm of the minimum k-cut-balanced orientation returns a solution of cost 0 if and only if $\vec{F}$ can be completed into a k-cut-balanced orientation.

Next we prove that it is NP-hard to approximate WNZF(k) within any finite factor for every finite $k \geq 3$. Letting $\beta = 1$ in Lemma 9, we obtain that for every finite k, WNZF(k) has a finite factor approximation algorithm if and only if WCBO(k) has a finite factor approximation algorithm. Therefore, the hardness of approximating WNZF(k) within any finite factor follows from that of WCBO(k). □

5 Bicriteria approximations for WNZF(k) and WCBO(k)

In this section, we design bicriteria approximations for WNZF(k) and WCBO(k). Our algorithms are based on an LP-relaxation for both problems.

We begin with a unified LP-relaxation for both problems. Let $G = (V, E)$ be the input graph, $\vec{G} = (V, E^+ \cup E^-)$ be its bidirected graph, and $c : E^+ \cup E^-$ be the given cost. For each edge $e \in E$, we introduce indicator variables $y(e^+)$ and $y(e^-)$ to indicate the orientation to be used for the edge and non-negative integer variables $z(e^+)$ and $z(e^-)$ for the flow value on the two orientations of the edge. For the y variables that indicate orientation, we impose the constraint that $y(e^+) + y(e^-) = 1$ for each edge $e \in E$, since each edge is to be oriented in exactly one of the two directions. For the flow variables z, we impose the constraint that $z(\delta^+_{\vec{G}}(v)) = z(\delta^-_{\vec{G}}(v))$ for each vertex $v \in V$ to conserve flow. We also impose the constraint that $y(e) \leq z(e) \leq (k-1)y(e)$ for every arc $e \in E^+ \cup E^-$ since we want the flow variables z to send flow only along the direction of the oriented arc and the capacity of the flow to be at most $k-1$. Together, we obtain the LP-relaxation given in (1). Thus, it follows from Lemma 8 that every integral feasible solution $y, z \in \mathbb{Z}^{E^+ \cup E^-}$ to (1) corresponds to a k-cut-balanced orientation and a nowhere-zero k-flow, respectively. For WCBO(k), we let $c_y = c$ and $c_z = 0$. For WNZF(k), we let $c_y = 0$ and $c_z = c$.

$$\begin{aligned} &\min c_y^\top y + c_z^\top z \\ &s.t.\ z(\delta^+_{\vec{G}}(v)) = z(\delta^-_{\vec{G}}(v)) \quad \forall v \in V \\ &\qquad y(e^+) + y(e^-) = 1 \quad \forall e \in E \\ &\qquad y \leq z \leq (k-1)y \\ &\qquad y \geq 0. \end{aligned} \tag{1}$$

In Sections 5.1 and 5.2, we focus on WNZF(k): we consider the projection of (1) on the z variables and prove certain extreme point properties of the projected LP in Section 5.1. We use these properties to design a bicriteria approximation for WNZF(k) in Section 5.2 thereby proving Theorem 2. Lemma 9 and Theorem 2 imply a $(6(k-1), 6)$-approximation algorithm for WCBO(k). Next, we improve on this bicriteria approximation to achieve a $(k, 6)$-approximation for WCBO(k). In Sections 5.3 and 5.4, we focus on WCBO(k): we consider the projection of (1) on the y variables and prove certain extreme point properties of the projected LP in Section 5.3. We use these properties to design a bicriteria approximation for WCBO(k) in Section 5.4, thereby proving Theorem 4.

5.1 LP relaxation for WNZF(k) and its extreme point structure

We can project the feasible region of (1) on the z variables, which yields an LP relaxation for WNZF(k).

$$\begin{aligned} &\min c^\top z \\ &s.t.\ z(\delta^+_{\vec{G}}(v)) = z(\delta^-_{\vec{G}}(v)) \quad \forall v \in V \\ &\qquad 1 \leq z(e^+) + z(e^-) \leq k-1 \quad \forall e \in E \\ &\qquad z \geq 0. \end{aligned} \tag{$\mathcal{P}_k$}$$

Lemma 11. *The polytope in $(\mathcal{P}_k)$ is the projection of the polytope in (1) to the z variables.*

Proof. We note that for every (y, z) feasible for (1), z is feasible for $(\mathcal{P}_k)$. Conversely, let z be feasible for $(\mathcal{P}_k)$. We define $y(e^+) := \frac{z(e^+)}{z(e^+)+z(e^-)}$ and $y(e^-) := \frac{z(e^-)}{z(e^+)+z(e^-)}$ $\forall e \in E$. It holds that

$$\frac{1}{k-1} z(e^+) \leq \frac{z(e^+)}{z(e^+) + z(e^-)} \leq z(e^+).$$

Thus, $\frac{1}{k-1} z(e^+) \leq y(e^+) \leq z(e^+)$. The same inequality holds for e^-. Moreover, $y(e^+) + y(e^-) = 1$. Thus, (y, z) is feasible for (1). □

Remark 1. *If we drop the upper bound constraint $z(e^+) + z(e^-) \leq k-1$ for every $e \in E$ from $(\mathcal{P}_k)$, then the resulting LP is an LP-relaxation of the asymmetric postman problem and it has been shown to be half-integral in several earlier works [26, 47, 36, 49].*

We show that $(\mathcal{P}_k)$ is half-integral. Our proof is inspired by extreme point results for postman problems on mixed graphs [49]. In fact, we show additional stronger properties as stated in the following theorem.

Theorem 12. *Let z^* be an extreme point optimal solution to $(\mathcal{P}_k)$. Then,*

(1) for every $e \in E$, if $e^+, e^- \in \operatorname{supp}(z^*)$, then $z^*(e^+) + z^*(e^-) = 1$,

(2) for every $e \in E$, $z^*(e^+) \in \mathbb{Z}$ if and only if $z^*(e^-) \in \mathbb{Z}$,

(3) for every $e \in E^+ \cup E^-$, if $z^*(e)$ is non-integral, then $z^*(e) = \frac{1}{2}$, and

(4) the integral arcs of z^* form a k-flow.

Proof. We first prove (1). Suppose $e^+, e^- \in \operatorname{supp}(z^*)$ and $z^*(e^+) + z^*(e^-) > 1$. Decreasing both $z^*(e^+)$ and $z^*(e^-)$ by some sufficiently small $\epsilon > 0$ gives a feasible solution whose cost is at most $c^\intercal z^*$, contradicting the fact that z^* is an extreme point optimal solution.

We show an intermediate claim that will help prove the remaining three statements of the theorem. Let $F := \{e \in E : z^*(e^+) \notin \mathbb{Z} \text{ or } z^*(e^-) \notin \mathbb{Z}\}$, which we call the *fractional edges* w.r.t. z^*. We claim that F is acyclic. Suppose not. Let $C \subseteq F$ be an undirected cycle. Let $\vec{C}$ be the corresponding directed cycle, where all edges are oriented clockwise. Let $\overleftarrow{C} := (\vec{C})^{-1}$ be corresponding directed cycle oriented counterclockwise. It follows from (1) that for every $e \in F$, if $z(e^+) \notin \mathbb{Z}$, then either $z^*(e^-) \notin \mathbb{Z}$ or $z^*(e^-) = 0$. Therefore, there are three types of arcs $e \in \vec{C} \cup \overleftarrow{C}$ with $z^*(e) > 0$: $C^0 := \{e \in \vec{C} \cup \overleftarrow{C} : z^*(e), z^*(e^{-1}) \notin \mathbb{Z}\}$, $C^+ := \{e \in \vec{C} : z^*(e) \notin \mathbb{Z}, z^*(e^{-1}) = 0\}$, and $C^- := \{e \in \overleftarrow{C} : z^*(e) \notin \mathbb{Z}, z^*(e^{-1}) = 0\}$. For a sufficiently small $\epsilon > 0$, let

$$z'(e) = \begin{cases} z^*(e) + \epsilon, & e \in C^+ \\ z^*(e) - \epsilon, & e \in C^- \\ z^*(e) + \epsilon/2, & e \in C^0 \cap \vec{C} \\ z^*(e) - \epsilon/2, & e \in C^0 \cap \overleftarrow{C} \\ z^*(e), & o/w. \end{cases} \quad \text{and} \quad z''(e) = \begin{cases} z^*(e) - \epsilon, & e \in C^+ \\ z^*(e) + \epsilon, & e \in C^- \\ z^*(e) - \epsilon/2, & e \in C^0 \cap \vec{C} \\ z^*(e) + \epsilon/2, & e \in C^0 \cap \overleftarrow{C} \\ z^*(e), & o/w. \end{cases}$$

We claim that both z' and z'' are feasible to $(\mathcal{P}_k)$: we prove this for z' and the statement for z'' follows by symmetric arguments. Flow conservation holds because we obtained z' from z^* by pushing ϵ units of flow along $\vec{C}$ (it is equivalent to view pushing $-\epsilon$ flow along $e \in \overleftarrow{C}$ as pushing ϵ flow along $e \in \vec{C}$). Moreover, for $e \in C^+ \cup C^-$, since $z^*(e) \notin \mathbb{Z}$ and $z^*(e^{-1}) = 0$, we have that $1 < z^*(e) + z^*(e^{-1}) < k - 1$. Thus, for sufficiently small $\epsilon > 0$, $1 \leq z'(e) + z'(e^{-1}) \leq k - 1$. Finally, $z'(e) \geq 0$ follows from the fact that for every $e \in C^+ \cup C^- \cup C^0$, $z^*(e) > 0$. This proves that both z' and, by symmetry, z'' are feasible to $(\mathcal{P}_k)$. However, $z^* = (z' + z'')/2$, which is a contradiction to the fact that z^* is an extreme point.

We next prove (2) and (3) together by induction. Since F is acyclic, it forms a forest. We remove the leaves of F one by one and delete non-integral edges along the way, maintaining the property that the flow z^* is integral on $E \setminus F$. Pick a leaf node v of F incident to a unique edge $e = (u, v) \in F$. If $z^*(e^+) \in \mathbb{Z}$, we immediately obtain that $z^*(e^-) \in \mathbb{Z}$ due to the flow conservation at v and the fact that every arc f incident to v other than e^- has integral flow $z^*(f)$. Therefore, we may assume that both $z^*(e^+), z^*(e^-) \notin \mathbb{Z}$. In this case, $z^*(e^+) - z^*(e^-) \in \mathbb{Z}$ by flow conservation at v. It follows from (1) that $z^*(e^+) + z^*(e^-) = 1$. Therefore, we conclude that $z^*(e^+) = z^*(e^-) = 1/2$. We delete e^+ and e^- together. Since $z^*(e^+) = z^*(e^-)$, the property that z^* is a circulation is preserved. This proves that (2) $z^*(e^+) \in \mathbb{Z}$ if and only if $z^*(e^-) \in \mathbb{Z}$; (3) if $z^*(e)$ is non-integral, then $z^*(e) = \frac{1}{2}$.

We finally prove (4). It follows from (3) that the integral arcs of z^* form a circulation. Moreover, $z^*(e) \leq k - 1 \ \forall e \in E^+ \cup E^-$. Furthermore, if both $z^*(e^+), z^*(e^-) \in \mathbb{Z}$, it follows from (1) that exactly one of them is in $\operatorname{supp}(z^*)$. Thus, $\operatorname{supp}(z^*)$ is a partial orientation which induces a k-flow. $\square$

5.2 Bicriteria approximation for WNZF(k)

In this section, we prove Theorem 2 by giving a bicriteria approximation algorithm for WNZF(k), for every finite integer $k \geq 6$ and for $k = \infty$. We recall that an algorithm is an (α, β)-approximation for WNZF(k) if we return a nowhere-zero βk-flow with cost at most $\alpha c(\mathrm{OPT})$ where $c(\mathrm{OPT})$ is the minimum cost of a nowhere-zero k-flow.

Proof of Theorem 2. Our algorithm proceeds as follows: Solve the LP relaxation ($\mathcal{P}_k$) and let z^* be an extreme point optimal solution. According to Theorem 12 (4), the integral arcs of z^* form a k-flow, denoted as f. Let g be an arbitrary nowhere-zero 6-flow of E, which can be computed in polynomial time [48]. Then, $-g$ is also a nowhere-zero 6-flow. Our algorithm returns $6f + g$ or $6f - g$, whichever has a smaller cost. Now, we bound its approximation guarantee.

According to Proposition 7, both $6f+g$ and $6f-g$ are nowhere-zero $6k$-flows. We prove that $\min\{c(6f+g), c(6f-g)\} \leq 6c(z^*)$. Let $E_1 := \mathrm{supp}(f)$ be the edges oriented by f, i.e., $\{e \in E^+ \cup E^- : f(e) \neq 0\}$. Let $\vec{E}_1 := \mathrm{supp}^+(f)$ be the orientation associated with f, i.e., $\{e \in E^+ \cup E^- : f(e) > 0\}$. Let $E_2 = E \setminus E_1$ and $\vec{E}_2$ be the orientation associated with g restricted to E_2. Then, $\overleftarrow{E}_2 = (\vec{E}_2)^{-1}$ is the orientation associated with $-g$ restricted to E_2. Let $\vec{E} := \vec{E}_1 \cup \vec{E}_2$, which will be the orientation associated with $6f + g$ according to Proposition 7. Similarly, let $\vec{E}' := \vec{E}_1 \cup \overleftarrow{E}_2$, which will be the orientation associated with $6f - g$. Then,

$$
\begin{aligned}
&\min\{c(6f+g), c(6f-g)\} \\
&\quad \leq \frac{1}{2}\Big(\sum_{e \in \vec{E}} c(e)(6f(e) + g(e)) + \sum_{e \in \vec{E}'} c(e)(6f(e) - g(e))\Big) \\
&\quad = \frac{1}{2}\sum_{e \in \vec{E}_1} c(e)\Big(\big(6f(e) + g(e)\big) + \big(6f(e) - g(e)\big)\Big) + \frac{1}{2}\sum_{e \in \vec{E}_2} c(e)g(e) + \frac{1}{2}\sum_{e \in \overleftarrow{E}_2} c(e)(-g)(e) \\
&\quad = 6\sum_{e \in \vec{E}_1} c(e)f(e) + \frac{1}{2}\sum_{e \in \vec{E}_2} \Big(c(e)g(e) + c(e^{-1})g(e)\Big) \\
&\quad \leq 6\sum_{e \in \vec{E}_1} c(e)f(e) + \frac{1}{2}\sum_{e \in \vec{E}_2} 5\Big(c(e) + c(e^{-1})\Big) \\
&\quad = 6\sum_{e \in \vec{E}_1} c(e)z^*(e) + \sum_{e \in \vec{E}_2} 5\Big(c(e)z^*(e) + c(e^{-1})z^*(e^{-1})\Big) \\
&\quad \leq 6c^\top z^*,
\end{aligned}
$$

where the second equality follows from the definition of $-g$, in which $(-g)(e) = -g(e) = g(e^{-1})$. The second inequality follows from the fact that g is a nowhere-zero 6-flow and thus $g(e) \leq 5\ \forall e \in \vec{E}_2$. The third equality follows from the fact that $f(e) = z^*(e)\ \forall e \in \vec{E}_1$ and that $z^*(e) = z^*(e^{-1}) = \frac{1}{2}\ \forall e \in E_2$ according to Theorem 12 (3).

□

5.3 LP relaxation for $\mathrm{WCBO}(k)$ and its extreme point structure

We can project the feasible region of (1) on the y variables, which yields an LP relaxation for $\mathrm{WCBO}(k)$.

$$\begin{aligned} &\min c^\intercal y \\ &s.t.\ y(\delta^+_{\vec{G}}(U)) \le \frac{k-1}{k}|\delta_E(U)| \quad \forall U \subseteq V \\ &\qquad y(e^+) + y(e^-) = 1 \quad \forall e \in E \\ &\qquad y \ge 0. \end{aligned} \tag{Q_k}$$

Lemma 13. *The polytope in* (Q_k) *is the projection of the polytope in* (1) *on the* y *variables.*

Proof. Let (y, z) be a feasible solution to (1). Then, for every $U \subseteq V$,

$$\begin{aligned} y(\delta^+_{\vec{G}}(U)) \le z(\delta^+_{\vec{G}}(U)) =& z(\delta^-_{\vec{G}}(U)) \le (k-1)y(\delta^-_{\vec{G}}(U)), \\ y(\delta^+_{\vec{G}}(U)) + y(\delta^-_{\vec{G}}(U)) =& \sum_{e \in \delta_E(U)} \big(y(e^+) + y(e^-)\big) = |\delta_E(U)|. \end{aligned}$$

Therefore,

$$|\delta_E(U)| = y(\delta^+_{\vec{G}}(U)) + y(\delta^-_{\vec{G}}(U)) \ge y(\delta^+_{\vec{G}}(U)) + \frac{1}{k-1} y(\delta^+_{\vec{G}}(U)) = \frac{k}{k-1} y(\delta^+_{\vec{G}}(U)).$$

Thus, y is feasible for (Q_k).

Let y be a feasible solution to (Q_k). Suppose the least common multiple of the coordinates of y is M. Construct a graph $G' = (V, E')$ with M copies of each edge $e \in E$ and an orientation $\vec{E}'$ such that $My(e^+)$ arcs are oriented the same as e^+ and the remaining $M - My(e^+) = My(e^-)$ arcs are oriented the same as e^-, denoted by $\vec{E}'(e^+)$ and $\vec{E}'(e^-)$, respectively. Since every arc is multiplied by a same factor from the fractional y, the new graph satisfies

$$|\delta^+_{\vec{E}'}(U)| \le \frac{k-1}{k}|\delta_{E'}(U)| \quad \forall U \subseteq V,$$

which means $\vec{E}'$ is a k-cut-balanced orientation of G'. By Lemma 8, $\vec{E}'$ induces a nowhere-zero k-flow $z' : \vec{E}' \to \{1, 2, ..., k-1\}$. Let $z(e^+) := \frac{1}{M}\sum_{e' \in \vec{E}'(e^+)} z'(e')$ and $z(e^-) := \frac{1}{M}\sum_{e' \in \vec{E}'(e^-)} z'(e')$ $\forall e \in E$. It follows from z' being a flow that z is a flow. Moreover, for every $e \in E^+ \cup E^-$, $y(e) = \frac{1}{M}|\vec{E}'(e)| \le z(e) = \frac{1}{M}\sum_{e' \in \vec{E}'(e)} z'(e') \le \frac{1}{M}(k-1)|\vec{E}'(e)| = (k-1)y(e)$. Thus, (y, z) is feasible for (1). □

To prove extreme point properties of (Q_k), we need the following seminal theorem of Edmonds and Giles [8].

Theorem 14 ([8])**.** *Let $D = (V, A)$ be a digraph and $f : 2^V \to \mathbb{R}$ be a submodular function. Consider the* submodular flow *polyhedron defined as follows:*

$$P(f) := \big\{y : A \to \mathbb{R} \mid y(\delta^+(U)) - y(\delta^-(U)) \le f(U) \quad \forall U \subseteq V\big\}.$$

If f is integral, then $P(f)$ is box-integral*, i.e., $P(f) \cap \{x : l \le x \le u\}$ is integral for every $l, u \in \mathbb{Z}$.*

The following theorem proves that after projecting (Q_k) onto $\{y(e^+) : e^+ \in E^+\}$, we obtain a submodular flow where f is $1/k$-integral.

Lemma 15. *The extreme points of* (Q_k) *are* $1/k$*-integral.*

Proof. For $y \in \mathbb{R}^{E^+ \cup E^-}$, denote by $y|_{E^+}$ the restriction of y to E^+. We claim that the projection of (Q_k) onto $y|_{E^+}$ is

$$P := \Big\{ y \in [0,1]^{E^+} : y(\delta^+_{E^+}(U)) - y(\delta^-_{E^+}(U)) \le \frac{1}{k}\Big((k-1)|\delta^+_{E^+}(U)| - |\delta^-_{E^+}(U)|\Big) \quad \forall U \subseteq V \Big\}. \tag{2}$$

To see this, let y be a feasible solution for (Q_k), Then, for $U \subseteq V$, we have that

$$\begin{aligned} y(\delta^+_{\vec{G}}(U)) =& y(\delta^+_{E^+}(U)) + y(\delta^+_{E^-}(U)) = y(\delta^+_{E^+}(U)) + \sum_{e^- \in \delta^+_{E^-}(U)} y(e^-) \\ &= y(\delta^+_{E^+}(U)) + \sum_{e^+ \in \delta^-_{E^+}(U)} (1 - y(e^+)) = y(\delta^+_{E^+}(U)) + |\delta^-_{E^+}(U)| - y(\delta^-_{E^+}(U)), \end{aligned}$$

where the third equality follows from $y(e^+) + y(e^-) = 1 \ \forall e \in E$. Thus, it follows from $y(\delta^+_{\vec{G}}(U)) \le \frac{k-1}{k}|\delta_E(U)|$ that

$$\begin{aligned} &y(\delta^+_{E^+}(U)) - y(\delta^-_{E^+}(U)) \le \frac{k-1}{k}|\delta_E(U)| - |\delta^-_{E^+}(U)| \\ =& \frac{k-1}{k}\Big(|\delta^+_{E^+}(U)| + |\delta^-_{E^+}(U)|\Big) - |\delta^-_{E^+}(U)| = \frac{1}{k}\Big((k-1)|\delta^+_{E^+}(U)| - |\delta^-_{E^+}(U)|\Big). \end{aligned}$$

This implies that $y|_{E^+} \in P$, where P is defined as (2). On the other hand, for every $y \in P$, define $\tilde{y} \in \mathbb{R}^{E^+ \cup E^-}$ as $\tilde{y}(e^+) := y(e^+)$ and $\tilde{y}(e^-) := 1 - y(e^+) \ \forall e \in E$. The same argument shows that $\tilde{y}$ is feasible to (Q_k). Therefore, P is the projection of (Q_k) onto $y|_{E^+}$.

Let $f : 2^V \to \mathbb{R}$ be defined as for $U \subseteq V$,

$$f(U) := \frac{1}{k}\Big((k-1)|\delta^+_{E^+}(U)| - |\delta^-_{E^+}(U)|\Big) = \frac{1}{k}\Big(\big(|\delta^+_{E^+}(U)| - |\delta^-_{E^+}(U)|\big) + (k-2)|\delta^+_{E^+}(U)|\Big).$$

Since $|\delta^+_{E^+}(U)| - |\delta^-_{E^+}(U)| = \sum_{v \in U}(|\delta^+_{E^+}(v)| - |\delta^-_{E^+}(v)|)$ is modular, $|\delta^+_{E^+}(U)|$ is submodular, and $k \ge 2$, we conclude that f is a $1/k$-integral submodular function. By Theorem 14, since $k \cdot f$ is integral, P is $1/k$-integral. Thus, for every $y \in P$, $y = \sum_{i=1}^t \lambda_i y_i$ for some $\sum_{i=1}^t \lambda_i = 1$, $\lambda \ge 0$ and $1/k$-integral $y_1, \dots, y_t \in P$. This implies that for every $\tilde{y} = (y, 1-y) \in \mathbb{R}^{E^+} \times \mathbb{R}^{E^-}$ feasible for (Q_k),

$$\tilde{y} = (y, 1-y) = \Big(\sum_{i=1}^t \lambda_i y_i, 1 - \sum_{i=1}^t \lambda_i y_i\Big) = \sum_{i=1}^t \lambda_i (y_i, 1 - y_i) = \sum_{i=1}^t \lambda_i \tilde{y}_i,$$

where $\tilde{y}_i$ is $1/k$-integral and feasible for (Q_k). This implies that the extreme points of (Q_k) are $1/k$-integral. □

5.4 Bicriteria approximation algorithms for WCBO(k)

Before going into the algorithm, we need one more definition. Given a graph $G = (V, E)$, a *partial k-cut-balanced orientation* is an orientation $\vec{F}$ of a subset $F \subseteq E$ of edges such that $|\delta^+_{\vec{F}}(U)| \le \frac{k-1}{k}|\delta_F(U)| \ \forall U \subseteq V$. We emphasize that the number of outgoing arcs is upper bounded by a fraction of the number of arcs in F instead of E.

In the following lemma, we give a characterization for the existence of a partial k-cut-balanced orientation that extends a partial orientation. The same characterization for $k = 2$ was given by Ford and Fulkerson [10] (see also [26]).

Lemma 16. *Given a graph $G = (V, E)$, suppose a subset $E_1 \subseteq E$ of edges has been oriented as $\vec{E}_1$. Then, $\vec{E}_1$ can be extended to a partial k-cut-balanced orientation if and only if*

$$|\delta^+_{\vec{E}_1}(U)| \leq \frac{k-1}{k}|\delta_E(U)| \quad \forall U \subseteq V. \tag{3}$$

Moreover, if $\vec{E}_1$ satisfies (3)*, then such a partial k-cut-balanced orientation can be found in polynomial time.*

Proof. "Only if" follows from the definition of partial k-cut-balanced orientations and the fact that $E_1 \subseteq E$. We prove the "if" direction. Bidirect the edges in $E_2 := E \setminus E_1$, denoted as $E_2^+ \cup E_2^-$. In order to complete $\vec{E}_1$ into a directed Eulerian subgraph of $(V, \vec{E}_1 \cup E_2^+ \cup E_2^-)$, we need a flow induced on $\vec{E}_1 \cup E_2^+ \cup E_2^-$, which has capacity lower bound 1 and upper bound $k-1$ for $e \in \vec{E}_1$, together with capacity lower bound 0 and upper bound $k-1$ for $e \in E_2^+ \cup E_2^-$. By Hoffman's circulation theorem [19], there exists such a flow if and only if

$$|\delta^+_{\vec{E}_1}(U)| \leq (k-1)|\delta^-_{\vec{E}_1}(U)| + (k-1)\Big(|\delta_E(U)| - |\delta^-_{\vec{E}_1}(U)| - |\delta^+_{\vec{E}_1}(U)|\Big) \quad \forall U \subseteq V,$$

i.e.,

$$|\delta^+_{\vec{E}_1}(U)| \leq \frac{k-1}{k}|\delta_E(U)| \quad \forall U \subseteq V.$$

Thus, by the given condition and Hoffman's circulation theorem, there exists a circulation f satisfying $0 \leq f(e) \leq k-1 \; \forall e \in E^+ \cup E^-$, and $\vec{E}_1 \subseteq \operatorname{supp}(f)$. Finally, if there is an edge $e \in E_2$ such that both e^+ and e^- have positive flow, then we decrease them by the same amount so that at least one of them has flow value zero. Thus, we may assume that for every edge $e \in E$, at most one of $f(e^+), f(e^-)$ is nonzero. Therefore, f is a k-flow. Let $F := \operatorname{supp}(f)$ be the edges oriented by f and $\vec{F} := \operatorname{supp}^+(f)$ be the partial orientation associated with f. Applying Lemma 8 to the graph (V, F), we conclude that $\vec{F}$ is a partial k-cut-balanced orientation. Moreover, we can use a polynomial-time circulation algorithm to construct f (see e.g. [10, 3]), and thus $\vec{F}$ can be constructed in polynomial time. □

Remark 2. *The function $f : 2^V \to \mathbb{R}$ defined as, for $U \subseteq V$,*

$$f(U) := \frac{k-1}{k}|\delta_E(U)| - |\delta^+_{\vec{E}_1}(U)| = \frac{1}{k}\Big(|\delta^-_{\vec{E}_1}(U)| - |\delta^+_{\vec{E}_1}(U)|\Big) + \frac{k-2}{k}|\delta^-_{\vec{E}_1}(U)| + \frac{k-1}{k}|\delta_{E\setminus E_1}(U)|$$

is submodular, since $|\delta^-_{\vec{E}_1}(U)| - |\delta^+_{\vec{E}_1}(U)|$ is modular, $|\delta^-_{\vec{E}_1}(U)|$ and $|\delta_{E\setminus E_1}(U)|$ are submodular, and $k \geq 2$. By Lemma 16, checking whether $\vec{E}_1$ can be extended to a partial k-cut-balanced orientation is equivalent to checking whether $f(U) \geq 0 \; \forall U \subseteq V$, which can be done in polynomial time using submodular minimization (see e.g. [38, 18, 21]). In contrast, according to Theorem 10, checking whether $\vec{E}_1$ can be extended to a (complete) k-cut-balanced orientation is NP-hard.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Our algorithm proceeds as follows: Solve the LP relaxation (Q_k) and let y^* be an extreme point optimal solution. Let $\vec{E}_1 := \{e \in E^+ \cup E^- : y^*(e) = 1\}$ and E_1 be its undirected counterpart. It follows that

$$|\delta^+_{\vec{E}_1}(U)| \leq y^*(\delta^+_{\vec{E}}(U)) \leq \frac{k-1}{k}|\delta_E(U)| \quad \forall U \subseteq V.$$

By Lemma 16, $\vec{E}_1$ can be extended to a partial k-cut-balanced orientation $\vec{F}$ in polynomial time. Let $\vec{E}_2 := \vec{F} \setminus \vec{E}_1$ and E_2 be its undirected counterpart. Since $\vec{F}$ is k-cut-balanced, it induces a partial k-flow f by Lemma 8. Let g be an arbitrary nowhere-zero 6-flow of (V, E), which can be constructed in polynomial time [48]. Our algorithm returns the orientation $\vec{E}$ associated with $6f + g$. Now, we bound its performance guarantee as stated in the theorem.

According to Proposition 7, $6f + g$ is a nowhere-zero $6k$-flow. Therefore, $\vec{E}$ is a $6k$-cut-balanced orientation. Next, we prove that $c(\vec{E}) \le kc(y^*)$. Let $E_3 = E \setminus (E_1 \cup E_2)$ and $\vec{E}_3$ be the orientation associated with g restricted to E_3. According to Proposition 7, $\vec{E} = \vec{E}_1 \cup \vec{E}_2 \cup \vec{E}_3$. Therefore,

$$c(\vec{E}) = \sum_{e\in\vec{E}} c(e) = \sum_{e\in\vec{E}_1} c(e) + \sum_{e\in\vec{E}_2\cup\vec{E}_3} c(e) \le \sum_{e\in\vec{E}_1} c(e)y^*(e) + \sum_{e\in\vec{E}_2\cup\vec{E}_3} c(e)\cdot ky^*(e) \le kc^\top y^*,$$

where the first inequality follows from the fact that $y^*(e) = 1\ \forall e \in E_1$, together with the fact that for every $e \in \vec{E}_2 \cup \vec{E}_3$, $y^*(e) \in (0,1)$ and $y^*(e)$ is $1/k$-integral by Lemma 15, which implies $y^*(e) \ge 1/k$. □

6 Min cost Symmetric Nowhere-Zero Flows

In this section, we study SWNZF(k), the weighted nowhere-zero k-flow with symmetric costs. We prove the NP-hardness of SWNZF(k) in Section 6.1 and give a 3-approximation algorithm for every $k \ge 6$ and $k = \infty$ in Section 6.2.

6.1 Hardness of SWNZF(k)

We prove Theorem 5 in this subsection. We first prove the NP-hardness of SWNZF(∞). We reduce from the NP-complete problem *not-all-equal 3-SAT (NAE3SAT)* [37]: the input is a collection of n variables and m clauses, where each clause is a disjunction of 3 variables or their negation. The goal is to determine if there is an assignment of Boolean values to variables such that the Boolean values assigned to the three variables in each clause are not all equal to each other (in other words, at least one is positive, and at least one is negative). Our construction is inspired by [50].

Theorem 17. SWNZF(∞) *for unit costs is NP-hard.*

Proof. Let the input instance to NAE3SAT consist of variables $x_1, ..., x_n$ and clauses $C_1, ..., C_m$. Suppose x_i appears a_i times positively and a'_i times negatively for each $i \in [n]$ and let $d_i = \max\{a_i, a'_i\}$ for each $i \in [n]$. We construct an undirected graph $G = (V, E)$ in the following way. For each $i \in [n]$, construct a cycle R_i of length $2d_i$ with vertices $u^i_1, u^i_2 ..., u^i_{2d_i}$ in cyclic order. For each clause C_j, there is a node v_j corresponding to it. Let $(u^i_{2s-1}, v_j) \in E$ if x_i appears positively in C_j for some $s \in [d_i]$; let $(u^i_{2s}, v_j) \in E$ if x_i appears negatively in C_j for some $s \in [d_i]$. We choose s in a way that each v_j is adjacent to a distinct u^i_t, $t \in [2d_i]$. For those u^i_t not adjacent to any v_j, we connect them to a special node v_0. Finally, we add an edge (v_j, v_0) for each $j \in [m]$ (see Figure 3 (1) for details and Figure 4 for an example). All arc costs are unit, i.e., $c(e^+) = c(e^-) = 1\ \forall e \in E$.

We claim that the NAE3SAT instance is satisfiable if and only if the min total value of a nowhere-zero flow of G equals $|E| + \sum_{i=1}^n d_i$. First, observe that the total value of a nowhere-zero flow is at least $|E| + \sum_{i=1}^n d_i$. Indeed, the degree of each u^i_t is 3 $\forall i \in [n], t \in [2d_i]$. By flow conservation, at least one arc adjacent to u^i_t has flow value at least 2. Thus, the number of arcs having flow value at least 2 is at least $\frac{1}{2}(\sum_{i=1}^n 2d_i) = \sum_{i=1}^n d_i$, where we divide by 2 because u^i_t and u^i_{t+1} may share an arc of flow at least 2 (we assume $u^i_{2d_i+1} := u^i_1$). Therefore, every nowhere-zero flow has value at least $|E| + \sum_{i=1}^n d_i$.

Suppose $x_1, ..., x_n$ is a feasible assignment such that each clause has at least one positive and one negative value. We construct a nowhere-zero flow $(\vec{E}, f)$ of total value $|E| + \sum_{i=1}^n d_i$ in the following way. Orient each cycle R_i as $u^i_1 \to u^i_2 \to \cdots \to u^i_{2d_i} \to u^i_1$. If $x_i = 1$, orient $(u^i_{2s-1}, v_j) \in \vec{E}$ and assign $f(u^i_{2s-1}, v_j) = 1$ for every $s \in [d_i], j \in \{0, 1, ..., m\}$ such that $(u^i_{2s}, v_j) \in E$; orient $(v_j, u^i_{2s}) \in \vec{E}$ and assign $f(v_j, u^i_{2s}) = 1$ for every $s \in [d_i], j \in \{0, 1, ..., m\}$ such that $(u^i_{2s}, v_j) \in E$. Let $f(u^i_{2s-1}, u^i_{2s}) = 1$, $f(u^i_{2s}, u^i_{2s+1}) = 2\ \forall s \in [d_i]$ (see Figure 3 (2)). If $x_i = 0$, orient $(v_j, u^i_{2s-1}) \in \vec{E}$ and assign $f(v_j, u^i_{2s-1}) = 1$

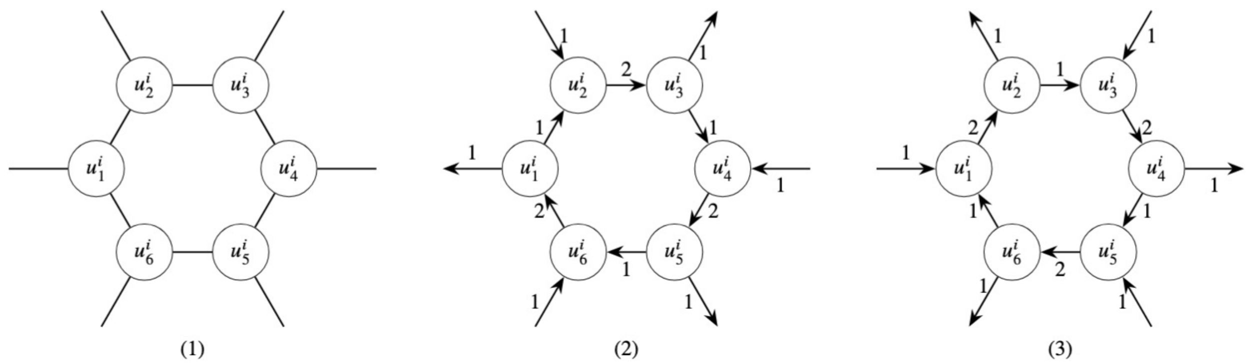

Figure 3: (1) Cycle R_i corresponding to variable x_i with $d_i = 3$. (2) Part of a nowhere-zero $(\vec{E}, f)$ when $x_i = 1$. (3) Part of a nowhere-zero $(\vec{E}, f)$ when $x_i = 0$.

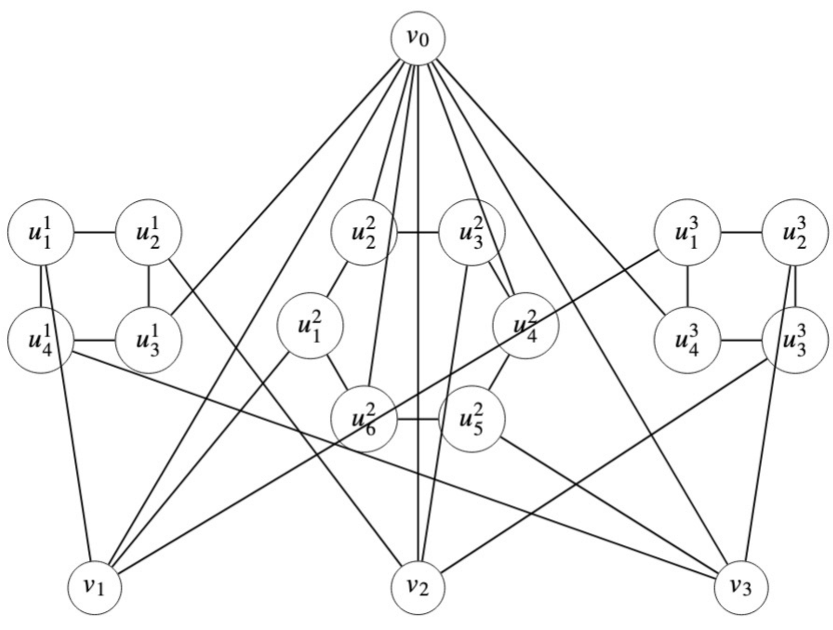

Figure 4: The graph G for NAE3SAT instance $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$.

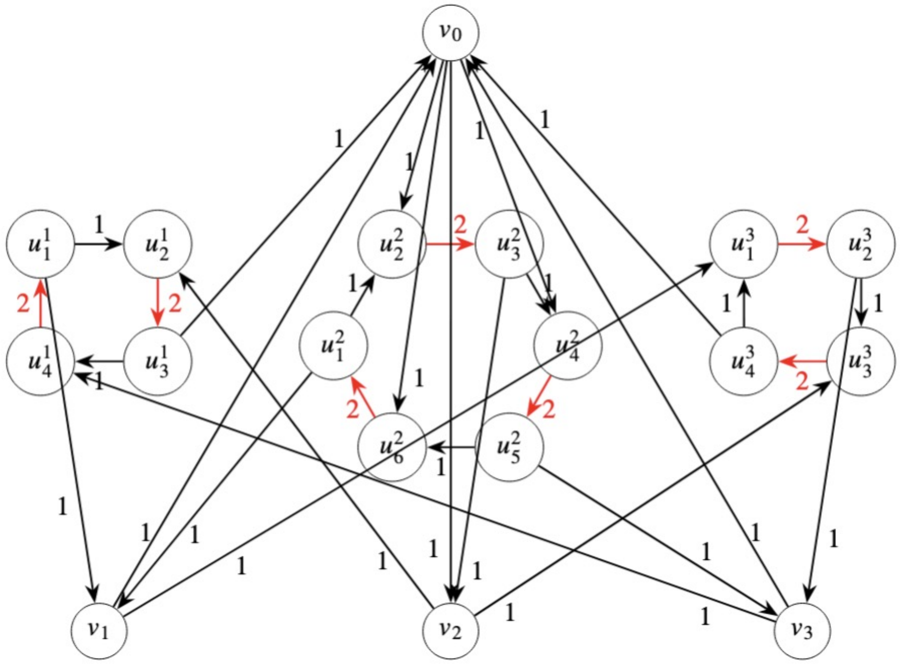

Figure 5: The nowhere-zero flow of total value $|E| + \sum_{i=1}^{n} d_i$ corresponding to a feasible assignment $x_1 = 1, x_2 = 1, x_3 = 0$ for NAE3SAT instance $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$. The arcs of flow value 2 are colored red, which are the even arcs of R_1, the even arcs of R_2, and the odd arcs of R_3.

for every for every $s \in [d_i], j \in \{0, 1, ..., m\}$ such that $(u^i_{2s-1}, v_j) \in E$; orient $(u^i_{2s}, v_j) \in \vec{E}$ and assign $f(u^i_{2s}, v_j) = 1$ for every for every $s \in [d_i], j \in \{0, 1, ..., m\}$ such that $(u^i_{2s}, v_j) \in E$. Let $f(u^i_{2s-1}, u^i_{2s}) = 2$, $f(u^i_{2s}, u^i_{2s+1}) = 1 \ \forall s \in [d_i]$ (see Figure 3 (3)). Finally, for each C_j, there are either two positive and one negative values or one positive and two negative values. In the former case we orient $(v_j, v_0) \in \vec{E}$ and assign $f(v_j, v_0) = 1$, while in the latter case we orient $(v_0, v_j) \in \vec{E}$ and assign $f(v_0, v_j) = 1$ (see Figure 5). This is indeed a nowhere-zero flow of total value $|E| + \sum_{i=1}^{n} d_i$, since the only arcs having flow value 2 are the even arcs or the odd arcs in the cycles.

We are left to prove the other direction: if the min total value of a nowhere-zero flow $(\vec{E}, f)$ equals $|E| + \sum_{i=1}^{n} d_i$, then the instance is satisfiable. It follows from the proof of $c(f) \geq |E| + \sum_{i=1}^{n} d_i$ that the equality holds if and only if the flow values are all 1 or 2, and the value 2 arcs are the even arcs or the odd arcs in each cycle R_i. Note that the arcs in R_i have to be oriented consistently, since the flow values of arcs of the form (u^i_t, v_j) or (v_j, u^i_t) are all 1. Moreover, the edges of the form (u^i_t, v_j) are oriented in a way that either $(u^i_{2s-1}, v_j) \in \vec{E}$, $(v_j, u^i_{2s}) \in \vec{E} \ \forall s \in [d_i]$ or $(v_j, u^i_{2s-1}) \in \vec{E}$, $(u^i_{2s}, v_j) \in \vec{E} \ \forall s \in [d_i]$, depending on the value 2 arcs are the even or odd arcs of R_i. In the former case we assign $x_i = 1$, while in the latter case we assign $x_i = 0$. This way, the value of x_i is positive in C_j if and only if $(u^i_t, v_j) \in \vec{E}$ for some $t \in [2d_i]$. This is indeed a feasible assignment, because for each clause v_j, $|\delta^+_{\vec{E}}(v_j)| = |\delta^-_{\vec{E}}(v_j)| = 2$, since the arcs adjacent to v_j all have flow values 1. This implies that C_j has at least one positive value and one negative value. This completes the proof. $\square$

In fact, the nowhere-zero flows we use in the proof of Theorem 17 are nowhere-zero 3-flows. Thus, the proof also implies that SWNZF(k) is NP-hard for every $k \geq 3$. Moreover, it is NP-hard to approximate SWNZF(k) within any finite factor for $k = 3, 4$, and for $k = 5$ if Tutte's 5-flow conjecture is false. Indeed, setting the costs $c = 0$ recovers the feasibility problem of deciding whether a graph has a nowhere-zero

k-flow for $k = 3, 4, 5$ which are NP-complete. Hence, Theorem 5 follows.

6.2 Approximation algorithms

As we discussed earlier, every nowhere-zero 6-flow is a 5-approximation to SWNZF(k) for every finite integer $k \geq 6$ and for $k = \infty$. Indeed, let $(\vec{E}, f)$ be an arbitrary nowhere-zero 6-flow, which is also a nowhere-zero k-flow. Let OPT be a min cost nowhere-zero k-flow. Then,

$$c(f) = \sum_{e \in \vec{E}} c(e) f(e) \leq \sum_{e \in \vec{E}} 5c(e) \leq 5c(\mathrm{OPT}),$$

where the first inequality follows from the fact that f is a 6-flow. The second inequality follows from the fact that OPT is nowhere-zero.

We provide an improved 3-approximation algorithm by finding a cheaper nowhere-zero 6-flow. Given a nowhere-zero 6-flow $(\vec{E}, f)$, for an arbitrary directed cycle $C \in \vec{E}$, if $\sum_{e \in C} c(e) f(e) > \sum_{e \in C} c(e)(6 - f(e))$, we can push 6 units flow along the reverse direction of C. This way, we reduce the cost of the flow along the cycle while maintaining a nowhere-zero 6-flow. This observation inspires the definition of locally optimal nowhere-zero 6-flow: we say that a nowhere-zero 6-flow is *locally optimal* if for every directed cycle C (wrt the flow),

$$\sum_{e \in C} c(e) f(e) \leq \sum_{e \in C} c(e)(6 - f(e)),$$

i.e.,

$$\sum_{e \in C} c(e) f(e) \leq 3 \sum_{e \in C} c(e). \tag{4}$$

We prove the following lemma.

Lemma 18. *For every finite integer $k \geq 6$ and for $k = \infty$, every locally optimal nowhere-zero 6-flow is a 3-approximation to* SWNZF(k).

Proof. Let $(\vec{E}, f)$ be a locally optimal nowhere-zero 6-flow. We recall that a circulation f can be decomposed into directed cycles $\mathscr{C}$, i.e., $f = \sum_{C \in \mathscr{C}} \chi(C)$, where $\chi(C)$ is the characteristic vector of C. Then,

$$\begin{aligned}
\sum_{e \in \vec{E}} c(e) f(e) &= \sum_{e \in \vec{E}} c(e) \sum_{C \in \mathscr{C}} \mathbf{1}\{e \in C\} = \sum_{C \in \mathscr{C}} \sum_{e \in C} c(e) \\
&\geq \frac{1}{3} \sum_{C \in \mathscr{C}} \sum_{e \in C} c(e) f(e) = \frac{1}{3} \sum_{e \in \vec{E}} c(e) f(e)^2 \geq \frac{1}{3} \frac{(\sum_{e \in \vec{E}} c(e) f(e))^2}{\sum_{e \in \vec{E}} c(e)},
\end{aligned}$$

where the first inequality follows from (4) and the second inequality follows from the Cauchy-Schwarz inequality. Let OPT be the min cost nowhere-zero k-flow. It follows that

$$c(f) = \sum_{e \in \vec{E}} c(e) f(e) \leq 3 \sum_{e \in \vec{E}} c(e) \leq 3c(\mathrm{OPT}).$$

$\square$

Next, we show that a local optimum exists by giving a pseudo-polynomial algorithm to find it.

Proposition 19. *A locally optimal nowhere-zero 6-flow exists and there is a pseudo-polynomial algorithm to find it.*

Proof. Start from an arbitrary nowhere-zero 6-flow $(\vec{E}, f)$, which can be found in polynomial time [48]. Check if there is a directed cycle $C \subseteq \vec{E}$ violating (4), i.e., $\sum_{e\in C} c(e)(3 - f(e)) < 0$. This can be done using an algorithm such as the Bellman-Ford algorithm that detects negative cycles in a weighted digraph $\vec{E}$ with weights $c(e)(3 - f(e))\ \forall e \in \vec{E}$. If so, we push 6 units flow along the reverse direction of C. This way, we obtain a new nowhere-zero 6-flow, and then we repeat the above step.

We show that the algorithm terminates. In each step, the total cost of the flow $\sum_{e\in\vec{E}} c(e)f(e)$ reduces by $\sum_{e\in C} c(e)\big(f(e) - (6 - f(e))\big) = \sum_{e\in C} 2c(e)(f(e) - 3) > 0$. Since $c \in \mathbb{Z}_{\geq 0}$, it always reduces by an integer at least 1. Thus, the algorithm terminates in at most $\sum_{e\in\vec{E}} c(e)f(e) \leq 5\sum_{e\in\vec{E}} c(e)$ steps. □

We note that to arrive at a locally optimal nowhere-zero 6-flow f' starting with f in the above proof, we add a collection of cycles pushing 6 units flow. This sum of cycles each pushing 6 units flow gives a feasible solution to a circulation problem where all the nonzero flows in the circulation have value 6. We formulate the task of finding such a circulation of minimum cost to obtain a strongly polynomial time algorithm for finding a locally optimal nowhere-zero 6-flow.

Let $(\vec{E}, f)$ be an arbitrary nowhere-zero 6-flow. We are now allowed to flip the direction of some arcs of $\vec{E}$ while maintaining a nowhere-zero 6-flow. We restrict ourselves to always pushing 6 units flow along the reverse direction of some directed cycle C. This way, for every arc $e \in \vec{E}$ we flip, the flow value $f(e)$ assigned to it becomes $6 - f(e)$. Thus, the change in the cost by flipping e becomes $c(e)\big((6 - f(e)) - f(e)\big) = 2c(e)(3 - f(e))$. This motivates us to formulate the following min cost circulation problem.

Let $D = (V, \overleftarrow{E})$ be the directed graph consisting of arcs in $\overleftarrow{E} := (\vec{E})^{-1}$. For each $e \in \overleftarrow{E}$, let cost $c'(e) := c(e)(3 - f(e^{-1}))$ and capacity $l(e) = 0,\ u(e) = 1$. An integral $g : \overleftarrow{E} \to \mathbb{Z}$ is feasible if it is a circulation, i.e., $g(\delta^+_{\overleftarrow{E}}(v)) = g(\delta^-_{\overleftarrow{E}}(v))\ \forall v \in V$, and it satisfies the capacity constraints $l(e) \leq g(e) \leq u(e)\ \forall e \in \overleftarrow{E}$. Let g be a min cost integral circulation of the instance. Return $f + 6g$ (notice that g is a 2-flow, which allows us to define $f + 6g$ the same way as in Section 2). In the lemma below, we show that $f + 6g$ is a locally optimal nowhere-zero 6-flow.

Lemma 20. *Let $(\vec{E}, f)$ be a nowhere-zero 6-flow and $D = (V, \overleftarrow{E})$ be a digraph with costs $c'(e) = c(e)(3 - f(e))\ \forall e \in \overleftarrow{E}$ and capacities $l = 0, u = 1$. Then, $g : \overleftarrow{E} \to \mathbb{Z}$ is a min cost circulation if and only if $f + 6g$ is a locally optimal nowhere-zero 6-flow.*

Proof. First, we claim that for an arbitrary $g : \overleftarrow{E} \to \mathbb{Z}$, g is feasible, i.e., a circulation obeying capacity constraints $0 \leq g(e) \leq 1\ \forall e \in \overleftarrow{E}$, if and only if $f + 6g$ is a nowhere-zero 6-flow. If g is feasible, then $g(e) \in \{0, 1\}\ \forall e \in \overleftarrow{E}$. For an arbitrary $e \in \overleftarrow{E}$ with $g(e) = 1$, one has $1 = (-5) + 6 \leq f(e) + 6g(e) \leq (-1) + 6 = 5$, where we use the facts that $f(e) = -f(e^{-1})$ and $1 \leq f(e^{-1}) \leq 5$. Otherwise, $g(e) = 0$ and thus $f(e) + 6g(e) = f(e)$, which implies $-5 \leq f(e) + 6g(e) \leq -1$. This implies that $f + 6g$ is a nowhere-zero 6-flow. Conversely, if g is not feasible, we show that $f + 6g$ is not a nowhere-zero 6-flow. If g is not a circulation, then $f + 6g$ is not a circulation and thus not a nowhere-zero flow. If g violates capacity constraints, then there exists $e \in \overleftarrow{E}$ such that $g(e) \geq 2$ or $g(e) \leq -1$, then $f(e) + 6g(e) \geq (-5) + 6 \cdot 2 = 7$ or $f(e) + 6g(e) \leq (-1) + 6 \cdot (-1) = -7$, respectively, which implies that $f + 6g$ is not a nowhere-zero 6-flow.

For convenience, from now on we will work with the following variant of the problem. For an arbitrary flow $g : \overleftarrow{E} \to \mathbb{Z}$, we extend its domain to $E^+ \cup E^-$ by letting $g(e) := -g(e^{-1})\ \forall e \in \vec{E}$. For each $e \in \vec{E}$, let cost $c'(e) := -c'(e^{-1}) = -c(e)(3 - f(e))$. Now, the cost of g becomes $\sum_{e\in\vec{E}\cup\overleftarrow{E}} c(e)g(e) = 2\sum_{e\in\overleftarrow{E}} c(e)g(e)$. Since for circulations g and g', $\sum_{e\in\vec{E}\cup\overleftarrow{E}} c'(e)g(e) \geq \sum_{e\in\vec{E}\cup\overleftarrow{E}} c'(e)g'(e)$ if and only if $\sum_{e\in\overleftarrow{E}} c(e)g(e) \geq \sum_{e\in\overleftarrow{E}} c(e)g'(e)$, the min cost circulation stay unchanged.

Now, we prove the "only if" direction. Suppose g is a min cost circulation. Then, $g(e) \in \{0, 1\}\ \forall e \in \overleftarrow{E}$. Moreover, let $\vec{E}' := \mathrm{supp}^+(f + 6g)$ be the orientation associated with $f + 6g$, i.e., $\{e \in E^+ \cup E^- : (f + 6g)(e) > 0\}$. It follows from Proposition 7 that for an arbitrary $e \in \vec{E}'$, $e \in \overleftarrow{E}$ if and only if $g(e) = 1$. For the sake of contradiction assume that $f + 6g$ is not a local optimum. Then, by (4), there exists some cycle $C \subseteq \vec{E}'$, satisfying

$$\begin{aligned} 0 > \sum_{e \in C} c(e)(3 - (f + 6g)(e)) &= \sum_{e \in C \cap \vec{E}} c(e)(3 - f(e)) + \sum_{e \in C \cap \overleftarrow{E}} c(e)(3 - (-f(e^{-1}) + 6)) \\ &= \sum_{e \in C \cap \vec{E}} c(e)(3 - f(e)) + \sum_{e \in C \cap \overleftarrow{E}} c(e)(f(e^{-1}) - 3) = -\sum_{e \in C} c'(e). \end{aligned} \tag{5}$$

Let χ_C be the indicator vector of C defined as $\chi_C(e) = 1\ \forall e \in C$, $\chi_C(e) = -1\ \forall e \in C^{-1}$, and $\chi_C(e) = 0$, otherwise. Let $g' = g - \chi_C$. Observe that $f + 6g' = f + 6g - 6\chi_C$ is a nowhere-zero 6-flow since $C \subseteq \vec{E}'$. Thus, it follows from the claim at the beginning of the proof that g' is feasible. However, $c'(g') - c'(g) = -c'(C) < 0$, a contradiction to the optimality of g.

We then prove "if" direction. Suppose $f + 6g$ is a locally optimal nowhere-zero 6-flow. By the claim, g is feasible. For the sake of contradiction, assume g is not a min cost circulation. Let g^* be a min cost circulation. The difference $g - g^*$ is also a circulation, and thus can be decomposed into cycles. Since $c'(g) > c'(g^*)$, there exists some cycle C with $c'(C) > 0$ such that $g' := g - \chi_C$ is also a feasible circulation. Thus, it follows from the claim that $f + 6g - 6\chi_C = f + 6g'$ is also a nowhere-zero 6-flow. This implies $C \subseteq \vec{E}'$. By (5), $\sum_{e \in C} c(e)(3 - (f + 6g)(e)) = -c'(C) < 0$. This contradicts the condition (4) that has to be satisfied when $f + 6g$ is a local optimum. □

Lemmas 18 and 20 together complete the proof of Theorem 6.

Proof of Theorem 6. Start from an arbitrary nowhere-zero 6-flow, which can be computed in polynomial time using Younger's algorithm [48]. Lemma 20 implies that a locally optimal nowhere-zero 6-flow can be found in strongly polynomial time using min cost circulation algorithms (see e.g. [10, 3]). Lemma 18 implies that this is a 3-approximation to SWNZF(k) $\forall k \geq 6$ and $k = \infty$. □

Remark 3. *The approximation ratio* 3 *is tight for this algorithm. To see this, consider the entire graph to be a cycle C, with unit costs. An optimal nowhere-zero flow is $(\vec{C}, f)$ with $f(e) = 1\ \forall e \in \vec{C}$, whose cost is $|C|$. However, if our algorithm starts from the nowhere-zero* 6*-flow $(\vec{C}, f')$ with $f'(e) = 3\ \forall e \in \vec{C}$, this is already a local optimum, so we only obtain a nowhere-zero flow of cost* $3|C|$.

7 Conclusion

We conclude with some interesting directions for future research. Firstly, is it possible to obtain a bicriteria approximation for min-cost well-balanced orientations? In particular, given an undirected graph $G = (V, E)$, is it possible to construct an orientation $\vec{E}$ such that (i) the cost of the orientation is at most a constant factor of a min-cost well-balanced orientation and (ii) the orientation $\vec{E}$ is constant-approximately well-balanced, i.e., $\lambda_{\vec{E}}(u, v) \geq \lfloor \lambda_E(u, v)/\rho \rfloor$ for some constant ρ? Secondly, can we improve on the $(6, 6)$-bicriteria approximation for WNZF(k), on the $(k, 6)$-bicriteria approximation for WCBO(k), or on the 3-factor approximation for SWNZF(k) for finite integer $k \geq 6$?

Acknowledgements The first two authors would like to thank Emléktábla workshop in July 2024 and ICERM reunion workshop on *Discrete Optimization: Mathematics, Algorithms, and Computation* in August 2024, where this work was initiated. We would like to thank Kristóf Bérczi and Gérard Cornuéjols for initial discussions.

References

[1] Attila Bernáth, Satoru Iwata, Tamás Király, Zoltán Király, and Zoltán Szigeti. Recent results on well-balanced orientations. *Discrete Optimization*, 5(4):663–676, 2008.

[2] Attila Bernáth and Gwenaël Joret. Well-balanced orientations of mixed graphs. *Information Processing Letters*, 106(4):149–151, 2008.

[3] Li Chen, Rasmus Kyng, Yang P Liu, Richard Peng, Maximilian Probst Gutenberg, and Sushant Sachdeva. Maximum flow and minimum-cost flow in almost-linear time. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 612–623, 2022.

[4] N Christofides, E Benavent, V Campos, A Corberán, and E Mota. An optimal method for the mixed postman problem. In *System Modelling and Optimization: Proceedings of the 11th IFIP Conference Copenhagen, Denmark, July 25–29, 1983*, pages 641–649. Springer, 1984.

[5] Gérard Cornuéjols, Siyue Liu, and R. Ravi. Approximately packing dijoins via nowhere-zero flows. In *Proceedings of Integer Programming and Combinatorial Optimization (IPCO)*, pages 71–84, 2024.

[6] Marek Cygan, Guy Kortsarz, and Zeev Nutov. Steiner forest orientation problems. *SIAM Journal on Discrete Mathematics*, 27(3):1503–1513, 2013.

[7] Matt DeVos and Kathryn Nurse. A short proof of seymour's 6-flow theorem. *arXiv preprint arXiv:2307.04768*, 2023.

[8] Jack Edmonds and Rick Giles. A min-max relation for submodular functions on graphs. In *Annals of Discrete Mathematics*, volume 1, pages 185–204. Elsevier, 1977.

[9] Jack Edmonds and Ellis L Johnson. Matching, Euler tours and the Chinese postman. *Mathematical Programming*, 5:88–124, 1973.

[10] Lester Randolph Ford and Delbert Ray Fulkerson. *Flows in networks*. Princeton university press, 2015.

[11] András Frank. Applications of submodular functions. *Surveys in combinatorics*, 187:85–136, 1993.

[12] András Frank. *Connections in combinatorial optimization*, volume 38. Oxford University Press Oxford, 2011.

[13] András Frank and Tamás Király. Combined connectivity augmentation and orientation problems. *Discrete Applied Mathematics*, 131(2):401–419, 2003.

[14] Greg N Frederickson. Approximation algorithms for some postman problems. *Journal of the ACM (JACM)*, 26(3):538–554, 1979.

[15] Michael R Garey, David S Johnson, and Larry Stockmeyer. Some simplified NP-complete problems. In *Proceedings of the sixth annual ACM symposium on Theory of computing*, pages 47–63, 1974.

[16] Luis A Goddyn. Some open problems I like. `https://www.sfu.ca/~goddyn/Problems/problems.html`.

[17] Luis A Goddyn, Michael Tarsi, and Cun-Quan Zhang. On (k, d)-colorings and fractional nowhere-zero flows. *Journal of Graph Theory*, 28(3):155–161, 1998.

[18] Martin Grötschel, László Lovász, and Alexander Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197, 1981.

[19] Alan J Hoffman. Some recent applications of the theory of linear inequalities to extremal combinatorial analysis. In *Selected Papers Of Alan J Hoffman: With Commentary*, pages 244–258. World Scientific, 2003.

[20] Ian Holyer. The NP-completeness of edge-coloring. *SIAM Journal on computing*, 10(4):718–720, 1981.

[21] Satoru Iwata, Lisa Fleischer, and Satoru Fujishige. A combinatorial strongly polynomial algorithm for minimizing submodular functions. *Journal of the ACM (JACM)*, 48(4):761–777, 2001.

[22] François Jaeger. On nowhere-zero flows in multigraphs. *Proceedings, Fifth British Combinatorial Conference, Aberdeen*, pages 373–378, 1975.

[23] François Jaeger. Balanced valuations and flows in multigraphs. *Proceedings of the American Mathematical Society*, 55(1):237–242, 1976.

[24] François Jaeger. Flows and generalized coloring theorems in graphs. *Journal of combinatorial theory, series B*, 26(2):205–216, 1979.

[25] François Jaeger. Nowhere-zero flow problems. *Selected Topics in Graph Theory*, 3:71–95, 1988.

[26] Charles H Kappauf and Gary J Koehler. The mixed postman problem. *Discrete Applied Mathematics*, 1(1-2):89–103, 1979.

[27] Sanjeev Khanna, Joseph Naor, and F Bruce Shepherd. Directed network design with orientation constraints. *SIAM Journal on Discrete Mathematics*, 19(1):245–257, 2005.

[28] Martin Kochol. Hypothetical complexity of the nowhere-zero 5-flow problem. *Journal of Graph Theory*, 28(1):1–11, 1998.

[29] Claudio L Lucchesi and Daniel Haven Younger. A minimax theorem for directed graphs. *Journal of the London Mathematical Society*, 2(3):369–374, 1978.

[30] Wolfgang Mader. A reduction method for edge-connectivity in graphs. In *Annals of Discrete Mathematics*, volume 3, pages 145–164. Elsevier, 1978.

[31] Edward Minieka. The Chinese postman problem for mixed networks. *Management Science*, 25(7):643–648, 1979.

[32] C St JA Nash-Williams. On orientations, connectivity and odd-vertex-pairings in finite graphs. *Canadian Journal of Mathematics*, 12:555–567, 1960.

[33] Christos H Papadimitriou. On the complexity of edge traversing. *Journal of the ACM (JACM)*, 23(3):544–554, 1976.

[34] Balaji Raghavachari and Jeyakesavan Veerasamy. A 3/2-approximation algorithm for the mixed postman problem. *SIAM Journal on Discrete Mathematics*, 12(4):425–433, 1999.

[35] Balaji Raghavachari and Jeyakesavan Veerasamy. Approximation algorithms for the asymmetric postman problem. In *Proceedings of the tenth annual ACM-SIAM symposium on Discrete algorithms*, pages 734–741, 1999.

[36] Ted K Ralphs. On the mixed Chinese postman problem. *Operations Research Letters*, 14(3):123–127, 1993.

[37] Thomas J Schaefer. The complexity of satisfiability problems. In *Proceedings of the tenth annual ACM symposium on Theory of computing*, pages 216–226, 1978.

[38] Alexander Schrijver. A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *Journal of Combinatorial Theory, Series B*, 80(2):346–355, 2000.

[39] Paul D Seymour. Nowhere-zero 6-flows. *Journal of Combinatorial Theory, series B*, 30(2):130–135, 1981.

[40] Mohit Singh and László A Végh. Approximating minimum cost connectivity orientation and augmentation. *SIAM Journal on Computing*, 47(1):270–293, 2018.

[41] Carsten Thomassen. The weak 3-flow conjecture and the weak circular flow conjecture. *Journal of Combinatorial Theory, Series B*, 102(2):521–529, 2012.

[42] Craig A Tovey. A simplified NP-complete satisfiability problem. *Discrete Applied Mathematics*, 8(1):85–89, 1984.

[43] William Thomas Tutte. A contribution to the theory of chromatic polynomials. *Canadian Journal of Mathematics*, 6:80–91, 1954.

[44] William Thomas Tutte. On the algebraic theory of graph colorings. *Journal of Combinatorial Theory*, 1(1):15–50, 1966.

[45] Jeyakesavan Veerasamy. *Approximation algorithms for postman problems*. The University of Texas at Dallas, 1999.

[46] Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[47] Zaw Win. On the windy postman problem on Eulerian graphs. *Mathematical Programming*, 44:97–112, 1989.

[48] Daniel Haven Younger. Integer flows. *Journal of Graph Theory*, 7(3):349–357, 1983.

[49] Francisco Javier Zaragoza Martinez. *Postman problems on mixed graphs*. University of Waterloo, 2003.

[50] Francisco Javier Zaragoza Martinez. Complexity of the mixed postman problem with restrictions on the edges. In *2006 Seventh Mexican International Conference on Computer Science*, pages 3–10, 2006.