ARRIVAL: Recursive Framework & ℓ_1-Contraction

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─── **Abstract** ───

ARRIVAL is the problem of deciding which out of two possible destinations will be reached first by a token that moves deterministically along the edges of a directed graph, according to so-called switching rules. It is known to lie in $\mathsf{NP} \cap \mathsf{CoNP}$, but not known to lie in P. The state-of-the-art algorithm due to Gärtner et al. (ICALP '21) runs in time $2^{\mathcal{O}(\sqrt{n}\log n)}$ on an n-vertex graph.

We prove that ARRIVAL can be solved in time $2^{\mathcal{O}(k\log^2 n)}$ on n-vertex graphs of treewidth k. Our algorithm is derived by adapting a simple recursive algorithm for a generalization of ARRIVAL called G-ARRIVAL. This simple recursive algorithm acts as a framework from which we can also rederive the subexponential upper bound of Gärtner et al.

Our second result is a reduction from G-ARRIVAL to the problem of finding an approximate fixed point of an ℓ_1-contracting function $f : [0,1]^n \to [0,1]^n$. Finding such fixed points is a well-studied problem in the case of the ℓ_2-metric and the ℓ_∞-metric, but little is known about the ℓ_1-case.

Both of our results highlight parallels between ARRIVAL and the Simple Stochastic Games (SSG) problem. Concretely, Chatterjee et al. (SODA '23) gave an algorithm for SSG parameterized by treewidth that achieves a similar bound as we do for ARRIVAL, and SSG is known to reduce to ℓ_∞-contraction.

1 Introduction

ARRIVAL is a computational problem first introduced by Dohrau et al. [11]. It can be described as a deterministic process (or zero-player game) on a directed graph with a designated origin o and two designated destinations d and $\bar{d}$. Every vertex in ARRIVAL has out-degree two, and exactly one outgoing edge at every vertex is marked (we also call it the *even* edge). We additionally assume that both destinations are reachable from every vertex in the graph. A token is placed on o and moved along the edges of the graph according to the following rule: At every vertex, the token continues along the outgoing edge that was used least so far. In case of a tie, the token uses the even edge. This effectively means that the token will alternate between the two outgoing edges at every vertex, starting with the even edge. The task is to decide which of the two destinations d or $\bar{d}$ will be visited first by the token.

Dohrau et al. [11] proved that ARRIVAL is contained in $\mathsf{NP} \cap \mathsf{CoNP}$. Naturally, they then asked whether it is also in P. This open problem has received some attention in recent years and the best algorithm to date, due to Gärtner et al. [16], runs in time $2^{\mathcal{O}(\sqrt{n}\log n)}$ on a graph with n vertices.

We present two new results for ARRIVAL: Our first result is an algorithm for ARRIVAL that runs in time $2^{\mathcal{O}(k\log^2 n)}$ on graphs with n vertices and treewidth k. Note that this bound is quasi-polynomial for graphs with bounded treewidth. Our algorithm is obtained by

adapting a simple recursive algorithm for G-ARRIVAL, a generalization of ARRIVAL that allows arbitrarily many origins, destinations, and tokens: The recursive algorithm solves an instance with ℓ destinations and origins by making recursive calls on instances with $\ell + 1$ destinations and origins. In other words, in each recursive call, a new vertex is made into a destination and origin vertex. By choosing this pivot vertex carefully, we can exploit the underlying graph structure.

It turns out that this simple recursive algorithm for G-ARRIVAL can also be seen as a framework for other algorithms for G-ARRIVAL. Concretely, we explain how the state-of-the-art upper bound $2^{\mathcal{O}(\sqrt{n}\log n)}$ due to Gärtner et al. [16] can be derived in this framework as well.

Our second result is a reduction from G-ARRIVAL to the problem of finding an approximate fixed point of a ℓ_1-contracting function $f : [0,1]^n \to [0,1]^n$. Concretely, we say that a function $f : [0,1]^n \to [0,1]^n$ is contracting with parameter $\lambda \in [0,1)$ if we have $\|f(x) - f(y)\|_1 \le \lambda \|x - y\|_1$ for all $x, y \in [0,1]^n$. Such a function is guaranteed to have a unique fixed point by Banach's fixed point theorem [4]. An ε-approximate fixed point $x \in [0,1]^n$ has to satisfy $\|f(x) - x\|_1 \le \varepsilon$. Given our reduction, any algorithm that can find an ε-approximate fixed point of f in time poly $\left(\log \frac{1}{\varepsilon}, \log \frac{1}{1-\lambda}, n\right)$ would imply a polynomial-time algorithm for G-ARRIVAL.

Finding approximate fixed points is a well-studied problem in the case of the ℓ_2-metric and the ℓ_∞-metric (see e.g. [24, 8]). In particular, efficient algorithms in the case of the ℓ_2-metric have been known since 1993 [23], and Chen et al. [8] only recently gave the first polynomial query upper bound for the ℓ_∞-metric. Even more recently, a generalization of the result by Chen et al. to ℓ_p-contractions for every $p \in [1, \infty]$ was announced [18]. Concretely, this means that an approximate fixed point of an ℓ_1-contraction can be found with polynomially many queries to the contraction map.

Unfortunately, the algorithm for ℓ_1-contractions in [18] is only query-efficient (and not time-efficient). Thus, our reduction currently does not imply better algorithms for G-ARRIVAL. We still think that our reduction is interesting: To the best of our knowledge, it provides the first concrete application for the ℓ_1-contraction problem. Moreover, the polynomial query upper bound for ℓ_1-contractions [18] suggests that better (maybe even time-efficient) algorithms could be obtained for ℓ_1-contraction and hence G-ARRIVAL in the future. For this, it might also be interesting to observe that the ℓ_1-contraction map obtained through our reduction from G-ARRIVAL has the additional property of being monotone with respect to the coordinate-wise partial order. Given recent work of Batziou et al. [5] on monotone ℓ_∞-contractions, it seems plausible that ℓ_1-contraction and monotonicity could maybe be exploited simultaneously as well.

1.1   Related Work

Since ARRIVAL is contained in NP ∩ CoNP, it also naturally fits into the complexity class TFNP, which contains total problems with efficiently verifiable solutions. In fact, after a series of results for containment in subclasses of TFNP [20, 15], we now know that ARRIVAL is contained in UEOPL [14].

Going into a slightly different direction, Gärtner et al. [15] proved that ARRIVAL is also contained in UP and CoUP, the analogues of NP and CoNP with unique solutions. In fact, it would not be hard to rederive this using our reduction to ℓ_1-contraction: The idea is that the fixed point acts as an efficient certificate for both YES- and NO-instances and it must be unique due to the contraction property.

It is also known that ARRIVAL can be solved in polynomial time on some restricted graph classes. For example, a result due to Priezzhev et al. [22] implies that ARRIVAL can be solved by simulation in polynomial time on Eulerian graphs. Other results include polynomial-time algorithms on tree-like multigraphs [1] and path-like multigraphs with many tokens [2]. More recently, algorithms running in quasi-polynomial-time were given for the case of tree-like multigraphs with many tokens [17]. Our algorithm effectively generalizes this result to all graphs of bounded treewidth.

Finally, further variants of ARRIVAL have been studied in the past, including a stochastic variant [25], a recursive variant [26], as well as variants with one or two players [13].

Comparison to SSG

As mentioned before, both our results show parallels between ARRIVAL and Simple Stochastic Games (SSG). We will briefly discuss the connections between the two problems. Similarly to ARRIVAL, SSG is contained in NP∩CoNP [9] and even UP∩CoUP [6], but no polynomial-time algorithm is known. The state-of-the-art algorithm due to Ludwig [21] runs in randomized subexponential time $2^{\mathcal{O}(\sqrt{n}\log n)}$.

Dohrau et al. [11] already wondered about similarities between ARRIVAL and SSG when they first introduced ARRIVAL. Since then, both problems were shown to reduce to the problem of finding a Tarski fixed point [12, 16], and to be contained in the complexity class UEOPL [14]. Both our results for ARRIVAL further extend this list of similarities: Our upper bound of $2^{\mathcal{O}(k\log^2 n)}$ for ARRIVAL on n-vertex graphs of treewidth k is comparable to a similar bound for SSG due to Chatterjee et al. [7]. Moreover, SSG reduces to finding an approximate fixed point of an ℓ_∞-contracting function [9], which is analogous to our reduction from ARRIVAL to ℓ_1-contraction.

Both ARRIVAL and SSG also admit polynomial-time algorithms on graphs with a bounded feedback vertex set [16, 3]. In fact, in the case of ARRIVAL, we will discuss this in more detail in Section 3.2.

1.2 Outline

As explained above, our results are actually obtained for a generalization of ARRIVAL called G-ARRIVAL. Thus, we will start Section 2 by formally introducing G-ARRIVAL. We also use Section 2 to recall further terminology and notation from the literature that will be useful for our arguments.

Note that instead of formally introducing the notion of treewidth and tree decompositions, we will directly work with so-called balanced separators instead. The reason for this choice of exposition is that our parameterized algorithm actually exploits the existence of small balanced separators (and not tree decompositions themselves), which are guaranteed to exist in graphs of small treewidth (see Section 2.1 for more details).

We describe our parameterized algorithm in Section 3. We start the exposition with our simple recursive algorithm for G-ARRIVAL (Section 3.1), which provides a framework from which we will derive the parameterized algorithm in Section 3.3. In Section 3.2, we additionally explain how the subexponential upper bound due to Gärtner et al. [16] and polynomial-time upper bounds on graphs with a bounded feedback vertex set [16] can be rederived from our framework.

Finally, Section 4 contains our reduction to the problem of finding an approximate fixed point of a ℓ_1-contraction map. Crucially, given an instance of G-ARRIVAL, we define a function $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ that is contracting and thus has a unique fixed point. We then

prove that a reasonably good approximation of this fixed point will give away the solution to the G-ARRIVAL-instance. The function can be restricted to a compact subset of $\mathbb{R}^n_{\geq 0}$ and scaled to fit into $[0, 1]^n$, if desired.

2 Preliminaries

We start by recalling G-ARRIVAL, which was first formally defined by Hoang [19]. Note that our formulation slightly deviates from the one by Hoang, but is easily seen to be equivalent.

A switch graph is a directed graph $G = (V, E, s_0, s_1)$ with $s_0, s_1 : V \to V$ and $E = \{(v, s_0(v)) \mid v \in V\} \cup \{(v, s_1(v)) \mid v \in V\}$. We consider E to be a multiset, and two edges $(v, s_0(v)), (v, s_1(v)) \in E$ to be distinct objects even if we have $s_0(v) = s_1(v)$. Given a switch graph, we call $s_0(v)$ and $s_1(v)$ the even and odd successor of $v \in V$, respectively. Similarly, we refer to edges induced by s_0 as even edges, and to edges induced by s_1 as odd edges.

In G-ARRIVAL, many tokens traverse the directed graph simultaneously, starting and ending at special vertices that we call terminals or terminal vertices. Concretely, a G-ARRIVAL-instance consists of a switch graph as well as a non-empty subset $T \subseteq V$ of terminals. For each terminal $v \in T$, we also get a natural number t_v^+ of tokens starting at v. We always assume that at least one terminal is reachable from each non-terminal vertex $v \in V \setminus T$ (otherwise, tokens could loop indefinitely without ever reaching a terminal). We will also always denote the total number of tokens by $t^+ := \sum_{v \in T} t_v^+$ and assume $t^+ \geq 1$.
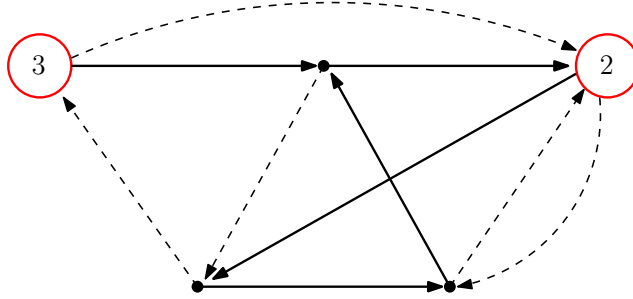
Figure 1 An instance of G-ARRIVAL. Even edges are bold while odd edges are dashed. Terminals are marked in red. Three tokens start at the left terminal, and two tokens start at the right terminal.

Now consider the following non-deterministic procedure. Initially, for every terminal $v \in T$, move $\lceil \frac{t_v^+}{2} \rceil$ tokens from v to $s_0(v)$, and $\lfloor \frac{t_v^+}{2} \rfloor$ tokens from v to $s_1(v)$ (we do this for every terminal simultaneously). Then, while there exists a token on a non-terminal $v \in V \setminus T$, non-deterministically choose one such token and move it along the out-edge of v that was used fewer times so far. In case of a tie, the token must use the even out-edge $(v, s_0(v)) \in E$. In other words, the even and odd out-edges at v will be used in an alternating fashion, starting with the even out-edge. This procedure stops once all tokens have reached a terminal. The goal of G-ARRIVAL is to predict the number of tokens t_v^- arriving at each terminal $v \in T$.

In order for G-ARRIVAL to be well-defined, we need to make sure that the above procedure terminates and that it always produces the same values t_v^- for all $v \in T$ (independently of the non-deterministic choices). Indeed, both of these properties hold, and this follows by generalizing the arguments of Dohrau et al. [11] to work with multiple tokens, multiple destinations, and multiple origins, as explained by Gärtner et al. [16] and Hoang [19]. To explain this, we first need to recall the concept of switchings flows from the literature.

Given a switch graph $G = (V, E, s_0, s_1)$ with terminals $T \subseteq V$ and starting tokens $(t_v^+)_{v \in T}$, a function $x : E \to \mathbb{R}_{\geq 0}$ satisfying the three constraints

$$x(v, s_0(v)) - x(v, s_1(v)) \in \{0, 1\} \qquad (\forall v \in V)$$

$$\underbrace{\sum_{u:(v,u)\in E} x(v, u)}_{=:x^+(v)} - \underbrace{\sum_{u:(u,v)\in E} x(u, v)}_{:=x^-(v)} = 0 \qquad (\forall v \in V \setminus T)$$

$$\underbrace{\sum_{u:(v,u)\in E} x(v, u)}_{=:x^+(v)} = t_v^+ \qquad (\forall v \in T)$$

is called a switching flow. We will refer to the first set of constraints above as switching behavior, and to the second set of constraints as flow conservation.

Dohrau et al. [11] proved the following theorem in the case of ARRIVAL, but we directly state its generalization for G-ARRIVAL (see also [16, 19]).

▶ **Theorem 1** (Integral Switching Flows are Certificates [11]). *Given a switch graph $G = (V, E, s_0, s_1)$ with terminals $\varnothing \neq T \subseteq V$ and starting tokens $(t_v^+)_{v \in T}$ with $t^+ \geq 1$, the number of tokens t_v^- arriving at terminal v is well-defined and any integral switching flow $x : E \to \mathbb{N}_0$ satisfies $x^-(v) = t_v^-$, for all $v \in T$.*

Concretely, Theorem 1 states that G-ARRIVAL is well-defined and that it can be solved by finding any integral switching flow. Observe that by simulating the non-deterministic procedure outlined before and recording the number of times that each edge is traversed by a token, one can obtain a special integral switching flow that we call the run profile. While the run profile is unique (i.e. it does not depend on non-deterministic choices) [16], Dohrau et al. [11] already observed that in general, integral switching flows are not unique. Concretely, there may be integral switching flows other than the run profile, but Theorem 1 tells us that they still predict the values $(t_v^-)_{v \in T}$ correctly.
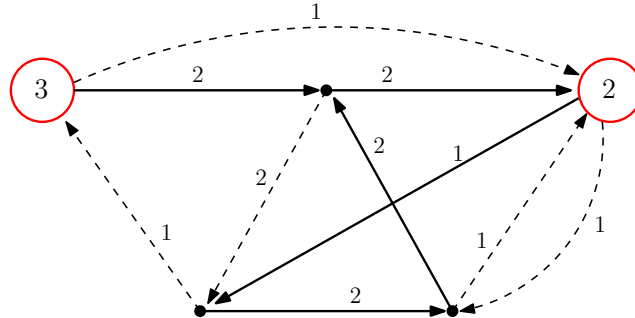
■ **Figure 2** The number on the edges indicate an integral switching flow. Note that this is not the run profile, but it still certifies (by Theorem 1) that four tokens arrive at the right terminal while only one token arrives at the left terminal. To get the run profile, one would have to decrease the flow on the edges of the directed triangle formed by the three non-terminal vertices by one each.

Finally, we will need an upper bound on the total flow in any integral switching flow. Similar upper bounds were used in previous work as well (see e.g. [11, 15, 16, 19]). We sketch a short proof.

▶ **Lemma 2** (Upper Bound on Integral Switching Flow [11]). *Let $x : E \to \mathbb{N}_0$ be an arbitrary integral switching flow for the G-ARRIVAL-instance on a switch graph $G = (V, E, s_0, s_1)$ with terminals $\varnothing \neq T \subseteq V$ and starting tokens $(t_v^+)_{v \in T}$ with $t^+ \geq 1$. Then we must have $t^+ = \sum_{v \in T} t_v^+ = \sum_{v \in T} t_v^- = \sum_{v \in T} x^-(v)$ and $x(e) < 2^{|V|} t^+$ for all $e \in E$.*

Proof. TOPROVE 0 ◀

2.1 Treewidth and Balanced Separators

Treewidth is a well-established graph parameter that plays an important role in parameterized algorithms, and it is intimately related to the notion of balanced separators (see e.g. [10, Chapter 7]). As is the case with many applications on graphs of bounded treewidth, our algorithm actually exploits the existence of balanced separators and can be formulated without computations of tree decompositions. Hence, we will refrain from formally introducing tree decompositions and instead focus on balanced separators.

Given an undirected graph $G = (V, E)$ and a subset $S \subseteq V$ of its vertices, we use $G - S$ to denote the graph resulting from deleting the vertices in S and their incident edges from G. We call S a balanced separator if each connected component in $G - S$ contains at most $\frac{1}{2}|V|$ vertices. Note that this does not necessarily imply that $G - S$ must have more than one connected component and it could even be an empty graph (despite what the term separator may suggest): Concretely, any set S of size at least $\frac{1}{2}|V|$ is a balanced separator, and thus there always exists a balanced separator. Using a brute-force approach, we can find a smallest balanced separator S in G in time $|V|^{\mathcal{O}(|S|)}$.

The following connection between treewidth and balanced separators is crucial for us.

▶ **Lemma 3** (Balanced Separators and Treewidth [10, Lemma 7.19]). *If G has treewidth at most k, then for every $S \subseteq V$, the subgraph $G - S$ has a balanced separator of size at most $k + 1$.*

This lemma allows us to use the treewidth of G to infer the existence of small balanced separators in all induced subgraphs of G.

All of the previous concepts are defined on undirected graphs. Since we will be working exclusively with directed graphs, we will adopt the following convention: When used on a directed graph, the terms treewidth and balanced separators are to be interpreted with respect to the underlying simple undirected graph.

2.2 Non-Expansive, Contracting, and Monotone Functions

We will be interested in the Manhattan distance, which is induced by the ℓ_1-norm. Concretely, we use $\|x\| := \sum_{i=1}^n |x_i|$ to denote the ℓ_1-norm of a vector $x \in \mathbb{R}^n$. The Manhattan distance of $x, y \in \mathbb{R}^n$ is then given by $\|x - y\|$.

We are mainly concerned with the non-negative orthant $\mathbb{R}_{\geq 0}^n \subseteq \mathbb{R}^n$. A function $f : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$ is called a λ-contraction (or is λ-contracting) for some $\lambda \in [0, 1)$ if and only if $\|f(x) - f(y)\| \leq \lambda \|x - y\|$ for all $x, y \in \mathbb{R}_{\geq 0}^n$. Banach's fixed point theorem [4] implies that such a contracting function admits a unique fixed point. If f only satisfies the weaker property $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in \mathbb{R}_{\geq 0}^n$, we call it non-expansive instead.

In Section 4, we reduce G-ARRIVAL to the following computational problem: Given access to a λ-contraction $f : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$, find an ε-approximate fixed point of f, i.e. a point $x \in \mathbb{R}_{\geq 0}^n$ such that $\|f(x) - x\| \leq \varepsilon$. We will also point out how the domain of our function f can be restricted to $[0, 1]^n$, if desired.

Another property that we need in some of our proofs is monotonicity with respect to the coordinate-wise partial order. Concretely, we call a function $f : X \subseteq \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$ monotone if and only if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in X$, where $\leq$ denotes coordinate-wise comparison.

▶ **Lemma 4** (Monotonicity in G-ARRIVAL [16]). *Consider a switch graph $G = (V, E, s_0, s_1)$ with terminals $\varnothing \neq T \subseteq V$. Consider the function $f : \mathbb{N}_0^{|T|} \to \mathbb{N}_0^{|T|}$ that maps the vector $(t_v^+)_{v \in T}$ of starting tokens to the vector $(t_v^-)_{v \in T}$ of tokens ending at terminals. The function f is monotone with respect to the coordinate-wise partial order.*

3   A Family of Recursive Algorithms

The main goal of this section is to give a parameterized algorithm for G-ARRIVAL that runs in time $2^{\mathcal{O}\left(k \log n \log(n + \log t^+)\right)}$ on graphs with n vertices, treewidth k, and a total of t^+ starting tokens. Note that for the sake of simplicity, our algorithm recurses by doing a binary search over one vertex at a time. One could instead use algorithms for the so-called Tarski-problem (see e.g. [12]) to recurse by searching over multiple vertices (e.g. all vertices of a balanced separator) at a time. However, with the current best algorithms for Tarski, this would not yield any asymptotic improvements.

3.1   A Simple Recursive Algorithm

We start by explaining a simple recursive algorithm for G-ARRIVAL that is inspired by the approach of Gärtner et al. [16]: The algorithm chooses an arbitrary non-terminal $p \in V \setminus T$ that we call the pivot, and makes a guess $a \in \mathbb{N}_0$ for the outflow of p in an integral switching flow. In order to verify the guess, p is converted to a terminal and $t_p^+ := a$ tokens are assigned to start at p. In this way, we obtain again an instance of G-ARRIVAL with one more terminal. After solving this subinstance, we can check our guess by looking at the number of tokens t_p^- that arrive at p in the subinstance. If we find that $t_p^- = a = t_p^+$, the integral switching flow obtained for the subinstance is also an integral switching flow for the original instance where p is not a terminal. Otherwise, we have $t_p^- < a$ or $t_p^- > a$ and we use this information to adjust our guess in a binary search fashion. We make this precise in Algorithm 1 and use the remainder of this section to prove that Algorithm 1 is correct.

▶ **Lemma 5** (Analysis of Algorithm 1). *Given an arbitrary G-ARRIVAL-instance consisting of $G = (V, E, s_0, s_1)$ with terminals $\varnothing \neq T \subseteq V$ and starting tokens $(t_v^+)_{v \in T}$ with $t^+ \geq 1$, Algorithm 1 correctly returns an integral switching flow in time $2^{\mathcal{O}\left(|V \setminus T| \log(|V| + \log t^+)\right)}$.*

Proof. TOPROVE 1 ◀

3.2   Subexponential Upper Bound

Picking the pivot $p \in V \setminus T$ in Algorithm 1 arbitrarily seems quite naive. In this section, we explain how applying the ideas of Gärtner et al. [16] yields a subexponential upper bound.

▶ **Lemma 6** (Algorithm with Diameter-Like Bound [16]). *Consider an arbitrary G-ARRIVAL-instance consisting of $G = (V, E, s_0, s_1)$ with non-empty $T \subseteq V$ and starting tokens $(t_v^+)_{v \in T}$ with $t^+ \geq 1$. Let $\ell := \max_{v \in V \setminus T} dist(v, T)$, where $dist(v, T)$ denotes the shortest path distance from v to any vertex in T. There is an algorithm that solves G-ARRIVAL in time $2^\ell \operatorname{poly}(|V|, \log t^+)$.*

■ **Algorithm 1**

```
Find-Switching-Flow-1(G = (V, E, s_0, s_1), T ⊆ V, (t_v^+)_{v∈T})
```
 2 **if** $T = V$ // Base Case
 3 **then**
 4 $\quad\Big|\quad x(v, s_0) \leftarrow \lceil\frac{t_v^+}{2}\rceil$ for all $v \in V = T$
 5 $\quad\Big|\quad x(v, s_1) \leftarrow \lfloor\frac{t_v^+}{2}\rfloor$ for all $v \in V = T$
 6 $\quad\Big|\quad$ **return** x
 7 choose arbitrary $p \in V \setminus T$ // Binary Search Case
 8 $T' \leftarrow T \cup \{p\}$
 9 $\ell \leftarrow 0$
10 $r \leftarrow 2^{|V|}t^+$
11 **while** $\ell < r$ **do**
12 $\quad\Big|\quad t_p^+ \leftarrow \lceil\frac{\ell+r}{2}\rceil$
13 $\quad\Big|\quad x \leftarrow$ Find-Switching-Flow-1$(G, T', (t_v^+)_{v∈T'})$
14 $\quad\Big|\quad$ **if** $x^-(p) < t_p^+$ **then**
15 $\quad\Big|\quad\quad\Big\lfloor\quad r \leftarrow t_p^+ - 1$
16 $\quad\Big|\quad$ **if** $x^-(p) = t_p^+$ **then**
17 $\quad\Big|\quad\quad\Big\lfloor\quad$ **return** x
18 $\quad\Big|\quad$ **if** $x^-(p) > t_p^+$ **then**
19 $\quad\Big|\quad\quad\Big\lfloor\quad \ell \leftarrow t_p^+ + 1$

▶ **Lemma 7** (Decomposition Lemma [16]). *Let $G = (V, E, s_0, s_1)$ be an arbitrary switch graph with a non-empty set $T \subseteq V$ of terminals. There is an algorithm that finds a set $S \subseteq V \setminus T$ of size $\mathcal{O}(\sqrt{|V|})$ satisfying $\max_{v \in V \setminus (T \cup S)} dist(v, T \cup S) \leq \mathcal{O}(\sqrt{|V|} \log |V|)$ in time $\mathcal{O}(|V|)$.*

Given those two observations, it is now not hard to adapt Algorithm 1 to run in subexponential-time: We first precompute the set S from Lemma 7 in linear time. Then, we run Algorithm 1 with two changes: In the recursive step, we make sure to always pick a pivot p from S instead of all of $V \setminus T$. Further, we add a new base case that applies the algorithm from Lemma 6 as soon as the parameter ℓ from Lemma 6 has shrunk to $\mathcal{O}(\sqrt{|V|} \log |V|)$, which is guaranteed to happen at the latest once all of the vertices in S have been turned into terminals. With these two changes, the recursion depth will become $\mathcal{O}(\sqrt{|V|})$, and since the base case also runs in time exponential only in $\mathcal{O}(\sqrt{|V|} \log |V|)$, we get an overall subexponential runtime.

As observed by Gärtner et al. [16], one can do better on graphs with a small feedback vertex set: The crucial ingredient is that G-ARRIVAL can be solved efficiently on acyclic graphs by greedy simulation of the tokens. Using this as a base case and choosing pivots from a feedback vertex set yields yet another instantiation of the framework provided by the recursive algorithm. Concretely, this yields a polynomial-time algorithm on graphs with a bounded feedback vertex set.

3.3 Exploiting Balanced Separators

We now describe an adaptation of Algorithm 1 that works well on graphs of small treewidth. The general idea is again to pick the pivot $p \in V \setminus T$ appropriately. Concretely, as mentioned in Section 2.1, small treewidth ensures small balanced separators in all induced subgraphs of

our input graph. Thus, it seems intuitive that in each step, we should pick the pivot p from a balanced separator of the graph induced by the remaining non-terminals. Eventually, this should disconnect the induced graph into independent subinstances, each with a significantly smaller number of non-terminals. Recursing on all subinstances independently yields the desired speedup. We make this precise in Algorithm 2 and analyse the algorithm in the remainder of this section.

▶ **Lemma 8** (Correctness of Algorithm 2). *Given an arbitrary G-ARRIVAL-instance consisting of $G = (V, E, s_0, s_1)$ with non-empty $T \subseteq V$, starting tokens $(t_v^+)_{v \in T}$ with $t^+ \geq 1$, and a balanced separator S for $G - T$, Algorithm 2 correctly returns an integral switching flow.*

Proof. TOPROVE 2 ◀

▶ **Lemma 9.** *Assume that we run Algorithm 2 on a G-ARRIVAL-instance consisting of $G = (V, E, s_0, s_1)$ with non-empty $T \subseteq V$ and starting tokens $(t_v^+)_{v \in T}$ with $t^+ \geq 1$. Further assume that we input a smallest balanced separator S of the subgraph $G - T$, and assume that G has treewidth at most k. Then the algorithm cannot reach recursion depth $k + 2$ without at least once recursing in a splitting case. Moreover, if a splitting case is reached, then each connected component $C \subseteq V \setminus (T \cup S)$ satisfies $|C| \leq \frac{|V \setminus T|}{2}$.*

Proof. TOPROVE 3 ◀

▶ **Theorem 10.** *Given a G-ARRIVAL-instance consisting of $G = (V, E, s_0, s_1)$ with non-empty $T \subseteq V$, starting tokens $(t_v^+)_{v \in T}$ with $t^+ \geq 1$, and a smallest balanced separator S of $G - T$, Algorithm 2 computes an integral switching flow x in time $2^{\mathcal{O}\left(k \log(|V \setminus T|) \log(|V| + \log t^+)\right)}$, where k is the treewidth of G.*

Proof. TOPROVE 4 ◀

4 Reduction to ℓ_1-Contraction

The goal of this section is to prove that G-ARRIVAL reduces to finding an approximate fixed point of an ℓ_1-contracting function. For this, we will frequently use the functions $h_0, h_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$h_0(x) := \min\left\{ x - \left\lfloor \frac{x}{2} \right\rfloor, \left\lceil \frac{x}{2} \right\rceil \right\} \quad \text{and} \quad h_1(x) := \max\left\{ \left\lfloor \frac{x}{2} \right\rfloor, x - \left\lceil \frac{x}{2} \right\rceil \right\}$$

for all $x \in \mathbb{R}_{\geq 0}$. Observe that both h_0 and h_1 are continuous and monotone, and that we have $h_0(x) + h_1(x) = x$ as well as $h_1(x) \leq h_0(x) \leq h_1(x) + 1$ for all $x \in \mathbb{R}_{\geq 0}$.

4.1 One-Step Update

Consider a vector $x \in \mathbb{R}_{\geq 0}^n$ and think of it as token mass that is distributed among the vertices of a switch graph, with x_v token mass currently occupying vertex v. We want to define a notion of moving all (possibly fractional) tokens by one step each while respecting the switching rules. The following one-step update function captures this idea, with the intuition that $f(x)_v$ is the amount token mass that $v \in V$ receives from its predecessors.

▶ **Definition 11** (One-Step Update). *Let $G = (V, E, s_0, s_1)$ be a switch graph. The one-step update function $f : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$ associated with this switch graph is defined as*

$$f(x)_v := \sum_{u : s_0(u) = v} h_0(x_u) + \sum_{u : s_1(u) = v} h_1(x_u)$$

for all $v \in V$.

■ **Algorithm 2** The main difference to Algorithm 1 is that we choose our pivot from a smallest balanced separator S that is passed through the recursive calls. As soon as all vertices in the separator have been turned into terminals, we can split the instance into independent subinstances and proceed from there.

Find-Switching-Flow-2($G = (V, E, s_0, s_1), T \subseteq V, (t_v^+)_{v \in T}, S \subseteq V \setminus T$)

2 **if** $T = V$ // Base Case
3 **then**
4 $\quad\mid\quad$ $x(v, s_0) \leftarrow \lceil \frac{t_v^+}{2} \rceil$ for all $v \in V = T$
5 $\quad\mid\quad$ $x(v, s_1) \leftarrow \lfloor \frac{t_v^+}{2} \rfloor$ for all $v \in V = T$
6 $\quad\mid\quad$ **return** x

7 **if** $|S| > 0$ // Binary Search Case
8 **then**
9 $\quad\mid\quad$ choose arbitrary $p \in S$
10 $\quad\mid\quad$ $T' \leftarrow T \cup \{p\}$
11 $\quad\mid\quad$ $S' \leftarrow S \setminus \{p\}$
12 $\quad\mid\quad$ $\ell \leftarrow 0$
13 $\quad\mid\quad$ $r \leftarrow 2^{|V|} t^+$
14 $\quad\mid\quad$ **while** $\ell < r$ **do**
15 $\quad\mid\quad\quad\mid\quad$ $t_p^+ \leftarrow \lceil \frac{\ell+r}{2} \rceil$
16 $\quad\mid\quad\quad\mid\quad$ $x \leftarrow$ Find-Switching-Flow-2($G, T', (t_v^+)_{v \in T'}, S'$)
17 $\quad\mid\quad\quad\mid\quad$ **if** $x^-(p) < t_p^+$ **then**
18 $\quad\mid\quad\quad\mid\quad\quad\mid\quad$ $r \leftarrow t_p^+ - 1$
19 $\quad\mid\quad\quad\mid\quad$ **if** $x^-(p) = t_p^+$ **then**
20 $\quad\mid\quad\quad\mid\quad\quad\mid\quad$ **return** x
21 $\quad\mid\quad\quad\mid\quad$ **if** $x^-(p) > t_p^+$ **then**
22 $\quad\mid\quad\quad\mid\quad\quad\mid\quad$ $\ell \leftarrow t_p^+ + 1$

23 **else**   // Splitting Case
24 $\quad\mid\quad$ **for** *every connected component $C \subseteq V \setminus T$ of $G - T$ (undirected)* **do**
25 $\quad\mid\quad\quad\mid\quad$ let G_C be the graph (directed) obtained from G by removing all vertices not in $C \cup T$, their outgoing edges, and replacing edges leaving the set $C \cup T$ by self-loops.
26 $\quad\mid\quad\quad\mid\quad$ find a smallest balanced separator $S_C \subseteq C$ of $G_C - T$ (undirected)
27 $\quad\mid\quad\quad\mid\quad$ $x^{(C)} \leftarrow$ Find-Switching-Flow-2($G_C, T, (t_v^+)_{v \in T}, S_C$)
28 $\quad\mid\quad$ combine solutions of each connected component to x (see Lemma 8 for details)
29 $\quad\mid\quad$ **return** x
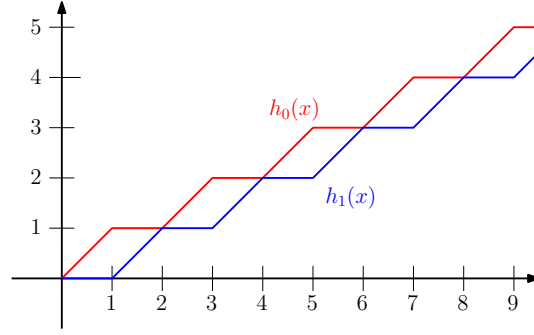
Figure 3 The functions h_0 and h_1.

▶ **Lemma 12** (Non-Expansiveness and Monotonicity). *The one-step update* $f : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$ *associated with the switch graph* $G = (V, E, s_0, s_1)$ *is monotone and non-expansive.*

Proof. TOPROVE 5 ◀

Next, we consider what happens if we reintroduce terminals. Concretely, we want to fix $x_v = t_v^+$ for all $v \in T$ and consider the resulting function on the non-terminal vertices.

▶ **Definition 13** (Extension and Projection). *Let* $f : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$ *be the one-step update associated with the switch graph* $G = (V, E, s_0, s_1)$. *Assume that we are given terminals* $T \subseteq V$ *with starting tokens* $(t_v^+)_{v \in T}$. *For arbitrary* $x \in \mathbb{R}_{\geq 0}^{|V \setminus T|}$, *let the extension* $x' \in \mathbb{R}_{\geq 0}^n$ *of* x *denote the vector*

$$x'_v = \begin{cases} t_v^+ & \text{if } v \in T \\ x_v & \text{otherwise} \end{cases}$$

obtained by filling in the values t_v^+ *for terminals* $v \in T$. *Moreover, we define the projection* $g : \mathbb{R}_{\geq 0}^{|V \setminus T|} \to \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *of* f *to non-terminals as* $g(x)_v := f(x')_v$ *for all* $v \in V \setminus T$ *and* $x \in \mathbb{R}_{\geq 0}^{|V \setminus T|}$.

Observe that the projected one-step update g is still non-expansive and monotone: In particular, non-expansiveness can be obtained by

$$\|g(x) - g(y)\| \leq \|f(x') - f(y')\| \leq \|x' - y'\| = \|x - y\|$$

for all $x, y \in \mathbb{R}_{\geq 0}^{|V \setminus T|}$.

The next lemma says that the fixed points of the projected one-step update reveal the solution to the given G-ARRIVAL-instance.

▶ **Lemma 14** (Interpretation of One-Step Update). *Let* $f : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$ *be the one-step update associated with the switch graph* $G = (V, E, s_0, s_1)$. *Assume that we are given terminals* $T \subseteq V$ *with starting tokens* $(t_v^+)_{v \in T}$, *and let* $g : \mathbb{R}_{\geq 0}^{|V \setminus T|} \to \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *be the projection of* f *to non-terminals. Let* $x \in \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *be arbitrary and consider its extension* $x' \in \mathbb{R}_{\geq 0}^n$. *Finally, define*

$$y(v, s_0(v)) := h_0(x'_v) \quad y(v, s_1(v)) := h_1(x'_v)$$

for all $v \in V$. *Then the following two statements are true:*

- *If* x *is a fixed point of* g *and* $y(e)$ *is fractional for some edge* $e \in E$, *then there exists a directed fractional cycle* C *containing* e.
- x *is an integral fixed point of* g *if and only if* y *is an integral switching flow.*

Proof. TOPROVE 6 ◀

4.2 Discounted One-Step Update

As we have seen, the one-step update function and its projection are non-expansive with respect to the Manhattan distance. In this section, we make them contracting by artificially introducing a contraction factor.

▶ **Definition 15** (Discounted One-Step Update)**.** *Let* $f : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n$ *be the one-step update function and* $g : \mathbb{R}_{\geq 0}^{|V \setminus T|} \to \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *its projection associated with the switch graph* $G = (V, E, s_0, s_1)$ *with terminals* $T \subseteq V$ *and token numbers* $(t_v^+)_{v \in T}$. *For* $\lambda \in [0, 1)$, $f^{(\lambda)} := \lambda f$ *is the* λ-*discounted one-step update function and* $g^{(\lambda)} := \lambda g$ *its* λ-*discounted projection.*

▶ **Corollary 16** (Contracting and Monotone Discounted One-Step Update)**.** *Let* $\lambda \in [0, 1)$ *be arbitrary. The* λ-*discounted one-step update function* $f^{(\lambda)}$ *and its* λ-*discounted projection* $g^{(\lambda)}$ *associated with the switch graph* $G = (V, E, s_0, s_1)$ *are monotone and contracting with parameter* λ.

Proof. TOPROVE 7   ◀

▶ **Lemma 17.** *Let* $\lambda \in [0, 1)$ *be arbitrary. Assume that* $g : \mathbb{R}_{\geq 0}^{|V \setminus T|} \to \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *is the projected one-step update and* $g^{(\lambda)} : \mathbb{R}_{\geq 0}^{|V \setminus T|} \to \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *its discounted version associated with the switch graph* $G = (V, E, s_0, s_1)$ *with terminals* $T \subseteq V$ *and token numbers* $(t_v^+)_{v \in T}$. *Assume that* $x^* \in \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *is the unique fixed point of* $g^{(\lambda)}$. *Every fixed point* $x \in \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *of* g *satisfies* $x^* \leq x$.

Proof. TOPROVE 8   ◀

▶ **Lemma 18.** *Let* $x^\star \in \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *be the unique fixed point of the* λ-*discounted projected one-step update function* $g^{(\lambda)} : \mathbb{R}_{\geq 0}^{|V \setminus T|} \to \mathbb{R}_{\geq 0}^{|V \setminus T|}$ *associated with the switch graph* $G = (V, E, s_0, s_1)$ *with terminals* T *and token numbers* $(t_v^+)_{v \in T}$. *Let* $\lambda \in (1 - \frac{1}{t^+ + ||x^*||}, 1)$. *With*

$$y(v, s_0(v)) := \lambda h_0(x_v^\star) \quad and \quad y(v, s_1(v)) := \lambda h_1(x_v^\star)$$

for all $v \in V \setminus T$, *as well as*

$$y(v, s_0(v)) := \lambda h_0(t_v^+) \quad and \quad y(v, s_1(v)) := \lambda h_1(t_v^+)$$

for all $v \in T$, *we must have* $\sum_{v \in T} y^-(v) > t^+ - 1$.

Proof. TOPROVE 9   ◀

▶ **Theorem 19.** *Deciding an instance of G-ARRIVAL on* n *vertices with* $t^+ \geq 1$ *starting tokens reduces to finding a fixed point of the* λ-*discounted projected one-step update with* $\lambda \in (1 - \frac{1}{t^+(1 + n 2^n)}, 1)$.

Proof. TOPROVE 10   ◀

Observe that the bound in Lemma 18 can be improved to $\sum_{v \in T} y^-(v) > t^+ - \delta$ by choosing $\lambda > 1 - \delta \frac{1}{t^+(1 + n 2^n)}$. Furthermore, any ε-approximate fixed point $\hat{x}$ satisfies

$$\|\hat{x} - x^\star\| \leq \|\hat{x} - g^{(\lambda)}(\hat{x})\| + \|g^{(\lambda)}(\hat{x}) - x^\star\| \leq \varepsilon + \|g^{(\lambda)}(\hat{x}) - g^{(\lambda)}(x^\star)\| \leq \varepsilon + \lambda \|\hat{x} - x^\star\|$$

and therefore $\|g^{(\lambda)}(\hat{x}) - g^{(\lambda)}(x^\star)\| \leq \|\hat{x} - x^\star\| \leq \frac{\varepsilon}{1 - \lambda}$. Thus, we get

$$\sum_{v \in T} \left| g(\hat{x})_v - t_v^- \right| \leq \sum_{v \in T} |g(\hat{x})_v - g(x^\star)_v| + \sum_{v \in T} |g(x^\star)_v - t_v^-| \leq \frac{\varepsilon}{1 - \lambda} + \delta,$$

which is enough to derive $(t_v^-)_{v \in T}$ from $\hat{x}$ if $\frac{\varepsilon}{1-\lambda} + \delta < \frac{1}{2}$.

Finally, capping the function $g^{(\lambda)}$ in each coordinate to make it map $[0, t^+ 2^n]^{|V \setminus T|}$ to itself preserves the contraction property. By Lemma 2, the unique fixed point lies inside $[0, t^+ 2^n]^{|V \setminus T|}$. Scaling everything by a factor of $\frac{1}{t^+ 2^n}$ yields a ℓ_1-contracting function that maps $[0, 1]^{|V \setminus T|}$ to itself.

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