The Converse of the Real Orthogonal Holant Theorem

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Abstract

The Holant theorem is a powerful tool for studying the computational complexity of counting problems. Due to the great expressiveness of the Holant framework, a converse to the Holant theorem would itself be a very powerful *counting indistinguishability theorem*. The most general converse does not hold, but we prove the following, still highly general, version: if any two sets of real-valued signatures are Holant-indistinguishable, then they are equivalent up to an orthogonal transformation. This resolves a partially open conjecture of Xia (2010). Consequences of this theorem include the well-known result that homomorphism counts from all graphs determine a graph up to isomorphism, the classical sufficient condition for simultaneous orthogonal similarity of sets of real matrices, and a combinatorial characterization of simultaneosly orthogonally decomposable (odeco) sets of symmetric tensors.

1 Introduction

Holant problems. Holant problems were introduced by Cai, Lu, and Xia [9] as a highly expressive framework for studying the computational complexity of counting problems. The problem Holant($\mathcal{F}$) is defined by a set $\mathcal{F}$ of *signatures*, where a signature F of *arity* n on *domain* $[q] := \{0, 1, \ldots, q-1\}$ is a tensor in $(\mathbb{C}^q)^{\otimes n}$. Given a *signature grid* Ω – a multigraph in which every degree-n vertex is assigned a n-ary signature from $\mathcal{F}$ – the problem is to compute the *Holant value* of Ω, which is the value of the contraction of Ω as a tensor network (see Section 2.1 for formal definitions). For various $\mathcal{F}$, Holant($\mathcal{F}$) captures a wide variety of natural counting problems on graphs, including counting partial or perfect matchings, graph homomorphisms, proper vertex or edge-colorings, or Eulerian orientations. Major complexity dichotomies classifying Holant($\mathcal{F}$) as either polynomial-time tractable or #P-hard, depending on $\mathcal{F}$, have been proved for various combinations of restrictions on $\mathcal{F}$ – for example, requiring that the signatures in $\mathcal{F}$ be real- or nonnegative-real-valued, symmetric (invariant under reordering of their inputs), or on the Boolean domain $q = 2$ [20, 8, 6, 21, 33].

Holant problems were motivated by Valiant's technique of *holographic transformations* [37]. In particular, Valiant's *Holant theorem* (Theorem 2.2 below) states roughly that two any signature sets $\mathcal{F} \cup \mathcal{F}'$ and $\mathcal{G} \cup \mathcal{G}'$ that are equivalent up to a certain linear transformation are *Holant-indistinguishable*, meaning that each signature grid Ω has the same Holant value whether its vertices are assigned signatures from $\mathcal{F} \cup \mathcal{F}'$ or from $\mathcal{G} \cup \mathcal{G}'$. Many problems which do not otherwise appear tractable are in fact tractable under a Holographic transformation to a known tractable problem [36]. Xia [40] conjectured the converse of the Holant theorem: if $\mathcal{F} \cup \mathcal{F}'$ and $\mathcal{G} \cup \mathcal{G}'$ are Holant-indistinguishable, then they are equivalent up to linear transformation. Xia's general conjecture is false (see Section 2.4), but one case highlighted by Xia was left open. This paper proves that case, which is as follows.

Theorem (Theorem 2.3, informal)**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be sets of real-valued signatures. Then $\mathcal{F}$ and $\mathcal{G}$ are equivalent under a real orthogonal transformation if and only if $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable.*

Vertex and edge coloring models. This work uses and generalizes ideas from the theory of vertex coloring models and edge coloring models, two well-studied classes of Holant problems. De la Harpe and Jones [19] defined vertex and edge coloring models as extensions of statistical mechanics models (e.g. the Ising model), calling them "spin models" and "vertex models", respectively. A vertex coloring model is defined by a graph X with edge and possibly vertex weights. Given an input graph K, one aims to compute the *partition*

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function, the number of (weighted) homomorphisms from K to X. As discussed in Section 2.1, computing the partition function of a vertex coloring model without vertex weights is equivalent to Holant($A_X \cup \mathcal{EQ}$), where A_X is the weighted adjacency matrix of X and $\mathcal{EQ}$ is the set of *equality* signatures (we can model vertex weights by replacing $\mathcal{EQ}$ with $\mathcal{GEQ}$, the set of weighted equalities). An edge coloring model is defined by a set $\mathcal{F}$ of symmetric signatures containing exactly one signature of each arity, and the problem of computing its partition function is equivalent to Holant($\mathcal{F}$) (this restriction on $\mathcal{F}$ ensures that edge coloring models take ordinary graphs, rather than signature grids, as input).

One thread of prior work on vertex and edge coloring models characterizes which graph parameters (scalar-valued functions defined on isomorphism classes of graphs) are expressible as vertex coloring models [17, 31] or as edge coloring models [35, 30, 13, 26]. Another, related, line of works compute the rank of *connection matrices* for vertex coloring models [24] and edge coloring models [28, 14]. See Regts' thesis [27] for an overview of many of the above results. Following Freedman, Lovász, and Schrijver [17], these works all use some form of (labeled) *quantum graphs*, algebras of formal linear combinations of graphs equipped with labeled vertices or "half edges" incident to a single vertex. Each labeled quantum graph defines a tensor by evaluating its partition function when its labeled vertices are fixed to input values. All such constructions are special cases of our *quantum gadgets* below (see Definition 3.4). Many of these works also apply techniques from invariant theory, either of the symmetric group in the case of vertex coloring models [31], or, as in this work, of the orthogonal group $O(q)$ in the case of edge-coloring models [35, 30, 13, 28, 14, 26].

Counting Indistinguishability Theorems. Theorem 2.3 is a very general and powerful algebraic *counting indistinguishability theorem*. Such a theorem proves that two signatures, or sets of signatures, are indistinguishable as parameters for a counting problem if and only if they are equivalent under an algebraic transformation. These theorems exist for both vertex and edge coloring models, as well as other counting problems [12, 18, 15]. Since the Holant framework captures a wide variety of counting problems, including both vertex and edge coloring models, many such theorems are special cases of Theorem 2.3 (see Section 5). If the counting problem in question is a generalized vertex coloring model (Holant($\mathcal{F} \cup \mathcal{EQ}$) for some $\mathcal{F}$), then the algebraic transformation is isomorphism, and if the counting problem is, as in this work, a generalized edge coloring model, then it is orthogonal. The first counting indistinguishability theorem, proved by Lovász [23], states that two graphs are isomorphic if and only if they admit the same number of homomorphisms from all graphs. Much later, Lovász [24] extended this theorem to vertex coloring models with nonnegative real weights, followed by extensions to complex edge weights by Schrijver [31], and to weights from any field of characteristic zero by Cai and Govorov [7]. Young [41] extended Cai and Govorov's proof to #CSP, or Holant($\mathcal{F} \cup \mathcal{EQ}$) for any $\mathcal{F}$.

For edge coloring models, Schrijver [30] showed that $\mathcal{F}$ and $\mathcal{G}$ define indistinguishable real edge coloring models if and only if $\mathcal{F}$ and $\mathcal{G}$ are equivalent under a real orthogonal transformation. This is a special case of our Theorem 2.3. Schrijver's proof exploits the specific nature of edge coloring models – that $\mathcal{F}$ and $\mathcal{G}$ consist of symmetric signatures and exactly one signature per arity – to transform input graphs into polynomials expressible in variables $y_1, \dots, y_q$ (for $\mathcal{F}$ and $\mathcal{G}$ on domain $[q]$), where a monomial with variable multiset $\{y_{i_1}, \dots, y_{i_n}\}$ corresponds to the $\{i_1, \dots, i_n\}$-entry of the unique n-ary signature in $\mathcal{F}$. Another form of this result (in which the orthogonal matrix is allowed to be complex) follows from Regts' proof of [26, Lemma 5], which similarly encodes $\mathcal{F}$ and $\mathcal{G}$ as polynomials.

Mančinska and Roberson [25] introduced a new form of counting indistinguishability theorem, showing that two graphs are *quantum isomorphic* – an abstract relaxation of isomorphism – if and only if they admit the same number of homomorphisms from all planar graphs. Cai and Young [11] translated Mančinska and Roberson's proof into the Holant framework (we adopt many of Cai and Young's definitions and notations in this paper) and extended it to planar #CSP (or Pl-Holant($\mathcal{F} \cup \mathcal{EQ}$), where Pl-Holant restricts to planar signature grids), showing that real-valued $\mathcal{F}$ and $\mathcal{G}$ are planar-#CSP-indistinguishable if and only if they are quantum isomorphic.

Odeco signature sets A real-valued symmetric signature (tensor) is *orthogonally decomposable*, or *odeco* [29], if it is orthogonally transformable to a signature in $\mathcal{GEQ}$, the set of *generalized equality* signatures, which take nonzero values only when all of their inputs are equal. Hence odeco tensors generalize diagonalizable matrices. Call a set $\mathcal{F}$ of signatures odeco if the signatures are simultaneously odeco (there is a single orthogonal transformation mapping $\mathcal{F}$ into $\mathcal{GEQ}$). In counting complexity, if $\mathcal{F}$ is odeco, then Holant($\mathcal{F}$) is polynomial-time tractable, as $\mathcal{F}$ maps into $\mathcal{GEQ}$, a trivially tractable set, under an orthogonal holographic transformation. Indeed, the tractability of *Fibonacci* signature sets [10] can, with one exception, be explained by such sets being simultaneously odeco (see e.g. [3, Section 2.2]). Fibonacci sets constitute almost all nontrivial tractable cases of Holant* problems (an important variant of Holant in which all unary signatures are assumed present) for symmetric signatures on the Boolean domain [9]. Simultaneously odeco sets provide a natural starting point for extending Fibonacci signatures to higher domains [22], where no full complexity dichotomy for Holant* is known.

Boralevi, Draisma, Horobeţ, and Robeva [1], resolving a conjecture of Robeva [29], showed using techniques from algebraic geometry that a single tensor F is odeco if and only if the signature of a certain F-gadget is symmetric. Using Theorem 2.3, we in Theorem 5.1 extend this characterization to sets of simultaneously odeco signatures: $\mathcal{F}$ is odeco if and only if every connected $\mathcal{F}$-gadget has a symmetric signature. This deepens the connection between Fibonacci and odeco signatures, as the original proof of tractability of any Fibonacci signature set $\mathcal{F}$ [10] relied on the fact that every connected $\mathcal{F}$-gadget has a signature which is itself Fibonacci (in particular, is symmetric). One can view the (iii) $\implies$ (ii) result in Theorem 5.1 as a general-domain version of this proof.

Overview. We introduce the necessary preliminaries and formally state Theorem 2.3 in Section 2. In Section 3, we use an invariant-theoretic result of Schrijver [32] (Theorem 3.1) to prove a combinatorial-algebraic duality (Theorem 3.2) showing that combinatorial quantum $\mathcal{F}$-gadgets exactly capture all tensors invariant under the algebraic action of $\text{Stab}(\mathcal{F})$, the group of orthogonal transformations stabilizing $\mathcal{F}$. The proof of Theorem 3.2 entails characterizing quantum $\mathcal{F}$-gadgets as the algebra generated by a few fundamental gadgets under operations which respect the action of $O(q)$. It follows and generalizes a proof of a similar result of Regts [28] for edge coloring models. By unifying the perspective of Regts with that of Mančinska and Roberson and Cai and Young [25, 11, 41] we find that our proof is also strongly analogous to proofs of similar results in the latter line of work (see Remark 3.1).

However, Mančinska and Roberson and Cai and Young's proofs of their ensuing counting indistinguishability theorems use the *orbits* and/or *orbitals* of the action of the symmetric or quantum symmetric group on the domain set $[q]$, which do not exist for $O(q)$. Instead, we apply a novel method: induction on the domain size q. We show in Lemma 4.3 that, if $\mathcal{F}$ and $\mathcal{G}$ contain a binary signature represented by a nontrivial (i.e., not a multiple of I) diagonal matrix, then we can separate $[q]$ into smaller subdomains and apply induction to obtain successive orthogonal transformations between the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to each subdomain, producing a full orthogonal transformation from $\mathcal{F}$ to $\mathcal{G}$. Using Theorem 3.2, we show that there is some nonzero matrix D intertwining $\mathcal{F}$ and $\mathcal{G}$. Exploiting the power of diagonalization afforded by orthogonal transformations, we may assume this D is diagonal. Either $D = \pm I$, giving a trivial orthogonal transformation between $\mathcal{F}$ and $\mathcal{G}$, or D is not a multiple of I, in which case we use the fact that D intertwines $\mathcal{F}$ and $\mathcal{G}$ to add D to both $\mathcal{F}$ and $\mathcal{G}$ while preserving their Holant-indistinguishability, then apply induction.

In Section 5, we show that Theorem 2.3 encompasses a wide range of existing counting indistinguishability theorems, and yields some novel variations of these theorems. We also prove our combinatiorial characterization of odeco signature sets.

Finally, in Section 6, we discuss two possible variations of Theorem 2.3. First for complex-valued signatures, and second for an extension of the results of Mančinska and Roberson [25] and Cai and Young [11] to planar-Holant-indistinguishability and quantum orthogonal transformations.

2 Preliminaries, Background, and the Main Theorem

2.1 Holant Problems

Let $\mathbb{N}$ be the set of natural numbers, including 0. A *signature* F of finite *arity* $n \in \mathbb{N}$ on finite *domain* $V(F)$ is function $V(F)^n \to \mathbb{R}$. We will often take $V(F) = [q] := \{0, 1, \ldots, q-1\}$, in which case we also view F as a tensor in $(\mathbb{R}^q)^{\otimes n}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in V(F)^n$, abbreviate $F_{\mathbf{x}} := F(x_1, \ldots, x_n) \in \mathbb{R}$. Signature F is *symmetric* if its value depends only on the multiset of inputs, not on their order. Any time we consider a set $\mathcal{F}$ of signatures, we assume that all signatures in $\mathcal{F}$ have the same domain, denoted $V(\mathcal{F})$.

In the context of a signature set $\mathcal{F}$, a *signature grid* (or *$\mathcal{F}$-grid*) Ω consists of an underlying multigraph with vertex set V and edge set E, and an assignment of an $\deg(v)$-ary signature $F_v \in \mathcal{F}$ to each $v \in V$, along with an ordering of the edges $\delta(v) \subset E$ incident to v to serve as the $\deg(v)$ input variables to F. That is, there is an ordering $e_1, \ldots, e_{\deg(v)}$ of v's incident edges such that, if $\sigma : E \to V(\mathcal{F})$ is an assignment of a value in $V(\mathcal{F})$ to each edge of Ω, then F_v evaluates to $F_v(\sigma|_{\delta(v)}) := F_v(\sigma(e_1), \ldots, \sigma(e_{\deg(v)}))$. Somewhat unusually, we also allow E to contain *vertexless loops*, edges whose two ends are connected to each other, with no incident vertices. The problem $\text{Holant}(\mathcal{F})$ is defined as follows: given an $\mathcal{F}$-grid Ω with vertex set V and edge set E, compute the *Holant value*

$$\text{Holant}_\Omega(\mathcal{F}) := \sum_{\sigma: E \to V(\mathcal{F})} \prod_{v \in V} F_v(\sigma|_{\delta(v)}). \tag{2.1}$$

When $\mathcal{F}$ is clear from context, we abbreviate $\text{Holant}_\Omega(\mathcal{F})$ as Holant_Ω. Each vertexless loop of Ω contributes a global factor $|V(\mathcal{F})|$ to Holant_Ω. More generally, the Holant value of a disconnected signature grid is the product of the Holant values of its connected components. For signature sets $\mathcal{F}$ and $\mathcal{F}'$, define $\text{Holant}(\mathcal{F} \mid \mathcal{F}')$ as the Holant problem whose input is a bipartite $(\mathcal{F} \sqcup \mathcal{F}')$-grid Ω, with the vertices in the two bipartitions assigned signatures in $\mathcal{F}$ and $\mathcal{F}'$, respectively.

For example, if $\mathcal{F}$ is on the Boolean domain $\{0, 1\}$ and consists of, for each arity n, a symmetric signature which evaluates to 1 on input strings of Hamming weight 1 and 0 on all other input strings, then $\text{Holant}_\Omega(\mathcal{F})$ equals the number of perfect matchings in the multigraph underlying Ω. For another example, let $A_X \in \mathbb{R}^{q \times q}$ be the adjacency matrix of q-vertex weighted graph X, and define the set $\mathcal{EQ} = \{=_n \mid n \geq 1\}$ of *equality* signatures, where $=_n (x_1, \ldots, x_n)$ is 1 if $x_1 = \ldots = x_n$, and is 0 otherwise. Consider $\text{Holant}(A_X \mid \mathcal{EQ})$. We can think of any edge assignment σ with a nonzero contribution to the Holant value as a map from vertices in Ω assigned $\mathcal{EQ}$ signatures to values in $[q]$, or equivalently vertices of X. This map must send the two $\mathcal{EQ}$ vertices adjacent to every A_X vertex to an edge in A_X, so, if K is the graph resulting from ignoring (i.e. treating as edges) the degree-two vertices assigned A_X in the underlying graph of Ω, then $\text{Holant}_\Omega(A_X \mid \mathcal{EQ})$ equals the number of graph homomorphisms from K to X. See Figure 2.1.

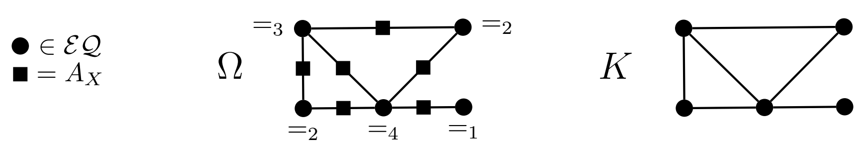


Figure 2.1: An $(A_X \mid \mathcal{EQ})$-grid Ω and the graph K such that $\text{Holant}_\Omega(A_X \mid \mathcal{EQ})$ counts the number of homomorphisms from K to X.

2.2 Gadgets and Signature Matrices

Instead of viewing a signature F as a tensor in $(\mathbb{R}^q)^{\otimes n}$ or function in $\mathbb{R}^{[q]^n}$, we can partition its inputs in two to view it naturally as a matrix.

Definition 2.1 ($F^{m,d}, f$)**.** For $F \in (\mathbb{R}^q)^{\otimes n}$ and $m, d \in \mathbb{N}$ with $m + d = n$, define the (m, d)-*signature matrix*, or *flattening*, $F^{m,d} \in \mathbb{R}^{q^m \times q^d}$ of F by, for $\mathbf{x} \in [q]^m$ and $\mathbf{y} \in [q]^d$,

$$(F^{m,d})_{\mathbf{x},\mathbf{y}} = F(x_0, \ldots, x_{m-1}, y_{d-1}, \ldots, y_0),$$

where we use the standard isomorphism $[q^n] \cong [q]^n$ to index $F^{m,d}$.

Abbreviate $f = F^{n,0} \in \mathbb{R}^{q^n}$ – the (column) *signature vector* of F.

We will often identify binary signatures in $(\mathbb{R}^q)^{\otimes 2}$ with their $1, 1$ signature matrices in $\mathbb{R}^{q \times q}$.

Definition 2.2 ($\mathfrak{G}_\mathcal{F}, \mathfrak{G}_\mathcal{F}(m, d)$)**.** For signature set $\mathcal{F}$, an $\mathcal{F}$-*gadget* is a $\mathcal{F}$-grid equipped with an ordered set of *dangling edges* with zero or one endpoints.

Define $\mathfrak{G}_\mathcal{F}$ to be the set of all $\mathcal{F}$-gadgets, and $\mathfrak{G}_\mathcal{F}(m, d) \subset \mathfrak{G}_\mathcal{F}$ to be the set of gadgets with $m + d$ dangling edges $\ell_0, \ldots, \ell_{m-1}, r_{d-1}, \ldots, r_0$ drawn with dangling ends in counterclockwise cyclic order around the gadget, with $\ell_0, \ldots, \ell_{m-1}$ and $r_0, \ldots, r_{d-1}$ on the left and right, respectively, from top to bottom.

See Figure 2.2 for examples of gadgets. A gadget in $\mathfrak{G}_{\mathcal{F}}(m,d)$ defines a $(m+d)$-ary signature in flattened form, with dangling edges representing inputs, as follows (cf. (2.1)):

Definition 2.3 ($M(\mathbf{K})$)**.** Define the *signature matrix* $M(\mathbf{K}) \in \mathbb{R}^{q^m \times q^d}$ of $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}(m,d)$ by

$$M(\mathbf{K})_{\mathbf{x},\mathbf{y}} = \sum_{\substack{\sigma: E(\mathbf{K}) \to [q] \\ \forall i: \sigma(\ell_i) = x_i \\ \forall j: \sigma(r_j) = y_j}} \prod_{v \in V} F_v(\sigma|_{\delta(v)}) \quad \text{for } \mathbf{x} \in [q]^m \text{ and } \mathbf{y} \in [q]^d.$$

If $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}(m,d)$ is a gadget consisting of a single vertex assigned F with m and d incident left and right dangling edges, respectively, ordered to match the input order of F, then $M(\mathbf{K}) = F^{m,d}$. In general, for $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}(m,d)$, there is a unique $F \in (\mathbb{R}^q)^{\otimes m+d}$, called the *signature of* $\mathbf{K}$, such that $M(\mathbf{K}) = F^{m,d}$. Note that F does not depend on the particular left/right partition (i.e. choice of m and d) of a fixed cyclic order of $\mathbf{K}$'s dangling edges.

Definition 2.4 ($\circ, \otimes, \top$)**.** Define the following three operations on gadgets:

- For $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}(m,d)$ and $\mathbf{L} \in \mathfrak{G}_{\mathcal{F}}(d,r)$, construct $\mathbf{K} \circ \mathbf{L} \in \mathfrak{G}_{\mathcal{F}}(m,r)$ by connecting the dangling ends of $r_i \in E(\mathbf{K})$ and $\ell_i \in E(\mathbf{L})$ for $i \in [d]$.
- For $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}(m_1,d_1)$ and $\mathbf{L} \in \mathfrak{G}_{\mathcal{F}}(m_2,d_2)$, construct $\mathbf{K} \otimes \mathbf{L} \in \mathfrak{G}_{\mathcal{F}}(m_1+m_2, d_1+d_2)$ as the disjoint union of $\mathbf{K}$ and $\mathbf{L}$, placing $\mathbf{K}$ above $\mathbf{L}$. Interleave $\mathbf{K}$ and $\mathbf{L}$'s dangling edges into an overall cyclic order: $\mathbf{K}$ left, then $\mathbf{L}$ left, then $\mathbf{L}$ right, then $\mathbf{K}$ right.
- For $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}(m,d)$, construct $\mathbf{K}^\top \in \mathfrak{G}_{\mathcal{F}}(d,m)$ by exchanging the roles of left and right dangling edges and reversing the overall cyclic dangling edge order (visually, horizontally reflect $\mathbf{K}$).

See Figure 2.2. As one might expect, the gadget operations $\circ, \otimes, \top$ induce the respective operations – composition, Kronecker product, transpose – on their signature matrices. See e.g. [3, Section 1.3].

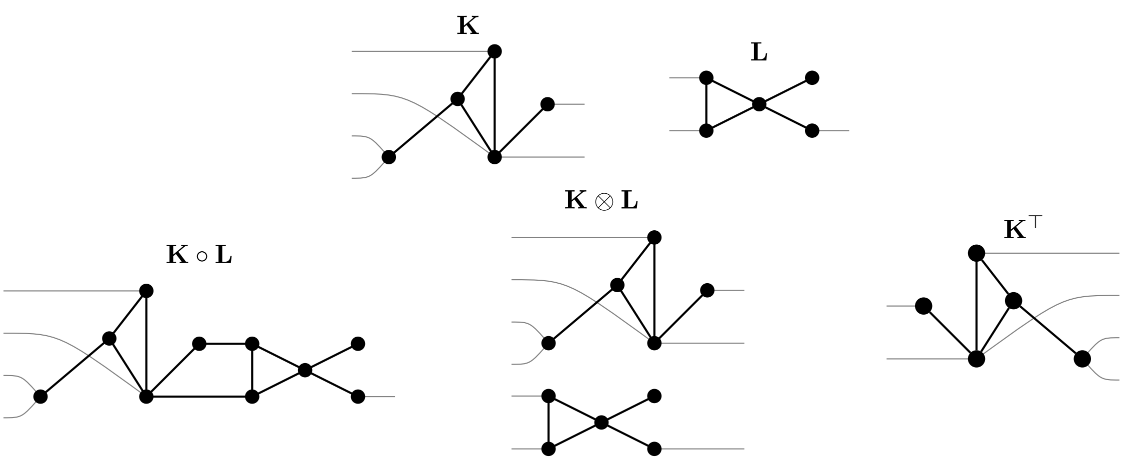


Figure 2.2: Operations on gadgets $\mathbf{K} \in \mathfrak{G}(4,2)$ and $\mathbf{L} \in \mathfrak{G}(2,1)$. Dangling edges are drawn thinner than internal edges.

Definition 2.5 ($\langle \cdot, \cdot \rangle, \| \cdot \|$)**.** For n-ary gadgets $\mathbf{K}$ and $\mathbf{L}$, construct the signature grid $\langle \mathbf{K}, \mathbf{L} \rangle$ by connecting the ith dangling edges of $\mathbf{K}$ and $\mathbf{L}$, for $i \in [n]$. If $\mathbf{K}$ and $\mathbf{L}$ have signatures K and L, then define $\langle K, L \rangle :=$ $\text{Holant}_{\langle \mathbf{K}, \mathbf{L} \rangle} = \langle K^{n,0}, L^{n,0} \rangle$ (the standard inner product on $\mathbb{R}^{q^n}$). Define $\|F\| := \sqrt{\langle F, F \rangle} = \sqrt{\sum_{\mathbf{x}} F_{\mathbf{x}}^2}$.

2.3 Signature Transformations and Invariance

Definition 2.6 ($T(\mathbb{R}^q)$)**.** Following Schrijver [32], define

$$T(\mathbb{R}^q) := \bigcup_{n\in\mathbb{N}} (\mathbb{R}^q)^{\otimes n}$$

to be the set of all signatures on domain $[q]$.

Definition 2.7 ($HF, H\mathcal{F}$)**.** Define an action of the group $\mathrm{GL}_q(\mathbb{R})$ of invertible $q \times q$ matrices on the set $T(\mathbb{R}^q)$ by, for $H \in \mathrm{GL}_q(\mathbb{R})$ and $F \in (\mathbb{R}^q)^{\otimes n}$, letting $HF \in (\mathbb{R}^q)^{\otimes n}$ be the signature whose signature vector is $H^{\otimes n} f$ – that is, $(HF)^{n,0} = H^{\otimes n} f$.

For a signature set $\mathcal{F} \subset T(\mathbb{R}^q)$, define $H\mathcal{F} := \{HF \mid F \in \mathcal{F}\}$.

We usually have $H \in O(q)$, the $q \times q$ (real) orthogonal group. We will use the following notations from invariant theory [32, 28].

Definition 2.8 ($T(\mathbb{R}^q)^Q$, $\mathrm{Stab}(\mathcal{F})$)**.** For a subgroup $Q \subset O(q)$, define

$$T(\mathbb{R}^q)^Q := \{F \in T(\mathbb{R}^q) \mid HF = F \text{ for every } H \in Q\} \subset T(\mathbb{R}^q).$$

Dually, for a signature set $\mathcal{F} \subset T(\mathbb{R}^q)$, define

$$\mathrm{Stab}(\mathcal{F}) := \{H \in O(q) \mid HF = F \text{ for every } F \in \mathcal{F}\} \subset O(q).$$

A two-sided dangling edge, or *wire*, has no incident vertices. As a gadget with one left-facing and one right-facing dangling end, a wire has siganture matrix I (the identity), as its left and right inputs must agree. Connecting the two ends of a wire, we obtain a vertexless loop.

Definition 2.9 ($\mathcal{W}$)**.** Let $\mathcal{W} \subset T(\mathbb{R}^q)$ be the signatures of gadgets with no vertices – that is, the signatures of gadgets consisting of only wires and vertexless loops.

A gadget consisting of n wires has a $2n$-ary signature in $\mathcal{W}$, which uniquely corresponds, via the order of the gadget's dangling edges, to a partition (or matching) of $[2n]$ into two-element subsets. Since the gadgets defining $\mathcal{W}$ have no vertices, they belong to $\mathfrak{G}_{\mathcal{F}}$ for any $\mathcal{F}$. This view of $\mathcal{W}$ as a set of universal signatures is reinforced by the following classical theorem from representation theory, called the (tensor) *First Fundamental Theorem* for $O(q)$, proved by Weyl [38], and stated in this form by Regts [27, Theorem 4.3]. Define $\langle \mathcal{F} \rangle_+$ as the set of all $\mathbb{R}$-linear combinations of (matching-arity) signatures in $\mathcal{F}$.

Theorem 2.1 (FFT for $O(q)$)**.** $T(\mathbb{R}^q)^{O(q)} = \langle \mathcal{W} \rangle_+$.

We will not need the $\subseteq$ direction of Theorem 2.1, although it follows directly from our Theorem 3.2 below. The $\supseteq$ direction follows from a simple calculation and has the geometric intuition shown in Figure 2.3.

2.4 The Holant Theorem

Throughout this work, we will be considering pairs of signature sets, usually denoted $\mathcal{F}$ and $\mathcal{G}$. We assume such pairs are *similar*, meaning they have the same domain size q and there is a bijection $\mathcal{F} \to \mathcal{G}$ such that, for n-ary $F \in \mathcal{F}$, the image $G \in \mathcal{G}$ of F, called the signature *corresponding* to F and denoted by $F \leftrightsquigarrow G$, has the same arity n.

Definition 2.10 ($\mathbf{K}_{\mathcal{F}\to\mathcal{G}}, \Omega_{\mathcal{F}\to\mathcal{G}}$)**.** For (similar) sets $\mathcal{F}$ and $\mathcal{G}$ and gadget $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}$, define the gadget $\mathbf{K}_{\mathcal{F}\to\mathcal{G}} \in \mathfrak{G}_{\mathcal{G}}$ by replacing every $F \in \mathcal{F}$ assigned to a vertex in $\mathbf{K}$ with the corresponding $G \in \mathcal{G}$. If $\mathbf{K}$ has zero dangling edges then it is an $\mathcal{F}$-grid Ω, and is transformed to a $\mathcal{G}$-grid $\Omega_{\mathcal{F}\to\mathcal{G}}$.

Holant problems were originally motivated by the following theorem, a powerful reduction tool proved by Valiant [37]. For $F \in (\mathbb{R}^q)^{\otimes n}$ and $A \in \mathrm{GL}_q(\mathbb{R})$, define FA similarly to AF, by $(FA)^{0,n} = F^{0,n} A^{\otimes n}$.

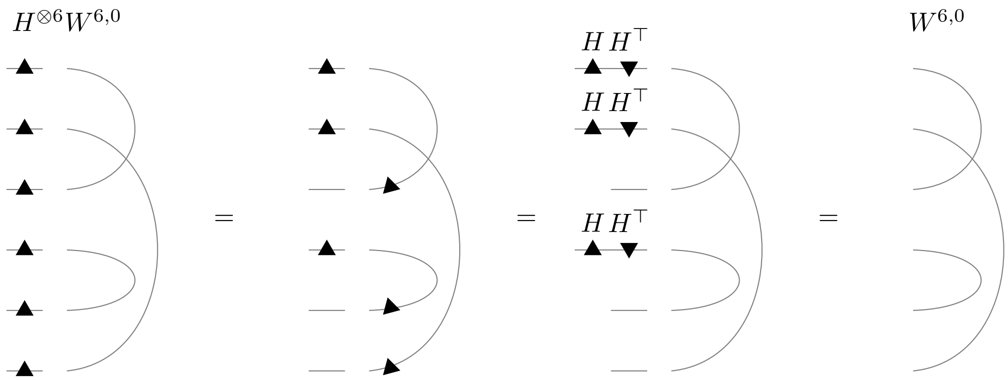


Figure 2.3: Demonstrating $H^{\otimes 6}W^{6,0} = W^{6,0}$ for 6-ary $W \in \mathcal{W}$ and orthogonal H. Each lower H is moved along its wire to the top, and transposed in the process (its left and right dangling edges are switched).

Theorem 2.2 (The Holant Theorem)**.** *For any $(\mathcal{F} \mid \mathcal{F}')$-grid Ω and matrix $A \in GL_q(\mathbb{R})$,*

$$\mathrm{Holant}_{\Omega}(\mathcal{F} \mid \mathcal{F}') = \mathrm{Holant}_{\Omega'}(A\,\mathcal{F} \mid \mathcal{F}'\,A^{-1}),$$

where $\Omega' = \Omega_{(\mathcal{F} \sqcup \mathcal{F}') \to (A\,\mathcal{F} \sqcup \mathcal{F}'\,A^{-1})}$.

Xia [40] conjectured that the converse of Theorem 2.2 holds as long as one of $\mathcal{F}$ or $\mathcal{F}'$ contain a signature with arity greater than one – that is, if $\mathrm{Holant}_{\Omega}(\mathcal{F} \mid \mathcal{F}') = \mathrm{Holant}_{\Omega_{(\mathcal{F} \sqcup \mathcal{F}') \to (\mathcal{G} \sqcup \mathcal{G}')}}(\mathcal{G} \mid \mathcal{G}')$ for every $(\mathcal{F} \mid \mathcal{F}')$-grid Ω, then there is an $A \in \mathrm{GL}_q(\mathbb{C})$ such that $\mathcal{G} = A\,\mathcal{F}$ and $\mathcal{G}' = \mathcal{F}'\,A^{-1}$. However, Cai and Chen [3, Section 7.1.1] observe that this conjecture is false. They consider the problems

$$\mathrm{Holant}([0,1,0] \mid [a,b,1,0,0]) \quad \text{and} \quad \mathrm{Holant}([0,1,0] \mid [0,0,1,0,0])$$

for any a, b not both 0, where $[0,1,0]$ and $[a,b,1,0,0]$ are symmetric signatures on the Boolean domain $\{0,1\}$ of arity $n = 2, 4$, respectively, specified by their values on input strings of Hamming weight 0 through n. For both problems, $[0,1,0]$ is only nonzero on edge assignment σ if σ assigns its two incident edges opposite values. Thus, unless its contribution to the total Holant value is 0, σ must assign 0 to exactly half the edges in Ω, and 1 to the other half. Then, since every $[a,b,1,0,0]$ evaluates to zero whenever it receives more 1 inputs than 0 inputs, no nonzero assignment σ assigns any $[a,b,1,0,0]$ more 0 inputs than 1 inputs. Therefore, in this context, $[a,b,1,0,0]$ is indistinguishable from $[0,0,1,0,0]$, so the hypothesis of Xia's conjecture is satisfied. However, there is no $A \in \mathrm{GL}_2(\mathbb{C})$ satisfying $A[0,1,0] = [0,1,0]$ and $[a,b,1,0,0]A^{-1} = [0,0,1,0,0]$. To see this, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then, with the signature vector of $[0,1,0]$ being $(0,1,1,0)^{\top}$,

$$A^{\otimes 2}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \iff ab = cd = 0 \text{ and } ad + bc = 1 \implies \begin{cases} a = d = 0 & \text{or} \\ b = c = 0 \end{cases}.$$

If $b = c = 0$ then A simply rescales a signature's entries, and if $a = d = 0$ then A rescales a signature's entries and exchanges the roles of 0 and 1, which has the effect of reversing the entries in the $[\cdot, \ldots, \cdot]$ notation. Thus $[0,0,1,0,0]A = [0,0,*,0,0] \neq [a,b,1,0,0]$.

This counterexample exists due to the bipartiteness enforced by the definition of $\mathrm{Holant}(\mathcal{F} \mid \mathcal{G})$, because

$$\langle [0,0,1,0,0], [0,0,1,0,0] \rangle = \|[0,0,1,0,0]\|^2 \neq \|[a,b,1,0,0]\|^2 = \langle [a,b,1,0,0], [a,b,1,0,0] \rangle\,,$$

so $[a,b,1,0,0]$ and $[0,0,1,0,0]$ are not indistinguishable on general (non-bipartite) signature grids. To avoid such counterexamples, we consider the following well-known form of Theorem 2.2 that applies to non-bipartite grids. An edge in an $\mathcal{F}$-grid Ω, viewed on its own, is a wire, with signature I. Therefore we can, without changing the Holant value, replace each edge in Ω by a binary gadget with a single vertex assigned I, effectively splitting the edge into a path of length two. Then Ω is a signature grid in the context of $\text{Holant}(\mathcal{F} \mid I)$, so, by Theorem 2.2, for any invertible H, $\text{Holant}_\Omega(\mathcal{F} \mid I) = \text{Holant}_{\Omega'}(H\,\mathcal{F} \mid IH^{-1})$. If H is orthogonal, then, by Theorem 2.1, $I^{0,2}H^{-1} = (HI^{2,0})^\top = (I^{2,0})^\top = I^{0,2}$, so $\text{Holant}_\Omega(\mathcal{F} \mid I) = \text{Holant}_{\Omega'}(H\,\mathcal{F} \mid I)$. Then $\text{Holant}(H\,\mathcal{F} \mid I)$ is again equivalent to $\text{Holant}(H\,\mathcal{F})$, so we have proved the following.

Corollary 2.1 (The Orthogonal Holant Theorem)**.** *If $\mathcal{G} = H\,\mathcal{F}$ for orthogonal matrix H, then* $\text{Holant}_\Omega(\mathcal{F}) = \text{Holant}_{\Omega_{\mathcal{F}\to\mathcal{G}}}(\mathcal{G})$ *for every $\mathcal{F}$-grid Ω.*

Xia [40] also considers the converse of Corollary 2.1, and proves that it holds for specific $\mathcal{F}$ and $\mathcal{G}$ consisting of real-valued symmetric signatures with small domain and/or arity. The main result of this work is the converse of Corollary 2.1 for any sets $\mathcal{F}$ and $\mathcal{G}$ of real-valued signatures, with no restrictions.

Theorem 2.3 (Main Result)**.** *Let $\mathcal{F}$, $\mathcal{G}$ be sets of real-valued signatures. Then the following are equivalent.*

(i) $\text{Holant}_\Omega(\mathcal{F}) = \text{Holant}_{\Omega_{\mathcal{F}\to\mathcal{G}}}(\mathcal{G})$ *for every $\mathcal{F}$-grid Ω.*

(ii) There is a real orthogonal matrix H such that $H\,\mathcal{F} = \mathcal{G}$.

Call $\mathcal{F}$ and $\mathcal{G}$ satisfying (i) *Holant-indistinguishable*, and call $\mathcal{F}$ and $\mathcal{G}$ satisfying (ii) *ortho-equivalent*. The following two properties follow directly from the definitions and Corollary 2.1, but, as they will prove useful throughout the proof of Theorem 2.3, we state them explicitly.

Proposition 2.1. *For any orthogonal H_1 and H_2, $\mathcal{F}$ and $\mathcal{G}$ are ortho-equivalent/Holant-indistinguishable if and only if $H_1\,\mathcal{F}$ and $H_2\,\mathcal{G}$ are ortho-equivalent/Holant-indistinguishable, respectively.*

Proposition 2.2. *For any $\mathcal{F}$ and $\mathcal{G}$, and any additional pair $\mathcal{F}'$ and $\mathcal{G}'$ of signature sets, if $\mathcal{F} \sqcup \mathcal{F}'$ and $\mathcal{G} \sqcup \mathcal{G}'$ are ortho-equivalent (where the $\mathcal{F}'$ signatures in $\mathcal{F} \sqcup \mathcal{F}'$ correspond to the $\mathcal{G}'$ signatures in $\mathcal{G} \sqcup \mathcal{G}'$), then $\mathcal{F}$ and $\mathcal{G}$ are ortho-equivalent.*

2.5 Block Matrices and Signatures

Definition 2.11. Let $\mathcal{I}$ be an index/domain set, and $X \sqcup Y = \mathcal{I}$ be a nontrivial partition of $\mathcal{I}$.

1. For a matrix $H \in \mathbb{R}^{\mathcal{I}\times\mathcal{I}}$ and $R, C \in \{X, Y\}$, let $H|_{R,C} \in \mathbb{R}^{R\times C}$ be the submatrix of H with rows indexed by R and columns indexed by C. Up to row and column reordering, H is the block matrix
$$H = \begin{bmatrix} H|_{X,X} & H|_{X,Y} \\ H|_{Y,X} & H|_{Y,Y} \end{bmatrix}.$$

2. More generally, for a signature/tensor $F \in \mathbb{R}^{\mathcal{I}^n}$ and $\mathbf{R} \in \{X, Y\}^n$, define $F|_{\mathbf{R}} \in \mathbb{R}^{\mathbf{R}}$ (where we identify $\mathbf{R}$ with the set $\prod_{i=1}^n R_i$) to be the subtensor of F with ith input restricted to R_i. For signature set $\mathcal{F}$, let $\mathcal{F}|_{\mathbf{R}} := \{F|_{\mathbf{R}} : F \in \mathcal{F}\}$.

   Abbreviate $F|_X := F|_{X^n}$ and $\mathcal{F}|_X := \mathcal{F}|_{X^n}$.

3. Let $F^{m,d} \in \mathbb{R}^{\mathcal{I}^m \times \mathcal{I}^d}$ be a signature matrix, and let $\mathbf{R} \in \{X, Y\}^m$ and $\mathbf{C} \in \{X, Y\}^d$. Define $F^{m,d}|_{\mathbf{R},\mathbf{C}} \in \mathbb{R}^{\mathbf{R}\times\mathbf{C}}$ as the submatrix of $F^{m,d}$ with rows indexed by $\mathbf{R}$ and columns indexed by $\mathbf{C}$ (in other words, $F^{m,d}|_{\mathbf{R},\mathbf{C}} = (F|_{\mathbf{R}\,\mathbf{C}})^{m,d}$, where $\mathbf{R}\,\mathbf{C}$ is the concatenation of $\mathbf{R}$ and $\mathbf{C}$).

Definition 2.12 ($\oplus$)**.** Let F, G be n-ary signatures on domains $V(F)$, $V(G)$, both of size q. The *direct sum* $F \oplus G$ of F and G is an n-ary signature on domain $V(F) \sqcup V(G)$ defined by

$$(F \oplus G)_{\mathbf{x}} = \begin{cases} F_{\mathbf{x}} & \mathbf{x} \in V(F)^n \\ G_{\mathbf{x}} & \mathbf{x} \in V(G)^n \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \mathbf{x} \in (V(F) \sqcup V(G))^n.$$

For signature sets $\mathcal{F}$ and $\mathcal{G}$, define $\mathcal{F} \oplus \mathcal{G} = \{F \oplus G \mid \mathcal{F} \ni F \leftrightsquigarrow G \in \mathcal{G}\}$.

We will frequently apply Definition 2.11 to the partition $V(\mathcal{F}) \sqcup V(\mathcal{G})$ of the domain of $\mathcal{F} \oplus \mathcal{G}$. It follows directly from the definitions that, for $\mathcal{F} \ni F \leftrightsquigarrow G \in \mathcal{G}$ and any $m + d = \text{arity}(F)$,

$$(F \oplus G)^{m,d}|_{\mathbf{R},\mathbf{C}} = \begin{cases} F^{m,d} & \mathbf{R} = V(\mathcal{F})^m \wedge \mathbf{C} = V(\mathcal{F})^d \\ G^{m,d} & \mathbf{R} = V(\mathcal{G})^m \wedge \mathbf{C} = V(\mathcal{G})^d \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Appendix A states and proves some additional necessary results involving actions of block matrices on block signatures. These additional results confirm that these block structures interact as one would expect, analogously to ordinary block matrix algebra.

3 Quantum Gadget Duality

In this section, we prove a duality (Theorem 3.2) between tensors invariant under the action of $\text{Stab}(\mathcal{F})$ and the signatures of $\mathcal{F}$-gadgets. Theorem 3.2 and its proof are extensions of Regts' [28, Theorem 3] from edge-coloring models to general signature sets $\mathcal{F}$, largely translated into the language of signature matrices (see Lemma 3.1). This translation reveals that Theorem 3.2 and its proof are also nonplanar/nonquantum, orthogonal versions of similar results of Mančinska, Roberson, Cai and Young [25, 11, 41] – see Remark 3.1.

Say $\mathcal{F} \subset T(\mathbb{R}^q)$ is a *graded subalgebra* of $T(\mathbb{R}^q)$ if $\mathcal{F} \cap (\mathbb{R}^q)^{\otimes n}$ (the signatures of arity n) is a vector space over $\mathbb{R}$ (closed under $\mathbb{R}$-linear combinations) and $\mathcal{F}$ is closed under $\otimes$. Say $\mathcal{F}$ is *contraction-closed* if, for every n-ary $F \in \mathcal{F}$, the $(n-2)$-ary signature resulting from *contracting* any two inputs of F (connecting the corresponding dangling edges of the n-ary gadget consisting of a single vertex assigned F) is also in $\mathcal{F}$. The following theorem, due to Schrijver, is the key connection between algebra and combinatorics underlying our proof of Theorem 3.2 and, eventually, Theorem 2.3.

Theorem 3.1 ([32, Corollary 1e])**.** *Let $\mathcal{F} \subset T(\mathbb{R}^q)$. Then there is a subgroup $Q \subset O(q)$ with $\mathcal{F} = T(\mathbb{R}^q)^Q$ if and only if $\mathcal{F}$ is a contraction-closed graded subalgebra of $T(\mathbb{R}^q)$ containing I.*

Schrijver's statement of Theorem 3.1 requires that $\mathcal{F}$ be "nondegenerate", but it can be seen from the proof that it suffices to assume that $\mathcal{F}$ contains I (cf. Regts' restatement [28, Theorem 4]).

Definition 3.1 ($M(\mathcal{F}), M(T(\mathbb{R}^q))$)**.** Let $\mathcal{F} \subset T(\mathbb{R}^q)$ be a signature set. Define

$$M(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} \left(\bigcup_{\substack{m,d \in \mathbb{N} \\ m+d = \text{arity}(F)}} F^{m,d} \right)$$

to be the set of all flattenings of all tensors in $\mathcal{F}$. In particular,

$$M(T(\mathbb{R}^q)) = \bigcup_{m,d \in \mathbb{N}} \mathbb{R}^{q^m \times q^d}.$$

For a subset $P \subset M(T(\mathbb{R}^q))$, define $\langle P \rangle_{+,\circ,\otimes,\top} \subset M(T(\mathbb{R}^q))$ to be the set generated by P under $\circ, \otimes, \top$, and $\mathbb{R}$-linear combinations of matrices with matching dimensions.

Definition 3.2 (S_σ, S)**.** For permutation $\sigma \in S_n$, define the $2n$-ary *braid* signature $S_\sigma \in \mathcal{W}$ by $(S_\sigma)_{\mathbf{x}\mathbf{y}} = 1$ iff $x_i = y_{\sigma(i)}$ for every $i \in [n]$, and $(S_\sigma)_{\mathbf{x}\mathbf{y}} = 0$ otherwise.

Define the 4-ary 'swap' signature $S := S_{(01)}$. See Figure 3.1

Every permutation is a product of adjacent transpositions, so $S_\sigma^{n,n} \in \left\langle I, S^{2,2} \right\rangle_{\circ,\otimes,\top}$ [41, Lemma 3]. The following is a central object of study in the works of Mančinska, Roberson, Cai and Young [25, 11, 41].

Definition 3.3. For $C \subset M(T(\mathbb{R}^q))$, define $C(m,d) := C \cap \mathbb{R}^{q^m \times q^d}$. C is a *tensor category with duals* (TCWD) if it satisfies the following properties:

(i) For fixed m and d, $C(m,d)$ is a vector space over $\mathbb{R}$,

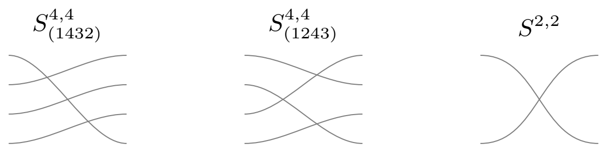


Figure 3.1: Braid gadgets and their signature matrices.

(ii) C is closed under $\circ, \otimes, \top$,

(iii) $I \in C(1,1)$,

(iv) $I^{0,2} \in C(0,2)$.

C is a *symmetric* tensor category with duals if it also satisfies

(v) $S^{2,2} \in C(2,2)$.

We next translate Theorem 3.1 into the language of TCWDs by flattening $\mathcal{F}$.

Lemma 3.1. *A subset $\mathcal{F} \subset T(\mathbb{R}^q)$ is a contraction-closed graded subalgebra of $T(\mathbb{R}^q)$ containing I if and only if $M(\mathcal{F})$ is a symmetric tensor category with duals.*

Proof. ($\Longrightarrow$): Let $\mathcal{F}$ be a contraction-closed graded subalgebra of $T(\mathbb{R}^q)$ containing I. If $F, G \in \mathcal{F}$ then the tensor/signature $F \otimes G$ has signature vector $f \otimes g$. Also, as observed by Schrijver [32], for any n-ary $F \in \mathcal{F}$, we can apply any permutation $\sigma \in S_n$ to the inputs of F by constructing $F \otimes I^{\otimes n}$, then contracting the ith input of F with one input of the $\sigma(i)$th copy of I.

First, $I, I^{0,2} \in M(\mathcal{F})$ because $I \in \mathcal{F}$. We can construct $S \in \mathcal{F}$ by contracting inputs to the first and third copies of I and second and fourth copies of I in $I^{\otimes 4} \in \mathcal{F}$, so $S^{2,2} \in M(\mathcal{F})$. Item (i) of Definition 3.3 holds because $\mathcal{F}$ is graded. Finally, we show that $M(\mathcal{F})$ satisfies item (ii). See Figure 3.2 (a),(b),(c), respectively.

- $\circ$: Let $F^{m,d}, G^{d,k} \in M(\mathcal{F})$. Construct the signature $K \in \mathcal{F}$ by starting with $F \otimes G \in \mathcal{F}$ and contracting the d lowest-indexed inputs of F (the right/column inputs in $F^{m,d}$) with the d highest-indexed inputs of G, in reverse order. Then $F^{m,d}G^{d,k} = K^{m,k} \in M(\mathcal{F})$.
- $\otimes$: Let $F^{m,d}, G^{m',d'} \in M(\mathcal{F})$. Construct the signature $K \in \mathcal{F}$ by starting with $F \otimes G \in \mathcal{F}$ and reordering the inputs of K as follows: the first m inputs to F, then all inputs to G, then the final d inputs to F. Then $F^{m,d} \otimes G^{m',d'} = K^{m+m',d'+d} \in M(\mathcal{F})$.
- $\top$: Let $F^{m,d} \in M(\mathcal{F})$. Construct the signature $K \in \mathcal{F}$ by reversing the input order of $F \in \mathcal{F}$. Then $(F^{m,d})^\top = K^{d,m} \in M(\mathcal{F})$.

($\Longleftarrow$): Let $M(\mathcal{F})$ be a symmetric TCWD, so $S, I \in \mathcal{F}$ and $\mathcal{F}$ is a subalgebra of $T(\mathbb{R}^q)$ because $M(\mathcal{F})$ is closed under $\otimes$ (and $f \otimes g = (F \otimes G)^{n,0}$) and is graded by part (i) of Definition 3.3. Recall that S_σ is constructible from I and S, so $S^{n,n}_\sigma \in M(\mathcal{F})$. Let $F \in \mathcal{F}$ have arity n. To contract inputs i and j of F, define $\sigma \in S_n$ to move inputs i and j to positions 1 and 2, and shift all other inputs down while maintaining their order. Then connect inputs i and j using $I^{0,2}$:

$$(I^{0,2} \otimes I^{\otimes n-2}) \circ S^{n,n}_\sigma \circ f \in M(\mathcal{F}) \tag{3.1}$$

is the signature vector of the ij-contraction of F. See Figure 3.3. $\square$

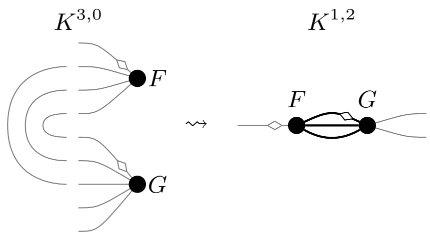
Define an action of $O(q)$ on $M(T(\mathbb{R}^q))$ by

$$HF^{m,d} := H^{\otimes m} F^{m,d} (H^\top)^{\otimes d}.$$

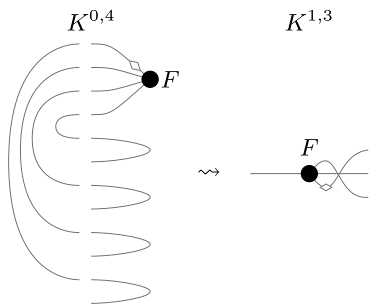
Using this action, define the matrix version of Definition 2.8: for $Q \subset O(q)$ and $\mathcal{F} \subset T(\mathbb{R}^q)$,

$$M(T(\mathbb{R}^q))^Q := \{F^{m,d} \in M(T(\mathbb{R}^q)) \mid H^{\otimes m} F^{m,d} (H^\top)^{\otimes d} = F^{m,d} \text{ for every } H \in Q\} \text{ and}$$
$$\mathrm{Stab}(M(\mathcal{F})) := \{H \in O(q) \mid H^{\otimes m} F^{m,d} (H^\top)^{\otimes d} = F^{m,d} \text{ for every } F^{m,d} \in M(\mathcal{F})\}.$$

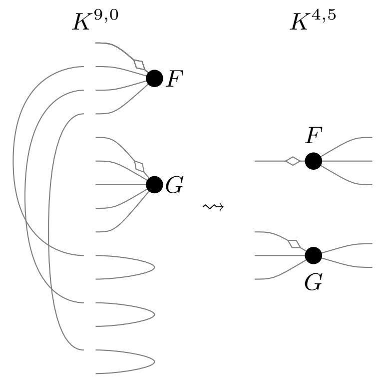
The next proposition shows that the matrix and signature versions of these definitions agree.



(a) Constructing $F^{1,3} \circ G^{3,2}$



(c) Constructing $(F^{3,1})^\top$



(b) Constructing $F^{1,3} \otimes G^{3,2}$

Figure 3.2: Realizing signature matrix $\circ, \otimes, \top$ using contractions, represented by a wire between two dangling edges. Signature inputs are in counterclockwise order, with a diamond indicating the first input.

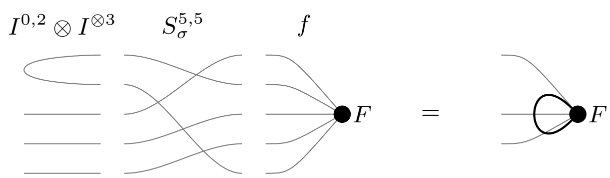


Figure 3.3: Contracting the 2nd and 5th inputs of F as in (3.1).

Proposition 3.1.

1. *For any subgroup $Q \subset O(q)$, $M(T(\mathbb{R}^q))^Q$ is a symmetric tensor category with duals.*
2. *For any signature F and orthogonal H, $(HF)^{m,d} = H^{\otimes m} F^{m,d} (H^\top)^{\otimes d}$ for any m, d.*
3. $\mathrm{Stab}(\mathcal{F}) = \mathrm{Stab}(M(\mathcal{F}))$.

Proof. If $F^{m,d}, G^{d,k} \in M(T(\mathbb{R}^q))^Q$ and $H \in Q$ then

$$H^{\otimes m}(F^{m,d}G^{d,k})(H^\top)^{\otimes k} = (H^{\otimes m}F^{m,d}(H^\top)^{\otimes d})(H^{\otimes d}G^{d,k}(H^\top)^{\otimes k}) = F^{m,d}G^{d,k},$$

so $F^{m,d}G^{d,k} \in M(T(\mathbb{R}^q))^Q$. Closure under the other operations $+, \otimes, \top$ follows similarly, so $M(T(\mathbb{R}^q))^Q$ satisfies items (i) and (ii) of Definition 3.3. Since $I \in \mathcal{W}$, Theorem 2.1 gives $I \in T(\mathbb{R}^q)^Q$, or $I^{2,0} \in M(T(\mathbb{R}^q))^Q$. Then $I^{0,2} = (I^{2,0})^\top \in M(T(\mathbb{R}^q))^Q$, and $I = I^{1,1} \in M(T(\mathbb{R}^q))^Q$ because $HIH^\top = I$ for $H \in Q \subset O(q)$, giving items (iii) and (iv). Hence $M(T(\mathbb{R}^q))^Q$ is a TCWD.

We now prove part 2. Let $G = HF$. Define the orthogonal antidiagonal block matrix B on $V(F) \sqcup V(G) = [2q]$ by

$$B := \begin{bmatrix} 0 & H^\top \\ H & 0 \end{bmatrix}. \tag{3.2}$$

Since $H^{\otimes n} f = g$ (so also $(H^\top)^{\otimes n} g = f$), Corollary A.2 with $m := n$ and $d := 0$ gives $f \oplus g = (F \oplus G)^{n,0} \in M(T(\mathbb{R}^{2q}))^{\langle B \rangle}$, which, by the previous paragraph, is a TCWD. Now, as in [11, Lemma 3], we use I and $I^{0,2}$ to 'pivot' any number of inputs of $f \oplus g$ from left to right (which preserves the cyclic dangling edge order) to show $(F \oplus G)^{m,d} \in M(T(\mathbb{R}^{2q}))^{\langle B \rangle}$ for any $m+d = n$. See Figure 3.4. Thus $B^{\otimes m}(F \oplus G)^{m,d} = (F \oplus G)^{m,d} B^{\otimes d}$,

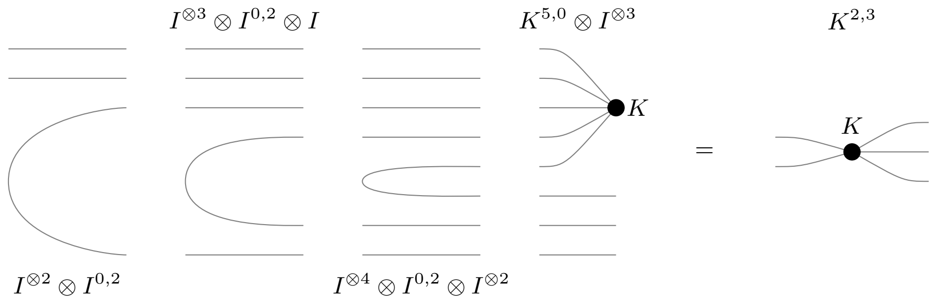


Figure 3.4: Pivoting three dangling edges of $K = F \oplus G$ from left to right using I and $I^{0,2}$.

so applying the other direction of Corollary A.2 gives $H^{\otimes m} F^{m,d} = G^{m,d} H^{\otimes d} = (HF)^{m,d} H^{\otimes d}$, and part 2 follows.

Part 3 follows directly from part 2 and the definitions. Finally, $S \in T(\mathbb{R}^q)^Q$ by Theorem 2.1 (as $S \in \mathcal{W}$), so $Q \subset \mathrm{Stab}(S) = \mathrm{Stab}(M(\{S\}))$ by part 3. Then $S^{2,2} \in M(T(\mathbb{R}^q))^Q$, giving item (v) of Definition 3.3 and proving part 1. ☐

That part 2 of Proposition 3.1 holds for any, not necessarily orthogonal, matrix A is a well-known fact, with a geometric intuition similar to Figure 2.3. We gave the above proof to mirror the proof of Cai and Young's similar [11, Lemma 6] for quantum orthogonal matrices, and to highlight the role of $I^{0,2}$ in the invariance of a TCWD under edge pivoting (or taking *duals* – see e.g. [11, Appendix D]).

Definition 3.4 **($\mathfrak{Q}_{\mathcal{F}}(m,d), \overline{\mathcal{F}}$).** A *$(m,d)$-quantum $\mathcal{F}$-gadget* is a formal (finite) $\mathbb{R}$-linear combination of gadgets in $\mathfrak{G}_{\mathcal{F}}(m,d)$. Let $\mathfrak{Q}_{\mathcal{F}}(m,d)$ be the collection of all (m,d)-quantum $\mathcal{F}$-gadgets, and $\mathfrak{Q}_{\mathcal{F}} = \bigcup_{m,d} \mathfrak{Q}_{\mathcal{F}}(m,d)$.

Extend the signature matrix function M linearly to $\mathfrak{Q}_{\mathcal{F}}$. Then define the *quantum gadget closure* $\overline{\mathcal{F}}$ of $\mathcal{F}$ as the set of signatures obtainable from $\mathcal{F}$ by quantum gadget construction:

$$\overline{\mathcal{F}} = \bigsqcup_{F^{m,d} \in M(\mathfrak{Q}_{\mathcal{F}})} F.$$

We use the disjoint union $\sqcup$ because different quantum gadgets could yield the same signature, and so that, if sets $\mathcal{F}$ and $\mathcal{G}$ are similar, then $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are similar, with the signature of $\mathbf{K} \in \mathfrak{Q}_{\mathcal{F}}$ corresponding to the signature of $\mathbf{K}_{\mathcal{F} \to \mathcal{G}} \in \mathfrak{Q}_{\mathcal{G}}$.

Since $I, S \in \mathcal{W} \subset \overline{\mathcal{F}}$ for any $\mathcal{F}$, and $\mathfrak{G}_{\mathcal{F}}$ is closed under gadget $\circ, \otimes$, and $\top$, $M(\mathfrak{Q}_{\mathcal{F}}) = M(\overline{\mathcal{F}})$ is a symmetric TCWD. Specifically, the next proposition, which is very similar to [41, Theorem 3] (the main difference being that we do not assume $\mathcal{EQ} \subset \mathcal{F}$), shows that $M(\mathfrak{Q}_{\mathcal{F}})$ is the symmetric TCWD generated by $M(\mathcal{F})$.

Proposition 3.2. $M(\mathfrak{Q}_{\mathcal{F}}) = \left\langle \left\{ I, I^{0,2}, S^{2,2} \right\} \cup \left\{ f \mid F \in \mathcal{F} \right\} \right\rangle_{+,\circ,\otimes,\top}$.

Proof. $I, S \in \mathcal{W} \subset \overline{\mathcal{F}}$, so $I^{1,1}, I^{0,2}, S^{2,2} \in M(\mathfrak{Q}_{\mathcal{F}})$. The $\supseteq$ direction follows. For the $\subseteq$ direction, it suffices to show that $M(\mathfrak{G}_{\mathcal{F}}) \subseteq \left\langle \left\{ I, I^{0,2}, S^{2,2} \right\} \cup \left\{ f \mid F \in \mathcal{F} \right\} \right\rangle_{\circ,\otimes,\top}$. Let $\mathbf{K} \in \mathfrak{G}_{\mathcal{F}}(m,d)$ be a gadget. By using $I^{2,0} = (I^{0,2})^\top$ to pivot dangling edges to the left as in the proof of Proposition 3.1, we may assume $d = 0$. Suppose $\mathbf{K}$ has p vertices, assigned signatures $F_1, \ldots, F_p$. Break all internal edges of $\mathbf{K}$ and orient the resulting dangling edges to the left to create a gadget with signature

$$\bigotimes_{i=1}^{p} f_i \in \langle f \mid F \in \mathcal{F} \rangle_{\circ,\otimes,\top}. \tag{3.3}$$

Now we reconstruct $\mathbf{K}$ by reconnecting (contracting) the broken internal edges. Using the formula (3.1) for arbitrary contraction, we see that $\mathbf{K} \in \left\langle \left\{ I, I^{0,2}, S^{2,2} \right\} \cup \left\{ f \mid F \in \mathcal{F} \right\} \right\rangle_{+,\circ,\otimes,\top}$. ☐

Proposition 3.3. $\mathrm{Stab}(\mathcal{F}) = \mathrm{Stab}(M(\mathfrak{Q}_{\mathcal{F}})) = \mathrm{Stab}(\overline{\mathcal{F}})$.

Proof. By part 3 of Proposition 3.1, $\mathrm{Stab}(M(\mathfrak{Q}_{\mathcal{F}})) = \mathrm{Stab}(M(\overline{\mathcal{F}})) = \mathrm{Stab}(\overline{\mathcal{F}})$. Since $M(\mathfrak{Q}_{\mathcal{F}})$ contains (the vector forms of) $\mathcal{F}$, we have $\mathrm{Stab}(M(\mathfrak{Q}_{\mathcal{F}})) \subseteq \mathrm{Stab}(\mathcal{F})$. By Proposition 3.2, $M(\mathfrak{Q}_{\mathcal{F}})$ is the symmetric TCWD generated by $\mathcal{F}$ (in particular, is the smallest symmetric TCWD containing $\mathcal{F}$). By part 1 of Proposition 3.1, $M(T(\mathbb{R}^q))^{\mathrm{Stab}(\mathcal{F})}$ is a symmetric TCWD containing $\mathcal{F}$, so we must have $M(\mathfrak{Q}_{\mathcal{F}}) \subseteq M(T(\mathbb{R}^q))^{\mathrm{Stab}(\mathcal{F})}$, which implies $\mathrm{Stab}(\mathcal{F}) \subseteq \mathrm{Stab}(M(\mathfrak{Q}_{\mathcal{F}}))$. □

Now we come to the main result of this section. We follow Regts' proof for the special case of edge coloring models [27, Theorem 6.8].

Theorem 3.2. *Let $\mathcal{F}$ be a set of real-valued signatures on domain $[q]$. Then*

$$T(\mathbb{R}^q)^{\mathrm{Stab}(\mathcal{F})} = \overline{\mathcal{F}}.$$

Equivalently, by flattening both sides and applying part 3 of Proposition 3.1, we have, for any m, d,

$$\left(\mathbb{R}^{q^m \times q^d}\right)^{\mathrm{Stab}(\mathcal{F})} = M(\mathfrak{Q}_{\mathcal{F}}(m, d)).$$

Proof. By Proposition 3.2, $M(\overline{\mathcal{F}}) = M(\mathfrak{Q}_{\mathcal{F}})$ is a TCWD and $S \in \overline{\mathcal{F}}$, so $\overline{\mathcal{F}}$ is a contraction-closed graded subalgebra of $T(\mathbb{R}^q)$ containing I by Lemma 3.1. Thus, by Theorem 3.1, there is a subgroup $G \subseteq O_q(\mathbb{R})$ such that $\overline{\mathcal{F}} = T(\mathbb{R}^q)^G$. This gives $G \subseteq \mathrm{Stab}(\overline{\mathcal{F}})$, so $T(\mathbb{R}^q)^{\mathrm{Stab}(\overline{\mathcal{F}})} \subseteq T(\mathbb{R}^q)^G$, and also $T(\mathbb{R}^q)^G = \overline{\mathcal{F}} \subseteq T(\mathbb{R}^q)^{\mathrm{Stab}(\overline{\mathcal{F}})}$. Therefore, applying Proposition 3.3,

$$T(\mathbb{R}^q)^{\mathrm{Stab}(\mathcal{F})} = T(\mathbb{R}^q)^{\mathrm{Stab}(\overline{\mathcal{F}})} = T(\mathbb{R}^q)^G = \overline{\mathcal{F}}.$$ □

A direct consequence is the FFT for $O(q)$. Schrijver [32] gives another proof based on Theorem 3.1.

Proof of Theorem 2.1. Putting $\mathcal{F} = \varnothing$ (the empty set) in Theorem 3.2 gives

$$T(\mathbb{R}^q)^{O(q)} = T(\mathbb{R}^q)^{\mathrm{Stab}(\varnothing)} = \overline{\varnothing} = \langle \mathcal{W} \rangle_+,$$

as $\varnothing$-gadgets must have no vertices. □

Remark 3.1. The reasons we translated from Schrijver and Regts [32, 28]'s language of contraction-closed graded subalgebras to Mančinska, Roberson, Cai and Young [25, 11, 41]'s language of symmetric TCWDs are twofold. The first is to highlight the similarities between this section's results and the results of the latter group. Replacing $O(q)$ with S_q (the symmetric group of permutation matrices) or S_q^+ (the *quantum symmetric group*), enforcing that the TCWDs contain $M(\mathcal{EQ})$ (corresponding to quantum gadgets in the context of #CSP), and dropping the symmetry condition on the TCWDs in the second case (corresponding to quantum gadget planarity), we find our constructions and theorems in this section analogous to those of [41] or [25, 11], respectively. In particular, $\mathrm{Stab}(\mathcal{F})$ becomes $\mathrm{Aut}(\mathcal{F})$ (the automorphism group of $\mathcal{F}$ [41]) or $\mathrm{Qut}(\mathcal{F})$ (the *quantum automorphism group* of $\mathcal{F}$ [25, 11]), respectively. Then $M(T(\mathbb{R}^q))^{\mathrm{Stab}(\mathcal{F})}$ becomes the *intertwiner space* of $\mathrm{Qut}(\mathcal{F})$ or $\mathrm{Aut}(\mathcal{F})$. Applying Lemma 3.1, we find Theorem 3.1 perfectly analogous to the *Tannaka-Krein duality* used by [41] and [25, 11] to prove results analogous to Theorem 3.2. As in this work, [25, 11, 41] use their versions of Theorem 3.2 to prove their respective versions of Theorem 2.3, but due to the nature of $O(q)$ – in particular, $O(q)$ has no construction analogous to the *orbits* and *orbitals* of S_q and S_q^+ – we take a different route for the rest of the proof of Theorem 2.3.

The second reason is that, just as the main result of [11] is a planar, quantum version of the main result of [41], our Theorem 2.3 should have a planar, quantum version (see Conjecture 6.1 below). Arbitrary contractions do not respect planarity, but, by simply removing the symmetry condition (i.e. removing $S^{2,2}$ in Proposition 3.2), we easily enforce planarity in the language of TCWDs.

The following lemma is this section's main contribution to proving Theorem 2.3, and is the only nonconstructive step in the proof.

Lemma 3.2. *If $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable, then there is an $H \in \mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$ with $H|_{V(\mathcal{F}),V(\mathcal{G})} \neq 0$ or $H|_{V(\mathcal{G}),V(\mathcal{F})} \neq 0$.*

Proof. Suppose every $H \in \mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$ has $H|_{V(\mathcal{F}),V(\mathcal{G})} = H|_{V(\mathcal{G}),V(\mathcal{F})} = 0$ (i.e. is block diagonal). Then the block diagonal matrix $A = \begin{bmatrix} I & 0 \\ 0 & 2I \end{bmatrix}$ satisfies $HA = AH$, hence $HAH^T = A$, for every $H \in \mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$. Thus, by Theorem 3.2 with $m = d = 1$, A is realizable as the signature matrix of a binary quantum gadget – that is, there exist binary $(\mathcal{F} \oplus \mathcal{G})$-gadgets $\mathbf{K}^1, \ldots \mathbf{K}^p \in \mathfrak{G}_{\mathcal{F}}(1,1)$ and $c_1, \ldots, c_p \in \mathbb{R}$ such that

$$A = \sum_{i=1}^{p} c_i M(\mathbf{K}^i). \tag{3.4}$$

Any connected component of a gadget $\mathbf{K}^i$ disconnected from the component(s) of $\mathbf{K}^i$ containing the two dangling edges contributes only an overall multiplicative factor; by absorbing this factor into c_i, we may assume each $\mathbf{K}^i$ has no components without a dangling edge. By definition of $\oplus$, inputting an $x \in V(\mathcal{F})$ along a dangling edge of $\mathbf{K}^i$ forces any edge assignment with nonzero value to assign an element of $V(\mathcal{F})$ to every edge in the connected component. So, for any $x, y \in V(\mathcal{F})$, we have $M(\mathbf{K}^i)_{x,y} = M(\mathbf{K}^i_{(\mathcal{F} \oplus \mathcal{G}) \to \mathcal{F}})_{x,y}$. Similar reasoning applies to $\mathcal{G}$, so the $V(\mathcal{F}), V(\mathcal{F})$ and $V(\mathcal{G}), V(\mathcal{G})$ blocks of (3.4) are

$$I = \sum_{i=1}^{p} c_i M(\mathbf{K}^i_{(\mathcal{F} \oplus \mathcal{G}) \to \mathcal{F}}) \text{ and } 2I = \sum_{i=1}^{p} c_i M(\mathbf{K}^i_{(\mathcal{F} \oplus \mathcal{G}) \to \mathcal{G}}), \tag{3.5}$$

respectively. Let Ω^i be the $\mathcal{F}$-grid resulting from connecting the two dangling edges of $\mathbf{K}^i_{(\mathcal{F} \oplus \mathcal{G}) \to \mathcal{F}}$. Then taking the trace of the equations in (3.5) gives

$$\sum_{i=1}^{p} c_i \operatorname{Holant}_{\Omega^i}(\mathcal{F}) = q \neq 2q = \sum_{i=1}^{p} c_i \operatorname{Holant}_{\Omega^i_{\mathcal{F} \to \mathcal{G}}}(\mathcal{G}),$$

so there is some i for which $\operatorname{Holant}_{\Omega^i}(\mathcal{F}) \neq \operatorname{Holant}_{\Omega^i_{\mathcal{F} \to \mathcal{G}}}(\mathcal{G})$. $\square$

4 Domain Induction: The Proof of Theorem 2.3

The following definition and its applications below borrow a simple but powerful idea of Shao and Cai [33, Section 8.2]: isolating all vertices of an $\mathcal{F} \cup \{F\}$-grid Ω assigned F, the rest of Ω is an $\mathcal{F}$-gadget. If adding certain F to $\mathcal{F}$ and G to $\mathcal{G}$ simplifies the proof of ortho-equivalence, we use the subgadget perspective to obtain Holant-indistinguishability of $\mathcal{F} \cup \{F\}$ and $\mathcal{G} \cup \{G\}$ from the Holant-indistinguishability of $\mathcal{F}$ and $\mathcal{G}$.

Definition 4.1 (Subgadget, $\overline{\mathbf{K}}$)**.** Let $\mathbf{J}$ be a gadget. A *subgadget* $\mathbf{K} \subset \mathbf{J}$ induced by a subset $U \subset V(\mathbf{J})$ of vertices of $\mathbf{J}$ is a gadget composed of the vertices in U and *all* of their incident edges: internal edges of $\mathbf{J}$ incident to exactly one vertex in U become new dangling edges of $\mathbf{K}$. For any $\mathbf{K} \subset \mathbf{J}$, there is a unique (up to left/right dangling edge pivoting) $\overline{\mathbf{K}} \subset \mathbf{J}$, induced by $V(J) \setminus U$, called the *complement* of $\mathbf{K}$, such that, upon reconnecting the new dangling edges of $\mathbf{K}$ and $\overline{\mathbf{K}}$, we recover $\mathbf{J}$.

We often take $\mathbf{J}$ to be a signature grid (0-ary gadget) Ω, in which case $\Omega = \langle \mathbf{K}, \overline{\mathbf{K}} \rangle$.

Say $\mathcal{F}$ is *quantum-gadget-closed* if $\mathcal{F} = \overline{\mathcal{F}}$. The following proposition, a nonplanar, orthogonal version of [11, Lemmas 31 and 32], lets us assume $\mathcal{F}$ and $\mathcal{G}$ are quantum-gadget-closed when proving Theorem 2.3.

Lemma 4.1. *For any signature sets $\mathcal{F}$ and $\mathcal{G}$,*

1. *$\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable iff $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are Holant-indistinguishable.*
2. *For any orthogonal H, $H\mathcal{F} = \mathcal{G}$ iff $H\overline{\mathcal{F}} = \overline{\mathcal{G}}$.*

Proof. 1: Any $\mathcal{F}$-grid or $\mathcal{G}$-grid is also a $\overline{\mathcal{F}}$-grid or $\overline{\mathcal{G}}$-grid, respectively, giving the ($\Longleftarrow$) direction. For ($\Longrightarrow$), we can express any $\overline{\mathcal{F}}$-grid Ω as a quantum $\mathcal{F}$-grid by, for every vertex v in Ω assigned a signature $F^v \in \overline{\mathcal{F}} \setminus \mathcal{F}$, replacing the subgadget of Ω induced by v by the quantum $\mathcal{F}$-gadget with signature F^v, then linearly expanding to obtain a quantum $\mathcal{F}$-grid. Do the same for $\Omega_{\overline{\mathcal{F}} \to \overline{\mathcal{G}}}$. By assumption, the resulting corresponding quantum $\mathcal{F}$ and $\mathcal{G}$-grids have the same values, so $\operatorname{Holant}_\Omega = \operatorname{Holant}_{\Omega_{\overline{\mathcal{F}} \to \overline{\mathcal{G}}}}$.

2: ($\Longleftarrow$) is direct from $\mathcal{F} \subset \overline{\mathcal{F}}$ and $\mathcal{G} \subset \overline{\mathcal{G}}$. For ($\Longrightarrow$), suppose $H\,\mathcal{F} = \mathcal{G}$ for orthogonal H. Consider $\mathcal{F} \oplus \mathcal{G}$ and the block matrix B in (3.2). As in the proof of Proposition 3.1, $B \in \mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$. Then, by Proposition 3.3, $B \in \mathrm{Stab}(\overline{\mathcal{F} \oplus \mathcal{G}})$. Now consider n-ary $\overline{\mathcal{F}} \ni F' \rightsquigarrow G' \in \overline{\mathcal{G}}$, where F' is the signature of $\mathbf{K} \in \mathfrak{Q}_{\mathcal{F}}(n, 0)$, and G' is the signature of $\mathbf{K}_{\mathcal{F}\to\mathcal{G}}$. We may assume no gadget in $\mathbf{K}$ has a component without dangling edges by absorbing the factors from such components (which must be equal for $\mathbf{K}$ and $\mathbf{K}_{\mathcal{F}\to\mathcal{G}}$ by Corollary 2.1) into the gadget's coefficient, so, by definition of $\oplus$, $M(\mathbf{K}_{\mathcal{F}\to\mathcal{F}\oplus\mathcal{G}})|_{V(\mathcal{F})^n} = M(\mathbf{K}) = f'$ and $M(\mathbf{K}_{\mathcal{F}\to\mathcal{F}\oplus\mathcal{G}})|_{V(\mathcal{G})^n} = M(\mathbf{K}_{\mathcal{F}\to\mathcal{G}}) = g'$ (other blocks are not necessarily 0 if $\mathbf{K}$ is still disconnected). Then, setting $K^{n,0} := M(\mathbf{K}_{\mathcal{F}\to\mathcal{F}\oplus\mathcal{G}}) \in M(\overline{\mathcal{F} \oplus \mathcal{G}})$ and considering the equation $K^{n,0} = B^{\otimes n} K^{n,0}$ in block form as in (A.2), we have

$$\begin{bmatrix} f' \\ * \\ \vdots \\ * \\ g' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & (H^\top)^{\otimes n} \\ 0 & 0 & \dots & * & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & * & \dots & 0 & 0 \\ H^{\otimes n} & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} f' \\ * \\ \vdots \\ * \\ g' \end{bmatrix} = \begin{bmatrix} (H^\top)^{\otimes n} g' \\ * \\ \vdots \\ * \\ H^{\otimes n} f' \end{bmatrix}.$$

In particular, $g' = H^{\otimes n} f'$. Therefore $H\,\overline{\mathcal{F}} = \overline{\mathcal{G}}$. ☐

In the context of #CSP, $\mathcal{EQ}$ enables the gadget-construction of entrywise products of arbitrary signatures. This is a basis for Vandermonde interpolation, a powerful technique in the study of counting problems. In the general Holant setting, we cannot construct the entrywise product of arbitrary signatures, but, for binary signatures whose matrix forms are diagonal, composition is equivalent to entrywise product. We use the following form of Vandermonde interpolation, which follows directly from the similar [18, Lemma 2.3].

Proposition 4.1. *Let $V \subset \mathbb{R}^q$ be a vector space closed under entrywise product and containing the all-ones vector. For $v \in V$ and $a \in \mathbb{R}$, define the indicator vector $v^a \in \{0,1\}^q$ by $(v^a)_x = 1$ if $v_x = a$ and $(v^a)_x = 0$ otherwise. Then $v^a \in V$ for every $v \in V$ and $a \in \mathbb{R}$.*

Proposition 4.1 will give us the following matrices, whose diagonals we identify with $\mathbb{R}^q$: For $Z \subset [q]$, define the diagonal matrix (binary signature) $\mathbb{1}_Z \in \{0,1\}^{q\times q}$ by $(\mathbb{1}_Z)_{x,x} = 1$ if $x \in Z$, and $(\mathbb{1}_Z)_{x,x} = 0$ otherwise.

Lemma 4.2. *Let $\mathcal{F}$ and $\mathcal{G}$, on domain $[q]$, be Holant-indistinguishable, and let $Z \subset [q]$. If $\mathcal{F}$ and $\mathcal{G}$ contain corresponding copies of $\mathbb{1}_Z$, then $\mathcal{F}|_Z$ and $\mathcal{G}|_Z$ are Holant-indistinguishable.*

Proof. Let Ω be a $\mathcal{F}|_Z$-grid. Construct a $\mathcal{F}$-grid Ω' from Ω by replacing every signature $F|_Z \in \mathcal{F}|_Z$ with the corresponding $F \in \mathcal{F}$, then replacing every edge with a degree-2 vertex assigned $\mathbb{1}_Z \in \mathcal{F}$. Since $\mathbb{1}_Z$ is 0 when given any input not in Z, any edge assignment sending an edge outside of Z contributes 0 to $\mathrm{Holant}_{\Omega'}(\mathcal{F})$. Furthermore, on inputs from Z, $\mathbb{1}_Z$ acts identically to an edge (as I) in a $\mathrm{Holant}(\mathcal{F}|_Z)$ grid. Similar reasoning applies to $\mathcal{G}$ and $\mathcal{G}|_Z$, so

$$\mathrm{Holant}_\Omega(\mathcal{F}|_Z) = \mathrm{Holant}_{\Omega'}(\mathcal{F}) = \mathrm{Holant}_{\Omega'_{\mathcal{F}\to\mathcal{G}}}(\mathcal{G}) = \mathrm{Holant}_{\Omega_{\mathcal{F}|_Z\to\mathcal{G}|_Z}}(\mathcal{G}|_Z).$$ ☐

Lemma 4.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be signature sets on domain $[q]$, and suppose Theorem 2.3 holds for all $\mathcal{F}'$, $\mathcal{G}'$ on domain smaller than q. If $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable and contain corresponding copies of a diagonal matrix (binary signature) $D \notin \mathrm{span}(I)$, then $\mathcal{F}$ and $\mathcal{G}$ are ortho-equivalent.*

Proof. By Lemma 4.1, we may replace $\mathcal{F}$ and $\mathcal{G}$ with $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ to assume $\mathcal{F}$ and $\mathcal{G}$ are quantum-gadget-closed. Since $D \notin \mathrm{span}(I)$, there exist $x, y \in [q]$ such that $D_{x,x} \neq D_{y,y}$, so the sets

$$X = \{z \in [q] : D_{z,z} = D_{x,x}\} \text{ and } Y = [q] \setminus X$$

are are nontrivial partition of $[q]$. Since $\mathcal{F}$ is quantum-gadget-closed, it contains $I \in \mathcal{W}$. Consider the subalgebra $\langle D, I\rangle_{+,\circ} \subset \overline{\mathcal{F}} = \mathcal{F}$. Since D and I are diagonal, composition $\circ$ is equivalent to entrywise multiplication in $\langle D, I\rangle_{+,\circ}$. Therefore, by Proposition 4.1 (identifying the matrix diagonals with $\mathbb{R}^q$), we have $\mathbb{1}_X, \mathbb{1}_Y \in \langle D, I\rangle_{+,\circ} \subset \mathcal{F}$. Applying the same interpolation in $\mathcal{G}$, we obtain corresponding copies of $\mathbb{1}_X, \mathbb{1}_Y \in$

$\mathcal{G}$. Now, applying Lemma 4.2 with $Z := X$, we conclude $\mathcal{F}|_X$ and $\mathcal{G}|_X$ are Holant-indistinguishable. Furthermore, $|X| < q$, so, by assumption, there is an orthogonal matrix $H_X \in \mathbb{R}^{X\times X}$ satisfying

$$H_X\,\mathcal{F}|_X = \mathcal{G}|_X. \tag{4.1}$$

For $b \in [q]$, let $\Delta_b \in \mathbb{R}^q$ be the unary *pinning signature* defined by $\Delta_b(x) = \begin{cases} 1 & b = x \\ 0 & b \neq x \end{cases}$ for $x \in [q]$. Call signatures in $\{\Delta_b \mid b \in X\}$ *X-pins*. Let I_Y be the identity operator on $\mathbb{R}^{Y\times Y}$ and define

$$\mathcal{F}' = \mathcal{F} \sqcup \{\Delta_b \mid b \in X\} \text{ and } \mathcal{G}' = \big((H_X^{-1} \oplus I_Y)\,\mathcal{G}\big) \sqcup \{\Delta_b \mid b \in X\}. \tag{4.2}$$

Claim 4.1. $\mathcal{F}'$ and $\mathcal{G}'$ are Holant-indistinguishable.

To see this, let Ω be a connected $\mathcal{F}'$-grid. If Ω contains no X-pins then it is an $\mathcal{F}$-grid, and $\Omega_{\mathcal{F}'\to\mathcal{G}'}$ is the corresponding $(H_X^{-1} \oplus I_Y)\,\mathcal{G}$-grid, so by assumption and Corollary 2.1, $\text{Holant}_\Omega = \text{Holant}_{\Omega_{\mathcal{F}'\to\mathcal{G}'}}$. If Ω contains two adjacent X-pins, then, since Ω is connected, its underlying multigraph consists only of these two vertices. X-pins in $\mathcal{F}'$ correspond to identical X-pins in $\mathcal{G}'$, so again $\text{Holant}_\Omega = \text{Holant}_{\Omega_{\mathcal{F}'\to\mathcal{G}'}}$. Otherwise, Ω contains p pairwise non-adjacent vertices assigned X-pins. Let $\mathbf{K} \subset \Omega$ be a subgadget induced by these p vertices, so $\mathbf{K}$'s signature is $\bigotimes_{i=1}^p \Delta_{b_i}$ for some $b_1, \dots, b_p \in X$. Since $\mathbf{K}$ includes *all* the vertices in Ω assigned X-pins, its complement $\overline{\mathbf{K}}$ is an $\mathcal{F}$-gadget. See Figure 4.1. Hence the signature F of $\overline{\mathbf{K}}$ is in $\mathcal{F}$, as

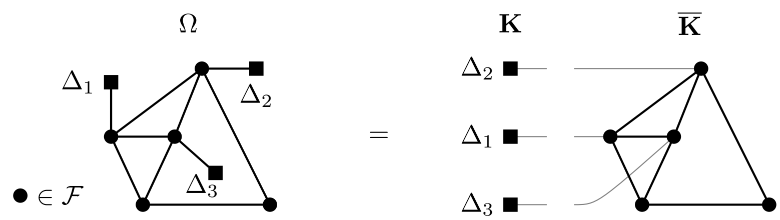


Figure 4.1: An $\mathcal{F}'$-grid Ω and its pin-induced subgadget $\mathbf{K}$ and complement $\overline{\mathbf{K}}$

$\mathcal{F}$ is quantum-gadget-closed. From $\Omega = \langle \mathbf{K}, \overline{\mathbf{K}} \rangle$, we obtain

$$\text{Holant}_\Omega = \left\langle \bigotimes_{i=1}^p \Delta_{b_i}, F \right\rangle = F_{\mathbf{b}},$$

where $\mathbf{b} = b_1 b_2 \dots b_p$. A similar calculation, using the fact that $\mathcal{G}$ is also quantum-gadget-closed, shows $\text{Holant}_{\Omega_{\mathcal{F}'\to\mathcal{G}'}} = ((H_X^{-1} \oplus I_Y)G)_{\mathbf{b}}$ for $\mathcal{G} \ni G \leftrightsquigarrow F$. Then, since $\mathbf{b} \in X^p$, we have, using (4.1),

$$\text{Holant}_\Omega = F_{\mathbf{b}} = (F|_X)_{\mathbf{b}} = (H_X^{-1} G|_X)_{\mathbf{b}} = (((H_X^{-1} \oplus I_Y)G)|_X)_{\mathbf{b}} = ((H_X^{-1} \oplus I_Y)G)_{\mathbf{b}} = \text{Holant}_{\Omega_{\mathcal{F}'\to\mathcal{G}'}}$$

(where the fourth equality uses Corollary A.1 with $K := G$ and $H := H_X^{-1} \oplus I_Y$). So $\mathcal{F}'$ and $\mathcal{G}'$ are Holant-indistinguishable, completing the proof of Claim 4.1.

By Lemma 4.1, $\overline{\mathcal{F}'}$ and $\overline{\mathcal{G}'}$ are also Holant-indistinguishable and, by Proposition 2.1 and Proposition 2.2, it suffices to show $\overline{\mathcal{F}'}$ and $\overline{\mathcal{G}'}$ are ortho-equivalent to complete the proof. Note that $\overline{\mathcal{F}'}$ and $\overline{\mathcal{G}'}$ still contain $\mathbb{1}_Y$ (as $\mathbb{1}_Y$, being 0 on X, is unaffected by the transform $H_X^{-1} \oplus I_Y$), so we may apply Lemma 4.2 to conclude $\big(\overline{\mathcal{F}'}\big)\big|_Y$ and $\big(\overline{\mathcal{G}'}\big)\big|_Y$ are Holant-indistinguishable. Again, since $|Y| < q$, there is an orthogonal $H_Y \in \mathbb{R}^{Y\times Y}$ such that $H_Y\big(\overline{\mathcal{F}'}\big)\big|_Y = \big(\overline{\mathcal{G}'}\big)\big|_Y$. Define

$$\mathcal{F}'' = \overline{\mathcal{F}'} \sqcup \{\Delta_b \mid b \in Y\} \text{ and } \mathcal{G}'' = \big((I_X \oplus H_Y^{-1})\overline{\mathcal{G}'}\big) \sqcup \{\Delta_b \mid b \in Y\}. \tag{4.3}$$

By Proposition 2.1 and Proposition 2.2, it suffices to show that $\mathcal{F}''$ and $\mathcal{G}''$ are ortho-equivalent. As $\overline{\mathcal{F}'}$ and $\overline{\mathcal{G}'}$ are, like $\mathcal{F}$ and $\mathcal{G}$, quantum-gadget-closed and Holant-indistinguishable, we repeat the proof of Claim 4.1, with with (4.3) in place of (4.2), to show that $\mathcal{F}''$ and $\mathcal{G}''$ are Holant-indistinguishable. Observe that, by definition, $\mathcal{F}''$ and $\mathcal{G}''$ contain corresponding copies of all Y-pins $\{\Delta_b \mid b \in Y\}$. Furthermore, $\mathcal{F}'' \supset \overline{\mathcal{F}'} \supset \mathcal{F}' \supset \{\Delta_b \mid b \in X\}$ and

$$\mathcal{G}'' \supset (I_X \oplus H_Y^{-1})\overline{\mathcal{G}'} \supset (I_X \oplus H_Y^{-1})\,\mathcal{G}' \supset (I_X \oplus H_Y^{-1})\{\Delta_b \mid b \in X\} = \{\Delta_b \mid b \in X\},$$

where the final equality holds because the X-pins are zero on Y, so are unaffected by the transform $(I_X \oplus H_Y^{-1})$. Thus $\mathcal{F}''$ and $\mathcal{G}''$ contain corresponding copies of all pins Δ_b for $b \in [q]$. We claim that this implies that $\mathcal{F}'' = \mathcal{G}''$. To see this, consider any $\mathcal{F}'' \ni F \leftrightsquigarrow G \in \mathcal{G}''$ of common arity n. For any $\mathbf{x} \in [q]^n$, since $\mathcal{F}''$ and $\mathcal{G}''$ are Holant-indistinguishable, we have

$$F_{\mathbf{x}} = \left\langle F, \bigotimes_{i=1}^{n} \Delta_{x_i} \right\rangle = \left\langle G, \bigotimes_{i=1}^{n} \Delta_{x_i} \right\rangle = G_{\mathbf{x}}.$$

Thus $F = G$ for every $\mathcal{F}'' \ni F \leftrightsquigarrow G \in \mathcal{G}''$, so $\mathcal{F}'' = \mathcal{G}''$. □

For any set S of matrices and matrices A, B, let $A \circ S \circ B := \{AQB \mid Q \in S\}$.

Proposition 4.2. *For any $\mathcal{F}$ and orthogonal matrix H,* $\mathrm{Stab}(H\,\mathcal{F}) = H \circ \mathrm{Stab}(\mathcal{F}) \circ H^\top$.

Proof. By parts 2 and 3 of Proposition 3.1, we have

$$\begin{aligned}
\mathrm{Stab}(H\,\mathcal{F}) &= \mathrm{Stab}(M(H\,\mathcal{F})) \\
&= \{A \in O(q) \mid A^{\otimes m}(HF)^{m,d}(A^\top)^{\otimes m} = (HF)^{m,d} \text{ for all } (HF)^{m,d} \in M(H\,\mathcal{F})\} \\
&= \{A \in O(q) \mid A^{\otimes m}H^{\otimes m}F^{m,d}(H^\top)^{\otimes d}(A^\top)^{\otimes d} = H^{\otimes m}F^{m,d}(H^\top)^{\otimes d} \text{ for all } F^{m,d} \in M(\mathcal{F})\} \\
&= \{A \in O(q) \mid (H^\top AH)^{\otimes m}F^{m,d}((H^\top AH)^\top)^{\otimes d} = F^{m,d} \text{ for all } F^{m,d} \in M(\mathcal{F})\} \\
&= \{A \in O(q) \mid H^\top AH \in \mathrm{Stab}(M(\mathcal{F}))\} \\
&= H \circ \mathrm{Stab}(M(\mathcal{F})) \circ H^\top = H \circ \mathrm{Stab}(\mathcal{F}) \circ H^\top.
\end{aligned}$$
□

The final step is to realize the diagonal matrix D in the statement of Lemma 4.3, and apply induction.

Proof of Theorem 2.3. $(ii) \implies (i)$ is Corollary 2.1 (which also follows directly from part 2 of Lemma 4.1: viewing Ω as a 0-ary $\mathcal{F}$-gadget, we have $\mathrm{Holant}_\Omega = M(\Omega) = H^{\otimes 0}M(\Omega) = M(\Omega_{\mathcal{F}\to\mathcal{G}}) = \mathrm{Holant}_{\Omega_{\mathcal{F}\to\mathcal{G}}}$). We show $(i) \implies (ii)$. Let $\mathcal{F}, \mathcal{G}$ be Holant-indistinguishable. We proceed by induction on the domain size q. If $q = 1$ then the only orthogonal matrices in $\mathbb{R}^{q\times q}$ are $\pm I$ and every signature $F \in \mathbb{R}^{[1]^n}$, regardless of arity n, has a single entry, which we denote by $F_0 \in \mathbb{R}$. Let $F \leftrightsquigarrow G$ have even arity n. Contracting $\frac{n}{2}$ pairs of inputs of both F and G, we obtain corresponding signature grids with values F_0 and G_0, respectively, so $F_0 = G_0$, hence $F = G$. For $F \leftrightsquigarrow G$ of odd arity n,

$$F_0^2 = \|F\|^2 = \langle F, F\rangle = \langle G, G\rangle = \|G\|^2 = G_0^2,$$

so $F_0 = \pm G_0$. Let $F \leftrightsquigarrow G$ have odd arity n and $F' \leftrightsquigarrow G'$ have odd arity n', all nonzero. Then $F^{\otimes n'} \otimes (F')^{\otimes n}$ has even arity $2nn'$; contracting its nn' pairs of inputs gives a $\mathcal{F}$-grid with value $F_0^{n'}(F_0')^n$. The corresponding $\mathcal{G}$-grid has value $G_0^{n'}(G_0')^n$. Let $G_0 = (-1)^a F_0$ and $G_0' = (-1)^{a'} F_0'$ for $a, a' \in \{0, 1\}$. Then

$$F_0^{n'}(F_0')^n = G_0^{n'}(G_0')^n = (-1)^{an' + a'n} F_0^{n'}(F_0')^n = (-1)^{a+a'} F_0^{n'}(F_0')^n$$

(where in the final equality we used that n, n' are odd), so $a = a'$. Thus there is a common $a \in \{0, 1\}$ such that $((-1)^a)^n F = G$ for every n-ary $F \leftrightsquigarrow G$, so $(-I)^a\,\mathcal{F} = \mathcal{G}$.

Now assume $q > 1$. By Lemma 4.1, we may assume $\mathcal{F}$ and $\mathcal{G}$ are quantum-gadget-closed. By Lemma 3.2, there is an $H \in \mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$ with either $H|_{\mathcal{F},\mathcal{G}} \neq 0$ or $H|_{\mathcal{G},\mathcal{F}} \neq 0$. Assume WLOG that $H|_{\mathcal{G},\mathcal{F}} \neq 0$. Let $H|_{\mathcal{G},\mathcal{F}} = U^\top DV$ be the singular value decomposition of $H|_{\mathcal{G},\mathcal{F}}$, with U, V orthogonal and $D \neq 0$ diagonal, all real. By Proposition 2.1, we may replace $\mathcal{F}$ with $V\,\mathcal{F}$ and $\mathcal{G}$ with $U\,\mathcal{G}$. This has the effect of replacing $\mathcal{F} \oplus \mathcal{G}$ with $(V\,\mathcal{F}) \oplus (U\,\mathcal{G}) = (V \oplus U)(\mathcal{F} \oplus \mathcal{G})$ (by (A.1)), which, by Proposition 4.2, has the effect of replacing $\mathrm{Stab}(\mathcal{F} \oplus \mathcal{G})$ with $(V \oplus U) \circ \mathrm{Stab}(\mathcal{F} \oplus \mathcal{G}) \circ (V \oplus U)^\top$. In particular, H is replaced with

$$\begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} H|_{\mathcal{F},\mathcal{F}} & H|_{\mathcal{F},\mathcal{G}} \\ U^\top DV & H|_{\mathcal{G},\mathcal{G}} \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ 0 & U^\top \end{bmatrix} = \begin{bmatrix} VH|_{\mathcal{F},\mathcal{F}}V^\top & VH|_{\mathcal{F},\mathcal{G}}U^\top \\ U(U^\top DV)V^\top & UH|_{\mathcal{G},\mathcal{G}}U^\top \end{bmatrix} = \begin{bmatrix} * & * \\ D & * \end{bmatrix}.$$

To summarize, after transforming $\mathcal{F}$ by V and $\mathcal{G}$ by U, we have $H = \begin{bmatrix} * & * \\ D & * \end{bmatrix} \in \operatorname{Stab}(\mathcal{F} \oplus \mathcal{G})$ for nonzero diagonal D. We consider two cases for D: either $D \in \operatorname{span}(I)$ or $D \notin \operatorname{span}(I)$. First, suppose $D \in \operatorname{span}(I)$, so $D = cI$ for $c \neq 0$. Let $F \leftrightsquigarrow G$ be nonzero n-ary signatures (note that $F = 0 \iff G = 0$ because $\|F\|^2 = \langle F, F\rangle = \langle G, G\rangle = \|G\|^2$). By part 3 of Proposition 3.1, we have

$$H^{\otimes n-1}(F \oplus G)^{n-1,1} = (F \oplus G)^{n-1,1} H. \tag{4.4}$$

Now, by Proposition A.1 with $K := F \oplus G$ and (2.2), we can write (4.4) as the block matrix equation

$$\begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & \iddots & \vdots \\ * & * & \cdots & * \\ D^{\otimes n-1} & * & \cdots & * \end{bmatrix} \begin{bmatrix} F^{n-1,1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & G^{n-1,1} \end{bmatrix} = \begin{bmatrix} F^{n-1,1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & G^{n-1,1} \end{bmatrix} \begin{bmatrix} * & * \\ D & * \end{bmatrix}. \tag{4.5}$$

The bottom-left block of (4.5) is $D^{\otimes n-1} F^{n-1,1} = G^{n-1,1} D$; using $D = cI$, this is equivalent to

$$c^{n-2} F = G. \tag{4.6}$$

Then

$$\|F\|^2 = \langle F, F\rangle = \langle G, G\rangle = c^{2(n-2)} \|F\|^2. \tag{4.7}$$

If all signatures in $\mathcal{F}$ and $\mathcal{G}$ have arity $n = 2$, then applying (4.6) to every pair $F \leftrightsquigarrow G$ gives $\mathcal{F} = \mathcal{G}$. Otherwise some pair $F \leftrightsquigarrow G$ have arity $n \neq 2$, so (4.7) gives $c = \pm 1$. Now again applying (4.6) to any n-ary pair $F \leftrightsquigarrow G$ gives $c^n F = c^{n-2} F = G$, so $(cI)\mathcal{F} = \mathcal{G}$ and we see $\mathcal{F}$ and $\mathcal{G}$ are ortho-equivalent.

Otherwise, $D \notin \operatorname{span}(I)$. We will show $\mathcal{F} \cup \{D\}$ and $\mathcal{G} \cup \{D\}$ are Holant-indistinguishable, then apply Lemma 4.3. The proof is illustrated in Figure 4.2. Consider a $\mathcal{F} \cup \{D\}$-grid Ω with at least one vertex

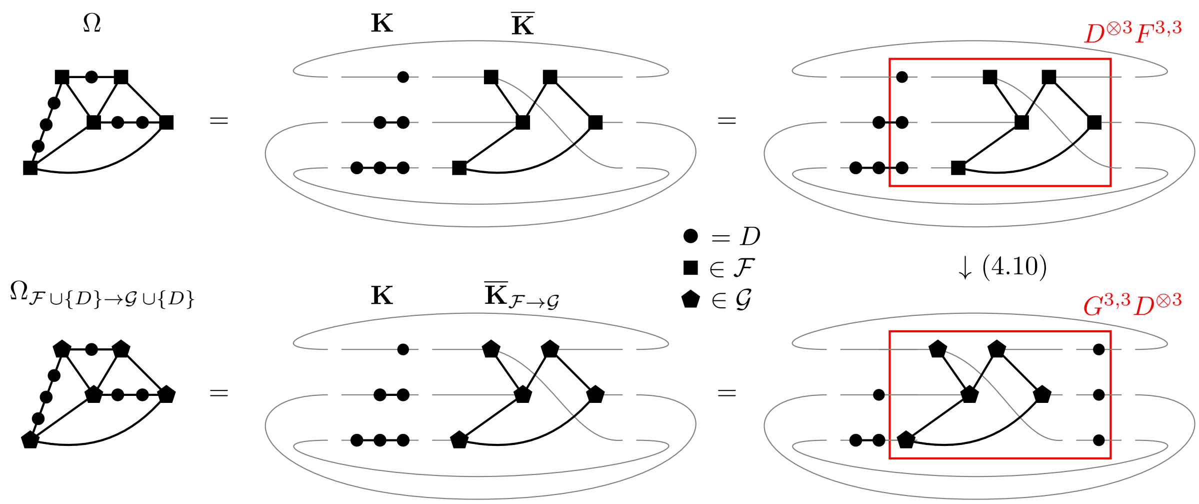


Figure 4.2: The Holant-value-preserving transformation from Ω to $\Omega_{\mathcal{F}\cup\{D\}\to\mathcal{G}\cup\{D\}}$ in the $D \notin \operatorname{span}(I)$ case. The transition from the bottom right grid to the bottom center grid wraps the three D vertices on the right around to the left along their respective wires.

assigned D (if Ω has no such vertex then we are done, as $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable). Let $\mathbf{K} \subset \Omega$ be the subgadget induced by all vertices of Ω assigned signature D. Any connected component of $\mathbf{K}$ is either a cycle or a binary path gadget with signature D^m for some m. The multiplicative factors from corresponding D-cycles in Ω and $\Omega_{\mathcal{F}\cup\{D\}\to\mathcal{G}\cup\{D\}}$ cancel, so, disregarding its cycle components, $\mathbf{K}$ consists of p disconnected path gadgets for some p. By rearranging the dangling edges of $\mathbf{K}$ and $\overline{\mathbf{K}}$, we may assume $\mathbf{K} \in \mathfrak{G}_{\{D\}}(p,p)$ with $M(\mathbf{K}) = \bigotimes_{i=1}^{p} D^{m_i}$ for $m_1, \ldots, m_p \geq 1$, and furthermore that $\overline{\mathbf{K}} \in \mathfrak{G}_{\mathcal{F}}(p,p)$, and that connecting the ith left input and ith right input of $\mathbf{K} \circ \overline{\mathbf{K}}$, for $i \in [p]$, reconstructs Ω (see Figure 4.2). Since $\overline{\mathbf{K}}$ is an $\mathcal{F}$-gadget and $\mathcal{F}$ is quantum-gadget-closed, $\overline{\mathbf{K}}$ has signature F for some $F \in \mathcal{F}$. Then

$$\mathrm{Holant}_{\Omega} = \mathrm{tr}\left(M(\mathbf{K})M(\overline{\mathbf{K}})\right) = \mathrm{tr}\left(\left(\bigotimes_{i=1}^{p} D^{m_i}\right) F^{p,p}\right). \tag{4.8}$$

For $G \leftrightsquigarrow F$, we similarly have

$$\mathrm{Holant}_{\Omega_{\mathcal{F}\cup\{D\}\to\mathcal{G}\cup\{D\}}} = \mathrm{tr}\left(\left(\bigotimes_{i=1}^{p} D^{m_i}\right) G^{p,p}\right). \tag{4.9}$$

As in (4.4), part 3 of Proposition 3.1 gives

$$H^{\otimes p}(F \oplus G)^{p,p} = (F \oplus G)^{p,p} H^{\otimes p},$$

which has block form

$$\begin{bmatrix} * & * & \dots & * \\ \vdots & \vdots & \iddots & \vdots \\ * & * & \dots & * \\ D^{\otimes p} & * & \dots & * \end{bmatrix} \begin{bmatrix} F^{p,p} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & G^{p,p} \end{bmatrix} = \begin{bmatrix} F^{p,p} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & G^{p,p} \end{bmatrix} \begin{bmatrix} * & * & \dots & * \\ \vdots & \vdots & \iddots & \vdots \\ * & * & \dots & * \\ D^{\otimes p} & * & \dots & * \end{bmatrix}.$$

The bottom left block of this equation is

$$D^{\otimes p} F^{p,p} = G^{p,p} D^{\otimes p}. \tag{4.10}$$

Now (4.8), (4.10), and (4.9) give

$$\begin{aligned} \mathrm{Holant}_{\Omega} &= \mathrm{tr}\left(\left(\bigotimes_{i=1}^{p} D^{m_i}\right) F^{p,p}\right) \\ &= \mathrm{tr}\left(\left(\bigotimes_{i=1}^{p} D^{m_i-1}\right) D^{\otimes p} F^{p,p}\right) \\ &= \mathrm{tr}\left(\left(\bigotimes_{i=1}^{p} D^{m_i-1}\right) G^{p,p} D^{\otimes p}\right) \\ &= \mathrm{tr}\left(D^{\otimes p}\left(\bigotimes_{i=1}^{p} D^{m_i-1}\right) G^{p,p}\right) \\ &= \mathrm{tr}\left(\left(\bigotimes_{i=1}^{p} D^{m_i}\right) G^{p,p}\right) \\ &= \mathrm{Holant}_{\Omega_{\mathcal{F}\cup\{D\}\to\mathcal{G}\cup\{D\}}}. \end{aligned}$$

Thus $\mathcal{F} \cup \{D\}$ and $\mathcal{G} \cup \{D\}$ are Holant-indistinguishable. This fact, along with the induction hypothesis, lets us apply Lemma 4.3 to conclude that $\mathcal{F} \cup \{D\}$ and $\mathcal{G} \cup \{D\}$ are ortho-equivalent. Therefore, by Proposition 2.2, $\mathcal{F}$ and $\mathcal{G}$ are ortho-equivalent. ☐

5 Consequences of Theorem 2.3

In this section, we exploit the expressiveness of the Holant framework to show that Theorem 2.3 encompasses a variety of existing results, and derive a few novel consequences.

5.1 Counting CSP and graph homomorphisms

For signature set $\mathcal{F}$, define the *counting constraint satisfaction problem* $\#\mathrm{CSP}(\mathcal{F})$ with *constraint function* set $\mathcal{F}$ to be the problem $\mathrm{Holant}(\mathcal{F} \mid \mathcal{EQ})$. Vertices assigned signatures in $\mathcal{F}$ and $\mathcal{EQ}$ are *constraints* and *variables*, respectively, and a $(\mathcal{F} \mid \mathcal{EQ})$-grid Ω is a constraint-variable incidence graph, where a variable appears in all of its incident constraints. Then $\mathrm{Holant}_\Omega(\mathcal{F} \mid \mathcal{EQ})$ is the sum over all variable assignments of the product of the constraint evaluations. Like Holant, #CSP is a well-studied problem in counting complexity, with dichotomy theorems classifying $\#\mathrm{CSP}(\mathcal{F})$ as either tractable or #P-hard proved for increasingly broad classes of constraint function sets [2, 16, 5, 4].

By inserting a dummy degree-2 constraint vertex assigned $I = (=_2) \in \mathcal{EQ}$ between adjacent variable vertices and combining adjacent constraint vertices assigned $=_a$ and $=_b$ into a single constraint vertex assigned $=_{a+b-2}$, we see that $\mathrm{Holant}(\mathcal{F} \cup \mathcal{EQ})$ is equivalent to $\mathrm{Holant}(\mathcal{F} \mid \mathcal{EQ})$. Using a standard Vandermonde interpolation argument, Xia [40] shows that H satisfies $H\,\mathcal{EQ} = \mathcal{EQ}$ if and only H is a permutation matrix. If $H\,\mathcal{F} = \mathcal{G}$ for some permutation matrix H, then $\mathcal{F}$ and $\mathcal{G}$ are the same up to relabeling of their domains, so are isomorphic. Now, applying Theorem 2.3 to $\mathcal{F} \cup \mathcal{EQ}$ and $\mathcal{G} \cup \mathcal{EQ}$, we obtain the main result of Young [41] for real-valued constraint functions. Say that $\mathcal{F}$ and $\mathcal{G}$ are #CSP-indistinguishable if $\mathcal{F} \cup \mathcal{EQ}$ and $\mathcal{G} \cup \mathcal{EQ}$ are Holant-indistinguishable (in other words, every #CSP instance has the same value whether we use constraint functions from $\mathcal{F}$ or from $\mathcal{G}$).

Corollary 5.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be sets of real-valued constraint functions. Then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic if and only if $\mathcal{F}$ and $\mathcal{G}$ are* #CSP*-indistinguishable.*

As discussed in Section 2.1, $\mathrm{Holant}(A_X \mid \mathcal{EQ}) = \#\mathrm{CSP}(A_X)$ counts the number of homomorphisms to graph X. Therefore Corollary 5.1 is a generalization of the classical theorem of Lovász [23] that two graphs are isomorphic if and only if they admit the same number of homomorphisms from every graph.

Let $\mathcal{EQ}_2 \subset \mathcal{EQ}$ be the set of equality signatures of even arity. Schrijver [32] shows[1] that H satisfies $H\,\mathcal{EQ}_2 = \mathcal{EQ}_2$ if and only if H is a signed permutation matrix (a matrix with entries in $\{0, \pm 1\}$ and exactly one nonzero entry in each row and column). As above, $\mathrm{Holant}(\mathcal{F} \cup \mathcal{EQ}_2)$ is equivalent to $\mathrm{Holant}(\mathcal{F} \mid \mathcal{EQ}_2)$ (critically, if $=_a \in \mathcal{EQ}_2$ and $=_b \in \mathcal{EQ}_2$, then $=_{a+b-2} \in \mathcal{EQ}_2$). Then, defining $\#\mathrm{CSP}^2(\mathcal{F}) := \mathrm{Holant}(\mathcal{F} \mid \mathcal{EQ}_2)$ as $\#\mathrm{CSP}(\mathcal{F})$ restricted to instances in which every variable appears an even number of times [6, 20], we have

Corollary 5.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be sets of real-valued constraint functions. Then there is a signed permutation matrix P satisfying $\mathcal{G} = P\,\mathcal{F}$ if and only if $\mathcal{F}$ and $\mathcal{G}$ are* $\#\mathrm{CSP}^2$*-indistinguishable.*

In particular, since (unweighted) graph adjacency matrices A_X and A_Y have entries in $\{0, 1\}$, we have $P^{\otimes 2}(A_X)^{2,0} = (A_Y)^{2,0} \implies (P')^{\otimes 2}(A_X)^{2,0} = (A_Y)^{2,0}$, where P' is the permutation matrix created by flipping every -1 entry of P to 1. Therefore we have the following sharpening of Lovász's theorem.

Corollary 5.3. *Graphs X and Y are isomorphic if and only if they admit the same number of homomorphisms from those graphs in which all vertices have even degree.*

5.2 Simultaneous matrix similarity

Let $\mathcal{F}$ and $\mathcal{G}$ be sets of binary signatures, thought of as matrices. Any connected $\mathcal{F}$-grid Ω is a cycle. Breaking an edge of the cycle, we obtain a binary path gadget with signature matrix $\prod_{i=1}^{c} F_i$, where, depending on its orientation, each $F_i \in \mathcal{F}$ or $F_i^\top \in \mathcal{F}$. Connecting the path's two dangling edges, we reform Ω, which thus has Holant value $\mathrm{tr}\left(\prod_{i=1}^{c} F_i\right)$. Let $\Gamma_\mathcal{F}$ be the set of all finite products of matrices in $\mathcal{F}$ and $\mathcal{F}^\top := \{F^\top \mid F \in \mathcal{F}\}$. Define $\Gamma_\mathcal{G}$ similarly and, for a word $w \in \Gamma_\mathcal{F}$, construct $w_{\mathcal{F}\to\mathcal{G}} \in \Gamma_\mathcal{G}$ by replacing every character F or $F^\top$ in w by the corresponding G or $G^\top$, respectively. For orthogonal H, we have $H\,\mathcal{F} = \mathcal{G} \iff HF^{1,1} = G^{1,1}H$ for every $F \leftrightsquigarrow G$ (by part 2 of Proposition 3.1), so, in this setting, Theorem 2.3 is equivalent to the following real-valued case of a classical theorem from representation theory, due to Specht [34] and Wiegmann [39]. Grohe, Rattan, and Seppelt [18] also give a combinatorial proof.

[1] The First Fundamental Theorem for $S_q^{\pm} \subset O(q)$ (the group of signed permutation matrices) states that $T(\mathbb{R}^q)^{S_q^{\pm}} = \overline{\mathcal{EQ}_2}$ (cf. Theorem 2.1, and recall that $\langle \mathcal{W} \rangle_+ = \overline{\varnothing}$). It follows as in the proof of Theorem 3.2 that $\mathrm{Stab}(\mathcal{EQ}_2) = S_q^{\pm}$. The fact that $\mathrm{Stab}(\mathcal{EQ}) = S_q \subset O(q)$ (the group of permutation matrices) similarly follows from the First Fundamental Theorem for S_q, which states that $T(\mathbb{R}^q)^{S_q} = \overline{\mathcal{EQ}}$.

Corollary 5.4. *Let $\mathcal{F}, \mathcal{G} \subset \mathbb{R}^{q \times q}$. Then there is an $H \in O(q)$ such that $HF = GH$ for every $\mathcal{F} \ni F \leftrightsquigarrow G \in \mathcal{G}$ if and only if $\mathrm{tr}(w) = \mathrm{tr}(w_{\mathcal{F} \to \mathcal{G}})$ for every $w \in \Gamma_{\mathcal{F}}$.*

Suppose $\mathcal{F} = \{A_X\}$ and $\mathcal{G} = \{A_Y\}$ for graphs X and Y. Transform an A_X-grid Ω to a $(A_X \mid \mathcal{EQ})$-grid Ω' by inserting a dummy degree-2 vertex assigned $I = (=_2) \in \mathcal{EQ}$ between every consecutive pair of vertices in the cycle. Recall from Section 2.1 that $\mathrm{Holant}_{\Omega'}(A_X \mid \mathcal{EQ})$ counts the number of homomorphisms from graph K to X, where K is the graph obtained from Ω' by ignoring the vertices assigned A_X. Here K is a cycle, so we have the following well-known result, an alternate formulation of this case of Corollary 5.4.

Corollary 5.5. *Let X and Y be graphs. Then there is an orthogonal matrix H satisfying $HA_X = A_Y H$ if and only if X and Y admit the same number of homomorphisms from all cycles.*

A matrix H is *pseudo-stochastic* if all of its rows and columns sum to 1. Dell, Grohe, and Rattan [12] proved that graphs X and Y admit the same number of homomorphisms from all paths if and only if there is a pseudo-stochastic matrix H such that $HA_X = A_Y H$. Using Theorem 2.3, we combine this result with Corollary 5.5, which also reproduces a combinatorial explanation for the connection between pseudo-stochastic matrices and homomorphisms from paths [18].

Corollary 5.6. *Let X and Y be graphs. Then there is a pseudo-stochastic orthogonal matrix H satisfying $HA_X = A_Y H$ if and only if X and Y admit the same number of homomorphisms from all cycles and paths.*

Proof. Consider $\mathrm{Holant}(A_X \cup \{=_1\})$. Any $A_X \cup \{=_1\}$-grid is a disjoint union of cycles composed of signatures assigned A_X and paths with degree-2 internal vertices assigned A_X and degree-1 endpoints assigned $=_1 \in \mathcal{EQ}$. As discussed before Corollary 5.5, every cycle A_X-grid Ω has the same Holant value as $\Omega_{A_X \to A_Y}$ if and only if X and Y admit the same number of homomorphisms from every cycle. Similarly inserting dummy vertices assigned $=_2$ between every pair of A_X vertices in a path component, we produce a $(A_X \mid \mathcal{EQ})$-grid whose Holant value equals the number of homomorphisms to X from the underlying path. Thus X and Y admit the same number of homomorphisms from all cycles and all paths if and only if $A_X \cup \{=_1\}$ and $A_Y \cup \{=_1\}$ are Holant-indistinguishable. By Theorem 2.3, this is equivalent to the existence of an orthogonal H satisfying $HA_X = A_Y H$ and $H(=_1) = (=_1)$. The vector form of $=_1$ is the all-ones vector, so $H(=_1) = (=_1)$ if and only if the rows of H sum to 1 and (since $H(=_1) = (=_1) \iff H^\top(=_1) = (=_1)$) the columns of H sum to 1. $\square$

5.3 Odeco signature sets

In this section, we derive a consequence of Theorem 2.3 that is not a counting indistinguishability theorem, but a combinatorial characterization of signatures that are simultaneosly 'diagonalizable'.

Definition 5.1 ($\mathcal{GEQ}$, odeco)**.** Define the set of *general equalities* (or *weighted equalities*) on domain $[q]$ as $\mathcal{GEQ} = \{=_n^{\mathbf{a}} \mid n \in \mathbb{N}, \mathbf{a} \in \mathbb{R}^{[q]}\}$, where $=_n^{\mathbf{a}}$ is the symmetric n-ary signature defined by

$$(=_n^{\mathbf{a}})_{\mathbf{x}} = \begin{cases} a_q & x_1 = \ldots = x_n = q \\ 0 & \text{otherwise.} \end{cases}$$

A set $\mathcal{F}$ of symmetric signatures is *orthogonally decomposable*, or *odeco*, if it is ortho-equivalent to a general equality set – that is, there exists an $H \in O(q)$ such that $H\,\mathcal{F} \subset \mathcal{GEQ}$.

The term "odeco" was coined by Robeva [29] to refer to individual symmetric tensors (signatures) which are ortho-equivalent to a general equality. A binary $\mathcal{GEQ}$ signature has a diagonal signature matrix, so the spectral theorem states that every (real) symmetric binary signature is odeco (recall part 2 of Proposition 3.1). Any nonzero edge assignment for a connected $\mathcal{GEQ}$-gadget $\mathbf{K}$ must assign all edges, including dangling edges, the same domain element, so, if $\mathbf{K}$ has arity n and is composed of vertices assigned signatures with weights $\mathbf{a}^1, \ldots, \mathbf{a}^p$, then $\mathbf{K}$ has signature $=_n^{\mathbf{a}^1 \bullet \ldots \bullet \mathbf{a}^p} \in \mathcal{GEQ}$, where $\bullet$ denotes entrywise product. In particular, if $\mathbf{K}$ is a $\mathcal{GEQ}$-grid Ω, then $\mathrm{Holant}_\Omega = \sum_{i=1}^q (\mathbf{a}^1 \bullet \ldots \bullet \mathbf{a}^p)_i$. Thus, if $\mathcal{F}$ is odeco, then the computational problem $\mathrm{Holant}(\mathcal{F})$ is polynomial-time tractable via orthogonal Holographic transformation.

Definition 5.2 ($*$)**.** For symmetric signatures $F_1, F_2 \in \mathcal{F}$ of arity n_1 and n_2, respectively, construct the $(n_1 + n_2 - 2)$-ary signature $F_1 * F_2 \in \overline{\mathcal{F}}$ from $F_1 \otimes F_2$ by contracting an input of F_1 and an input of F_2.

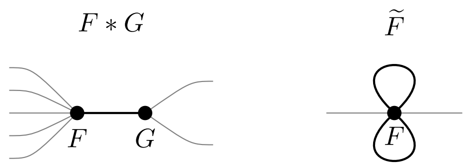


Figure 5.1: Illustrating (the gadgets with signatures) $F * G$ and $\widetilde{F}$ for 6-ary F and 3-ary G.

See Figure 5.1. $F_1 * F_2$ doesn't depend on which inputs we connect, as F_1 and F_2 are symmetric. For $\mathbf{x} \in [q]^{n_1-1}$ and $\mathbf{y} \in [q]^{n_2-1}$, we have (with vectors viewed as input lists)

$$(F_1 * F_2)(\mathbf{x}, \mathbf{y}) = \sum_{z \in [q]} F_1(\mathbf{x}, z) F_2(\mathbf{y}, z).$$

Proposition 5.1. *For any $H \in O(q)$, we have $HF \in \mathcal{GEQ}$ if and only if $H(F * F) \in \mathcal{GEQ}$.*

Proof. ($\Longrightarrow$): If $HF = E$ for $H \in O(q)$ and $E \in \mathcal{GEQ}$, then by part 2 of Lemma 4.1, $H(F * F) = (HF) * (HF) = E * E \in \mathcal{GEQ}$.

($\Longleftarrow$): Let $F \in \mathbb{R}^{[q]^n}$. Every unary signature is in $\mathcal{GEQ}$, so if $n = 1$ then we are done. If $n = 2$ then $F * F = F^2$ (a matrix product) and F is a real symmetric matrix, so if H diagonalizes F^2 then H diagonalizes F. Now assume $n \geq 3$. Let $\mathcal{GEQ} \ni E = H(F * F) = (HF) * (HF)$ for $H \in O(q)$. Suppose toward contradiction that $HF \notin \mathcal{GEQ}$, so there is a $\mathbf{x} \in [q]^n$ such that $(HF)(\mathbf{x}) \neq 0$ but $\exists i, j$ such that $x_i \neq x_j$. Assume WLOG that $i, j \neq n$. Construct $\mathbf{x}' \in [q]^{n-1}$ by deleting the nth (last) entry of $\mathbf{x}$. Then

$$E(\mathbf{x}', \mathbf{x}') = ((HF) * (HF))(\mathbf{x}', \mathbf{x}') = \sum_{z \in [q]} (HF)(\mathbf{x}', z)^2 \geq (HF)(\mathbf{x})^2 > 0,$$

contradicting $E \in \mathcal{GEQ}$. Thus $HF \in \mathcal{GEQ}$. $\square$

Theorem 5.1. *Let $\mathcal{F}$ be a set of real-valued symmetric signatures (tensors). The following are equivalent.*

(i) $\mathcal{F}$ is odeco.

(ii) Every connected $\mathcal{F}$-gadget has a symmetric signature.

(iii) For every $F_1, F_2 \in \mathcal{F}$, $F_1 * F_2$ is symmetric.

Robeva [29] conjectured the equivalence of items (i) and (iii) when $\mathcal{F}$ contains a single signature. Boralevi, Draisma, Horobeţ, and Robeva [1] confirmed this conjucture using techniques from algebraic geometry. We use Theorem 2.3 to give a combinatorial proof, generalized to arbitrary symmetric signature sets.

Remark 5.1. If $\mathcal{F}$ is a set of symmetric binary signatures, then $F_1 * F_2 = F_1 \circ F_2 = F_1 F_2$ (a matrix product) for $F_1, F_2 \in \mathcal{F}$. In general, symmetric matrices commute if and only if their product is symmetric (as if $F_1 F_2$ is symmetric then $F_1 F_2 = (F_1 F_2)^\top = F_2^\top F_1^\top = F_2 F_1$ and if F_1 and F_2 commute then $(F_1 F_2)^\top = F_2^\top F_1^\top = F_2 F_1 = F_1 F_2$). Therefore Theorem 5.1 encompasses the extension of the spectral theorem which states that commuting symmetric matrices are simultaneously diagonalizable. We use this fact in the proof below.

Proof of Theorem 5.1. (i) $\Longrightarrow$ (ii),(iii): Suppose $H\mathcal{F} \subset \mathcal{GEQ}$ for some $H \in O(q)$. Let $K \in \overline{\mathcal{F}}$ be the signature of a connected $\mathcal{F}$-gadget (e.g. $K = F_1 * F_2$). By part 2 of Lemma 4.1, $HK = J$, where J is the signature of a connected $\mathcal{GEQ}$-gadget. Then $J \in \mathcal{GEQ}$, so J, and hence $K = H^{-1}J$, are symmetric.

(ii) $\Longrightarrow$ (i): First, replace every non-unary odd-arity $F \in \mathcal{F}$ by $F * F$ (every unary signature is in $\mathcal{GEQ}$, so simply remove all unaries from $\mathcal{F}$). This does not change the fact that $\mathcal{F}$ satisfies item (ii), and, by Proposition 5.1, does not change whether $\mathcal{F}$ satisfies item (i). Thus we may assume all signatures in $\mathcal{F}$ have even arity. For $F \in \mathcal{F}$, let $\widetilde{F}$ be the matrix of the binary signature constructed by contracting all but one pair of inputs of F (see Figure 5.1). Since F is symmetric, $\widetilde{F}$ doesn't depend on how we pair up F's inputs. Every $\widetilde{F}$, and every composition $\widetilde{F_1} \circ \widetilde{F_2}$ for $F_1, F_2 \in F$, is the signature of a connected $\mathcal{F}$-gadget, so is symmetric by assumption. Therefore, as in Remark 5.1, the matrices $\widetilde{F}$ for $F \in \mathcal{F}$ all commute.

Claim 5.1. If $\mathbf{K}$ is a connected binary $\mathcal{F}$-gadget with p vertices, assigned signatures $F_1, \dots, F_p \in \mathcal{F}$, then $M(\mathbf{K}) = \prod_{i=1}^{p} \widetilde{F_i}$.

We prove Claim 5.1 by induction on p. For $p = 1$, by the symmetry of F_1, every connected F_1-gadget with a single vertex has signature $\widetilde{F_1}$. Now suppose $p \geq 2$. Every connected multigraph contains a vertex whose removal does not disconnect the multigraph (take the final vertex visited by e.g. breadth first search). Let v be such a vertex in the underlying multigraph of $\mathbf{K}$, and assume WLOG that v is assigned signature F_p. Construct $\mathbf{K}'$ from $\mathbf{K}$ by breaking all but one edge of v incident to other vertices. Each broken edge becomes two new dangling edges – one incident to v and one incident to another vertex. $\mathbf{K}'$ is connected, as $\mathbf{K}$ would remain connected upon removing v entirely, so by assumption its signature is symmetric. The number of dangling edges of $\mathbf{K}'$ incident to v is odd, as v has even degree and exactly one edge to another vertex (loops on v do not affect the parity). Since $\mathbf{K}'$ has an even number of dangling edges (two plus twice the number of edges broken), there are an odd number of dangling edges incident to the other vertices of $\mathbf{K}'$. Now create a binary gadget $\mathbf{K}''$ from $\mathbf{K}'$ by arbitrarily pairing up and connecting all but one dangling edge incident to v, and similarly pairing up and connecting all but one dangling edge incident to the other vertices of $\mathbf{K}'$. We may also recover $\mathbf{K}$ from $\mathbf{K}'$ by connecting possibly different pairs of dangling edges (reforming the edges broken to create $\mathbf{K}'$) and, since the signature of $\mathbf{K}'$ is symmetric, the signature of a gadget produced by connecting dangling edges of $\mathbf{K}'$ does not depend on which pairs of dangling edges we connect (although the underlying graphs of the gadgets differ). Therefore $M(\mathbf{K}'') = M(\mathbf{K})$. See Figure 5.2

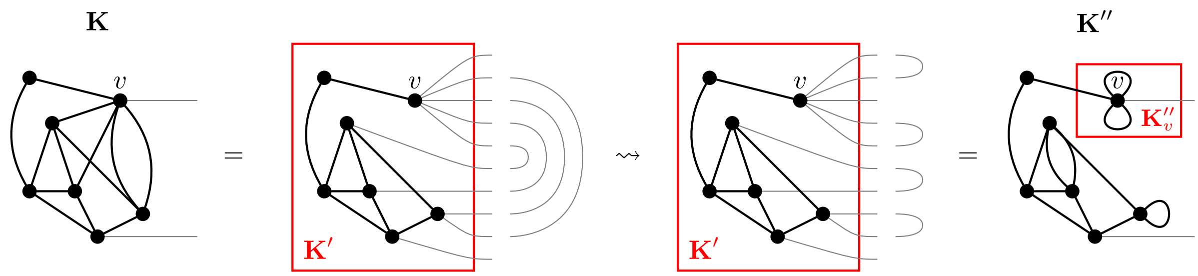


Figure 5.2: Illustrating the proof of Claim 5.1 when v has a single incident dangling edge in $\mathbf{K}$. The cases where v has zero or two incident dangling edges are similar.

Let $\mathbf{K}''_v$ be the subgadget of $\mathbf{K}''$ induced by v. $\mathbf{K}''_v$ has two dangling edges, one dangling in $\mathbf{K}''$, and one which, in $\mathbf{K}''$, connects v to a vertex in $\overline{\mathbf{K}''_v}$ (the complement of $\mathbf{K}''_v$), with the remaining edges incident to v paired into loops. Thus $M(\mathbf{K}''_v) = \widetilde{F_p}$. Similarly, $\overline{\mathbf{K}''_v}$ has two dangling edges, one dangling in $\mathbf{K}''$ and one incident to to v in $\mathbf{K}''$, so we recover $\mathbf{K}''$ by reconnecting the edge between $\mathbf{K}''_v$ and $\overline{\mathbf{K}''_v}$. Thus, applying induction to $\overline{\mathbf{K}''_v}$, which has $p-1$ vertices, we obtain

$$M(\mathbf{K}) = M(\mathbf{K}'') = M(\mathbf{K}''_v) \circ M(\overline{\mathbf{K}''_v}) = \widetilde{F_p} \circ \prod_{i=1}^{p-1} \widetilde{F_i} = \prod_{i=1}^{p} \widetilde{F_i}$$

(recall that the $\widetilde{F_i}$ commute). This completes the proof of Claim 5.1.

The matrices $\widetilde{F}$ for $F \in \mathcal{F}$ are symmetric and commute, so they are simultaneously diagonalizable: there is an $H \in O(q)$ such that, for every $F \in \mathcal{F}$, $H\widetilde{F}H^{\top} = (=_2^{\mathbf{a}^F})$ for some $\mathbf{a}^F \in \mathbb{R}^q$. By Proposition 2.1 and part 2 of Proposition 3.1, we may replace $\mathcal{F}$ with $H\mathcal{F}$ to assume each $\widetilde{F} = (=_2^{\mathbf{a}^F})$. Define

$$\mathcal{G} = \{=_{\text{arity}(F)}^{\mathbf{a}^F} \mid F \in \mathcal{F}\} \subset \mathcal{GEQ}\,.$$

Let Ω be a $\mathcal{F}$-grid, containing signatures $F_1, \dots, F_p$. Ω is not a tree, as it contains no degree-1 vertices, so we can break some edge of Ω to produce a connected binary $\mathcal{F}$-gadget $\mathbf{K}$. By Claim 5.1,

$$M(\mathbf{K}) = \prod_{i=1}^{p} \widetilde{F_i} = \prod_{i=1}^{p} (=_2^{\mathbf{a}^{F_i}}) = \left(=_2^{\mathbf{a}^{F_1} \bullet \dots \bullet \mathbf{a}^{F_p}}\right) = M(\mathbf{K}_{\mathcal{F}\to\mathcal{G}}).$$

Now, reconnecting the dangling edges of $\mathbf{K}$, we find $\mathrm{Holant}_\Omega = \mathrm{Holant}_{\Omega_{\mathcal{F}\to\mathcal{G}}}$. Thus $\mathcal{F}$ and $\mathcal{G}$ are Holant-indistinguishable, so, by Theorem 2.3, $\mathcal{F}$ and $\mathcal{G}$ are ortho-equivalent. Hence $\mathcal{F}$ is odeco.

(iii) $\implies$ (ii): Let K be the n-ary signature of a connected $\mathcal{F}$-gadget $\mathbf{K}$. Every unary signature is trivially symmetric, so assume $n \geq 2$. It suffices to show that, for any fixed partial input $\mathbf{z} \in [q]^{n-2}$ to K and any $x, y \in [q]$, we have $K(x, y, \mathbf{z}) = K(y, x, \mathbf{z})$ (where we assume WLOG that x and y are the first two inputs to K by reordering the dangling edges of $\mathbf{K}$). Let u and w be the vertices of $\mathbf{K}$ incident to the first and second dangling edges of $\mathbf{K}$ (after reordering). If $u = w$ then we are done, as every $F \in \mathcal{F}$ is symmetric. Otherwise, since $\mathbf{K}$ is connected, it contains a path $P = (u = v_0, v_1, \ldots, v_{p-2}, v_{p-1} = w)$ from u to w, where v_i is assigned signature $F_i \in \mathcal{F}$, for $i \in [p]$. Let $E(P) := \{e_0, e_1, \ldots, e_{p-1}, e_p\}$ be the edges of P, including the dangling edges e_0 and e_p incident to u and w, respectively. Then e_i and e_{i+1} are inputs to F_i for all $i \in [p]$. For any fixed assignment $\sigma : E(\mathbf{K}) \setminus E(P) \to [q]$, define the matrix $F_i^\sigma \in \mathbb{R}^{q\times q}$ by $(F_i^\sigma)_{a,b} := F_i(\sigma|_{\delta(v_i)}, a, b)$ (that is, fix the inputs to F_i from edges outside of $E(P)$). Then

$$K(x, y, \mathbf{z}) = \sum_{\substack{\sigma: E(\mathbf{K})\setminus E(P)\to[q] \\ \sigma(D)=\mathbf{z}}} \left(\prod_{v\in V(\mathbf{K})\setminus P} F_v(\sigma|_{\delta(v)}) \right) F_P^\sigma(x, y), \tag{5.1}$$

where D is the ordered list of the last $n-2$ dangling edges of $\mathbf{K}$ and

$$F_P^\sigma(x, y) = \sum_{\substack{\phi: E(P)\to[q] \\ \phi(e_0)=x, \phi(e_p)=y}} \prod_{i=0}^{p-1} F_i(\sigma|_{\delta(v_i)}, \phi(e_i), \phi(e_{i+1})) = \sum_{\substack{\phi: E(P)\to[q] \\ \phi(e_0)=x, \phi(e_p)=y}} \prod_{i=0}^{p-1} (F_i^\sigma)_{\phi(e_i),\phi(e_{i+1})} = \left(\prod_{i=0}^{p-1} F_i^\sigma \right)_{x,y}.$$

On the RHS of (5.1), x and y appear only in $F_P^\sigma(x, y)$. Thus it suffices to show that, for any fixed σ, $F_P^\sigma(x, y) = F_P^\sigma(y, x)$. For any $i, j \in [p]$ and $a, b \in [q]$,

$$\begin{aligned}(F_i^\sigma F_j^\sigma)_{a,b} &= \sum_{z\in[q]} (F_i^\sigma)_{a,z} (F_j^\sigma)_{z,b} = \sum_{z\in[q]} F_i(\sigma|_{\delta(v_i)}, a, z) F_j(\sigma|_{\delta(v_j)}, b, z) = (F_i * F_j)(\sigma|_{\delta(v_i)}, a, \sigma|_{\delta(v_j)}, b) \\ &= (F_i * F_j)(\sigma|_{\delta(v_i)}, b, \sigma|_{\delta(v_j)}, a) = \sum_{z\in[q]} F_i(\sigma|_{\delta(v_i)}, b, z) F_j(\sigma|_{\delta(v_j)}, a, z) = \sum_{z\in[q]} (F_i^\sigma)_{b,z} (F_j^\sigma)_{z,a} = (F_i^\sigma F_j^\sigma)_{b,a},\end{aligned}$$

where the fourth equality uses the assumption that $F_i * F_j$ is symmetric. Thus $F_i^\sigma F_j^\sigma$ is symmetric. Both F_i^σ and F_j^σ are symmetric, as F_i and F_j are symmetric, so, as in Remark 5.1, F_i^σ and F_j^σ commute. Therefore

$$F_P^\sigma(x, y) = \left(\prod_{i=0}^{p-1} F_i^\sigma \right)_{x,y} = \left(\prod_{i=0}^{p-1} F_i^\sigma \right)^\top_{y,x} = \left(\prod_{i=0}^{p-1} (F_{p-1-i}^\sigma)^\top \right)_{y,x} = \left(\prod_{i=0}^{p-1} F_i^\sigma \right)_{y,x} = F_P^\sigma(y, x). \qquad \square$$

6 Possible Variations of Theorem 2.3

6.1 Complex-valued signatures

Although we have focused on real-valued signatures and matrices, the general and orthogonal Holant Theorems hold for complex-valued signatures and matrices. However, Theorem 2.3 does not hold for general sets $\mathcal{F}$ and $\mathcal{G}$ of complex-valued signatures, even when we allow H to be complex. For example, Draisma and Regts [14] consider the *vanishing* unary signature $F \in \mathbb{R}^{[2]^1}$ defined by $F_0 = 1$ and $F_1 = i$. Any F-grid Ω with at least one vertex satisfies $\mathrm{Holant}_\Omega(F) = 0$, as Ω is a disjoint union of K_2 complete graphs, with each component having value $f^\top f = [1, i]^\top [1, i] = 0$. Thus F is Holant-indistinguishable from 0, but there is no orthogonal matrix H, real or complex, satisfying $Hf = H[1, i]^\top = [0, 0]^\top$.

Draisma and Regts also observe that a direct extension of Theorem 3.1 to the complex orthogonal group $O_q(\mathbb{C})$ is impossible because $O_q(\mathbb{C})$ is not compact. However, they also provide some techniques for handling complex-valued signatures in edge coloring models, including a version [14, Theorem 3] of Theorem 3.2 for edge-coloring models over any algebraically closed field of characteristic zero. Cai and Young [11] and Young [41] prove their counting indistinguishability theorems for complex-valued signature sets with the assumption that the sets are *conjugate-closed*, meaning they must contain the entrywise conjugate of each of their complex signatures. It is feasible that Theorem 2.3 could similarly hold for complex-valued conjugate-closed $\mathcal{F}$ and $\mathcal{G}$ and complex orthogonal H (observe that this invalidates the above counterexample, as $[1, i]^\top [1, -i] \neq 0$).

6.2 Quantum orthogonal matrices and planar signature grids

A *quantum orthogonal matrix* U is, roughly, a matrix whose entries come from an abstract, not necessarily commutative, C^*-algebra, satisfying the relation $UU^\top = U^\top U = I \otimes \mathbf{1}$, where $\mathbf{1}$ is the identity element of the C^*-algebra. Just as a permutation matrix is an orthogonal matrix that stabilizes $\mathcal{EQ}$, a *quantum permutation matrix* is a quantum orthogonal matrix that stabilizes $\mathcal{EQ}$ (see e.g. [11, Equation 27]). The main theorem of Cai and Young [11], extending the result of Mančinska and Roberson [25], is a planar, quantum version of Corollary 5.1: $\mathcal{F}$ and $\mathcal{G}$ are Pl-Holant$(\cdot \cup \mathcal{EQ})$-indistinguishabile (planar-#CSP-indistinguishable) if and only if there is a quantum permutation matrix U satisfying $U\mathcal{F} = \mathcal{G}$ ($\mathcal{F}$ and $\mathcal{G}$ are *quantum isomorphic*). Removing $\mathcal{EQ}$, we should obtain the following planar, quantum version of Theorem 2.3.

Conjecture 6.1. *Let $\mathcal{F}$, $\mathcal{G}$ be sets of real-valued signatures. Then the following are equivalent.*

(i) $\text{Pl-Holant}_\Omega(\mathcal{F}) = \text{Pl-Holant}_{\Omega_{\mathcal{F}\to\mathcal{G}}}(\mathcal{G})$ *for every planar $\mathcal{F}$-grid Ω.*

(ii) There is a quantum orthogonal matrix U such that $U\mathcal{F} = \mathcal{G}$.

Cai and Young [11, Theorem 5] prove (ii) $\implies$ (i) when U is a quantum permutation matrix; however, their proof only relies on U being a quantum orthogonal matrix. Therefore, to prove Conjecture 6.1, it suffices to show (i) $\implies$ (ii). The *Tannaka-Krein duality* for the quantum symmetric group [11, Theorem 3] has a more general version for the quantum orthogonal group O_q^+ [25, Theorem 2.13]. This version is analogous to Theorem 3.1 above, but concerning quantum subgroups of O_q^+ and asymmetric (planar) tensor categories with duals. Then, following the proof of [11, Theorem 4], but ommitting the gadgets $E^{1,0}$ and $E^{1,2}$ used to construct $\mathcal{EQ}$, we will obtain a quantum analogue of our Theorem 3.2 for planar quantum $\mathcal{F}$-gadgets, giving a quantum analogue of Lemma 3.2. The rest of the proof of Theorem 2.3, however, involves nonplanar signature grid manipulations (e.g. in Figure 4.2 it is in general impossible to embed Ω such that every instance of D lies on the outer face) and, more critically, relies on the existence of the singular value decomposition of a submatrix of a real orthogonal matrix, then on viewing the resulting diagonal matrix as a signature. It is yet unclear whether the same or similar reasoning applies to a submatrix of a quantum orthogonal matrix.

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A Block Signature Actions

In this appendix, we prove several technical results which state that the action of a block matrix H on a block signature K follows block matrix multiplication rules as one would expect.

Proposition A.1. *Let $\mathcal{I} = X \sqcup Y$. For $H \in \mathbb{R}^{\mathcal{I}\times\mathcal{I}}$, $K^{m,d} \in \mathbb{R}^{\mathcal{I}^m \times \mathcal{I}^d}$ and any $\mathbf{R} \in \{X, Y\}^m$ $\mathbf{C} \in \{X, Y\}^d$,*

$$(H^{\otimes m} K^{m,d})|_{\mathbf{R},\mathbf{C}} = \sum_{\mathbf{J}\in\{X,Y\}^m} \left(\bigotimes_{i=1}^{m} H|_{R_i,J_i} \right) K^{m,d}|_{\mathbf{J},\mathbf{C}}$$

(with $H^{\otimes m} K^{m,d} \in \mathbb{R}^{\mathcal{I}^m \times \mathcal{I}^d}$ indexed as in part 3 of Definition 2.11) and similarly

$$(K^{m,d} H^{\otimes d})|_{\mathbf{R},\mathbf{C}} = \sum_{\mathbf{J}\in\{X,Y\}^d} K^{m,d}|_{\mathbf{R},\mathbf{J}} \left(\bigotimes_{i=1}^{d} H|_{J_i,C_i} \right).$$

That is, with $H^{\otimes m}|_{\mathbf{R},\mathbf{J}} = \bigotimes_{i=1}^m H|_{R_i,J_i}$, we can compute $H^{\otimes m}K^{m,d}$ as a block matrix product with

$$H^{\otimes m} = \begin{bmatrix} H^{\otimes m}|_{X^m,X^m} & H^{\otimes m}|_{X^m,X^{m-1}Y} & H^{\otimes m}|_{X^m,X^{m-2}YX} & \dots & H^{\otimes m}|_{X^m,Y^m} \\ H^{\otimes m}|_{X^{m-1}Y,X^m} & H^{\otimes m}|_{X^{m-1}Y,X^{m-1}Y} & H^{\otimes m}|_{X^{m-1}Y,X^{m-2}YX} & \dots & H^{\otimes m}|_{X^{m-1}Y,Y^m} \\ \vdots & \vdots & \vdots & & \vdots \\ H^{\otimes m}|_{Y^{m-1}X,X^m} & H^{\otimes m}|_{Y^{m-1}X,X^{m-1}Y} & H^{\otimes m}|_{Y^{m-1}X,X^{m-2}YX} & \dots & H^{\otimes m}|_{Y^{m-1}X,Y^m} \\ H^{\otimes m}|_{Y^m,X^m} & H^{\otimes m}|_{Y^m,X^{m-1}Y} & H^{\otimes m}|_{Y^m,X^{m-2}YX} & \dots & H^{\otimes m}|_{Y^m,Y^m} \end{bmatrix}$$

and

$$K^{m,d} = \begin{bmatrix} K^{m,d}|_{X^m,X^d} & K^{m,d}|_{X^m,X^{d-1}Y} & \dots & K^{m,d}|_{X^m,Y^{d-1}X} & K^{m,d}|_{X^m,Y^d} \\ K^{m,d}|_{X^{m-1}Y,X^d} & K^{m,d}|_{X^{m-1}Y,X^{d-1}Y} & \dots & K^{m,d}|_{X^{m-1}Y,Y^{d-1}X} & K^{m,d}|_{X^{m-1}Y,Y^d} \\ K^{m,d}|_{X^{m-2}YX,X^d} & K^{m,d}|_{X^{m-2}YX,X^{d-1}Y} & \dots & K^{m,d}|_{X^{m-2}YX,Y^{d-1}X} & K^{m,d}|_{X^{m-2}YX,Y^d} \\ \vdots & \vdots & & \vdots & \vdots \\ K^{m,d}|_{Y^m,X^d} & K^{m,d}|_{Y^m,X^{d-1}Y} & \dots & K^{m,d}|_{Y^m,Y^{d-1}X} & K^{m,d}|_{Y^m,Y^d}. \end{bmatrix}.$$

Proof. We prove the first statement. The second is proved similarly. Let $\mathbf{r} \in \mathbf{R}$ and $\mathbf{c} \in \mathbf{C}$. Then

$$\begin{aligned} ((H^{\otimes m}K^{m,d})|_{\mathbf{R},\mathbf{C}})_{\mathbf{r},\mathbf{c}} = (H^{\otimes m}K^{m,d})_{\mathbf{r},\mathbf{c}} &= \sum_{\mathbf{j}\in\mathcal{I}^m} \left(\prod_{i=1}^m H_{r_i,j_i}\right) K^{m,d}_{\mathbf{j},\mathbf{c}} \\ &= \sum_{\mathbf{J}\in\{X,Y\}^m} \sum_{\mathbf{j}\in\mathbf{J}} \left(\prod_{i=1}^m H_{r_i,j_i}\right) K^{m,d}_{\mathbf{j},\mathbf{c}} \\ &= \sum_{\mathbf{J}\in\{X,Y\}^m} \sum_{\mathbf{j}\in\mathbf{J}} \left(\bigotimes_{i=1}^m H|_{R_i,J_i}\right)_{\mathbf{r},\mathbf{j}} (K^{m,d}|_{\mathbf{J},\mathbf{C}})_{\mathbf{j},\mathbf{c}} \\ &= \sum_{\mathbf{J}\in\{X,Y\}^m} \left(\left(\bigotimes_{i=1}^m H|_{R_i,J_i}\right) K^{m,d}|_{\mathbf{J},\mathbf{C}}\right)_{\mathbf{r},\mathbf{c}} \\ &= \left(\sum_{\mathbf{J}\in\{X,Y\}^m} \left(\bigotimes_{i=1}^m H|_{R_i,J_i}\right) K^{m,d}|_{\mathbf{J},\mathbf{C}}\right)_{\mathbf{r},\mathbf{c}}. \end{aligned}$$

□

We will only need Proposition A.1 as written, for partitions of $\mathcal{I}$ into two blocks, but it is not hard to see that it extends naturally to partitions of $\mathcal{I}$ into more than two blocks.

We will apply Proposition A.1 for two special types of H.

Corollary A.1. *If*

$$H = H_X \oplus H_Y = \begin{bmatrix} H_X & 0 \\ 0 & H_Y \end{bmatrix}$$

is block-diagonal, then, for any $K \in \mathbb{R}^{\mathcal{I}^n}$, the block form of $H^{\otimes n}K^{n,0}$ is

$$\begin{bmatrix} H_X^{\otimes n} & 0 & \dots & 0 & 0 \\ 0 & * & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & * & 0 \\ 0 & 0 & \dots & 0 & H_Y^{\otimes n} \end{bmatrix} \begin{bmatrix} (K|_X)^{n,0} \\ * \\ \vdots \\ * \\ (K|_Y)^{n,0} \end{bmatrix} = \begin{bmatrix} H_X^{\otimes n}(K|_X)^{n,0} \\ * \\ \vdots \\ * \\ H_Y^{\otimes n}(K|_Y)^{n,0} \end{bmatrix}.$$

Proof. Here, $H|_{R_i,J_i} = H_{R_i}$ if $R_i = J_i$ and $H|_{R_i,J_i} = 0$ if $R_i \neq J_i$, so H takes the claimed block form in Proposition A.1. □

If $X = V(F)$, $Y = V(G)$, and $K = F \oplus G$, then by (2.2), all blocks of K are 0 except $K|_X = F$ and $K|_Y = G$, so, by Corollary A.1,

$$(H_{V(F)} \oplus H_{V(G)})(F \oplus G) = (H_{V(F)}F) \oplus (H_{V(G)}G). \tag{A.1}$$

Corollary A.2. *If*

$$B = \begin{bmatrix} 0 & H^\top \\ H & 0 \end{bmatrix}$$

is block-antidiagonal, with blocks indexed by $X = V(F)$ and $Y = V(G)$, then, for any m, d,

$$H^{\otimes m} F^{m,d} = G^{m,d} H^{\otimes d} \quad \text{and} \quad (H^\top)^{\otimes m} G^{m,d} = F^{m,d} (H^\top)^{\otimes d} \iff B^{\otimes m} (F \oplus G)^{m,d} = (F \oplus G)^{m,d} B^{\otimes d}.$$

Proof. $B|_{R_i, J_i} = 0$ if $R_i = J_i$, so, by Proposition A.1 and (2.2), we compute blockwise with

$$B^{\otimes m}(F \oplus G)^{m,d} \text{ as } \begin{bmatrix} 0 & 0 & \dots & 0 & (H^\top)^{\otimes m} \\ 0 & 0 & \dots & * & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & * & \dots & 0 & 0 \\ H^{\otimes m} & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} F^{m,d} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & G^{m,d} \end{bmatrix} \tag{A.2}$$

and

$$(F \oplus G)^{m,d} B^{\otimes d} \text{ as } \begin{bmatrix} F^{m,d} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & G^{m,d} \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 & (H^\top)^{\otimes d} \\ 0 & 0 & \dots & * & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & * & \dots & 0 & 0 \\ H^{\otimes d} & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{A.3}$$

The result follows from comparing the bottom left and upper right blocks of (A.2) and (A.3). ☐

References

[1] Ada Boralevi, Jan Draisma, Emil Horobeţ, and Elina Robeva. "Orthogonal and unitary tensor decomposition from an algebraic perspective". en. In: *Israel Journal of Mathematics* 222.1 (Oct. 2017), pp. 223–260. DOI: 10.1007/s11856-017-1588-6.

[2] Andrei A. Bulatov. "The Complexity of the Counting Constraint Satisfaction Problem". In: *J. ACM* 60.5 (Oct. 2013). DOI: 10.1145/2528400.

[3] Jin-Yi Cai and Xi Chen. *Complexity Dichotomies for Counting Problems.* Vol. 1. Cambridge University Press, 2017, pp. 1–34. DOI: 10.1017/9781107477063.002.

[4] Jin-Yi Cai and Xi Chen. "Complexity of Counting CSP with Complex Weights". In: *J. ACM* 64.3 (June 2017). DOI: 10.1145/2822891.

[5] Jin-Yi Cai, Xi Chen, and Pinyan Lu. "Nonnegative Weighted #CSP: An Effective Complexity Dichotomy". In: *SIAM J. Comput.* 45.6 (2016), pp. 2177–2198. DOI: 10.1137/15M1032314.

[6] Jin-Yi Cai, Zhiguo Fu, Heng Guo, and Tyson Williams. "A Holant dichotomy: is the FKT algorithm universal?" In: *2015 IEEE 56th Annual Symposium on Foundations of Computer Science.* IEEE. 2015, pp. 1259–1276.

[7] Jin-Yi Cai and Artem Govorov. "On a Theorem of Lovász That (., H) Determines the Isomorphism Type of H". In: *ACM Trans. Comput. Theory* 13.2 (2021). DOI: 10.1145/3448641.

[8] Jin-Yi Cai, Heng Guo, and Tyson Williams. "A Complete Dichotomy Rises from the Capture of Vanishing Signatures". In: *SIAM Journal on Computing* 45.5 (2016), pp. 1671–1728. DOI: 10.1137/15M1049798.

[9] Jin-Yi Cai, Pinyan Lu, and Mingji Xia. "Computational Complexity of Holant Problems". In: *SIAM Journal on Computing* 40.4 (2011), pp. 1101–1132. DOI: 10.1137/100814585.

[10] Jin-Yi Cai, Pinyan Lu, and Mingji Xia. "Holographic algorithms by Fibonacci gates and holographic reductions for hardness". In: *2008 49th Annual IEEE Symposium on Foundations of Computer Science.* IEEE. 2008, pp. 644–653.

[11] Jin-Yi Cai and Ben Young. "Planar #CSP Equality Corresponds to Quantum Isomorphism – A Holant Viewpoint". In: *ACM Transactions on Computation Theory* 16.3 (Sept. 2024).

[12] Holger Dell, Martin Grohe, and Gaurav Rattan. "Lovász Meets Weisfeiler and Leman". In: *45th International Colloquium on Automata, Languages, and Programming (ICALP 2018)*. Ed. by Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella. Vol. 107. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018, 40:1–40:14. DOI: 10.4230/LIPIcs.ICALP.2018.40.

[13] Jan Draisma, Dion C. Gijswijt, László Lovász, Guus Regts, and Alexander Schrijver. "Characterizing partition functions of the vertex model". In: *Journal of Algebra* 350.1 (Jan. 2012), pp. 197–206. DOI: 10.1016/j.jalgebra.2011.10.030.

[14] Jan Draisma and Guus Regts. "Tensor invariants for certain subgroups of the orthogonal group". en. In: *Journal of Algebraic Combinatorics* 38.2 (Sept. 2013), pp. 393–405. DOI: 10.1007/s10801-012-0408-7.

[15] Zdeněk Dvořák. "On recognizing graphs by numbers of homomorphisms". en. In: *Journal of Graph Theory* 64.4 (2010), pp. 330–342. DOI: 10.1002/jgt.20461.

[16] Martin Dyer and David Richerby. "An Effective Dichotomy for the Counting Constraint Satisfaction Problem". In: *SIAM Journal on Computing* 42.3 (2013), pp. 1245–1274. DOI: 10.1137/100811258.

[17] Michael Freedman, László Lovász, and Alexander Schrijver. "Reflection Positivity, Rank Connectivity, and Homomorphism of Graphs". In: *Journal of the American Mathematical Society* 20.1 (2007), pp. 37–51.

[18] Martin Grohe, Gaurav Rattan, and Tim Seppelt. "Homomorphism Tensors and Linear Equations". In: *49th International Colloquium on Automata, Languages, and Programming (ICALP 2022)*. Vol. 229. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 70:1–70:20. DOI: 10.4230/LIPIcs.ICALP.2022.70. arXiv: 2111.11313 [math.CO]. URL: https://arxiv.org/abs/2111.11313.

[19] P. de la Harpe and V. F. R. Jones. "Graph invariants related to statistical mechanical models: examples and problems". In: *J. Comb. Theory Ser. B* 57.2 (Mar. 1993), pp. 207–227. DOI: 10.1006/jctb.1993.1017.

[20] Sangxia Huang and Pinyan Lu. "A dichotomy for real weighted Holant problems". In: *Computational Complexity* 25 (2016), pp. 255–304.

[21] Jiabao Lin and H. Wang. "The Complexity of Boolean Holant Problems with Nonnegative Weights". In: *SIAM Journal on Computing* 47.3 (2018), pp. 798–828. DOI: 10.1137/17M113304X.

[22] Yin Liu. *A combinatorial view of Holant problems on higher domains*. 2024. arXiv: 2403.14150 [cs.CC]. URL: https://arxiv.org/abs/2403.14150.

[23] László Lovász. "Operations with structures". In: *Acta Mathematica Hungarica* 18.3-4 (1967), pp. 321–328.

[24] László Lovász. "The rank of connection matrices and the dimension of graph algebras". In: *European Journal of Combinatorics* 27.6 (2006), pp. 962–970.

[25] Laura Mančinska and David E. Roberson. "Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs". In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. 2020, pp. 661–672. DOI: 10.1109/FOCS46700.2020.00067. arXiv: 1910.06958 [quant-ph]. URL: https://arxiv.org/abs/1910.06958.

[26] Guus Regts. "A characterization of edge-reflection positive partition functions of vertex-coloring models". en. In: *The Seventh European Conference on Combinatorics, Graph Theory and Applications*. CRM Series. Pisa: Scuola Normale Superiore, 2013, pp. 305–311. ISBN: 978-88-7642-475-5. DOI: 10.1007/978-88-7642-475-5_49.

[27] Guus Regts. "Graph parameters and invariants of the orthogonal group". PhD thesis. Universiteit van Amsterdam, 2013.

[28] Guus Regts. "The rank of edge connection matrices and the dimension of algebras of invariant tensors". In: *European Journal of Combinatorics* 33.6 (Aug. 2012), pp. 1167–1173. DOI: 10.1016/j.ejc.2012.01.014.

[29] Elina Robeva. "Orthogonal Decomposition of Symmetric Tensors". In: *SIAM Journal on Matrix Analysis and Applications* 37.1 (Jan. 2016), pp. 86–102. DOI: 10.1137/140989340.

[30] Alexander Schrijver. "Graph Invariants in the Edge Model". In: *Building Bridges: Between Mathematics and Computer Science*. Ed. by Martin Grötschel, Gyula O. H. Katona, and Gábor Sági. Berlin, Heidelberg: Springer, 2008, pp. 487–498. DOI: 10.1007/978-3-540-85221-6_16.

[31] Alexander Schrijver. "Graph invariants in the spin model". In: *Journal of Combinatorial Theory, Series B* 99.2 (2009), pp. 502–511. DOI: https://doi.org/10.1016/j.jctb.2008.10.003.

[32] Alexander Schrijver. "Tensor subalgebras and first fundamental theorems in invariant theory". In: *Journal of Algebra* 319.3 (Feb. 2008), pp. 1305–1319. DOI: 10.1016/j.jalgebra.2007.10.039.

[33] Shuai Shao and Jin-Yi Cai. "A Dichotomy for Real Boolean Holant Problems". In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. 2020, pp. 1091–1102. DOI: 10.1109/FOCS46700.2020.00105. arXiv: 2005.07906 [cs.CC]. URL: https://arxiv.org/abs/2005.07906.

[34] Wilhelm Specht. "Zur Theorie der Matrizen. II." ger. In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 50 (1940), pp. 19–23.

[35] Balázs Szegedy. "Edge coloring models and reflection positivity". en. In: *Journal of the American Mathematical Society* 20.4 (May 2007), pp. 969–988. DOI: 10.1090/S0894-0347-07-00568-1.

[36] Leslie G Valiant. "Accidental algorthims". In: *2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*. IEEE. 2006, pp. 509–517.

[37] Leslie G. Valiant. "Holographic algorithms". In: *SIAM Journal on Computing* 5 (2008), pp. 1565–1594.

[38] Hermann Weyl. *The Classical Groups: Their Invariants and Representations*. Princeton University Press, 1966. ISBN: 978-0-691-05756-9. DOI: 10.2307/j.ctv3hh48t.

[39] NA Wiegmann. "Necessary and sufficient conditions for unitary similarity". In: *Journal of the Australian Mathematical Society* 2.1 (1961), pp. 122–126.

[40] Mingji Xia. "Holographic reduction: A domain changed application and its partial converse theorems". In: *Automata, Languages and Programming: 37th International Colloquium, ICALP 2010, Bordeaux, France, July 6-10, 2010, Proceedings, Part I 37*. Springer. 2010, pp. 666–677.

[41] Ben Young. *Equality on all #CSP Instances Yields Constraint Function Isomorphism via Interpolation and Intertwiners*. 2022. arXiv: 2211.13688 [cs.DM].