Three-cuts are a charm: acyclicity in 3-connected cubic graphs*

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Abstract

Let G be a bridgeless cubic graph. In 2023, the three authors solved a conjecture (also known as the S_4-Conjecture) made by Mazzuoccolo in 2013: there exist two perfect matchings of G such that the complement of their union is a bipartite subgraph of G. They actually show that given any 1^+-factor F (a spanning subgraph of G such that its vertices have degree at least 1) and an arbitrary edge e of G, there exists a perfect matching M of G containing e such that $G \setminus (F \cup M)$ is bipartite. This is a step closer to comprehend better the Fan–Raspaud Conjecture and eventually the Berge–Fulkerson Conjecture. The S_4-Conjecture, now a theorem, is also the weakest assertion in a series of three conjectures made by Mazzuoccolo in 2013, with the next stronger statement being: there exist two perfect matchings of G such that the complement of their union is an acyclic subgraph of G. Unfortunately, this conjecture is not true: Jin, Steffen, and Mazzuoccolo later showed that there exists a counterexample admitting 2-cuts. Here we show that, despite of this, every cyclically 3-edge-connected cubic graph satisfies this second conjecture.

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1 Introduction

In 2013, Giuseppe Mazzuoccolo [?] proposed three beguiling conjectures about bridgeless cubic graphs. His first conjecture, implied by the Berge–Fulkerson Conjecture [?], is the following.

Conjecture 1.1 (Mazzuoccolo, 2013 [?])**.** *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is a bipartite graph.*

This conjecture, which is no longer open, has been solved by the three authors. More precisely they prove the following stronger statement.

Theorem 1.2 (Kardoš, Máčajová & Zerafa, 2023 [?])**.** *Let G be a bridgeless cubic graph. Let F be a 1^+-factor of G and let $e \in E(G)$. Then, there exists a perfect matching M of G such that $e \in M$, and $G \setminus (F \cup M)$ is bipartite.*

We note that a 1^+-*factor* of G is the edge set of a spanning subgraph of G such that its vertices have degree 1, 2 or 3. Theorem **??** not only shows the existence of two perfect matchings of G whose deletion leaves a bipartite subgraph of G, but that for every perfect matching of G there exists a second one such that the deletion of the two leaves a bipartite subgraph of G. In particular, Theorem **??** also implies that for every collection of disjoint odd circuits of G, there exists a perfect matching which intersects at least one edge from each odd circuit (this was posed as an open problem by Mazzuoccolo and the last author in [?], see also [?]).

Mazzuoccolo moved on to propose two stronger conjectures, with Conjecture **??** being the strongest of all three.

Conjecture 1.3 (Mazzuoccolo, 2013 [?])**.** *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is an acyclic graph.*

Conjecture 1.4 (Mazzuoccolo, 2013 [?])**.** *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is an acyclic graph, whose components are of order 2 or 3.*

Clearly, these last two conjectures are true for 3-edge-colourable cubic graphs, and Janos Hägglund verified the strongest of these conjectures (Conjecture **??**) by computer for all non-trivial snarks (non 3-edge-colourable cubic graphs) of order at most 34 [?]. However, 5 years later, Jin, Steffen, and Mazzuoccolo [?] gave a counterexample to Conjecture **??**. Their counterexample contains a lot of 2-edge-cuts and the authors state that the conjecture "could hold true for 3-connected or cyclically 4-edge-connected cubic graphs". In fact, as in real life, being more connected has its own benefits, and in this paper we show the following stronger statement.

Theorem 1.5. *Let G be a cyclically 3-edge-connected cubic graph, which is not a Klee-graph. Then, for any $e \in E(G)$ and any 1^+-factor F of G, there exists a perfect matching M of G containing e such that $G \setminus (F \cup M)$ is acyclic.*

We remark that Klee-graphs (see Definition **??**), which are to be discussed further in Section **??**, are 3-edge-colourable cubic graphs and so are not a counterexample to Conjecture **??**. However, the stronger statement given in Theorem **??** does not hold for this class of graphs, and this is the reason why we exclude them.

Although Theorem **??** is not a direct consequence of the Berge–Fulkerson Conjecture, we believe that the results presented here and in [**?**] are valuable steps towards trying to decipher long-standing conjectures such as the Fan–Raspaud Conjecture [**?**], and the Berge–Fulkerson Conjecture itself.

In fact, we will prove the following statement, which is equivalent to Theorem **??**.

Theorem 1.6. *Let G be a cyclically 3-edge-connected cubic graph, which is not a Klee-graph. Then, for any $e \in E(G)$ and any collection of disjoint circuits $\mathcal{C}$, there exists a perfect matching M of G containing e such that every circuit in $\mathcal{C}$ contains an edge from M.*

Indeed, given a collection of disjoint circuits $\mathcal{C}$, its complement is a 1^+-factor, say $F_{\mathcal{C}}$. A perfect matching M containing e such that $G \setminus (F_{\mathcal{C}} \cup M)$ is acyclic must contain an edge from every circuit in $\mathcal{C}$. On the other hand, given a 1^+-factor F, its complement is a collection of disjoint paths and circuits, and so it suffices to consider the collection $\mathcal{C}_F$ of circuits disjoint from F. A perfect matching M containing e such that every circuit in $\mathcal{C}_F$ contains an edge from M, clearly makes $G \setminus (F \cup M)$ acyclic.

1.1 Important definitions and notation

Graphs considered in this paper are simple, that is, they cannot contain parallel edges and loops, unless otherwise stated.

Let G be a graph and (V_1, V_2) be a partition of its vertex set, that is, $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$. Then, by $E(V_1, V_2)$ we denote the set of edges having one endvertex in V_1 and one in V_2; we call such a set an *edge-cut*. An edge which itself is an edge-cut of size one is a *bridge*. A graph which does not contain any bridges is said to be *bridgeless*.

An edge-cut $X = E(V_1, V_2)$ is called *cyclic* if both graphs $G[V_1]$ and $G[V_2]$, obtained from G after deleting X, contain a *circuit* (a 2-regular connected subgraph). The *cyclic edge-connectivity* of a graph G is defined as the smallest size of a cyclic edge-cut in G if G admits one; it is defined as $|E(G)| - |V(G)| + 1$, otherwise. For cubic graphs, the latter only concerns K_4, $K_{3,3}$, and the graph consisting of two vertices joined by three parallel edges, whose cyclic edge-connectivity is thus 3, 4, and 2, respectively. An *acyclic* graph is a graph which does not contain any circuits.

Let G be a bridgeless cubic graph. A 1^+*-factor* of G is the edge set of a spanning subgraph of G such that its vertices have degree 1, 2 or 3. In particular, a *perfect matching* and a 2*-factor* of G are 1^+-factors whose vertices have exactly degree 1 and 2, respectively.

2 Klee-graphs

Definition 2.1 ([**?**]). A graph G is a Klee-graph if G is the complete graph on 4 vertices K_4 or there exists a Klee-graph G_0 such that G can be obtained from G_0 by replacing a vertex by a triangle (see Figure **??**).

Figure 1: Examples of Klee-graphs on 4 upto 12 vertices, left to right.

For simplicity, if a graph G is a Klee-graph, we shall sometimes say that G is Klee. We note that there is a unique Klee-graph on 6 vertices (the graph of a 3-sided prism), and a unique Klee-graph on 8 vertices. As we will see in Section **??**, these two graphs are Klee ladders, and shall be respectively denoted as KL_6 and KL_8.

Lemma 2.2 ([**?**]). *The edge set of any Klee-graph can be uniquely partitioned into three pairwise disjoint perfect matchings. In other words, any Klee-graph is 3-edge-colourable, and the colouring is unique up to a permutation of the colours.*

Since Klee-graphs are 3-edge-colourable, they easily satisfy the statement of Conjecture **??**.

Proposition 2.3. *Let G be a Klee-graph. Then, G admits two perfect matchings M_1 and M_2 such that $G \setminus (M_1 \cup M_2)$ is acyclic.*

The new graph obtained after expanding a vertex of a Hamiltonian graph (not necessarily Klee) into a triangle is still Hamiltonian, and so, since K_4 is Hamiltonian, all Klee-graphs are Hamiltonian. Hamiltonian cubic graphs have the following distinctive property.

Proposition 2.4. *Let G be a Hamiltonian cubic graph. Then, for any collection of disjoint circuits $\mathcal{C}$ of G there exists a perfect matching M of G which intersects at least one edge of every circuit in $\mathcal{C}$.*

Proof. Since G is Hamiltonian, it admits three disjoint perfect matchings M_1, M_2, M_3 covering $E(G)$ such that at least two of them induce a Hamiltonian circuit. Without loss of generality, assume that $M_2 \cup M_3$ induce a Hamiltonian circuit. Let $\mathcal{C}$ be a collection of disjoint circuits of G for which the statement of the proposition does not hold. In particular, this implies that M_1 does not intersect all the circuits in $\mathcal{C}$ — since the complement of M_1 is a Hamiltonian circuit, $\mathcal{C}$ consists of exactly one circuit. However, this means that M_2 (or M_3) intersects the only circuit in $\mathcal{C}$, contradicting our initial assumption. □

Corollary 2.5. *For any collection of disjoint circuits $\mathcal{C}$ of a Klee-graph G there exists a perfect matching M of G which intersects at least one edge of every circuit in $\mathcal{C}$.*

On the other hand, we have to exclude Klee-graphs from Theorem **??** (and Theorem **??**) since for some Klee-graphs there are edges contained in a unique perfect matching, as we will see in the following subsection.

2.1 Other results about Klee-graphs

Lemma 2.6 ([**?**]). *Let G be a Klee-graph on at least 6 vertices. Then, G has at least two triangles and all its triangles are vertex-disjoint.*

Indeed, expanding a vertex into a triangle can only destroy triangles containing the vertex to be expanded.

We will now define a series of particular Klee-graphs, which we will call *Klee ladders*. Let KL_4 be the complete graph on 4 vertices, and let u_4v_4 be an edge of KL_4. For any even $n \geq 4$, let KL_{n+2} be the Klee-graph obtained from KL_n by expanding the vertex u_n into a triangle. In the resulting graph KL_{n+2}, we denote the vertex corresponding to v_n by v_{n+2}, and denote the vertex of the new triangle adjacent to v_{n+2} by u_{n+2}.

In other words, the graph KL_{2k+2} consists of the Cartesian product $P_2 \square P_k$ (where P_t denotes a path on t vertices) with two additional vertices u_{2k+2} and v_{2k+2} adjacent to each other, such that u_{2k+2} (v_{2k+2}) is adjacent to the two vertices in the first (last, respectively) copy of P_2 in $P_2 \square P_k$ (see Figure **??**).

Klee ladders can be used to illustrate why we have to exclude Klee-graphs from our main result. For a given Klee ladder G there exists an edge e such that e is contained in a unique perfect matching of G, and therefore there is no hope for a statement like Theorem **??** to be true.

Figure 2: An example of a Klee ladder, KL_{12}. There is a unique perfect matching (here depicted using dotted lines) containing the edge e. The complement of this perfect matching is a Hamiltonian circuit.

We will frequently use the following structural property of certain Klee-graphs.

Lemma 2.7. *Let G be a Klee-graph on at least 8 vertices having exactly two (disjoint) triangles. Then,*

(i) exactly one edge of each triangle lies on a 4-circuit; and

(ii) if G admits an edge joining the two triangles, then G is a Klee ladder.

Proof. We prove this by induction. Claim (i) is obvious for KL_8, the only Klee-graph on 8 vertices, so let G be a Klee-graph on $n \geq 10$ vertices. By definition, it can be obtained from a smaller one, say G_0, by expanding a vertex into a triangle. Since G only has two triangles, this operation must have destroyed a (single) triangle of G_0, which in turn gives rise to a 4-circuit containing exactly one of the edges of the new triangle.

Moreover, if G admits an edge e joining the two triangles, then the corresponding edge e_0 in G_0 joins a triangle to a vertex contained in a (distinct) triangle, so it joins the two triangles of G_0. By induction, G_0 is a Klee ladder, say KL_{n-2}, for some $n \geq 8$, and the edge e_0 is the edge $u_{n-2}v_{n-2}$ (see the definition of Klee ladders above). Claim (ii) follows immediately. $\square$

3 Proof of Theorem ??

Proof. Let G be a minimum counterexample to the statement of Theorem **??**. Since K_4 is Klee, G has at least six vertices. There are only two 3-connected cubic graphs on six vertices, namely KL_6 and $K_{3,3}$. The former is Klee. For the latter, $K_{3,3}$, a collection of disjoint circuits can only contain one circuit on either four or six vertices and in both cases it is easy to check that every edge is contained in a perfect matching intersecting the prescribed circuit. Therefore, G has at least eight vertices.

Let $e \in E(G)$ be an edge of G such that there exists a collection of disjoint circuits such that for every perfect matching M containing e there is a circuit in the collection containing no edge from M. Amongst all such collections, we can choose an inclusion-wise minimal one, denoted by $\mathcal{C}$. By the choice of $\mathcal{C}$, we may assume that $e \notin C$ for any $C \in \mathcal{C}$.

In the sequel, we will prove progressively a series of structural properties of G. Before that, we need to define three additional graph families. Let KL_{2k-2} be the Klee ladder on $2k-2$ vertices with $k \geq 4$; let u_{2k-2} and v_{2k-2} be the two vertices contained in the two triangles, say $u_{2k-2}u_1u_2$ and $v_{2k-2}v_1v_2$, which are adjacent to each other. Moreover, we may assume that $KL_{2k-2} \setminus \{u_{2k-2}, v_{2k-2}\}$ contains two disjoint paths of length $k-3$, one from u_1 to v_1 and the other from u_2 to v_2.

We remove the vertices u_{2k-2} and v_{2k-2} and replace them by four vertices, say u_1', u_2', v_1', and v_2', adjacent to u_1, u_2, v_1, and v_2, respectively, and we add a 4-cycle passing through the four new vertices. In fact, we can see the last operation as adding a complete graph on 4 vertices and removing a perfect matching. Up to symmetry, only three outcomes are possible.

- A *ladder* L_{2k} is obtained if the edges $u_1'v_2'$ and $u_2'v_1'$ are missing.
- A *Möbius ladder* ML_{2k} is obtained if the edges $u_1'v_1'$ and $u_2'v_2'$ are missing.
- A *quasi-ladder* QL_{2k} is obtained if the edges $u_1'u_2'$ and $v_1'v_2'$ are missing.

Observe that the ladder L_{2k} is the graph of a k-sided prism. Observe that ladders and Möbius ladders are vertex-transitive.

KL_{2k-2} L_{2k}

ML_{2k-2} QL_{2k}

Figure 3: An illustration of a Klee ladder, a ladder, a Möbius ladder, and a quasi-ladder.

Claim 1. The graph G is not a ladder, a Möbius ladder, nor a quasi-ladder.
Proof of Claim 1. We proceed by contradiction. Suppose that $G \in \{L_{2n}, ML_{2n}, QL_{2n} : n \geq 4\}$, and let $\mathcal{C}$ be a collection of disjoint circuits in G. We prove that for every edge e there exists a perfect matching M_e containing e such that its complement is a Hamiltonian circuit, say C_e; moreover, there exists yet another perfect matching M_e' containing e. The first perfect matching can be used to prove Theorem **??** unless $\mathcal{C} = \{C_e\}$. If this is the case, then we can use M_e'.

In most of the cases, the second perfect matching M_e' can be obtained from M_e by the following operation: we find a 4-circuit consisting of the edges e_1, e_2, e_3, e_4 (in this cyclic order) avoiding e and containing exactly two edges from M_e, say e_1 and e_3. We then set $M_e' = (M_e \setminus \{e_1, e_3\}) \cup \{e_2, e_4\}$. In other words, M_e' is obtained as the symmetric difference of M_e and a suitable 4-circuit.

If G is a ladder or a Möbius ladder, then G is vertex-transitive, and there are only two edge orbits. It suffices to distinguish between e being an edge contained in two 4-circuits (vertical according to Figure **??**) or in a single one (horizontal or diagonal). An example of a pair of perfect matchings M_e and M'_e having the desired properties is depicted in Figure **??**.

Figure 4: An example of a Hamiltonian circuit C_e (drawn using double lines) avoiding a given edge e whose complement is a perfect matching M_e containing e, for both possible positions of the prescribed edge e in a ladder (top line) or a Möbius ladder (bottom line). A second perfect matching M'_e can be obtained by the symmetric difference with the grey 4-circuit.

Let $G = QL_{2k}$ for some $k \geq 4$. If e is an edge of the subgraph $P_2 \square P_{k-2}$ or an edge of the 4-circuit $u'_1 v'_1 u'_2 v'_2$ (see the definition of a quasi-ladder for the notation), then a pair of perfect matchings M_e and M'_e having the desired properties can be found in a same way as in the previous case, see Figure **??** for an illustration.

Otherwise, let $e = u_1 u'_1$ (for the remaining three edges the situation is symmetric). There is a unique Hamiltonian circuit C_e avoiding e and containing $u_2 u'_2$, see Figure **??** for an illustration. In this case, there is another perfect matching M'_e containing $\{u_1 u'_1, u_2 u'_2, v_1 v'_1, v_2 v'_2\}$ and all the vertical edges of the subgraph $P_2 \square P_{k-2}$ except for the first and the last one. ■

Figure 5: An example of a Hamiltonian circuit C_e avoiding a given edge e (drawn using double lines) whose complement is a perfect matching M_e containing e, for an edge e contained in the grid $P_2 \square P_{k-2}$ (top line) and in the 4-circuit outside the grid (bottom line) of a quasi-ladder. A second perfect matching M'_e can be obtained by the symmetric difference with the grey 4-circuit.

Figure 6: An example of a Hamiltonian circuit C_e avoiding a given edge e (drawn using double lines) whose complement is a perfect matching M_e containing e, for an edge e joining a vertex in the grid $P_2 \square P_{k-2}$ to a vertex of the 4-circuit outside the grid in a quasi-ladder (top line, two cases depending on the parity of the length of the grid). A second perfect matching M'_e (bottom).

Claim 2. The graph G does not have any cyclic 3-edge-cuts.
Proof of Claim 2. Suppose that G admits a cyclic 3-edge-cut $E(V', V'')$ with $E(V', V'') = \{f_1, f_2, f_3\} =: X$, where each $f_i = v'_i v''_i$, for some $v'_1, v'_2, v'_3 \in V'$ and $v''_1, v''_2, v''_3 \in V''$. Since G has no 2-edge-cuts, the vertices $v'_1, v'_2, v'_3, v''_1, v''_2, v''_3$ are all distinct.

Either there is no circuit in $\mathcal{C}$ intersecting X, or the cut X is intersected by a unique circuit C_X in $\mathcal{C}$. Without loss of generality, we shall assume that when C_X exists, $X \cap C_X = \{f_2, f_3\}$.

Let G' and G'' be the two graphs obtained from G after deleting X and joining the vertices v'_i to a new vertex v', and the vertices v''_i to a new vertex v''. For each $i \in [3]$, let $e'_i = v'_i v'$ and $e''_i = v''_i v''$.

Figure 7: The graphs G' and G'' when G admits a cyclic 3-edge-cut $\{f_1, f_2, f_3\}$.

Let

$$\mathcal{C}' = \begin{cases} \{C \in \mathcal{C} \setminus \{C_X\} : C \cap E(G') \neq \emptyset\} \cup \{(C_X \cap E(G')) \cup \{e'_2, e'_3\}\} & \text{if } C_X \text{ exists,} \\ \{C \in \mathcal{C} : C \cap E(G') \neq \emptyset\} & \text{otherwise.} \end{cases}$$

Similarly, let

$$\mathcal{C}'' = \begin{cases} \{C \in \mathcal{C} \setminus \{C_X\} : C \cap E(G'') \neq \emptyset\} \cup \{(C_X \cap E(G'')) \cup \{e''_2, e''_3\}\} & \text{if } C_X \text{ exists,} \\ \{C \in \mathcal{C} : C \cap E(G'') \neq \emptyset\} & \text{otherwise.} \end{cases}$$

It is not hard to see that $\mathcal{C}'$ ($\mathcal{C}''$) is a collection of disjoint circuits in G' (in G'', respectively). Every circuit $C \neq C_X$ in $\mathcal{C}$ corresponds to a circuit either in $\mathcal{C}'$ or in $\mathcal{C}''$. The circuit C_X (if it exists) corresponds to two circuits C'_X and C''_X in $\mathcal{C}'$ and $\mathcal{C}''$, respectively.

Case A. We first consider the case when G does not admit any triangles, and claim that G' (similarly G'') is not Klee. For, suppose that G' is Klee. Since G has no triangles, $|V(G')| \geq 6$, and so, by Lemma **??**, G' must admit two disjoint triangles. This is impossible since any triangle in G' must contain the vertex v'. Hence, when G does not admit any triangles, G' and G'' are both not Klee.

Without loss of generality, we can also assume that at least one of the endvertices of e corresponds to a vertex in V'. We consider two cases, depending on the existence of C_X.

Case A1. First, consider the case when C_X does not exist. When $e \in X$, say $e = f_1$, then, by minimality of G, there exists a perfect matching M' of G' (M'' of G'') containing e'_1 (e''_1), intersecting every circuit in $\mathcal{C}'$ (in $\mathcal{C}''$, respectively). Consequently, $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$ is a perfect matching of G containing $e = f_1$, intersecting every circuit in $\mathcal{C}$.

It remains to consider the case when $e \notin X$, and so the endvertices of e both correspond to vertices in G'. Once again, for simplicity, we shall refer to this edge as e. Let M' be a perfect matching of G' containing e intersecting every circuit in $\mathcal{C}'$. Without loss of generality, assume that $e'_1 \in M'$. Let M'' be a perfect matching of G'' containing e''_1 intersecting every circuit in $\mathcal{C}''$. Let $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$. This is a perfect matching of G containing e, intersecting every circuit in $\mathcal{C}$, a contradiction.

Case A2. Suppose that C_X exists. When $e \in X$, we have that $e = f_1$ by the choice of C_X, and so, by the minimality of G, there exists a perfect matching M' of G' (M'' of G'') containing e'_1 (e''_1), intersecting every circuit in $\mathcal{C}'$ (in $\mathcal{C}''$, respectively). Consequently, $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$ is a perfect matching of G containing $e = f_1$. Clearly, every circuit in $\mathcal{C} \setminus \{C_X\}$ is intersected by M. The circuit C_X must be intersected by M since C'_X (C''_X) contains an edge of M' (M''), not incident to v' (v'', respectively).

When $e \notin X$, the endvertices of e both correspond to vertices in G'. Once again, for simplicity, we shall refer to this edge as e. Let M' be a perfect matching of G' containing e intersecting every circuit in $\mathcal{C}'$. We have $e'_i \in M'$ for some $i \in [3]$. Let M'' be a perfect matching of G'' containing e''_i intersecting every circuit in $\mathcal{C}''$. Let $M = M' \cup M'' \cup \{f_i\} \setminus \{e'_i, e''_i\}$. This is a perfect matching of G containing e. As before, M intersects every circuit in $\mathcal{C}$ unless $i = 1$ and no edge of G' or G'' corresponding to an edge of C_X is in M' or M'', which is impossible since C'_X (C''_X) is a circuit in $\mathcal{C}'$ ($\mathcal{C}''$), so it contains an edge of M' (M''), not incident to v' (v'', respectively).

Case B. What remains to be considered is the case when G admits a triangle. Consequently, without loss of generality, we can assume that G'' is K_4. We note that in this case, G' cannot be Klee because otherwise G itself would be Klee. Thus, the inductive hypothesis can only be applied to G' but not to G''. As in Case A, we can assume that at least one of the endvertices of e corresponds to a vertex in V', since if the endvertices of e both belong to V'', say $e = v''_i v''_j$, a perfect matching of G contains e if and only if it contains f_k, where $\{i, j, k\} = [3]$. We proceed as in Case A and note that the perfect matching M' containing (the edge corresponding to) e intersecting every circuit in $\mathcal{C}'$ obtained after applying the inductive hypothesis to G' can be easily extended to a perfect matching M of G containing e. What remains to show is that M intersects every circuit in $\mathcal{C}$. The only circuit possibly not intersected by M is C_X, if it exists. However, this can only happen if $i = 1$, and, if this is the case, then, in particular, C'_X is a circuit in G' and so contains an edge of M' not incident to v'. This implies that C_X contains the corresponding edge of M in G, a contradiction. ■

Claim 3. The graph G does not have any cyclic 4-edge-cuts.

Proof of Claim 3. Suppose first that, in particular, G has a 4-circuit $C = (v''_1, v''_2, v''_3, v''_4)$. Let v'_1, v'_2, v'_3, v'_4 be the vertices in $G - C$ respectively adjacent to $v''_1, v''_2, v''_3, v''_4$, let $f_i = v'_i v''_i$ for $i \in \{1, 2, 3, 4\}$ and let $X = \{f_1, f_2, f_3, f_4\}$. The vertices v'_i are pairwise distinct since G does not have any cyclic 3-cuts.

Let $\{i, j, k\} = \{2, 3, 4\}$. We denote by G_{1i} the graph obtained after adding two new vertices x and y to $G - C$, such that:

- x and y are adjacent;
- v'_1 and v'_i are adjacent to x; and
- v'_j and v'_k are adjacent to y.

It is known that the graph G_{1i} is 3-connected whenever G is cyclically 4-edge-connected [**?**]. We claim that G_{1i} is not Klee, for any $i \in \{2, 3, 4\}$. For, suppose not. Since G does not admit any cyclic 3-cuts, by Lemma **??**, the only two possible triangles in G_{1i} are (v'_1, v'_i, x) and (v'_j, v'_k, y). Moreover, since x is adjacent to y, by Lemma **??**, G_{1i} is a Klee ladder. For every $i \in \{2, 3, 4\}$, this implies that G is a graph isomorphic to a ladder, a Möbius ladder, or a quasi-ladder — this is a contradiction.

We proceed by considering whether e belongs to $E(C)$, X, or $E(G - C)$.

Case A. When $e \in E(C)$, then for every $i \in \{2, 3, 4\}$, every perfect matching of G_{1i} containing $e' = xy$ extends to a perfect matching of G containing e. The cut X contains an even number of edges belonging to some circuit in $\mathcal{C}$. In particular, $E(C)$ can contain at most three circuit edges belonging to some circuit in $\mathcal{C}$, and so $C \notin \mathcal{C}$.

If X contains no circuit edges, then we can set $\mathcal{C}' = \mathcal{C}$ and apply induction on any $G' = G_{1i}$ to find a perfect matching M' containing e' intersecting every circuit in $\mathcal{C}'$, which readily extends to a perfect matching M containing e intersecting every circuit in $\mathcal{C}$. Note that the circuit C, in particular, is always intersected by at least one edge of M.

If there is a single circuit intersecting X exactly twice, say C_X passing through the edges f_j and f_k, then we apply induction on the graph $G' = G_{1i}$ where $\{1, i\} = \{j, k\}$ (if $1 \in \{j, k\}$), or $|\{1, i, j, k\}| = 4$ (otherwise). The circuit C'_X in G' corresponding to C_X contains two edges both incident to either x or y. Hence, if a perfect matching M' containing $e' = xy$ intersects every circuit in $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C'_X\}$, then it extends to a perfect matching containing e intersecting every circuit in $\mathcal{C}$, since C'_X contains an edge in M' not incident to x (nor y).

If there are two distinct circuits each intersecting X twice, say C_X passing through the edges f_1 and f_2, and D_X passing through the edges f_3 and f_4, or if there is a single circuit intersecting X four times, say C_X passing through the vertices v'_1, v''_1, v''_2, v'_2, and also v'_3, v''_3, v''_4, v'_4, then we can apply induction on the graph G_{12} with $e' = xy$ just like in the previous case.

Case B. When $e \in X$, say $e = f_1$, then every perfect matching of $G' = G_{13}$ containing $e' = xu_1$ extends to a perfect matching of G containing e in a unique way. If there is no circuit in $\mathcal{C}$ intersecting X, then we can set $\mathcal{C}' = \mathcal{C} \setminus \{C\}$ and apply induction directly. If there is a circuit in $\mathcal{C}$ intersecting X, say C_X, then $|C_X \cap X| = 2$. The corresponding circuit C'_X in G' is well-defined: it always contains y and eventually also x (when $C_X \cap X \neq \{f_2, f_4\}$). A perfect matching M' in G' containing e' intersecting every circuit in $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C'_X\}$ intersects C'_X at a cut-edge incident to y or an edge of $G - C$. In both cases, the corresponding perfect matching M in G containing e intersects every circuit in $\mathcal{C}$, since M intersects C_X at an edge in X or an edge of $G - C$.

Case C. It remains to consider the case when $e \in E(G - C)$. Let $G' = G_{13}$ and let e' be the edge of G' corresponding to e in G. Every perfect matching M' of G' containing e' and not containing xy extends to a perfect matching M of G containing e in a unique way; every perfect matching M' of G' containing e' and xy extends to a perfect matching M of G in two distinct ways, whose symmetric difference is the 4-circuit C. In all the cases, we obtain a perfect matching M of G containing at least one edge of C.

If X contains no edges belonging to any circuit in $\mathcal{C}$, then we can set $\mathcal{C}' = \mathcal{C} \setminus \{C\}$ and apply induction directly. The circuit C in particular (if it is in $\mathcal{C}$) is always intersected by at least one edge of M.

If there is a single circuit intersecting X exactly twice, say C_X, passing through the edge f_1 and f_i for some $i \in \{2, 3, 4\}$, then the corresponding circuit C'_X in G' is well-defined: it always contains x and eventually also y (when $C_X \cap X \neq \{f_1, f_3\}$). We can set $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C'_X\}$ and apply induction. If M' contains an edge of C'_X not incident to x nor y, then M contains an edge of C_X not incident to any vertex of C. If M' contains the edge xy, then amongst the two possible extensions of M' into M we can always choose one that contains at least one edge of C_X. If M' contains an edge incident to x or to y distinct from xy, then M contains the corresponding edge in X. In all the cases, it is possible to extend a perfect matching M' of G' containing e' and intersecting every circuit in $\mathcal{C}'$ into a perfect matching M of G containing e and intersecting every circuit in $\mathcal{C}$.

If there are two distinct circuits each intersecting X twice, say C_X passing through the edges f_1 and f_2 and D_X passing through the edges f_3 and f_4, then we apply induction on G' with $\mathcal{C}' = \mathcal{C} \setminus \{C_X, D_X\}$. If the perfect matching M' containing e' and intersecting every circuit in $\mathcal{C}'$ obtained by induction also contains xy, then we can choose M to contain both $v_1''v_2''$ and $v_3''v_4''$, and so it intersects both C_X and D_X as well. If M' does not contain xy, then $|M \cap \{f_1, f_2, f_3, f_4\}| = 2$. If M contains exactly one of f_1 and f_2 then it also contains one of f_3 and f_4, and so M intersects both C_X and D_X. If $\{f_1, f_2\} \subset M$, then $v_3''v_4'' \in M$; similarly, if $\{f_3, f_4\} \subset M$, then $v_1''v_2'' \in M$. In all the cases M intersects both C_X and D_X, as desired.

If there is a single circuit intersecting X four times, say C_X passing through $v_1'v_1''v_2''v_2'$ and also $v_3'v_3''v_4''v_4'$, then we can apply induction on the graph G' with $\mathcal{C}' = \mathcal{C} \setminus \{C_X\}$ just like in the previous case.

From this point on we may assume that G does not contain any 4-circuits. In particular, for every cyclic 4-edge-cut $E(V', V'')$ both sides have at least six vertices. Suppose that G admits a cyclic 4-edge-cut $E(V', V'')$ with $E(V', V'') = \{f_1, f_2, f_3, f_4\} =: X$, where each $f_i = v_i'v_i''$, for some $v_1', v_2', v_3', v_4' \in V'$ and $v_1'', v_2'', v_3'', v_4'' \in V''$. Since G has no 3-edge-cuts, the vertices $v_1', v_2', v_3', v_4', v_1'', v_2'', v_3'', v_4''$ are all distinct.

We define graphs G_{1i}' and G_{1i}'' for $i \in \{2, 3, 4\}$ analogously as in the previous part. We denote by x' and y' (x'' and y'') the two new vertices in G_{1i}' (in G_{1i}''), and by e_1', e_2', e_3', e_4' (e_1'', e_2'', e_3'', e_4'') the edges of G_{1i}' (of G_{1i}'', respectively) corresponding to f_1, f_2, f_3, f_4, respectively, for $i \in \{2, 3, 4\}$. These graphs are all 3-connected [**?**]. None of these graphs can be a Klee-graph: if this was the case, it would have to be a Klee ladder on at least eight vertices, but there are no 4-circuits at all in G, so this is impossible.

Consider first the case when $e \in X$, say $e = f_1$. If there is a circuit C_X in $\mathcal{C}$ intersecting X, then $e \notin C_X$ and $|C_X \cap X| = 2$. We may assume that $C_X \cap X = \{f_2, f_3\}$. We consider all the three graphs G_{12}', G_{13}', and G_{14}' (and all the three graphs G_{12}'', G_{13}'', and G_{14}'') at the same time. The circuit C_X (if it exists) corresponds to a circuit C_X' (C_X'') in each of them in a natural way, covering either one or two vertices amongst x' and y' (x'' and y'', respectively). If C_X does not exist, we shall proceed in the same manner, but letting C_X, C_X', and C_X'' be equal to $\emptyset$. We apply induction with $e' = e_1'$ ($e'' = e_1''$) and $\mathcal{C}' = ((\mathcal{C} \setminus \{C_X\}) \cap E(G_{1i}')) \cup \{C_X'\}$ ($\mathcal{C}'' = ((\mathcal{C} \setminus \{C_X\}) \cap E(G_{1i}'')) \cup \{C_X''\}$, respectively). Let M_i' (M_i'') be a perfect matching in G_{1i}' (G_{1i}'') containing e' (e'') intersecting every circuit in $\mathcal{C}'$ (in $\mathcal{C}''$, respectively). Every perfect matching amongst M_2', M_3', and M_4' contains exactly one edge e_k' corresponding to a cut edge f_k for some $k \in \{2, 3, 4\}$ (besides the edge e' corresponding to f_1) and the three values of k cannot all be the same for the three perfect matchings. The same thing holds for the other three perfect matchings M_2'', M_3'', and M_4''. Therefore, for some $k \in \{2, 3, 4\}$ there exist two perfect matchings M_i' and M_j'' containing the edge e_k' and e_k'', respectively. We can combine them together into a perfect matching M containing e and f_k, intersecting every circuit in $\mathcal{C}$. In particular, if C_X exists, then it can only be avoided by M if $k = 4$, but then M_i' (M_j'') cannot contain any edge of C_X' (C_X'') incident to x' or to y' (to x'' or to y''), so it intersects C_X' inside $G[V']$ (C_X'' inside $G[V'']$, respectively). Consequently, M intersects C_X inside $G[V']$ and

$G[V'']$.

Consider next the case where $e \notin X$. We may assume that $e \in G[V']$. Let $\mathcal{C}_X$ be the set of circuits in $\mathcal{C}$ intersecting X. We have $|\mathcal{C}_X| \leq 2$, and even if there is a single circuit in $\mathcal{C}_X$, it may contain all four edges of X. Let $\mathcal{C}'_0$ ($\mathcal{C}''_0$) be the set of circuits from $\mathcal{C}$ within $G[V']$ ($G[V'']$, respectively). Given $G' = G'_{1i}$ ($G'' = G''_{1i}$) for some arbitrary $i \in \{2, 3, 4\}$, let $\mathcal{C}'_X$ ($\mathcal{C}''_X$) be the set of circuits obtained from the subpaths of circuits in $\mathcal{C}_X$ contained in $G[V']$ (in $G[V'']$) by adding the necessary edges from $\{e'_1, e'_2, e'_3, e'_4\}$ (from $\{e''_1, e''_2, e''_3, e''_4\}$) and eventually also the edge $x'y'$ ($x''y''$, respectively), if needed. Observe that $|\mathcal{C}'_X| = 2$ ($|\mathcal{C}''_X| = 2$) is possible when $|\mathcal{C}_X| = 1$, and vice-versa. Finally, let $\mathcal{C}' = \mathcal{C}'_0 \cup \mathcal{C}'_X$ and $\mathcal{C}'' = \mathcal{C}''_0 \cup \mathcal{C}''_X$.

Let e' be the edge in G' corresponding to e in G. By induction, we obtain a perfect matching M' containing e' intersecting every circuit in $\mathcal{C}'$.

Consider first the case when $x'y' \in M'$. We apply induction to obtain a perfect matching M'' of $G'' = G_{1i}$, for any $i \in \{2, 3, 4\}$, containing $x''y''$ intersecting every circuit in $\mathcal{C}''$. Then, $M = (M' \setminus \{x'y'\}) \cup (M'' \setminus \{x''y''\})$ is a perfect matching of G containing e. It is easy to check that M intersects every circuit in $\mathcal{C}'_0$ and in $\mathcal{C}''_0$; it remains to certify that M intersects all the circuits in $\mathcal{C}_X$. If $|\mathcal{C}_X| \leq 1$, then we choose G''_{1i} in such a way that $x''y''$ does not belong to any circuit in $\mathcal{C}''_X$, and so M'' contains at least one edge (not incident to x'' or y'') of every circuit in $\mathcal{C}''_X$, and so the circuit in $\mathcal{C}_X$ will contain at least one edge from M. If $|\mathcal{C}_X| = 2$, then it suffices to choose G''_{1i} in such a way that $\mathcal{C}''_X$ contains two distinct circuits (avoiding $x''y''$), and then each of them will contain at least one edge (not incident to x'' or y'') from M'', and thus each circuit in $\mathcal{C}_X$ will contain at least one edge from M, as desired.

It remains to consider the case when for every choice of $G' = G'_{1i}$, a perfect matching M'_i of G' containing e' and intersecting every circuit in $\mathcal{C}'$ never contains the edge $x'y'$. Without loss of generality, we may assume that for G'_{12} the perfect matching M'_2 contains the edges e'_1 and e'_3. We then consider G'_{13}. Again, without loss of generality, the perfect matching M'_3 contains the edges e'_1 and e'_2. Finally, we apply induction on $G'' = G''_{14}$ with $e'' = e''_1$. Every perfect matching M'' of G'' containing e'' contains either e''_2 or e''_3, so it can be combined with either M'_2 or M'_3 to give a perfect matching M of G containing e. We may assume that $e''_2 \in M''$. It is easy to check that such a perfect matching M intersects all the circuits in $\mathcal{C}'_0$ and in $\mathcal{C}''_0$; it remains to certify that M intersects all the circuits in $\mathcal{C}_X$. The only circuit from $\mathcal{C}_X$ potentially not intersected by M is the one containing the edges f_3 and f_4, say C_X. However, the corresponding circuits C'_X and C''_X in $\mathcal{C}'_X$ and $\mathcal{C}''_X$ (there is exactly one on each side) respectively contain an edge of M'_3 (not incident to x' or y') and an edge of M'' (not incident to x'' or y''). Therefore, M intersects C_X at least twice, which is more than what is desired. ∎

From this point on we may assume that G is cyclically 5-edge-connected. We now consider the edges at distance 2 from e (distance measured as the length of a shortest path joining corresponding vertices in the line graph of G).

Claim 4. Let f be an edge at distance 2 from e. Then, $f \notin C$ for any $C \in \mathcal{C}$.
Proof of Claim 4. We will use a procedure that transforms a cubic graph G into a cubic graph G' smaller than G, such that every perfect matching of G' containing a certain edge

can be extended into a perfect matching of G containing the corresponding edge.

This operation was already used by Voorhoeve [**?**] to study perfect matchings in bipartite cubic graphs and it is one of the main tools used for counting perfect matchings in general in [**?**]. This technique is also used by the three authors in [**?**] to prove Theorem **??**.

Let $f = uv$, let the neighbours of u distinct from v be α and γ, and let the neighbours of v distinct from u be β and δ. In particular, since G is cyclically 5-edge-connected, these four vertices are all distinct and non-adjacent. Without loss of generality, we may assume that α is an endvertex of e.

Figure 8: The vertices $\alpha, \beta, \gamma, \delta$ and an $(\alpha\beta : \gamma\delta)_{uv}$-reduction.

As shown in Figure **??**, we obtain a smaller graph by deleting the endvertices of f (together with all edges incident to them) and adding the edges $\alpha\beta$ and $\gamma\delta$. Let this resulting graph be G'. We shall say that G' is obtained after an $(\alpha\beta : \gamma\delta)_{uv}$-reduction. It is well-known that when applying this operation, the cyclic edge-connectivity of a cubic graph can drop by at most 2. Since G is cyclically 5-edge-connected, G' is cyclically 3-edge-connected.

Let the edge in G' corresponding to e, and the vertices in G' corresponding to $\alpha, \beta, \gamma, \delta$ be denoted by the same name. We recall that any perfect matching of G' which contains e can be extended to a perfect matching of G containing the edge e (see also Figure **??**). In fact, let M' be a perfect matching of G' containing e. This is extended to a perfect matching M of G containing e as follows:

$$M = \begin{cases} M' \cup \{u\gamma, v\delta\} \setminus \{\gamma\delta\} & \text{if } \gamma\delta \in M', \\ M' \cup \{f\} & \text{otherwise.} \end{cases}$$

Figure 9: Extending a perfect matching of G' containing e to a perfect matching of G containing e. Dotted lines represent edges in M or M'.

Suppose that for some edge f at distance 2 from e, f is in some circuit C_f in $\mathcal{C}$. This means that exactly one of $u\alpha$ and $u\gamma$, and exactly one of $v\beta$ and $v\delta$ belong to C_f. Without loss of generality, we may assume that $u\alpha \in C_f$ if and only if $v\beta \in C_f$ (otherwise, we rename β and δ). Let G' be the graph obtained from G after an $(\alpha\beta : \gamma\delta)_{uv}$-reduction. Let C'_f be the circuit in G' corresponding to C_f in G obtained by replacing the 3-edge path passing through u and v by a single edge. Since G is of girth 5, C_f is a circuit of length at least 3. Let $\mathcal{C}' = (\mathcal{C} \setminus \{C_f\}) \cup \{C'_f\}$ be the collection of disjoint circuits of G' obtained by this reduction. This is portrayed in Figure **??**.

Figure 10: If $f \in C_f$, then we apply induction on G' — the graph obtained from G after an $(\alpha\beta : \gamma\delta)_{uv}$-reduction. Dashed lines represent edges outside $\mathcal{C}$ or $\mathcal{C}'$, respectively.

Let us first assume that G' is not a Klee-graph. Since G' is cyclically 3-edge-connected and its order is strictly less than G, it is not a counterexample. Let M' be a perfect matching of G' containing e intersecting all the circuits in $\mathcal{C}'$. We extend this perfect matching to a perfect matching M of G containing e as described above (see Figure **??**), and claim that it intersects all the circuits in $\mathcal{C}$. Every circuit $C' \neq C'_f$ in $\mathcal{C}'$ is hit by an edge of M' in G', and so the corresponding circuit C is hit by the corresponding edge of M in G. The circuit C'_f is hit by an edge M' in G', and so the corresponding circuit C_f is hit by the corresponding edge in G, unless $\gamma\delta \in E(C_f)$ and the hitting edge is $\gamma\delta$, but then C_f is hit by both edges γu and $v\delta$. Observe that M' cannot contain $\alpha\beta$ because $e \in M'$.

Therefore, G' must be Klee. Since G' is obtained after an $(\alpha\beta : \gamma\delta)_{uv}$-reduction, and G is cyclically 5-edge-connected, by Lemma **??**, the graph G' admits exactly two (disjoint) triangles T_ℓ and T_r such that $V(T_\ell) = \{v_\ell, \alpha, \beta\}$ and $V(T_r) = \{v_r, \gamma, \delta\}$, for some v_ℓ and v_r in G'. Let a, b, c, d be the vertices in $G' - \{v_\ell, v_r, \alpha, \beta, \gamma, \delta\}$ which are adjacent to $\alpha, \beta, \gamma, \delta$, respectively (see Figure **??**). Furthermore, since G is cyclically 5-edge-connected, by Lemma **??** the edge $\alpha\beta$ ($\gamma\delta$) is the only edge in T_ℓ (in T_r) which lies on a 4-circuit. Therefore, (α, β, b, a) and (γ, δ, d, c) are 4-circuits in G'. Next we show that a, b, c, d are pairwise distinct. Clearly, $a \neq b$, and $c \neq d$. Moreover, $a \neq c$, and $b \neq d$, otherwise G would admit a 4-circuit. What remains to show is that $a \neq d$, and $b \neq c$. We first note that since G is cubic, and $a = d$ if and only if $b = c$. Indeed, if $a = d$ and $b \neq c$, then, a is adjacent to α, b, c, δ, a contradiction. Moreover, since G is cyclically 4-edge-connected, if $a = d$, G would be the Petersen graph. However, it is an easy exercise to check that the Petersen graph is not a counterexample.

Figure 11: If for some graph G, the graph G' obtained after an $(\alpha\beta : \gamma\delta)_{uv}$-reduction is a Klee graph, then the graph G'' obtained after an $(\alpha\delta : \beta\gamma)_{uv}$-reduction is not.

Hence, a, b, c, d are four distinct vertices. Consequently, if we apply an $(\alpha\delta : \beta\gamma)_{uv}$-reduction to G we can be sure that $\alpha\delta$ and $\beta\gamma$ do not lie on a triangle in the resulting graph G''. In particular, G'' is not Klee. Let $\mathcal{C}'' = \mathcal{C} \setminus \{C_f\}$. By the inductive hypothesis, G'' admits a perfect matching M'' containing e intersecting every circuit in $\mathcal{C}''$. By extending the perfect matching M'' to a perfect matching M of G containing e as described above (see Figure **??**), we can deduce that M intersects every circuit in $\mathcal{C}$, because, in particular, M contains exactly one edge from $E(C_f) \cap \{u\gamma, uv, v\beta\}$. ■

From this point on we may assume that no edge f at distance 2 from e is contained in a circuit in $\mathcal{C}$. As a consequence, we have that no edge at distance at most 2 from e is contained in a circuit in $\mathcal{C}$.

Claim 5. Every vertex at distance 2 from e is traversed by a circuit in $\mathcal{C}$.
Proof of Claim 5. Once again, let us consider an edge $f = uv$ at distance 2 from e, with vertices denoted α, β, γ, δ as above. In particular, there can be a circuit in $\mathcal{C}$ passing through an endvertex of f only if it is passes through the edges βv and $v\delta$.

Suppose that there is no such circuit. As we have seen in the above claim, at least one of the the resulting graphs obtained by an $(\alpha\beta : \gamma\delta)_{uv}$-reduction or an $(\alpha\delta : \gamma\beta)_{uv}$-reduction is not Klee, and so, without loss of generality, we can assume that the graph G' obtained after an $(\alpha\beta : \gamma\delta)_{uv}$-reduction is not Klee. In G', there is a perfect matching M' containing e and intersecting every circuit in $\mathcal{C}' = \mathcal{C}$. It is easy to see that the perfect matching M (of G) containing e obtained as an extension of M' still intersects all the circuits in $\mathcal{C}$, a contradiction. ■

As a consequence of Claim 5, we have the following.

Claim 6. The edge e does not belong to a 5-circuit.
Proof of Claim 6. Suppose that e belongs to a 5-circuit $C = (t_1, t_2, t_3, t_4, t_5)$. Let the vertices in $G - V(C)$ which are adjacent to some vertex in C be v_1, v_2, v_3, v_4, v_5, such that v_i is adjacent to t_i and $e = t_1t_2$. Since G is cyclically 5-edge-connected, the v_is are pairwise distinct. Moreover, by Claim 4, no edge in C can be contained in a circuit of $\mathcal{C}$, but by Claim 5, the vertex t_4 must be traversed by a circuit of $\mathcal{C}$, which is clearly impossible. ■

We also show that e cannot be at distance 2 from a 5-circuit.

Claim 7. Edges at distance 2 from e do not belong to a 5-circuit.
Proof of Claim 7. Suppose the above assertion is false and let $C = (t_1, t_2, t_3, t_4, t_5)$ be such a 5-circuit, with t_1 being an endvertex of an edge adjacent to e. We obtain a smaller graph G' by deleting the edge t_3t_4, and smooth the vertices t_3 and t_4. Let the resulting graph be denoted by G'. It can be easily seen that G' is cyclically 3-edge-connected and that it does not admit any triangles (and therefore note Klee), because otherwise, G would contain 4-circuits. For each $i \in [5]$, let the vertex in $V(G) - V(C)$ adjacent to t_i be denoted by t_i'. We proceed by first showing that a perfect matching M' of G' containing e can be extended to a perfect matching M of G containing e. Without loss of generality, assume that $t_1t_2 \in M'$. We extend this to a perfect matching of G as follows:

$$M = \begin{cases} M' \cup \{t_3t_4\} & \text{if } t_5t_5' \in M', \\ M' \cup \{t_1t_5, t_2t_3, t_4t_4'\} \setminus \{t_1t_2, t_4't_5\} & \text{otherwise.} \end{cases}$$

Next, since in G, no edge at distance at most 2 from e belongs to a circuit in $\mathcal{C}$, and every vertex at distance 2 from an endvertex of e is traversed by one, we have that t_2t_3 and t_4t_5 belong to some circuit in $\mathcal{C}$ (possibly the same). We consider two cases depending on whether the edge t_3t_4 is in a circuit edge or not.

(i) When t_3t_4 is a circuit edge, then the vertices t_2, t_3, t_4, t_5 are consecutive vertices on some circuit C_X in $\mathcal{C}$. In this case, we let $E(C_X') = E(C_X) \cup \{t_1t_2, t_1t_5\} \setminus \{t_2t_3, t_3t_4, t_4t_5\}$ and $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C_X'\}$ to be a collection of disjoint circuits in G'. By the inductive hypothesis there exists a perfect matching M' containing e intersecting every circuit in $\mathcal{C}'$. The perfect matching M of G containing e obtained from M' as explained above clearly intersects every circuit in $\mathcal{C}$ (it contains either t_2t_3 or t_3t_4). This contradicts our initial assumption and so we must have the following case.

(ii) When t_3t_4 is not a circuit edge, we let C_X and C_Y (with X not necessarily distinct from Y) to be the circuits containing the edges t_2t_3 and t_4t_5, respectively. The corresponding circuits C_X' and C_Y' in G' are obtained by smoothing out the vertices

t_3 and t_4. We then set $\mathcal{C}' = (\mathcal{C}\setminus\{C_X, C_Y\})\cup\{C'_X, C'_Y\}$. Note that the edges $t_2t'_2$ and $t_5t'_5$ belong to distinct circuits in $\mathcal{C}'$ if and only if they belong to distinct circuits in $\mathcal{C}$. By the inductive hypothesis, there exists a perfect matching M' of G' containing e intersecting every circuit in $\mathcal{C}'$. Without loss of generality, assume that $t_1t_2 \in M'$. The perfect matching M' contains either $t_5t'_5$ or $t_5t'_4$, and consequently, so does the perfect matching M obtained from M' as explained above. This implies that C_Y is intersected by M. If $C'_X \neq C'_Y$, then M' contains an edge of C'_X not incident to t_2 in G', and so M contains the corresponding edge of C_X in G. Altogether, the perfect matching M obtained from M' as shown above intersects every circuit in $\mathcal{C}$. This is again a contradiction to our initial assumption that G is a counterexample — thus proving our claim. ■

Let's get back to analysing an edge $f = uv$ at distance 2 from e. We cannot use the reduction portrayed in Figure **??** as we do not have a guarantee that we can obtain a perfect matching M intersecting the circuit in $\mathcal{C}$ containing the edges $v\beta$ and $v\delta$, which we shall denote by C_v. Since G is cyclically 5-edge-connected, this latter circuit is of length at least 5. Let δ, v, β, y, z be consecutive and distinct vertices on this circuit (see Figure **??**). Moreover, let w and x be the vertices in G respectively adjacent to β and y, such that $w\beta, xy \notin E(C_v)$. We proceed by applying an $(\alpha\beta : \gamma\delta)_{uv}$-reduction followed by an $(\alpha x : wz)_{\beta y}$-reduction as portrayed in Figure **??**.

Figure 12: An $(\alpha\beta : \gamma\delta)_{uv}$-reduction followed by an $(\alpha x : wz)_{\beta y}$-reduction. Dashed edges represent edges outside $\mathcal{C}$ or $\mathcal{C}'$.

Let the resulting graph after these two reductions be denoted by G', and let $\mathcal{C}' = \mathcal{C} \setminus \{C_v\}$. Since G' is obtained by applying twice the reduction at an edge at distance 2 from e, a perfect matching of G' containing e can always be extended to a perfect matching of G containing e (recall that βy cannot be adjacent to e, since $\beta y \in C_v$). Moreover, any such matching contains either the edge βy or the edge yz, and so it also contains at least one edge of the circuit C_v. Therefore, as long as G' is cyclically 3-edge-connected and not Klee, by minimality of G, there exists a perfect matching M' of G' containing e intersecting every circuit in $\mathcal{C}'$, which extends to a perfect matching M of G containing e intersecting every circuit in $\mathcal{C}$. This contradicts our initial assumption that G is a counterexample.

Therefore, G' is either Klee or admits a (cyclic) edge-cut of size at most 2 (more details after proof of Claim 8).

Claim 8. The graph G' is not Klee.

Proof of Claim 8. Suppose that G' is Klee. The edge $\gamma\delta$ cannot be on a triangle otherwise the edges uv and $u\gamma$ at distance 2 from e belong to a 5-circuit, contradicting Claim 7. Since G is cyclically 5-edge-connected, if G' is Klee then we must have that αx and wz each lie on a triangle (see Lemma **??**). Therefore, in particular, if G' is Klee, α and x must have a common neighbour (in both G' and G, so it is not u). This common neighbour cannot be u', the neighbour of α not incident to e and distinct from u, either, since x would then be a vertex at distance 2 from the endvertex α of e (via u') and so, by Claim 5, it would be traversed by a circuit in $\mathcal{C}$. However, the edges xu' and xy are not in any circuit in $\mathcal{C}$, a contradiction. Therefore, the common neighbour of x and α is α', the other endvertex of e. By Lemma **??**, one edge of the triangle (x, α, α') lies on a 4-circuit in G', which is not present in G. First, consider the case when exactly one of $\alpha' x$ and $\alpha'\alpha$ lie on a 4-circuit, say (α', s, t, x) or (α', s, t, α) accordingly. Since the edges $\alpha' s, xt, \alpha' x, \alpha'\alpha, \alpha t$ all belong to G, s and t cannot be adjacent in G, and so $\{s, t\}$ is equal to $\{w, z\}$ or $\{\gamma, \delta\}$. If $\{s, t\} = \{\gamma, \delta\}$, then we either have that $\alpha'\gamma \in E(G)$, implying that $(\alpha', \gamma, u, \alpha)$ is a 4-circuit in G, or that $\alpha'\delta \in E(G)$, implying that $(\alpha', \alpha, u, v, \delta)$ is a 5-circuit in G containing e, both a contradiction. Hence, $\{s, t\} = \{w, z\}$. Since, G is cyclically 5-edge-connected, x cannot be adjacent to z nor w, and so, we must have that the edge lying on the 4-circuit with w and z is $\alpha'\alpha$. However, this is impossible since z cannot be adjacent to an endvertex of e. Consequently, we must have that the edge of the triangle (x, α, α') lying on a 4-circuit in G' is αx. In this case, st cannot be an edge in G, otherwise $(\alpha', \alpha, s, t, x)$ would be a 5-circuit in G containing e, contradicting Claim 6. Thus, $\{s, t\}$ is equal to $\{w, z\}$ or $\{\gamma, \delta\}$ once again. As before, x cannot be adjacent to z or w, implying that α being adjacent to γ or δ, respectively giving rise to (α, u, γ) or (α, u, v, δ) in G, a contradiction. Therefore, G' is not Klee. ∎

Consequently, G' must admit some (cyclic) edge-cut of size at most 2. Whenever G is cyclically 4-edge-connected, by the analysis done at the end of the main result in [**?**], we know the graph G' is bridgeless, so if G' is not (cyclically) 3-edge-connected, it admits a 2-edge-cut. We next show that this cannot be the case, that is, if G' admits a 2-edge-cut, then G is not a counterexample to our statement.

Claim 9. G' is cyclically 3-edge-connected, unless α is adjacent to x in G.
Proof of Claim 9. Suppose that G' admits a 2-edge-cut $X_2 = \{g_1, g_2\}$. Let $\Omega_1 = \{\alpha, \gamma, \delta, w, x, z\}$ and let $\Omega_2 = \{u, v, \beta, y\}$.

We label the vertices of $G' \setminus X_2$ with labels A and B depending in which connected component of $G' \setminus X_2$ they belong to. Consequently, G' has exactly two edges which are not monochromatic: g_1 and g_2. We consider different cases depending on the number of vertices in Ω_1 labelled A in G', and show that, in each case, a 2-edge-cut in G' would imply that G is not cyclically 5-edge-connected or not a counterexample to our statement. Without loss of generality, we shall assume that the number of vertices in Ω_1 labelled A is at least the number of vertices in Ω_1 labelled B. We consider four cases.

(B0) All the vertices in Ω_1 are labelled A in G'.

First, we extend this labelling of $V(G')$ to a partial labelling of $V(G)$ by giving to the vertices in $V(G) - \Omega_2$ the same label they had in G'. We then give label A to all the vertices in Ω_2. However, this means that G has exactly two edges, corresponding to the edges in X_2, which are not monochromatic, a contradiction, since G does not admit any 2-edge-cuts.

(B1) Exactly 5 vertices in Ω_1 are labelled A in G'.

This means that exactly one of the edges in $\{\alpha x, wz, \gamma\delta\}$ belongs to X_2, say g_1, without loss of generality. Once again, we extend this labelling to a partial labelling of G, and then give label A to all the vertices in Ω_2. However, this means that

G has an edge which has exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A, which together with the edge g_2 gives a 2-edge-cut in G, a contradiction once again.

(B2) Exactly 4 vertices in Ω_1 are labelled A in G'.

We consider two cases depending on whether there is one or three monochromatic edges in $\{\alpha x, wz, \gamma\delta\}$. First, consider the case when $\{\alpha x, wz, \gamma\delta\}$ has exactly one monochromatic edge, meaning that $X_2 \subset \{\alpha x, wz, \gamma\delta\}$. As in the previous cases, we extend this labelling to a partial labelling of $V(G)$, and then give label A to all the vertices in Ω_2. However, this means that G has exactly two edges each having exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A, meaning that G admits a 2-edge-cut, a contradiction.

Therefore the edges $\{\alpha x, wz, \gamma\delta\}$ are all monochromatic: two edges with all their endvertices coloured A, and one edge with its endvertices coloured B. We extend this labelling to a partial labelling of $V(G)$, and then give label A to all the vertices in Ω_2. This gives rise to exactly two edges each having exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A. These two edges together with the two edges in X_2 form a 4-edge-cut X_4 of G. Since the latter is cyclically 5-edge-connected this 4-edge-cut is not cyclic — it separates two adjacent vertices from the rest of the graph. Since $w \neq z$ and $\gamma \neq \delta$ (otherwise there would be a 3-circuit in G) and also $\alpha \neq x$ (otherwise C_v would contain edges at distance 1 from e), these two adjacent vertices in G are endvertices of exactly one of αx, wz, or $\gamma\delta$ in G', and the 2-edge-cut in G' separates a 2-circuit from the rest of the graph. However, G can contain neither the edge wz nor $\gamma\delta$, since G has no 4-circuits. Thus, we must have that α is adjacent to x in G.

(B3) Exactly 3 vertices in Ω_1 are labelled A in G'.

Since G' has exactly two edges which are not monochromatic, there is exactly one edge in $\{\alpha x, wz, \gamma\delta\}$ which is not monochromatic. The latter corresponds to one of the edges in X_2, say g_1, without loss of generality. As before, we extend this labelling to a partial labelling of $V(G)$, and then give label A to all the vertices in Ω_2. This gives rise to exactly three edges each having exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A, which together with the edge g_2 from X_2 form a 4-edge-cut X_4 of G. As in the previous case, X_4 separates two adjacent vertices from the rest of the graph — G has exactly two vertices labelled B. As in the previous case, the endvertices of the monochromatic edge belonging to $\{\alpha x, wz, \gamma\delta\}$ (in G') which are labelled B must be either equal or adjacent in G, which is only possible if α is adjacent to x in G. ■

Therefore, given any edge $f = uv$ at distance 2 from e, applying an $(\alpha\beta : \gamma\delta)_{uv}$-reduction followed by an $(\alpha x : wz)_{\beta y}$-reduction would lead to α being adjacent to x in G. Let y' and z' be the two consecutive vertices on C_v such that y' is adjacent to δ (note that y' or z' are possibly equal to z). Let w' and x' respectively be the vertices in $G - C_v$ adjacent to δ and y', and let α' be the other endvertex of e. Applying an $(\alpha\delta : \gamma\beta)_{uv}$-reduction followed by an $(\alpha x' : w'z')_{\delta y'}$-reduction leads to α being adjacent to x'. Therefore, x' can be equal to α', u or x. If $x' = \alpha'$, then the edge $\delta y'$ would be an edge belonging to C_v at distance 2 from the edge e, and if $x' = u$, then (u, v, δ, y') would be a 4-circuit in G, with both cases leading to a contradiction. Therefore, $x = x'$.

Let G' be the graph obtained after an $(\alpha\beta : \gamma\delta)_{uv}$-reduction and let $\mathcal{C}' = \mathcal{C} \setminus \{C_v\}$. By induction, there exists a perfect matching M' containing e intersecting all the circuits in $\mathcal{C}'$, and it can be extended into a perfect matching M of G containing e. Suppose that M does not intersect C_v (it is the only circuit in $\mathcal{C}$ that M could possibly avoid). To cover vertices y and y', we must have $\{xy, xy'\} \subset M$ — which is impossible. $\square$

Here are some consequences of Theorem **??**. Corollary **??** follows by the above result and Corollary **??**.

Corollary 3.1. *Let G be a cyclically 3-edge-connected cubic graph and let $\mathcal{C}$ be a collection of disjoint circuits of G. Then, there exists a perfect matching M such that $M \cap E(C) \neq \emptyset$, for every $C \in \mathcal{C}$.*

Corollary 3.2. *Let G be a cyclically 3-edge-connected cubic graph. For every perfect matching M_1 of G, there exists a perfect matching M_2 of G such that $G \setminus (M_1 \cup M_2)$ is acyclic.*

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