Algebraic Language Theory with Effects

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Abstract

Regular languages – the languages accepted by deterministic finite automata – are known to be precisely the languages recognized by finite monoids. This characterization is the origin of algebraic language theory. In this paper, we generalize the correspondence between automata and monoids to automata with generic computational effects given by a monad, providing the foundations of an *effectful* algebraic language theory. We show that, under suitable conditions on the monad, a language is computable by an effectful automaton precisely when it is recognizable by (1) an effectful monoid morphism into an effect-free finite monoid, and (2) a monoid morphism into a monad-monoid bialgebra whose carrier is a finitely generated algebra for the monad, the former mode of recognition being conceptually completely new. Our prime application is a novel algebraic approach to languages computed by probabilistic finite automata. Additionally, we derive new algebraic characterizations for nondeterministic probabilistic finite automata and for weighted finite automata over unrestricted semirings, generalizing previous results on weighted algebraic recognition over commutative rings.

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1 Introduction

The algebraic approach to finite automata rests on the observation that regular languages (the languages accepted by finite automata) coincide with the languages recognized by *finite monoids*. This provides the basis for investigating complex properties of regular languages with semigroup-theoretic methods. A prime example is the celebrated result by McNaughton, Papert, and Schützenberger [45,59] that a regular language is definable in first-order logic over finite words iff its syntactic monoid (the minimal monoid recognizing the language) is aperiodic. Since the latter property is easy to verify, this implies decidability of first-order definability. Similar characterizations and decidability results are known for numerous subclasses of regular languages [26, 49, 51]. The correspondence between automata and monoids has been generalized beyond regular languages, for example to ω-regular languages [48, 69], languages of words over linear orders [11], data languages [18], tree languages [7, 13, 19, 58], cost functions [24], and weighted languages over commutative rings [55].

In this paper, we show that the equivalence between automata and monoids can be

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established at the more general level of automata with generic computational effects given by a monad $\mathbb{T}$ [33, 62]. This class of automata forms a common generalization of a wide range of automata models such as probabilistic automata [54], nondeterministic probabilistic automata [67], weighted automata [27], and even pushdown automata and Turing machines [33]. We introduce two modes of algebraic recognition for $\mathbb{T}$-effectful languages, both of which are natural effectful generalizations of the classical recognition by finite monoids: recognition by (1) $\mathbb{T}$-*effectful monoid morphisms* into finite effect-free monoids, and (2) ordinary monoid morphisms into finitely generated $\mathbb{T}$-algebras equipped with an additional monoid structure (plus optional compatibility conditions). We subsequently identify suitable conditions on the monad $\mathbb{T}$ ensuring that (1) and (2) capture precisely the languages computed by finite $\mathbb{T}$-automata; this yields an *effectful automata/monoid correspondence* (Theorems 4.11 and 4.22). Our results fundamentally exploit the double role played by monads in computation, namely as abstractions of both computational effects [46] and algebraic theories [43].

The investigation of algebraic recognition at the present level of generality has several benefits. First and foremost, the abstract perspective provided by generic effects naturally motivates and isolates conceptual ideas that would be easily missed for concrete instantiations of the monad $\mathbb{T}$. Notably, recognition mode (1) above is conceptually completely new, and mode (2) generalizes earlier work on algebraic recognition over commutative varieties [2] to arbitrary (non-commutative) effects. In this way, our theory leads to novel algebraic characterizations of several important automata models.

Our prime application is a characterization of languages computed by probabilistic finite automata (PFAs) [54]. PFAs extend classical deterministic finite automata by probabilistic effects, and thus serve as a natural model of state-based computations that involve uncertainty or randomization. These arise in many application domains [9, 47], as witnessed for instance by the wide range and success of probabilistic model checkers [35, 40]. On the theoretical side, PFAs share some remarkable similarities with finite automata; in particular, machine-independent characterizations of PFA-computable languages in terms of probabilistic regular expressions [20, 56] and probabilistic monadic second-order logic [68] are known. However, the fundamental algebraic perspective on finite automata has thus far withstood a probabilistic generalization. We fill this gap by establishing two different modes of *probabilistic algebraic recognition* (Theorems 3.10 and 3.18) for PFA-computable languages which instantiate the two above-mentioned modes, namely (1) recognition by finite monoids via *probabilistic monoid morphisms*; (2) recognition by *convex monoids* carried by a finitely generated convex set. The first mode emphasizes the effectful nature of probabilistic languages, while the second one relies on traditional universal algebra. These characterizations may serve as a starting point for the algebraic investigation of PFA-computable languages.

Further notable instances of the general theory include algebraic characterizations of nondeterministic probabilistic automata and weighted automata. For the latter, a weighted automata/monoid correspondence was only known for weights from a *commutative ring* [55]; our version applies to general semirings and thus captures additional types of weighted automata such as min-plus and max-plus automata [50, Ch. 5] and transducers [50, Ch. 3].

Related Work. The use of category theory to unify results for different automata models has a long tradition [6, 8, 31]. $\mathbb{T}$-automata were first studied by Goncharov et al. [33] as an instance of effectful *coalgebras* [57, 62]; they also fit into the framework of *functor automata* by Colcombet and Petrişan [25]. On the algebra side, Bojańczyk [15] used monads to unify notions of algebraic recognition. This abstract perspective on algebraic language theory has led to a series of works, including a uniform theory of Eilenberg-type correspondences [66] and an abstract account of logical definability of languages [14]. However, while each of the above works studies either automata-based or algebraic recognition individually, formal connections between both approaches are far less explored at categorical generality. The only results in this direction appear in the work of Adámek et al. [2] on the automata/monoid correspondence in commutative varieties. Our approach takes the step to non-commutative monads and on the way develops the entirely new concept of effectful language recognition.

2 Algebraic Recognition of Regular Languages

To set the scene for our algebraic approach to effectful languages, we recall the classical correspondence between finite automata and finite monoids as recognizers for regular languages [51, 53]. Let us settle some notation used in the sequel:

▶ **Notation 2.1.** Fix a finite alphabet Σ, and let Σ^* denote the set of finite words over Σ, with empty word $\varepsilon \in \Sigma^*$. We put $1 = \{*\}$ and $2 = \{\bot, \top\}$. A map $x\colon 1 \to X$ is identified with the element $x(*) \in X$, which by abuse of notation is also denoted by $x \in X$. Maps $p\colon X \to 2$ (*predicates*) are identified with subsets of X; in particular, languages are presented as predicates $L\colon \Sigma^* \to 2$. We denote the composite of two maps $f\colon X \to Y$, $g\colon Y \to Z$ by $f\,;g\colon X \to Z$ (note the order!), and the identity map on X by $\mathsf{id}_X\colon X \to X$. We use $\mapsto$ to define anonymous functions. Lastly, $X \to Y$ denotes the set of all maps from X to Y.

Finite Automata. A *deterministic finite automaton* (*DFA*) $\mathcal{A} = (Q, i, \delta, o)$ consists of a finite set Q of *states* and maps representing an *initial state*, *transitions*, and *final states*:

$$i\colon 1 \longrightarrow Q, \qquad \delta\colon Q \times \Sigma \longrightarrow Q, \qquad o\colon Q \longrightarrow 2,$$

Let $\bar{\delta}\colon \Sigma \to (Q \to Q)$, $\bar{\delta}(a) = \delta(-, a)$, denote the curried form of δ, and define the *iterated transition map* $\bar{\delta}^*\colon \Sigma^* \to (Q \to Q)$ and the *language* $L\colon \Sigma^* \to 2$ *computed by* $\mathcal{A}$ by

$$\bar{\delta}^*(\varepsilon) = \mathsf{id}_Q, \quad \bar{\delta}^*(wa) = \bar{\delta}^*(w)\,; \bar{\delta}^*(a) \quad \text{and} \quad L(w) = i\,; \bar{\delta}^*(w)\,; o \quad \text{for } w \in \Sigma^*. \tag{2.1}$$

Monoids. A *monoid* $(M, \cdot, e)$ is a set M equipped with an associative multiplication $M \times M \xrightarrow{\cdot} M$ and a neutral element $e \in M$; that is, the equations $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $e \cdot x = x = x \cdot e$ hold for all $x, y, z \in M$. A map $h\colon N \to M$ between monoids $(N, \cdot, n)$ and $(M, \cdot, e)$ is a *monoid morphism* if $h(n) = e$ and $h(x \cdot y) = h(x) \cdot h(y)$ for all $x, y \in N$.

▶ **Example 2.2.** **(1)** For every set X, the set $X \to X$ of endomaps forms a monoid with multiplication given by composition ; and neutral element $\mathsf{id}_X\colon X \to X$.

(2) The set Σ^* of words, with concatenation as multiplication and neutral element ε, is the *free monoid* on Σ: for every monoid $(M, \cdot, e)$ and every map $h_0\colon \Sigma \to M$, there exists a unique monoid morphism $h\colon \Sigma^* \to M$ such that $h_0(a) = h(a)$ for all $a \in \Sigma$. The morphism h, the *free extension* of h_0, is given by $h(a_1 \cdots a_n) = h_0(a_1) \cdot \cdots \cdot h_0(a_n)$ for $a_1, \ldots, a_n \in \Sigma$. For instance, the map $\bar{\delta}^*\colon \Sigma^* \to (Q \to Q)$ defined in (2.1) is the free extension of $\bar{\delta}\colon \Sigma \to (Q \to Q)$.

Monoids enable an algebraic notion of language recognition. A monoid M *recognizes* the language $L\colon \Sigma^* \to 2$ if there exists a monoid morphism $h\colon \Sigma^* \to M$ and a predicate $p\colon M \to 2$ such that $L = h\,; p$. Monoid recognition captures precisely the regular languages:

▶ **Theorem 2.3** [53, Thm. 1]**.** *For every language $L\colon \Sigma^* \to 2$, there exists a DFA computing L iff there exists a finite monoid recognizing L.*

Proof sketch. If a DFA $\mathcal{A} = (Q, i, \delta, o)$ computes L, the monoid $Q \to Q$ recognizes L via the morphism $\bar{\delta}^* : \Sigma^* \to (Q \to Q)$ and predicate $p : (Q \to Q) \to 2$ defined by $p(f) = i \,;\, f \,;\, o \in 2$.

Conversely, given a finite monoid $(M, \cdot, e)$ that recognizes L via a monoid morphism $h : \Sigma^* \to M$ and a predicate $p : M \to 2$, we can turn M into a DFA $\mathcal{A} = (M, e, \delta, p)$ computing L with transitions defined by $\delta(m, a) = m \cdot h(a)$. ◀

3 Algebraic Recognition of Probabilistic Languages

Before we present the correspondence of automata and monoids on the categorical level of general effectful automata in Section 4, we whet the reader's appetite by illustrating the important special case of probabilistic languages. These are languages computed by *probabilistic finite automata* [54], whose computational effects are finite probability distributions. All results in this section are instances of those in Section 4; however, for the convenience of the reader we provide sketches of the concrete arguments our general proofs instantiate to.

3.1 Probability Distributions and Probabilistic Channels

A *finite probability distribution* on a set X is a map $d : X \to [0, 1]$ whose *support* $\mathrm{supp}(d) := \{x \in X \mid d(x) \neq 0\}$ is finite and which satisfies $\sum_{x \in X} d(x) = 1$. A distribution can be represented as a finite formal sum $\sum_{i \in I} r_i x_i$ where $x_i \in X$, $r_i \in [0, 1]$, $\sum_{i \in I} r_i = 1$ and $d(x) = \sum_{i \in I : x_i = x} r_i$ for $x \in X$. The set of all distributions on X is denoted by $\mathcal{D}X$.

A map of the form $f : X \to \mathcal{D}Y$, denoted by $f : X \multimap\!\!\!\to Y$, is a *probabilistic channel* or *Markov kernel* from X to Y. Intuitively, f is a map from X to Y that assigns to a given input x the output y with probability $f(x)(y)$. We write $X \multimap\!\!\!\to Y$ for the set of all probabilistic channels from X to Y. Two probabilistic channels $f : X \multimap\!\!\!\to Y$ and $g : Y \multimap\!\!\!\to Z$ can be composed to yield a probabilistic channel $f \fatsemi g : X \multimap\!\!\!\to Z$ given by $(f \fatsemi g)(x) = \big(z \mapsto \sum_{y \in Y} f(x)(y) \cdot g(y)(z)\big)$. The *unit* at X is the probabilistic channel $\eta_X : X \multimap\!\!\!\to X$ sending $x \in X$ to the *Dirac distribution* $\delta_x \in \mathcal{D}X$ defined by $\delta_x = (y \mapsto 1 \text{ if } x = y \text{ else } 0)$. Using sum notation, composition of probabilistic channels is given by $(f \fatsemi g)(x) = \sum_{i \in I} \sum_{j \in J_i} r_i r_{ij} z_{ij}$, where $f(x) = \sum_{i \in I} r_i y_i$ and $g(y_i) = \sum_{j \in J_i} r_{ij} z_{ij}$, and the unit is $\eta_X(x) = 1x$. Composition $\fatsemi$ of probabilistic channels is associative and has η as an identity: $(f \fatsemi g) \fatsemi h = f \fatsemi (g \fatsemi h)$ and $\eta_X \fatsemi f = f \fatsemi \eta_Y$ for $f : X \multimap\!\!\!\to Y, g : Y \multimap\!\!\!\to Z$ and $h : Z \multimap\!\!\!\to W$. A channel $f : X \multimap\!\!\!\to 2$ is a *probabilistic predicate* $f : X \to [0, 1]$ on X, where we identify $\mathcal{D}2$ with the unit interval. A map $f : X \to Y$ induces the *pure* channel $f \,;\, \eta_Y : X \multimap\!\!\!\to Y$, which we also denote by $f : X \to Y$ by abuse of notation.

3.2 Probabilistic Automata

Probabilistic automata, due to Rabin [54], generalize deterministic automata. Transitions are no longer given by a unique successor state $\delta(q, a)$ for every state q and input a, but a probability distribution over possible successor states. Accordingly, probabilistic automata compute *probabilistic languages*, which are simply probabilistic predicates $\Sigma^* \multimap\!\!\!\to 2$. Formally:

▶ **Definition 3.1.** A *probabilistic finite automaton* (*PFA*) $\mathcal{A} = (Q, i, \delta, o)$ consists of a finite set Q of *states* and the following channels for an *initial distribution*, the *transition distributions*, and an *acceptance predicate*, respectively:

$$i : 1 \multimap\!\!\!\longrightarrow Q, \qquad \delta : Q \times \Sigma \multimap\!\!\!\longrightarrow Q, \qquad o : Q \multimap\!\!\!\longrightarrow 2.$$

We denote by $\bar{\delta} : \Sigma \to (Q \multimap\!\!\!\to Q)$ the curried form of δ and define the *iterated transition map* $\bar{\delta}^* : \Sigma^* \to (Q \multimap\!\!\!\to Q)$ and the *language* $L : \Sigma^* \multimap\!\!\!\to 2$ *computed by* $\mathcal{A}$, respectively, by

$$\bar{\delta}^*(\varepsilon) = \eta_Q, \quad \bar{\delta}^*(wa) = \bar{\delta}^*(w) \fatsemi \bar{\delta}^*(a) \quad \text{and} \quad L(w) = i \fatsemi \bar{\delta}^*(w) \fatsemi o \quad \text{for } w \in \Sigma^*.$$

Comparing with the definition of DFAs in Section 2, we see that DFAs are precisely PFAs where i, δ, o are pure maps.

▶ **Remark 3.2.** We think of $i(q)$ as the probability that $\mathcal{A}$ starts in state q, of $\delta(q,a)(q')$ as the probability that $\mathcal{A}$ transitions from q to q' on input a, and of $o(q) \in \mathcal{D}2 \cong [0,1]$ as the probability that q is accepting. Unravelling the above definition, the language $L\colon \Sigma^* \multimap\!\!\!\to 2$ computed by $\mathcal{A}$ is given by the explicit formula

$$L(w) = \textstyle\sum_{\vec{q}\in Q^{n+1}} i(q_0)\cdot \big(\prod_{k=1}^{n} \delta(q_{k-1},a_k)(q_k)\big)\cdot o(q_n) \quad \text{for } w = a_1\cdots a_n. \tag{3.1}$$

The summand for $\vec{q} \in Q^{n+1}$ is the probability that, on input w, the automaton takes the path $\vec{q}$ and accepts w. Hence, $L(w)$ is the total acceptance probability.

▶ **Remark 3.3.** **(1)** Rabin's original notion of PFA [54] features an initial state and a set of final states, which amounts to restricting $i\colon 1 \multimap\!\!\!\to Q$ and $o\colon Q \multimap\!\!\!\to 2$ in Definition 3.1 to pure channels. Except for the behaviour on the empty word, the two versions are expressively equivalent [22, Lemmas 3.1.1 and 3.1.2].

(2) The PFA model used here is often called *reactive* in the literature, as opposed to *generative* PFAs, whose transition map is of type $Q \multimap\!\!\!\to 1 + Q\times\Sigma$, computing *stochastic languages*, which are (not necessarily finite!) distributions over Σ^*. Generative PFA are not instances of $\mathbb{T}$-automata (Definition 4.3) and therefore not considered in this paper.

Our aim is to understand PFA-computable probabilistic languages in terms of recognition by algebraic structures, in the same way that finite monoids recognize regular languages. To this end, we introduce two modes of probabilistic algebraic recognition and prove that they capture precisely the PFA-computable languages.

3.3 Recognizing Probabilistic Languages by Finite Monoids

For our first mode of probabilistic algebraic recognition, we stick with finite monoids as recognizing structures, and only add probabilistic effects to the recognizing monoid morphisms. First we need some auxiliary machinery:

▶ **Definition 3.4.** *For all finite sets X and all sets Y, Z we define the maps*

$$\xi_{X,Y}\colon \mathcal{D}(X\to Y)\to (X\to \mathcal{D}Y), \qquad \xi_{X,Y}(d)(x) = \big(y \mapsto \textstyle\sum_{f\colon X\to Y, f(x)=y} d(f)\big), \tag{3.2}$$

$$\lambda_{X,Y}\colon (X\to\mathcal{D}Y)\to\mathcal{D}(X\to Y), \qquad \lambda_{X,Y}(g) = \big(f\mapsto \textstyle\prod_{x\in X} g(x)(f(x))\big), \tag{3.3}$$

$$\pi_{Y,Z}\colon \mathcal{D}Y\times\mathcal{D}Z\to\mathcal{D}(Y\times Z), \qquad \pi_{Y,Z}(d,e) = \big((y,z)\mapsto d(y)\cdot e(z)\big). \tag{3.4}$$

Intuitively, $\xi_{X,Y}(d)(x)(y)$ is the probability of picking some f according to d satisfying $f(x) = y$. For a channel $g\colon X \multimap\!\!\!\to Y$, the probability of $f\colon X\to Y$ in the distribution $\lambda_{X,Y}(g)$ provides a measure of how compatible f is to g. The map $\pi_{Y,Z}$ sends two distributions on Y and Z to their *product distribution* on $Y\times Z$. Well-definedness of ξ and λ and the next lemma can be shown by calculation; conceptually, they follow from $\mathcal{D}$ being an *affine monad*.

▶ **Lemma 3.5.** *We have $\lambda_{X,Y}\,;\xi_{X,Y} = \mathsf{id}$. In particular, the map $\xi_{X,Y}$ is surjective.*

▶ **Notation 3.6.** Given channels $f_1\colon X_1 \multimap\!\!\!\to Y_1$ and $f_2\colon X_2 \multimap\!\!\!\to Y_2$, we define the channel $f_1 \bar{\times} f_2\colon X_1\times X_2 \multimap\!\!\!\to Y_1\times Y_2$ as the composite in diagram in Figure 1.

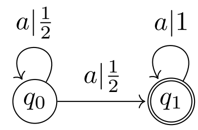
▶ **Definition 3.7.** A *probabilistic monoid morphism* from a monoid $(N,\cdot,n)$ to a monoid $(M,\cdot,e)$ is a channel $h\colon N \multimap\!\!\!\to M$ making the diagram in Figure 2 commute, where the maps $\cdot$, n, e are regarded as pure channels.

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "f_1 \times f_2"] \arrow[dr, "f_1 \bar{\times} f_2"'] & \mathcal{D}Y_1 \times \mathcal{D}Y_2 \arrow[d, "\pi_{Y_1,Y_2}"] \\ & Y_1 \times Y_2 \end{tikzcd}$$

■ **Figure 1** Definition of $\bar{\times}$

$$\begin{tikzcd} N \times N \arrow[r, "\cdot"] \arrow[d, "h \bar{\times} h"'] & N \arrow[d, "h"] & 1 \arrow[l, "n"'] \arrow[dl, "e"] \\ M \times M \arrow[r, "\cdot"] & M \end{tikzcd}$$

■ **Figure 2** Probabilistic monoid morphism



■ **Figure 3** A PFA

▶ **Remark 3.8.** The universal property of Σ^* extends to the probabilistic case: For every monoid $(M, \cdot, e)$ and every channel $h_0 \colon \Sigma \mathrel{-\!\!\circ\!\!\to} M$, there exists a unique probabilistic monoid morphism $h \colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} M$ with $h(a) = h_0(a)$ for all $a \in \Sigma$. It is given by

$$h(w) = \big(m \mapsto \textstyle\sum_{m=m_1\cdot\dots\cdot m_n} \prod_{i=1}^n h_0(a_i)(m_i)\big) \qquad \text{for } w = a_1 \cdots a_n \in \Sigma^*.$$

▶ **Definition 3.9.** A monoid M *probabilistically recognizes* a probabilistic language $L \colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} 2$ if there exists a probabilistic monoid morphism $h \colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} M$ and a probabilistic predicate $p \colon M \mathrel{-\!\!\circ\!\!\to} 2$ such that $L = h \mathbin{⨾} p$.

We stress that, in Definition 3.9, probabilistic effects only appear in the channels h and p, while the monoid M itself is pure. At first sight, it may seem more natural to use ‘probabilistic monoids’, with proper channels as neutral element $1 \mathrel{-\!\!\circ\!\!\to} M$ and multiplication $M \times M \mathrel{-\!\!\circ\!\!\to} M$, as recognizers. Remarkably, we need not require this additional generality:

▶ **Theorem 3.10.** *For every probabilistic language $L \colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} 2$, there exists a PFA computing L iff there exists a finite monoid probabilistically recognizing L.*

Proof sketch. Given a PFA $\mathcal{A} = (Q, \delta, i, o)$ computing L, the finite monoid $Q \to Q$ probabilistically recognizes L via the probabilistic monoid morphism $h \colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} (Q \to Q)$ that freely extends the probabilistic channel $\bar{\delta} \mathbin{⨾} \lambda_{Q,Q} \colon \Sigma \mathrel{-\!\!\circ\!\!\to} (Q \to Q)$, and the probabilistic predicate $p \colon (Q \to Q) \mathrel{-\!\!\circ\!\!\to} 2$ defined by $p(f) = i \mathbin{⨾} f \mathbin{⨾} o \in \mathcal{D}2$. Explicitly, the maps h and p are given by

$$h(w) = \big(f \mapsto \textstyle\sum_{f=f_1;\dots;f_n} \prod_{i=1}^n \prod_{q\in Q} \delta(q, a_i)(f_i(q))\big) \quad \text{and} \quad p(f) = \sum_{q\in Q} i(q)\cdot o(f(q)),$$

for $w = a_1 \cdots a_n$. A lengthy calculation using Lemma 3.5 shows that $L = h \mathbin{⨾} p$.

Conversely, if a finite monoid $(M, \cdot, e)$ probabilistically recognizes L via $h \colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} M$ and $p \colon M \mathrel{-\!\!\circ\!\!\to} 2$, then the PFA $\mathcal{A} = (M, \delta, e, p)$ computes L, with $\delta \colon M \times \Sigma \mathrel{-\!\!\circ\!\!\to} M$ defined by

$$\delta(m, a) = \big(n \mapsto \textstyle\sum_{m' \,:\, m\cdot m'=n} h(a)(m')\big). \qquad ◀$$

▶ **Example 3.11.** The probabilistic monoid morphism induced by the channel $h_0 \colon \{a\} \mathrel{-\!\!\circ\!\!\to} (\{0,1\}, \vee, 0)$ sending $a \mapsto \frac{1}{2}0 + \frac{1}{2}1$, together with the pure predicate $\mathsf{id} \colon \{0,1\} \to \{0,1\}$ recognizes the language $L(a^n) = 1 - \frac{1}{2^n}$. The corresponding PFA due to Theorem 3.10 is shown in Figure 3.

3.4 Recognizing Probabilistic Languages by Convex Monoids

The presence of probabilistic effects in the recognizing morphisms places the above mode of probabilistic recognition outside standard universal algebra. In this section we develop an equivalent, purely algebraic approach based on the theory of convex sets.

A *convex set* [64] is a set X equipped with a family of binary operations $+_r \colon X \times X \to X$ ($r \in [0,1]$) subject to following equations, where $s' = r + s - rs \neq 0$ and $r' = \frac{r}{s'}$:

$$x +_r x = x, \quad x +_0 y = y, \quad x +_r y = y +_{1-r} x, \quad x +_r (y +_s z) = (x +_{r'} y) +_{s'} z$$

A map $f \colon X \to Y$ between convex sets is *affine* if $f(x +_r x') = f(x) +_r f(x')$ for $x, x' \in X$ and $r \in [0,1]$.

▶ **Example 3.12.** **(1)** The prototypical convex sets are convex subsets $X \subseteq \mathbb{R}^\kappa$ (for a cardinal κ) with the operations $\vec{x} +_r \vec{y} := r \cdot \vec{x} + (1-r) \cdot \vec{y}$. Up to affine isomorphism, these are precisely the *cancellative* convex sets [64], which are those satisfying

$$x +_r y = x +_r z \implies y = z \qquad \text{for all } x, y, z \in X \text{ and } r \in (0,1).$$

(2) The set $\mathcal{D}X$ of distributions on a set X is a convex set with structure given by $d +_r e = \big(x \mapsto r \cdot d(x) + (1-r) \cdot e(x)\big)$ for $d, e \in \mathcal{D}X$. This is the *free convex set on* X: every map $h\colon X \to Y$ to a convex set Y extends uniquely to an affine map $h^\#\colon \mathcal{D}X \to Y$ such that $h = \eta_X \,;\, h^\#$. Concretely, $h^\#$ can be defined by $h^\#(\sum_{i=1}^n r_i x_i) = h(x_1) +_{r_1} h^\#(\sum_{i=2}^n \frac{r_i}{1-r_1} x_i)$.

(3) For all sets X and Y, the set $X \multimap Y$ forms a convex set with the operations $f +_r g = \big(x \mapsto f(x) +_r g(x)\big)$ for $f, g\colon X \multimap Y$, $x \in X$ and $r \in [0,1]$.

Reutenauer [55] showed that finite-dimensional $\mathbb{R}$-*algebras* – real vector spaces equipped with a compatible monoid structure – precisely recognize rational power series $\Sigma^* \to \mathbb{R}$. For algebraic recognition of probabilistic languages, we generalize $\mathbb{R}$-algebras to convex monoids, which are monoids with an additional convex structure that is respected by the multiplication:

▶ **Definition 3.13.** A *convex monoid* is a convex set M equipped with a monoid structure $(M, \bullet, e)$ whose multiplication $\bullet\colon M \times M \to M$ satisfies

$$(m +_r m') \bullet n = m \bullet n +_r m' \bullet n \quad \text{and} \quad m \bullet (n +_r n') = m \bullet n +_r m \bullet n'. \tag{3.5}$$

▶ **Example 3.14.** **(1)** The convex set $\mathcal{D}\Sigma^*$ with multiplication $(\sum_{i \in I} r_i v_i) \bullet (\sum_{j \in J} s_j w_j) = \sum_{i \in I,\, j \in J} r_i s_j v_i w_j$ and neutral element $\eta_{\Sigma^*}(\varepsilon) = 1\varepsilon$ is the *free convex monoid* on Σ: every map $h_0\colon \Sigma \to M$ to a convex monoid M extends to a unique affine monoid morphism $h\colon \mathcal{D}\Sigma^* \to M$ such that $h(1a) = h_0(a)$.

(2) For every set X, the convex set $X \multimap X$ from Example 3.123 forms a convex monoid with channel composition $\fatsemi$ as multiplication and unit η_X as neutral element.

▶ **Definition 3.15.** A convex monoid M *recognizes* a language $L\colon \Sigma^* \multimap 2$ if there exists a monoid morphism $h\colon \Sigma^* \to M$ and an affine map $p\colon M \to \mathcal{D}2$ such that $L = h \,;\, p$.

For a correspondence between PFA-computable languages and convex monoids, we need to impose a suitable finiteness restriction on the latter. Finite convex monoids are not sufficient; instead, we shall work with a more permissive notion of finiteness:

▶ **Definition 3.16.** A convex set X is *finitely generated* if there exists an affine surjection $s\colon \mathcal{D}G \twoheadrightarrow X$ for some finite set G. A convex monoid is *fg-carried* if its underlying convex set is finitely generated.

Intuitively, this definition says that X is the convex hull of a finite subset $s[G] \subseteq X$: every element in X is a convex combination of the elements $s(g), g \in G$.

▶ **Example 3.17.** **(1)** A convex subset of $\mathbb{R}^n$ is finitely generated iff it is a *bounded convex polytope* [34], that is, it is compact and has finitely many extremal points.

(2) For all finite sets X, the convex monoid $X \multimap X$ from Example 3.123 is fg-carried. This is witnessed by the map $\xi_{X,X}\colon \mathcal{D}(X \to X) \twoheadrightarrow (X \to \mathcal{D}X)$ of (3.2), which is surjective by Lemma 3.5, and affine, since it is the free extension of the map $f \mapsto f \,;\, \eta_X$.

Fg-carried convex monoids give rise to our second algebraic characterization of PFAs:

▶ **Theorem 3.18.** *For every probabilistic language $L\colon \Sigma^* \multimap 2$, there exists a PFA computing L iff there exists an fg-carried convex monoid recognizing L.*

Proof sketch. Given a PFA $\mathcal{A} = (Q, \delta, i, f)$ computing L, the fg-carried convex monoid $Q \multimap Q$ from Example 3.172 recognizes L via the morphism $\bar{\delta}^*\colon \Sigma^* \to (Q \multimap Q)$ and the affine map $p\colon (Q \multimap Q) \to \mathcal{D}2$ given by $p(f) = i \,;\, f \,;\, o \in \mathcal{D}2$.

Conversely, suppose that $(M, \cdot, e)$ is an fg-carried convex monoid (witnessed by an affine surjection $s\colon \mathcal{D}Q \twoheadrightarrow M$) recognizing L via $h\colon \Sigma^* \to M$ and $p\colon M \to \mathcal{D}2$. Define the PFA $\mathcal{A}_Q = (Q, \delta, i, o)$ where $o = \eta_Q \,;\, s \,;\, p$, the transition distribution δ is chosen such that $s(\delta(g, a)) = s(g) \cdot h(a)$ for all $g \in Q$ and $a \in \Sigma$, and the initial distribution i is chosen such that $s(i) = e$; such choices exist by surjectivity of s. Then $\mathcal{A}_Q$ computes L. ◀

A remarkable property of fg-carried convex monoids is that they always admit a finite presentation, despite not necessarily being finite themselves. To see this, let us recall some terminology from universal algebra. Given an equational class $\mathcal{V}$ of algebras over a finitary signature Λ, a *finite presentation* of an algebra $A \in \mathcal{V}$ is given by (1) a finite set G of *generators*, (2) a finite set R of *relations* $s_i = t_i$ $(i = 1, \ldots, n)$ where $s_i, t_i \in T_\Lambda G$ are Λ-terms in variables from G, and (3) a surjective Λ-algebra morphism $q\colon T_\Lambda G \twoheadrightarrow A$ satisfying $q(s_i) = q(t_i)$ for all i, subject to the universal property that every morphism $h\colon T_\Lambda G \to B$, where $B \in \mathcal{V}$ and $h(s_i) = h(t_i)$ for all i, factorizes through q. In particular, we have the notions of *finitely presentable convex set* and *finitely presentable convex monoid.*

By a non-trivial result due to Sokolova and Woracek [63], finitely generated convex sets are finitely presentable (the converse holds trivially). This is the key to the following theorem; a categorical version of it is later proved in Theorem 4.17.

▶ **Theorem 3.19.** *Every fg-carried convex monoid is finitely presentable.*

Proof sketch. Let M be an fg-carried convex monoid. Choose a finite presentation (G, R, q) of M as a convex set. Since the multiplication of M is fully determined by its action on $q[G]$ by (3.5), this extends to a finite presentation of the convex monoid M by adding a relation $g \cdot g' = t_{g,g'}$ for each $g, g' \in G$, where $t_{g,g'}$ is any term in the signature of convex sets such that $q(t_{g,g'}) = q(g) \cdot q(g')$. ◀

We will see in Example 3.24 that the converse of Theorem 3.19 does not hold.

3.5 Syntactic Convex Monoids

Compared to ordinary monoids, convex monoids are fairly complex structures. The benefit of the additional complexity is the existence of *canonical* recognizers for probabilistic languages. Recall that every regular language $L\colon \Sigma^* \to 2$ has a canonical recognizer, the *syntactic monoid* $\mathrm{Syn}(L)$. It is given by the quotient $\Sigma^*/{\approx_L}$ of the free monoid Σ^* modulo the *syntactic congruence* $\approx_L \subseteq \Sigma^* \times \Sigma^*$, where $v \approx_L w$ iff $L(xvy) = L(xwy)$ for all $x, y \in \Sigma^*$. The projection $h_L\colon \Sigma^* \twoheadrightarrow \mathrm{Syn}(L)$, sending $w \in \Sigma^*$ to its congruence class $[w]$, recognizes L via $p\colon \mathrm{Syn}(L) \to 2$ with $p([w]) = 1$ iff $w \in L$. Moreover, h_L factorizes through every surjective morphism $h\colon \Sigma^* \twoheadrightarrow M$ recognizing L, thus h_L is the 'smallest' surjective morphism recognizing L. For probabilistic languages, this notion generalizes as follows:

▶ **Definition 3.20.** Given a probabilistic language $L\colon \Sigma^* \multimap 2$, a *syntactic convex monoid* of L is a convex monoid $\mathrm{Syn}(L)$ with a surjective affine monoid morphism $h_L\colon \mathcal{D}\Sigma^* \twoheadrightarrow \mathrm{Syn}(L)$ such that $\eta_{\Sigma^*} \,;\, h_L$ recognizes L and, moreover, h_L factorizes through every surjective affine monoid morphism h such that $\eta_{\Sigma^*} \,;\, h$ recognizes L:

$$
\begin{tikzcd}
\Sigma^* \arrow[r, "\eta_{\Sigma^*}"] \arrow[rrr, bend left, "L"] & \mathcal{D}\Sigma^* \arrow[r, two heads, "\forall h"] \arrow[dr, two heads, "h_L"'] & M \arrow[r, "p"] \arrow[d, dashed, "\exists"'] & {[0,1]} \\
& & \mathrm{Syn}(L) \arrow[ur, "p_L"'] &
\end{tikzcd}
$$

Syntactic structures for formal languages are well-studied from a categorical perspective [2, 15, 66]. The following result is an instance of [2, Thm. 3.14]:

▶ **Theorem 3.21.** *Every probabilistic language $L\colon \Sigma^* \multimap\!\!\!\rightarrow 2$ has a syntactic convex monoid, unique up to isomorphism. It is presented by generators Σ and relations given by the* syntactic congruence $\approx_L \subseteq \mathcal{D}\Sigma^* \times \mathcal{D}\Sigma^*$ *defined in Equation* (3.6)*. The maps $h_L\colon \mathcal{D}\Sigma^* \to \mathrm{Syn}(L)$ and $p_L\colon \mathrm{Syn}(L) \to [0,1]$ are given by $h_L(\sum_i r_i v_i) = [\sum_i r_i v_i]$ and $p_L([\sum_i r_i v_i]) = \sum_i r_i L(v_i)$.*

$$\textstyle\sum_i r_i v_i \approx_L \sum_j s_j w_j \quad \textit{iff} \quad \forall x, y \in \Sigma^*\colon\ \sum_i r_i L(x v_i y) = \sum_j s_j L(x w_j y) \tag{3.6}$$

Alternatively, one can construct the syntactic convex monoid as the *transition monoid* of the *minimal $\mathcal{D}$-automaton* (see the full version [41]), entailing a restriction on the convex structure of syntactic convex monoids:

▶ **Theorem 3.22.** *For every language $L\colon \Sigma^* \multimap\!\!\!\rightarrow 2$, the convex set* $\mathrm{Syn}(L)$ *is cancellative.*

Syntactic convex monoids are useful as a descriptional tool for characterizing (and potentially deciding) properties of languages in algebraic terms. Here is a simple illustration:

▶ **Example 3.23.** A probabilistic language $L\colon \Sigma^* \multimap\!\!\!\rightarrow 2$ is *commutative* if $L(a_1 \cdots a_n) = L(a_{\pi(1)} \cdots a_{\pi(n)})$ for all $a_1, \ldots, a_n \in \Sigma$ and all permutations π of $\{1, \ldots, n\}$. One easily verifies that L is commutative iff $\mathrm{Syn}(L)$ is a commutative convex monoid.

Let us note that while every PFA-computable language is recognized by some fg-carried convex monoid (Theorem 3.18), its *syntactic* convex monoid is generally not fg-carried:

▶ **Example 3.24.** Consider the PFA from Example 3.11 computing the probabilistic language $L(a^n) = 1 - 2^{-n}$. Its syntactic convex monoid $\mathrm{Syn}(L)$ is isomorphic to the half-open interval $(0,1] \subseteq \mathbb{R}$ with the usual convex structure and multiplication of reals; indeed, the map $i\colon \mathcal{D}\Sigma^*/{\approx_L} \to (0,1]$ given by $[\sum_k r_k a^{n_k}] \mapsto \sum_k r_k \cdot 2^{-n_k}$ is easily seen to be an isomorphism. Since the finitely generated convex subsets of $\mathbb{R}$ are closed intervals, $\mathrm{Syn}(L)$ is not fg-carried. However, despite Theorem 3.19 not applying here, the convex monoid $(0,1]$ can be shown to be finitely presentable, with the finite presentation given by a single generator a and a single relation $e +_{\frac{1}{3}} a \cdot a = a$. The proof is somewhat intricate; see the full version [41] for details.

▶ **Open Problem.** *Is* $\mathrm{Syn}(L)$ *finitely presentable for every PFA-computable language L?*

4 Algebraic Recognition of Effectful Languages

We now turn to the main results of our paper: two novel modes of algebraic recognition for languages computed by effectful automata. All results are parametric in the computational effect, which is modelled by a monad satisfying a suitable condition. Our results instatiate to the characterizations of PFA-computable probabilistic languages from Section 3. Other instances of our results yield new algebraic characterizations for languages recognized by weighted automata and automata that combine nondeterministic and probabilistic branching.

4.1 Monads

In the following, familiarity with basic category theory is assumed; see Mac Lane [42] for a gentle introduction. We recall some concepts from the theory of monads [43] to fix our terminology and notation. We write Set for the category of sets and functions. A *monad* $\mathbb{T} = (T, \eta, \mu)$ on Set is a triple consisting of an endofunctor $T\colon \mathsf{Set} \to \mathsf{Set}$ and two natural transformations $\eta\colon \mathsf{Id} \to T$ (the *unit*) and $\mu\colon TT \to T$ (the *multiplication*), satisfying the laws $T\mu \,;\, \mu = \mu T \,;\, \mu$ and $T\eta \,;\, \mu = \mathsf{Id}_T = \eta T \,;\, \mu$.

The *Kleisli category* $\mathcal{K}\ell(\mathbb{T})$ has sets as objects, and a morphism from X to Y, denoted $f\colon X \multimap\!\!\to Y$, is a map $f\colon X \to TY$. The composite of $f\colon X \multimap\!\!\to Y$ and $g\colon Y \multimap\!\!\to Z$ is denoted by $f \fatsemi g\colon X \multimap\!\!\to Z$ and defined by $f \fatsemi g = f \,;\, Tg \,;\, \mu_Z$. The identity morphism on X is the component $\eta_X\colon X \multimap\!\!\to X$ of the unit. Intuitively, a Kleisli morphism is a function with computational effects given by the monad $\mathbb{T}$ [46]. A map $f\colon X \to Y$ is identified with the Kleisli morphism $f \,;\, \eta_Y\colon X \multimap\!\!\to Y$; such Kleisli morphisms are said to be *pure*, since they correspond to effect-free computations. Note that there is no need for operator precedence between ; and $\fatsemi$ since $(f \,;\, g) \fatsemi h = f \,;\, (g \fatsemi h)$ and $(g \fatsemi h) \,;\, k = g \fatsemi (h \,;\, k)$ for all Kleisli morphisms $g\colon X \to TY, h\colon Y \to TZ$ and pure maps $f\colon W \to X, k\colon TZ \to U$.

Further, a $\mathbb{T}$-*algebra* (A, a) consists of set A (the *carrier*) and a map $a\colon TA \to A$ (the *structure*) satisfying the *associative law* $Ta \,;\, a = \mu_A \,;\, a$ and the *unit law* $\eta_A \,;\, a = \mathsf{id}_A$. A *morphism* from (A, a) to a $\mathbb{T}$-algebra (B, b) (a $\mathbb{T}$-*morphism* for short) is a map $h\colon A \to B$ such that $a \,;\, h = Th \,;\, b$. We write $\mathsf{Alg}(\mathbb{T})$ for the category of $\mathbb{T}$-algebras and $\mathbb{T}$-morphisms. Products in $\mathsf{Alg}(\mathbb{T})$ are formed in Set: the product algebra of (A_i, a_i), $i \in I$, has the structure $T(\prod_i A_i) \xrightarrow{\langle Tp_i \rangle_i} \prod_i TA_i \xrightarrow{\prod_i a_i} \prod_i A_i$, where $p_i\colon \prod_i A_i \to A_i$ is the ith product projection.

The forgetful functor from $\mathsf{Alg}(\mathbb{T})$ to Set has a left adjoint sending a set X to the *free* $\mathbb{T}$-algebra $\mathbb{T}X = (TX, \mu_X)$ over X. For every $\mathbb{T}$-algebra (A, a) and every map $h\colon X \to A$, there exists a unique $\mathbb{T}$-morphism $h^\#\colon \mathbb{T}X \to (A, a)$ such that $\eta_X \,;\, h^\# = h$; that $\mathbb{T}$-morphism $h^\#$ is the *free extension* of h. The Kleisli category $\mathcal{K}\ell(\mathbb{T})$ is equivalent to a full subcategory of $\mathsf{Alg}(\mathbb{T})$ via the embedding $X \mapsto \mathbb{T}X$ and $h \mapsto h^\#$.

Monads provide a categorical view of universal algebra. Every finitary algebraic theory (Λ, E), given by a signature Λ of finitary operation symbols and a set E of equations between Λ-terms, induces a monad $\mathbb{T}$ on Set, where TX is the carrier of the free (Λ, E)-algebra (viz. the set of all Λ-terms over X modulo the equations in E), and $\eta_X\colon X \to TX$ and $\mu_X\colon TTX \to TX$ are given by inclusion of variables and flattening of terms over terms. Then $\mathbb{T}$-algebras bijectively correspond to Λ-algebras satisfying all equations in E, that is, algebras of the *variety* specified by (Λ, E). Monads $\mathbb{T}$ induced by finitary algebraic theories are precisely the *finitary* monads, that is, those preserving directed colimits [4].

Every monad $\mathbb{T}$ has a canonical *left strength*, the natural transformation

$$\mathsf{ls}_{X,Y}\colon X \times TY \to T(X \times Y) \qquad \text{defined by} \qquad \mathsf{ls}_{X,Y}(x, t) = T(y \mapsto (x, y))(t).$$

Its *right strength* $\mathsf{rs}_{X,Y}\colon TX \times Y \to T(X \times Y)$ is defined analogously. The monad $\mathbb{T}$ is *commutative* if $\mathsf{rs}_{X,TY} \fatsemi \mathsf{ls}_{X,Y} = \mathsf{ls}_{TX,Y} \fatsemi \mathsf{rs}_{X,Y}$ in $\mathcal{K}\ell(\mathbb{T})$. For every monad (be it commutative or not), we denote the left-hand composite of this equation – a *double strength* – by

$$\pi_{X,Y} := (TX \times TY \xrightarrow{\mathsf{rs}_{X,TY}} T(X \times TY) \xrightarrow{T\mathsf{ls}_{X,Y}} TT(X \times Y) \xrightarrow{\mu_{X \times Y}} T(X \times Y)). \tag{4.1}$$

Commutative finitary monads are precisely those induced by a *commutative* algebraic theory (Λ, E). This means that all operations commute with each other; for example, for all binary operations $\alpha, \beta \in \Lambda$, we have $\alpha(\beta(x_{1,2}, x_{1,2}), \beta(x_{2,1}, x_{2,2})) = \beta(\alpha(x_{1,1}, x_{2,1}), \alpha(x_{1,2}, x_{2,2}))$, and similarly for every pair of operations of other (not necessarily equal) arities.

▶ **Example 4.1.** In our applications we shall encounter the following monads:

(1) The *distribution monad* $\mathcal{D}$ sends a set X to the set $\mathcal{D}X$ of all finite probability distributions on X (Section 3.1), and a map $f\colon X \to Y$ to the map $\mathcal{D}f\colon \mathcal{D}X \to \mathcal{D}Y$ defined by $\mathcal{D}f(d) = \big(y \mapsto \sum_{x\in X,\, f(x)=y} d(x)\big)$; in sum notation, $\mathcal{D}f(\sum_i r_i x_i) = \sum_i r_i f(x_i)$. Its unit $\eta_X\colon X \to \mathcal{D}X$ is given by $\eta_X(x) = \delta_x$, and the multiplication $\mu_X\colon \mathcal{D}\mathcal{D}X \to \mathcal{D}X$ is defined by $\mu_X(e) = \big(x \mapsto \sum_{d\in\mathcal{D}X} e(d)\cdot d(x)\big)$. In sum notation, $\eta_X(x) = 1x$ and $\mu_X(\sum_{i\in I} r_i(\sum_{j\in J_i} r_{ij}x_{ij})) = \sum_{i\in I}\sum_{j\in J_i} r_i r_{ij} x_{ij}$. The Kleisli category of $\mathcal{D}$ is the category of sets and probabilistic channels, and algebras for $\mathcal{D}$ are precisely the convex sets [65]. The monad $\mathcal{D}$ is commutative; the natural transformation (4.1) is concretely given by (3.4).

(2) The *convex power set of distributions monad* $\mathcal{C}$ sends a set X to the set $\mathcal{C}X$ of non-empty, finitely generated convex subsets of $\mathcal{D}X$, and a map $f\colon X \to Y$ to $\mathcal{C}f\colon \mathcal{C}X \to \mathcal{C}Y$ defined by $\mathcal{C}f(S) = \{\mathcal{D}f(d) \mid d \in S\}$. Its unit $\eta_X\colon X \to \mathcal{C}X$ is $\eta_X(x) = \{\delta_x\}$, and the multiplication $\mu_X\colon \mathcal{C}\mathcal{C}X \to \mathcal{C}X$ is defined by

$$\mu_X(S) = \textstyle\bigcup_{\Phi\in S}\{\sum_{U\in\mathrm{supp}(\Phi)} \Phi(U)\cdot d_U \mid \forall U \in \mathrm{supp}(\Phi)\colon d_U \in U\} \qquad \text{for every } S \in \mathcal{C}\mathcal{C}X.$$

Algebras for $\mathcal{C}$ are precisely *convex semilattices* [21], which are convex sets A carrying an additional *semilattice* (i.e. a commutative idempotent semigroup) structure $(A, +)$ satisfying $(x + y) +_r z = (x +_r z) + (y +_r z)$ for $x, y, z \in A, r \in [0, 1]$. The monad $\mathcal{C}$ is not commutative.

(3) A *semiring* is a set S equipped with both the structure of a monoid $(S, \cdot, 1)$ and of a commutative monoid $(S, +, 0)$ such that multiplication distributes over addition. Every semiring S induces a monad $\mathcal{S}$ sending a set X to $\mathcal{S}X = \{f\colon X \to S \mid f(x) \neq 0 \text{ for finitely many } x \in X\}$, and a map $f\colon X \to Y$ to the map $\mathcal{S}f\colon \mathcal{S}X \to \mathcal{S}Y$ defined by $\mathcal{S}f(g) = \big(y \mapsto \sum_{f(x)=y} g(x)\big)$. The unit $\eta_X\colon X \to \mathcal{S}X$ is given by $\eta_X(x) = (y \mapsto 1 \text{ if } x = y \text{ else } 0)$, and the multiplication $\mu_X\colon \mathcal{S}\mathcal{S}X \to \mathcal{S}X$ is defined by $\mu_X(e) = \big(x \mapsto \sum_{d\in\mathcal{S}X} e(d)\cdot d(x)\big)$. Algebras for $\mathcal{S}$ correspond precisely to S-*semimodules*, that is, commutative monoids $(M, +, 0)$ with an associative scalar multiplication $S \times M \to M$ that distributes over the additive structures of S and M. The monad $\mathcal{S}$ is commutative iff the multiplication of S is commutative. The distribution monad $\mathcal{D}$ is a submonad of $\mathcal{S}$ for the semiring $S = \mathbb{R}$ of reals with the usual operations.

(4) The monad $\mathcal{M}$ sends a set X to $\mathcal{M}X = X^*$ (finite words over X), and a map $f\colon X \to Y$ to $\mathcal{M}f = f^*\colon X^* \to Y^*$ defined by $f^*(x_1\cdots x_n) = f(x_1)\cdots f(x_n)$. The unit $\eta_X\colon X \to X^*$ is given by $\eta_X(x) = x$ and the multiplication $\mu_X\colon (X^*)^* \to X^*$ by flattening (concatenation) of words. Algebras for $\mathcal{M}$ correspond precisely to monoids.

4.2 Automata with Effects

Automata with computational effects in a monad were introduced in previous work [33, 62]. PFAs as in Definition 3.1 are the instance where the monad is $\mathcal{D}$ and the output algebra is the free $\mathcal{D}$-algebra $\mathcal{D}2$.

▶ **Assumption 4.2.** We fix a monad $\mathbb{T} = (T, \eta, \mu)$ on Set and a $\mathbb{T}$-algebra O.

▶ **Definition 4.3.** A *finite* $\mathbb{T}$-*automaton* ($\mathbb{T}$-*FA*) $\mathcal{A} = (Q, \delta, i, f)$ consists of a finite set Q of *states* together with two $\mathcal{K}\ell(\mathbb{T})$-morphisms and a map as shown below:

$$i\colon 1 \longrightarrow\!\!\circ\; Q, \qquad \delta\colon Q\times\Sigma \longrightarrow\!\!\circ\; Q, \qquad o\colon Q \longrightarrow O.$$

They represent an *initial state*, *transitions*, and *outputs*, respectively. We define the curried version $\bar{\delta}\colon \Sigma \to (Q \multimap Q)$ of δ and the extended transition map $\bar{\delta}^*\colon \Sigma^* \to (Q \multimap Q)$ just like in Definition 3.1. The language $L\colon \Sigma^* \to O$ *computed* by $\mathcal{A}$ is given by $L(w) = i \,\mathbin{;}\, \bar{\delta}^*(w) \,;\, o^{\#}$.

▶ **Remark 4.4.** **(1)** Earlier works [33, 62] model $\mathbb{T}$-automata as coalgebras $Q \to O \times (TQ)^\Sigma$, and their semantics is defined via the final coalgebra for the functor $FX = O \times X^\Sigma$, carried by the set O^{Σ^*} of languages [57]. Definition 4.3 is equivalent to the coalgebraic one – modulo initial state and currying of transitions – yet better suited for our algebraic constructions.

(2) $\mathbb{T}$-automata and their language semantics are also instances of the framework of functor automata by Colcombet and Petrişan [25] interpreted either in the Kleisli category for $\mathbb{T}$ for free output algebras $O = \mathbb{T}O_0$, or in the Eilenberg-Moore category for $\mathbb{T}$ for general output algebras and the state object being the free algebra $\mathbb{T}Q$.

(3) Every $\mathbb{T}$-FA is equivalent to a $\mathbb{T}$-FA with a pure initial state: one may simply add a new pure initial state simulating the behaviour of a non-pure one [67, Rem. 1]. Hence, the essence of the effectful nature of $\mathbb{T}$-FAs lies in the transitions.

▶ **Example 4.5.** **(1)** $\mathcal{D}$-FAs with the convex output set $O = \mathcal{D}2$ are precisely PFAs.

(2) $\mathcal{C}$-FAs are *nondeterministic probabilistic finite automata* (*NPFAs*) [67]. They combine nondeterministic and probabilistic branching and as such are closely related to Segala systems [61] and Markov decision processes [50, Ch. 36]. We take the output convex semilattice $O = [0,1]_{\max}$ given by the interval $[0,1] \subseteq \mathbb{R}$ with its usual convex structure and taking maxima as the semilattice operation. An NPFA $\mathcal{A} = (Q, \delta, i, o)$ consists of a finite state set Q and maps $i\colon 1 \to \mathcal{C}Q$, $\delta\colon Q \times \Sigma \to \mathcal{C}Q$, and $o\colon Q \to [0,1]$. If $\mathcal{A}$ is in state q and receives the input letter a, then it chooses a distribution $d \in \delta(q,a)$ and transitions to the state q' with the probability $d(q')$. The choices are made with the goal of maximizing the acceptance probability of the input. Formally, $\mathcal{A}$ computes the probabilistic language $L_{\max}\colon \Sigma^* \to [0,1]$ defined for $w = a_1 \cdots a_n$ by

$$w \mapsto \max\{\textstyle\sum_{\vec{q} \in Q^{n+1}} d_0(q_0) \cdot (\prod_{k=1}^n d_{q_{k-1},k}(q_k)) \cdot o(q_n) \mid d_0 \in i, \forall q. \forall k. d_{q,k} \in \delta(q, a_k)\}.$$

Under this semantics, NPFAs are more expressive than PFAs [67]. Two alternative semantics emerge by modifying the output convex semilattice: for $O = [0,1]_{\mathsf{min}}$, the interval $[0,1]$ with the minimum operation, an NPFA computes the language $L_{\min}\colon \Sigma^* \to [0,1]$ of minimal acceptance probabilities. For $O = \mathcal{C}2$, the convex semilattice of closed subintervals of $[0,1]$, it computes the language $L_{\mathrm{int}}\colon \Sigma^* \to \mathcal{C}2$ sending $w \in \Sigma^*$ to the interval $[L_{\min}(w), L_{\max}(w)]$. See van Heerdt et al. [67] for a detailed coalgebraic account of the different semantics.

(3) $\mathcal{S}$-FAs with the output semimodule $O = \mathcal{S}1 \cong S$, are precisely *weighted finite automata* (*WFAs*) [27] over the semiring S. A WFA computes a *weighted language* (or *formal power series*) $L\colon \Sigma^* \to S$, which is given by the formula (3.1), with sums and products formed in S. Interesting choices for the semiring S include:

(1) the semiring $\mathbb{R}$ of reals – note that PFAs are a special case of WFAs over $\mathbb{R}$;

(2) the *min-plus* or *max-plus semiring* of natural numbers with the operation min (resp. max) as addition and the operation + as multiplication; WFAs then correspond to *min-plus* and *max-plus automata* computing shortest (resp. longest) paths [50, Ch. 5].

(3) the semiring of regular languages over an alphabet Γ w.r.t. union and concatenation; WFAs are equivalent to *transducers* computing relations $R \subseteq \Sigma^* \times \Gamma^*$ [50, Ch. 3].

4.3 Recognizing Effectful Languages by Bialgebras

We now introduce two modes of algebraic recognition for languages computed by $\mathbb{T}$-automata. In contrast to the exposition in Section 3 for the instance $\mathbb{T} = \mathcal{D}$, we swap the order of presentation and first consider the recognition by algebraic structures and subsequently the recognition by effectful homomorphisms in Section 4.4, since from the categorical viewpoint the latter is best understood in the context of the former.

▶ **Definition 4.6.** A $(\mathbb{T}, \mathcal{M})$-*bialgebra* is a set M equipped with both a $\mathbb{T}$-algebra structure $a\colon TM \to M$ and the structure of a monoid $(M, \boldsymbol{\cdot}, e)$. It is a $\mathbb{T}$-*monoid* if the monoid multiplication $M \times M \xrightarrow{\cdot} M$ is a $\mathbb{T}$-*bimorphism*: for every $m \in M$ the maps $m\boldsymbol{\cdot}(-), (-)\boldsymbol{\cdot}m\colon M \to M$ are $\mathbb{T}$-endomorphisms on (M, a).

▶ **Remark 4.7.** Both $(\mathbb{T}, \mathcal{M})$-bialgebras and $\mathbb{T}$-monoids admit a categorical view:

(1) $(\mathbb{T}, \mathcal{M})$-bialgebras correspond to algebras for the coproduct [1] of the monads $\mathbb{T}$ and $\mathcal{M}$.

(2) If $\mathbb{T}$ is commutative, then $\mathbb{T}$-monoids correspond to algebras for the composite monad $\mathbb{T}\mathcal{M}$ induced by the canonical distributive law of $\mathcal{M}$ over $\mathbb{T}$ [44, Thm 4.3.4]. They also correspond to monoid objects in the closed monoidal category $(\mathsf{Alg}(\mathbb{T}), \otimes, \mathbb{T}1)$ whose tensor product represents $\mathbb{T}$-bimorphisms (i.e. $\mathbb{T}$-morphisms $A \otimes B \to C$ correspond to $\mathbb{T}$-bimorphisms $A \times B \to C$) [10,38,60]. The internal hom of $A, B \in \mathsf{Alg}(\mathbb{T})$ is the algebra $[A, B]$ of all $\mathbb{T}$-morphisms from A to B, viewed as a subalgebra of the product $B^{|A|}$. In particular, composition $[A, B] \times [B, C] \xrightarrow{;} [A, C]$ is a $\mathbb{T}$-bimorphism, so $([A, A], ;, \mathsf{id}_A)$ is a $\mathbb{T}$-monoid.

▶ **Example 4.8.** **(1)** A $(\mathcal{D}, \mathcal{M})$-bialgebra is a convex set carrying an additional monoid structure (without any interaction between the two structures). A $\mathcal{D}$-monoid is a convex monoid, that is, a $(\mathcal{D}, \mathcal{M})$-bialgebra whose monoid multiplication satisfies (3.5).

(2) For every set X, the set $X \multimap\!\!\to X$ is a $(\mathbb{T}, \mathcal{M})$-bialgebra with monoid structure given by Kleisli composition and $\mathbb{T}$-algebra structure given by the product $(\mathbb{T}X)^X$ of the free $\mathbb{T}$-algebra $\mathbb{T}X$. If $\mathbb{T}$ is commutative, then $X \multimap\!\!\to X$ is a $\mathbb{T}$-monoid; in fact, it is isomorphic to the $\mathbb{T}$-monoid $[\mathbb{T}X, \mathbb{T}X]$ (Remark 4.72).

Both the notion of recognition of probabilistic languages by convex monoids and the respective finiteness condition (Definitions 3.15 and 3.16) are instances of the following:

▶ **Definition 4.9.** A $(\mathbb{T}, \mathcal{M})$-bialgebra M *recognizes* the language $L\colon \Sigma^* \to O$ if $L = h \mathbin{;} p$ for some monoid morphism $h\colon \Sigma^* \to M$ and some $\mathbb{T}$-morphism $p\colon M \to O$.

▶ **Definition 4.10.** A $\mathbb{T}$-algebra (A, a) is *finitely generated* if there exists a surjective $\mathbb{T}$-morphism $\mathbb{T}G \twoheadrightarrow (A, a)$ for some finite set G. A $(\mathbb{T}, \mathcal{M})$-bialgebra is *fg-carried* if its underlying $\mathbb{T}$-algebra is finitely generated.

▶ **Theorem 4.11.** *Suppose that $X \multimap\!\!\to X$ is a finitely generated $\mathbb{T}$-algebra for every finite set X. Then for every language $L\colon \Sigma^* \to O$, the following are equivalent:*

(1) *There exists a $\mathbb{T}$-FA computing L.*

(2) *There exists an fg-carried $(\mathbb{T}, \mathcal{M})$-bialgebra recognizing L.*

If the monad $\mathbb{T}$ is commutative, then these statements are equivalent to:

(3) *There exists an fg-carried $\mathbb{T}$-monoid recognizing L.*

▶ **Remark 4.12.** The condition of the theorem is equivalent to finitely generated $\mathbb{T}$-algebras being closed under finite products.

$\mathbb{T}$	$\mathbb{T}$-FA	M	$\xi_0\colon M \to (X \mathrel{-\!\!\circ\!\!\!\to} X)$
$\mathcal{D}$	PFA	$X \to X$	$\xi_0(f) = \big(x \mapsto \eta_X(f(x))\big)$
$\mathcal{C}$	NPFA	$X \to X$	$\xi_0(f) = \big(x \mapsto \eta_X(f(x))\big)$
$\mathcal{S}$	WFA	$X \rightharpoonup X$	$\xi_0(f) = \big(x \mapsto \eta_X(f(x))$ if $f(x)$ is defined, else $0 \in \mathcal{S}X\ \big)$

Figure 4 Witnesses for $X \mathrel{-\!\!\circ\!\!\!\to} X$ being (monoidally) finitely generated

Proof sketch. (1)⇒(2) Let $\mathcal{A} = (Q, \delta, i, o)$ be a $\mathbb{T}$-FA computing L. Then L is recognized by the fg-carried $(\mathbb{T}, \mathcal{M})$-bialgebra $Q \mathrel{-\!\!\circ\!\!\!\to} Q$ via the monoid morphism $\bar{\delta}^*\colon \Sigma^* \to (Q \mathrel{-\!\!\circ\!\!\!\to} Q)$ and the $\mathbb{T}$-morphism $p\colon (Q \mathrel{-\!\!\circ\!\!\!\to} Q) \to O$ which is given by $p(f) = i \fatsemi f \mathbin{;} o^{\#}$. The proof that p is a $\mathbb{T}$-morphism requires the initial state i to be pure, which we may assume due to Remark 4.4.

(2)⇒(1) Let M be an fg-carried $(\mathbb{T}, \mathcal{M})$-bialgebra recognizing L via the monoid morphism $h\colon \Sigma^* \to M$ and the $\mathbb{T}$-morphism $p\colon M \to O$. Since M is finitely generated as a $\mathbb{T}$-algebra, there exists a surjective $\mathbb{T}$-morphism $s\colon \mathbb{T}Q \twoheadrightarrow M$ for some finite set Q. Let $(M, \cdot, e)$ denote the monoid structure, and let $s_0\colon Q \to M$ and $h_0\colon \Sigma \to M$ be the domain restrictions of s and h. We construct a $\mathbb{T}$-FA $\mathcal{A} = (Q, \delta, i, o)$ computing L as follows: we choose $i\colon 1 \mathrel{-\!\!\circ\!\!\!\to} Q$ and $\delta\colon Q \times \Sigma \mathrel{-\!\!\circ\!\!\!\to} Q$ such that the first two diagrams below commute – these choices exist because s is surjective. Moreover, we define $o\colon Q \to O$ by $o := s_0 \mathbin{;} p$ as in the third diagram.

$$
\begin{tikzcd}
1 \arrow[r, "i"] \arrow[dr, "e"'] & TQ \arrow[d, two heads, "s"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \times \Sigma \arrow[rr, "\delta"] \arrow[d, "s_0 \times \mathsf{id}"'] & & TQ \arrow[d, two heads, "s"] \\
M \times \Sigma \arrow[r, "\mathsf{id} \times h_0"] & M \times M \arrow[r, "\cdot"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "o"] \arrow[d, "s_0"'] & O \\
M \arrow[ur, "p"'] &
\end{tikzcd}
$$

One can prove that the $\mathbb{T}$-FA $\mathcal{A}$ computes the language L.

Now suppose that the monad $\mathbb{T}$ is commutative. Then (1)⇒(3) is shown like (1)⇒(2), adding the observation that the recognizing $(\mathbb{T}, \mathcal{M})$-bialgebra $Q \mathrel{-\!\!\circ\!\!\!\to} Q$ is a $\mathbb{T}$-monoid as in Example 4.82. The implication (3)⇒(2) is trivial. ◀

▶ **Example 4.13.** The condition of Theorem 4.11 holds for $\mathbb{T} \in \{\mathcal{D}, \mathcal{C}, \mathcal{S}\}$ from Example 4.1. We present for each finite set X a finite set M and a map $\xi_0\colon M \to (X \mathrel{-\!\!\circ\!\!\!\to} X)$ such that $\xi = \xi_0^{\#}\colon TM \twoheadrightarrow (X \mathrel{-\!\!\circ\!\!\!\to} X)$ is surjective. The respective witnesses are given in Figure 4, where $X \rightharpoonup X$ denotes the set of all partial functions on X. For all three monads, the condition amounts to the observation that every effectful function $f\colon X \mathrel{-\!\!\circ\!\!\!\to} X$ can be built from (partial or total) *effect-free* functions using $\mathbb{T}$-operations.

▶ **Remark 4.14.** Using the abstract theory of syntactic structures [2, 15, 66], it is possible to associate a canonical algebraic recognizer to every language $L\colon \Sigma^* \to O$, namely its *syntactic* $(\mathbb{T}, \mathcal{M})$*-bialgebra* and, in the commutative case, its *syntactic* $\mathbb{T}$*-monoid*. The syntactic convex monoid (Theorem 3.21) is an instance of the latter.

Next, we show that, under conditions on $\mathbb{T}$, fg-carried $\mathbb{T}$-monoids are finitely presentable.

▶ **Definition 4.15.** Let $\mathbb{S}$ be a monad. An $\mathbb{S}$-algebra is *finitely presentable* if it is the coequalizer in $\mathsf{Alg}(\mathbb{S})$ of some pair $p, q\colon \mathbb{S}X \to \mathbb{S}Y$ of $\mathbb{S}$-morphisms for finite sets X and Y.

▶ **Remark 4.16.** For the monad $\mathbb{S}$ associated to an algebraic theory (Λ, E), an $\mathbb{S}$-algebra is finitely presentable iff its corresponding Λ-algebra in the variety defined by (Λ, E) admits a finite presentation by generators and relations [5, Prop. 11.28]. Instantiating $\mathbb{S}$ to the monad corresponding to the algebraic theory of $\mathbb{T}$-monoids, where $\mathbb{T}$ is finitary, we obtain a notion of *finitely presentable* $\mathbb{T}$*-monoid*.

▶ **Theorem 4.17.** *If $\mathbb{T}$ is finitary and commutative, and finitely generated $\mathbb{T}$-algebras are finitely presentable, then every fg-carried $\mathbb{T}$-monoid is finitely presentable.*

Note that Theorem 3.19 is the instance of this result for $\mathbb{T} = \mathcal{D}$.

4.4 Recognizing Effectful Languages by Finite Monoids

Our second mode of recognition of $\mathbb{T}$-FA-computable languages uses effectful monoid morphisms to finite monoids; recognition by probabilistic channels from Section 3.3 is an instance.

▶ **Notation 4.18.** Given $\mathcal{K}\ell(\mathbb{T})$-morphisms $f_i\colon X_i \mathrel{-\!\!\circ\!\!\to} Y_i$ $(i = 1, 2)$, we define the Kleisli morphism $f_1 \bar{\times} f_2\colon X_1 \times X_2 \mathrel{-\!\!\circ\!\!\to} Y_1 \times Y_2$ as in Notation 3.6, with $\mathbb{T}$ in lieu of $\mathcal{D}$.

▶ **Definition 4.19.** A ($\mathbb{T}$-)*effectful monoid morphism* from a monoid $(N, \cdot, n)$ to a monoid $(M, \cdot, e)$ is a $\mathcal{K}\ell(\mathbb{T})$-morphism $h\colon N \mathrel{-\!\!\circ\!\!\to} M$ such that the diagram in Figure 2 commutes. The monoid M *effectfully recognizes* the language $L\colon \Sigma^* \to O$ if there exists an effectful monoid morphism $h\colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} M$ and a map $p\colon M \to O$ such that $L = h\,;p^{\#}$.

The key condition on the monad $\mathbb{T}$ that makes effectful recognition work is isolated in Theorem 4.22. This requires some technical preparation:

▶ **Remark 4.20.** The natural transformations $\pi_{X,Y}\colon TX \times TY \to T(X \times Y)$ of (4.1) and $\eta_X\colon X \to TX$ make T a *monoidal functor*, that is, π and η satisfy coherence laws w.r.t. the natural isomorphisms $(X \times Y) \times Z \cong X \times (Y \times Z)$ and $1 \times X \cong X \cong X \times 1$ [36, Thm. 2.1]. Consequently, T preserves monoid structures: for every monoid $(M, \cdot, e)$, the set TM forms a monoid with neutral element and multiplication, respectively, defined by

$$1 \xrightarrow{e} M \xrightarrow{\eta_M} TM \qquad \text{and} \qquad TM \times TM \xrightarrow{\pi_{M,M}} T(M \times M) \xrightarrow{T\cdot} TM.$$

It is folklore that if $\mathbb{T}$ is commutative, then, for every $\mathbb{T}$-monoid N, the extension $h^{\#}\colon TM \to N$ of every monoid morphism $h\colon M \to N$ is also a monoid morphism. For non-commutative monads $\mathbb{T}$, this is not true in general.

▶ **Definition 4.21.** A $(\mathbb{T}, \mathcal{M})$-bialgebra N is *monoidally finitely generated* if there exists a finite monoid M and a monoid morphism $\xi_0\colon M \to N$ whose free extension $\xi = \xi_0^{\#}\colon TM \to N$ is a surjective monoid morphism.

We are now ready to state the desired general result on effectful recognition. Note that its condition implies that of Theorem 4.11; for a separating example see the full version [41].

▶ **Theorem 4.22.** *Suppose that for every finite set X the $(\mathbb{T}, \mathcal{M})$-bialgebra $X \mathrel{-\!\!\circ\!\!\to} X$ from Example 4.82 is monoidally finitely generated. Then for every language $L\colon \Sigma^* \to O$, there exists a $\mathbb{T}$-FA computing L iff there exists a finite monoid $\mathbb{T}$-effectfully recognizing L.*

Proof sketch. ($\Rightarrow$) Suppose that $\mathcal{A} = (Q, \delta, i, f)$ is a $\mathbb{T}$-FA computing L. The proof of Theorem 4.11 shows that L is recognized by the $(\mathbb{T}, \mathcal{M})$-bialgebra $Q \mathrel{-\!\!\circ\!\!\to} Q$; say $L = g\,;p$ for some monoid morphism $g\colon \Sigma^* \to (Q \mathrel{-\!\!\circ\!\!\to} Q)$ and some $\mathbb{T}$-morphism $p\colon (Q \mathrel{-\!\!\circ\!\!\to} Q) \to O$. (The specific choices of g and p in that proof are not relevant here.) Since $Q \mathrel{-\!\!\circ\!\!\to} Q$ is monoidally finitely generated, there exists a monoid morphism $\xi_0\colon M \to (Q \mathrel{-\!\!\circ\!\!\to} Q)$ such that M is finite and $\xi = \xi_0^{\#}\colon TM \twoheadrightarrow (Q \mathrel{-\!\!\circ\!\!\to} Q)$ is a surjective monoid morphism. By the universal property of the free monoid Σ^* and the surjectivity of ξ, we obtain a (not necessarily unique) monoid morphism $h\colon \Sigma^* \to TM$ with $g = h\,;\xi$ to fill the commutative diagram (4.2). Moreover, one can show that $h\colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} M$ is an *effectful* monoid morphism. Then the finite monoid M effectfully recognizes L via h and $\xi_0 \,;\, p$ because $L = g \,;\, p = h \,;\, (\xi_0 \,;\, p)^{\#}$.

$$\begin{tikzcd} \Sigma^* \arrow[r, dashed, "h"] \arrow[dr, "g"'] & TM \arrow[r, "(\xi_0;p)^{\#}"] \arrow[d, "\xi"] & O \\ & Q \multimap Q \arrow[ur, "p"'] & \end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd} M \times \Sigma \arrow[r, "\eta_M \times h_0"] \arrow[d, "\delta"] & TM \times TM \arrow[d, "\pi_{M,M}"] \\ M & M \times M \arrow[l, "\cdot"'] \end{tikzcd} \tag{4.3}$$

($\Leftarrow$) Suppose that the finite monoid $(M, \cdot, e)$ effectfully recognizes L via $h\colon \Sigma^* \multimap M$ and $p\colon M \to O$. Then the $\mathbb{T}$-FA $\mathcal{A} = (M, \delta, e, p)$ whose transition map $\delta\colon M \times \Sigma \multimap M$ is given by the composite (4.3) (where $h_0(a) = h(a)$ for $a \in \Sigma$) computes L. ◀

▶ **Example 4.23.** The condition of Theorem 4.22 holds for $\mathbb{T} \in \{\mathcal{D}, \mathcal{C}, \mathcal{S}\}$, as the maps $\xi_0\colon M \to (X \multimap X)$ from Figure 4 witnessing that $X \multimap X$ is finitely generated also witness that $X \multimap X$ is *monoidally* finitely generated. We note that the verification that $\xi\colon TM \to (X \multimap X)$ is a monoid morphism is non-trivial unless $\mathbb{T}$ is commutative (see Remark 4.20). Proposition 4.24 below simplifies the computations for non-commutative $\mathbb{T}$, and also explains conceptually *why* the non-commutative monads $\mathcal{C}$ and $\mathcal{S}$ satisfy the condition: for a finite set X, the $(\mathbb{T}, \mathcal{M})$-bialgebra $X \multimap X$ is generated by *central* morphisms.

Recall that a Kleisli morphism $f\colon X \multimap Y$ is *central* [23, 52] if for all Kleisli morphisms $f'\colon X' \multimap Y'$, the following diagram commutes in $\mathcal{K}\ell(\mathbb{T})$.

$$\begin{tikzcd}[column sep=small] & TY \times X' \arrow[r, "\mathsf{rs}_{Y,X'}"] & Y \times X' \arrow[r, "1\times f'"] & Y \times TY' \arrow[dr, "\mathsf{ls}_{Y,Y'}"] & \\ X \times X' \arrow[ur, "f\times 1"] \arrow[dr, "1\times f'"'] & & & & Y \times Y' \\ & X \times TY' \arrow[r, "\mathsf{ls}_{X,Y'}"] & X \times Y' \arrow[r, "f\times 1"] & TY \times Y' \arrow[ur, "\mathsf{rs}_{Y,Y'}"'] & \end{tikzcd} \tag{4.4}$$

▶ **Proposition 4.24.** *If $\xi_0\colon M \to (X \multimap X)$ is a monoid morphism whose uncurried form $X \times M \multimap X$ is central, then its free extension $\xi\colon TM \to (X \multimap X)$ is a monoid morphism.*

▶ **Remark 4.25.** The witness $\xi_0\colon (X \to X) \to (X \multimap X)$ defined by $f \mapsto f \,;\, \eta_X$ for the monad $\mathcal{D}$ from Definition 3.4 works more generally for all *affine monads* [37] (i.e. monads $\mathbb{T}$ satisfying $T1 \cong 1$). Hence, our Theorems 4.11 and 4.22 apply to all affine monads $\mathbb{T}$.

4.5 Applications

We proceed to instantiate Theorems 4.11 and 4.22 to derive algebraic characterizations for several effectful automata models. For $\mathbb{T} = \mathcal{D}$, we recover Theorems 3.10 and 3.18 for PFAs:

▶ **Theorem 4.26.** *A probabilistic language is PFA-computable iff it is $\mathcal{D}$-effectfully recognized by a finite monoid iff it is recognized by an fg-carried convex monoid.*

For $\mathbb{T} = \mathcal{C}$, we obtain a new algebraic characterization of NPFAs. Note that a $(\mathcal{C}, \mathcal{M})$-*bialgebra* is a convex semilattice with an additional monoid structure.

▶ **Theorem 4.27.** *A probabilistic language is NPFA-computable iff it is $\mathcal{C}$-effectfully recognized by a finite monoid iff it is recognized by an fg-carried $(\mathcal{C}, \mathcal{M})$-bialgebra.*

Finally, $\mathbb{T} = \mathcal{S}$ gives a new algebraic characterization of WFAs. Note that an $(\mathcal{S}, \mathcal{M})$-*bialgebra* is an S-semimodule with an additional monoid structure. For a commutative semiring S, the notion of an $\mathcal{S}$-monoid is that of an *(associative) S-algebra*.

▶ **Theorem 4.28.** *An S-weighted language is WFA-computable iff it is $\mathcal{S}$-effectfully recognized by a finite monoid iff it is recognized by an fg-carried $(\mathcal{S}, \mathcal{M})$-bialgebra, and, for a commutative semiring S, iff it is recognized by an fg-carried S-algebra.*

For the case where S is a commutative ring (i.e. has additive inverses), we recover the correspondence between WFAs and fg-carried S-algebras due to Reutenauer [55]. However, Theorem 4.28 applies to every semiring, so it also yields novel algebraic characterizations of min- and max-plus automata and transducers (Example 4.53) as special instances.

5 Conclusions and Future Work

We have developed the foundations of an algebraic theory of automata with generic computational effects. Under suitable conditions on the effect monad $\mathbb{T}$, we characterized $\mathbb{T}$-FA-computable languages by algebraic modes of effectful recognition. As special cases, this entails the first algebraic characterizations of probabilistic automata and weighted automata over unrestricted semirings. We proceed to give some prospects for future work.

Proving finite presentability of syntactic convex monoids is challenging even for simple probabilistic languages, as Example 3.24 illustrates. Identifying conditions on the language ensuring this property is thus a natural question.

We aim to extend our theory beyond the category of sets. Of particular interest are *nominal sets*, which would allow us to capture effectful (e.g. probabilistic or weighted) register automata [16]. The main technical hurdle lies in the construction $Q \mapsto (Q \to Q)$ of function spaces, which does not preserve orbit-finite (i.e. finitely presentable) nominal sets. This might be overcome via the recently proposed restriction to *single-use* functions [17, 18].

An orthogonal generalization of our theory concerns effectful languages beyond finite words. This requires switching from monoids to algebras for a monad $\mathbb{S}$ [15]. For example, taking the monad $\mathbb{S}$ corresponding to *ω-semigroups* [48] could lead to algebraic recognition of effectful languages over *infinite words*. For a generalization of the recognition by $\mathbb{T}$-monoids, we expect that the interaction between the monads $\mathbb{S}$ and $\mathbb{T}$ be given by some form of distributive law similar to the interaction of $\mathcal{M}$ and $\mathbb{T}$.

Lastly, our effectful automata/monoid correspondence could pave the way to a *topological* account of effectful languages based on effectful profinite monoids. A first glimpse in this direction is given by Fijalkow [28] who analyzes the value-1 problem for PFAs in terms of a notion of *free prostochastic monoid*. We aim to study effectful versions of the duality theory of profinite monoids [12, 29] and its applications, notably variety theorems [30, 66].

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Appendix

This appendix provides proofs and additional details omitted from the main text for space reasons. In Appendix A we recall some concepts from category theory and establish a few auxiliary results that will be used in subsequent proofs. In Appendix B we explain the construction of syntactic convex monoids via transition monoids briefly mentioned in Section 3.5. In Appendix C we present full proofs of all theorems in the paper, as well as more details for examples.

A Categorical Background

The Monoidal Category of $\mathbb{T}$-Algebras

We the closed monoidal structure on the category $\mathsf{Alg}(\mathbb{T})$ for a commutative monad $\mathbb{T}$ on Set (Remark 4.7). We remark that the constructions shown here are not specific to Set, but can be developed for monads on general symmetric monoidal closed categories with enough limits and colimits [38, 39, 60].

Given algebras $A, B, C \in \mathsf{Alg}(\mathbb{T})$, a $\mathbb{T}$-*bimorphism* from A, B to C is a map $f\colon A \times B \to C$ such that all maps $f(a, -)\colon B \to C$ and $f(-, b)\colon A \to C$ are $\mathbb{T}$-morphisms for $a \in A$ and $b \in B$. We write $\mathsf{Alg}(\mathbb{T})(A, B; C)$ for the set of all $\mathbb{T}$-bimorphisms from A, B to C.

The *tensor product* $A \otimes B$ of two $\mathbb{T}$-algebras (A, a) and (B, b) is defined as the coequalizer of the following pair of $\mathbb{T}$-morphisms, with π given by (4.1):

$$T(TA \times TB) \overset{T\pi_{A,B}\,;\,\mu_{A\times B}}{\underset{a\times b}{\rightrightarrows}} T(A \times B) \xrightarrow{c_{A,B}} \mathrel{\!\!\to} A \otimes B$$

It has the following universal property: The map

$$t_{A,B} = \eta_{A\times B}\,;c_{A,B}\colon A \times B \to A \otimes B$$

is a *universal* $\mathbb{T}$-*bimorphism* for A, B, which means that (1) $t_{A,B}$ itself is a bimorphism, (2) for every $\mathbb{T}$-bimorphism $f\colon A \times B \to C$ there exists a unique $\mathbb{T}$-morphism $h\colon A \otimes B \to C$ such that $f = t_{A,B}\,; h$.

The tensor product $\otimes$ makes $\mathsf{Alg}(\mathbb{T})$ a symmetric monoidal closed category (with tensor unit $\mathbb{T}1$). The right adjoint of $- \otimes A\colon \mathsf{Alg}(\mathbb{T}) \to \mathsf{Alg}(\mathbb{T})$ is given by $[A, -]\colon \mathsf{Alg}(\mathbb{T}) \to \mathsf{Alg}(\mathbb{T})$; recall that the *internal hom* $[A, B] = \mathsf{Alg}(\mathbb{T})(A, B)$ is carried by the set of $\mathbb{T}$-morphisms from A to B, with the subalgebra structure of the product algebra $B^{|A|}$, i.e. with the $\mathbb{T}$-algebra structure defined pointwise from the structure of B. We thus have the following natural isomorphisms:

$$\mathsf{Alg}(\mathbb{T})(A, B; C) \cong \mathsf{Alg}(\mathbb{T})(A \otimes B, C) \cong \mathsf{Alg}(\mathbb{T})(A, [B, C]).$$

Since the tensor product has $\mathbb{T}1$ as tensor unit and satisfies $\mathbb{T}(A \times B) \cong \mathbb{T}A \otimes \mathbb{T}B$, we have that the free algebra functor $\mathbb{T}\colon \mathsf{Set} \to \mathsf{Alg}(\mathbb{T})$ is strong monoidal. Recall that a *monoid* in a monoidal category $\mathcal{C}$ (with tensor product $\otimes$ and unit I) is given by an object M with a unit $e\colon I \to M$ and multiplication $\bullet\colon M \otimes M \to M$ satisfying diagrammatic versions of the associative and unit laws. A *morphism* $h\colon (M, \bullet, e) \to (M', \bullet, e')$ of monoids is a $\mathcal{C}$-morphism $h\colon M \to M'$ that preserves that monoid structure. We write $\mathsf{Mon}(\mathcal{C})$ for the category of monoids and their morphisms. Since in $\mathcal{C} = \mathsf{Alg}(\mathbb{T})$ the tensor product $\otimes$ represents $\mathbb{T}$-bimorphisms and $I = \mathbb{T}1$, the category $\mathsf{Mon}(\mathsf{Alg}(\mathbb{T}))$ is isomorphic to the category of $\mathbb{T}$-monoids, that is, $\mathbb{T}$-algebras with an additional monoid structure on the underlying set whose multiplication is a $\mathbb{T}$-bimorphism (Definition 4.6). Morphisms of $\mathbb{T}$-monoids are maps that are simultaneously $\mathbb{T}$-morphisms and monoid morphisms.

Finitary Functors

A diagram $D\colon I \to \mathcal{C}$ in a category $\mathcal{C}$ is *directed* if its scheme I is a directed poset, that is, every finite subset of I has an upper bound. A *directed colimit* is a colimit of a directed diagram. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *finitary* if it preserves directed colimits.

▶ **Proposition A.1.** *A functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is finitary if and only if it is finitary in both arguments.*

Proof. We first note that for a directed set I the diagonal $\Delta\colon I \to I \times I$ is *final*, which means for $(i,j) \in I \times I$ the slice $(i,j) \downarrow \Delta$ is non-empty and connected. Indeed, for non-emptiness take an upper bound k of i,j, then $(i,j) \leq (k,k) = \Delta k$. For connectivity assume that $(i,j) \leq (k,k)$ and $(i,j) \leq (k',k')$, then any upper bound $k,k' \leq k''$ joins $(i,j) \leq \Delta k, \Delta k' \leq \Delta k''$. Finality of Δ implies that any two diagrams D, D' with $D = \Delta \,;\, D'$ have the same colimit [42, Section IX.3].

Now let $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor, and suppose that F is finitary in both arguments. Let $D = \langle D_1, D_2\rangle\colon I \to \mathcal{C} \times \mathcal{D}$ be a directed diagram. Then we can write D as the composite functor $D = \Delta \,;\, (D_1 \times D_2)$. We prove that F is finitary:

$$\begin{aligned}
\operatorname*{colim}_{i\in I} FD(i) &= \operatorname*{colim}_{i\in I} F(D_1 \times D_2)\Delta(i) && \\
&\cong \operatorname*{colim}_{(i,j)\in I\times I} F(D_1(i), D_2(j)) && \Delta \text{ final} \\
&\cong \operatorname*{colim}_{i\in I} \operatorname*{colim}_{j\in I} F(D_1(i), D_2(j)) && \text{colimits commute with colimits} \\
&\cong F(\operatorname*{colim}_{i\in I} D_1(i), \operatorname*{colim}_{j\in I} D_2(j)) && F \text{ finitary in both arguments} \\
&\cong F(\operatorname*{colim}_{(i,j)\in I\times I} (D_1 \times D_2)(i,j)) && \text{colimits in } \mathcal{C}\times\mathcal{D} \text{ computed pointwise} \\
&\cong F(\operatorname*{colim}_{i\in I} (D_1 \times D_2)\Delta(i)) && \Delta \text{ final} \\
&\cong F(\operatorname*{colim}_{i\in I} D(i))
\end{aligned}$$

Conversely, if F is finitary, then for every $C \in \mathcal{C}$ the functor $F(C,-)$ is finitary, being the composite of F with the finitary functor $\langle C, \mathsf{Id}\rangle\colon \mathcal{D} \to \mathcal{C} \times \mathcal{D}$. Analogously, $F(-,D)$ is finitary for every $D \in \mathcal{D}$. So F is finitary in both arguments. ◀

Finitely Presentable Algebras

The idea of presenting algebras by generators and relations can be generalized to the level of objects of arbitrary categories. An object X of a category $\mathcal{C}$ is *finitely presentable* (*fp*) if the hom functor $\mathcal{C}(X,-)\colon \mathcal{C} \to \mathsf{Set}$ is finitary. In more explicit terms, this means that every morphism f from X into a directed colimit $\operatorname{colim}_{i\in I} C_i$ factorizes *essentially uniquely* through some colimit injections $\kappa_i\colon C_i \to \operatorname{colim} C_i$. This means that

(1) there exists $i \in I$ and a morphism $f'\colon X \to C_i$ such that $f' \,;\, \kappa_i = f$, and

(2) given two such factorizations $f' \,;\, \kappa_i = f'' \,;\, \kappa_i$ of f, there exists a connecting morphism $\kappa_{i,j}\colon C_i \to C_j$ in the diagram such that $f' \,;\, \kappa_{i,j} = f'' \,;\, \kappa_{i,j}$.

We denote the class of all finitely presentable object of $\mathcal{C}$ by $\mathcal{C}_{\mathsf{fp}}$.

For $\mathcal{C} = \mathsf{Alg}(\mathbb{T})$, the categories of algebras for a finitary monad $\mathbb{T}$ on Set induced by an algebraic theory (Λ, E), the abstract categorical notion of finite presentability coincides with the usual algebraic one. More precisely, the following statements are equivalent for every $A \in \mathsf{Alg}(\mathbb{T})$, see e.g. [5, Prop. 11.28]:

(1) A is a finitely presentable object of $\mathsf{Alg}(\mathbb{T})$.
(2) A is a coequalizer in $\mathsf{Alg}(\mathbb{T})$ of a pair $p, q\colon \mathbb{T}X \to \mathbb{T}Y$ where X, Y are finite.
(3) A has a finite presentation by generators and equations as an algebra of the variety specified by Σ and E.

▶ **Proposition A.2.** *For every finitary commutative monad $\mathbb{T}$ on Set, the tensor product on $\mathsf{Alg}(\mathbb{T})$ preserves finitely presentable algebras.*

Proof. We first show that every finitely presentable algebra A is also *internally finitely presentable*: for every directed diagram $C_i, i \in I$ of algebras with colimiting cocone $\kappa_i\colon C_i \to \operatorname{colim}_i C_i$ the canonical bijection

$$\begin{array}{llcl} \varphi\colon & \operatorname*{colim}_i \mathsf{Alg}(\mathbb{T})(A, C_i) & \cong & \mathsf{Alg}(\mathbb{T})(A, \operatorname*{colim}_i C_i) \\ & [f_i\colon A \to C_i] & \mapsto & (f_i \,;\kappa_i\colon A \to C_i \to \operatorname*{colim}_i C_i), \end{array}$$

which is extending the cocone $\mathsf{Alg}(\mathbb{T})(A, \kappa_i)$, is an algebra homomorphism. Consider the following diagram in Set, where a denotes the respective algebra structures and $\iota_i = \mathsf{Alg}(\mathbb{T})(A, \kappa_i); \varphi^{-1}\colon \mathsf{Alg}(\mathbb{T})(A, C_i) \to \operatorname{colim}_i \mathsf{Alg}(\mathbb{T})(A, C_i)$.

$$\begin{tikzcd}
T\mathsf{Set}(A, C_i) \arrow[rr, "T\mathsf{Set}(A,\kappa_i)"] \arrow[ddd, bend right, "a"'] & & T\mathsf{Set}(A, \operatorname{colim}_i C_i) \arrow[ddd, bend left, "a"] \\
T\mathsf{Alg}(\mathbb{T})(A, C_i) \arrow[u, "T(\hookrightarrow)"] \arrow[r, "T\iota_i"] \arrow[d, "a"] & T(\operatorname{colim}_i \mathsf{Alg}(\mathbb{T})(A, C_i)) \arrow[r, "T\varphi"] \arrow[d, "a"] & T\mathsf{Alg}(\mathbb{T})(A, \operatorname{colim}_i C_i) \arrow[u, "T(\hookrightarrow)"'] \arrow[d, "a"] \\
\mathsf{Alg}(\mathbb{T})(A, C_i) \arrow[r, "\iota_i"] \arrow[d, hook] & \operatorname{colim}_i \mathsf{Alg}(\mathbb{T})(A, C_i) \arrow[r, "\varphi"] & \mathsf{Alg}(\mathbb{T})(A, \operatorname{colim}_i C_i) \arrow[d, hook] \\
\mathsf{Set}(A, C_i) \arrow[rr, "\mathsf{Set}(A,\kappa_i)"] & & \mathsf{Set}(A, \operatorname{colim}_i C_i)
\end{tikzcd}$$

Then the top and bottom squares commute by definition of φ; the outer left and right parts commute because the inclusions are $\mathbb{T}$-morphisms; the center left square by definition of the algebra structure on the filtered colimit. This shows that the center right square commutes when pre- and postcomposed with $T\iota_i$ and the inclusion, respectively. The former are collectively epic and the latter is monic, thus we get the desired commutativity of the center right square.

This entails that $\otimes$ preserves finitely presentable algebras. Indeed, given finitely presentable algebras A, B, the tensor $A \otimes B$ is finitely presentable because its hom functor preserves filtered colimits:

$$\begin{aligned} & \mathsf{Alg}(\mathbb{T})(A \otimes B, \operatorname*{colim}_i C_i) \\ \cong\ & \mathsf{Alg}(\mathbb{T})(A, \mathsf{Alg}(\mathbb{T})(B, \operatorname*{colim}_i C_i)) \\ \cong\ & \mathsf{Alg}(\mathbb{T})(A, \operatorname*{colim}_i \mathsf{Alg}(\mathbb{T})(B, C_i)) \\ \cong\ & \operatorname*{colim}_i \mathsf{Alg}(\mathbb{T})(A, \mathsf{Alg}(\mathbb{T})(B, C_i)) \\ \cong\ & \operatorname*{colim}_i \mathsf{Alg}(\mathbb{T})(A \otimes B, C_i) \end{aligned}$$ ◀

▶ **Lemma A.3.** *Every fully faithful finitary functor $U\colon \mathcal{C} \to \mathcal{D}$ reflects fp objects.*

Proof. Let C be an object of $\mathcal{C}$ such that UC is fp in $\mathcal{D}$. Then for every directed diagram $C_i, i \in I$ in $\mathcal{C}$ we have

$$\begin{aligned}
\mathcal{C}(C, \operatorname*{colim}_i C_i) &\cong \mathcal{D}(UC, U \operatorname*{colim}_i C_i) && U \text{ fully faithful} \\
&\cong \mathcal{D}(UC, \operatorname*{colim}_i UC_i) && U \text{ finitary} \\
&\cong \operatorname*{colim}_i \mathcal{D}(UC, UC_i) && UC \text{ fp} \\
&\cong \operatorname*{colim}_i \mathcal{C}(C, C_i) && U \text{ fully faithful.}
\end{aligned}$$

This proves that $C \in \mathcal{C}_{\mathsf{fp}}$. ◀

Recall that an *algebra* for an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ (short: an *F-algebra*) is a pair (A, a) of an object $A \in \mathcal{C}$ and a morphism $a\colon FA \to A$. A *morphism* $h\colon (A, a) \to (B, b)$ of F-algebras is a morphism $h\colon A \to B$ of $\mathcal{C}$ such that $a\,;h = Fh\,;b$. We write $\mathsf{Alg}(F)$ for the category of F-algebras and their morphisms.

▶ **Definition A.4.** Let $\mathcal{C}$ be a category and let $\mathcal{D} \in \{\mathsf{Alg}(\mathbb{T}), \mathsf{Alg}(F), \mathsf{Mon}(\mathcal{C})\}$. We call an object of $\mathcal{D}$ *fp-carried* if its underlying $\mathcal{C}$-object is finitely presentable.

▶ **Proposition A.5.** *Suppose that $\mathcal{C}$ has directed colimits and that $F\colon \mathcal{C} \to \mathcal{C}$ is finitary and preserves finitely presentable objects. Then every fp-carried F-algebra is finitely presentable.*

The analogous statement for F-*co*algebras is known [3, Lemma 3.2], and the proof follows a similar pattern.

Proof. Let (A, a) be an F-algebra such that $A \in \mathcal{C}_{\mathsf{fp}}$. To prove that (A, a) is finitely presentable as an algebra, let (A_i, a_i), $i \in I$ be a directed diagram in $\mathsf{Alg}(F)$. The colimit is created by the forgetful functor $\mathsf{Alg}(F) \to \mathcal{C}$ since F is finitary (see [57, Thm 4.6] for the dual result for coalgebras). Therefore, the colimit is given by $(\operatorname{colim} A_i, a)$ for the unique F-algebra structure $a\colon F(\operatorname{colim} A_i) \to \operatorname{colim} A_i$ such that the colimit injections $\kappa_i\colon A_i \to \operatorname{colim} A_i$ are algebra morphisms. Now let $f\colon (A, a) \to (\operatorname{colim} A_i, a)$ be an algebra morphism. We have to show that it factorizes essentially uniquely through a colimit injection. Since A is finitely presentable in $\mathcal{C}$, the underlying $\mathcal{C}$-morphism $f\colon A \to \operatorname{colim} A_i$ factorizes in $\mathcal{C}$ through a colimit injection as $f = f_i\,;\kappa_i$ for some morphism $f_i\colon A \to A_i$. We do not claim that f_i is an algebra morphism. But we see that κ_i merges $a\,;f_i$ and $Ff\,;Fa_i$ using that f is an algebra morphism:

$$(a\,;f_i)\,;\kappa_i = a\,;f = Ff\,;a = Ff_i\,;F\kappa_i\,;a = (Ff_i\,;Fa_i)\,;\kappa_i.$$

Since F preserves fp objects, we have that $FA \in \mathcal{C}_{\mathsf{fp}}$. Thus, there exists a connecting morphism $a_{i,j}$ merging $a\,;f_i$ and $Ff_i\,;a_i$. The composite $f_j = f_i\,;a_{i,j}$ is a morphism of F-algebras; indeed, the outside of the diagram below commutes since so do the upper and lower parts as well as the right-hand square, whereas the left-hand square commutes when post-composed with $a_{i,j}$:

$$\begin{tikzcd}
A \arrow[r, "f_i"] \arrow[rr, bend left, "f_j"] & A_i \arrow[r, "a_{i,j}"] & A_j \\
FA \arrow[u, "a"] \arrow[r, "Ff_i"] \arrow[rr, bend right, "Ff_j"'] & FA_i \arrow[u, "a_i"] \arrow[r, "Fa_{i,j}"] & FA_j \arrow[u, "a_j"]
\end{tikzcd}$$

Moreover, f_j is the desired factorization: $f_j \,;\, \kappa_j = f_i \,;\, a_{i,j} \,;\, \kappa_j = f_i \,;\, \kappa_i = f$. Essential uniqueness of the factorization is clear since it already holds in $\mathcal{C}$. This proves that (A, a) is finitely presentable. ◀

The above proposition extends from functor algebras to monoids:

▶ **Theorem A.6.** *Let $(\mathcal{C}, \otimes, I)$ be a closed monoidal category with directed colimits such that $\otimes$ preserves finitely presentable objects and and I is fp. Then every fp-carried monoid is finitely presentable.*

Proof. Since the category $\mathcal{C}$ is closed, the tensor preserves arbitrary colimits in both arguments and thus is finitary by Proposition A.1. Therefore the functor $FX = I + X \otimes X$ is finitary, and it is fp-preserving by assumption. (Note that fp objects are closed under finite coproducts in every category.) Consider the functor $U \colon \mathsf{Mon}(\mathcal{C}) \to \mathsf{Alg}(F)$ that turns a monoid $(M, \cdot, e)$ into an F-algebra $I + M \otimes M \xrightarrow{[e, \cdot]} M$, and is identity on morphisms. If $M \in \mathcal{C}_{\mathsf{fp}}$, then $U(M, \cdot, e)$ is finitely presentable in $\mathsf{Alg}(F)$ by Proposition A.5. Since U is fully faithful and finitary, Lemma A.3 shows that $(M, \cdot, e)$ is finitely presentable in $\mathsf{Mon}(\mathcal{C})$. ◀

By Proposition A.2 the condition of the above theorem is satisfied in the category of algebras for a finitary commutative monad. Therefore, we conclude:

▶ **Corollary A.7.** *Let $\mathbb{T}$ be a finitary commutative monad on* Set *and let M be a $\mathbb{T}$-monoid. If the $\mathbb{T}$-algebra underlying M is finitely presentable, then M is finitely presentable as a $\mathbb{T}$-monoid.*

B Syntactic Convex Monoids and Transition Monoids

We present an alternative construction of the syntactic convex monoid of a probabilistic language (Definition 3.20) as a *transition monoid*. For this purpose we first need a probabilistic automaton model that admits *canonical* minimal automata, which PFA do not. We instead consider deterministic automata in the category Conv of convex sets and affine maps.

▶ **Definition B.1.** Let Σ be a finite alphabet. A *convex automaton* consists of a convex state set Q, an inital state $q_0 \in Q$, an affine output map $o \in Q \to [0, 1]$ and a transition map $d \colon Q \times \Sigma \to Q$ such that every $d(-, a) \colon Q \to Q$ is affine.

Note that convex automata are simply algebras for the endofunctor $F = 1 + - \otimes \mathcal{D}\Sigma$ on Conv with an affine output map. Equivalently, convex automata are pointed coalgebras for the endofunctor $G = [0, 1] \times (-)^{\Sigma}$ on Conv. The free convex monoid $\mathcal{D}\Sigma^*$ naturally carries the structure of an F-algebra by right multiplication $\mathcal{D}\Sigma^* \otimes \mathcal{D}\Sigma \cong \mathcal{D}(\Sigma^* \times \Sigma) \to \mathcal{D}\Sigma^*$ and with $\eta_{\Sigma^*}(\varepsilon) \colon 1 \to \mathcal{D}\Sigma^*$ as initial state. Dually, the convex set of all probabilistic languages $[0, 1]^{\Sigma^*} \cong \mathsf{Conv}(\mathcal{D}\Sigma^*, [0, 1])$ is a G-coalgebra with transition map the curryfication of

$$\mathsf{Conv}(\mathcal{D}\Sigma^*, [0, 1]) \otimes \mathcal{D}\Sigma \otimes \mathcal{D}\Sigma^* \xrightarrow{\mathsf{id} \otimes m} \mathsf{Conv}(\mathcal{D}\Sigma^*, [0, 1]) \otimes \mathcal{D}\Sigma^* \xrightarrow{\mathsf{ev}} [0, 1],$$

where m is monoid multiplication in $\mathcal{D}\Sigma^*$. The output map is evaluation at the empty word.

▶ **Corollary B.2** [2, 32]**.** **(1)** *The F-algebra $\mathcal{D}\Sigma^*$ is the initial F-algebra.*
(2) *The G-coalgebra $[0, 1]^{\Sigma^*}$ is the final G-coalgebra.*

A convex automaton A is *reachable* if the unique F-algebra morphism $\mathcal{D}\Sigma^* \to A$ is surjective and it is *simple* if the unique G-coalgebra morphism $A \to [0, 1]^{\Sigma^*}$ is injective; A is *minimal* if it is both reachable and simple. Given a probabilistic language L we can turn the initial F-algebra $\mathcal{D}\Sigma^*$ and the final G-coalgebra $[0,1]^{\Sigma^*}$ into convex automata by choosing L as output map and initial state, respectively, to get the initial and final automata recognizing L. Then the *minimal convex automaton* $\mathrm{Min}(L)$ is then given by the image of the unique automaton morphism $\mathcal{D}\Sigma^* \to [0,1]^{\Sigma^*}$ (obtained via its surjective-injective factorization), see [32]. The *transition monoid* $\mathrm{Tr}(A)$ of a convex automaton $A = (Q, d, i)$ is the image of the monoid homomorphism $\mathcal{D}\Sigma^* \to \mathsf{Conv}(Q, Q)$ induced by $a \mapsto d(-, a)$.

▶ **Corollary B.3** [2]. *Let $L\colon \Sigma^* \to [0,1]$ be a probabilistic language, then the syntactic convex monoid is the transition monoid of the minimal automaton of L:*

$$\mathrm{Syn}(L) \cong \mathrm{Tr}(\mathrm{Min}(L)).$$

C Omitted Proofs and Details

Details for Definition 3.4

The map $\xi_{X,Y} : \mathcal{D}(X \to Y) \to (X \to \mathcal{D}Y)$ is the free extension $\xi_{X,Y} = s^{\#}$ of

$$s\colon (X \to Y) \to (X \to \mathcal{D}Y), \quad f \mapsto f \,;\eta_Y$$

and therefore well-defined.

It is a standard result that the product distribution $\pi_{Y,Z}$ is a probability distribution on $Y \times Z$. Iterating the product distribution $|X|$-times for a fixed set Y (i.e. $Z := Y$) and identifying such $|X|$-tuples with maps $f\colon X \to Y$ yields the definition of $\lambda_{X,Y}$, which is therefore a probability distribution, too.

Proof of Lemma 3.5

The equation $\lambda_{X,Y} ; \xi_{X,Y} = \mathsf{id}$ holds for every affine monad, as shown in the Details for Remark 4.25 (where the morphism $\xi_{X,Y}$ is denoted by $h^{\#}$).

Proof Sketch for Theorem 3.19

[For a full proof, see the proof of the general Theorem 4.17 for $\mathbb{T} = \mathcal{D}$.]

Let M be an fg-carried convex monoid. Choose a finite presentation (G, R, q) of M as a convex set. Since the multiplication of M is fully determined by its action on $q[G]$ by (3.5), this extends to a finite presentation of the convex monoid M by adding a relation $g \bullet g' = t_{g,g'}$ for each $g, g' \in G$, where $t_{g,g'}$ is any term in the signature of convex sets such that $q(t_{g,g'}) = q(g) \bullet q(g')$.

Details for Example 3.23

We prove that for every probabilistic language $L\colon \Sigma^* \multimap\!\!\to 2$,

$$L \text{ is commutative} \quad \text{iff} \quad \mathrm{Syn}(L) \text{ is commutative.}$$

If $\mathrm{Syn}(L)$ is commutative, then we have for all $a_1, \ldots, a_n \in \Sigma$ and all permutations π of

$\{1, \ldots, n\}$:

$$\begin{aligned} L(a_1 \cdots a_n) &= p_L(h_L(a_1 \cdots a_n)) \\ &= p_L(h_L(a_1) \cdots h_L(a_n)) \\ &= p_L(h_L(a_{\pi(1)}) \cdots h_L(a_{\pi(n)})) \\ &= p_L(h_L(a_{\pi(1)} \cdots a_{\pi(n)})) \\ &= L(a_{\pi(1)} \cdots a_{\pi(n)}); \end{aligned}$$

the third equality uses commutativity of $\mathrm{Syn}(L)$. Thus, L is commutative.

Conversely, if L is commutative, then in its syntactic monoid all pairs of elements $[\sum_i r_i v_i], [\sum_j s_j w_j] \in \mathrm{Syn}(L)$ satisfy

$$\begin{aligned} \big[\textstyle\sum_i r_i v_i\big]\big[\textstyle\sum_j s_j w_j\big] &= \Big[\sum_i \sum_j r_i s_j v_i w_j\Big] \\ &= \Big[\sum_j \sum_i s_j r_i w_j v_i\Big] \\ &= \Big[\sum_j s_j w_j\Big]\Big[\sum_i r_i v_i\Big]; \end{aligned}$$

the second equality uses commutativity of L. Thus, $\mathrm{Syn}(L)$ is commmutative.

Proof of Theorem 3.22

Note first that the full subcategory of Conv given by cancellative convex sets is closed under arbitrary products and convex subsets; this is because the cancellation property is defined via Horn implications of equations, so cancellative convex sets form a *quasivariety*. Let $L\colon \Sigma^* \to [0,1]$ be a language. The minimal convex automaton $\mathrm{Min}(L)$ of L is the image of the unique automaton morphism from the inital to the final convex automaton for L (Appendix B). Its carrier A_L is cancellative since it is a convex subset of $[0,1]^{\Sigma^*}$. By Corollary B.3 we get

$$\mathrm{Syn}(L) \cong \mathrm{Tr}(\mathrm{Min}(L)) \hookrightarrow \mathsf{Conv}(A_L, A_L) \hookrightarrow A_L^{|A_L|}.$$

Thus, $\mathrm{Syn}(L)$ is cancellative, being a convex subset of a power of A_L.

Details for Example 3.24

A convex monoid M is finitely presentable iff there exists a finite set G and a surjective affine monoid morphism $e\colon \mathcal{D}G^* \twoheadrightarrow M$ whose kernel congruence $\{(\varphi, \varphi') \mid e(\varphi) = e(\varphi')\}$ is finitely generated. If $G = \{g_1, \ldots, g_m\}$ and the kernel is generated by the pairs $(\varphi_1, \varphi_1'), \ldots, (\varphi_n, \varphi_n')$, we say that M is finitely presented by the generators $g_1, \ldots, g_m$ and equations $\varphi_1 = \varphi_1', \ldots, \varphi_n = \varphi_n'$. We prove the following claim:

▶ **Proposition C.1.** *The convex monoid* $((0,1], \cdot, 1)$ *is presented by a single generator* a *and the equation* $a^0 +_{\frac{1}{3}} a^2 = a^1$.

Proof. The map $\{a\} \mapsto (0,1]$, $a \mapsto \frac{1}{2}$, extends to the affine monoid morphism $E = E_X\colon \mathcal{D}a^* \to (0,1]$ that sends a finite distribution φ over a^* to its expected value $E_X\varphi$ with respect to the random variable

$$X\colon a^* \to \mathbb{R}, \quad a^n \mapsto \frac{1}{2^n}, \qquad \text{i.e.} \qquad E(\varphi) = \sum_{n\in\mathbb{N}} \frac{\varphi(n)}{2^n}.$$

It is easy to see that E is surjective: every $x \in (0,1]$ has a unique representation as $x = \frac{r}{2^n} + \frac{1-r}{2^{n+1}}$ for unique $n \in \mathbb{N}, r \in (0,1]$. This representation gives a canonical preimage of x under E given by the distribution $\varphi_x = a^n +_r a^{n+1} \in \mathcal{D}a^*$. We call φ_x the *canonical representative* of $x \in (0,1]$.

To prove the proposition, it suffices to show that the kernel of E coincides with the convex monoid congruence $\sim \subseteq \mathcal{D}a^* \times \mathcal{D}a^*$ generated by $a^0 +_{\frac{1}{3}} a^2 \sim a^1$:

$$\forall \varphi, \varphi' \in \mathcal{D}a^*.\, \varphi \sim \varphi' \iff E(\varphi) = E(\varphi').$$

For the $\Rightarrow$ direction we only need to check that the pair $(a^0 +_{\frac{1}{3}} a^2, a^1)$ is contained in the kernel congruence of E, which follows from

$$E(a^0 +_{\frac{1}{3}} a^2) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{2} = E(a^1).$$

For the $\Leftarrow$ direction it suffices to show that $\varphi \sim \varphi_{E(\varphi)}$ for every $\varphi \in \mathcal{D}a^*$; then $E(\varphi) = E(\varphi')$ implies $\varphi \sim \varphi_{E(\varphi)} \sim \varphi_{E(\varphi')} \sim \varphi'$ as required. For the proof of $\varphi \sim \varphi_{E(\varphi)}$, let us first note that $a^n +_{\frac{1}{3}} a^{n+2} \sim a^{n+1}$ for all n since $\sim$ is closed under multiplication. We claim that for all λ with $3\lambda + \sum_n r_n = 1$ we have

$$\begin{aligned}
&(\sum_{i \in \mathbb{N} \setminus \{n,n+1,n+2\}} r_i a^i) + (r_n + \lambda) a^n + r_{n+1} a^{n+1} + (r_{n+2} + 2\lambda) a^{n+2} \\
\sim \; & \\
&(\sum_{i \in \mathbb{N} \setminus \{n,n+1,n+2\}} r_i a^i) + r_n a^n + (r_{n+1} + 3\lambda) a^{n+1} + r_{n+2} a^{n+2}.
\end{aligned} \tag{C.1}$$

To see this, assume we have a sequence $(r_i)_{i \in \mathbb{N}}$ with $3\lambda + \sum_i r_i = 1$ and $\lambda < \frac{1}{3}$ (if $\lambda = \frac{1}{3}$ the statement is trivial). Let $\chi = \frac{\lambda}{1-3\lambda}$ (so $\lambda = \frac{\chi}{3\chi+1}$) and $s_i = r_i(3\chi + 1)$. Note that

$$\sum_i s_i = (3\chi + 1) \sum_i r_i = (3\chi + 1)(1 - 3\lambda) = 1,$$

and $\chi > 0$ since $\lambda < \frac{1}{3}$. so $\sum_i s_i a^i$ is an element of $\mathcal{D}a^*$. We compute

$$
\begin{aligned}
&\Big(\sum_{i\in\mathbb{N}\setminus\{n,n+1,n+2\}} r_i a^i\Big) + (r_n+\lambda)a^n + r_{n+1}a^{n+1} + (r_{n+2}+2\lambda)a^{n+2}\\
&= \Big(\sum_{i\in\mathbb{N}\setminus\{n,n+1,n+2\}} \frac{s_i}{3\chi+1} a^i\Big) + \frac{s_n+\chi}{3\chi+1}a^n + \frac{s_{n+1}}{3\chi+1}a^{n+1} + \frac{s_{n+2}+2\chi}{3\chi+1}a^{n+2}\\
&= \Big(\sum_{i\in\mathbb{N}\setminus\{n,n+1,n+2\}} \frac{s_i}{3\chi+1} a^i\Big) + \frac{s_n}{3\chi+1}a^n + \frac{s_{n+1}}{3\chi+1}a^{n+1} + \frac{s_{n+2}}{3\chi+1}a^{n+2}\\
&\qquad + \frac{3\chi}{3\chi+1}\frac{1}{3}a^n + \frac{3\chi}{3\chi+1}\frac{2}{3}a^{n+2}\\
&= \frac{1}{3\chi+1}\Big(\sum_{i\in\mathbb{N}} s_i a^i\Big) + \Big(1-\frac{1}{3\chi+1}\Big)\Big(\frac{1}{3}a^n + \frac{2}{3}a^{n+2}\Big)\\
&= \frac{1}{3\chi+1}\Big(\sum_{i\in\mathbb{N}} s_i a^i\Big) + \Big(1-\frac{1}{3\chi+1}\Big)\big(a^n +_{\frac{1}{3}} a^{n+2}\big)\\
&\sim \frac{1}{3\chi+1}\Big(\sum_{i\in\mathbb{N}} s_i a^i\Big) + \Big(1-\frac{1}{3\chi+1}\Big)a^{n+1}\\
&= \Big(\sum_{i\in\mathbb{N}\setminus\{n,n+1,n+1\}} \frac{s_i}{3\chi+1} a^i\Big) + \frac{s_n}{3\chi+1}a^n + \frac{s_{n+1}+3\chi}{3\chi+1}a^{n+1} + \frac{s_{n+2}}{3\chi+1}a^{n+2}\\
&= \Big(\sum_{i\in\mathbb{N}\setminus\{n,n+1,n+2\}} r_i a^i\Big) + r_n a^n + (r_{n+1}+3\lambda)a^{n+1} + r_{n+2}a^{n+2}.
\end{aligned}
$$

This proves (C.1). We now consider the following *one-player game*:

▶ **Game C.2.** *A game position is given by a distribution $\varphi = \sum_n r_n \cdot a^n \in \mathcal{D}a^*$, denoted by an indexed row vector*

$$
\begin{bmatrix} \cdots & n & n+1 & n+2 & \cdots \\ \cdots & r_n & r_{n+1} & r_{n+2} & \cdots \end{bmatrix}.
$$

The game has only one bidirectional rule corresponding to (C.1)*: given positions*

$$
\varphi = (r_{n+1}+3\lambda)a^{n+1} + \sum_{i\neq n+1} r_i a^i \quad \textit{and} \quad \varphi' = (r_n+\lambda)a^n + (r_{n+2}+2\lambda)a^{n+2} + \sum_{i\neq n,n+2} r_i a^i
$$

for some λ with $3\lambda + \sum_i r_i = 1$, the player can move between φ and φ' as indicated by

$$
\begin{bmatrix} \cdots & n & n+1 & n+2 & \cdots \\ \cdots & r_n & r_{n+1}+3\lambda & r_{n+2} & \cdots \\ \cdots & r_n+\lambda & r_{n+1} & r_{n+2}+2\lambda & \cdots \end{bmatrix}. \tag{RULE}
$$

We say that the rule has been applied at index n. The winning positions *are the following positions for $r_n \in [0,1)$, corresponding to canonical distributions:*

$$
\begin{bmatrix} \cdots & n-1 & n & n+1 & n+2 & \cdots \\ \cdots & 0 & r_n & 1-r_n & 0 & \cdots \end{bmatrix}.
$$

By construction, we have $\varphi \sim \varphi_{E(\varphi)}$ iff the player can reach a (necessarily unique) winning position when starting in the position corresponding to φ. So we have to prove that the player has a winning strategy for every starting position. We introduce some auxiliary terminology:

▶ **Definition C.3.** A *hole* in a position $\varphi \in \mathcal{D}a^*$ is given by a pair $[n,k] \in \mathbb{N}$ with $k \geq 2$ such that $\varphi(n) \neq 0, \varphi(n+k) \neq 0$ and $\forall 1 \leq j < k \colon \varphi(n+j) = 0$. The *range* of a position $\varphi \in \mathcal{D}a^*$ is defined as

$$\operatorname{rng}\varphi = (\max_{\varphi(n)\neq 0} n) - (\min_{\varphi(n)\neq 0} n) + 1.$$

Note that a position is winning iff it has range 1 or 2. Our strategy rests on the following two lemmas.

▶ **Lemma C.4** (Spreading)**.** *From every position the player can reach a position without holes.*

Proof. Let $\varphi \in \mathcal{D}a^*$ be the current position. Let $[n,k]$ be a hole in φ, we show how to "patch" it, i.e. how to get to a position that has one less hole. This means that φ looks locally like the first position in

$$\begin{bmatrix} \cdots & n-1 & n & n+1 & \cdots & n+k-2 & n+k-1 & n+k & n+k+1 & \cdots \\ \cdots & r_{n-1} & r_n & 0 & \cdots & 0 & 0 & r_{n+k} & r_{n+k+1} & \cdots \\ \cdots & r_{n-1} & r_n & 0 & \cdots & 0 & \frac{r_{n+k}}{4} & \frac{r_{n+k}}{4} & r_{n+k+1} + \frac{r_{n+k}}{2} & \cdots \end{bmatrix}.$$

We apply (RULE) at $n+k-1$ to split the coefficient at $n+k$ with $\lambda = \frac{r_{n+k}}{4}$, which makes the hole smaller (i.e. decreases k) but does not create any new holes. If $k = 2$ the hole is fixed, otherwise we repeat this process with new, smaller hole $[n, k-1]$ iteratively. ◀

▶ **Lemma C.5** (Sweeping)**.** *From every non-winning position without holes the player can reach a position without holes that has a strictly smaller range.*

Proof. Let $\varphi \in \mathcal{D}a^*$ be a non-winning position without holes. Let the minimal and maximal indices with non-zero coefficients be n and $n+k$, respectively, so that $\operatorname{rng}\varphi = k+1 > 2$ as φ is non-winning. Set $\lambda = \min(\{r_{n+i} \mid i < k\} \cup \{\frac{r_{n+k}}{2}\})$. We now "sweep" from left to right, applying (RULE) in sequence from right to left, starting with index $n+k-2$ and stopping at index n.

$$\begin{bmatrix} n & n+1 & n+2 & \cdots & n+k-3 & n+k-2 & n+k-1 & n+k \\ r_n & r_{n+1} & r_{n+2} & \cdots & r_{n+k-3} & r_{n+k-2} & r_{n+k-1} & r_{n+k} \\ r_n & r_{n+1} & r_{n+2} & \cdots & r_{n+k-3} & r_{n+k-2}-\lambda & r_{n+k-1}+3\lambda & r_{n+k}-2\lambda \\ r_n & r_{n+1} & r_{n+2} & \cdots & r_{n+k-3}-\lambda & r_{n+k-2}+2\lambda & r_{n+k-1}+\lambda & r_{n+k}-2\lambda \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_n & r_{n+1}-\lambda & r_{n+2}+2\lambda & \cdots & r_{n+k-3} & r_{n+k-2} & r_{n+k-1}+\lambda & r_{n+k}-2\lambda \\ r_n-\lambda & r_{n+1}+2\lambda & r_{n+2} & \cdots & r_{n+k-3} & r_{n+k-2} & r_{n+k-1}+\lambda & r_{n+k}-2\lambda \end{bmatrix}$$

We observe (1) that this constitutes a valid sequence of applications of (RULE), i.e. it does not happen that any index is < 0 by definition of λ; (2) after one such sweep all fields with index in $\{n+1, \ldots, n+k-1\}$ did not decrease in value; (3) the fields at n and $n+k$ strictly decreased in value. We continue sweeping $j-1$ times with the same λ until $r_n - j\lambda \leq \lambda$ or $r_{n+k} - 2\lambda \leq 2j\lambda$. Note that (2) above ensures that we can indeed continue with the same λ. The new position now looks like this:

$$\begin{bmatrix} n & n+1 & n+2 & \cdots & n+k-3 & n+k-2 & n+k-1 & n+k \\ r_n - j\lambda & r_{n+1}+2j\lambda & r_{n+2} & \cdots & r_{n+k-3} & r_{n+k-2} & r_{n+k-1}+j\lambda & r_{n+k}-2j\lambda \end{bmatrix}.$$

If now $r_n - j\lambda = 0$ or $r_{n+k} - 2j\lambda = 0$ we are done: the new position has a strictly smaller range. Otherwise, we perform one last sweep with $\lambda' = \min(r_n - j\lambda,\, r_{n+k}/2 - j\lambda) \leq \lambda$.

Afterwards either the field at n or at $n+k$ (or both) have value 0, so this position has a range smaller than φ, and we are done. ◀

Now given a starting position $\varphi \in \mathcal{D}a^*$ we first use Spreading to reach a position without holes. We then repeatedly apply Sweeping until we reach a position that has range at most 2. This is a winning position. ◀

Proof of Theorem 4.11

We start with a technical remark:

▶ **Remark C.6.** **(1)** Given a $\mathbb{T}$-algebra A, the map $(X \mathrel{-\!\circ\!\!\to} Y) \xrightarrow{-;g^{\#}} (X \to A)$ is a $\mathbb{T}$-morphism for every $g\colon Y \to A$, as it is just the product map $(\mathbb{T}Y)^X \xrightarrow{(g^{\#})^X} A^X$.

(2) The map $(Y \mathrel{-\!\circ\!\!\to} Z) \xrightarrow{f ⨾ -} (X \mathrel{-\!\circ\!\!\to} Z)$ is a $\mathbb{T}$-morphism for every *pure* map $f\colon X \to Y$. (For commutative $\mathbb{T}$, this also holds for non-pure $f\colon X \mathrel{-\!\circ\!\!\to} Y$.)

Now we turn to the proof of the theorem.

Proof. (1)⇒(2) Let $\mathcal{A} = (Q, \delta, i, o)$ be a $\mathbb{T}$-FA computing L. We may assume that the initial state i is pure (Remark 4.4). Then L is recognized by the fg-carried $(\mathbb{T}, \mathcal{M})$-bialgebra $Q \mathrel{-\!\circ\!\!\to} Q$ via the monoid morphism $\bar{\delta}^*\colon \Sigma^* \to (Q \mathrel{-\!\circ\!\!\to} Q)$ and the predicate $p\colon (Q \mathrel{-\!\circ\!\!\to} Q) \to O$ given by $p(f) = (i ⨾ f\,; o^{\#})$. Note that p is a $\mathbb{T}$-morphism by Remark C.6. Indeed, $L = \bar{\delta}^*\,; p$ holds because for all $w \in \Sigma^*$,

$$L(w) = (i ⨾ \bar{\delta}^*(w)\,; o^{\#}) = p(\bar{\delta}^*(w)).$$

(2)⇒(1) Let M be an fg-carried $(\mathbb{T}, \mathcal{M})$-bialgebra recognizing L via a monoid morphism $h\colon \Sigma^* \to M$ and a $\mathbb{T}$-morphism $p\colon M \to O$. Since M is finitely generated as a $\mathbb{T}$-algebra, there exists a surjective $\mathbb{T}$-morphism $q\colon TG \twoheadrightarrow M$ for some finite set G. Let $(M, \bullet, e)$ denote the monoid structure and $q_0\colon G \to M$ and $h_0\colon \Sigma \to M$ the respective domain restrictions of q and h. We construct a $\mathbb{T}$-FA $\mathcal{A} = (G, \delta, i, o)$ computing L as follows. Choose $i\colon 1 \mathrel{-\!\circ\!\!\to} G$ and $\delta\colon G \times \Sigma \mathrel{-\!\circ\!\!\to} G$ such that the first two diagrams below commute; such choices exist because q is surjective. Moreover, define $o\colon G \to O$ by $o := q_0\,; p$ as in the third diagram.

$$\begin{tikzcd} 1 \arrow[r, "i"] \arrow[dr, "e"'] & TG \arrow[d, two heads, "q"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} G \times \Sigma \arrow[rr, "\delta"] \arrow[d, "q_0 \times \mathsf{id}"'] & & TG \arrow[d, two heads, "q"] \\ M \times \Sigma \arrow[r, "\mathsf{id} \times h_0"] & M \times M \arrow[r, "\bullet"] & M \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "o"] \arrow[d, "q_0"'] & O \\ M \arrow[ur, "p"'] \end{tikzcd}$$

To show that $\mathcal{A}$ computes the language L, fix $w = a_1 \cdots a_n \in \Sigma^*$. Then the two diagrams below (in $\mathcal{K}\ell(\mathbb{T})$ and Set, respectively) commute:

$$\begin{tikzcd} & G \arrow[r, "\bar{\delta}(a_1)"] \arrow[d, "q_0"] & G \arrow[d, "q_0"] & \cdots & G \arrow[r, "\bar{\delta}(a_n)"] \arrow[d, "q_0"] & G \arrow[d, "q_0"] \\ 1 \arrow[ur, "i"] \arrow[dr, "e"'] & & & & & \\ & M \arrow[r, "- \bullet h_0(a_1)"] & M & \cdots & M \arrow[r, "- \bullet h_0(a_n)"] & M \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[dr, "o^{\#}"] \arrow[dd, "q = q_0^{\#}"'] & \\ & O \\ M \arrow[ur, "p"'] & \end{tikzcd}$$

Since $h(w) = e \bullet h_0(a_1) \bullet \cdots \bullet h_0(a_n)$, we see that

$$L(w) = (h\,; p)(w) = (i ⨾ \bar{\delta}^*(w)\,; o^{\#}).$$

This proves that the automaton $\mathcal{A}$ computes the language L, as claimed.

Now suppose that the monad $\mathbb{T}$ is commutative. Then (1)⇒(3) is shown like (1)⇒(2), together with the observation that the recognizing $(\mathbb{T}, \mathcal{M})$-bialgebra $Q \mathrel{-\!\circ\!\!\to} Q$ is a $\mathbb{T}$-monoid (Example 3.172). The implication (3)⇒(2) is trivial. ◀

Details for Remark 4.12

The condition of Theorem 4.11, which states that for every finite set X the $\mathbb{T}$-algebra $(TX)^X$ is finitely generated, is equivalent to the simpler condition that finitely generated $\mathbb{T}$-algebras are closed under finite products.

This simpler condition trivially implies the original one. To show that it is equivalent, suppose A, B are finitely generated $\mathbb{T}$-algebras, we have to show that their product is finitely generated. If one of A, B is empty, then so is their product $A \times B$, which is thus also finitely generated. Otherwise, there exist surjective $\mathbb{T}$-algebra morphisms $e_A : TX \twoheadrightarrow A$ and $e_B : TY \twoheadrightarrow B$ for finite, non-empty sets X, Y. Choose a set Z of cardinality larger than those of X, Y, which contains at least two elements, and surjections $f : Z \twoheadrightarrow X, g : Z \twoheadrightarrow Y$. By assumption the $\mathbb{T}$-algebra $(TZ)^Z$ is finitely generated, so there exists a finite set K and a surjection $h : TK \twoheadrightarrow (TZ)^Z$. Since every endofunctor on Set preserves surjections the following map is a surjective $\mathbb{T}$-algebra morphism, where the morphism p is the projection $(TZ)^Z \cong (TZ)^2 \times TZ^{|Z|-2}$ on the first two components – it exists since $|Z| \geq 2$.

$$TK \xrightarrow{h} (TZ)^Z \xrightarrow{p} (TZ)^2 \cong TZ \times TZ \xrightarrow{Tf \times Tg} TX \times TY \xrightarrow{e_A \times e_B} A \times B.$$

This shows that the $\mathbb{T}$-algebra $A \times B$ is indeed finitely generated.

Details for Example 4.13

See ‘Details for Examples 4.13 and 4.23’ further below.

Proof of Theorem 4.17

The statement is immediate from Corollary A.7 and the assumption that finitely generated and finitely presentable $\mathbb{T}$-algebras coincide. ◀

Details for Remark 4.20

It is folklore that if $\mathbb{T}$ is commutative, then, for every $\mathbb{T}$-monoid N, the extension $h^{\#} : TM \to N$ of every monoid morphism $h : M \to N$ is also a monoid morphism. This can be seen as follows: the condition that N is a $\mathbb{T}$-monoid can be expressed equivalently in purely diagrammatical terms by requiring commutativity of Diagram C.2.

$$\begin{tikzcd}
TN \times TN \arrow[r, "\pi_{N,N}"] \arrow[d, "n \times n"] & T(N \times N) \arrow[r, "T(\cdot)"] & TN \arrow[d, "n"] \\
N \times N \arrow[rr, "\cdot"] & & N
\end{tikzcd} \tag{C.2}$$

In the terminology of [60], the multiplication of N is a $\mathbb{T}$-*bimorphism*. [Note that this condition makes sense for any – not necessarily non-commutative – double-strength $\pi_{X,Y}$.]

The general statement is as follows: Let $(N, n, \cdot)$ be a $(\mathbb{T}, \mathcal{M})$-bialgebra satisfying Diagram C.2. Then the extension $f^{\#} : TM \to N$ of every monoid morphism $f : M \to N$ into $(N, \cdot)$ is also a monoid morphism. Here TM is equipped with the obvious multiplication $TM \times TM \to T(M \times M) \to TM$ induced by the double strength, and in general it does not satisfy Diagram C.2 in general. The fact that $f^{\#}$ is a monoid morphism is witnessed by the following commutative diagram:

$$
\begin{tikzcd}
TM \times TM \arrow[r, "\pi_{M,M}"] \arrow[d, "Tf \times Tf"] \arrow[dd, bend right=60, "f^{\#} \times f^{\#}"'] & T(M \times M) \arrow[r, "T\cdot"] \arrow[d, "T(f\times f)"] & TM \arrow[d, "Tf"] \arrow[dd, bend left=60, "f^{\#}"] \\
TN \times TN \arrow[r, "\pi_{N,N}"] \arrow[d, "n \times n"] & T(N \times N) \arrow[r, "T(\cdot)"] & TN \arrow[d, "n"] \\
N \times N \arrow[rr, "\cdot"] & & N
\end{tikzcd}
\tag{C.3}
$$

The top left and right squares commute by naturality of π and since f is a monoid morphism, respectively, and the bottom rectangle is just Diagram C.2 again.

Proof of Theorem 4.22

▶ **Remark C.7.** In the proof we use that the free monoid Σ^* is *projective*: Given a surjective monoid morphism $e\colon M \twoheadrightarrow N$ and a morphism $g\colon \Sigma^* \to N$, there exists a morphism $h\colon \Sigma^* \to M$ such that $g = h \mathbin{;} e$. Indeed, we can choose h to be the free extension of $g_0 \mathbin{;} s$, where $g_0\colon \Sigma \to N$ is the domain restriction of h and $s\colon N \to M$ is a splitting of the surjection e, that is, a map such that $s \mathbin{;} e = \mathsf{id}_N$.

Proof. ($\Rightarrow$) Suppose that $\mathcal{A}$ is a $\mathbb{T}$-FA with states Q computing L. From the proof of Theorem 4.11 we know that L is recognized by the $(\mathbb{T}, \mathcal{M})$-bialgebra $Q \mathrel{-\!\!\circ\!\!\to} Q$; say $L = g \mathbin{;} p$ for some monoid morphism $g\colon \Sigma^* \to (Q \mathrel{-\!\!\circ\!\!\to} Q)$ and some $\mathbb{T}$-morphism $p\colon (Q \mathrel{-\!\!\circ\!\!\to} Q) \to O$. (The specific choices of g and p in that proof are not relevant for the present argument.) Since $Q \mathrel{-\!\!\circ\!\!\to} Q$ is monoidally finitely generated, there exists a monoid morphism $e\colon M \to (Q \mathrel{-\!\!\circ\!\!\to} Q)$ such that M is a finite monoid and $e^{\#}\colon TM \twoheadrightarrow (Q \mathrel{-\!\!\circ\!\!\to} Q)$ is a surjective monoid morphism. By projectivity of Σ^*, there exists a monoid morphism $h\colon \Sigma^* \to TM$ such that $g = h \mathbin{;} e^{\#}$. Putting $q := e \mathbin{;} p$, we thus have the following commutative diagram:

$$
\begin{tikzcd}
\Sigma^* \arrow[r, "h"] \arrow[dr, "g"'] & TM \arrow[r, "q^{\#}"] \arrow[d, two heads, "e^{\#}"] & O \\
& Q \mathrel{-\!\!\circ\!\!\to} Q \arrow[ur, "p"'] &
\end{tikzcd}
$$

Since h is monoid morphism, the diagram below in Set commutes.

$$
\begin{tikzcd}
\Sigma^* \times \Sigma^* \arrow[rr, "\cdot"] \arrow[d, "h \times h"'] & & \Sigma^* \arrow[d, "h"] & 1 \arrow[l, "\varepsilon"'] \arrow[d, "e"] \\
TM \times TM \arrow[r, "\pi_{M,M}"] & T(M \times M) \arrow[r, "T\cdot"] & TM & M \arrow[l, "\eta_M"']
\end{tikzcd}
\tag{C.4}
$$

This is equivalent to commutativity of the following diagram in $\mathcal{K}\ell(\mathbb{T})$, which states that $h\colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} M$ is an effectful monoid morphism:

$$
\begin{tikzcd}
\Sigma^* \times \Sigma^* \arrow[r, "\cdot"] \arrow[d, "h \bar{\times} h"'] & \Sigma^* \arrow[d, "h"'] & 1 \arrow[l, "\varepsilon"'] \arrow[dl, "e"] \\
M \times M \arrow[r, "\cdot"] & M &
\end{tikzcd}
\tag{C.5}
$$

The finite monoid M recognizes L via h and q because $L = g \mathbin{;} p = h \mathbin{;} q^{\#}$.

($\Longleftarrow$) Suppose that the finite monoid $(M, \cdot, e)$ effectfully recognizes the language L via the effectful monoid morphism $h\colon \Sigma^* \mathrel{-\!\!\circ\!\!\to} M$ and the map $p\colon M \to O$. Then (C.5) or equivalently

(C.4) commutes, so h is a monoid morphism from Σ^* to the monoid TM. Thus $\mathbb{T}M$ is an fg-carried $(\mathbb{T},\mathcal{M})$-bialgebra recognizing L via h and $p^\#$. From Theorem 4.11 we conclude that there exists a $\mathbb{T}$-FA computing L. Specifically, by instantiating the proof of (2)⇒(1) to the $(\mathbb{T},\mathcal{M})$-bialgebra $\mathbb{T}M$ and the choices $G = M$ and $q = \mathsf{id}_{TM}$, we see that a $\mathbb{T}$-FA computing L is given by $\mathcal{A} = (M, \delta, e, p)$ with the transitions

$$(M \times \Sigma \xrightarrow{\delta} M) \;=\; (M \times \Sigma \xrightarrow{\eta_M \times h_0} TM \times TM \xrightarrow{\pi_{M,M}} T(M \times M) \xrightarrow{T\cdot} TM).$$

◀

Details for Example 4.13 and Example 4.23

We verify that for $\mathbb{T} \in \{\mathcal{D}, \mathcal{C}, \mathcal{S}\}$ and each finite set X the maps $\xi_0 \colon M \to (X \multimap X)$ of Figure 4 freely extend to $\mathbb{T}$-morphisms $\xi = \xi_0^\# \colon M \twoheadrightarrow (X \multimap X)$ that are both surjective and monoid morphisms, and thus witness that the $\mathbb{T}$-bialgebra $X \multimap X$ is monoidally finitely generated. For the non-commutative monads $\mathcal{C}$ and $\mathcal{S}$, we show that ξ is a monoid morphism using Proposition 4.24.

Distribution monad $\mathcal{D}$: Since $\mathcal{D}$ is an affine monad ($\mathcal{D}1 \cong 1$), the affine map

$$\xi \colon \mathcal{D}(X \to X) \to (X \to \mathcal{D}X)$$

extending the map $\xi_0 \colon (X \to X) \to (X \to \mathcal{D}X)$ given by $\xi_0(f) = f \,;\, \eta_X$ is a surjective monoid morphism by Remark 4.25.

S-semimodule monad $\mathcal{S}$: Analogous to probability distributions, we may represent an element f of $\mathcal{S}X$ as a formal finite sum $\sum_{i\in I} s_i x_i$ where $x_i \in X$, $s_i \in S$ and $f(x) = \sum_{i\in I\colon s_i = x} s_i$ for all $x \in X$. Consider the map $\xi_0 \colon (X \rightharpoonup X) \to (X \to \mathcal{S}X)$ as in Figure 4.

(1) ξ is surjective: Let $n = |X|$. Then the $(\mathcal{S},\mathcal{M})$-bialgebra $n \multimap n$ is, up to isomorphism, the set of $n \times n$-matrices over the semiring S with the usual multiplication, addition and scalar multiplication of matrices. Moreover $n \rightharpoonup n$ is the submonoid of matrices with all entries in $\{0, 1\}$ where each row contains at most one 1 entry. Then ξ_0 is just the inclusion, and ξ is given by $\sum_i s_i A_i \mapsto \sum_i s_i A_i$, i.e. a formal linear combination of matrices in $n \rightharpoonup n$ is mapped to the actual linear combination in the semimodule $n \multimap n$. Since each $n \times n$-matrix over S can be expressed as a linear combination of matrices in $n \rightharpoonup n$ (in fact, matrices with a single entry 1, and 0 otherwise), we see that ξ is surjective.

(2) ξ is a monoid morphism: By Proposition 4.24 we only need to show that the uncurried form $f \colon X \times (X \rightharpoonup X) \to \mathcal{S}X$ of ξ_0 is central. The map f is given by $f(x, k) = 1 \cdot k(x) \in \mathcal{S}X$ if $k(x)$ is defined, and $0 \in \mathcal{S}X$ otherwise. Instantiating (4.4) to f and $f' \colon X' \to \mathcal{S}Y'$, wee see that the two maps $X \times (X \rightharpoonup X) \times X' \to \mathcal{S}(X \times Y')$ of (4.4) both send (x, k, x') to $\sum_i s_i(k(x), y'_i)$ (where $f'_i(x') = \sum_i s_i y'_i$) if $k(x)$ is defined, and to $0 \in \mathcal{S}(X \times Y')$ otherwise. Thus, f is central.

Convex powerset of distributions monad $\mathcal{C}$: The free extension of $\xi_0 \colon (X \to X) \to (X \to \mathcal{C}X)$ as in Figure 4 is given by

$$\xi \colon \mathcal{C}(X \to X) \to (X \to \mathcal{C}X), \qquad \xi(U)(x) = \{\sum_i r_i \cdot f_i(x) \mid \sum_i r_i \cdot f_i \in U\}.$$

(1) ξ is surjective: We give a categorical proof using affinity of the non-empty finite power set monad $\mathcal{P}_\mathrm{f}^+$ and the finite distribution monad $\mathcal{D}$. For a convex set A the set of its finitely generated non-empty convex subsets is itself a convex set CA, and this yields a monad on $\mathsf{Conv} \cong \mathsf{Alg}(\mathcal{D})$. Then $\mathcal{C} = \mathcal{D} \,;\, C \,;\, |-|$ is the composite of the monad C with the adjunction $\mathcal{D} \dashv |-|$, where $|-|\colon \mathsf{Conv} \to \mathsf{Set}$ denotes the forgetful functor. In [67, Lemma 2] it is shown that taking convex hulls is a natural transformation $\langle - \rangle_A \colon |\mathcal{P}_\mathrm{f}^+ A| \to |\mathcal{C}A|$. [Note that it is not natural for the endofunctors $\mathcal{P}_\mathrm{f}^+, \mathcal{C}$ on Conv since $\langle - \rangle$ is not affine.] We then get the following diagram in Set, where $\xi^{\mathcal{P}_\mathrm{f}^+}$ and $\xi^{\mathcal{D}}$ are the free extensions of the respective inclusion maps $(X \to X) \to (X \multimap X)$.

$$\begin{tikzcd}
\mathcal{P}_\mathrm{f}^+\mathcal{D}(X \to X) \arrow[r, two heads, "\mathcal{P}_\mathrm{f}^+\xi^{\mathcal{D}}"] \arrow[d, two heads, "\langle - \rangle"] & \mathcal{P}_\mathrm{f}^+(X \to \mathcal{D}X) \arrow[r, two heads, "\xi^{\mathcal{P}_\mathrm{f}^+}"] \arrow[d, two heads, "\langle - \rangle"] & (X \to \mathcal{P}_\mathrm{f}^+\mathcal{D}X) \arrow[d, two heads, "\langle - \rangle"] \\
C\mathcal{D}(X \to X) \arrow[r, "C\xi^{\mathcal{D}}"] \arrow[d, equal] & C(X \to \mathcal{D}X) \arrow[r, "\langle C\,\mathrm{ev}_x \rangle_{x \in X}"] & (X \to C\mathcal{D}X) \arrow[d, equal] \\
\mathcal{C}(X \to X) \arrow[rr, "h^{\#}"] & & (X \to \mathcal{C}X)
\end{tikzcd}$$

The two upper squares commute by naturality of $\langle - \rangle$ and the bottom rectangle commutes since it does if we postcompose with the projections ev_x:

$$C\xi^{\mathcal{D}} \,;\, \langle C\,\mathrm{ev}_x \rangle_{x \in X} \,;\, \mathrm{ev}_x = C\xi^{\mathcal{D}} \,;\, C\,\mathrm{ev}_x = C(\xi^{\mathcal{D}} \,;\, \mathrm{ev}_x) = C\mathcal{D}\,\mathrm{ev}_x = \mathcal{C}\,\mathrm{ev}_x = \xi \,;\, \mathrm{ev}_x\,.$$

The composition $\langle - \rangle \,;\, \xi$ is surjective since the top right path is surjective by affinity of $\mathcal{P}_\mathrm{f}^+$ and $\mathcal{D}$, see Remark 4.25). In particular, ξ is surjective.

(2) ξ is a monoid morphism: By Proposition 4.24 we only need to show that the uncurried form $f\colon X \times (X \to X) \to \mathcal{C}X$ of h is central. The map f is given by $f(x,k) = \{1 \cdot k(x)\} \in \mathcal{C}X$. Instantiating (4.4) to f and $f'\colon X' \to \mathcal{C}Y'$, wee see that the two maps $X \times (X \to X) \times X' \to \mathcal{C}(X \times Y')$ of (4.4) both send (x,k,x') to

$$\{\sum_{i \in I_j} s_i^j(k(x), y_i^j) \mid j \in J\} \in \mathcal{C}(X \times Y') \qquad \text{where } f'(x') = \{\sum_{i \in I_j} s_i^j y_i^j \mid j \in J\}.$$

A Separating Example for Theorems 4.11 and 4.22

While all monads considered in this paper satisfy both Theorems 4.11 and 4.22 (even the list monad $\mathcal{M}$), we show here that the condition of Theorem 4.11 is indeed strictly stronger than that of Theorem 4.22 by giving an example of a monad $\mathbb{T}$ satisfying the conditions of the former but not of the latter. Consider the monad $\mathbb{T}$ corresponding to the algebraic variety (Λ, E) given by operations $\Lambda = \{+/2, e/1, \alpha/1\}$ with equations

$$E = \{x + e = x, e + x = x\}. \tag{C.6}$$

▶ **Lemma C.8.** *The monad $\mathbb{T}$ satisfies the condition of Theorem 4.11.*

Proof. We have to show that for every finite set X the $\mathbb{T}$-algebra $X \to TX$ is finitely generated. Since $(X \multimap X) \cong (TX)^X$ is a finite product of finitely generated free $\mathbb{T}$-algebras, it suffices to show that finitely generated $\mathbb{T}$-algebras are closed under finite products. Given finitely generated $\mathbb{T}$-algebras A, B with generating sets $A_0 \subseteq A, B_0 \subseteq B$, it is easy to see that $A \times B$ is generated by the set $A_0 \times \{e\} \cup \{e\} \times B_0$, since $(a,b) = (a+e, e+b) = (a,e)+(e,b)$. ◀

▶ **Lemma C.9.** *The monad $\mathbb{T}$ does not satisfy the condition of Theorem 4.22.*

Note in particular that Proposition 4.24 is not satisfied, the conceptual reason being that all central morphisms in $\mathcal{K}\ell(\mathbb{T})$ are pure.

Proof. We show that for $X = 2$ the $(\mathbb{T}, \mathcal{M})$-bialgebra $X \to TX$ is not monoidally finitely generated. Concretely, this means that for every finite monoid M and a monoid morphism $e \colon M \to (2 \to T2)$ the extension $e^{\#} \colon TM \twoheadrightarrow (2 \to T2)$ cannot be both surjective and a monoid morphism Note that we may assume, without loss of generality, that M is a submonoid of $2 \to T2$ and $e = \iota$ is its inclusion, since we can always factorize e through its image $\iota \colon \mathrm{Im}(e) \hookrightarrow (2 \to T2)$. So we have to show that for every submonoid $\iota \colon M \hookrightarrow (2 \to T2)$, if $\iota^{\#}$ is surjective the following diagram does not commute:

$$\begin{tikzcd}
TM \times TM \arrow[r, "\pi_{M,M}"] \arrow[d, "T\iota \times T\iota"] \arrow[dd, bend right=60, "\iota^{\#} \times \iota^{\#}"'] & T(M \times M) \arrow[r, "T\fatsemi"] & TM \arrow[d, "T\iota"] \arrow[dd, bend left=60, "\iota^{\#}"] \\
T(2 \to T2) \times T(2 \to T2) \arrow[d, "\mu_2 \times \mu_2"] & & T(2 \to T2) \arrow[d, "\mu_2"] \\
(2 \to T2) \times (2 \to T2) \arrow[rr, "\fatsemi"] & & 2 \to T2
\end{tikzcd} \tag{C.7}$$

(1) First we show that a subset $\iota \colon M \hookrightarrow (2 \to T2)$ generating the $\mathbb{T}$-algebra $2 \to T2$ must contain some non-pure map $f \colon 2 \to T2$, viz. a map f that is not of the form $f' \mathbin{;} \eta_2$ for some $f' \colon 2 \to 2$. This is the case, since even the full subset $M = (2 \to 2) \hookrightarrow (2 \to T2)$ of all pure maps does not generate the algebra $2 \to T2$. The algebra morphism $\iota^{\#} \colon T(2 \to 2) \to (2 \to T2)$ is not surjective, as it concretely is defined as

$$\begin{aligned}
&\iota^{\#} \colon T(2 \to 2) \to (2 \to T2) \\
&\quad t(f_0, \dots, f_3) \mapsto (x \mapsto t(f_0(x), \dots, f_3(x))),
\end{aligned}$$

where $t(f_0, \dots, f_3)$ is a Λ-term in the variables $f_i \in (2 \to 2)$. It is easy to see that the Kleisli morphism sending $0 \mapsto e, 1 \mapsto \alpha$ cannot be in the image of $\iota^{\#}$, as it would require both $e = t$ and $t = \alpha$.

(2) We show that if M contains a non-pure map $f \colon 2 \to T2$ then diagram C.7 does not commute. More specifically, we show for every non-pure Kleisli map $f \colon 2 \to T2$ there exists a term $v \in TM$ such that the pair $(\eta_M(f), v(f, f)) \in TM \times TM$ is sent by the upper right and lower left path of Diagram C.7 to different elements of $2 \to T2$. Let us first simplify the notation. We identify a Kleisli map $f \colon 2 \to T2$ with its graph tuple $[s(0,1), t(0,1)] \in (T2)^2$, where $f(0) = s(0,1) \in T2, f(1) = t(0,1) \in T2$ are Λ-terms in $0, 1 \in 2$. The Kleisli composite $f \fatsemi f$ corresponds to the tuple $[s(s(0,1), t(0,1)), t(s(0,1), t(0,1))]$. Suppose that $v(f) \in TM$ is a Λ-term with variable $f \in M$. The pair $(\eta_M(f), v(f)) \in TM \times TM$ is sent by the upper right path of Diagram C.7 to $\iota^{\#}(\eta_M(f) \cdot v(f)) \in (2 \to T2)$, which is given by

$$[v(s(s(0,1), t(0,1))), v(t(s(0,1), t(0,1)))]. \tag{C.8}$$

By the lower left path of Diagram C.7 it is sent to $\iota^{\#}(\eta_M(f)) \cdot \iota^{\#}(v(f, f))$, given by

$$[s(v(s(0,1)), v(t(0,1))), t(v(s(0,1)), v(t(0,1)))] \tag{C.9}$$

We now give for every possible value of $f(0) \in T2$ a term $v(f) \in TM$ such $\iota^{\#}(\eta(f) \cdot v(f)) \neq \iota^{\#}(\eta(f)) \cdot \iota^{\#}(v(f))$. If $f(0) = s(0,1) = e$ then we set $v(f) = \alpha$ to get

$$\iota^{\#}(\eta(f) \cdot v(f))(0) = (\text{C.8})(0) = \alpha \neq e = (\text{C.9})(0) = \big(\iota^{\#}(\eta(f)) \cdot \iota^{\#}(v(f))\big)(0).$$

Similarly if $f(0) = s(0,1) = \alpha$ then we set $v(f) = e$ to get

$$\iota^{\#}(\eta(f) \cdot v(f))(0) = (\text{C.8})(0) = e \neq \alpha = (\text{C.9})(0) = \big(\iota^{\#}(\eta(f)) \cdot \iota^{\#}(v(f))\big)(0).$$

Finally, if $f(0) = s(0,1) = c(0,1) + d(0,1)$ for terms $c(0,1), d(0,1) \in T2$, then may assume by Equation (C.6) that $c(0,1) \neq e$ and $d(0,1) \neq e$, and we set $v(f) = \alpha$. Suppose now for the sake of contradiction that we have an equality

$$\iota^{\#}(\eta(f) \cdot v(f))(0) = (\text{C.8})(0) = \alpha = c(\alpha,\alpha) + d(\alpha,\alpha) = (\text{C.9})(0) = \big(\iota^{\#}(\eta(f)) \cdot \iota^{\#}(v(f))\big)(0).$$

The term $c(\alpha,\alpha) + d(\alpha,\alpha)$ is a ground Λ-term, i.e., it does not contain any variables from M. It is clear from the equations of the theory that for all ground terms x, y we have $x + y = \alpha$ if and only if $x = \alpha$ and $y = e$ or $x = e$ and $y = \alpha$. But this means that either $c(\alpha,\alpha) = e$ or $d(\alpha,\alpha) = e$, and w.l.o.g. we assume the former. If $c(\alpha,\alpha) = e$ then c can not contain any occurence of the variables $0, 1$. This implies that $c(0,1) = e$, a contradiction to our assumptions $c(\alpha,\alpha) \neq e \neq d(\alpha,\alpha)$.

We conclude that there does not exist a submonoid $\iota\colon M \hookrightarrow (2 \to T2)$ such that the extension is both surjective and a monoid morphism, so $2 \to T2$ is not monoidally finitely generated. ◀

Proof of Proposition 4.24

Proof. The left and right strength of the monad $\mathbb{T}$ entail that $\mathcal{K}\ell(\mathbb{T})$ has the structure of a *strict premonoidal category*: For ev ery set X there exist functors $X \lhd (-), (-) \rhd X\colon \mathcal{K}\ell(\mathbb{T}) \to \mathcal{K}\ell(\mathbb{T})$ with $X \lhd Y = X \times Y = X \rhd Y$ that are defined on morphisms $f\colon X \multimap Y$ by

$$X' \lhd f = (\mathsf{id}_{X'} \times f) \mathbin{⨟} \mathsf{ls}_{X',Y}\colon X' \times X \to X' \times TY \multimap X' \times Y$$

and

$$f \rhd X' = (f \times \mathsf{id}_{X'}) \mathbin{⨟} \mathsf{rs}_{X',Y}\colon X \times X' \to TY \times X' \multimap Y \times X',$$

and the following equations hold:

$$\begin{aligned} \mathsf{id}_X \rhd X' &= \mathsf{rs}_{X,X'}, \\ X' \lhd \mathsf{id}_X &= \mathsf{ls}_{X',X}, \\ (X \lhd \mathsf{id}_Y) \rhd Z &= X \lhd (\mathsf{id}_Y \rhd Z). \end{aligned}$$

Now it is equivalent to say that a morphism $f\colon X \multimap Y$ in $\mathcal{K}\ell(\mathbb{T})$ is central iff for all $f'\colon X' \multimap Y'$ the following square commutes:

$$\begin{tikzcd} X \times X' \arrow[r, "f \rhd X'"] \arrow[d, "X \lhd f'"'] & Y \times X' \arrow[d, "Y \lhd f'"] \\ X \times Y' \arrow[r, "f \rhd Y'"] & Y \times Y' \end{tikzcd}$$

Now assume that $\bar{f}\colon M \to (TX)^X$ is a monoid morphism such that $f\colon X \times M \to TX$ is central and let $\bar{f}^{\#}\colon TM \to (TX)^X$ be the extension of f to a $\mathbb{T}$-morphism. Then the three squares below commute, where ev is the evaluation morphism and m is the multiplication of M.

$$\begin{tikzcd} X \times TM \arrow[r, "X \lhd \mathsf{id}"] \arrow[d, "\mathsf{id} \times \bar{f}^{\#}"'] & X \times M \arrow[d, "f"] \\ X \times (TX)^X \arrow[r, "\mathrm{ev}"] & X \end{tikzcd} \qquad \begin{tikzcd} X \times (TX)^X \times (TX)^X \arrow[r, "\mathsf{id} \times ⨟"] \arrow[d, "\mathrm{ev} \rhd (TX)^X"'] & X \times (TX)^X \arrow[d, "\mathrm{ev}"] \\ X \times (TX)^X \arrow[r, "\mathrm{ev}"] & X \end{tikzcd}$$

$$
\begin{tikzcd}
X \times M \times M \arrow[r, "\mathsf{id} \times m"] \arrow[d, "f \rhd \mathsf{id}"'] & X \times M \arrow[d, "f"] \\
X \times M \arrow[r, "f"] & X
\end{tikzcd}
$$

Indeed, the first square commutes because $X \lhd \mathsf{id} = \mathsf{ls}_{X,M}$ and the definition of ls. The middle square commutes by definition of Kleisli composition $\fatsemi$, and the right square commutes since f is a monoid morphism. We therefore get that the following diagram commutes in $\mathcal{K}\ell(\mathbb{T})$:

$$
\begin{tikzcd}
X \times TM \times TM \arrow[rr, bend left, "X \lhd \pi_{M,M}"] \arrow[r, shift left, "X \lhd (\mathsf{id} \rhd TM)"] \arrow[r, shift right, "(X \lhd \mathsf{id}) \rhd TM"'] \arrow[d, "\mathsf{id} \times \bar{f}^{\#} \times \mathsf{id}"] \arrow[dd, bend right=60, "\mathsf{id} \times (\bar{f}^{\#} \times \bar{f}^{\#})"'] & X \times M \times TM \arrow[r, "(X \times M) \lhd \mathsf{id}"] \arrow[d, "f \rhd TM"] & X \times M \times M \arrow[d, "f \rhd \mathsf{id}"] \arrow[dr, "X \lhd m"] & \\
X \times (TX)^X \times TM \arrow[r, "\mathrm{ev} \rhd TM"] \arrow[d, "\mathsf{id} \times \mathsf{id} \times \bar{f}^{\#}"] & X \times TM \arrow[r, "X \lhd \mathsf{id}"] \arrow[d, "\mathsf{id} \times \bar{f}^{\#}"] & X \times M \arrow[d, "f"] & X \times M \arrow[dl, "f"] \\
X \times (TX)^X \times (TX)^X \arrow[r, "\mathrm{ev} \rhd (TX)^X"] \arrow[dr, "\mathsf{id} \times \fatsemi"] & X \times (TX)^X \arrow[r, "\mathrm{ev}"] & X & \\
& X \times (TX)^X \arrow[ur, "\mathrm{ev}"] & &
\end{tikzcd}
$$

Here the upper right square uses that f is central. The curried forms of the outer paths precisely yield that $\bar{f}^{\#}$ is a monoid homomorphism. ◀

Details for Remark 4.25

A commutative monad $\mathbb{T}$ is called *affine* if $T1 \cong 1$. By [37, Thm. 2.1], this is equivalent to commutativity of the following triangle for all $X, Y \in \mathsf{Set}$, where p_1 and p_2 are the product projections:

$$
\begin{tikzcd}
TX \times TY \arrow[r, tail, "\pi_{X,Y}"] \arrow[dr, equal] & T(X \times Y) \arrow[d, two heads, "\langle Tp_1, Tp_2 \rangle"] \\
& TX \times TY
\end{tikzcd}
\tag{C.10}
$$

For every finite set X, we have $(X \to X) \cong \prod_{x \in X} X$ and $(X \multimap\!\!\to X) \cong \prod_{x \in X} TX$ and so repeated application of (C.10) yields a commutative triangle of the following form, where $\xi_0 \colon (X \to X) \to (X \multimap\!\!\to X)$ is the monoid morphism given by $\xi_0(f) = f \mathbin{;} \eta_X$ and $\xi = \xi_0^{\#}$:

$$
\begin{tikzcd}
(X \multimap\!\!\to X) \arrow[r, tail, "\lambda_{X,Y}"] \arrow[dr, equal] & T(X \to X) \arrow[d, two heads, "\xi_{X,Y}"] \\
& (X \multimap\!\!\to X)
\end{tikzcd}
$$

Thus ξ is surjective, and it is a monoid morphism because $\mathbb{T}$ is commutative.