Sets of r-graphs that color all r-graphs

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Abstract

An r-regular graph is an r-graph, if every odd set of vertices is connected to its complement by at least r edges. Let G and H be r-graphs. An *H-coloring* of G is a mapping $f\colon E(G) \to E(H)$ such that each r adjacent edges of G are mapped to r adjacent edges of H. For every $r \geq 3$, let $\mathcal{H}_r$ be an inclusion-wise minimal set of connected r-graphs, such that for every connected r-graph G there is an $H \in \mathcal{H}_r$ which colors G.

We show that $\mathcal{H}_r$ is unique and characterize $\mathcal{H}_r$ by showing that $G \in \mathcal{H}_r$ if and only if the only connected r-graph coloring G is G itself.

The Petersen Coloring Conjecture states that the Petersen graph P colors every bridgeless cubic graph. We show that if true, this is a very exclusive situation. Indeed, either $\mathcal{H}_3 = \{P\}$ or $\mathcal{H}_3$ is an infinite set and if $r \geq 4$, then $\mathcal{H}_r$ is an infinite set. Similar results hold for the restriction on simple r-graphs.

By definition, r-graphs of class 1 (i.e. those having edge-chromatic number equal to r) can be colored with any r-graph. Hence, our study will focus on those r-graphs whose edge-chromatic number is bigger than r, also called r-graphs of class 2. We determine the set of smallest r-graphs of class 2 and show that it is a subset of $\mathcal{H}_r$.

Keywords: perfect matchings, regular graphs, factors, r-graphs, edge-coloring, class 2 graphs, Petersen Coloring Conjecture, Berge-Fulkerson Conjecture.

1 Introduction

All graphs considered in this paper are finite and may have parallel edges but no loops. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. A graph is *r-regular* if

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every vertex has degree r. An r-regular graph is an *r-graph*, if $|\partial_G(X)| \geq r$ for every $X \subseteq V(G)$ of odd cardinality, where $\partial_G(X)$ denotes the set of edges that have precisely one vertex in X.

Let G be a graph and S be a set. An *edge-coloring* of G is a mapping $f\colon E(G) \to S$. It is a *k-edge-coloring* if $|S| = k$, and it is *proper* if $f(e) \neq f(e')$ for any two adjacent edges e and e'. The smallest integer k for which G admits a proper k-edge-coloring is the *edge-chromatic number* of G, which is denoted by $\chi'(G)$. A *matching* is a set $M \subseteq E(G)$ such that no two edges of M are adjacent. Moreover, M is said to be *perfect* if every vertex of G is incident with an edge of M.

If $\chi'(G)$ equals the maximum degree of G, then G is said to be *class* 1; otherwise G is *class* 2. If $\chi'(G) = r$, then r is the minimum number such that $E(G)$ decomposes into r matchings, which are perfect matchings in case of r-regular graphs. For $r \geq 1$, let $\mathcal{T}_r$ be the set of the smallest r-graphs of class 2. For example, the only element of $\mathcal{T}_3$ is the Petersen graph, which is denoted by P throughout this paper.

The generalized Berge-Fulkerson Conjecture [**?**] states that every r-graph has $2r$ perfect matchings such that every edge is in precisely two of them. For $r = 3$ the conjecture was attributed to Berge and Fulkerson [**?**], who put it into print (cf. [**?**]). As a unifying approach to study some hard conjectures on cubic graphs, Jaeger [**?**] introduced colorings with edges of another graph. To be precise, let G and H be graphs. An *H-coloring* of G is a mapping $f\colon E(G) \to E(H)$ such that

- if $e_1, e_2 \in E(G)$ are adjacent, then $f(e_1) \neq f(e_2)$,
- for every $v \in V(G)$ there exists a vertex $u \in V(H)$ with $f(\partial_G(v)) = \partial_H(u)$.

If such a mapping exists, then we write $H \prec G$ and say H *colors* G. A set $\mathcal{A}$ of connected r-graphs such that for every connected r-graph G there is an element $H \in \mathcal{A}$ which colors G is said to be *r-complete*. For every $r \geq 3$, let $\mathcal{H}_r$ be an inclusion-wise minimal r-complete set.

For $r = 3$, Jaeger [**?**] conjectured that the Petersen graph colors every bridgeless cubic graph. If true, this conjecture would have far reaching consequences. For instance, it would imply that the Berge-Fulkerson Conjecture and the 5-Cycle Double Cover Conjecture (see [**?**]) are also true. The Petersen Coloring Conjecture is a starting point for research in several directions. Different aspects of it are studied and partial results are proved, see for instance [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

Analogously to the case $r = 3$, if all elements of $\mathcal{H}_r$ would satisfy the generalized Berge-Fulkerson Conjecture, then every r-graph would satisfy it. Mazzuoccolo et al. [**?**] asked whether there exists a connected r-graph H such that $H \prec G$ for every (simple) r-graph G, for all $r \geq 3$. We show that $\mathcal{H}_r$ is unique and that it is an infinite set when $r \geq 4$. Furthermore, if $r = 3$, then either $\mathcal{H}_3 = \{P\}$ (if the Petersen Coloring Conjecture is true) or $\mathcal{H}_3$ is an infinite set. More precisely, in Section **??** we characterize $\mathcal{H}_r$ and provide constructions for infinite subsets of $\mathcal{H}_r$. Similar results are proved for simple r-graphs.

By definition, any r-graph G of class 1 can be colored with any r-graph H. Indeed, let $M_1, \dots, M_r$ be r pairwise disjoint perfect matchings of G and v a vertex of H with $\partial_H(v) = \{e_1, \dots, e_r\}$. Every edge of M_i of G can be mapped to e_i in H. Hence, the aforementioned questions and conjectures reduce to r-graphs of class 2. In Section **??** we determine the set $\mathcal{T}_r$ of the smallest r-graphs of class 2 and prove that $|\mathcal{T}_r| \geq p'(r-3, 6)$, where $p'(r-3,6)$ is the number of partitions of $r-3$ into at most 6 parts. Furthermore, we show that if $r \geq 4$, then $\mathcal{T}_r$ is a proper subset of $\mathcal{H}_r$.

The Petersen Coloring Conjecture has also been studied in the context of quasi-orders on the set of graphs, see [**?**, **?**]. In Section **??** we briefly put our results in this context. We conclude the paper with some open questions.

1.1 Definitions and basic results

Let G be a graph. For any subset X of $V(G)$, we use $G - X$ to denote the graph obtained from G by deleting all vertices of X and all incident edges. Similarly, for $F \subseteq E(G)$, denote by $G - F$ the graph obtained by deleting all edges of F from G. In particular, we simply write $G - x$ and $G - e$ for $G - X$ and $G - F$, respectively, when $X = \{x\}$ and $F = \{e\}$. The subgraph of G induced by the vertex set X is denoted by $G[X]$. Moreover, the graph obtained from G by identifying all vertices of X and deleting all resulting loops is denoted by G/X; we denote the new vertex by w_X. Let Y be a subset of $V(G)$ with $X \cap Y = \emptyset$. We use $[X,Y]_G$ to denote the set of all edges of G with one vertex in X and the other one in Y. Furthermore, if $Y = X^c = V(G) \setminus X$ and $[X,Y]_G$ is nonempty, then we call it an *edge-cut* of G and denote it by $\partial_G(X)$. If X or Y consists of one vertex, we skip the set-brackets notation. In addition, $|\partial_G(x)|$ is called the *degree* of $x \in V(G)$ and it is denoted by $d_G(x)$. If G is an r-graph, then $\partial_G(X)$ is *tight* if $|X|$ is odd and $|\partial_G(X)| = r$. A tight edge-cut is *trivial* if X or X^c consists of a single vertex. Moreover, for $v \in V(G)$ we denote by $N_G(v)$ the set of neighbors of v.

A 1-*factor* of a graph G is a spanning 1-regular subgraph of G, and its edge set is a perfect matching. A connected 2-regular graph is called a *circuit*. A circuit of length k is called a k-*circuit* and it is denoted by C_k.

For two graphs G and H, if there are two bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $e = uv \in E(G)$ if and only if $\phi(e) = \theta(u)\theta(v) \in E(H)$, then we say that G and H are *isomorphic*, denoted by $G \cong H$, and call the pair of mappings (θ, ϕ) an *isomorphism* between G and H. In particular, an *automorphism* of a graph is an isomorphism of the graph to itself.

Let $H_1, \dots, H_t$ be a sequence of graphs such that $V(H_i) \subseteq V(H_1)$ for each $i \in \{2, \dots, t\}$. Denote by $H_1 + E(H_2) + \ldots + E(H_t)$ the graph obtained from H_1 by adding a copy of every edge of H_i for every $i \in \{2, \dots, t\}$. Let $\mathcal{M}$ be a finite multiset of perfect matchings of the Petersen graph P. The graph $P + \sum_{M \in \mathcal{M}} M$ is denoted by $P^{\mathcal{M}}$.

Lemma 1.1 ([?]). *For every finite multiset $\mathcal{M}$ of perfect matchings of the Petersen graph P, the graph $P^{\mathcal{M}}$ is class 2.*

The following observation will frequently be used without reference.

Observation 1.2. *Let $r \geq 3$, let G be an r-graph and let $X \subseteq V(G)$. If $|X|$ is even, then $|\partial_G(X)|$ is even. If $|X|$ is odd, then $|\partial_G(X)|$ has the same parity as r.*

One major fact that we use in this paper is that every r-graph can be decomposed into a k-graph which is class 1 and an $(r-k)$-regular graph, for a suitable $k \in \{1, \dots, r\}$. For every r-graph G let $\pi(G)$ be the largest integer t such that G has t pairwise disjoint perfect matchings. Let $r \geq 3$ and $k \in \{1, \dots, r\}$ be integers. Let $\mathcal{G}(r,k) = \{G\colon G \text{ is an } r\text{-graph with } \pi(G) = k\}$. Note that $\mathcal{G}(r, r-1) = \emptyset$, since every r-graph with $r-1$ pairwise disjoint perfect matchings is a class 1 graph and thus, it has r pairwise disjoint perfect matchings. If $k \leq r-2$, then the elements of $\mathcal{G}(r,k)$ are class 2 graphs and $\mathcal{G}(r,i) \cap \mathcal{G}(r,j) = \emptyset$, if $1 \leq i \neq j \leq r-2$. We are interested in the subset of $\mathcal{G}(r,k)$ consisting of all such graphs with the smallest order. This set is denoted by $\mathcal{T}(r,k)$. By definition, $\mathcal{T}_r \subseteq \bigcup_{i=1}^{r-2} \mathcal{T}(r,i)$.

2 Smallest r-graphs of class 2

2.1 Determination of $\mathcal{T}_r$

The following theorem extends Lemma **??** and characterizes the perfect matchings M on $V(P)$ such that $P+M$ is a class 2 graph.

Theorem 2.1. *Let P be the Petersen graph and H be a 1-regular graph on $V(P)$ with edge set M. Then $P+M$ is class 2 if and only if $M \subseteq E(P)$.*

Proof. TOPROVE 0 □

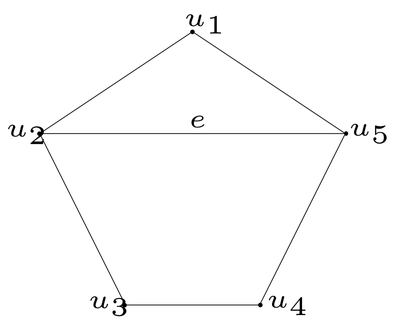

Figure 1: The 5-circuit C_5^1 with the edge e.

Theorem 2.2. *For all $r \geq 3$, $\mathcal{T}_r = \mathcal{T}(r, r-2) = \{P^{\mathcal{M}}\colon \mathcal{M}$ is a multiset of $r-3$ perfect matchings of the Petersen graph $P\}$.*

Proof. TOPROVE 1 □

2.2 Lower bounds for $|\mathcal{T}_r|$

The following lemma is a direct consequence of the fact that the Petersen graph is 3-arc-transitive, see e.g. Corollary 1.8 in [**?**]. That is, for any two paths of length 3 of P there is an automorphism of P which maps one to the other.

Lemma 2.3. *Let $M_1, \dots, M_6$ be the six perfect matchings of the Petersen graph P. Moreover, let $N_1, N_2, N_3 \in \{M_1, \dots, M_6\}$ and $g\colon \{N_1, N_2, N_3\} \to \{M_1, \dots, M_6\}$ be an injective function. There is an automorphism (θ, ϕ) of P such that, for all $i \in \{1, 2, 3\}$, $\phi(N_i) = g(N_i)$.*

Proof. TOPROVE 2 □

We now consider partitions of integers, which are ways of writing an integer as a sum of positive integers, see e.g. [**?**]. We are interested in partitions of an integer into a fixed number of parts. We allow 0 to be a part of a partition. A *partition* of an integer n into k parts is a multiset of k integers $n_1, \dots, n_k$ with $n_i \geq 0$ for $i \in \{1, \dots, k\}$ such that $n = \sum_{i=1}^{k} n_i$. Two partitions of n are equal if they yield the same multiset, i.e. if they differ only in the order of their elements. For two positive integers $k \leq n$, let $p'(n, k)$ be the number of partitions of n into k parts. Set $p'(0, k) = 1$.

Theorem 2.4. *If $3 \leq r \leq 8$, then $|\mathcal{T}_r| = p'(r-3, 6)$, and if $r \geq 9$, then $|\mathcal{T}_r| > p'(r-3, 6)$.*

Proof. TOPROVE 3 □

3 Complete sets

In this section we give the following characterization of $\mathcal{H}_r$: $G \in \mathcal{H}_r$ if and only if the only connected r-graph coloring G is G itself. Moreover, we show that $\mathcal{H}_r$ is an infinite set when $r \geq 4$. For $r = 3$ it turns out that, if the Petersen Coloring Conjecture is false, then $\mathcal{H}_3$ is an infinite set too. We prove similar results for the restriction on simple r-graphs.

We start with some preliminary technical results. In particular, we introduce a lifting operation for r-graphs.

3.1 Substructures and lifting

Let G be a graph and $F \subseteq E(G)$. We say that F *induces* a subgraph H of G if $E(H) = F$ and $V(H)$ contains all vertices of G which are incident with an edge of F. We denote such a subgraph H by $G[F]$. A spanning subgraph G' of G is a $\{K_{1,1}, C_m : m \geq 3\}$-factor if each component of G' is isomorphic to an element of $\{K_{1,1}, C_m : m \geq 3\}$, where $K_{s,t}$ is the complete bipartite graph with two partition sets of sizes s and t. Some of the following observations appear also in [**?**].

Observation 3.1. *Let H and G be graphs and let f be an H-coloring of G.*

(i) $\chi'(G) \leq \chi'(H)$.

(ii) *If $M_1, \ldots, M_k$ are k pairwise disjoint perfect matchings in H, then $f^{-1}(M_1), \ldots, f^{-1}(M_k)$ are k pairwise disjoint perfect matchings in G.*

(iii) *If C is a 2-regular subgraph of H, then $f^{-1}(E(C))$ induces a 2-regular subgraph in G.*

(iv) *If H' is a $\{K_{1,1}, C_m \colon m \geq 3\}$-factor in H, then $f^{-1}(E(H'))$ induces a $\{K_{1,1}, C_m \colon m \geq 3\}$-factor in G.*

Proof. TOPROVE 4 □

Let G be a graph and let $x \in V(G)$ with $|N_G(x)| \geq 2$. A *lifting* (of G) at x is the following operation: Choose two distinct neighbors y and z of x, delete an edge e_1 connecting x with y, delete an edge e_2 connecting x with z and add a new edge e connecting y with z; additionally, if e_1 and e_2 were the only two edges incident with x, then delete the vertex x in the new graph. We say e_1 and e_2 are *lifted to* e; the new graph is denoted by $G(e_1, e_2)$.

We will make use of the following fact. Let G be a graph, then $|\partial_G(X \cap Y)| + |\partial_G(X \cup Y)| \leq |\partial_G(X)| + |\partial_G(Y)|$ for every $X, Y \subseteq V(G)$.

Lemma 3.2. *Let $r \geq 2$ be an integer and let G be a connected graph of order at least 2 with a vertex $x \in V(G)$ such that*

- *$d_G(v) = r$ for all $v \in V(G) \setminus \{x\}$, and*
- *if $|V(G)|$ is even, then $d_G(x) \neq r$, and*
- *$|\partial_G(S)| \geq r$ for every $S \subseteq V(G) \setminus \{x\}$ of odd cardinality.*

Then, for every labeling $\partial_G(x) = \{e_1, \ldots, e_{d_G(x)}\}$ there exists an $i \in \mathbb{Z}_{d_G(x)}$ such that $G(e_i, e_{i+1})$ is a connected graph with $|\partial_{G(e_i,e_{i+1})}(S')| \geq r$ for every $S' \subseteq V(G(e_i, e_{i+1})) \setminus \{x\}$ of odd cardinality.

Proof. TOPROVE 5 □

The previous lemma can be used in r-graphs as follows.

Theorem 3.3. *Let $r \geq 2$ be an integer, let G be a connected r-graph and let X be a non-empty proper subset of $V(G)$. If $|X|$ is even, then G/X can be transformed into a connected r-graph by applying $\frac{1}{2}|\partial_G(X)|$ lifting operations at w_X. If $|X|$ is odd, then G/X can be transformed into a connected r-graph by applying $\frac{1}{2}(|\partial_G(X)| - r)$ lifting operations at w_X.*

Proof. TOPROVE 6 □

Note that the previous lifting operations can be applied so that they preserve embeddings of graphs in surfaces.

3.2 Characterization of $\mathcal{H}_r$

Let f be an H-coloring of G. The subgraph of H induced by the edge set $Im(f)$ is denoted by H_f. Observe that H_f also colors G. Furthermore, if H has no two vertices u_1, u_2 with $\partial_H(u_1) = \partial_H(u_2)$, then f induces a mapping $f_V : V(G) \to V(H)$, where every $v \in V(G)$ is mapped to the unique vertex $u \in V(H)$ with $f(\partial_G(v)) = \partial_H(u)$. Note that f_V is well defined if H is a connected graph with $|V(H)| > 2$. A vertex of $V(H) \setminus Im(f_V)$ is called *unused*.

Theorem 3.4. *Let $r \geq 3$ and let G be an r-graph of class 2 that cannot be colored by an r-graph of smaller order. If H is a connected r-graph and f is an H-coloring of G, then (f_V, f) is an isomorphism, i.e. $H \cong G$.*

Proof. TOPROVE 7 □

In [?], Mkrtchyan proves that if a connected 3-graph H colors the Petersen graph P, then $H \cong P$. The following result is implied by Theorem **??** together with Observation **??** (ii) and gives a generalization of Mkrtchyan's result in the r-regular case.

Corollary 3.5. *Let $r \geq 3$ and let G be an r-graph of class 2 such that $\pi(G') > \pi(G)$ for every r-graph G' with $|V(G')| < |V(G)|$. If H is a connected r-graph with $H \prec G$, then $H \cong G$.*

By Theorem **??**, $\mathcal{T}_r = \mathcal{T}(r, r-2) = \{P^{\mathcal{M}} : \mathcal{M}$ is a set of $r-3$ perfect matchings of the Petersen graph $P\}$. Hence, with Corollary **??** we obtain the following theorem.

Theorem 3.6. *Let $r \geq 3$, let H be a connected r-graph and let $G \in \mathcal{T}(r, r-2) \cup \mathcal{T}(r, 1)$. If $H \prec G$, then $H \cong G$.*

Theorem 3.7. *Let $r \geq 3$ and let G be a connected r-graph. The following statements are equivalent.*

1) $G \in \mathcal{H}_r$.

2) The only connected r-graph coloring G is G itself.

3) G cannot be colored by a smaller r-graph.

Proof. TOPROVE 8 □

Corollary 3.8. *For every $r \geq 3$, there exists only one inclusion-wise minimal r-complete set, i.e. $\mathcal{H}_r$ is unique.*

For $r = 3$, we have $\mathcal{T}(r, r-2) = \mathcal{T}(r, 1) = \{P\}$. The Petersen Coloring Conjecture states that $\mathcal{H}_3 = \{P\}$. This situation is very exclusive as we show in the following subsection.

3.3 Infinite subsets of $\mathcal{H}_r$

Lemma 3.9. *Let $r \geq 3$, let G and H be two connected r-graphs and let f be an H-coloring of G. For any 2-edge-cut $F = \{e_1, e_2\} \subseteq E(G)$, either $|f(F)| = 1$ or $f(F)$ is a 2-edge-cut of H.*

Proof. TOPROVE 9 □

Let G, H be two graphs, let $f\colon E(G) \to E(H)$, $g\colon V(G) \to V(H)$ and let G' be a subgraph of G. The restriction of f to $E(G')$ is denoted by $f|_{G'}$; the restriction of g to $V(G')$ is denoted by $g|_{G'}$.

Lemma 3.10. *Let G and H be two r-graphs, where $r \geq 3$, and let f be an H-coloring of G. Let $\mathcal{M}$ be a multiset of $r-3$ perfect matchings of P and let $e_0 \in E(P^{\mathcal{M}})$. Let G' be an induced subgraph of G isomorphic to $P^{\mathcal{M}} - e_0$ and H' be the subgraph of H induced by $f(E(G'))$. Then, $(f_V|_{G'}, f|_{G'})$ is an isomorphism between G' and H', i.e. $H' \cong G'$.*

Proof. TOPROVE 10 □

Let G and G' be two disjoint r-graphs of class 2 with $e \in E(G)$ and $e' \in E(G')$. Denote by $(G, e)|(G', e')$ the set of all graphs obtained from G by replacing the edge e of G by (G', e'), that is, deleting e from G and e' from G', and then adding two edges between $V(G)$ and $V(G')$ such that the resulting graph is regular (see Figure **??**).

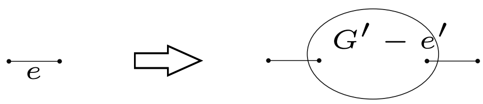

Figure 2: A replacement of the edge e by (G', e').

In fact, any graph in $(G, e)|(G', e')$ is an r-graph of class 2. Furthermore, we use $G|(G', e')$ to denote the set of all graphs obtained from G by replacing each edge of G by (G', e').

Theorem 3.11. *Let $\mathcal{M}$ be a multiset of $r-3$ perfect matchings of P, where $r \geq 3$, and let $e_0 \in E(P^{\mathcal{M}})$. Let G be an r-graph such that $G \ncong P^{\mathcal{M}}$. If $G \in \mathcal{H}_r$, then $G|(P^{\mathcal{M}}, e_0) \subset \mathcal{H}_r$.*

Proof. TOPROVE 11 □

The following corollary answers the question of [**?**] whether for each $r \geq 4$, there exists a connected r-graph H with $H \prec G$ for every r-graph G.

Corollary 3.12. *Either $\mathcal{H}_3 = \{P\}$ or $\mathcal{H}_3$ is an infinite set. Moreover, if $r \geq 4$, then $\mathcal{H}_r$ is an infinite set.*

Proof. TOPROVE 12 □

3.4 Simple r-graphs

In [?] the authors also asked whether for every $r \geq 4$, there is a connected r-graph coloring all simple r-graph. In this section we answer this question by showing that there is no finite set of connected r-graphs $\mathcal{H}'_r$ such that every connected simple r-graph can be colored by an element of $\mathcal{H}'_r$.

Lemma 3.13 ([?])**.** *Let r be a positive integer, G be an r-graph and $F \subseteq E(G)$. If $|F| \leq r-1$, then $G - F$ has a 1-factor.*

Recall that, for an r-graph G and an odd set $X \subseteq V(G)$, an edge-cut $\partial_G(X)$ is *tight* if it consists of exactly r edges.

Lemma 3.14. *Let $r \geq 3$, let G, H be connected r-graphs and let f be an H-coloring of G. If $F \subseteq E(G)$ is a tight edge-cut in G, then $f(F)$ is a tight edge-cut in H.*

Proof. TOPROVE 13 □

Lemma 3.15. *Let $r \geq 3$, let G and H be two r-graphs, and let X be a subset of $V(H)$ such that $\partial_H(X)$ is a tight cut and $\chi'(H/X^c) = r$. If $H \prec G$, then $H/X \prec G$.*

Proof. TOPROVE 14 □

For any graph G, the number of isolated vertices of G is denoted by $iso(G)$. A simple graph H is *regularizable* if we can obtain a regular graph from H by replacing each edge of H by a nonempty set of parallel edges. We need the following lemma, which follows from two results of [?] and [?]. The equivalence of the first two statements is shown in [?]; the equivalence of the first and the third statement is shown in [?].

Lemma 3.16. *Let G be a simple connected graph which is not bipartite with two partition sets of the same cardinality. The following statements are equivalent:*

- *$iso(G - S) < |S|$, for all $S \subseteq V(G)$.*
- *G is regularizable [?].*
- *for every $v \in V(G)$, both $G - v$ and G have a $\{K_{1,1}, C_m : m \geq 3\}$-factor [?].*

Lemma 3.17. *Let $r \geq 3$, let G and H be r-graphs, where H is connected, and let $S \subseteq V(G)$ such that $\partial_G(S)$ is a tight cut and $G[S]$ has no $\{K_{1,1}, C_m : m \geq 3\}$-factor. If G has an H-coloring $f : E(G) \to E(H)$ and $\partial_H(X) = f(\partial_G(S))$ for an $X \subseteq V(H)$, then H/X or H/X^c is a bipartite graph with two partition sets of the same cardinality.*

Proof. TOPROVE 15 □

Let G be an r-regular graph with a vertex $v \in V(G)$. A *Meredith extension* of G at v is the following operation. Delete the vertex v from G and add a copy K of the complete bipartite graph $K_{r,r-1}$. Finally add r edges between $V(G-v)$ and $V(K)$ such that the resulting graph is r-regular.

Lemma 3.18 (Rizzi [?])**.** *Let G be a graph and $X \subseteq V(G)$ with $|X|$ odd. If G/X and G/X^c are both r-graphs, then G is an r-graph.*

Theorem 3.19. *Let $r \geq 3$ and let $\mathcal{H}$ be a set of connected r-graphs such that every $H \in \mathcal{H}$ does not contain a non-trivial tight edge-cut $\partial_H(X)$ such that H/X or H/X^c is class 1. If every connected simple r-graph can be colored by an element of $\mathcal{H}$, then every connected r-graph can be colored by an element of $\mathcal{H}$.*

Proof. TOPROVE 16 □

We obtain the main result of this section as a corollary.

Corollary 3.20. *Let $r \geq 3$ and let $\mathcal{H}'_r$ be a set of connected r-graphs such that every connected simple r-graph can be colored by an element of $\mathcal{H}'_r$.*

i) If the Petersen Coloring Conjecture is false, then $\mathcal{H}'_3$ is an infinite set.

ii) If $r \geq 4$, then $\mathcal{H}'_r$ is an infinite set.

Proof. TOPROVE 17 □

4 Concluding remarks

4.1 Quasi-ordered sets

Jaeger [?] initiated the study of the Petersen Coloring Conjecture in terms of partial ordered sets. DeVos, Nešetřil and Raspaud [?] studied cycle-continuous mappings and asked whether there is an infinite set $\mathcal{G}$ of bridgeless graphs such that every two of them are cycle-continuous incomparable, i.e. there is no cycle-continuous map between any two graphs in $\mathcal{G}$. Šámal [?] gave an affirmative answer to the above question by constructing such an infinite set $\mathcal{G}$ of bridgeless cubic graphs. In fact, he also mentioned that this result can be considered in view of a quasi-order induced by cycle-continuous mappings on the set of bridgeless cubic graphs. That is, this quasi-ordered set contains infinite antichains.

For every integer $r \geq 1$, H-colorings of r-graphs induce a quasi-order on the set of r-graphs. Then, our result on r-graphs can be restated as follows: for any $r \geq 4$, there is an infinite set $\mathcal{H}_r$ of r-graphs such that each of them is incomparable to any other r-graph, and such infinite set exists for $r = 3$ if the Petersen Coloring Conjecture is false. In particular, the set $\mathcal{H}_r$ is an infinite antichain.

4.2 Open problems

The edge connectivity of an r-graph is equal to r or it is an even number. We have shown that $\mathcal{T}(r, r-2) \cup \mathcal{T}(r, 1) \subseteq \mathcal{H}_r$. Thus, for $r \neq 5$, for each possible edge-connectivity t there is a t-edge-connected r-graph in $\mathcal{H}_r$. For $r = 5$, we do not know any 5-edge-connected 5-graph with this property, see [**?**] for a discussion of this topic. However, we know only a finite number of t-edge-connected r-graphs of $\mathcal{H}_r$ if $t \geq 3$.

Problem 4.1. *For $r, t \geq 3$, does $\mathcal{H}_r$ contain infinitely many t-edge-connected r-graphs?*

It is also not clear whether $\mathcal{H}_r$ contains elements of $\mathcal{T}(r, k)$ for $k \in \{2, \ldots, r-3\}$. So far, these sets are not determined for $k \in \{1, \ldots, r-3\}$. Indeed, we even do not know the order of their elements. Let $o(r, k)$ be the order of the graphs of $\mathcal{T}(r, k)$.

Problem 4.2. *For all $r \geq 3$ and $k \in \{1, \ldots, r-2\}$: Determine $o(r, k)$.*

By our results, $o(r, r-2) = 10$. By results of Rizzi [**?**], $o(r, 1) \leq 2 \times 5^{r-2}$. We conjecture the following to be true.

Conjecture 4.3. *For all $r \geq 3$ and $k \in \{2, \ldots, r-2\}$: $o(r, k-1) \geq o(r, k)$.*

If Conjecture **??** would be true, then it would follow with Corollary **??** that $\mathcal{T}(r, k) \subset \mathcal{H}_r$ for each $k \in \{1, \ldots, r-2\}$.

Similar problems arise for simple r-graphs. Let $o_s(r, k)$ be the smallest order of a simple r-graph G with $\pi(G) = k$. Small simple r-graphs of class 2 can be obtained as follows. Consider a perfect matching M of P and the graph $G = P + (r-3)M$. Let H be a simple r-graph of smallest order and $v \in V(H)$. Then, H is class 1 and $|V(H)| = r+1$ if r is odd and $|V(H)| = r+2$ if r is even. Now, replace appropriately five vertices of G by $H - v$ to obtain a simple r-graph G'. Since H is class 1 and $\pi(G) = r-2$, we have $\pi(G') = r-2$. Therefore, if r is odd, then $o_s(r, r-2) \leq 5(r+1)$ and if r is even, then $o_s(r, r-2) \leq 5(r+2)$. Furthermore, bounds for $o_s(r, k)$ can be obtained by using Meredith extensions, since if G' is a Meredith extension of an r-graph G, then $\pi(G') = \pi(G)$.

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