Supersaturation beyond color-critical graphs

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Abstract

The supersaturation problem for a given graph F asks for the minimum number $h_F(n,q)$ of copies of F in an n-vertex graph with $\mathrm{ex}(n,F)+q$ edges. Subsequent works by Rademacher, Erdős, and Lovász and Simonovits determine the optimal range of q (which is linear in n) for cliques F such that $h_F(n,q)$ equals the minimum number $t_F(n,q)$ of copies of F obtained from a maximum F-free n-vertex graph by adding q new edges. A breakthrough result of Mubayi extends this line of research from cliques to color-critical graphs F, and this was further strengthened by Pikhurko and Yilma who established the equality $h_F(n,q)=t_F(n,q)$ for $1\le q\le \epsilon_F n$ and sufficiently large n. In this paper, we present several results on the supersaturation problem that extend beyond the existing framework. Firstly, we explicitly construct infinitely many graphs F with restricted properties for which $h_F(n,q)<q\cdot t_F(n,1)$ holds when $n\gg q\ge 4$, thus refuting a conjecture of Mubayi. Secondly, we extend the result of Pikhurko-Yilma by showing the equality $h_F(n,q)=t_F(n,q)$ in the range $1\le q\le \epsilon_F n$ for any member F in a diverse and abundant graph family (which includes color-critical graphs, disjoint unions of cliques K_r, and the Petersen graph). Lastly, we prove the existence of a graph F for any positive integer s such that $h_F(n,q)=t_F(n,q)$ holds when $1\le q\le \epsilon_F n^{1-1/s}$, and $h_F(n,q)<t_F(n,q)$ when $n^{1-1/s}/\epsilon_F\le q\le \epsilon_F n$, indicating that $q=\Theta(n^{1-1/s})$ serves as the threshold for the equality $h_F(n,q)=t_F(n,q)$. We also discuss some additional remarks and related open problems.

1 Introduction

Let F be a graph. A graph is *F-free* if it does not contain F as a subgraph. The *Turán number* $\mathrm{ex}(n,F)$ of F denotes the maximum number of edges in an n-vertex F-free graph. An n-vertex graph is called an *extremal graph for F* if it is F-free and has the maximum number $\mathrm{ex}(n,F)$ of edges. In this paper, we study the *supersaturation problem for F*, that is, to determine the minimum number $h_F(n,q)$ of copies of F in an n-vertex graph with $\mathrm{ex}(n,F)+q$ edges. A related concept is the minimum number $t_F(n,q)$ of copies of F in graphs obtained from an n-vertex extremal graph for F by adding q new edges. It is worth noting that $h_F(n,q)\le t_F(n,q)$, and extensive research has been conducted in the literature to establish the equality $h_F(n,q)=t_F(n,q)$ under certain circumstances. This paper presents results on the supersaturation problem that go beyond the existing framework, showcasing intricate and unexpected relations between $h_F(n,q)$, $q\cdot t_F(n,1)$, and $t_F(n,q)$ in particular.

The celebrated Turán theorem [32] (the case $r=2$ was first proved by Mantel [17]) states that any n-vertex graph with $t_r(n)+1$ edges contains at least one copy of K_{r+1}, where $t_r(n)$ denotes the number of edges in the Turán graph $T_r(n)$, i.e., the complete r-partite n-vertex graph. In 1941, Rademacher proved that any n-vertex graph with $t_2(n)+1$ edges contains at least $\lfloor n/2\rfloor$ copies of K_3. Stated in the above context, we have the equality $h_{K_3}(n,1)=\lfloor n/2\rfloor=t_{K_3}(n,1)$. This result is often recognized as the starting point for the study on the

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supersaturation problem in extremal graph theory. In subsequent papers [1, 2], Erdős extended this by showing that: there exists a constant $\epsilon_3 > 0$ so that

$$h_{K_3}(n,q) = t_{K_3}(n,q) \text{ holds for any } 1 \le q < \epsilon_3 n.$$

Later Lovász and Simonovits [15] determined the optimal value of ϵ_3 as $n \to \infty$, confirming a longstanding conjecture of Erdős. In a subsequent work, Lovász and Simonovits [16] extended their result from the triangle K_3 to every clique K_r, establishing the equality $h_{K_r}(n,q) = t_{K_r}(n,q)$ for any $1 \le q < \epsilon_r n$ with the best constant ϵ_r. In fact Lovász and Simonovits [16] completely solved the supersaturation problem for cliques K_r with $r \ge 3$ when $q = o(n^2)$. The case $q = \Omega(n^2)$ of the supersaturation problem for cliques K_r has also been extensively studied, see [6, 7, 13, 22, 24, 25] and the references therein.

The supersaturation problems were also investigated for general graphs beyond just cliques. For bipartite graphs, the captivating conjecture put forth by Erdős-Simonovits [30] and Sidorenko [27] has received significant attention and extensive research efforts. However, in the scope of this paper, we will not delve into a detailed discussion of this conjecture, and instead, we will focus on non-bipartite graphs. Now let F be a non-bipartite non-clique graph. By the number of copies of F in a given graph G, we mean the number of edge subsets $A \subseteq E(G)$ which induces an copy of F. This also equals the number of edge-preserving injections from $V(F)$ to $V(G)$ divided by $\mathrm{Aut}(F)$, where $\mathrm{Aut}(F)$ denotes the number of automorphisms of F. A graph is *color-critical* if it contains an edge whose deletion reduces its chromatic number. The family of color-critical graphs plays an important role in the development of extremal graph theory. A classic theorem of Simonovits [28] states that the Turán graph $T_r(n)$ is the unique extremal graph for any color-critical graph F with chromatic number $r+1$ when n is sufficiently large. In other words, he proved that if n is sufficiently large then any n-vertex graph with $t_r(n)+1$ edges contains at least one copy of such F. In a breakthrough paper, Mubayi [19] extended Simonovits' theorem using a novel and unified approach for color-critical graphs. Throughout this paper, for any graph F, let $c(n,F)$ be the minimum number of copies of F obtained from an n-vertex extremal graph for F by adding one edge.[1]

Theorem 1.1 (Mubayi [19]). *For every color-critical graph F with chromatic number $r+1$, there exists a constant $\delta = \delta_F > 0$ such that if n is sufficiently large and $1 \le q \le \delta n$, then any n-vertex graph with $t_r(n)+q$ edges contains at least $q \cdot c(n,F)$ copies of F. That is, $h_F(n,q) \ge q \cdot c(n,F)$.*

One significant aspect of this result is its utilization of the Graph Removal Lemma (see e.g. [11]) and the Erdos-Simonovits Stability Theorem [4, 5, 28] to accurately count substructures in graphs. We point out that provided $1 \le q \le \delta n$, the lower bound $h_F(n,q) \ge q \cdot c(n,F)$ is sharp for many color-critical graphs F (including cliques, odd cycles, and the graph obtained from K_4 by deleting an edge); moreover, it is asymptotically tight for any color-critical graph F due to the following fact:

$$q \cdot c(n,F) \le t_F(n,q) \le (1+o(1))q \cdot c(n,F) \quad \Longrightarrow \quad (1-o(1))t_F(n,q) \le h_F(n,q) \le t_F(n,q) \text{ for } 1 \le q \le \delta n.$$

This line of research on color-critical graphs was further enhanced by Pikhurko and Yilma [23]. Among other results, they proved the following strengthening of Theorem 1.1.

Theorem 1.2 (Pikhurko and Yilma [23]). *For every color-critical graph F, there exists a constant $\delta = \delta_F > 0$ such that if n is sufficiently large n and $1 \le q \le \delta n$, then $h_F(n,q) = t_F(n,q)$.*

The authors [23] also determined $h_F(n,q)$ asymptotically for any color-critical graph F in the case $q = o(n^2)$, by reducing to some optimization problems (see Theorems 3.10-3.11 in [23]). Of particular interest to them is identifying a threshold for when graphs obtained from extremal graphs for F by adding q new edges are optimal or asymptotically optimal in the range $q = O(n)$ (e.g., equations (3) and (4) in [23]). We will explore this intriguing question, showing that such thresholds can be rather sophisticated.

To the best of our knowledge, the study of supersaturation problems for non-bipartite graphs, specifically excluding color-critical cases, has only recently been undertaken for the "bowtie" graph, which consists of two

[1] Note that for any graph F, we have $c(n,F) = t_F(n,1)$ and $t_F(n,q) \ge q \cdot c(n,F)$ for any $q \ge 1$.

copies of K_3 merged at a vertex, as explored by Kang, Makai and Pikhurko in [9]. On the other hand, the powerful approach utilizing the graph removal lemma and the Erdos-Simonovits stability theorem, as introduced in [19], was effectively employed in the proof of the aforementioned Theorem 1.2 of [23], and subsequently extended to hypergraph settings in [20, 21]. These results "suggest that whenever one can obtain stability and exact results for an extremal problem, one can also obtain counting results", cited from [20]. In an effort to unify this approach, Mubayi [20] formulated a conjecture as follows. An r-uniform hypergraph (i.e., an *r-graph* in short) F is **stable** if $\mathrm{ex}(n, F)$ is achieved by a unique n-vertex r-graph $H(n)$ for sufficiently large n, and every n-vertex F-free r-graph with $(1 - o(1))\mathrm{ex}(n, F)$ edges can be obtained from $H(n)$ by changing at most $o(n^r)$ edges.

Conjecture 1.3 (Mubayi, Conjecture 5.1 in [20])**.** *Let $r \geq 2$ and let F be a non r-partite stable r-graph. For every positive integer q, if n is sufficiently large, then $h_F(n, q) \geq q \cdot c(n, F)$.*[2]

In this paper, we investigate supersaturation problems beyond color-critical graphs while exploring the corresponding natural enumerative parameters. Our first result refutes Conjecture 1.3 in the graph case by providing a counterexample for every integer $q \geq 4$, in the following strong form.

Theorem 1.4. *There exists a non-bipartite stable graph F such that the following holds. There exist a small constant $\delta = \delta_F > 0$ and an integer $n_0 = n_0(F)$ such that for any integers $n \geq n_0$ and $4 \leq q \leq \delta n$, it holds that*

$$\frac{h_F(n, q)}{q \cdot c(n, F)} \leq 1 - \delta.$$

The proof of this result actually yields infinitely many counterexamples F with arbitrary chromatic number at least four to Conjecture 1.3. Additionally, since $t_F(n, q) \geq q \cdot c(n, F)$, this implies that for such F,

$$h_F(n, q) < t_F(n, q) \text{ holds for any fixed } q \geq 4 \text{ and sufficiently large } n.$$

To the best of our knowledge, these examples represent the first instances with the above property for general graphs. We will discuss more about related problems in the concluding remarks.

Our second main result extends Theorem 1.2 to a diverse and abundant family of graphs. The precise definition of this family requires the introduction of some technical notations, which we will defer until Definition 5.2. We mention here that this family includes color-critical graphs, Kneser graphs $K(t, 2)$, disjoint unions of cliques K_r, and many others (see the remarks following Definition 5.2). In the subsequent statement, we focus solely on the Kneser graphs $K(t, 2)$, which are the graphs with the vertex set $\binom{[t]}{2}$ where two vertices A and B in $\binom{[t]}{2}$ are adjacent if and only if $A \cap B = \emptyset$; we refer to Subsection 5.2 for a detailed discussion on extremal results concerning the Kneser graphs $K(t, 2)$.

Theorem 1.5. *For any Kneser graph $K = K(t, 2)$ with $t \geq 5$, there exists a constant $\delta > 0$ such that for any sufficiently large integer n and any integer $1 \leq q \leq \delta n$, we have $h_K(n, q) = t_K(n, q)$.*

A notable case is the Petersen graph $\mathbf{P}$, which corresponds to the Kneser graph $K(5, 2)$. As a prompt corollary, one can deduce from Theorem 1.5 and an old result of Simonovits on $\mathrm{ex}(n, \mathbf{P})$ [29] that for sufficiently large n,

$$h_{\mathbf{P}}(n, 1) = c(n, \mathbf{P}) = 96\binom{\lceil \frac{n}{2} \rceil - 3}{2}\binom{\lfloor \frac{n}{2} \rfloor - 1}{4} \approx \frac{n^6}{32}.$$

Our proof, similar to [19, 23], employs the graph removal lemma and the Erdos-Simonovits stability theorem as the main tools, while also requiring novel techniques for counting substructures in various scenarios. The full statement of this result can be found in Theorem 5.5.

Our final result explores the thresholds for the equality $h_F(n, q) = t_F(n, q)$ to hold as q varies as a function of n for graphs F. As noted previously, this question was examined in [23] for color-critical graphs. The following result indicates that for any positive integer s, this threshold can be achieved with $q = \Theta(n^{1-1/s})$ for some non-bipartite stable graph F.

[2]Here, these definitions for r-graphs F are analogously defined.

Theorem 1.6. *For any positive integer s, there exists a non-bipartite stable graph F such that the following holds. There is a constant $\epsilon > 0$ such that for every sufficiently large integer n,*

(1) if $1 \le q \le \epsilon n^{1-1/s}$, then $h_F(n,q) = t_F(n,q)$, and

(2) if $n^{1-1/s}/\epsilon \le q \le \epsilon n$, then $h_F(n,q) < t_F(n,q)$.

The organization of this paper is as follows: In Section 2, we provide preliminaries, including notations, key lemmas, and the definition of a graph family that plays a crucial role throughout this paper. Section 3 presents an explicit example to prove Theorem 1.4 and refute Conjecture 1.3. In Section 4, we establish quantitative and structural properties for graphs with the minimum number of copies of F, which are essential for the subsequent sections. Section 5 introduces a special family of graphs and demonstrates that for any graph F in this family, the equality $h_F(n,q) = t_F(n,q)$ holds for $1 \le q \le \epsilon_F n$ and sufficiently large n, implying Theorem 1.5. In Section 6, we complete the proof of Theorem 1.6. Finally, in the concluding section, we provide several remarks and discuss related problems.

2 Preliminaries

2.1 Notations

Let G be a given graph. The *neighborhood* of a vertex u in G is denoted by $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. By $N_G[u]$ we denote the set $N_G(u) \cup \{u\}$. The *degree* $d_G(u)$ of the vertex u in G is the size of $N_G(u)$. For an edge subset $A \subseteq E(G)$, we use $d_A(u)$ to denote the number of edges in A incident with u. For a vertex subset $X \subseteq V(G)$, let $N_X(u) = X \cap N_G(u)$ and $d_X(u) = |N_X(u)|$. We use $N_G(X)$ and $N_G[X]$ to denote $\left(\bigcup_{u\in X} N_G(u)\right)\setminus X$ and $\bigcup_{u\in X} N_G[u]$, respectively. We also write $e_G(X)$ to express the number of edges contained in the induced subgraph $G[X]$. We say X is *stable* if there is no edges of G contained in X. We often drop the subscript when the graph G is clear from the context. For a subset S of vertices or edges, let $G - S$ or $G \setminus S$ be the graph obtained from G by deleting every element in S. Denote by $\overline{G}$ the complement graph of G.

Let G and H be graphs and k be a positive integer. Denote by $G \cup H$ the vertex-disjoint union of G and H and by $k \cdot G$ the vertex-disjoint union of k copies of a graph G. Let $G + H$ be obtained from $G \cup H$ by adding all possible edges between $V(G)$ and $V(H)$. For graphs $H_1, \ldots, H_k$, it is connivent to use $H_1 + \ldots + H_k$ to express the graph $(H_1 + \ldots + H_{k-1}) + H_k$. For a set X of vertices, by $K[X]$ we mean the complete graph with the vertex set X. Let $K(V_1, \ldots, V_r)$ denote the complete r-partite graph with parts $V_1, \ldots, V_r$. For a graph F, we denote the number of copies of F in a graph G as $\mathcal{N}_F(G)$ (sometimes also written as $\#F(G)$).

We denote the independent set on k vertices by I_k, the star on k vertices by S_k, the path on k vertices by P_k, and the matching of k edges by M_k. For two functions $f, g : \mathbb{N}^+ \to \mathbb{R}^+$, by $f = \Omega(g)$ we mean $f \ge c \cdot g$ for a sufficiently large constant c, by $f = O(g)$ we mean $f \le d \cdot g$ for a fixed constant $d > 0$, and by $f = \Theta(g)$ we mean that $c_1 \cdot g \le f \le c_2 \cdot g$ for fixed constants $c_2 > c_1 > 0$. Throughout this paper, we write $[k]$ for the set $\{1, 2, \ldots, k\}$.

2.2 Extremal results

We introduce some classic theorems and useful lemmas needed in the following proofs. As we discussed in the introduction, the Graph Removal Lemma (see e.g., Theorem 2.9 in [11]) and the Erdős-Simonovits Stability Theorem are key to the proofs (of Theorems 1.5 and 1.6).

Theorem 2.1 (Graph Removal Lemma [11])**.** *Let F be a graph with f vertices. Then for every $\delta > 0$ there is $\epsilon > 0$ such that every graph with $n \ge 1/\epsilon$ vertices and at most ϵn^f copies of F can be made F-free by removing at most δn^2 edges.*

Theorem 2.2 (Erdős-Simonovits Stability Theorem [4, 5, 28]))**.** *Let $r \ge 2$ and F be a graph with chromatic number $r + 1$. Then for every $\delta > 0$ there is $\epsilon > 0$ such that every F-free graph H with $n \ge 1/\epsilon$ vertices and* *at least $t_r(n)-\epsilon n^2$ edges contains an r-partite subgraph with at least $t_r(n)-\delta n^2$ edges and moreover, H can be obtained from an extremal graph for F by changing at most δn^2 edges.*

Let $Z(m,n,a,b)$ be the maximum number of edges of $G \subseteq K(m,n)$ such that G does not contain a copy of $K_{a,b}$ with a vertices from the first class and b vertices from the second class of $K(m,n)$. In 1954, Kövári, Sós and Turán [12] proved the following classic result.

Theorem 2.3 (Kövári, Sós and Turán, [12])**.** *For any integers $m \geq a$ and $n \geq b$, it holds that*

$$Z(m,n,a,b) \leq (b-1)^{1/a} \cdot mn^{1-1/a} + (a-1)n$$

We need the following special form of Theorem 2.3.

Lemma 2.4. *For every real $\delta > 0$ and integer $m \geq 1$, there exists a real $\epsilon > 0$ such that the following holds. If G is an (m,n)-bipartite graph where each vertex in the partite set of size m has degree at least δn, then G contains a copy of $K_{\delta m, \epsilon n}$.*

Proof. TOPROVE 0 □

The next lemma provides a handy tool for counting matchings of given size.

Lemma 2.5. *Let $\epsilon \in (0,1)$ be a small constant. Let G be an n-vertex graph with $e(G) \geq 2k\epsilon n$ and maximum degree $\Delta(G) \leq \epsilon n$. Then $\mathcal{N}_{M_k}(G) \geq (k-1)!(2\epsilon n)^k$.*

Proof. TOPROVE 1 □

We also need the following useful lemma proved by Mubayi [19].

Lemma 2.6 (Mubayi, Lemma 4 in [19])**.** *Suppose that $r \geq 2$ is fixed, n is sufficiently large, $s < n$ and $n_1 + \ldots + n_r = n$. If $\sum_{1 \leq i < j \leq r} n_i n_j \geq t_r(n) - s$, then $\lfloor n/r \rfloor - s \leq n_i \leq \lceil n/r \rceil + s$ for all $i \in [r]$.*

2.3 Color-k-critical graphs

In this subsection, we introduce a significant family of graphs that plays a crucial role in our proofs: the color-k-critical graphs. We will also present an extremal result due to Simonovits for graphs in this family.

Definition 2.7. For any positive integer k, a graph G is called **color-k-critical** if

(i). there exist k suitable edges whose removal decreases its chromatic number, and

(ii). deleting any $k-1$ vertices does not decrease its chromatic number.

It is clear from the definition that any k edges whose removal decreases $\chi(G)$ must form a matching of size k. In particular, color-1-critical graphs are just color-critical graphs.[3]

In [29], Simonovits determined the unique extremal graph for every color-k-critical graph.

Definition 2.8. *Denote by $H(n,r,k) = K_{k-1} + T_r(n-k+1)$ the n-vertex graph obtained by joining each vertex of the Turán graph $T_r(n-k+1)$ to each vertex of a copy of K_{k-1}. Let $h(n,r,k) = e(H(n,r,k))$.*

Theorem 2.9 (Simonovits, Theorem 2.2 in [29])**.** *Let $k \geq 1$ and let F be a color-k-critical graph with $\chi(F) = r+1$. If n is sufficiently large, then $H(n,r,k)$ is the unique extremal graph for F.*

It is already known that (see, i.e., [29, 31]) the family of color-k-critical graphs is rich, including disjoint unions of cliques K_r, the Petersen graph, and the dodecahedron graph. In Subsection 5.2, we show that Kneser graphs $K(t,2)$ for every $t \geq 6$ are color-k-critical graphs for $k=3$ (and actually we show that they are color-3-critical with additional nice properties).

We conclude this section with the following lemma. It is easy to see that the only bipartite color-k-critical graph is the matching of size k.

Lemma 2.10. *Any non-bipartite color-k-critical graph is stable.*

Proof. TOPROVE 2 □

[3] This is why we refer to this family as color-k-critical, as it naturally extends the concept of color-critical graphs.

3 Counterexamples to Conjecture 1.3

In this section, we prove Theorem 1.4 by providing a counterexample to Conjecture 1.3 for every integer $q \geq 4$. As we shall see later, this proof in fact leads to infinitely many counterexamples to Conjecture 1.3. We will construct a non-bipartite stable graph F and show that there exists a small constant $b_F > 0$ such that for any sufficiently large integer n and any integer $4 \leq q \leq b_F n$,

$$\frac{h_F(n,q)}{q \cdot c(n,F)} \leq 1 - b_F.$$

Let $k \geq 2$ be any integer. Throughout this section, we let $A = M_k$ and $B = P_4 \cup M_{k-2}$ be two fixed graphs and define $F = A + B$ (See Figure 1 (a)). It is clear that $\chi(F) = 4$.

We first explain that F is a stable color-k-critical graph. If we delete any $k-1$ vertices from F, the resulting graph contains at least one edge in A and at least one edge in B and hence contains a copy of K_4. Moreover, it is easy to see that removing all k edges in A will decrease the chromatic number by one. Hence, F is indeed color-k-critical. By Lemma 2.10, we see that F is also stable.

Let $X \cup V_1 \cup V_2 \cup V_3$ be the partition of $V(H(n,3,k))$ such that X induces the clique of size $k-1$ and each V_i is an independent set of size n_i for $i \in [3]$, where $\lceil (n-k+1)/3 \rceil = n_1 \geq n_2 \geq n_3 = \lfloor (n-k+1)/3 \rfloor$. Let H_i be the graph obtained from $H(n,3,k)$ by adding one edge into V_i for $i \in [3]$.

Throughout the rest of the proof, let $\{i,j,\ell\} = \{1,2,3\}$. We now consider all possible embeddings of F in each H_i. Suppose that H_i contains a copy of $F = A + B$. We claim that

$$\text{either } V(A) \subseteq X \cup V_i \text{ and } V(B) \subseteq V_j \cup V_\ell, \text{ or } V(B) \subseteq X \cup V_i \text{ and } V(A) \subseteq V_j \cup V_\ell. \tag{1}$$

To see this, first suppose that $V_j \cup V_\ell$ contains some vertices $x \in A$ and $y \in B$. Then $H_i[V_j \cup V_\ell]$ cannot contain an edge from A or from B; otherwise this edge (say in A) together with the vertex y in B will form a triangle (by the definition of F) in $H_i[V_j \cup V_\ell]$, but $H_i[V_j \cup V_\ell]$ is bipartite, a contradiction. Hence $H_i[V_j \cup V_\ell]$ contains at most k vertices from A and at most k vertices from B. That says, $H_i[X \cup V_i]$ must contains at least k vertices from A and at least k vertices from B, and hence contains a copy of $K_{k,k}$, a contradiction. So $V_j \cup V_\ell$ has either (i) no vertices from A or (ii) no vertices from B. Suppose (i) occurs. If $V_j \cup V_\ell$ contains at most $2k-1$ vertices of B, then $H_i[X \cup V_i]$ contains all $2k$ vertices of A and at least one vertex of B. In particular, $H_i[X \cup V_i]$ contains a copy of $K_1 + M_k$, but this is a contradiction. Hence when (i) occurs, $V_j \cup V_\ell$ must contain all vertices from B, implying (1). The other case (ii) can be derived similarly. This proves (1).

Let $c_i(n,F)$ denote the number of copies of F in H_i. Using (1) we can compute $c_i(n,F)$ precisely. We note that the numbers of copies of A and B in $K_{k,k}$ are $k!$ and $k!k(k-1)$, respectively. Moreover, the numbers of copies of A and B in $K_{k-1} + (K_2 \cup I_{k-1})$ are $(k-1)!$ and $(\frac{3k}{2}-1)(k-1)(k-1)!$, respectively.[4] Following (1), there are only two ways of embedding F in H_i, which leads to

$$\begin{aligned} c_i(n,F) = &\left((k-1)!\binom{n_i-2}{k-1}\right) \cdot \left(k!k(k-1)\binom{n_j}{k}\binom{n_\ell}{k}\right) \\ &+ \left((3k/2-1)(k-1)(k-1)!\binom{n_i-2}{k-1}\right) \cdot \left(k!\binom{n_j}{k}\binom{n_\ell}{k}\right) \\ =&\frac{(k-1)(5k-2)}{2} \cdot (k-1)!k! \cdot \binom{n_i-2}{k-1}\binom{n_j}{k}\binom{n_\ell}{k}. \end{aligned}$$

Since $\binom{(x+1)-2}{k-1}\binom{x}{k} < \binom{x-2}{k-1}\binom{x+1}{k}$ for sufficiently large integers x, we have

$$c(n,F) = \min_{1 \leq i \leq 3} c_i(n,F) = c_1(n,F) = \frac{(k-1)(5k-2)}{2} \cdot (k-1)!k! \cdot \binom{n_1-2}{k-1}\binom{n_2}{k}\binom{n_3}{k}. \tag{2}$$

[4]The later one holds because that the numbers of copies of $B = P_4 \cup M_{k-2}$ in $K_{k-1} + (K_2 \cup I_{k-1})$ with the middle edge of P_4 lying inside K_{k-1}, between K_{k-1} and I_{k-1}, and between K_{k-1} and K_2 are $\binom{k-1}{2}(k-1)!$, $(k-1)(k-2)(k-1)!$, and $2(k-1)(k-1)!$, respectively, which add up to $(\frac{3k}{2}-1)(k-1)(k-1)!$.

In what follows, we will construct an n-vertex graph H^* with $\mathrm{ex}(n,F)+q = e(H(n,3,k))+q$ edges which contains at most $(1-b_F)\cdot q\cdot c(n,F)$ copies of F. As indicated in the beginning of this section, here we take n to be sufficiently large and q to be any integer at least 4 and at most $b_F\cdot n$ for some small constant $b_F>0$. To construct H^*, we first take $H' = I_{k-1}+T_3(n-k+1)$ and let V_1,V_2,V_3 be the three partite sets of $T_3(n-k+1)$ with $n_i=|V_i|$ and $n_1\geq n_2\geq n_3$. Let $t=q+\binom{k}{2}+1$. Now define H^* to be the graph obtained from H' by first adding a copy of the star S_t into V_1 and then removing the $k-1$ edges between the center of S_t and I_{k-1} of H'.

First observe that indeed H^* has $e(H')+(t-1)-(k-1) = e(H(n,3,k))-\binom{k-1}{2}+q+\binom{k}{2}-k = e(H(n,3,k))+q$ edges. Note that any copy of F in H^* using $w\geq 2$ edges of S_t must contain $w+1$ vertices of S_t, all vertices in I_{k-1}, and $3k-w$ other vertices. So the number of copies of F in H^* using at least two edges from S_t is $O_F(\sum_{w\geq 2} q^w n^{3k-w}) = O_F(q^2n^{3k-2})$, where the inequality holds as $q/n\leq b_F$. Next we consider the number $\mathcal{N}_1$ of copies of F in H^* using exactly one edge from S_t. We point out that every such F use all $k-1$ vertices of I_{k-1} and thus the claim (1) applies when counting $\mathcal{N}_1$. Since the k edges between the center of S_t and I_{k-1} are deleted in H^*, the number of copies of B in H^* using a fixed edge from S_t and $k-1$ fixed vertices of V_1 equals $(k-1)(k-2)(k-1)!+(k-1)(k-1)! = (k-1)^2(k-1)!$. Following (1) we have

$$\frac{\mathcal{N}_1}{t-1} = \Big(k!\cdot(k-1)^2(k-1)!+(k-1)!\cdot k!k(k-1)\Big)\cdot\binom{n_1-2}{k-1}\binom{n_2}{k}\binom{n_3}{k},$$

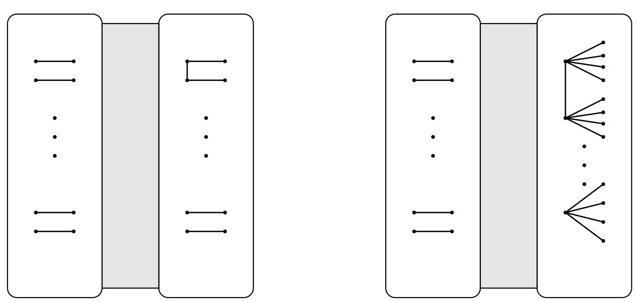
where $t-1=q+\binom{k}{2}$. Putting everything together, $\mathcal{N}_F(H^*) = \mathcal{N}_1+O_F(q^2n^{3k-2})$ which gives that

$$\mathcal{N}_F(H^*) = \left(q+\binom{k}{2}\right)\cdot(k-1)(2k-1)\cdot(k-1)!k!\cdot\binom{n_1-2}{k-1}\binom{n_2}{k}\binom{n_3}{k}+O_F(q^2n^{3k-2}). \tag{3}$$

Comparing with (2) and (3), we see that there exists a small constant $b_F>0$ such that $\mathcal{N}_F(H^*)\leq(1-b_F)\cdot q\cdot c(n,F)$ as long as $\left(q+\binom{k}{2}\right)\cdot(2k-1)<(1-b_F)\cdot q\cdot\frac{5k-2}{2}$ and $q\leq b_Fn$. Solving the inequality, this shows that there exists a small constant $b_F>0$ such that

$$\mathcal{N}_F(H^*)<(1-b_F)\cdot q\cdot c(n,F)\text{ whenever } q \text{ satisfies that } (k-1)(2k-1)<q\leq b_Fn.$$

In particular, if we take $k=2$ and $F=M_2+P_4$, then $h_F(n,q)\leq\mathcal{N}_F(H^*)<(1-b_F)\cdot q\cdot c(n,F)$ holds for any $4\leq q\leq b_Fn$ when n is sufficiently large. The proof of Theorem 1.4 is complete. ■



(a). For Theorem 1.4 (b). For Theorem 1.6

Figure 1. Examples for color-k-critical graphs

4 Properties on supersaturated graphs

In the rest of this paper, let F be a color-k-critical graph on f vertices with $\chi(F)=r+1$ where $r\geq 2$. This section aims to establish some quantitative and structural properties for graphs with the minimum number of copies of F subject to given numbers of vertices and edges.

4.1 Basic properties

We first present some lemmas on the minimum number of copies of F obtained from some well-characterized graphs by adding few edges, which generalize similar lemmas proved in [19, 23].

Recall Theorem 2.9 that for sufficiently large n, $H(n, r, k)$ is the unique n-vertex extremal graph for F. The coming two lemmas concerns quantitative properties of $c(n, F)$, which, in this case, denotes the minimum number of copies of F obtained from $H(n, r, k)$ by adding one new edge. Let $n_1, \ldots, n_r$ be positive integers satisfying $\sum_{i=1}^{r} n_i = n - k + 1$ and let $H(n_1, \ldots, n_r)$ be the graph obtained from $K_{k-1} + K(V_1, \ldots, V_r)$ by adding a new edge xy into V_1 where each $|V_i| = n_i$. Let $c(n_1, \ldots, n_r; F)$ be the number of copies of F contained in $H(n_1, \ldots, n_t)$.

Lemma 4.1. *There are positive constants α_F, β_F such that if n is sufficiently large, then*

$$|c(n, F) - \alpha_F n^{f-k-1}| < \beta_F n^{f-k-2}.$$

In particular, $\frac{1}{2}\alpha_F n^{f-k-1} < c(n, F) < 2\alpha_F n^{f-k-1}$.

Proof. TOPROVE 3 □

Lemma 4.2. *There exist constants θ_F and η_F such that the following holds for sufficiently large n. Let $\sum_{i=1}^{r} n_i = \sum_{i=1}^{r} n'_i = n - k + 1$ and $c(n, F) = c(n'_1, \ldots, n'_r; F)$. Let $a_i = n_i - n'_i$ for each $i \in [r]$ and $A = \max\{|a_i| : i \in [r]\}$. Then $|c(n_1, \ldots, n_r; F) - c(n, F) - \theta_F a_1 n^{f-k-2}| \leq \eta_F A^2 n^{f-k-3}$.*

Proof. TOPROVE 4 □

Let $d(n, F)$ be the minimum number of copies of F in the graph obtained from $T_r(n)$ by adding a copy of M_k to one partite set of $T_r(n)$. By a proof similar to that of Lemma 4.1, we can show the following lemma and in particular, $d(n, F)$ is a polynomial in n of degree $f - 2k$.

Lemma 4.3. *There are positive constants α'_F, β'_F such that if n is sufficiently large, then*

$$|d(n, F) - \alpha'_F n^{f-2k}| < \beta'_F n^{f-2k-1}.$$

In particular, $\frac{1}{2}\alpha'_F n^{f-2k} < d(n, F) < 2\alpha'_F n^{f-2k}$.

We say an n-vertex r-partite graph G with a partition $V(G) = \bigcup_{i=1}^{r} V_i$ is *δ-equivalence* if $|V_1| = \ldots = |V_r|$ and each vertex is adjacent to at least $(1 - \delta)n/r$ vertices in each of other partite sets.

Lemma 4.4. *Let $0 \leq \delta \ll 1$. Let G' be the graph obtained from an n-vertex δ-equivalence r-partite graph G by adding a copy of M_k into one partite set of G. Then there is a positive constant γ depending on F and δ such that $\mathcal{N}_F(G') \geq d(n, F) - \gamma n^{f-2k}$, where $\gamma \to 0$ as $\delta \to 0$.*

Proof. TOPROVE 5 □

4.2 Refined properties

Let F be a color-k-critical graph on f vertices with $\chi(F) = r + 1$ where $r \geq 2$. Throughout this subsection, we use the following constants satisfying the given hierarchy:

$$1 \gg \epsilon \gg \epsilon_{14} \gg \epsilon_{13} \gg \ldots \gg \epsilon_2 \gg \epsilon_1 \gg \delta \gg \frac{1}{n}.$$

Let $1 \leq q \leq \delta n$ and H be an n-vertex graph with $\mathrm{ex}(n, F) + q$ edges and minimum number of copies of F. We will show some refined properties on H, which are important in the coming sections.

Let $H(n, r, k, q)$ be the graph obtained from $H(n, r, k)$ by adding a copy of S_{q+1} into one part of $H(n, r, k)$ such that the number of copies of F using exactly one edge from S_{q+1} is $c(n, F)$. It is clear (by considering the definition of color-k-critical) that any copy of F in $H(n, r, k, q)$ must use the center of the S_{q+1} as well as the $k-1$ vertices of degree $n-1$ (call them the *top* vertices of $H(n,r,k)$). If a copy of F in $H(n,r,k,q)$ uses $t\geq 2$ edges of S_{q+1}, then except the $t+1$ vertices of S_{q+1} and the $k-1$ top vertices, it uses $f-k-t$ many other vertices. So the number of copies of F in $H(n,r,k,q)$ using at least two edges of S_{q+1} is at most $O_F\left(\sum_{t=2}^{q}\binom{q}{t}\cdot n^{f-k-t}\right)=O_F(q^2)\cdot n^{f-k-2}$, where we use $q/n\leq\delta$. Hence, we have

$$\mathcal{N}_F(H)\leq\mathcal{N}_F(H(n,r,k,q))=q\cdot c(n,F)+O_F(q^2)\cdot n^{f-k-2}\leq\epsilon_1 n^{f-k}=\left(\frac{\epsilon_1}{n^k}\right)n^f. \tag{4}$$

Since n is sufficiently large, by Theorem 2.1 there are at most $\epsilon_2 n^2$ edges of H whose removal results in a graph H' with no copies of F. Since $e(H')>t_r(n)-\epsilon_2 n^2$, by Theorem 2.2, we conclude that there is an r-partition of $V(H)=V(H')$ such that the total number of edges in H' (also in H) between two parts is at least $t_r(n)-\epsilon_3 n^2$.

Fix an r-partition $V(H)=V_1\cup\ldots\cup V_r$ which maximizes $|E(H)\cap E(K(V_1,\ldots,V_r))|$.[5] By the previous paragraph, we have $|E(H)\cap E(K(V_1,\ldots,V_r))|\geq t_r(n)-\epsilon_3 n^2$. Let $|V_i|=n_i$ for $i\in[r]$ with $n_1\geq\ldots\geq n_r$. Using Lemma 2.6, we can derive that

$$\left(\frac{1}{r}-\epsilon_4\right)n\leq|V_i|\leq\left(\frac{1}{r}+\epsilon_4\right)n. \tag{5}$$

Let $B=E(H)\setminus E(K(V_1,\ldots,V_r))$ and $M=E(K(V_1,\ldots,V_r))\setminus E(H)$. Then we have

$$|B|=e(H)-|E(H)\cap E(K(V_1,\ldots,V_r))|\leq(\mathrm{ex}(n,F)+q)-(t_r(n)-\epsilon_3 n^2)\leq\epsilon_4 n^2 \tag{6}$$

and

$$|M|=e(K(V_1,\ldots,V_r))-|E(H)\cap E(K(V_1,\ldots,V_r))|\leq t_r(n)-(t_r(n)-\epsilon_3 n^2)\leq\epsilon_4 n^2. \tag{7}$$

Claim I. There exist exactly $k-1$ vertices $x_1,\ldots,x_{k-1}$ of degree $d_H(x_i)\geq n-\epsilon_9 n$.

Proof. TOPROVE 6 $\square$

Throughout the rest of this section, we denote $X=\{x_1,\ldots,x_{k-1}\}$ from Claim I. Let $H^*=K[X]+K(V_1\setminus X,\ldots,V_r\setminus X)$. Let $B^*=E(H)\setminus E(H^*)$ and call edges in B^* **bad**. Let $M^*=E(H^*)\setminus E(H)$ and call edges in M^* **missing**. Then $|B^*|-|M^*|=e(H)-e(H^*)=h(n,r,k)+q-e(H^*)\geq q$. We point out that by definition, $B^*\subseteq B$ and thus $|B^*|\leq|B|\leq\epsilon_4 n^2$ by (6). The next claim gives a significant improvement on the upper bound of $|B^*|$.

Claim II. It holds that $|B^*|\leq\epsilon_5 n$.

Proof. TOPROVE 7 $\square$

Denote by $f(n,F)$ the minimum number of copies of F obtained from $I_{k-1}+T_r(n-k+1)$ by adding an edge (say e) to one class of $T_r(n-k+1)$ and removing all edges between $V(e)$ and I_{k-1}.[6] So we see $f(n,F)=(n/r)^{k-1}d(n,F)+O_F(n^{f-k-2})$ is a polynomial of degree $f-k-1$.

Claim III. Let $\omega=h(n,r,k)-e(H\setminus B^*)$. Then $|B^*|=q+\omega$, $|M^*|\leq\omega\leq\epsilon_5 n$, and there exists an absolute positive constant $c=c(F)$ such that for each $e\in B^*$, $F(e)\geq f(n,F)-c\cdot\omega\cdot n^{f-k-2}$.

Proof. TOPROVE 8 $\square$

[5]We will call such an r-partition of $V(H)$ as a *max-cut* of H.

[6]Note that here the edges between $V(e)$ and I_{k-1} are deleted, so there is a unique way of embedding F in the resulting graph, i.e., first finding a k-matching consisting of e and edges x_iy_i for $1\leq i\leq k-1$ where $y_1,\ldots,y_{k-1}$ are from the same partite set and then embedding F in the same way as in the definition of $d(n,F)$ (see the paragraph before Lemma 4.3).

5 Admissible color-k-critical graphs

In this section, we first introduce an ample subfamily of color-k-critical graphs (called **admissible**; see Definition 5.2), which include all color-ℓ-critical graphs for $\ell \in \{1,2\}$ and Kneser graphs $K(n,2)$. Subsequently, we demonstrate that for any graph F within this subfamily, there exists a constant $\delta > 0$ such that the equality $h_F(n,q) = t_F(n,q)$ holds for all sufficiently large n and all $1 \le q \le \delta n$ (see Theorem 5.5). These results collectively lead to the proof of Theorem 1.5.

5.1 Definitions and examples

We now define the subfamily of color-k-critical graphs as mentioned above. We begin by the following.

Definition 5.1. Let F be a graph with $\chi(F) = r+1 \ge 3$. Let $\mathcal{F} = (F_0, F_1, \ldots, F_r)$ be an ordered sequence of graphs.[7] Write $E(\mathcal{F}) = \bigcup_{i=0}^{r} E(F_i)$. If the graph $F_0 + F_1 + \ldots + F_r$ contains a copy of F as its spanning subgraph and this F contains all edges in $E(\mathcal{F})$, then we say $\mathcal{F}$ is an *embedding type* (or for short, a *type*) of F. Moreover, we let $\mathcal{F}_\alpha := F_0$ be the *top* of the type $\mathcal{F}$ and $\mathcal{F}_\beta := \bigcup_{i=1}^{r} F_i$ be the *bottom* of the type $\mathcal{F}$. If $|V(F_0)| = \ell$, then we also call $\mathcal{F}$ an ℓ-type.

The definition of types offers us a useful perspective for counting the number of copies F in a graph with a given partition of $r+1$ parts. For a graph G, let $\nu(G)$ be its *matching number*, i.e., the maximum size of a matching in G.

Definition 5.2. A color-k-critical graph F with $\chi(F) = r+1 \ge 3$ is called **admissible**, if for any embedding type $\mathcal{F} = (F_0, F_1, \ldots, F_r)$ of F, the following hold that

(A). $\nu(\bigcup_{i=1}^{r} F_i) \ge k - |V(F_0)|$, and

(B). if there is an edge in F_0, then $\nu(\bigcup_{i=1}^{r} F_i) \ge k + 1 - |V(F_0)|$.

The family of admissible color-k-critical graphs forms a diverse and abundant collection. In what follows, we will provide some notable examples and properties that showcase the richness of this family.

- All color-ℓ-critical graphs F for $\ell \in \{1,2\}$ are admissible. The case when $\ell = 1$ is trivial as both properties (A) and (B) are automatically satisfied. Now we consider the case when $\ell = 2$. First, the property (A) follows by the definition that F is color-2-critical. For (B), clearly it holds when $|V(F_0)| \ge 3$. So we may assume $|V(F_0)| \le 2$. Since there is an edge in F_0, we may assume F_0 is just an edge ab. We need to show $\nu(\bigcup_{i=1}^{r} F_i) \ge 1$, which again follows by the definition.

- In the coming subsection, we show that all Kneser graphs $K(n,2)$ belong to admissible color-k-critical graphs for $k = 3$.

- **Proposition.** If F_1 is an admissible color-k-critical graph and F_2 is an admissible color-ℓ-critical graph with $\chi(F_1) = \chi(F_2)$, then $F_1 \cup F_2$ is an admissible color-$(k+\ell)$-critical graph.

  Repeatedly using this proposition, we see that the disjoint union of cliques of the same size (or more generally, the disjoint union of color-ℓ_i-critical graphs F_i, where $\ell_i \in \{1,2\}$ for $i \in [t]$, of the same chromatic number) is an admissible color-k-critical graph for $k = \sum_{i\in[t]} \ell_i$.

5.2 Kneser graphs

Let n, t be positive integers with $n \ge 2t+1$. The *Kneser graph* $K(n,t)$ is the graph with the vertex set $\binom{[n]}{t}$, where any two vertices $A, B \in \binom{[n]}{t}$ are adjacent if and only if $A \cap B = \emptyset$. Answering a famous conjecture of Kneser [10], Lovász [14] proved that the chromatic number of $K(n,t)$ equals $n - 2t + 2$. For a permutation π

[7] These graphs F_i for $i \in \{0,1,...,r\}$ may be empty.

on $[n]$, we say a t-subset of $[n]$ is *π-stable* if it contains no pairs $\{\pi(i), \pi(i+1)\}$ with $1 \leq i < n$ nor the pair $\{\pi(1), \pi(n)\}$. Schrijver [26] proved that for any permutation π on $[n]$, the induced subgraph of $K(n,t)$ on the vertex set consisting of all π-stable t-subsets of $[n]$ has the same chromatic number $n - 2t + 2$.[8]

To the best of our knowledge, the cases $n \geq 6$ of the following lemma appear to be previously unestablished.

Lemma 5.3. *For any $n \geq 5$, the Kneser graph $K(n,2)$ is color-3-critical with chromatic number $n-2$.*

Proof. TOPROVE 9 $\square$

Combined with Theorem 2.9, this shows that for $K = K(t,2)$, we have $\mathrm{ex}(n,K) = e(H(n,t-3,3))$. The following lemma is the main result of this subsection.

Lemma 5.4. *For any $n \geq 5$, the Kneser graph $K(n,2)$ is an admissible color-3-critical graph.*

Proof. TOPROVE 10 $\square$

5.3 Supersaturation for admissible graphs

In the remainder, we present a proof of the main result of this section as follows. This, in conjunction with Lemma 5.4, provides a complete proof for Theorem 1.5.

Theorem 5.5. *For any admissible color-k-critical graph F, there exists a constant $\delta > 0$ such that for any sufficiently large integer n and any integer $1 \leq q \leq \delta n$, we have $h_F(n,q) = t_F(n,q)$.*

To prove this, we need two preliminary lemmas. The following lemma helps us to bound the number of $\mathcal{F}$-types of admissible color-k-critical graphs F.

Lemma 5.6. *Let G be a graph with m edges and F be an f-vertex graph with minimum degree at lease one. Then the number of copies of F in G is at most $O_F(m^{f-\nu(F)})$.*

Proof. TOPROVE 11 $\square$

Let F be a color-k-critical graph with $\chi(F) = r+1$. Let $\ell \geq 0$ be an integer and $\mathcal{F} = (F_0, F_1, \dots, F_r)$ be an ℓ-type of F. Fix disjoint sets V_i of size $n_i \geq |V(F_i)|$ for $i \in [r]$, where $n/2 \leq \sum_{i \in [r]} n_i \leq n$. Let $K_{\mathcal{F}}$ be obtained from $\overline{K}_\ell + K(V_1, \dots, V_r)$ by embedding $E(F_0)$ into $\overline{K}_\ell$ and $E(F_i)$ into V_i for $i \in [r]$. Denote by $c_{\mathcal{F}}(n_1, \dots, n_r)$ the number of copies of F in $K_{\mathcal{F}}$ containing all edges of $E(\mathcal{F})$.

The following lemma can be easily proven by the same argument as Lemmas 4.1 and 4.2, the details of which are omitted here. Let $i(G)$ be the number of isolated vertices of a graph G.

Lemma 5.7. *Let $\mathcal{F}$ be an ℓ-type of F and n be sufficiently large. Let $\sum_{i=1}^r n_i = \sum_{i=1}^r n_i' \in [n/2, n]$ where $\max_{i,j} |n_i' - n_j'| \leq 1$. Define $a_i = n_i - n_i'$ for $i \in [r]$ and $A = \max\{|a_i| : i \in \{1, \dots, r\}\}$. Then there exists a constant $\eta_{\mathcal{F}} > 0$ such that (recall $\mathcal{F}_\beta$ denotes the bottom of the type $\mathcal{F}$)*

$$|c_{\mathcal{F}}(n_1, \dots, n_r) - c_{\mathcal{F}}(n_1', \dots, n_r')| \leq \eta_{\mathcal{F}} \cdot A \cdot n^{i(\mathcal{F}_\beta)-1},$$

where $c_{\mathcal{F}}(n_1, \dots, n_r)$ is a multi-polynomial of degree $i(\mathcal{F}_\beta)$.

We are ready for the proof of Theorem 5.5.

Proof of Theorem 5.5. Fix an admissible color-k-critical graph F with $\chi(F) = r+1$ and $f = |V(F)|$. Let $1/n \ll \delta \ll \epsilon_1 \ll \epsilon_2 \ll \dots \ll \epsilon \ll 1$ be sufficiently small so that Claims I, II and III in Subsection 4.2 hold. Let H be an n-vertex graph on $h(n,r,k) + q$ edges with minimum number of copies of F, where $1 \leq q \leq \delta n$. Then we can partition $V(H) = X \cup V_1 \cup \ldots \cup V_r$ such that $|X| = k-1$ and the following hold. Let M be the set

[8] In fact, Schrijver [26] also proved that such an induced subgraph of $K(n,t)$ is *vertex-critical*, i.e., deleting any vertex will decrease the chromatic number.

of non-edges of H between $X, V_1, \ldots, V_r$, and let $B_i = E(H[V_i])$.[9] Let $m = |M|$, $b_i = |B_i|$, $b = \sum_{i=1}^{r} b_i$, and $\omega = b - q$ be from Claim III. Then $\epsilon n \geq b = q + \omega \geq q + m$.

To show the equality $h_F(n,q) = t_F(n,q)$, it suffices to prove that H contains $H(n,r,k)$ as a subgraph. We gradually achieve this. Initially, we establish a crucial inequality as indicated in (8). An edge $uv \in \bigcup_{i\in[r]} B_i$ is called *bad*. Denote by $\#F(uv)$ the number of copies of F of H containing uv as the unique bad edge. Then $\#F(uv) = \Omega_F(n^{f-k-1})$ by Claim III. For any $xy \in M$, define

$$\#F'(xy) = \#F(H + xy) - \#F(H)$$

to be the number of *transitional* copies of F associated with uv, that is, the number of copies of F generated by including the non-edge xy of H. Now we assert that

$$\#F'(xy) = \Omega_F(n^{f-k-1}) \text{ holds for all } xy \in M. \tag{8}$$

To see this, we point out that in fact, $\#F'(xy) \geq \#F(uv)$ for any $xy \in M$ and bad edge uv (as, otherwise, we can reduce the number of copies of F by deleting uv and adding xy, a contradiction).

For any copy of F contained in H, it corresponds to a unique type $\mathcal{F} = (F_0, F_1, ..., F_r)$, namely, where F_0 denotes the induced subgraph of this F within X and, for $i \in [r]$, F_i denotes the induced subgraph of this F within V_i. Let $n_i = |V_i|$ for $i \in [r]$. We can bound the number of copies of F in H from above by summarizing the number of edge-sets $E(\mathcal{F})$ multiplying $c_{\mathcal{F}}(n_1, \ldots, n_r)$ over all types $\mathcal{F}$. Utilizing this counting strategy, we will now proceed to demonstrate the following two claims.

For integers $\ell \geq 0$, let Y_ℓ be the collection of all ℓ-types of F and $\widehat{Y}_\ell \subseteq Y_\ell$ be the collection of ℓ-types of F whose top contains at least one edge.

Claim IV. There exist $k-1$ vertices of H with degree $n-1$.

Proof. TOPROVE 12 □

Claim V. We have $M = \emptyset$.

Proof. TOPROVE 13 □

By Claims IV and V, we see that $K_{k-1} + K(V_1, \ldots, V_r) \subseteq H$. Recall that $\epsilon n \geq \omega = b - q \geq 0$. To complete the proof of Theorem 5.5, it suffices to show that either $\omega = 0$ or $A := \max_{i,j} ||V_i| - |V_j|| \leq 1$ (if so, then we have $H(n,r,k) \subseteq H$, as desired). Suppose for a contradiction that $\omega \geq 1$ and $A \geq 2$. By Lemma 2.6, we have $A = O(\omega)$. Fix B_q to be a subset of $\bigcup_{i\in[r]} B_i$ consisting of q edges. Let H' be obtained from $H(n,r,k)$ by adding all edges of B_q. Note that H' has the same number of edges as H. Since F is admissible, for any $\mathcal{F} \in Y_\ell$, we have $\nu(\mathcal{F}_\beta) \geq k - \ell$. Then the number of copies of F in H' satisfies that (for $i \in [r]$ let $n_i' \in \{\lceil (n-k+1)/r \rceil, \lfloor (n-k+1)/r \rfloor\}$ such that $\sum_{i\in[r]} n_i' = n-k+1$)

$$\#F(H') \leq \sum_{\ell=0}^{k-1} \sum_{\mathcal{F}\in Y_\ell} O_F\big(|B_q|^{|V(\mathcal{F}_\beta)|-\nu(\mathcal{F}_\beta)-i(\mathcal{F}_\beta)}\big) \cdot c_{\mathcal{F}}(n_1', \ldots, n_r') \leq O_F(\epsilon \cdot n^{f-k}).$$

Let T be the number of copies of F in H only using bad edges from B_q. By Lemma 5.7, we see that

$$|T - \#F(H')| \leq \#F(H') \cdot O_F(A/n) \leq O_F\big(\epsilon \cdot A n^{f-k-1}\big).$$

There are $\omega = b - q$ edges in $\big(\bigcup_{i\in[r]} B_i\big) \backslash B_q$ not used in any copy of F in H contributed to T, hence

$$\begin{aligned}
\#F(H) &\geq T + \omega \cdot c(n_1, ..., n_r; F) \\
&\geq \big(\#F(H') - O_F\big(\epsilon \cdot A n^{f-k-1}\big)\big) + \omega \cdot \big(c(n,F) - O_F(A) n^{f-k-2} - O_F(A^2) n^{f-k-3}\big) \\
&\geq \#F(H') - O_F\big(\epsilon \cdot \omega \cdot n^{f-k-1}\big) + \Omega_F(\omega \cdot n^{f-k-1}) > \#F(H'),
\end{aligned}$$

[9]Using the terminologies M^* and B^* from Subsection 4.2 (see the paragraph before Claim II), here we have $M = M^* \backslash E[X]$ and $\bigcup_{i\in[r]} B_i = B^*$, where $E[X]$ consists of all edges with both vertices in X.

where the second inequality holds by Lemma 4.2 and the third inequality holds because $c(n, F)$ is a polynomial of degree $f - k - 1$ and $A = O(\omega) = O(\epsilon n)$. This contradicts the minimality of $\#F(H)$ and thus completes the proof of Theorem 5.5. ■

6 Proof of Theorem 1.6

The goal of this section is to prove Theorem 1.6. To explain and describe the intricate thresholds of Theorem 1.6, we need to get deeper into the structure of a color-k-critical graph F. In the rest of this section, we always assume that $k \geq 2, r \geq 2$ and F denotes a color-k-critical graph with $\chi(F) = r + 1$.

We begin by introducing some new parameters on F. Let $\lambda(F)$ denotes the minimum size of a subset $A \subseteq V(F)$ satisfying $\chi(F \setminus A) = r$. Let $\mathbb{X}(F) = \{A \subseteq V(F) : |A| = \lambda(F) \text{ and } \chi(F \setminus A) = r\}$ be the family of all *critical subsets* of F. For a critical subset $A \in \mathbb{X}(F)$, let $\mathcal{V}(A)$ denote the family of all possible partitions $\{U_1, \ldots, U_r\}$ of $V(F \setminus A)$ such that each U_i is stable. For $A \in \mathbb{X}(F)$ and any integer $\ell \geq 1$, if there exist $x \in A$ and $U_j \in \{U_1, \ldots, U_r\} \in \mathcal{V}(A)$ with $|N_F(x) \cap U_j| \geq \ell$, then let

$$\delta_\ell(A) = \min\{|N_F(x) \cap U_j| : x \in A,\ U_j \in \{U_1, \ldots, U_r\} \in \mathcal{V}(A) \text{ and } |N_F(x) \cap U_j| \geq \ell\};$$

otherwise, let $\delta_\ell(A) = \infty$. We now define two parameters playing crucial roles in this section. Let

$$t(F) = \min_{A \in \mathbb{X}(F) : A \text{ is stable}} \delta_2(A) \quad \text{and} \quad s(F) = \min_{A \in \mathbb{X}(F) : A \text{ is not stable}} \delta_1(A).$$ [10]

For example, if F consists of k vertex-disjoint copies of K_{r+1}, then $t(F) = \infty$ and $s(F) = \infty$.

Now we are able to state the main result of this section, which implies Theorem 1.6.

Theorem 6.1. *Let F be given as in the first paragraph of this section with additional properties that $t(F) \in [4, \infty)$ and $s(F) \geq 2$.*[11] *Then there exists $\epsilon > 0$ such that the following hold for sufficiently large n and any n-vertex graph H on $h(n, r, k) + q$ edges with minimum number of copies of F:*

(a) if $1 \leq q \leq \epsilon n^{1-1/s(F)}$, then H contains $H(n, r, k)$ as a subgraph, and

(b) if $n^{1-1/s(F)}/\epsilon \leq q \leq \epsilon n$, then H does not contain $H(n, r, k)$ as a subgraph.

Using Theorem 6.1, one can derive Theorem 1.6 promptly in the following.

Proof. TOPROVE 14 □

In what follows, we prove Theorem 6.1 by first establishing some useful properties. For the proof, we need to consider some special n-vertex graphs and use them to derive upper bounds on $h_F(n, q)$. Fix $1 \leq q \leq \epsilon n$ for some small real $\epsilon > 0$. Let $L = \{\ell_1, \ldots, \ell_r\}$ be a set of non-negative integers with $\sum_{i=1}^r \ell_i = q$. Denote by $H(L)$ the graph obtained from $H(n, r, k)$ by adding r stars with $\ell_1, \ldots, \ell_r$ edges into the r parts of $H(n, r, k)$ respectively. Let X be the vertex set of the clique K_{k-1} in $H(n, r, k)$ and let C be the set of centers of these embedded stars in $H(L)$.[12] Denote by $H'(L)$ the graph obtained from $H(L)$ by deleting all edges inside $X \cup C$. So $e(H'(L)) = e(H(L)) - \binom{k-1+\alpha_L}{2}$, where α_L denotes the number of positive integers in L.

The following propositions are useful for estimating the copies of F in the proof of Theorem 6.1.

Proposition 6.2. *Each copy of F in $H'(L)$ or $H(L)$ contains at least k vertices in $X \cup C$. Moreover, if a copy of F in $H(L)$ contains exactly k vertices and at least one edge in $X \cup C$, then inside this copy F, every $x \in V(F) \cap C$ is incident to at least $s(F)$ edges of the embedded star with center x.*

[10]Here if none of $A \in \mathbb{X}(F)$ satisfies the requirement, then the corresponding parameter is defined to be ∞.

[11]We can prove similar results for the case $t = 3$ (in this case, the extremal graphs for the supersaturation problem may be obtained by putting either a triangle or a star into each part of $H(n, r, k)$; see Lemma 6.4 for some hints), but it requires more effort, so we have decided not to pursue it. We would like to treat the case $t = \infty$ in a forthcoming paper.

[12]Note that we also use X to denote the set of the $k - 1$ vertices given by Claim I of Section 4. This may cause confusion at the first sight, but we would like to use X at both circumstances as they refer to the same set of vertices conceptually. If a star is a single edge, then one can choose any one of its vertices as its center.

Proof. TOPROVE 15 □

Proposition 6.3. *Let $t := t(F) < \infty$ and $s := s(F) \geq 2$. Then the following hold that*

$$\mathcal{N}_F(H'(L)) = q \cdot c(n,F) + \sum_{i=1}^{r} \beta(\ell_i) n^{f-k-t} + \sum_{i \neq j} O(\ell_i \ell_j n^{f-k-2}),$$

where $\beta(x) = ax^t + \Theta(x^{t-1})$ for some absolute constant $a > 0$, and

$$\mathcal{N}_F(H(L)) = \mathcal{N}_F(H'(L)) + \sum_{i=1}^{r} \Theta(\ell_i^s n^{f-k-s}) + \sum_{i \neq j} O(\ell_i \ell_j n^{f-k-3}).$$

Proof. TOPROVE 16 □

The following lemma is technical. In its simplest case $|I| = 1$, it provides a lower bound on a linear combination of the number of matchings of size two and the number of stars (say with t edges).

Lemma 6.4. *Fix a real $\alpha > 0$, integers $t \geq 3$, $r \geq 2$ and a non-empty set I of indexes with $|I| \leq r$. Let $1 \gg \delta \gg \epsilon \gg 1/n > 0$ be sufficiently small compared to α, t and r. For every $i \in I$, let G_i be a graph with $m_i \leq \epsilon n$ edges such that if $t = 3$ and $m_i = 3$, then G_i is not a triangle.*[13] *If there exists an index $j \in I$ with $\Delta(G_j) \leq (1-\delta) m_j$, then*

$$\alpha \cdot \sum_{i \in I} \mathcal{N}_{M_2}(G_i) n^{t|I|-2} \geq \prod_{i \in I} \binom{m_i}{t}.$$

If $\Delta(G_i) \geq (1-3\delta) m_i$ for every $i \in I$, then

$$\alpha \cdot \sum_{i \in I} \mathcal{N}_{M_2}(G_i) n^{t|I|-2} + \prod_{i \in I} \binom{\Delta(G_i)}{t} \geq \prod_{i \in I} \binom{m_i}{t}.$$

Proof. TOPROVE 17 □

We are ready to present the proof of Theorem 6.1.

Proof of Theorem 6.1. Fix $k \geq 2, r \geq 2$ and a color-k-critical graph F with $\chi(F) = r+1$ such that $t := t(F) \geq 4$ and $s := s(F) \geq 2$. Let $1 \gg \delta \gg \epsilon \gg 1/n > 0$ be sufficiently small to satisfy claims of Section 4 and Lemma 6.4 (where $1 \gg \delta \gg \epsilon > 0$ are from Lemma 6.4 and the constant $\alpha_{6.4}$ there will be determined later). Let H be an n-vertex graph on $h(n,r,k) + q$ edges with minimum number of copies of F, where $1 \leq q \leq \epsilon n$. Then using Claims I, II and III in Subsection 4.2, the following hold. One can partition $V(H) = X \cup V_1 \cup \ldots \cup V_r$, where $|X| = k-1$ and each vertex in X has degree at least $n - \epsilon n$. Let M be the set of missing edges of H between $X, V_1, \ldots, V_r$, and let $B_i = E(H[V_i])$. Let $m = |M|$, $b_i = |B_i|$, $b = \sum_{i=1}^{r} b_i$, and $\omega = b - q$ be from Claim III. Then $\epsilon n \geq b = q + \omega \geq q + m$.

We divide the proof into two parts depending on the range of q. Throughout this proof, we use $B^* = \bigcup_{i=1}^{r} B_i$ and $L_\ell = \{\ell, 0, \ldots, 0\}$. For a set of edges A, we denote by $\Delta(A)$ the maximum degree of the graph induced by the edges in A.

Case (A). $1 \leq q \leq \epsilon n^{1-1/s}$.

In this case our goal is to show $H(n,r,k) \subseteq H$. Recall the definition of $f(n,F)$ before Claim III, and note that $c(n,F) = f(n,F)$ whenever $s \geq 2$.

We first prove the case $s = 2$. This proof is straightforward and reveals the main proof idea, that is, to construct a "well-designed" graph with the same numbers of vertices and edges but with less copies of F than H. Let $a := \max\{q, \omega\} \leq \epsilon n$. By Claim III, if $\omega \geq 1$, then $\mathcal{N}_F(H) \geq \sum_{e \in B} F(e) \geq (q+\omega)(c(n,F) - \Theta(\omega) n^{f-k-2}) \geq qc(n,F) + \omega c(n,F) - a\Theta(\omega) n^{f-k-2} = qc(n,F) + \Omega(n^{f-k-1})$. By Proposition 6.3, we have $\mathcal{N}_F(H(L_q)) =$

[13] It is easy to see that this lemma does not hold if $t = 3$ and all G_i's are triangles.

$qc(n,F)+O(q^2)n^{f-k-2}$. Since $q \le \epsilon n^{1/2}$, we derive that $\mathcal{N}_F(H) \ge qc(n,F)+\Omega(n^{f-k-1}) > \mathcal{N}_F(H(L_q))$, a contradiction to the minimality of $\mathcal{N}_F(H)$. Thus $\omega = 0$, from which we can derive $H(n,r,k) \subseteq H$ (i.e., see the proof of Claim III), as desired.

From now on we consider the general case $s \ge 3$. Our proof strategy is (again) to show that whenever $\omega \ge 1$, one can construct an n-vertex graph with $h(n,r,k)+q$ edges whose number of copies of F is strictly smaller than $\mathcal{N}_F(H)$. For that, we need to estimate the number of copies of F more precisely and thus we introduce several notations in the following paragraphs. First, denote by $f(M_2)$ the minimum number of copies of F obtained from $I_{k-1}+T_r(n-k+1)$ by adding a copy of M_2 to one class of $T_r(n-k+1)$ and removing all edges between $V(M_2)$ and I_{k-1}. Let $i \ge 1, j \ge 0$ be integers with $i+j \le r$. Denote by $f_{i,j}(S_{t+1})$ the minimum number of copies of F obtained from $I_{k-1}+T_r(n-k+1)$ by adding a copy of S_{t+1} to each of i classes of $T_r(n-k+1)$, adding an edge into each of other j classes of $T_r(n-k+1)$, and removing all edges inside $C' \cup I_{k-1}$, where C' is the set of centers of embedding stars S_{t+1} and edges.[14] Denote by $f^*(M_\ell)$ the minimum number of copies of F obtained from $I_{k-1}+T_r(n-k+1)$ by adding one edge to each of ℓ classes of $T_r(n-k+1)$ and removing all edges inside $C'' \cup I_{k-1}$, where C'' consists of vertices of these ℓ edges. It is not hard to see that there exist reals $\alpha > 0, \beta_{i,j} \ge 0$, and $\gamma_\ell \ge 0$ satisfying $f(M_2) = \alpha n^{f-k-2} + O(n^{f-k-3})$, $f_{i,j}(S_{t+1}) = \beta_{i,j} n^{f-k-ti-j} + O(n^{f-k-ti-j-1})$ (for $t = \infty$, let $f_{i,j}(S_{t+1}) = 0$), and $f^*(M_\ell) = \gamma_\ell n^{f-k-\ell} + O(n^{f-k-\ell-1})$.

For comparison, we name analogous types of F in H. Let $i \in [r]$. For a copy of M_2 in B_i, let $F(M_2)$ be the number of copies of F in H containing this M_2 as the only edges from B^*. For disjoint $I, J \subseteq [r]$, denote by $F_{I,J}(S_{t+1})$ the number of copies of F in H containing a copy of S_{t+1}, whose leaves are completely adjacent to X in H, in every B_i for $i \in I$ and containing an edge in every B_j for $j \in J$.[15] For a copy of M_ℓ with at most one edge in each B_i, denote by $F^*(M_\ell)$ the number of copies of F in H containing this M_ℓ as the only edges in B^*. Repeating the proof of Claim III, one can similarly obtain the following estimations: there exists $c > 0$ such that

$$F(M_2) \ge f(M_2) - c \cdot \omega \cdot n^{f-k-3}, \tag{9}$$

$$F_{I,J}(S_{t+1}) \ge f_{|I|,|J|}(S_{t+1}) - c \cdot \omega \cdot n^{f-k-t|I|-|J|-1}, \text{where } |I| \ge 1, \tag{10}$$

$$F^*(M_\ell) \ge f^*(M_\ell) - c \cdot \omega \cdot n^{f-k-\ell-1}. \tag{11}$$

Recall $1 \gg \delta \gg \epsilon > 0$ from Lemma 6.4 which are defined in the beginning of this proof. We claim that there exist subsets $B_i^* \subseteq B_i$ for each $i \in [r]$ such that

(A1). $\sum_{i=1}^r b_i^* \ge b - m \ge q$ (where $b_i^* := |B_i^*|$), and

(A2). either $\Delta(B_i^*) < (1-\delta)b_i^*$, or there exists a vertex u with $\Delta(B_i^*) \ge d_{B_i^*}(u) \ge (1-3\delta)b_i^*$ and every vertex in $N_{B_i^*}(u)$ is completely adjacent to X.[16]

To see this, we run the following algorithm (within H) for each $i \in [r]$. Initially, let $B_i^* = B_i$ and $b_i^* = |B_i^*|$. If $\Delta(B_i^*) \ge (1-\delta)b_i^*$ and some vertex v of degree one in B_i^* is incident to X by a missing edge of H, then we delete the unique edge of v from B_i^*; repeat the above process until we cannot delete any edges. When this process ends, either $\Delta(B_i^*) < (1-\delta)b_i^*$, or $\Delta(B_i^*) \ge (1-\delta)b_i^*$. In the latter case, let u be the vertex with $d_{B_i^*}(u) = \Delta(B_i^*)$, and all vertices of degree one in B_i^* are complete to X. If there is some $v \in N_{B_i^*}(u)$ incident to X by a missing edge of H (which must have degree two in B_i^* and there are at most $2\delta b_i^*$ such vertices), then we delete uv from B_i^*. In the end, we see that $d_{B_i^*}(u) \ge (1-3\delta)b_i^*$ and every vertex in $N_{B_i^*}(u)$ is completely adjacent to X. Since the number of the deleted edges is at most m the number of missing edges in H, we have $\sum_{i=1}^r b_i^* \ge b - m \ge q$.

Hence there exist non-negative integers $\ell_1, \dots, \ell_r$ with $\sum_{i=1}^r \ell_i = q$ and $\ell_i \le b_i^* \le b_i$. Let $L = \{\ell_1, \dots, \ell_r\}$. Note that $H(L)$ is an n-vertex graph with $h(n,r,k)+q$ edges.

[14]Here, we view both vertices of an embedding edge as its centers.

[15]Here it is important to require that the leaves of stars S_{t+1} are complete to X, for the validation of (10). Also note that because of $s(F) \ge 3$, every copy of F counted in $F_{I,J}(S_{t+1})$ cannot contain any edge between X and the unique edge in B_j for every $j \in J$.

[16]The latter case is consistent with the definition of $F_{I,J}(S_{t+1})$.

In the remaining of the proof, we compare $\mathcal{N}_F(H)$ with $\mathcal{N}_F(H(L))$. First we consider $H(L)$. Let $1 \le \ell \le k$. The number of copies of F containing edges of ℓ embedding stars (no edges from other stars) and at least $k-\ell+1$ vertices of X in $H(L)$ is $O(q^\ell n^{f-k-\ell-1}) = o(n^{f-k-1})$. Now we bound the number of copies of F containing edges from ℓ embedding stars (no edges from other stars) and exactly $k-\ell$ vertices of X in $H(L)$; call them *standard*. If a standard copy of F contains an edge from $X \cup C$, then by Proposition 6.2, it must contain s edges from some embedding star, so the number of such standard copies is $O(q^s n^{f-k-s}) = O(\epsilon^s) \cdot n^{f-k-1}$, where we use $q \le \epsilon n^{1-1/s}$. It remains to consider standard copies of F containing none of the edges from $X \cup C$; call them *feasible*. Let

- $\mathcal{W}_\ell = \{$copies of F in $H(L)$ containing exactly ℓ independent edges from the embedding stars of $H(L)$ and containing exactly $k-\ell$ vertices of $X\}$, where $1 \le \ell \le k$,[17]
- $\mathcal{R}_{I,J} = \{$feasible copies of F in $H(L)$ containing a copy of S_{t+1} in each of the embedding stars of sizes ℓ_i for $i \in I$, containing an edge in each of the embedding stars of sizes ℓ_j for $j \in J$, and containing exactly $k - |I| - |J|$ vertices of $X\}$, where $I, J \subseteq [r], I \cap J = \emptyset$ and $|I| \ge 1$.

By the definition of $t(F)$, each feasible copy of F belongs to either $\bigcup_{\ell=1}^k \mathcal{W}_\ell$ or $\mathcal{R}_{I,J}$ for some $I, J \subseteq [r], I \cap J = \emptyset$ and $|I| \ge 1$. Putting these all together, we have the following estimation on $\mathcal{N}_F(H(L))$

$$q \cdot c(n, F) + \sum_{\ell=2}^{k} |\mathcal{W}_\ell| + \sum_{I,J\subseteq[r], I\cap J=\emptyset, |I|\ge 1} |\mathcal{R}_{I,J}| \ge \mathcal{N}_F(H(L)) - O(\epsilon^s) \cdot n^{f-k-1}. \tag{12}$$

For the purpose of comparison, we consider the following pairwise disjoint collections of copies of F in H (again by no mean of a partition; recall the sets B_i^* from the properties (A1) and (A2)):

- $\mathcal{W}_\ell^* = \{$copies of F containing exactly ℓ independent edges in ℓ parts V_i's as the only edges of $\bigcup_{\alpha\in[r]} B_\alpha^*$ and containing exactly $k-\ell$ vertices of $X\}$, where $1 \le \ell \le k$,
- $\mathcal{R}_{I,J}^* = \{$copies of F containing a copy of S_{t+1} in every B_i^* for $i \in I$ and an edge in every B_j^* for $j \in J$ as the only edges in $\bigcup_{\alpha\in[r]} B_\alpha^*$ and containing exactly $k - |I| - |J|$ vertices of $X\}$, where $I, J \subseteq [r], I \cap J = \emptyset$ and $|I| \ge 1$,
- $\mathcal{T}_i^* = \{$copies of F containing a copy of M_2 in B_i^* as the only edges in $\bigcup_{\alpha\in[r]} B_\alpha^*$, where $i \in [r]$.

Clearly, we have

$$\mathcal{N}_F(H) \ge \sum_{e\in B^*} F(e) + \sum_{\ell=2}^{k} |\mathcal{W}_\ell^*| + \sum_{I,J\subseteq[r], I\cap J=\emptyset, |I|\ge 1} |\mathcal{R}_{I,J}^*| + \sum_{i=1}^{r} |\mathcal{T}_i^*|. \tag{13}$$

In the following, we will show that assuming $\omega \ge 1$,

$$|\mathcal{W}_\ell^*| \ge |\mathcal{W}_\ell| - O(q^\ell)\omega n^{f-k-\ell-1} \text{ for each } 2 \le \ell \le k \tag{14}$$

and

$$\sum_{I,J\subseteq[r], I\cap J=\emptyset, |I|\ge 1} |\mathcal{R}_{I,J}^*| + \sum_{i=1}^{r} |\mathcal{T}_i^*| \ge \sum_{I,J\subseteq[r], I\cap J=\emptyset, |I|\ge 1} |\mathcal{R}_{I,J}| - \Theta(\omega\epsilon n^{f-k-1}). \tag{15}$$

Let us first show that to complete the proof for Case (A), it suffices to show (14) and (15). Indeed, by

[17] We point out that each copy F in $\mathcal{W}_\ell$ for $\ell \ge 1$ is feasible by definition (as $s(F) \ge 2$).

combining (13), (14), (15) with Claim III of Section 4, assuming $\omega \geq 1$ we have

$$
\begin{aligned}
\mathcal{N}_F(H) \geq & \sum_{e \in B^*} F(e) + \sum_{\ell=2}^{k} |\mathcal{W}_\ell^*| + \sum_{I,J \subseteq [r], I \cap J = \emptyset, |I| \geq 1} |\mathcal{R}_{I,J}^*| + \sum_{i=1}^{r} |\mathcal{T}_i^*| \\
\geq & (q+\omega)\left(c(n,F) - \Theta(\omega n^{f-k-2})\right) + \sum_{\ell=2}^{k} \left(|\mathcal{W}_\ell| - O(q^\ell \omega n^{f-k-\ell-1})\right) \\
& + \sum_{I,J \subseteq [r], I \cap J = \emptyset, |I| \geq 1} |\mathcal{R}_{I,J}| - \Theta(\omega \epsilon n^{f-k-1}) \\
\geq & qc(n,F) + \sum_{\ell=2}^{k} |\mathcal{W}_\ell| + \sum_{I,J \subseteq [r], I \cap J = \emptyset, |I| \geq 1} |\mathcal{R}_{I,J}| + \omega \cdot \Theta(n^{f-k-1}),
\end{aligned}
$$

where the last inequality follows from that $q \leq \epsilon n$ and $c(n,F)$ is a polynomial of degree $f-k-1$ with variable n. If $\omega \geq 1$, then we can derive the following contradiction that

$$
\mathcal{N}_F(H) \geq qc(n,F) + \sum_{\ell=2}^{k} |\mathcal{W}_\ell| + \sum_{I,J \subseteq [r], I \cap J = \emptyset, |I| \geq 1} |\mathcal{R}_{I,J}| + \Theta(n^{f-k-1}) > \mathcal{N}_F(H(L)),
$$

where the last equality follows by (12). Hence $\omega = 0$, which implies that H contains $H(n,r,k)$ as a subgraph, thus proving Case (A).

Turning back to (14) and (15), we will first prove (14). Since $\sum_{i=1}^{r}(b_i - \ell_i) = b - q = \omega \geq 1$, there is an integer β with $b_\beta - \ell_\beta \geq \omega/r$. Let $\ell_i' = \ell_i$ for each $i \in [r] \backslash \{\beta\}$ and $\ell_\beta' = \ell_\beta + \omega/r$ so that $b_i \geq \ell_i'$ for all $i \in [r]$. Fix $2 \leq j \leq k$. Recall the definition of $f^*(M_j)$, which equals $\gamma_j n^{f-k-j} + O(n^{f-k-j-1})$ for some $\gamma_j \geq 0$. Since $s(F) = s \geq 3$, we see that all copies of F in $\mathcal{W}_j$ (in $H(L)$) are contributed in the same way as counted in $f^*(M_j)$. So $|\mathcal{W}_j| \leq \sum_{k_1,\ldots,k_j \subseteq [r]} \ell_{k_1} \ldots \ell_{k_j}\left(\gamma_j n^{f-k-j} + O(n^{f-k-j-1})\right)$. If $\gamma_j = 0$, then (14) holds trivially. So assume $\gamma_j > 0$. By (11), we see that $|\mathcal{W}_j^*| - |\mathcal{W}_j|$ equals

$$
\begin{aligned}
& \sum_{k_1,\ldots,k_j \subseteq [r]} b_{k_1} \ldots b_{k_j} \left(\gamma_j n^{f-k-j} - c\omega n^{f-k-j-1}\right) - \sum_{k_1,\ldots,k_j \subseteq [r]} \ell_{k_1} \ldots \ell_{k_j}\left(\gamma_j n^{f-k-j} + O(n^{f-k-j-1})\right) \\
\geq & \sum_{k_1,\ldots,k_j \subseteq [r]} \ell_{k_1}' \ldots \ell_{k_j}' \left(\gamma_j n^{f-k-j} - c\omega n^{f-k-j-1}\right) - \sum_{k_1,\ldots,k_j \subseteq [r]} \ell_{k_1} \ldots \ell_{k_j} \gamma_j n^{f-k-j} - O(q^j n^{f-k-j-1}) \\
\geq & \omega/r \cdot \left(\sum_{k_1,\ldots,k_{j-1} \subseteq [r] \backslash \{\beta\}} \ell_{k_1} \ldots \ell_{k_{j-1}} \right) \cdot \gamma_j n^{f-k-j} \\
& - \left(\ell_\beta' \sum_{k_1,\ldots,k_{j-1} \subseteq [r] \backslash \{\beta\}} \ell_{k_1} \ldots \ell_{k_{j-1}} + \sum_{k_1,\ldots,k_j \subseteq [r] \backslash \{\beta\}} \ell_{k_1} \ldots \ell_{k_j} \right) c\omega n^{f-k-j-1} - O(q^j n^{f-k-j-1}) \\
\geq & -O(q^j)\omega n^{f-k-j-1},
\end{aligned}
$$

where the last inequality holds because $\ell_\beta' \leq b_\beta \leq b \leq \epsilon n$, $\ell_i \leq q \leq \epsilon n$ for any $i \in [r] \setminus \{\beta\}$, and ϵ is sufficiently small (i.e., $\epsilon \ll \gamma_j / c$). This proves (14).

Now we consider (15). Fix disjoint $I, J \subseteq [r]$ with $|I| \geq 1$. Then

$$
|\mathcal{T}_i^*| = \mathcal{N}_{M_2}(B_i^*) \cdot F(M_2) \quad \text{and} \quad |\mathcal{R}_{I,J}^*| \geq \left(\prod_{i \in I} \binom{\Delta(B_i^*)}{t} \prod_{j \in J} b_j^* \right) \cdot F_{I,J}(S_{t+1}),
$$

where $F(M_2) \geq (\alpha - o(1)) \cdot n^{f-k-2}$ and $F_{I,J}(S_{t+1}) = (1-o(1)) f_{|I|,|J|}(S_{t+1}) = (\beta_{|I|,|J|} - o(1)) \cdot n^{f-k-|I|t-|J|}$ from (9) and (10) respectively. Since $t \geq 4$, all B_i^* for $i \in I$ satisfy Lemma 6.4. Using the property (A2) and

Lemma 6.4 with $\alpha_{6.4} = \alpha/(2 \cdot 3^r \beta_{|I|,|J|} + 1)$,

$$\frac{1}{3^r}\sum_{i\in I}|\mathcal{T}_i^*| + |\mathcal{R}_{I,J}^*| \geq \left(\alpha_{6.4}\cdot\sum_{i\in I}\mathcal{N}_{M_2}(B_i^*)n^{t|I|-2} + \prod_{i\in I}\binom{\Delta(B_i^*)}{t}\right)\cdot\left(\prod_{j\in J} b_j^* \cdot F_{I,J}(S_{t+1})\right)$$
$$\geq \prod_{i\in I}\binom{b_i^*}{t}\cdot\left(\prod_{j\in J} b_j^*\cdot F_{I,J}(S_{t+1})\right).$$

Using $|\mathcal{R}_{I,J}| = \left(\prod_{i\in I}\binom{\ell_i}{t}\prod_{j\in J}\ell_j\right)\cdot f_{|I|,|J|}(S_{t+1})$, since $\ell_i \leq b_i^*$ and $\ell_i \leq q \leq \epsilon n$ for each i, we obtain

$$\frac{1}{3^r}\sum_{i\in I}|\mathcal{T}_i^*| + |\mathcal{R}_{I,J}^*| - |\mathcal{R}_{I,J}| \geq \prod_{i\in I}\binom{b_i^*}{t}\cdot\prod_{j\in J}b_j^*\cdot F_{I,J}(S_{t+1}) - \prod_{i\in I}\binom{\ell_i}{t}\cdot\prod_{j\in J}\ell_j\cdot f_{|I|,|J|}(S_{t+1})$$
$$\geq \left(\prod_{i\in I}\binom{\ell_i}{t}\prod_{j\in J}\ell_j\right)\cdot\left(F_{I,J}(S_{t+1}) - f_{|I|,|J|}(S_{t+1})\right) \geq -\Theta(\omega q^t n^{f-k-t-1}),$$

where the last inequality holds because of (10) and $|I| \geq 1$. Summing up the above inequalities for all $I, J \subseteq [r]$ with $I \cap J = \emptyset$ and $|I| \geq 1$ (there are at most 3^r many such inequalities), we can easily derive (15). The proof of Case (A) is complete.

Case (B). $n^{1-1/s}/\epsilon \leq q \leq \epsilon n$.

Suppose for a contradiction that H contains $H(n,r,k)$ as a subgraph. First consider the case $s = 2$. Without loss of generality, let $e(B_1) \geq q/r$. Note that $\mathcal{N}_{M_2}(B_1) + \mathcal{N}_{S_3}(B_1) \geq \binom{q/r}{2}$ and the number of copies of F contains exactly two edges of B_1 (which are incident or not) is $\Theta(n^{f-k-2})$. Let $q^* = q + \binom{k}{2}$. Then since $\epsilon n \geq q \geq n^{1/2}/\epsilon$ and $t \geq 4$, we have $\mathcal{N}_F(H) \geq qc(n,F) + \Theta(q^2)n^{f-k-2} \geq q^*c(n,F) + \Theta(q^t/\epsilon^2)n^{f-k-t} > \mathcal{N}_F(H'(L_{q^*}))$, where the last inequality holds by the first inequality of Proposition 6.3. This is a contradiction as $H'(L_{q^*})$ has the same numbers of vertices and edges as H.

Assume that $s \geq 3$. Let $L = \{\ell_1, \ldots, \ell_r\}$, where $\ell_i = b_i$ if $b_i \geq 4$ and $\ell_i = 0$ otherwise. We first compare $\mathcal{N}_F(H)$ with $\mathcal{N}_F(H(L))$. Fix $\alpha \in [r], I, J \subseteq [r], I \cap J = \emptyset$, $|I| \geq 1$ and $\tau \geq k - |I| - |J|$. Let $\eta = \min\{s,t\}$ if $\tau = k - |I| - |J|$ and $\eta = 2$ if $\tau \geq k - |I| - |J| + 1$. Define

- $\mathcal{R}'_{I,J,\tau} = \{$copies of F in $H(L)$ containing a star of size at least η in each of the embedding stars of sizes ℓ_i for $i \in I$, containing an edge in each of the embedding stars of sizes ℓ_j for $j \in J$, and containing exactly τ vertices of $X\}$,
- $\mathcal{R}^*_{I,J,\tau} = \{$copies of F in H containing a star of size at least η in B_i for $i \in I$, containing an edge in each B_j for $j \in J$, and containing exactly τ vertices of $X\}$,
- $\mathcal{T}^*_\alpha = \{$copies of F in H containing a copy of M_2 in B_i as the only edges in $\bigcup_{\beta\in[r]} B_\beta$.

We note that for $\tau \geq k - |I| - |J| + 1$, both $|\mathcal{R}'_{I,J,\tau}|$ and $|\mathcal{R}^*_{I,J,\tau}|$ become lower order terms than when $\tau = k - |I| - |J|$. Similar as the proof of (15) in Case (A), by Lemma 6.4, we can show that

$$\frac{1}{3^r k}\sum_{i\in I}|\mathcal{T}_i^*| + |\mathcal{R}^*_{I,J,\tau}| \geq |\mathcal{R}'_{I,J,\tau}|. \tag{16}$$

Note that H contains a copy of $H(n,r,k)$, so compared with the proof in Case (A), we have $\omega = 0$ and thus the counting proof here is easier. It is also clear that the number of copies of F in H containing exactly ℓ independent edges in ℓ parts V_i's as the only edges of $\bigcup_{\alpha\in[r]} B_\alpha$ is larger than the number of copies of F in $H(L)$ containing exactly ℓ independent edges in ℓ embedding stars as the only embedding edges. Therefore, putting all copies of F together (e.g., summing up (16) for all $I, J \subseteq [r]$ with $I \cap J = \emptyset$, $|I| \geq 1$ and τ), we can obtain

$$\mathcal{N}_F(H) \geq \mathcal{N}_F(H(L)).$$

Hence, using the second inequality of Proposition 6.3, we get

$$\mathcal{N}_F(H) - \mathcal{N}_F(H'(L)) \geq \mathcal{N}_F(H(L)) - \mathcal{N}_F(H'(L)) \geq \Theta(q^s)n^{f-k-s}.$$

Let $L^* = \{\ell_1 + \binom{k-1+\alpha_L}{2} + \sum_{i=1}^{r}(b_i - \ell_i), \ell_2 \ldots, \ell_r\}$, where α_L is the number of positive integers in L. Then $H'(L^*)$ has the same number of edges as H, and using Proposition 6.3 again, we obtain that $\mathcal{N}_F(H'(L^*)) = \mathcal{N}_F(H'(L)) + \Theta(n^{f-k-1})$. Finally putting the above all together, since $q \geq n^{1-1/s}/\epsilon$,

$$\mathcal{N}_F(H) \geq \mathcal{N}_F(H'(L)) + \Theta(q^s)n^{f-k-s} \geq \mathcal{N}_F(H'(L)) + \Theta(n^{f-k-1}/\epsilon^s) > \mathcal{N}_F(H'(L^*)),$$

a contradiction. The proof of Theorem 6.1 is complete. ■

7 Concluding remarks

In this paper, we explore the supersaturation problem and present several results, both positive and negative, that extend beyond the existing framework. These findings offer new insights into the complexity and intricate nature of this problem for general graphs. We now proceed to discuss some remarks and related problems.

Let F be a color-k-critical graph with $\chi(F) = r+1$. In Section 4, we establish several general properties for supersaturated graphs of F (that is, graphs of given order and size with the minimum number of copies of F). Using these properties, one can quickly prove a general lower bound on $h_F(n,q)$ as follows. Recall the definition of $f(n,F)$ (from Claim III of Subsection 4.2), which denotes the minimum number of copies of F obtained from $I_{k-1} + T_r(n-k+1)$ by adding an edge (say e) to one class of $T_r(n-k+1)$ and removing all edges between $V(e)$ and I_{k-1}.

Theorem 7.1. *Fix $k \geq 1$ and any color-k-critical graph F with $\chi(F) = r+1$. Then there exists a constant $\delta = \delta_F > 0$ such that if n is sufficiently large and $1 \leq q \leq \delta n$, then $h_F(n,q) \geq q \cdot f(n,F)$.*

Proof. TOPROVE 18 □

This result can be seen as an extension of Theorem 1.1 since the notation $f(n,F)$ corresponds to $c(n,F)$ when $k=1$.

Let F be a color-critical graph. As mentioned earlier, Pikhurko and Yilma [23] asymptotically determined $h_F(n,q)$ in the range $q = o(n^2)$. Investigating the asymptotic behavior of $h_F(n,q)$ when $q = \Omega(n^2)$ suggests by itself an challenging problem. A good starting point might be to examine the case when F is an odd cycle.

In Theorem 1.4, we show that Conjecture 1.3 does not hold in the graph case. As discussed after Theorem 1.4, assuming n is sufficiently large, there exist non-bipartite stable graphs F such that $h_F(n,q) < t_F(n,q)$ holds for any fixed integer $q \geq 4$. This leads us to inquire whether the same result holds for the cases $q \in \{1,2,3\}$. In contrast to our findings for $q \geq 4$, we speculate that Conjecture 1.3 holds in the intriguing case $q = 1$. Furthermore, we believe that the equality $h_F(n,1) = t_F(n,1)$ holds for the majority of graphs F, regardless of whether it is stable or bipartite. Consequently, we pose the following question.

Question 7.2. *Is it true that for any graph F containing a cycle and for sufficiently large n, the equality $h_F(n,1) = t_F(n,1)$ holds?*

Based on our current knowledge, all graphs for which the extremal graphs have been determined provide positive evidence for this question. Nevertheless, it remains an interesting problem to pursue Conjecture 1.3 in the context of hypergraphs or graphs with chromatic number three.

References

[1] P. Erdős, Some theorems on graphs, Riveon Lematematika 9 (1955), 13-17.

[2] P. Erdős, On a theorem of Rademacher-Turán, Illinois J. Math. 6 (1962), 122-127.

[3] P. Erdős, On the number of complete subgraphs contained in certain graphs, Magy. Tud. Acad. Mat. Kut. Int. Közl. 7 (1962), 459-474.

[4] P. Erdős, Some recent results on extremal problem in graph theory, Theory of Graphs (ed P. Rosenstiehl), (Internat. Sympos., Rome, 1966), Gordon and Breach, New York, and Dunod, Paris, 1967, pp. 117-123.

[5] P. Erdős, On some new inequalities concering extremal properties of graphs, Theory of Graphs (P. Erdős and G. Katona, Eds.), Academic Press, New. York, 1968, pp. 77-81.

[6] D. C. Fisher, Lower bounds on the number of triangles in a graph, J. Graph Theory 13(4) (1989), 505-512.

[7] D. C. Fisher and J. Ryan, Bounds on the number of complete subgraphs, Discrete Math. 103 (1992), no. 3, 313-320.

[8] Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems, Erdős centennial, 169-264, Bolyai Soc. Math. Stud., 25, János Bolyai Math. Soc., Budapest, 2013.

[9] M. Kang, T. Makai and O. Pikhurko, Supersaturation problem for the bowtie, European J. Combin. 88 (2020) 103107, 27 pp.

[10] M. Kneser, Aufgabe 300, Jahresber. Dtsch. Math.-Ver. 58 (1955).

[11] J. Komlós and M. Simonovits, Szemerédi's regularity lemma and its application in graph theory, in: D. Miklós, V. Sós, T. Szonyi (Eds.), Paul Erdős Is Eighty, vol. 2, Bolyai Mathematical Society, 1996, pp. 295-352.

[12] T. Kövári, V. T. Sós and P. Turán, On a problem of K. Zarankiewicz, Colloq. Math. 3 (1954), 50-57.

[13] H. Liu, O. Pikhurko and K. Staden, The exact minimum number of triangles in graphs with given order and size, Forum Math. Pi 8 (2020), e8, 144 pp.

[14] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, J. Comb. Theory, Ser. A 25 (1978), 319-324.

[15] L. Lovász and M. Simnonovits, On the number of complete subgraphs of a graph, in: Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aderdeen, 1975), pages 431-441. Congress Numerantium, No. XV. Utilitas Math., Winnipeg, Man., 1976.

[16] L. Lovász and M. Simnonovits, On the number of complete subgraphs of a graph II, in: Studies in Pure Mathematics, Birkhäuser, Basel, 1983, pages 459-495.

[17] W. Mantel, Problem 28, Wiskundige Opgaven 10 (1907), 60-61.

[18] J. Matoušek, Using the Borsuk-Ulam theorem, Springer, 2003.

[19] D. Mubayi, Counting substructures I: color critical graphs, Adv. Math., 225(5) (2010) 2731-2740.

[20] D. Mubayi, Counting substructures II: hypergraphs, Combinatorica 33 (2013), 591-612.

[21] D. Mubayi, Counting substructures III: quadruple systems, arXiv: 0905.4735

[22] V. Nikiforov, The number of cliques in graphs of given order ans size, Trans. Amer. Math. Soc., 363(3) (2001), 1599-1618.

[23] O. Pikhurko and Z. Yilma, Supersaturation problem for color-critical graphs, J. Combin. Theory Ser. B, 123 (2017), 148-185.

[24] A. A. Razborov, On the minimal density of triangles in graphs, Comnin. Probab. Comput., 17(4) (2008), 603-618.

[25] C. Reiher, The clique density theorem, Ann. of Math. (2), 184(3) (2016) 683-707.

[26] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, Nieuw Arch. Wiskd. (3) 26 (3) (1978), 454-461.

[27] A. F. Sidorenko, A correlation inequality for bipartite graphs, Graphs Combin., 9 (1993), 201-204.

[28] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, 1968, pages 279-319.

[29] M. Simonovits, Extremal graph problems with symmetrical extremal graphs, additionnal chromatic conditions, Discrete Math. 7 (1974), 349-376.

[30] M. Simonovits, Extremal graph problems, degenerate extremal problems and super-saturated graphs, in: Progress in Graph Theory, Waterloo, Ont., 1982, Academic Press, Tornoto, 1984, pages 419-437.

[31] M. Simonovits, How to solve a Turán type extremal graph problem? (linear decomposition), Contemporary trends in dicrete mathematics (Stirin Castle, 1997), pp. 283-305, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 49, Amer. Math. Soc., Providence, RI, 1999.

[32] P. Turán, On an extremal problem in graph theory (in Hungrarian), Mat. Fiz. Lapok. 48 (1941), 436-452.