Asymptotically Optimal Inapproximability of Maxmin k-Cut Reconfiguration

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Abstract

k-COLORING RECONFIGURATION is one of the most well-studied reconfiguration problems, which asks to transform a given proper k-coloring of a graph to another by repeatedly recoloring a single vertex. Its approximate version, MAXMIN k-CUT RECONFIGURATION, is defined as an optimization problem of maximizing the minimum fraction of bichromatic edges during the transformation between (not necessarily proper) k-colorings. In this paper, we prove that the optimal approximation factor of this problem is $1 - \Theta\left(\frac{1}{k}\right)$ for every $k \geq 2$. Specifically, we show the PSPACE-hardness of approximating the objective value within a factor of $1 - \frac{\varepsilon}{k}$ for some universal constant $\varepsilon > 0$, whereas we present a deterministic polynomial-time algorithm that achieves the approximation factor of $1 - \frac{2}{k}$.

To prove the hardness result, we develop a new probabilistic verifier that tests a "striped" pattern. Our polynomial-time algorithm is based on "random reconfiguration via a random solution," i.e., the transformation that goes through one random k-coloring.

Contents

1 Introduction

Reconfiguration is an emerging field in theoretical computer science, which studies reachability and connectivity problems over the space of solutions. A *reconfiguration problem* can be defined for any combinatorial problem Π and any transformation rule over the feasible solutions of Π. The problem Π is referred to as the *source problem* of a reconfiguration problem. For an instance I of Π and a pair of its feasible solutions, the reconfiguration problem asks if one solution can be transformed into the other by repeatedly applying the transformation rule while always preserving that every intermediate solution is feasible. Speaking differently, the reconfiguration problem concerns the reachability over the *configuration graph*, where each node corresponds to a feasible solution of the given instance I and each link represents that its endpoints are "adjacent" under the transformation rule. Such a sequence of feasible solutions that form a path on the configuration graph is called a *reconfiguration sequence*. Over the past twenty years, many reconfiguration problems have been defined from a variety of source problems, including Boolean satisfiability, constraint satisfaction problems, and graph problems.

The computational complexity of reconfiguration problems has been extensively studied; e.g., reconfiguration problems of 3-SAT [GKMP09], INDEPENDENT SET [HD05, HD09], and SET COVER [IDHPSUU11] are PSPACE-complete, whereas those of 2-SAT [GKMP09], MATCHING [IDHPSUU11], and SPANNING TREE [IDHPSUU11] belong to P. We refer the readers to the surveys by Bousquet, Mouawad, Nishimura, and Siebertz [BMNS24], Mynhardt and Nasserasr [MN19], Nishimura [Nis18], and van den Heuvel [van13] as well as the Combinatorial Reconfiguration wiki [Hoa23] for more algorithmic, hardness, and structural results of reconfiguration problems.

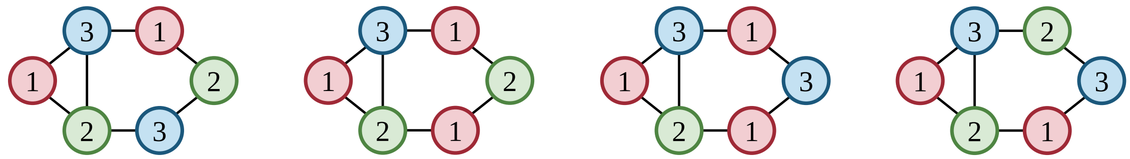
1.1 k-COLORING RECONFIGURATION and Its Approximate Version

One of the most well-studied reconfiguration problems, which we study in this paper, is k-COLORING RECONFIGURATION [BC09, Cer07, CvJ08, CvJ09, CvJ11], whose source problem is k-COLORING. Recall that k-COLORING is a graph coloring problem of deciding if a graph G is *k-colorable*; namely, there is a *proper* k-coloring $f\colon V(G) \to [k]$ of G, which renders every edge bichromatic.[1] In the k-COLORING RECONFIGURATION problem, for a k-colorable graph G and a pair of its proper k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}\colon V(G) \to [k]$, we seek a reconfiguration sequence from f_{start} to f_{end} consisting only of proper k-colorings of G, such that every pair of neighboring k-colorings differ in a single vertex. See Figures 1 and 2 for YES and NO instances of k-COLORING RECONFIGURATION. If the number k of available colors is sufficiently large (e.g., the maximum degree of G plus 2 or more [DFFV06, Jer95]), the answer to this problem is always YES. For a constant value of k, the following complexity results are known: If $k \leqslant 3$, then k-COLORING RECONFIGURATION belongs to P [CvJ11].[2] On the other hand, k-COLORING RECONFIGURATION is PSPACE-complete for every $k \geqslant 4$ [BC09]. Quite interestingly, 3-COLORING "becomes" easy in the reconfiguration regime even though 3-COLORING itself is NP-complete [GJS76, Lov73, Sto73]. Several existing work further investigate the parameterized complexity [BMNR14, JKKPP16] and the complexity for restricted graph classes [BB13, BJLPP11, BJLPP14, CvJ09, HIZ19, Wro18]. Note that the configuration graph of k-COLORING RECONFIGURATION is closely related to the *Glauber dynamics* [DFFV06, Jer95, Mol04]. See also Section 4 for related work.

In this paper, we study *approximability* of k-COLORING RECONFIGURATION. Since 2023, approximability of reconfiguration problems has been studied actively from both hardness and algorithmic sides

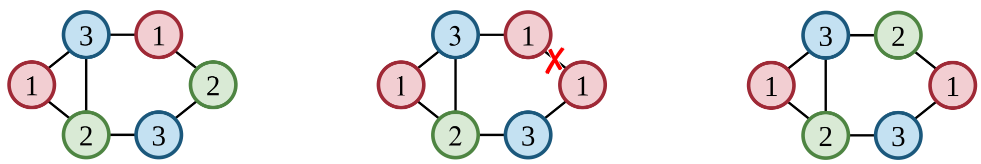
[1] An edge is *bichromatic* if its endpoints receive different colors.

[2] Moreover, a reconfiguration sequence for YES instances can be found in polynomial time.



(a) 3-coloring f_{start}. (b) 3-coloring $f^{(2)}$. (c) 3-coloring $f^{(3)}$. (d) 3-coloring f_{end}.

Figure 1: A YES instance of 3-COLORING RECONFIGURATION. There is a reconfiguration sequence $(f_{\mathsf{start}} = f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} = f_{\mathsf{end}})$ such that each 3-coloring is proper and is obtained by the previous one by recoloring a single vertex.



(a) 3-coloring g_{start}. (b) 3-coloring $g^{(2)}$. (c) 3-coloring g_{end}.

Figure 2: A NO instance of 3-COLORING RECONFIGURATION. There is no reconfiguration sequence from g_{start} to g_{end}, because g_{end} is "frozen" in that any vertex cannot recolored. Considering this input as an instance of MAXMIN 3-CUT RECONFIGURATION, we can transform g_{start} into g_{end} via $g^{(2)}$, which contains a single monochromatic edge.

[HO24a, HO24b, KM23, Ohs23, Ohs24a, Ohs24b, Ohs24c, Ohs25]. For a reconfiguration problem, its *approximate version* [IDHPSUU11] allows to relax the feasibility of intermediate solutions, but requires to optimize the "worst" feasibility during reconfiguration. For example, an approximate version of SET COVER RECONFIGURATION admits a 2-factor approximation algorithm [IDHPSUU11], which has been recently proven to be PSPACE-hard to approximate within a factor of $2 - o(1)$ [HO24a]. There are two natural approximate versions of k-COLORING RECONFIGURATION since k-COLORING has the following two approximate versions:

1. Maximizing the number of bichromatic edges: For a (not necessarily k-colorable) graph G, the first problem asks to find a k-coloring of G that makes as many edges as possible bichromatic. This problem is known by the names of MAX k-CUT and MAX k-COLORABLE SUBGRAPH [GS13, PY91].[3]

2. Minimizing the number of used colors: For a (not necessarily k-colorable) graph G, the second problem asks to find a proper coloring of G that uses as few colors as possible. This problem is known as CHROMATIC NUMBER and GRAPH COLORING.

In this paper, we study a reconfiguration analogue of the first problem k-CUT, which we call MAXMIN k-CUT RECONFIGURATION. In this problem, given a (not necessarily k-colorable) graph $G = (V, E)$ and a pair of its k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [k]$, we shall construct a reconfiguration sequence $\mathscr{F}$ from f_{start} to f_{end} consisting of any (not necessarily proper) k-colorings of G that maximizes the *minimum fraction* of bichromatic edges of G, where the minimum is taken over all k-colorings of $\mathscr{F}$.

[3] This problem is also called MAX k-COLORING [AOTW14, FK98] or MAX k-COLORABILITY [Pet94]. We do not use these names to avoid confusion with other graph coloring problems.

MAXMIN k-CUT RECONFIGURATION

Input:	a graph $G = (V, E)$ and a pair of k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [k]$ of G.
Output:	a reconfiguration sequence $\mathscr{F}$ from f_{start} to f_{end}.
Goal:	maximize the minimum fraction of bichromatic edges of G over all k-colorings of $\mathscr{F}$.

See Figure 2 for an example of MAXMIN k-CUT RECONFIGURATION. Solving this problem, we may be able to find a "reasonable" reconfiguration sequence, which consists of "almost" proper k-colorings, so that we can manage NO instances of k-COLORING RECONFIGURATION.

Here, we briefly review known results on MAXMIN k-CUT RECONFIGURATION. The PSPACE-hardness of exactly solving MAXMIN k-CUT RECONFIGURATION for every $k \geqslant 4$ follows from that of k-COLORING RECONFIGURATION [BC09]. For the PSPACE-hardness of approximation, the *Probabilistically Checkable Reconfiguration Proof* (PCRP) theorem due to Hirahara and Ohsaka [HO24b] and Karthik C. S. and Manurangsi [KM23], along with a series of gap-preserving reductions due to Bonsma and Cereceda [BC09] and Ohsaka [Ohs23], implies that MAXMIN 4-CUT RECONFIGURATION is PSPACE-hard to approximate within some constant factor.[4] However, the *asymptotic* behavior of approximability for MAXMIN k-CUT RECONFIGURATION with respect to the number k of available colors is not well understood.

1.2 Our Results

In this paper, we find out that the asymptotically optimal approximation factor of MAXMIN k-CUT RECONFIGURATION is $1 - \Theta\left(\frac{1}{k}\right)$. On the hardness side, we demonstrate the PSPACE-hardness of approximation within a factor of $1 - \Omega\left(\frac{1}{k}\right)$ for *every* $k \geqslant 2$.

Theorem 1.1 (informal; see Theorem 6.1)**.** *There exist universal constants $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$ such that for every $k \geqslant 2$, a multigraph G, and a pair of its k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}$, it is* PSPACE*-hard to distinguish between the following cases*:

- (*Completeness*) *There exists a reconfiguration sequence from f_{start} to f_{end} consisting of k-colorings that make at least $\left(1 - \frac{\varepsilon_c}{k}\right)$-fraction of edges of G bichromatic.*
- (*Soundness*) *Every reconfiguration sequence contains a k-coloring that makes more than $\frac{\varepsilon_s}{k}$-fraction of edges of G monochromatic.*

In particular, MAXMIN k-CUT RECONFIGURATION *is* PSPACE*-hard to approximate within a factor of $1 - \frac{\varepsilon}{k}$ for every $k \geqslant 2$ for some universal constant $\varepsilon \in (0,1)$.*

On the algorithmic side, we develop a deterministic $\left(1 - \frac{2}{k}\right)$-factor approximation algorithm for *every $k \geqslant$ 2*.[5]

Theorem 1.2 (informal; see Theorem 7.1)**.** *For every $k \geqslant 2$, there exists a deterministic $\left(1 - \frac{2}{k}\right)$-factor approximation algorithm for* MAXMIN k-CUT RECONFIGURATION.

To the best of our knowledge, this is the first non-trivial approximation algorithm for MAXMIN k-CUT RECONFIGURATION.

Theorems 1.1 and 1.2 provide asymptotically tight lower and upper bounds for approximability of MAXMIN k-CUT RECONFIGURATION.

[4] See Section 4 for other applications of the PCRP theorem in the PSPACE-hardness of approximating reconfiguration problems.
[5] Although $1 - \frac{2}{k} = 0$ if $k = 2$, the actual approximation factor can be arbitrarily close to $\frac{1}{4}$. See Section 7.

1.3 Organization

The rest of this paper is organized as follows. In Sections 2 and 3, we present an overview of the proof of Theorems 1.1 and 1.2, respectively. In Section 4, we review related work on variants of k-COLORING RECONFIGURATION, and approximability of MAX k-CUT and reconfiguration problems. In Section 5, we formally define k-COLORING RECONFIGURATION as well as MAXMIN k-CUT RECONFIGURATION. In Section 6, we prove that MAXMIN k-CUT RECONFIGURATION is PSPACE-hard to approximate within a factor of $1-\Omega\left(\frac{1}{k}\right)$ (Theorem 1.1). In Section 7, we develop a deterministic $\left(1-\frac{2}{k}\right)$-factor approximation algorithm for MAXMIN k-CUT RECONFIGURATION (Theorem 1.2). Some technical proofs are deferred to Appendix A.

1.4 Notations

For a nonnegative integer $n \in \mathbb{N}$, let $[n] \coloneqq \{1,2,\ldots,n\}$. We use the Iverson bracket $[\![\cdot]\!]$; i.e., $[\![P]\!]$ for a statement P is defined as 1 if P is true and 0 otherwise. A *sequence* $\mathscr{S}$ of a finite number of objects, $s^{(1)},\ldots,s^{(T)}$, is denoted by $(s^{(1)},\ldots,s^{(T)})$, and we write $s \in \mathscr{S}$ to indicate that s appears in $\mathscr{S}$. The symbol $\circ$ stands for a concatenation of two sequences or functions, and $\mathfrak{S}_n$ for the set of all permutations over $[n]$. For a set $\mathcal{S}$, we write $X \sim \mathcal{S}$ to mean that X is a random variable uniformly drawn from $\mathcal{S}$. For two functions $f,g\colon \mathcal{D} \to \mathcal{R}$ over a finite domain $\mathcal{D}$, the *relative Hamming distance* between f and g, denoted by $\mathrm{dist}(f,g)$, is defined as the fraction of positions on which f and g differ; namely,

$$\mathrm{dist}(f,g) \coloneqq \mathop{\mathbb{P}}_{x\sim\mathcal{D}}\Big[f(x) \neq g(x)\Big] = |\mathcal{D}|^{-1} \cdot \left|\Big\{x \in \mathcal{D} \;\Big|\; f(x) \neq g(x)\Big\}\right|. \tag{1.1}$$

We say that f is ε*-close to* g if $\mathrm{dist}(f,g) \leqslant \varepsilon$ and ε*-far from* g if $\mathrm{dist}(f,g) > \varepsilon$. Similar notations are used for a set of function G from $\mathcal{D}$ to $\mathcal{R}$; e.g., $\mathrm{dist}(f,G) \coloneqq \min_{g\in G} \mathrm{dist}(f,g)$ and f is ε-close to G if $\mathrm{dist}(f,G) \leqslant \varepsilon$.

2 Proof Overview of PSPACE-hardness of Approximation

In this section, we give an overview of the proof of Theorem 1.1; i.e., MAXMIN k-CUT RECONFIGURATION is PSPACE-hard to approximate within a factor of $1-\Omega\left(\frac{1}{k}\right)$. For a graph G and a pair of its k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}\colon V(G) \to [k]$, let $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}})$ denote the *optimal value* of MAXMIN k-CUT RECONFIGURATION; namely, the maximum of the minimum fraction of bichromatic edges of G, where the maximum is taken over all possible reconfiguration sequences from f_{start} to f_{end}. For any reals $0 \leqslant s \leqslant c \leqslant 1$, GAP$_{c,s}$ k-CUT RECONFIGURATION asks whether $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) \geqslant c$ or $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) < s$. See Section 5 for the formal definition.

Our starting point is the PSPACE-hardness of approximating MAXMIN 2-CUT RECONFIGURATION, whose proof is based on [BC09, HO24b, Ohs23].

Proposition 2.1 (informal; see Proposition 6.2)**.** *There exist universal constants $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$ such that* GAP$_{1-\varepsilon_c,1-\varepsilon_s}$ 2-CUT RECONFIGURATION *is* PSPACE*-hard.*

We construct the following two gap-preserving reductions from MAXMIN 2-CUT RECONFIGURATION to MAXMIN k-CUT RECONFIGURATION, the former for all sufficiently large k and the latter for finitely many k.

Lemma 2.2 (informal; see Lemma 6.3)**.** *For every reals $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$, there exist reals $\delta_c, \delta_s \in (0,1)$ with $\delta_c < \delta_s$ such that for all sufficiently large $k \geqslant k_0 := 10^3$, there exists a gap-preserving reduction from* GAP$_{1-\varepsilon_c,1-\varepsilon_s}$ 2-CUT RECONFIGURATION *to* GAP$_{1-\frac{\delta_c}{k},1-\frac{\delta_s}{k}}$ k-CUT RECONFIGURATION.

Lemma 2.3 (informal; see Lemma 6.4)**.** *For every integer $k \geqslant 3$ and every reals $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$, there exist universal constants $\delta_c, \delta_s \in (0,1)$ with $\delta_c < \delta_s$ such that there exists a gap-preserving reduction from* GAP$_{1-\varepsilon_c,1-\varepsilon_s}$ 2-CUT RECONFIGURATION *to* GAP$_{1-\delta_c,1-\delta_s}$ k-CUT RECONFIGURATION.

We obtain Theorem 1.1 as a corollary of Proposition 2.1 and Lemmas 2.2 and 2.3. Since the most technical part in the proof of Theorem 1.1 is Lemma 2.2, we will outline its proof in the remainder of this section. See Appendix A for the proofs of Proposition 2.1 and Lemma 2.3.

2.1 Failed Attempt: Why [GS13, KKLP97] Do Not Work for Proving Lemma 2.2

To prove Lemma 2.2, one might think of applying the existing proof techniques for the NP-hardness of approximating MAX k-CUT, which has the (asymptotically) same approximation threshold of $1-\Theta\left(\frac{1}{k}\right)$ as MAXMIN k-CUT RECONFIGURATION [AOTW14, FJ97, GS13, KKLP97] (see Section 4 for related work). However, this approach *does not work* for proving Lemma 2.2: when a gap-preserving reduction from MAX 2-CUT to MAX k-CUT due to [GS13, KKLP97] is used to reduce MAXMIN 2-CUT RECONFIGURATION to MAXMIN k-CUT RECONFIGURATION, the ratio between completeness and soundness becomes $1-O\left(\frac{1}{k^2}\right)$.

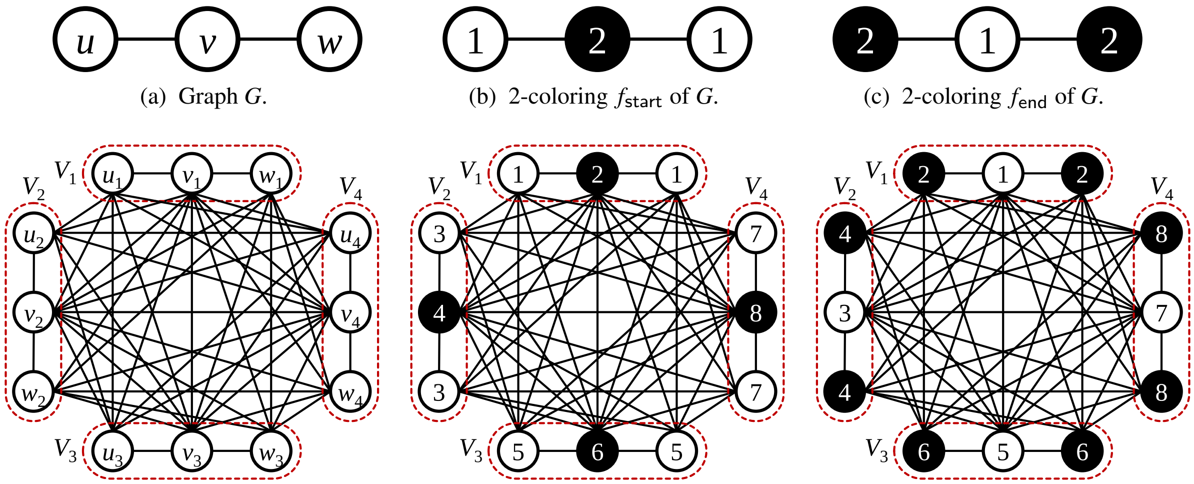
To explain the detail, we briefly review the gap-preserving reduction of Kann, Khanna, Lagergren, and Panconesi [KKLP97].[6] For a graph $G=(V,E)$ and a positive even integer k, a new weighted graph H is constructed as follows.

- Create fresh $\frac{k}{2}$ copies of each vertex v of G, denoted by $v_1,\ldots,v_{\frac{k}{2}}$.
- For each edge (v,w) of G and pair $i,j \in \left[\frac{k}{2}\right]$, create an edge (v_i,w_j) of weight 1.
- For each vertex v of G and pair $i \neq j \in \left[\frac{k}{2}\right]$, create an edge (v_i,v_j) of weight equal to the degree of v.

See Figures 3a and 3d for illustration. The total edge weight of H is equal to $\binom{k}{2}\cdot|E|$. By [KKLP97], this construction is an approximation-preserving reduction from MAX 2-CUT to MAX k-CUT, implying the NP-hardness of $\left(1-\Omega\left(\frac{1}{k}\right)\right)$-factor approximation for MAX k-CUT.

Let us apply the above reduction to reduce MAXMIN 2-CUT RECONFIGURATION to MAXMIN k-CUT RECONFIGURATION. Given a graph $G=(V,E)$ and a pair of its proper 2-colorings $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [2]$ as an instance of MAXMIN 2-CUT RECONFIGURATION, we construct an instance of MAXMIN k-CUT RECONFIGURATION as follows. First, create a weighted graph H from G according to [KKLP97]. Then, create a pair of k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}} \colon V(H) \to [k]$ of H in a natural manner such that $f'_{\mathsf{start}}(v_i) := f_{\mathsf{start}}(v)+2(i-1)$ and $f'_{\mathsf{end}}(v_i) := f_{\mathsf{end}}(v)+2(i-1)$ for each vertex v_i of H. See Figures 3b, 3c, 3e and 3f for illustration. For each $i \in \left[\frac{k}{2}\right]$, we define $V_i := \{v_i \mid v \in V\}$. Observe that f'_{start} and f'_{end} are proper if so are f_{start} and f_{end}, and every vertex of V_i is colored in $2i-1$ or $2i$. Here, we claim that $\mathsf{opt}_H(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}) \geqslant 1-\frac{2}{k(k-1)}$ independent of the value of $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}})$. Consider a reconfiguration sequence $\mathscr{F}'$ from f'_{start} to f'_{end} obtained by recoloring vertices of $V_1, V_2, \ldots, V_{\frac{k}{2}}$ in this order. Suppose we are on the way of recoloring the

[6] The reduction of Guruswami and Sinop [GS13] differs from that of [KKLP97] in that it starts from MAX 3-CUT to preserve the perfect completeness.



(a) Graph G. (b) 2-coloring f_{start} of G. (c) 2-coloring f_{end} of G.

(d) Graph H. (e) k-coloring f'_{start} of H. (f) k-coloring f'_{end} of H.

Figure 3: A failed attempt to reduce MAXMIN 2-CUT RECONFIGURATION to MAXMIN k-CUT RECONFIGURATION ($k = 8$) using [KKLP97]. Given a graph G and a pair of its 2-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}$, we construct a new graph H and a pair of its k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}}$. Consider a reconfiguration sequence $\mathscr{F}'$ from f'_{start} to f'_{end} obtained by recoloring vertices of $V_1, V_2, \ldots, V_{\frac{k}{2}}$ in this order. For any intermediate k-coloring of $\mathscr{F}'$, all but one induced subgraph $H[V_i]$ do not contain any monochromatic edges.

vertices of V_i. The subgraph $H[V_i]$ may contain (at most) $|E|$ monochromatic edges, but all other $(\frac{k}{2} - 1)$ subgraphs $H[V_j]$ for $j \neq i$ *do not* contain any monochromatic edges, deriving that

$$\mathsf{opt}_H\left(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\right) \geqslant 1 - \frac{1 \cdot |E| + \left(\frac{k}{2} - 1\right) \cdot 0}{\binom{k}{2} \cdot |E|} \geqslant 1 - \frac{2}{k(k-1)}. \tag{2.1}$$

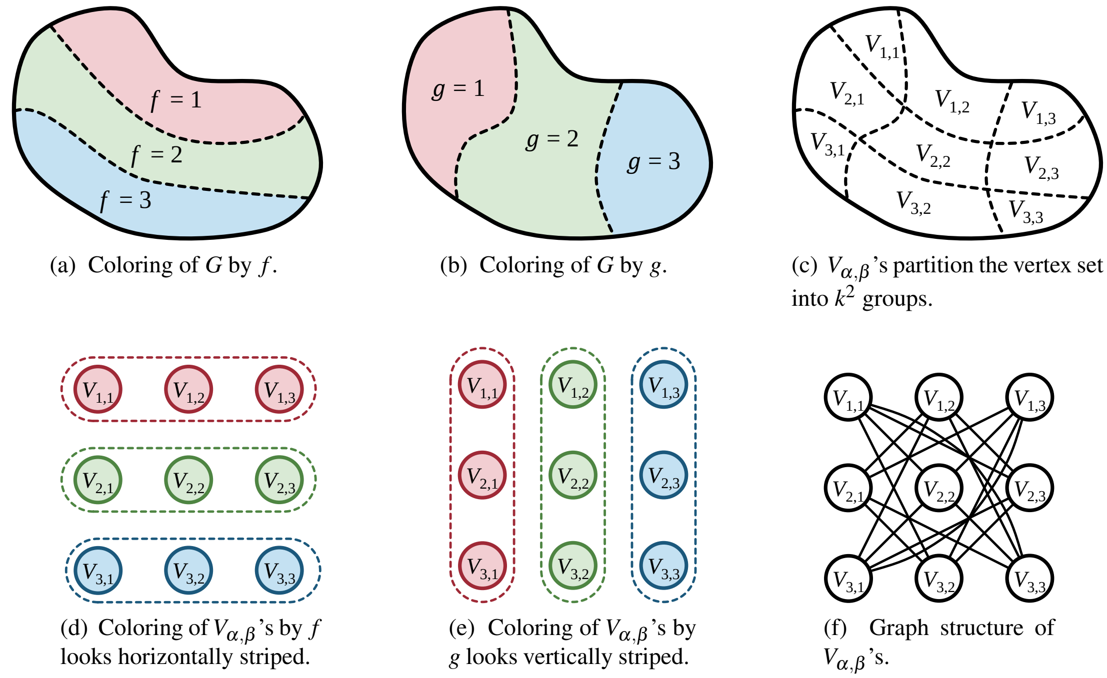
This is undesirable because the ratio between completeness and soundness is at least $1 - O\left(\frac{1}{k^2}\right)$.

2.2 Our Reduction in the Proof of Lemma 2.2

Our gap-preserving reduction from MAXMIN 2-CUT RECONFIGURATION to MAXMIN k-CUT RECONFIGURATION is completely different from those of [GS13, KKLP97]. Briefly speaking, we shall encode a 2-coloring of each vertex v of a graph G by a k-coloring of a $k \times k$ grid $[k]^2$. Our proposed encoding is motivated by the following scenario: Suppose that for a graph $G = (V, E)$ and a pair of its proper k-colorings $f, g \colon V \to [k]$, we would like to find an optimal reconfiguration sequence from f to g (see Figures 4a and 4b). For each pair of colors $\alpha, \beta \in [k]$, let $V_{\alpha,\beta}$ be the set of vertices in V colored α by f and β by g (see Figure 4c); namely,

$$V_{\alpha,\beta} \coloneqq \left\{ v \in V \;\middle|\; f(v) = \alpha \text{ and } g(v) = \beta \right\}. \tag{2.2}$$

If $V_{\alpha,\beta}$'s are placed on a $k \times k$ grid, f looks "horizontally striped" while g looks "vertically striped" (see Figures 4d and 4e). Since both f and g are proper, there may exist edges between V_{α_1,β_1} and V_{α_2,β_2} *only if* $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$ (see Figure 4f). On the other hand, any reconfiguration sequence from f to g seems to make a nonnegligible fraction of edges into monochromatic. The above structural observation motivates the following two ideas:



(a) Coloring of G by f.

(b) Coloring of G by g.

(c) $V_{\alpha,\beta}$'s partition the vertex set into k^2 groups.

(d) Coloring of $V_{\alpha,\beta}$'s by f looks horizontally striped.

(e) Coloring of $V_{\alpha,\beta}$'s by g looks vertically striped.

(f) Graph structure of $V_{\alpha,\beta}$'s.

Figure 4: Our proposed encoding and the stripe test are motivated by the graph structure formed by two different proper k-colorings.

Idea 1: Consider the "striped" pattern represented by a k-coloring of $[k]^2$ as if it were encoding $[2]$; i.e., the "horizontally striped" pattern represents 1, whereas the "vertically striped" pattern represents 2. This encoding can be thought of as a *very redundant* error-correcting code from $[2]$ to $[k]^{[k]^2}$.

Idea 2: Given a graph $G = (V,E)$ and a collection of $|V|$ k-colorings of $[k]^2$ for each vertex of G, we test if these k-colorings encode a *proper* 2-coloring of G. Specifically, we will design a probabilistic verifier that checks if (1) a k-coloring of $[k]^2$ associated with each vertex of G is close to a striped pattern, and (2) a pair of k-colorings of $[k]^2$ corresponding to each edge of G encode different colors. In the subsequent sections, we will introduce the following three auxiliary verifiers to achieve this requirement: **Stripe**, **consistency**, and **edge verifiers**.

We will say that a k-coloring $f\colon [k]^2 \to [k]$ is *horizontally striped* if $f(x,y) = \sigma(y)$ for all $(x,y) \in [k]^2$ for some permutation $\sigma \in \mathfrak{S}_k$, *vertically striped* if $f(x,y) = \sigma(x)$ for all $(x,y) \in [k]^2$ for some permutation $\sigma \in \mathfrak{S}_k$, and *striped* if it is horizontally or vertically striped.

2.2.1 Stripe Test (Section 6.2.1)

Our first, most important verifier is the *stripe verifier* $\mathcal{V}_{\text{stripe}}$, which checks if a k-coloring f of $[k]^2$ is close to a striped pattern. Specifically, $\mathcal{V}_{\text{stripe}}$ samples a pair of vertices from $[k]^2$ that forms a *diagonal line* in a $k \times k$ grid, and it accepts if they have different colors, as follows:

4	1	1	8	1	1	1	1
2	2	2	2	2	7	6	2
3	3	6	3	7	3	3	3
4	8	4	4	4	4	1	4
3	5	5	5	2	5	5	5
6	6	6	6	6	7	6	2
7	1	7	5	7	7	7	7
8	8	3	8	8	4	8	8

Figure 5: A k-coloring f of $[k]^2$ that is far from being striped. Obviously, f is closest to an 8×8 horizontally striped pattern but differs in 16 entries; thus, f is 0.25-far from being striped.

Stripe verifier $\mathcal{V}_{\text{stripe}}$.

Oracle access: a k-coloring $f\colon [k]^2 \to [k]$.

select $(x_1, y_1) \in [k]^2$ and $(x_2, y_2) \in [k]^2$ s.t. $x_1 \neq x_2$ and $y_1 \neq y_2$ uniformly at random.
if $f(x_1, y_1) = f(x_2, y_2)$ **then**
    declare reject.
else
    declare accept.

We say that a k-coloring $f\colon [k]^2 \to [k]$ is *ε-far from being striped* if f is ε-far from every striped k-coloring, and is *ε-close to being striped* if f is ε-close to some striped k-coloring. See Figure 5 for an example of a k-coloring of $[k]^2$ far from being striped.

The following lemma is the crux of the proof of Lemma 2.2, which bounds $\mathcal{V}_{\text{stripe}}$'s rejection probability with respect to the distance from f to the striped pattern:

Lemma 2.4 (informal; see Lemmas 6.6 and 6.7)**.** *The following hold*:

- *if f is striped, $\mathcal{V}_{\text{stripe}}$ accepts with probability* 1;
- *if f is ε-far from being striped, $\mathcal{V}_{\text{stripe}}$ rejects with probability* $\Omega\left(\frac{\varepsilon}{k}\right)$.

The rejection probability "$\Omega\left(\frac{\varepsilon}{k}\right)$" is critical for deriving a $\left(1 - \Omega\left(\frac{1}{k}\right)\right)$-factor gap between completeness and soundness. The latter statement of Lemma 2.4 presented in Section 6.4 involves the most technical proof in this paper, exploiting the nontrivial structure of a k-coloring of $[k]^2$ far from being striped.

Observe that $\mathcal{V}_{\text{stripe}}$ is only allowed to sample a pair (v, w) of vertices from $[k]^2$ (nonadaptively) and accepts (resp. rejects) if $f(v) \neq f(w)$ (resp. $f(v) = f(w)$). Thus, $\mathcal{V}_{\text{stripe}}$ can be "emulated" by a graph H such that

$$V(H) := [k]^2, \tag{2.3}$$

$$E(H) := \left\{ \big((x_1, y_1), (x_2, y_2)\big) \in \left([k]^2\right)^2 \;\middle|\; x_1 \neq x_2 \text{ and } y_1 \neq y_2 \right\}, \tag{2.4}$$

in a sense that for any k-coloring f of $[k]^2$, the probability that $\mathcal{V}_{\text{stripe}}$ accepts (resp. rejects) f is equal to the fraction of edges in H that are made bichromatic (resp. monochromatic) by f. In fact, the graph structure of Figure 4f coincides with H. The remaining two verifiers can also be emulated by (multi)graphs.

2.2.2 Consistency Test (Section 6.2.2)

Our next verifier is the *consistency verifier* $\mathcal{V}_{\mathrm{cons}}$, which checks if a pair of k-colorings f, g of $[k]^2$ share the *same* striped pattern (given that both f and g are close to being striped). Specifically, $\mathcal{V}_{\mathrm{cons}}$ runs the *row test* and *column test* with equal probability, the former for the horizontally striped pattern and the latter for the vertically striped pattern, as follows:

Consistency verifier $\mathcal{V}_{\mathrm{cons}}$.

Oracle access: two k-colorings $f, g\colon [k]^2 \to [k]$.

```
sample r ~ [0,1].
if 0 ⩽ r < 1/2 then                                              ▷ run the row test with probability 1/2.
│   select (x1, y1) ∈ [k]^2 and (x2, y2) ∈ [k]^2 s.t. y1 ≠ y2 uniformly at random.
else                                                            ▷ run the column test with probability 1/2.
└   select (x1, y1) ∈ [k]^2 and (x2, y2) ∈ [k]^2 s.t. x1 ≠ x2 uniformly at random.
if f(x1, y1) = g(x2, y2) then
│   declare reject.
else
└   declare accept.
```

A pair of k-colorings $f, g\colon [k]^2 \to [k]$ are said to be *consistent* if both f and g are closest to horizontally striped k-colorings or closest to vertically striped k-colorings, and *inconsistent* otherwise. The following lemma bounds $\mathcal{V}_{\mathrm{cons}}$'s rejection probability.

Lemma 2.5 (informal; see Lemmas 6.8 and 6.9)**.**

Suppose f and g are striped. Then, the following hold:

- *if $f = g$ (i.e., f and g have the same striped k-coloring), $\mathcal{V}_{\mathrm{cons}}$ rejects with probability exactly $\frac{1}{2k}$*;
- *if f and g are inconsistent, $\mathcal{V}_{\mathrm{cons}}$ rejects with probability exactly $\frac{1}{k}$*.

Suppose f and g are ε-close to being striped. Then, the following hold:

- *if f and g are consistent, $\mathcal{V}_{\mathrm{cons}}$ rejects with probability more than $(1-4\varepsilon)\cdot\frac{1}{2k}$*;
- *if f and g are inconsistent, $\mathcal{V}_{\mathrm{cons}}$ rejects with probability more than $(1-4\varepsilon)\cdot\frac{1}{k}$*.

Since Lemma 2.5 does not bound $\mathcal{V}_{\mathrm{cons}}$'s rejection probability from below if f and g are far from being striped, we will combine $\mathcal{V}_{\mathrm{stripe}}$ and $\mathcal{V}_{\mathrm{cons}}$ in the third test.

2.2.3 Edge Test (Section 6.2.3)

Our final verifier is the *edge verifier* $\mathcal{V}_{\mathrm{edge}}$, which checks if a pair of k-colorings f, g of $[k]^2$ are *close to* the same striped pattern. To this end, $\mathcal{V}_{\mathrm{edge}}$ calls the stripe verifier $\mathcal{V}_{\mathrm{stripe}}$ and the consistency verifier $\mathcal{V}_{\mathrm{cons}}$ with a carefully designed probability, as follows:[7]

[7] The value of ρ denotes the hidden constant in $\Omega\left(\frac{\varepsilon}{k}\right)$ of Lemma 2.4.

Edge verifier $\mathcal{V}_{\text{edge}}$.

Oracle access: two k-colorings $f, g\colon [k]^2 \to [k]$.
1: let $\rho := 10^{-8}$ **and** $Z := \frac{2}{\rho} + \frac{2}{\rho} + 1$.
2: sample $r \sim [0,1]$.
3: **if** $0 \leqslant r < \frac{2}{\rho Z}$ **then** ▷ *with probability* $\frac{2}{\rho Z}$
4: execute $\mathcal{V}_{\text{stripe}}$ on f.
5: **else if** $\frac{2}{\rho Z} \leqslant r < \frac{2}{\rho Z} + \frac{2}{\rho Z}$ **then** ▷ *with probability* $\frac{2}{\rho Z}$
6: execute $\mathcal{V}_{\text{stripe}}$ on g.
7: **else** ▷ *with probability* $\frac{1}{Z}$
8: execute $\mathcal{V}_{\text{cons}}$ on $f \circ g$.

The following lemma bounds $\mathcal{V}_{\text{edge}}$'s rejection probability.

Lemma 2.6 (informal; see Lemmas 6.10 to 6.12)**.** *The following hold*:

- *if f and g are striped and $f = g$ (i.e., f and g have the same striped k-coloring), $\mathcal{V}_{\text{edge}}$ rejects with probability at most $\frac{1}{2Z \cdot k}$*;
- *if f and g are inconsistent, $\mathcal{V}_{\text{edge}}$ rejects with probability at least $\frac{1}{Z \cdot k}$*;
- *$\mathcal{V}_{\text{edge}}$ always rejects with probability at least $\frac{1}{2Z \cdot k}$*.

2.2.4 Putting Them Together (Section 6.3)

We are now ready to reduce Maxmin 2-Cut Reconfiguration to Maxmin k-Cut Reconfiguration to accomplish the proof of Lemma 2.2. Given a graph $G = (V,E)$ and a pair of its 2-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}\colon V \to [2]$ as an instance of Maxmin 2-Cut Reconfiguration, we construct a new (multi)graph H and a pair of its k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}}\colon V(H) \to [k]$ as an instance of Maxmin k-Cut Reconfiguration as follows. For each vertex v of G, we create a fresh copy of a $k \times k$ grid $[k]^2$; namely, the vertex set of H is defined as

$$V(H) := V \times [k]^2. \tag{2.5}$$

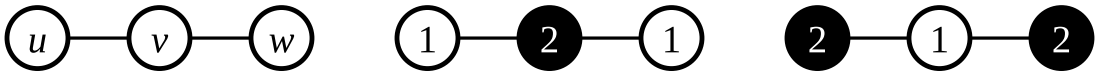
Since a k-coloring $f'\colon V \times [k]^2 \to [k]$ of H consists of $|V|$ k-colorings of $[k]^2$, we will think of it as a function $f'\colon V \to ([k]^2 \to [k])$ such that $f'(v)$ gives a k-coloring of $[k]^2$ associated with $v \in V$.

Consider the following verifier $\mathcal{V}_G$, given oracle access to a function $f'\colon V \to ([k]^2 \to [k])$, which samples an edge (v,w) from G and runs $\mathcal{V}_{\text{edge}}$ on $f'(v) \circ f'(w)^\top$:[8]

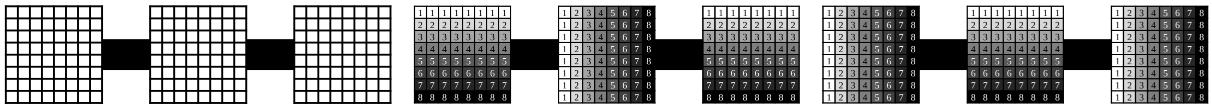
Overall verifier $\mathcal{V}_G$.

Input: a graph $G = (V,E)$.
Oracle access: a function $f'\colon V \to ([k]^2 \to [k])$.
1: select an edge (v,w) of G uniformly at random.
2: execute $\mathcal{V}_{\text{edge}}$ on $f'(v) \circ f'(w)^\top$.

[8] $f'(w)^\top$ is the *transposition* of $f'(w)$; i.e., $f'(w)^\top(x,y) = f'(w)(y,x)$ for all $(x,y) \in [k]^2$. The transposition comes from the design of $\mathcal{V}_{\text{edge}}$ to check the *consistency* between a pair of k-colorings, whereas we here need to check the *inconsistency*.



(a) Graph G. (b) 2-coloring f_{start} of G. (c) 2-coloring f_{end} of G.



(d) Graph H of Lemma 2.2. (e) k-coloring f'_{start} of H. (f) k-coloring f'_{end} of H.

Figure 6: Our reduction from Maxmin 2-Cut Reconfiguration to Maxmin k-Cut Reconfiguration in the proof of Lemma 2.2 ($k = 8$). Given a graph G and a pair of its 2-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}$, we construct a new (multi)graph H and a pair of its k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}}$, where the vertex set of H consists of $|V|$ $k \times k$ grids and the edge set of H emulates $\mathcal{V}_G$. (The edges are represented by the thick lines in the above figures because they are too complicated to be drawn). Each $k \times k$ grid is colored in either a horizontally or vertically striped pattern depending on f_{start} or f_{end}. If any reconfiguration sequence from f_{start} to f_{end} makes ε-fraction of edges G monochromatic, any reconfiguration sequence from f'_{start} to f'_{end} makes $\Omega\left(\frac{\varepsilon}{k}\right)$-fraction of edges of H monochromatic.

It is not hard to generate the set $E(H)$ of (parallel) edges between $V(H)$ so as to emulate $\mathcal{V}_G$ in that for any k-coloring $f' \colon V \times [k]^2 \to [k]$, the fraction of bichromatic edges in $E(H)$ is equal to the acceptance probability of $\mathcal{V}_G$. Construct a pair of k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}} \colon V \to ([k]^2 \to [k])$ of H such that for each vertex v of G, we define $f'_{\mathsf{start}}(v)$ (resp. $f'_{\mathsf{end}}(v)$) to be horizontally striped if $f_{\mathsf{start}}(v)$ (resp. $f_{\mathsf{end}}(v)$) is 1 and vertically striped if $f_{\mathsf{start}}(v)$ (resp. $f_{\mathsf{end}}(v)$) is 2. This completes the description of the reduction. See Figure 6 for illustration. Our reduction enjoys the following gap-preserving property:

Lemma 2.7 (informal; see Lemmas 6.13 and 6.14)**.** *The following hold*:

$$\mathsf{opt}_G\left(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\right) \geqslant 1 - \varepsilon_c \implies \mathsf{opt}_H\left(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\right) \geqslant 1 - \frac{1+\varepsilon_c}{2Z \cdot k} - o(1), \tag{2.6}$$

$$\mathsf{opt}_G\left(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\right) < 1 - \varepsilon_s \implies \mathsf{opt}_H\left(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\right) < 1 - \frac{1+\varepsilon_s}{2Z \cdot k}, \tag{2.7}$$

where $Z \coloneqq \frac{2}{\rho} + \frac{2}{\rho} + 1$ *and* $\rho \coloneqq 10^{-8}$.

The proof of Lemma 2.7 relies on Lemma 2.6, and the proof of Lemma 2.2 is almost immediate from Lemma 2.7.

Remark 2.8. Our reduction can also be used to reduce Max 2-Cut to Max k-Cut in a gap-preserving manner (Lemma 6.15), which reproves that Max k-Cut is NP-hard to approximate within a factor of $1 - \Omega\left(\frac{1}{k}\right)$ [GS13, KKLP97]. Since the existing reductions for Max k-Cut due to [GS13, KKLP97] do not work for Maxmin k-Cut Reconfiguration, the present study demonstrates the inherent difficulty in designing approximation-preserving reductions between reconfiguration problems.

3 Proof Overview of Approximation Algorithm

In this section, we present a highlight of the proof of Theorem 1.2, i.e., a deterministic $\left(1 - \frac{2}{k}\right)$-factor approximation algorithm for Maxmin k-Cut Reconfiguration. Our approximation algorithm is based on *a random reconfiguration via a random solution*. Let $G = (V, E)$ be a graph and $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [k]$ be a pair of its k-colorings. We assume f_{start} and f_{end} are proper for the sake of simplicity (see Lemma 7.2 for how to address when f_{start} and f_{end} contain many monochromatic edges). Let $F \colon V \to [k]$ be a *random* k-coloring of G, which makes $\left(1 - \frac{1}{k}\right)$-fraction of edges of G bichromatic in expectation. Consider now the following two *random* reconfiguration sequences:

- a reconfiguration sequence $\mathscr{F}_1$ from f_{start} to F obtained by recoloring vertices at which f_{start} and F differ in a random order, and
- a reconfiguration sequence $\mathscr{F}_2$ from F to f_{end} obtained by recoloring vertices at which F and f_{end} differ in a random order.

Concatenating $\mathscr{F}_1$ and $\mathscr{F}_2$, we obtain a random reconfiguration sequence $\mathscr{F}$ from f_{start} to f_{end}. It is easy to prove that for each edge e of G, *all* k-colorings of $\mathscr{F}$ make e bichromatic with probability at least $1 - \frac{9}{k}$ (Observation 7.9). In particular, $\mathscr{F}$ already achieves a $\left(1 - \frac{9}{k}\right)$-factor approximation for Maxmin k-Cut Reconfiguration in expectation. Note that Karthik C. S. and Manurangsi [KM23] used a similar strategy to approximate Maxmin 2-CSP Reconfiguration, which constructs a reconfiguration sequence that goes through a random assignment in a *greedy* manner.

Separately deriving *concentration bounds* for each $\mathscr{F}_1$ and $\mathscr{F}_2$, we further improve the approximation factor from $1 - \frac{9}{k}$ to $1 - \frac{2}{k}$. Our crucial insight for this purpose is to partition the vertex set of G into the low-degree and high-degree sets. We say that a vertex of G is *low degree* if its degree is less than $|E|^{\frac{2}{3}}$ and *high degree* otherwise.

- Suppose first G contains only low-degree vertices. By case analysis, we can show that each edge is always bichromatic within $\mathscr{F}_1$ with probability at least $\left(1 - \frac{1}{k}\right)^2 = 1 - \frac{2}{k} + \frac{1}{k^2}$. By applying the read-$k$ Chernoff bound [GLSS15] with parameter $|E|^{\frac{2}{3}}$, we obtain that every k-coloring of $\mathscr{F}_1$ makes at least $\left(1 - \frac{2}{k}\right)$-fraction of edges bichromatic with high probability. The same result holds for $\mathscr{F}_2$.
- Suppose now G contains high-degree vertices, for which a direct application of the read-k Chernoff bound does not yield useful concentration bounds. We resort to the following ad-hoc observations, which are reminiscent of those for Maxmin 2-CSP Reconfiguration due to [KM23]:
  1. Since there are "few" high-degree vertices, the number of edges between them is negligible.
  2. Each high-degree vertex has "many" low-degree neighbors, whose colors assigned by F are distributed almost evenly; thus, a nearly $\left(1 - \frac{1}{k}\right)$-fraction of edges between high-degree vertices and low-degree vertices are bichromatic with high probability.

  In light of the second observation, we generate a reconfiguration sequence $\mathscr{F}_1$ from f_{start} to F by first recoloring low-degree vertices followed by high-degree vertices, and a reconfiguration sequence $\mathscr{F}_2$ from F to f_{end} by first recoloring high-degree vertices followed by low-degree vertices.

The following randomized algorithm generates a random reconfiguration sequence $\mathscr{F}$ from f_{start} to f_{end}, which guarantees a $\left(1 - \frac{2}{k}\right)$-factor approximation for Maxmin k-Cut Reconfiguration with high probability:

Generating a random reconfiguration sequence $\mathscr{F}$ from f_{start} to f_{end}.

Input: a graph $G = (V, E)$ and two k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}} : V \to [k]$ of G.

1: sample a random k-coloring $F : V \to [k]$ of G.
2: ▷ *start from f_{start}.* ◁
3: recolor each low-degree vertex v from $f_{\mathsf{start}}(v)$ to $F(v)$ in a random order.
4: recolor each high-degree vertex v from $f_{\mathsf{start}}(v)$ to $F(v)$ in a random order.
5: ▷ *obtain F.* ◁
6: recolor each high-degree vertex v from $F(v)$ to $f_{\mathsf{end}}(v)$ in a random order.
7: recolor each low-degree vertex v from $F(v)$ to $f_{\mathsf{end}}(v)$ in a random order.
8: ▷ *end at f_{end}.* ◁

Our deterministic algorithm is obtained by derandomizing the above algorithm, which can be done by the method of conditional expectations [AS16].

4 Related Work

4.1 Variants of k-Coloring Reconfiguration

Other than reachability problems, there are several types of reconfiguration problems [Mou15, Nis18, van13]. One is *connectivity problems*, which ask if the configuration graph is connected. In the connectivity variant of k-Coloring Reconfiguration, we are asked to decide if *every* pair of proper k-colorings of a graph G are reconfigurable each other. Such a graph G is said to be k*-mixing*. On the complexity side, it is coNP-hard to decide if a graph is k-mixing for every $k \geqslant 3$ [Bou24, CvJ09]. The name of k-mixing comes from the relation to the (rapid) mixing of the Glauber dynamics [DFFV06, Jer95, Mol04]. The *Glauber dynamics* is a Markov Chain such that starting from a graph G and a proper k-coloring of G, we repeatedly recolor a random vertex with a random color (as long as it yields a proper k-coloring). The Glauber dynamics is *ergodic* only if G is k-mixing.

Other algorithmic and structural problems related to k-Coloring Reconfiguration include finding the shortest reconfiguration sequence [BHIKMMSW20, CvJ11, JKKPP16] and bounding the diameter of the configuration graph [BC09, BJLPP11, BJLPP14, CvJ11], respectively. See also Nishimura [Nis18, §6], van den Heuvel [van13, §3], and Mynhardt and Nasserasr [MN19].

4.2 Approximability of Max k-Cut

The Max k-Cut problem (a.k.a. Max k-Colorable Subgraph [GS13, PY91]) seeks a k-coloring of a graph that makes the maximum number of edges bichromatic. Observe easily that a random k-coloring makes a $\left(1 - \frac{1}{k}\right)$-fraction of edges bichromatic in expectation; moreover, Frieze and Jerrum [FJ97] developed a $\left(1 - \frac{1}{k} + \frac{2\ln k}{k^2}\right)$-factor approximation algorithm. On the hardness side, $\left(1 - \frac{1}{17k+O(1)}\right)$-factor approximation is NP-hard [AOTW14, GS13, KKLP97]. For the special case of $k = 2$, i.e., Max Cut, the current best approximation factor is ≈ 0.878 [GW95], which is proven to be optimal [KKMO07, MOO10] under the Unique Games Conjecture [Kho02].

4.3 Approximability of Reconfiguration Problems

Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, and Uno [IDHPSUU11] proved that several reconfiguration problems (e.g., MAXMIN SAT RECONFIGURATION) are NP-hard to approximate relying on the NP-hardness of approximating the source problems (e.g., MAX SAT). Since most reconfiguration problems are PSPACE-complete, NP-hardness results are not optimal. In fact, [IDHPSUU11] posed PSPACE-hardness of approximation as an open problem.

Motivated by PSPACE-hardness of approximation for reconfiguration problems, Ohsaka [Ohs23] postulated a reconfiguration analogue of the PCP theorem [ALMSS98, AS98], called the *Reconfiguration Inapproximability Hypothesis* (RIH). Under RIH, (approximate versions of) several reconfiguration problems are PSPACE-hard to approximate, including those of 3-SAT, INDEPENDENT SET, VERTEX COVER, and CLIQUE. Very recently, Hirahara and Ohsaka [HO24b] and Karthik C. S. and Manurangsi [KM23] independently gave a proof of RIH by establishing the *Probabilistically Checkable Reconfiguration Proof* (PCRP) theorem, which provides a new PCP-type characterization of PSPACE. The PCRP theorem, along with a series of gap-preserving reductions [HO24a, HO24b, Ohs23, Ohs24b], implies *unconditional* PSPACE-hardness of approximation results for many reconfiguration problems, thereby resolving the open problem of [IDHPSUU11] affirmatively.

One recent trend regarding approximability of reconfiguration problems is to prove an explicit factor of PSPACE-hardness of approximation. In the NP regime, the *parallel repetition theorem* [Raz98] can be used to derive explicit, strong inapproximability results [BGS98, Fei98, Hås01, Hås99, Zuc07]. Unfortunately, a naive parallel repetition does not reduce the soundness error of a reconfiguration analogue of two-prover games [Ohs25]. Ohsaka [Ohs24b] adapted Dinur's gap amplification [Din07, Rad06, RS07] to show that MAXMIN 2-CSP RECONFIGURATION and MINMAX SET COVER RECONFIGURATION are PSPACE-hard to approximate within a factor of 0.9942 and 1.0029, respectively. Subsequently, Karthik C. S. and Manurangsi [KM23] showed that MINMAX SET COVER RECONFIGURATION is NP-hard to approximate within a factor of $2-\varepsilon$ for every $\varepsilon > 0$. Hirahara and Ohsaka [HO24a] demonstrated that MINMAX SET COVER RECONFIGURATION is PSPACE-hard to approximate within a factor of $2-o(1)$, improving upon [KM23, Ohs24b]. Since MINMAX SET COVER RECONFIGURATION admits a 2-factor approximation algorithm [IDHPSUU11], this is the first optimal PSPACE-hardness result for approximability of any reconfiguration problem.

Approximation algorithms have been developed for several reconfiguration problems; e.g., MAXMIN 2-CSP RECONFIGURATION admits a $\left(\frac{1}{2}-\varepsilon\right)$-factor approximation [KM23], SUBSET SUM RECONFIGURATION admits a PTAS [ID14], and SUBMODULAR RECONFIGURATION admits a constant-factor approximation [OM22].

5 Preliminaries

5.1 k-COLORING RECONFIGURATION and MAXMIN k-CUT RECONFIGURATION

We formulate k-COLORING RECONFIGURATION and its approximate version. Throughout this paper, all graphs are *undirected*. For a graph $G=(V,E)$, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. For a vertex v of G, let $\mathcal{N}_G(v)$ denote the set of the neighbors of v and $d_G(v)$ denote the degree of v. For a vertex set $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. Unless otherwise stated, graphs appearing in this paper are *multigraphs*; namely, the edge set is a multiset consisting of *parallel* *edges*.

For a graph $G = (V,E)$ and a positive integer $k \in \mathbb{N}$, a *k-coloring* of G is a function $f\colon V \to [k]$ that assigns a color of $[k]$ to each vertex of G. We call $f(v)$ the *color* of v. An edge (v,w) of G is said to be *bichromatic* on f if $f(v) \neq f(w)$ and *monochromatic* on f if $f(v) = f(w)$. We say that a k-coloring f of G is *proper* if every edge of G is bichromatic on f. A graph G is said to be *k-colorable* if there is a proper k-coloring of G. The *value* of f is defined as the fraction of edges of G that are bichromatic on f; namely,

$$\mathsf{val}_G(f) \coloneqq \frac{1}{|E|} \cdot \left| \left\{ (v,w) \in E \;\middle|\; f(v) \neq f(w) \right\} \right|. \tag{5.1}$$

Recall that k-Coloring asks to decide if a graph G is k-colorable, and its approximate version called Max k-Cut (a.k.a. Max k-Colorable Subgraph [GS13, PY91][9]) requires to find a k-coloring f of G that maximizes $\mathsf{val}_G(f)$.

Subsequently, we formulate a reconfiguration version of k-Coloring as well as Max k-Cut. For a graph $G = (V,E)$ and a pair of its k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}\colon V \to [k]$, a *reconfiguration sequence from f_{start} to f_{end}* is any sequence $\mathscr{F} = (f^{(1)}, \ldots, f^{(T)})$ over k-colorings of G such that $f^{(1)} = f_{\mathsf{start}}$, $f^{(T)} = f_{\mathsf{end}}$, and every pair of adjacent k-colorings differ in at most one vertex. The k-Coloring Reconfiguration problem [BC09, Cer07, CvJ08, CvJ09, CvJ11] asks to decide if there is a reconfiguration sequence from f_{start} to f_{end} consisting only of proper k-colorings of G. Note that k-Coloring Reconfiguration belongs to P if $k \leqslant 3$ [CvJ11] whereas it becomes PSPACE-complete for every $k \geqslant 4$ [BC09].

Since we are concerned with approximability of k-Cut Reconfiguration, we formulate its approximate version. For a reconfiguration sequence $\mathscr{F} = (f^{(1)}, \ldots, f^{(T)})$ over k-colorings of G, let $\mathsf{val}_G(\mathscr{F})$ denote the *minimum fraction* of bichromatic edges over all $f^{(t)}$'s in $\mathscr{F}$; namely,

$$\mathsf{val}_G(\mathscr{F}) \coloneqq \min_{f^{(t)} \in \mathscr{F}} \mathsf{val}_G(f^{(t)}). \tag{5.2}$$

For a graph $G = (V,E)$ and a pair of its k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}$, the Maxmin k-Cut Reconfiguration problem requires to maximize $\mathsf{val}_G(\mathscr{F})$ subject to $\mathscr{F} = (f_{\mathsf{start}}, \ldots, f_{\mathsf{end}})$. Maxmin k-Cut Reconfiguration is PSPACE-hard because so is k-Coloring Reconfiguration. For a pair of k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}$ of G, let $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}})$ denote the maximum value of $\mathsf{val}_G(\mathscr{F})$ over all possible reconfiguration sequences $\mathscr{F}$ from f_{start} to f_{end}; namely,

$$\mathsf{opt}_G\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) \coloneqq \max_{\mathscr{F} = (f_{\mathsf{start}}, \ldots, f_{\mathsf{end}})} \mathsf{val}_G(\mathscr{F}). \tag{5.3}$$

Note that $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) \leqslant \min\{\mathsf{val}_G(f_{\mathsf{start}}), \mathsf{val}_G(f_{\mathsf{end}})\}$. The gap version of Maxmin k-Cut Reconfiguration is defined as follows:

Problem 5.1. For every reals $0 \leqslant s \leqslant c \leqslant 1$ and positive integer $k \in \mathbb{N}$, Gap$_{c,s}$ k-Cut Reconfiguration requires to determine for a graph G and a pair of its k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}$, whether $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) \geqslant c$ or $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) < s$. Here, c and s are respectively called *completeness* and *soundness*.

[9] In [GS13], Max k-Colorable Subgraph always refers to the perfect completeness case; i.e., G is promised to be k-colorable.

We say that a reconfiguration sequence $\mathscr{F} = (f^{(1)}, \ldots, f^{(T)})$ from f_{start} to f_{end} is *irredundant* if (1) no pair of adjacent k-colorings are identical, and (2) for each vertex v of G, there is a unique index $\tau_v \in [T]$ such that

$$f^{(t)}(v) = \begin{cases} f_{\mathsf{start}}(v) & \text{if } 1 \leqslant t \leqslant \tau_v, \\ f_{\mathsf{end}}(v) & \text{if } \tau_v < t \leqslant T. \end{cases} \tag{5.4}$$

Informally, irredundancy ensures that each vertex is recolored at most once; in particular, the length of $\mathscr{F}$ must be the number of vertices on which f_{start} and f_{end} differ. Let $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}})$ denote the set of all irredundant reconfiguration sequences from f_{start} to f_{end}. The size of $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}})$ is equal to $d!$, where d is the number of vertices on which f_{start} and f_{end} differ. For any ℓ k-colorings $f_1, f_2, \ldots, f_\ell$ of a graph G, let $\mathbb{F}(f_1 \leftrightsquigarrow f_2 \leftrightsquigarrow \cdots \leftrightsquigarrow f_\ell)$ denote the set of reconfiguration sequences obtained by concatenating any $\ell-1$ irredundant reconfiguration sequences of $\mathbb{F}(f_i \leftrightsquigarrow f_{i+1})$ for all $i \in [\ell-1]$, which can be defined recursively as follows:

$$\mathbb{F}\big(f_1 \leftrightsquigarrow f_2 \leftrightsquigarrow \cdots \leftrightsquigarrow f_\ell\big) := \Big\{\mathscr{F} \circ \mathscr{F}' \;\Big|\; \mathscr{F} \in \mathbb{F}\big(f_1 \leftrightsquigarrow f_2\big) \text{ and } \mathscr{F}' \in \mathbb{F}\big(f_2 \leftrightsquigarrow \cdots \leftrightsquigarrow f_\ell\big)\Big\}. \tag{5.5}$$

5.2 Some Concentration Inequalities

Here, we introduce some concentration inequalities. The Chernoff bound is first introduced below.

Theorem 5.2 (Chernoff bound)**.** *Let $X_1, \ldots, X_n$ be independent Bernoulli random variables, and $X := \sum_{i\in[n]} X_i$. Then, for any real $\varepsilon \in (0,1)$, it holds that*

$$\begin{aligned} \mathbb{P}\Big[X \geqslant (1+\varepsilon)\,\mathbb{E}[X]\Big] &\leqslant \exp\left(-\frac{\varepsilon^2 \cdot \mathbb{E}[X]}{3}\right), \\ \mathbb{P}\Big[X \leqslant (1-\varepsilon)\,\mathbb{E}[X]\Big] &\leqslant \exp\left(-\frac{\varepsilon^2 \cdot \mathbb{E}[X]}{3}\right). \end{aligned} \tag{5.6}$$

We then introduce a read-k family of random variables and a read-k analogue of the Chernoff bound due to Gavinsky, Lovett, Saks, and Srinivasan [GLSS15].

Definition 5.3. A family $X_1, \ldots, X_n$ of random variables is called a *read-k family* if there exist m independent random variables $Y_1, \ldots, Y_m$, n subsets $S_1, \ldots, S_n$ of $[m]$, and n Boolean functions $f_1, \ldots, f_n$ such that

- each X_i is represented as $X_i = f_i((Y_j)_{j\in S_i})$, and
- each j of $[m]$ appears in at most k of the S_i's.

Theorem 5.4 (Read-k Chernoff bound [GLSS15])**.** *Let $X_1, \ldots, X_n$ be a family of read-k Bernoulli random variables, and $X := \sum_{i\in[n]} X_i$. Then, for any real $\varepsilon > 0$, it holds that*

$$\begin{aligned} \mathbb{P}\Big[X \leqslant \mathbb{E}[X] - \varepsilon n\Big] &\leqslant \exp\left(-\frac{2\varepsilon \cdot n}{k}\right), \\ \mathbb{P}\Big[X \geqslant \mathbb{E}[X] + \varepsilon n\Big] &\leqslant \exp\left(-\frac{2\varepsilon \cdot n}{k}\right). \end{aligned} \tag{5.7}$$

6 PSPACE-hardness of $\left(1-\Omega\left(\frac{1}{k}\right)\right)$-factor Approximation for MAXMIN k-CUT RECONFIGURATION

In this section, we prove that MAXMIN k-CUT RECONFIGURATION is PSPACE-hard to approximate within a factor of $1-\Omega\left(\frac{1}{k}\right)$ for every $k \geqslant 2$.

Theorem 6.1. *There exist universal constants $\delta_c, \delta_s \in (0,1)$ with $\delta_c < \delta_s$ such that for all sufficiently large $k \geqslant k_0 := 10^3$, $\text{GAP}_{1-\frac{\delta_c}{k}, 1-\frac{\delta_s}{k}}$ k-CUT RECONFIGURATION is PSPACE-hard. Moreover, there exists a universal constant $\delta_0 \in (0,1)$ such that MAXMIN k-CUT RECONFIGURATION is PSPACE-hard to approximate within a factor of $1-\frac{\delta_0}{k}$ for every $k \geqslant 2$. The same hardness result holds even if the maximum degree of the input graph is $O(k^2)$.*

6.1 Outline of the Proof of Theorem 6.1

Here, we present an outline of the proof of Theorem 6.1. Our starting point is PSPACE-hardness of approximating MAXMIN 2-CUT RECONFIGURATION, whose proof is based on [BC09, HO24b, Ohs23] and deferred to Appendix A.1.

Proposition 6.2 ($*$)**.** *There exist universal constants $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$ such that $\text{GAP}_{1-\varepsilon_c, 1-\varepsilon_s}$ 2-CUT RECONFIGURATION is PSPACE-hard. Moreover, the same hardness result holds even if the maximum degree of input graphs is bounded by some constant $\Delta \in \mathbb{N}$.*

We then construct the following two gap-preserving reductions from MAXMIN 2-CUT RECONFIGURATION to MAXMIN k-CUT RECONFIGURATION, the former for all sufficiently large k and the latter for finitely many k.

Lemma 6.3. *For every reals $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$, there exist reals $\delta_c, \delta_s \in (0,1)$ with $\delta_c < \delta_s$ depending only on the values of ε_c and ε_s such that for all sufficiently large $k \geqslant k_0 := 10^3$ and any integer $\Delta \in \mathbb{N}$, there exists a gap-preserving reduction from $\text{GAP}_{1-\varepsilon_c, 1-\varepsilon_s}$ 2-CUT RECONFIGURATION on graphs of maximum degree Δ to $\text{GAP}_{1-\frac{\delta_c}{k}, 1-\frac{\delta_s}{k}}$ k-CUT RECONFIGURATION on graphs of maximum degree $O(\Delta \cdot k^2)$.*

Lemma 6.4 ($*$)**.** *For every integer $k \geqslant 3$, every reals $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$, and every integer $\Delta \in \mathbb{N}$, there exist universal constants $\delta_c, \delta_s \in (0,1)$ with $\delta_c < \delta_s$ such that there exists a gap-preserving reduction from $\text{GAP}_{1-\varepsilon_c, 1-\varepsilon_s}$ 2-CUT RECONFIGURATION on graphs of maximum degree Δ to $\text{GAP}_{1-\delta_c, 1-\delta_s}$ k-CUT RECONFIGURATION on graphs of maximum degree $O(\Delta + \text{poly}(k))$.*

Remark 6.5. The values of δ_c, δ_s in Lemma 6.4 depend on $\varepsilon_c, \varepsilon_s, k$ and quadratically decrease in k; i.e., $\delta_c, \delta_s = \Theta(k^{-2})$. We thus cannot use Lemma 6.4 to prove Theorem 6.1 for large k.

The proof of Lemma 6.4 is deferred to Appendix A.2. As a corollary of Proposition 6.2 and Lemmas 6.3 and 6.4, we obtain Theorem 6.1.

Proof of Theorem 6.1. By Proposition 6.2, $\text{GAP}_{1-\varepsilon_c, 1-\varepsilon_s}$ 2-CUT RECONFIGURATION on graphs of maximum degree Δ is PSPACE-hard for some constants $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$ and $\Delta \in \mathbb{N}$. By Lemma 6.3, there exist universal constants $\delta_c, \delta_s \in (0,1)$ with $\delta_c < \delta_s$ such that $\text{GAP}_{1-\frac{\delta_c}{k}, 1-\frac{\delta_s}{k}}$ k-CUT RECONFIGURATION is PSPACE-hard for every $k \geqslant k_0$. The ratio between completeness and soundness is evaluated as follows:

$$\frac{1-\frac{\delta_s}{k}}{1-\frac{\delta_c}{k}} = \frac{1-\frac{\delta_c}{k}+\frac{\delta_c-\delta_s}{k}}{1-\frac{\delta_c}{k}} = 1-\frac{\delta_s-\delta_c}{1-\frac{\delta_c}{3}}\cdot\frac{1}{k} \leqslant 1-\frac{\delta_s-\delta_c}{k}. \tag{6.1}$$

Therefore, Maxmin k-Cut Reconfiguration is PSPACE-hard to approximate within a factor of $1-\frac{\delta_s-\delta_c}{k}$ for every $k \geqslant k_0$. By applying Lemma 6.4 to Proposition 6.2 for each $k < k_0$, we obtain a universal constant $\delta' \in (0,1)$ such that Maxmin k-Cut Reconfiguration is PSPACE-hard to approximate within a factor of $1-\delta'$ for every $k < k_0$. Both results imply the existence of a universal constant $\delta_0 \in (0,1)$ such that Maxmin k-Cut Reconfiguration is PSPACE-hard to approximate within a factor of $1-\frac{\delta_0}{k}$ for every $k \geqslant 2$, accomplishing the proof. □

The remainder of this section is devoted to the proof of Lemma 6.3.

6.2 Three Tests

In this subsection, we introduce the key ingredients in the proof of Lemma 6.3. Consider a probabilistic verifier $\mathcal{V}$, given oracle access to a k-coloring $f\colon V \to [k]$, that is allowed to sample a pair (v,w) of distinct vertices from V (nonadaptively) and accepts (resp. rejects) if $f(v) \neq f(w)$ (resp. $f(v) = f(w)$). Observe easily that $\mathcal{V}$ can be emulated by a multigraph G on vertex set V in a sense that the acceptance (resp. rejection) probability of $\mathcal{V}$ is equal to the fraction of the bichromatic (resp. monochromatic) edges in G. Our reduction in Section 6.3 from Maxmin 2-Cut Reconfiguration to Maxmin k-Cut Reconfiguration will be described in the language of such verifiers.

Suppose we are given an instance $(G, f_{\mathsf{start}}, f_{\mathsf{end}})$ of Maxmin 2-Cut Reconfiguration. We shall encode a 2-coloring of each vertex v of G by using a k-coloring of $[k]^2$, denoted by $f'(v)\colon [k]^2 \to [k]$, whose motivation was described in Section 2.2. Specifically, $f'(v)$ is supposed to be "horizontally striped" if v's color is 1, and $f'(v)$ is supposed to be "vertically striped" if v's color is 2. We would like to check if these k-colorings $(f'(v))_{v\in V}$ are an encoding of a *proper* 2-coloring of G. For this purpose, we will implement the following three auxiliary verifiers:

- **Stripe verifier** $\mathcal{V}_{\text{stripe}}$, which checks if a k-coloring f of $[k]^2$ is close to a "striped" pattern.
- **Consistency verifier** $\mathcal{V}_{\text{cons}}$, which checks if a pair of k-coloring f,g of $[k]^2$ share the *same* striped pattern (given that both f and g are close to striped patterns).
- **Edge verifier** $\mathcal{V}_{\text{edge}}$, which checks if a pair of k-coloring f,g of $[k]^2$ are *closed to* the same striped pattern, by calling $\mathcal{V}_{\text{stripe}}$ and $\mathcal{V}_{\text{cons}}$ with a carefully designed probability.

We will say that a k-coloring $f\colon [k]^2 \to [k]$ is *horizontally striped* if $f(x,y) = \sigma(y)$ for all $(x,y) \in [k]^2$ for some permutation $\sigma \in \mathfrak{S}_k$, *vertically striped* if $f(x,y) = \sigma(x)$ for all $(x,y) \in [k]^2$ for some permutation $\sigma \in \mathfrak{S}_k$, and *striped* if it is horizontally or vertically striped. Throughout this subsection, we fix $k \geqslant k_0 \coloneqq 10^3$.

6.2.1 Stripe Test

We first introduce the *stripe verifier* $\mathcal{V}_{\text{stripe}}$, which tests if a k-coloring f of $[k]^2$ is close to being striped.

Stripe verifier $\mathcal{V}_{\text{stripe}}$.

Oracle access: a k-coloring $f\colon [k]^2 \to [k]$.

select $(x_1,y_1) \in [k]^2$ and $(x_2,y_2) \in [k]^2$ s.t. $x_1 \neq x_2$ and $y_1 \neq y_2$ uniformly at random.
if $f(x_1,y_1) = f(x_2,y_2)$ **then**
  declare reject.
else
  declare accept.

Observe easily that $\mathcal{V}_{\text{stripe}}$ always accepts f if and only if f is striped.

Lemma 6.6. *Let $f\colon [k]^2 \to [k]$ be any k-coloring. Then, $\mathcal{V}_{\text{stripe}}$ accepts f with probability* 1 *if and only if f is striped.*

Proof. Since the "if" direction is obvious, we show the "only-if" direction. Consider a graph G that emulates $\mathcal{V}_{\text{stripe}}$. Let $f\colon [k]^2 \to [k]$ be any k-coloring accepted by $\mathcal{V}_{\text{stripe}}$ with probability 1. Denoting by $(I_1,\ldots,I_k)$ a partition of $[k]^2$ such that $I_\alpha := \{(x,y) \in [k]^2 \mid f(x,y) = \alpha\}$, we find each I_α an independent set of G. Since $(x,x) \neq (y,y)$ do not belong to the same independent set, we can assume $(\sigma(\alpha),\sigma(\alpha)) \in I_\alpha$ for some permutation $\sigma\colon [k] \to [k]$. Observe that any maximal independent set is of the form either $\{(x,y) \in [k]^2 \mid y = \alpha, x \in [k]\}$ or $\{(x,y) \in [k]^2 \mid x = \alpha, y \in [k]\}$ for some $\alpha \in [k]$, implying that f must be either horizontally striped or vertically striped. $\square$

Let ▤ denote the set of all horizontally striped k-colorings, ▥ denote the set of all vertically striped k-colorings, and ▦ := ▤ $\cup$ ▥. We say that a k-coloring $f\colon [k]^2 \to [k]$ is *ε-far from being striped* if dist(f,▦) $> \varepsilon$ and *ε-close to being striped* if dist(f,▦) $\leqslant \varepsilon$. We now demonstrate that if a k-coloring $f\colon [k]^2 \to [k]$ is ε-far from being striped, $\mathcal{V}_{\text{stripe}}$ rejects f with probability $\Omega\left(\frac{\varepsilon}{k}\right)$, whose proof is rather complicated and deferred to Section 6.4.

Lemma 6.7. *There exists a universal constant $\rho := 10^{-8}$ such that for any k-coloring $f\colon [k]^2 \to [k]$ that is ε-far from being striped, $\mathcal{V}_{\text{stripe}}$ rejects f with probability more than*

$$\frac{\rho \cdot \varepsilon}{k}. \tag{6.2}$$

6.2.2 Consistency Test

We next proceed to the *consistency verifier* $\mathcal{V}_{\text{cons}}$, which tests if a pair of k-colorings f,g of $[k]^2$ share the same striped pattern. Specifically, $\mathcal{V}_{\text{cons}}$ runs the following two tests with equal probability:

- the *row test*, which accepts $f \circ g$ if they have the same horizontally striped pattern;
- the *column test*, which accepts $f \circ g$ if they have the same vertically striped pattern.

Consistency verifier $\mathcal{V}_{\mathrm{cons}}$.

Oracle access: two k-colorings $f, g \colon [k]^2 \to [k]$.

sample $r \sim [0,1]$.
if $0 \leqslant r < \frac{1}{2}$ **then** ▷ *perform the row test w.p. $\frac{1}{2}$.*
  select $(x_1, y_1) \in [k]^2$ and $(x_2, y_2) \in [k]^2$ s.t. $y_1 \neq y_2$ uniformly at random.
else ▷ *perform the column test w.p. $\frac{1}{2}$.*
  select $(x_1, y_1) \in [k]^2$ and $(x_2, y_2) \in [k]^2$ s.t. $x_1 \neq x_2$ uniformly at random.
if $f(x_1, y_1) = g(x_2, y_2)$ **then**
  declare reject.
else
  declare accept.

Let $\mathsf{dec}(f)$ indicate whether f is closest to being horizontally striped (denoted by 1) or vertically striped (denoted by 2); namely,

$$\mathsf{dec}(f) := \begin{cases} 1 & \text{if } \mathrm{dist}(f, \text{▤}) \leqslant \mathrm{dist}(f, \text{▥}), \\ 2 & \text{if } \mathrm{dist}(f, \text{▤}) > \mathrm{dist}(f, \text{▥}). \end{cases} \tag{6.3}$$

A pair of k-colorings $f, g \colon [k]^2 \to [k]$ are said to be *consistent* if $\mathsf{dec}(f) = \mathsf{dec}(g)$ (i.e., both f and g are closest to being horizontally striped or vertically striped), and *inconsistent* if $\mathsf{dec}(f) \neq \mathsf{dec}(g)$.

When f and g are striped, $\mathcal{V}_{\mathrm{cons}}$'s rejection probability can be calculated exactly as follows.

Lemma 6.8. *For any striped two k-colorings $f, g \colon [k]^2 \to [k]$, the following hold*:

- *if $f = g$ (in particular, f and g are consistent), $\mathcal{V}_{\mathrm{cons}}$ rejects $f \circ g$ with probability exactly $\frac{1}{2k}$*;
- *if f and g are inconsistent, $\mathcal{V}_{\mathrm{cons}}$ rejects $f \circ g$ with probability exactly $\frac{1}{k}$.*

Proof. To prove the first claim, assume $f = g$ is horizontally striped. The other case can be shown similarly. In the row test, it always holds that $y_1 \neq y_2$; i.e., we have $f(x_1, y_1) = g(x_2, y_2)$ with probability 0. In the column test, y_1 and y_2 are chosen *independently* and uniformly at random; thus, we have $f(x_1, y_1) = g(x_2, y_2)$ with probability exactly $\frac{1}{k}$. Therefore, the consistency verifier rejects $f \circ g$ with probability $\frac{1}{2}\left(0 + \frac{1}{k}\right) = \frac{1}{2k}$, as desired.

To prove the second claim, assume f is horizontally striped and g is vertically striped. The opposite case can be shown in the same way. In the row test, we have $f(x_1, y_1) = g(x_2, y_2)$ with probability $\frac{1}{k}$ because x_2 (and thus $g(x_2, y_2)$) is uniformly distributed over $[k]$. In the column test, we have $f(x_1, y_1) = g(x_2, y_2)$ with probability $\frac{1}{k}$ because y_1 (and thus $f(x_1, y_1)$) is uniformly distributed. Therefore, $\mathcal{V}_{\mathrm{cons}}$ rejects $f \circ g$ with probability $\frac{1}{2}\left(\frac{1}{k} + \frac{1}{k}\right) = \frac{1}{k}$, as desired. $\square$

Even when f and g are not striped, $\mathcal{V}_{\mathrm{cons}}$'s rejection probability can be bounded from below as follows.

Lemma 6.9. *For any two k-colorings $f, g \colon [k]^2 \to [k]$ such that f is ε_f-close to being striped and g is ε_g-close to being striped, the following hold*:

- *if f and g are inconsistent, $\mathcal{V}_{\text{cons}}$ rejects $f \circ g$ with probability more than*

$$\Big(1 - 2\varepsilon_f - 2\varepsilon_g\Big) \cdot \frac{1}{k}, \tag{6.4}$$

- *if f and g are consistent, $\mathcal{V}_{\text{cons}}$ rejects $f \circ g$ with probability more than*

$$\Big(1 - 2\varepsilon_f - 2\varepsilon_g\Big) \cdot \frac{1}{2k}. \tag{6.5}$$

Proof. Let $f^*, g^* \colon [k]^2 \to [k]$ be two striped k-colorings closest to f and g such that $\mathsf{dec}(f^*) = \mathsf{dec}(f)$ and $\mathsf{dec}(g^*) = \mathsf{dec}(g)$,[10] respectively. For each $i \in [k]$, let $\Delta_f(*, i)$ and $\Delta_g(*, i)$ denote the number of x's such that $f(x, i) \neq f^*(x, i)$ and $g(x, i) \neq g^*(x, i)$, respectively, and let $\Delta_f(i, *)$ and $\Delta_g(i, *)$ denote the number of y's such that $f(i, y) \neq f^*(i, y)$ and $g(i, y) \neq g^*(i, y)$, respectively. By assumption, we have

$$\sum_{x \in [k]} \Delta_f(x, *) \leqslant \varepsilon_f \cdot k^2 \;\text{ and }\; \sum_{x \in [k]} \Delta_g(x, *) \leqslant \varepsilon_g \cdot k^2, \tag{6.6}$$

$$\sum_{y \in [k]} \Delta_f(*, y) \leqslant \varepsilon_f \cdot k^2 \;\text{ and }\; \sum_{y \in [k]} \Delta_g(*, y) \leqslant \varepsilon_g \cdot k^2. \tag{6.7}$$

Suppose first $\mathsf{dec}(f) \neq \mathsf{dec}(g)$; we assume that $\mathsf{dec}(f) = 1$ and $\mathsf{dec}(g) = 2$ without loss of generality. By symmetry of $\mathcal{V}_{\text{cons}}$, the rows and columns can be rearranged so that $f^*(x, y) = y$ and $g^*(x, y) = x$ for all $(x, y) \in [k]^2$. We first bound the rejection probability of the row test. Let Q denote the set of all quadruples examined by the row test; namely,

$$Q \coloneqq \Big\{(x_1, y_1, x_2, y_2) \in [k]^4 \;\Big|\; y_1 \neq y_2\Big\}. \tag{6.8}$$

Note that $|Q| = k^3(k-1)$. Conditioned on the event that $y_1 = x_2 = i$ for some $i \in [k]$,

- there are $(k - \Delta_f(*, i))$ x_1's such that $f(x_1, i) = f^*(x_1, i) = i$;
- there are $(k - 1 - \Delta_g(i, *))$ y_2's such that $g(i, y_2) = g^*(i, y_2) = i$ and $y_2 \neq y_1 = i$;

namely, there are (at least) $(k - \Delta_f(*, i)) \cdot (k - 1 - \Delta_g(i, *))$ pairs (x_1, y_2) such that $f(x_1, y_1) = g(x_2, y_2)$. Taking the sum over all $i \in [k]$, we deduce that the number of quadruples (x_1, y_1, x_2, y_2) in Q such that $f(x_1, y_1) = g(x_2, y_2)$ is at least

$$\begin{aligned}
&\sum_{i \in [k]} (k - \Delta_f(*, i)) \cdot (k - 1 - \Delta_g(i, *)) \\
&= \sum_{i \in [k]} k(k-1) - (k-1) \cdot \underbrace{\sum_{i \in [k]} \Delta_f(*, i)}_{\leqslant \varepsilon_f \cdot k^2} - k \cdot \underbrace{\sum_{i \in [k]} \Delta_g(i, *)}_{\leqslant \varepsilon_g \cdot k^2} + \underbrace{\sum_{i \in [k]} \Delta_f(*, i) \cdot \Delta_g(i, *)}_{\geqslant 0} \\
&\geqslant k^2(k-1) - k^2(k-1) \cdot \varepsilon_f - k^3 \cdot \varepsilon_g.
\end{aligned} \tag{6.9}$$

[10] We need this condition for tie-breaking.

Since the row test draws a quadruple from Q uniformly at random, its rejection probability is at least

$$\begin{aligned}\frac{1}{|Q|}\cdot\Big(k^2(k-1)-k^2(k-1)\cdot\varepsilon_f-k^3\cdot\varepsilon_g\Big)&=\frac{1}{k}-\frac{\varepsilon_f}{k}-\frac{\varepsilon_g}{k-1}\\&=\frac{1}{k}\left(1-\varepsilon_f-\varepsilon_g-\frac{\varepsilon_g}{k-1}\right)\\&\geqslant\frac{1}{k}\Big(1-\varepsilon_f-2\varepsilon_g\Big).\end{aligned}\tag{6.10}$$

Similarly, the column test rejects with probability at least

$$\frac{1}{k}\Big(1-2\varepsilon_f-\varepsilon_g\Big).\tag{6.11}$$

Consequently, the rejection probability of $\mathcal{V}_{\text{cons}}$ is at least

$$\frac{1}{2}\left(\frac{1}{k}\Big(1-\varepsilon_f-2\varepsilon_g\Big)+\frac{1}{k}\Big(1-2\varepsilon_f-\varepsilon_g\Big)\right)\geqslant\frac{1}{k}\Big(1-2\varepsilon_f-2\varepsilon_g\Big),\tag{6.12}$$

as desired.

Suppose next $\mathsf{dec}(f)=\mathsf{dec}(g)$; we assume $\mathsf{dec}(f)=\mathsf{dec}(g)=1$ without loss of generality. We bound the rejection probability of the column test. Conditioned on the event that $f^*(\cdot,y_1)=g^*(\cdot,y_2)=i$ for some $i\in[k]$, there are (at least) $(k-\Delta_f(*,y_1))\cdot(k-1-\Delta_g(*,y_2))$ pairs (x_1,x_2) such that $f(x_1,y_1)=f^*(x_1,y_1)=i=g^*(x_2,y_2)=g(x_2,y_2)$. Taking the sum over all $i\in[k]$, we deduce that the number of quadruples $(x_1,y_1,x_2,y_2)\in Q$ such that $f(x_1,y_1)=g(x_2,y_2)$ is at least

$$\sum_{i\in[k]}(k-\Delta_f(*,y_1))\cdot(k-1-\Delta_g(*,y_2))\geqslant k^2(k-1)-k^2(k-1)\cdot\varepsilon_f-k^3\cdot\varepsilon_g.\tag{6.13}$$

The rejection probability of the column test is at least

$$\frac{1}{|Q|}\cdot\Big(k^2(k-1)-k^2(k-1)\cdot\varepsilon_f-k^3\cdot\varepsilon_g\Big)\geqslant\frac{1}{k}\Big(1-2\varepsilon_f-2\varepsilon_g\Big).\tag{6.14}$$

Consequently, the rejection probability of $\mathcal{V}_{\text{cons}}$ is at least

$$\frac{1}{2}\left(\frac{1}{k}\Big(1-2\varepsilon_f-2\varepsilon_g\Big)+0\right)=\frac{1}{2k}\Big(1-2\varepsilon_f-2\varepsilon_g\Big),\tag{6.15}$$

which completes the proof. □

6.2.3 Edge Test

We finally design the *edge verifier* $\mathcal{V}_{\text{edge}}$, which tests if a pair of k-colorings f,g of $[k]^2$ are close to the same stripe pattern. For this purpose, $\mathcal{V}_{\text{edge}}$ executes $\mathcal{V}_{\text{stripe}}$ on f with probability $\frac{2}{\rho Z}$, $\mathcal{V}_{\text{stripe}}$ on g with probability $\frac{2}{\rho Z}$, and $\mathcal{V}_{\text{cons}}$ on $f\circ g$ with probability $\frac{1}{Z}$, where $Z:=\frac{2}{\rho}+\frac{2}{\rho}+1$ and $\rho:=10^{-8}$ is the rejection rate of $\mathcal{V}_{\text{stripe}}$.

Edge verifier $\mathcal{V}_{\text{edge}}$.

Oracle access: two k-colorings $f, g \colon [k]^2 \to [k]$.

let $Z := \frac{2}{\rho} + \frac{2}{\rho} + 1$.
sample $r \sim [0,1]$.
if $0 \leqslant r < \frac{2}{\rho Z}$ **then** ▷ *with probability* $\frac{2}{\rho Z}$
  execute $\mathcal{V}_{\text{stripe}}$ on f.
else if $\frac{2}{\rho Z} \leqslant r < \frac{2}{\rho Z} + \frac{2}{\rho Z}$ **then** ▷ *with probability* $\frac{2}{\rho Z}$
  execute $\mathcal{V}_{\text{stripe}}$ on g.
else ▷ *with probability* $\frac{1}{Z}$
  execute $\mathcal{V}_{\text{cons}}$ on $f \circ g$.

When f and g are striped, $\mathcal{V}_{\text{edge}}$'s rejection probability is obtained immediately from Lemmas 6.6 and 6.8 as follows.

Lemma 6.10. *For any two striped k-colorings $f, g \colon [k]^2 \to [k]$, the following hold:*

- *if $f = g$ (in particular, f and g are consistent), $\mathcal{V}_{\text{edge}}$ rejects $f \circ g$ with probability exactly $\frac{1}{2Z \cdot k}$;*
- *if f and g are inconsistent, $\mathcal{V}_{\text{edge}}$ rejects $f \circ g$ with probability exactly $\frac{1}{Z \cdot k}$.*

Whenever f and g are inconsistent, $\mathcal{V}_{\text{edge}}$'s rejection probability is at least $\frac{1}{Z \cdot k}$ (regardless of the distance from being striped).

Lemma 6.11. *Let $f, g \colon [k]^2 \to [k]$ be any two inconsistent k-colorings. Then, $\mathcal{V}_{\text{edge}}$ rejects $f \circ g$ with probability at least $\frac{1}{Z \cdot k}$.*

Proof. Define $\varepsilon_f := \mathrm{dist}(f, \text{▦})$ and $\varepsilon_g := \mathrm{dist}(g, \text{▦})$. Since $\mathsf{dec}(f) \neq \mathsf{dec}(g)$, by Lemmas 6.7 and 6.9, $\mathcal{V}_{\text{edge}}$ rejects $f \circ g$ with probability at least

$$\begin{aligned}&\frac{2}{\rho Z} \cdot \frac{\rho \cdot \varepsilon_f}{k} + \frac{2}{\rho Z} \cdot \frac{\rho \cdot \varepsilon_g}{k} + \frac{1}{Z} \cdot \max\left\{ \left(1 - 2\varepsilon_f - 2\varepsilon_g\right) \frac{1}{k}, 0 \right\} \\ &= \frac{1}{Z \cdot k} \left(2\varepsilon_f + 2\varepsilon_g + \max\left\{ 1 - 2\varepsilon_f - 2\varepsilon_g, 0 \right\} \right).\end{aligned} \tag{6.16}$$

If $1 - 2\varepsilon_f - 2\varepsilon_g < 0$, this value is at least

$$\frac{1}{Z \cdot k} \left(2\varepsilon_f + 2\varepsilon_g \right) > \frac{1}{Z \cdot k}. \tag{6.17}$$

Otherwise, this value is at least

$$\frac{1}{Z \cdot k} \left(2\varepsilon_f + 2\varepsilon_g + \left(1 - 2\varepsilon_f - 2\varepsilon_g \right) \right) = \frac{1}{Z \cdot k}, \tag{6.18}$$

as desired. $\square$

Also, we give a lower bound $\frac{1}{2Z \cdot k}$ on $\mathcal{V}_{\text{edge}}$'s rejection probability for any two k-colorings.

Lemma 6.12. *Let $f, g \colon [k]^2 \to [k]$ be any two k-colorings. Then, $\mathcal{V}_{\text{edge}}$ rejects $f \circ g$ with probability at least $\frac{1}{2Z \cdot k}$.*

Proof. Owing to Lemma 6.11, it is sufficient to bound the rejection probability in the case of $\mathsf{dec}(f) = \mathsf{dec}(g)$. By Lemmas 6.7 and 6.9, $\mathcal{V}_{\mathrm{edge}}$ rejects $f \circ g$ with probability at least

$$\begin{aligned}
&\frac{2}{\rho Z} \cdot \frac{\rho \cdot \varepsilon_f}{k} + \frac{2}{\rho Z} \cdot \frac{\rho \cdot \varepsilon_g}{k} + \frac{1}{Z} \cdot \max\left\{ \left(1 - 2\varepsilon_f - 2\varepsilon_g\right) \frac{1}{2k}, 0 \right\} \\
&= \frac{1}{Z \cdot k} \left(2\varepsilon_f + 2\varepsilon_g + \max\left\{ \frac{1 - 2\varepsilon_f - 2\varepsilon_g}{2}, 0 \right\} \right)
\end{aligned} \tag{6.19}$$

If $1 - 2\varepsilon_f - 2\varepsilon_g < 0$, this value is at least

$$\frac{1}{Z \cdot k} \left(2\varepsilon_f + 2\varepsilon_g \right) > \frac{1}{Z \cdot k}. \tag{6.20}$$

Otherwise, this value is at least

$$\frac{1}{Z \cdot k} \left(2\varepsilon_f + 2\varepsilon_g + \frac{1 - 2\varepsilon_f - 2\varepsilon_g}{2} \right) \geqslant \frac{1}{2Z \cdot k}, \tag{6.21}$$

as desired. □

6.3 Putting Them Together: Proof of Lemma 6.3

Reduction. Our gap-preserving reduction from Maxmin 2-Cut Reconfiguration to Maxmin k-Cut Reconfiguration is described below. Fix $k \geqslant k_0$, $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$, and $\Delta \in \mathbb{N}$. Let $(G, f_{\mathsf{start}}, f_{\mathsf{end}})$ be an instance of Gap$_{1-\varepsilon_c, 1-\varepsilon_s}$ 2-Cut Reconfiguration, where $G = (V, E)$ is a graph of maximum degree $\Delta \in \mathbb{N}$, and $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [2]$ are a pair of 2-colorings of G. We construct an instance $(H, f'_{\mathsf{start}}, f'_{\mathsf{end}})$ of Maxmin k-Cut Reconfiguration as follows. For each vertex v of G, we create a fresh copy of $[k]^2$, denoted S_v; namely,

$$S_v \coloneqq \left\{ (v, x, y) \;\middle|\; (x, y) \in [k]^2 \right\}, \tag{6.22}$$

and we define

$$V(H) \coloneqq \bigcup_{v \in V} S_v = V \times [k]^2. \tag{6.23}$$

Since a k-coloring $f' \colon V \times [k]^2 \to [k]$ of $V(H)$ consists of a collection of $|V|$ k-colorings of $[k]^2$, we will think of it as $f' \colon V \to ([k]^2 \to [k])$ such that $f'(v)$ gives a k-coloring of S_v.

Consider the following verifier $\mathcal{V}_G$, given oracle access to a k-coloring $f' \colon V \to ([k]^2 \to [k])$:

Overall verifier $\mathcal{V}_G$.

Input: a graph $G = (V, E)$.
Oracle access: a k-coloring $f' \colon V \to ([k]^2 \to [k])$.

1: select an edge (v, w) of G uniformly at random.
2: execute $\mathcal{V}_{\mathrm{edge}}$ on $f'(v) \circ f'(w)^\intercal$, where $f'(w)^\intercal$ is the *transposition* of $f'(w)$; i.e., $f'(w)^\intercal(x, y) = f'(w)(y, x)$ for all $(x, y) \in [k]^2$.

Create the set $E(H)$ of parallel edges between $V(H)$ so as to emulate $\mathcal{V}_G$ in a sense that for any k-coloring $f' \colon V \times [k]^2 \to [k]$,

$$\mathsf{val}_H(f') = \mathbb{P}\left[\mathcal{V}_G \text{ accepts } f' \right]. \tag{6.24}$$

Since a pair of vertices (v,x_1,y_1) and (w,x_2,y_2) of H might be selected by $\mathcal{V}_G$ only if $(v,w) \in E$, the maximum degree of H can be bounded by $O(\Delta \cdot k^2)$. For a 2-coloring $f \colon V \to [2]$ of G, consider a k-coloring $f' \colon V \times [k]^2 \to [k]$ of H such that $f'(v)$ is horizontally striped if $f(v) = 1$ and vertically striped if $f(v) = 2$; namely,

$$f'(v,x,y) := \begin{cases} y & \text{if } f(v) = 1 \\ x & \text{if } f(v) = 2 \end{cases} \text{ for all } (v,x,y) \in V \times [k]^2. \tag{6.25}$$

Construct a pair of k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}} \colon V \times [k]^2 \to [k]$ of H from $f_{\mathsf{start}}, f_{\mathsf{end}}$ according to the above procedure, respectively. This completes the description of the reduction.

Correctness. We first show the following completeness.

Lemma 6.13. *The following holds*:

$$\mathsf{opt}_G\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) \geqslant 1 - \varepsilon_c \implies \mathsf{opt}_H\big(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\big) \geqslant 1 - \frac{1+\varepsilon_c}{2Z \cdot k} - \frac{\Delta}{|E|}. \tag{6.26}$$

Proof. Suppose $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) = 1 - \varepsilon_c$. It is sufficient to consider the case that f_{start} and f_{end} differ in a single vertex, say $v^\star$. Without loss of generality, we assume that $f_{\mathsf{start}}(v^\star) = 1$ and $f_{\mathsf{end}}(v^\star) = 2$. Note that f'_{start} and f'_{end} differ only in vertices of $S_{v^\star}$. Consider an irredundant reconfiguration sequence $\mathscr{F}'$ from f'_{start} to f'_{end}, which is obtained by recoloring (some) vertices of $S_{v^\star}$. For any intermediate k-coloring f' in $\mathscr{F}'$, the following hold:

- For at most $(1-\varepsilon_c)$-fraction of edges (v,w) of G, $f'(v)$ and $f'(w)$ are striped and $\mathsf{dec}(f'(v)) \neq \mathsf{dec}(f'(w))$. The edge verifier $\mathcal{V}_{\mathsf{edge}}$ rejects $f'(v) \circ f'(w)^\intercal$ for such (v,w) with probability $\frac{1}{2Z \cdot k}$ due to Lemma 6.10.
- For at most ε_c-fraction of edges (v,w) of G, $f'(v)$ and $f'(w)$ are striped and $\mathsf{dec}(f'(v)) = \mathsf{dec}(f'(w))$. The edge verifier $\mathcal{V}_{\mathsf{edge}}$ rejects $f'(v) \circ f'(w)^\intercal$ for such (v,w) with probability $\frac{1}{Z \cdot k}$ due to Lemma 6.10.
- Since $f'(v^\star)$, which is *in transition*, may be far from being striped, for at most $\frac{\Delta}{m}$-fraction of edges $(v^\star,w)$ of G, the edge verifier $\mathcal{V}_{\mathsf{edge}}$ may reject $f'(v^\star) \circ f'(w)^\intercal$ with probability at most 1.

Consequently, $\mathcal{V}_G$ rejects f' with probability at most

$$\begin{aligned} &(1-\varepsilon_c) \cdot \frac{1}{2Z \cdot k} + \varepsilon_c \cdot \frac{1}{Z \cdot k} + \frac{\Delta}{|E|} \cdot 1 = (1+\varepsilon_c) \cdot \frac{1}{2Z \cdot k} + \frac{\Delta}{|E|}, \\ &\implies \mathsf{opt}_H\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) \geqslant 1 - \frac{1+\varepsilon_c}{2Z \cdot k} - \frac{\Delta}{|E|}, \end{aligned} \tag{6.27}$$

which completes the proof. □

We then show the following soundness.

Lemma 6.14. *The following holds*:

$$\mathsf{opt}_G\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) < 1 - \varepsilon_s \implies \mathsf{opt}_H\big(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\big) < 1 - \frac{1+\varepsilon_s}{2Z \cdot k}. \tag{6.28}$$

Proof. Suppose $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) < 1-\varepsilon_s$. Let $\mathscr{F}' = (f'^{(1)}, \ldots, f'^{(T)})$ be any reconfiguration sequence from f'_{start} to f'_{end} such that $\mathsf{val}_H(\mathscr{F}') = \mathsf{opt}_H(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}})$. Construct then a new reconfiguration sequence $\mathscr{F} = (f^{(1)}, \ldots, f^{(T)})$ over 2-colorings of G from f_{start} to f_{end} such that $f^{(t)}(v) := \mathsf{dec}(f'^{(t)}(v))$ for all $v \in V$. Since $\mathscr{F}$ is a valid reconfiguration sequence, there exists $f^{(t^\star)}$ in $\mathscr{F}$ that makes more than ε_s-fraction of edges of G monochromatic, denoted $M \subset E$. Conditioned on the event that any edge of M is sampled, $\mathcal{V}_G$ rejects $f'^{(t^\star)}$ with probability $\frac{1}{Z\cdot k}$ by Lemma 6.11. On the other hand, regardless of the sampled edge, $\mathcal{V}_G$ rejects $f'^{(t^\star)}$ with probability $\frac{1}{2Z\cdot k}$ by Lemma 6.12. Consequently, $\mathcal{V}_G$ rejects $f'^{(t^\star)}$ with probability more than

$$
\begin{aligned}
&(1-\varepsilon_s)\cdot\frac{1}{2Z\cdot k} + \varepsilon_s \cdot \frac{1}{Z\cdot k} = \frac{1+\varepsilon_s}{2Z\cdot k}, \\
&\implies \mathsf{opt}_H\big(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\big) < 1 - \frac{1+\varepsilon_s}{2Z\cdot k},
\end{aligned}
\tag{6.29}
$$

which completes the proof. ☐

We are now ready to prove Lemma 6.3.

Proof of Lemma 6.3. Given a graph G of maximum degree Δ and a pair of its 2-colorings $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V(G) \to [2]$ as an instance of GAP$_{1-\varepsilon_c,1-\varepsilon_s}$ 2-CUT RECONFIGURATION, we create a multigraph H of maximum degree $O(\Delta\cdot k^2)$ and a pair of its k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}} \colon V(H) \to [k]$ as an instance of MAXMIN k-CUT RECONFIGURATION according to the above reduction. By Lemmas 6.13 and 6.14, it holds that

$$
\begin{aligned}
&\mathsf{opt}_G\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) \geqslant 1-\varepsilon_c \implies \mathsf{opt}_H\big(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\big) \geqslant 1 - \frac{1+\varepsilon_c}{2Z\cdot k} - \frac{\Delta}{|E|}, \\
&\mathsf{opt}_G\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) < 1-\varepsilon_s \implies \mathsf{opt}_H\big(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\big) < 1 - \frac{1+\varepsilon_s}{2Z\cdot k}.
\end{aligned}
\tag{6.30}
$$

Without loss of generality, we can assume that $|E|$ is sufficiently large so that

$$
\frac{1+\varepsilon_c}{2Z\cdot k} + \frac{\Delta}{|E|} < \frac{1+\frac{\varepsilon_c+\varepsilon_s}{2}}{2Z\cdot k}.
\tag{6.31}
$$

In fact, the above inequality holds when

$$
|E| > \frac{4\Delta Z\cdot k}{\varepsilon_s - \varepsilon_c}.
\tag{6.32}
$$

Consequently, we obtain the following:

$$
\begin{aligned}
&\mathsf{opt}_G\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) \geqslant 1-\varepsilon_c \implies \mathsf{opt}_H\big(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\big) \geqslant 1 - \frac{\delta_c}{k}, \\
&\mathsf{opt}_G\big(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}\big) < 1-\varepsilon_s \implies \mathsf{opt}_H\big(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}\big) < 1 - \frac{\delta_s}{k},
\end{aligned}
\tag{6.33}
$$

where δ_c and δ_s are defined as

$$
\delta_c := \frac{1+\frac{\varepsilon_c+\varepsilon_s}{2}}{2Z} \quad \text{and} \quad \delta_s := \frac{1+\varepsilon_s}{2Z}.
\tag{6.34}
$$

Note that δ_s and δ_c do not depend on k, and $\delta_c < \delta_s$, as desired. ☐

Our construction of H can also be used to derive the following gap-preserving reduction from MAX 2-CUT to MAX k-CUT, which reproves the NP-hardness of approximating MAX k-CUT within a factor of $1-\Omega\left(\frac{1}{k}\right)$ [GS13, KKLP97]:

Lemma 6.15. *For a graph G and a multigraph H generated by the above reduction, the following hold*:

$$\begin{aligned} &\exists f\colon V(G)\to[2],\ \mathsf{val}_G(f)\geqslant 1-\varepsilon_c \implies \exists f'\colon V(H)\to[k],\ \mathsf{val}_H(f')\geqslant 1-\frac{1+\varepsilon_c}{2Z\cdot k},\\ &\forall f\colon V(G)\to[2],\ \mathsf{val}_G(f)< 1-\varepsilon_s \implies \forall f'\colon V(H)\to[k],\ \mathsf{val}_H(f')< 1-\frac{1+\varepsilon_s}{2Z\cdot k}. \end{aligned} \tag{6.35}$$

Therefore, for every reals $\varepsilon_c,\varepsilon_s\in(0,1)$ with $\varepsilon_c<\varepsilon_s$, there exist reals $\delta_c,\delta_s\in(0,1)$ with $\delta_c<\delta_s$ such that for all sufficiently large $k\geqslant k_0:=10^3$, there exists a gap-preserving reduction from GAP$_{1-\varepsilon_c,1-\varepsilon_s}$ 2-CUT *to* GAP$_{1-\frac{\delta_c}{k},1-\frac{\delta_s}{k}}$ k-CUT.

Proof. See the proofs of Lemmas 6.13 and 6.14. □

6.4 Rejection Rate of the Stripe Test: Proof of Lemma 6.7

This subsection is devoted to the proof of Lemma 6.7. Some notations and definitions are introduced below. Fix $k\geqslant k_0=10^3$. Let $f\colon[k]^2\to[k]$ be a k-coloring of $[k]^2$ such that $\mathrm{dist}(f,\boxplus)=\varepsilon$ for some $\varepsilon<\varepsilon_0:=10^{-2}$. Each (x,y) in $[k]^2$ will be referred to as a *point*. Hereafter, let X_1,Y_1,X_2,Y_2 denote independent random variables uniformly chosen from $[k]$. The stripe verifier $\mathcal{V}_{\mathrm{stripe}}$ rejects f with probability

$$\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}}\text{ rejects }f\Big]:=\mathbb{P}\Big[f(X_1,Y_1)=f(X_2,Y_2)\;\Big|\;X_1\neq X_2\text{ and }Y_1\neq Y_2\Big]. \tag{6.36}$$

We say that $\mathcal{V}_{\mathrm{stripe}}$ *rejects f by color $\alpha\in[k]$* when $\mathcal{V}_{\mathrm{stripe}}$ draws (X_1,Y_1,X_2,Y_2) such that $f(X_1,Y_1)=f(X_2,Y_2)=\alpha$. Such an event occurs with probability

$$\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}}\text{ rejects }f\text{ by }\alpha\Big]:=\mathbb{P}\Big[f(X_1,Y_1)=f(X_2,Y_2)=\alpha\;\Big|\;X_1\neq X_2\text{ and }Y_1\neq Y_2\Big]. \tag{6.37}$$

Note that

$$\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}}\text{ rejects }f\Big]=\sum_{\alpha\in[k]}\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}}\text{ rejects }f\text{ by }\alpha\Big]. \tag{6.38}$$

For each color $\alpha\in[k]$, we use $f^{-1}(\alpha)$ to denote the set of (x,y)'s such that $f(x,y)=\alpha$; namely,

$$f^{-1}(\alpha):=\Big\{(x,y)\in[k]^2\;\Big|\;f(x,y)=\alpha\Big\}. \tag{6.39}$$

For each $x,y,\alpha\in[k]$, let $R_{y,\alpha}$ denote the number of x's such that $f(x,y)=\alpha$ and $C_{x,\alpha}$ denote the number of y's such that $f(x,y)=\alpha$; namely,

$$\begin{aligned} R_{y,\alpha}&:=\left|\Big\{x\in[k]\;\Big|\;f(x,y)=\alpha\Big\}\right|,\\ C_{x,\alpha}&:=\left|\Big\{y\in[k]\;\Big|\;f(x,y)=\alpha\Big\}\right|. \end{aligned} \tag{6.40}$$

Let f^* be a striped k-coloring of $[k]^2$ that is closest to f. Without loss of generality, we can assume that f^* is horizontally striped, and that the rows of f and f^* are rearranged so that $f^*(x,y)=y$ for all $(x,y)\in[k]^2$.

For each $y \in [k]$, let D_y denote the set of (x,y)'s at which f and f^* disagree, and let D denote the union of D_y's for all $y \in [k]$; namely,

$$\begin{aligned} D_y &:= \left\{ (x,y) \in [k]^2 \;\middle|\; x \in [k], f(x,y) \neq f^*(x,y) \right\}, \\ D &:= \bigcup_{y \in [k]} D_y = \left\{ (x,y) \in [k]^2 \;\middle|\; f(x,y) \neq f^*(x,y) \right\}. \end{aligned} \tag{6.41}$$

Note that $|D| = \varepsilon k^2$.

Define further Good and Bad as the set of y's such that $|D_y|$ is at most $0.99k$ and greater than $0.99k$, respectively; namely,

$$\begin{aligned} \mathsf{Good} &:= \left\{ y \in [k] \;\middle|\; |D_y| \leqslant 0.99k \right\}, \\ \mathsf{Bad} &:= \left\{ y \in [k] \;\middle|\; |D_y| > 0.99k \right\}. \end{aligned} \tag{6.42}$$

Observe that $|\mathsf{Bad}| < 1.02\varepsilon k$ because

$$\mathop{\mathbb{P}}_{Y \sim [k]}\Big[Y \in \mathsf{Bad}\Big] = \mathop{\mathbb{P}}_{Y \sim [k]}\Big[|D_Y| > 0.99k\Big] < \frac{\mathop{\mathbb{E}}_{Y \sim [k]}\Big[|D_Y|\Big]}{0.99k} < 1.02\varepsilon. \tag{6.43}$$

For each color $\alpha \in [k]$, let N_α denote the number of (x,y)'s in D such that $f(x,y) = \alpha$; namely,

$$N_\alpha := \left| \left\{ (x,y) \in D \;\middle|\; f(x,y) = \alpha \right\} \right|. \tag{6.44}$$

For a set $S \subseteq [k]$ of colors, we define $N(S)$ as the sum of N_α over $\alpha \in S$; namely,

$$N(S) := \sum_{\alpha \in S} N_\alpha. \tag{6.45}$$

Denote $N_{\mathsf{Good}} := N(\mathsf{Good})$ and $N_{\mathsf{Bad}} := N(\mathsf{Bad})$; note that $N([k]) = N_{\mathsf{Good}} + N_{\mathsf{Bad}} = |D|$.

Lastly, we show the probability of $\mathcal{V}_{\text{stripe}}$ rejecting by color α, depending on the number of occurrences of α per row and column, which will be used several times.

Lemma 6.16. *For a k-coloring $f\colon [k]^2 \to [k]$ such that color α appears at least $m \geqslant 100$ times, $R_{y,\alpha} \leqslant \vartheta k$ for all $y \in [k]$, and $C_{x,\alpha} \leqslant \vartheta k$ for all $x \in [k]$, it holds that*

$$\mathbb{P}\Big[\mathcal{V}_{\text{stripe}} \text{ rejects } f \text{ by } \alpha\Big] \geqslant \frac{m \cdot (m - \vartheta k)}{10^2 \cdot k^4}. \tag{6.46}$$

Proof. Consider the following case analysis: (1) $R_{y^*,\alpha} \geqslant \frac{m}{16}$ for some y^*, (2) $C_{x^*,\alpha} \geqslant \frac{m}{16}$ for some x^*, and (3) $R_{y,\alpha} < \frac{m}{16}$ for all y and $C_{x,\alpha} < \frac{m}{16}$ for all x.

Suppose first $R_{y^*,\alpha} \geqslant \frac{m}{16}$ for some y^*. Since $R_{y^*,\alpha} \leqslant \vartheta k$ by assumption, we have

$$\sum_{y \neq y^*} R_{y,\alpha} \geqslant m - \vartheta k, \tag{6.47}$$

and thus, $\mathcal{V}_{\text{stripe}}$ rejects f by α with the following probability:

$$
\begin{aligned}
&\mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\Big[f(X_1,Y_1) = f(X_2,Y_2) = \alpha\Big] \\
&\geqslant \mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\Big[f(X_1,Y_1) = f(X_2,Y_2) = \alpha \text{ and } Y_1 = y^* \text{ and } Y_2 \neq y^*\Big] \\
&= \mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\Big[f(X_1,Y_1) = f(X_2,Y_2) = \alpha \;\Big|\; Y_1 = y^* \text{ and } Y_2 \neq y^*\Big] \cdot \mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\Big[Y_1 = y^* \text{ and } Y_2 \neq y^*\Big] \\
&= \frac{1}{k} \cdot \mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_2 \neq y^*}}\Big[f(X_1,y^*) = f(X_2,Y_2) = \alpha\Big] \qquad (6.48) \\
&= \frac{1}{k} \cdot \underbrace{\mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_2 \neq y^*}}\Big[f(X_1,y^*) = \alpha \;\Big|\; f(X_2,Y_2) = \alpha\Big]}_{\geqslant \frac{\frac{m}{16}-1}{k-1}} \cdot \underbrace{\mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_2 \neq y^*}}\Big[f(X_2,Y_2) = \alpha\Big]}_{\frac{m-\vartheta k}{k(k-1)}} \\
&\geqslant \frac{(\frac{m}{16}-1)\cdot(m-\vartheta k)}{k^2(k-1)^2} \underbrace{\geqslant}_{m \geqslant 100} \frac{m\cdot(m-\vartheta k)}{20k^4}.
\end{aligned}
$$

Suppose next $C_{x^*,\alpha} \geqslant \frac{m}{16}$ for some x^*. Similarly to the first case, $\mathcal{V}_{\text{stripe}}$ rejects f by α with probability at least $\frac{m\cdot(m-\vartheta k)}{20k^4}$.

Suppose then $R_{y,\alpha} < \frac{m}{16}$ for all y and $C_{x,\alpha} < \frac{m}{16}$ for all x. Let $A_1,\ldots,A_{2k}$ be $2k$ independent random variables uniformly chosen from $[2]$. For each $(i,j) \in [2]^2$ and $(x,y) \in f^{-1}(\alpha)$, we define $B^{(i,j)}_{x,y}$ as

$$B^{(i,j)}_{x,y} := \Big[\!\!\Big[A_x = i \text{ and } A_{y+k} = j\Big]\!\!\Big]. \tag{6.49}$$

For each $(i,j) \in [2]^2$, we define $Z^{(i,j)}$ as

$$Z^{(i,j)} := \sum_{(x,y)\in f^{-1}(\alpha)} B^{(i,j)}_{x,y}. \tag{6.50}$$

Observe that for each $(i,j) \in [2]^2$, the collection of $B^{(i,j)}_{x,y}$'s is a read-$\frac{m}{16}$ family. By the read-k Chernoff bound (Theorem 5.4), it holds that for any $\delta > 0$,

$$\mathbb{P}\Big[Z^{(i,j)} \leqslant \mathbb{E}\Big[Z^{(i,j)}\Big] - \delta m\Big] \leqslant \exp\left(-\frac{2\delta m}{\frac{m}{16}}\right). \tag{6.51}$$

Since $\mathbb{E}[Z^{(i,j)}] = \frac{m}{4}$, we let $\delta := \frac{1}{8}$ to obtain

$$\mathbb{P}\Big[Z^{(i,j)} \leqslant \tfrac{m}{8}\Big] \leqslant \exp\left(-\frac{2\cdot\frac{m}{8}}{\frac{m}{16}}\right) = \exp(-4) < 0.02. \tag{6.52}$$

Taking a union bound, we derive

$$\mathbb{P}\Big[\exists (i,j) \in [2]^2 \text{ s.t. } Z^{(i,j)} \leqslant \tfrac{m}{8}\Big] \leqslant 4\cdot 0.02 < 1. \tag{6.53}$$

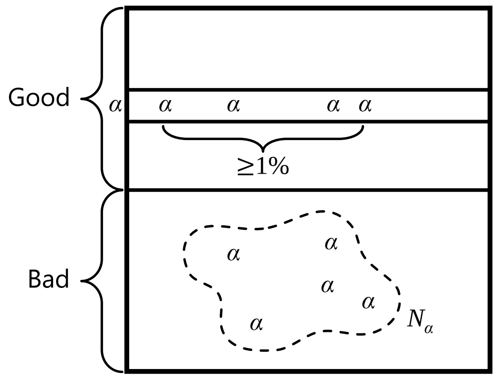


Figure 7: Illustration of the proof of Claim 6.17. Since each color α of Good appears $0.01k$ times on the α^{th} row and $N_{\mathsf{Good}} = \Omega(\varepsilon k^2)$, we can bound the rejection probability from below.

Therefore, there exist two partitions (P_1, P_2) and (Q_1, Q_2) of $[k]$ such that

$$\left|f^{-1}(\alpha) \cap (P_1 \times Q_1)\right| \geqslant \frac{m}{8} \text{ and } \left|f^{-1}(\alpha) \cap (P_2 \times Q_2)\right| \geqslant \frac{m}{8}. \tag{6.54}$$

Letting $S_{1,1} := f^{-1}(\alpha) \cap (P_1 \times Q_1)$ and $S_{2,2} := f^{-1}(\alpha) \cap (P_2 \times Q_2)$, we derive

$$\begin{aligned}
&\mathbb{P}\left[f(X_1, Y_1) = f(X_2, Y_2) = \alpha \;\middle|\; X_1 \neq X_2 \text{ and } Y_1 \neq Y_2\right] \\
&\geqslant \mathbb{P}\left[(X_1, Y_1) \in S_{1,1} \text{ and } (X_2, Y_2) \in S_{2,2} \;\middle|\; X_1 \neq X_2 \text{ and } Y_1 \neq Y_2\right] \\
&\geqslant \mathbb{P}\left[(X_1, Y_1) \in S_{1,1} \text{ and } (X_2, Y_2) \in S_{2,2}\right] \\
&\geqslant \frac{m}{8k^2} \cdot \frac{m}{8k^2} = \frac{m^2}{64k^4}.
\end{aligned} \tag{6.55}$$

Consequently, we get

$$\mathbb{P}\left[\mathcal{V}_{\text{stripe}} \text{ rejects } f \text{ by } \alpha\right] \geqslant \min\left\{\frac{m \cdot (m - \vartheta k)}{20k^4}, \frac{m^2}{64k^4}\right\} \geqslant \frac{m \cdot (m - \vartheta k)}{10^2 \cdot k^4}, \tag{6.56}$$

which completes the proof. $\square$

Hereafter, we present the proof of Lemma 6.7 by cases. We first divide into two cases according to N_{Good}.

$\boxed{\textbf{(Case 1) } N_{\mathsf{Good}} \geqslant 0.01\varepsilon k^2\textbf{.}}$ We show that $\mathcal{V}_{\text{stripe}}$'s rejection probability is $\Omega\left(\frac{N_{\mathsf{Good}}}{k^3}\right)$. See Figure 7 for illustration of its proof.

Claim 6.17. *It holds that*

$$\mathbb{P}\left[\mathcal{V}_{\text{stripe}} \textit{ rejects } f\right] \geqslant \frac{10^{-3}}{k^3} \cdot N_{\mathsf{Good}} \geqslant \frac{10^{-5} \cdot \varepsilon}{k}. \tag{6.57}$$

Proof. Observe that $\mathcal{V}_{\text{stripe}}$'s rejection probability is

$$\begin{aligned}
&\mathbb{P}\Big[f(X_1,Y_1)=f(X_2,Y_2)\;\Big|\;X_1\neq X_2 \text{ and } Y_1\neq Y_2\Big]\\
&=\frac{1}{k^2}\cdot\sum_{(x_1,y_1)\in[k]^2}\mathbb{P}_{\substack{X_2\neq x_1\\ Y_2\neq y_1}}\Big[f(x_1,y_1)=f(X_2,Y_2)\Big]\\
&\geqslant\frac{1}{k^2}\cdot\sum_{\substack{(x_1,y_1)\in D\\ f(x_1,y_1)\in\mathsf{Good}}}\underbrace{\mathbb{P}_{\substack{X_2\neq x_1\\ Y_2\neq y_1}}\Big[f(x_1,y_1)=f(X_2,Y_2)\text{ and }Y_2=f(x_1,y_1)\Big]}_{\bigstar:=}
\end{aligned}\tag{6.58}$$

For each point $(x_1,y_1)\in D$ such that $f(x_1,y_1)\in\mathsf{Good}$, we bound $\bigstar$ as follows.

$$\begin{aligned}
\bigstar&=\mathbb{P}_{\substack{X_2\neq x_1\\ Y_2\neq y_1}}\Big[f(x_1,y_1)=f(X_2,Y_2)\;\Big|\;Y_2=f(x_1,y_1)\Big]\cdot\mathbb{P}_{Y_2\neq y_1}\Big[Y_2=f(x_1,y_1)\Big]\\
&=\mathbb{P}_{X_2\neq x_1}\Big[f(x_1,y_1)=f(X_2,f(x_1,y_1))\Big]\cdot\frac{1}{k-1}\\
&=\mathbb{P}_{X_2\neq x_1}\Big[f^*(X_2,f(x_1,y_1))=f(X_2,f(x_1,y_1))\Big]\cdot\frac{1}{k-1}\\
&=\left(\frac{k-1-|D_{f(x_1,y_1)}|}{k-1}\right)\cdot\frac{1}{k-1}\\
&\geqslant\left(\frac{k-1-0.99k}{k}\right)\cdot\frac{1}{k-1}\\
&\underbrace{\geqslant}_{k\geqslant k_0}\frac{0.009}{k-1}\geqslant\frac{10^{-3}}{k}.
\end{aligned}\tag{6.59}$$

Consequently, $\mathcal{V}_{\text{stripe}}$'s rejection probability is at least

$$\frac{1}{k^2}\cdot\sum_{\substack{(x,y)\in D\\ f(x,y)\in\mathsf{Good}}}\frac{10^{-3}}{k}=\frac{10^{-3}}{k^3}\cdot N_{\mathsf{Good}}\geqslant\frac{10^{-3}}{k^3}\cdot 0.01\varepsilon k^2>\frac{10^{-5}\cdot\varepsilon}{k},\tag{6.60}$$

which completes the proof. □

$\boxed{\textbf{(Case 2) } N_{\mathsf{Good}}<0.01\varepsilon k^2\textbf{.}}$ Note that $N_{\mathsf{Bad}}>0.99\varepsilon k^2$ by assumption. We partition Bad into $\mathsf{Bad}^{\text{lot}}$ and $\mathsf{Bad}^{\text{few}}$ as follows:

$$\begin{aligned}
\mathsf{Bad}^{\text{lot}}&:=\Big\{\alpha\in\mathsf{Bad}\;\Big|\;N_\alpha\geqslant 1.01k\Big\},\\
\mathsf{Bad}^{\text{few}}&:=\Big\{\alpha\in\mathsf{Bad}\;\Big|\;N_\alpha< 1.01k\Big\}.
\end{aligned}\tag{6.61}$$

We will divide into two cases according to the size of $\mathsf{Bad}^{\text{lot}}$.

$\boxed{\textbf{(Case 2-1) } |\mathsf{Bad}^{\mathrm{lot}}| \geqslant 0.01\varepsilon k.}$ We show that $\mathcal{V}_{\mathrm{stripe}}$'s rejection probability is $\Omega\left(\frac{|\mathsf{Bad}^{\mathrm{lot}}|}{k^2}\right)$.

Claim 6.18. *It holds that*

$$\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}} \textit{ rejects } f\Big] \geqslant \frac{10^{-4}}{k^2} \cdot |\mathsf{Bad}^{\mathrm{lot}}| \geqslant \frac{10^{-6} \cdot \varepsilon}{k}. \tag{6.62}$$

Proof. By applying Lemma 6.16 with $\vartheta = 1$ to each color α of $\mathsf{Bad}^{\mathrm{lot}}$, we have

$$\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}} \text{ rejects } f \text{ at } \alpha\Big] \geqslant \frac{N_\alpha \cdot (N_\alpha - k)}{10^2 \cdot k^4} \geqslant \frac{1.01k \cdot 0.01k}{10^2 \cdot k^4} > \frac{10^{-4}}{k^2}. \tag{6.63}$$

Consequently, $\mathcal{V}_{\mathrm{stripe}}$'s rejection probability is at least

$$\sum_{\alpha \in \mathsf{Bad}^{\mathrm{lot}}} \mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}} \text{ rejects } f \text{ by } \alpha\Big] \geqslant \sum_{\alpha \in \mathsf{Bad}^{\mathrm{lot}}} \frac{10^{-4}}{k^2} = \frac{10^{-4}}{k^2} \cdot |\mathsf{Bad}^{\mathrm{lot}}| \geqslant \frac{10^{-6} \cdot \varepsilon}{k}, \tag{6.64}$$

completing the proof. □

$\boxed{\textbf{(Case 2-2) } |\mathsf{Bad}^{\mathrm{lot}}| < 0.01\varepsilon k.}$ We will divide into two cases according to $N(\mathsf{Bad}^{\mathrm{lot}})$.

$\boxed{\textbf{(Case 2-2-1) } N(\mathsf{Bad}^{\mathrm{lot}}) \geqslant 0.02\varepsilon k^2.}$ We show that $\mathcal{V}_{\mathrm{stripe}}$'s rejection probability is $\Omega\left(\frac{N(\mathsf{Bad}^{\mathrm{lot}})}{k^3}\right)$ for very small $|\mathsf{Bad}^{\mathrm{lot}}|$.

Claim 6.19. *It holds that*

$$\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}} \textit{ rejects } f\Big] \geqslant \frac{10^{-2}}{k^3} \cdot \Big(N(\mathsf{Bad}^{\mathrm{lot}}) - k \cdot |\mathsf{Bad}^{\mathrm{lot}}|\Big) \geqslant \frac{10^{-4} \cdot \varepsilon}{k}. \tag{6.65}$$

Proof. By applying Lemma 6.16 with $\vartheta = 1$ to each color α of $\mathsf{Bad}^{\mathrm{lot}}$, we have

$$\mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}} \text{ rejects } f \text{ by } \alpha\Big] \geqslant \frac{N_\alpha \cdot (N_\alpha - k)}{10^2 \cdot k^4} \geqslant \frac{1.01k \cdot (N_\alpha - k)}{10^2 \cdot k^4} > \frac{N_\alpha - k}{10^2 \cdot k^3}. \tag{6.66}$$

Consequently, $\mathcal{V}_{\mathrm{stripe}}$'s rejection probability is at least

$$\begin{aligned} \sum_{\alpha \in \mathsf{Bad}^{\mathrm{lot}}} \mathbb{P}\Big[\mathcal{V}_{\mathrm{stripe}} \text{ rejects } f \text{ by } \alpha\Big] &\geqslant \sum_{\alpha \in \mathsf{Bad}^{\mathrm{lot}}} \frac{N_\alpha - k}{10^2 \cdot k^3} \\ &\geqslant \frac{10^{-2}}{k^3} \cdot \Big(N(\mathsf{Bad}^{\mathrm{lot}}) - k \cdot |\mathsf{Bad}^{\mathrm{lot}}|\Big) \\ &> \frac{10^{-2}}{k^3} \cdot \Big(0.02\varepsilon k^2 - k \cdot 0.01\varepsilon k\Big) \\ &> \frac{10^{-4} \cdot \varepsilon}{k}, \end{aligned} \tag{6.67}$$

which completes the proof. □

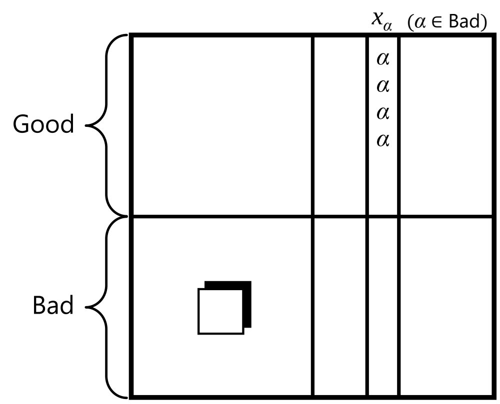


Figure 8: Illustration of $\square$, which exclude the $x_\alpha{}^{\text{th}}$ column for every $\alpha \in \mathsf{Bad}$.

$\boxed{\textbf{(Case 2-2-2)}\ N(\mathsf{Bad}^{\text{lot}}) < 0.02\varepsilon k^2\textbf{.}}$ By assumption, $N(\mathsf{Bad}^{\text{few}})$ and $|\mathsf{Bad}^{\text{few}}|$ can be bounded from below as follows:

$$\begin{aligned} &0.99\varepsilon k^2 < N_{\mathsf{Bad}} = N(\mathsf{Bad}^{\text{lot}}) + N(\mathsf{Bad}^{\text{few}}) < 0.02\varepsilon k^2 + N(\mathsf{Bad}^{\text{few}}), \\ &\implies N(\mathsf{Bad}^{\text{few}}) > 0.97\varepsilon k^2. \end{aligned} \tag{6.68}$$

$$\begin{aligned} &0.97\varepsilon k^2 < N(\mathsf{Bad}^{\text{few}}) = \sum_{\alpha \in \mathsf{Bad}^{\text{few}}} N_\alpha < 1.01k \cdot |\mathsf{Bad}^{\text{few}}|, \\ &\implies |\mathsf{Bad}^{\text{few}}| > 0.96\varepsilon k. \end{aligned} \tag{6.69}$$

For each color $\alpha \in [k]$, let x_α denote the column that includes the largest number of α's; namely,

$$x_\alpha := \operatorname*{argmax}_{x \in [k]} \left\{ C_{x,\alpha} \right\}. \tag{6.70}$$

Define $\square$ as a subset of $[k]^2$ obtained by excluding the $x_\alpha{}^{\text{th}}$ column for every $\alpha \in \mathsf{Bad}$ and the rows specified by Good; namely,

$$\square := \left([k] \setminus \left\{ x_\alpha \;\middle|\; \alpha \in \mathsf{Bad} \right\} \right) \times \left([k] \setminus \mathsf{Good} \right). \tag{6.71}$$

See Figure 8 for illustration. Note that the size of $\square$ is

$$\begin{aligned} |\square| &\geqslant (k - |\mathsf{Bad}|) \cdot |\mathsf{Bad}| \\ &\geqslant (k - |\mathsf{Bad}|) \cdot |\mathsf{Bad}^{\text{few}}| \\ &\underbrace{>}_{|\mathsf{Bad}| < 1.02\varepsilon k \ \&\ |\mathsf{Bad}^{\text{few}}| > 0.96\varepsilon k} (k - 1.02\varepsilon k) \cdot 0.96\varepsilon k \\ &\underbrace{>}_{\varepsilon < \varepsilon_0} 0.95 \cdot \varepsilon k^2. \end{aligned} \tag{6.72}$$

We show that most of the points of $\square$ are colored in $\mathsf{Bad}^{\text{few}}$.

Claim 6.20. *It holds that*

$$\left|f^{-1}(\mathsf{Bad}^{\text{few}})\cap \text{❒}\right| > 0.91\cdot \varepsilon k^2, \tag{6.73}$$

namely, more than $0.91\cdot\varepsilon k^2$ *points of* ❒ *are colored in* $\mathsf{Bad}^{\text{few}}$.

Proof. The number of points of ❒ *not* colored in $\mathsf{Bad}^{\text{few}}$ can be bounded as follows:

- Since ❒ does not include α^{th} row for any $\alpha\in\mathsf{Good}$, the number of points colored in Good is $N_{\mathsf{Good}} < 0.01\cdot\varepsilon k^2$.
- The number of points colored in $\mathsf{Bad}^{\text{lot}}$ that *disagree* with f^* is $N(\mathsf{Bad}^{\text{lot}}) < 0.02\cdot\varepsilon k^2$.
- The number of points colored in $\mathsf{Bad}^{\text{lot}}$ that *agree* with f^* is

$$\sum_{\alpha\in\mathsf{Bad}^{\text{lot}}}(k-|D_\alpha|) \underbrace{<}_{|D_\alpha|>0.99k} 0.01k\cdot|\mathsf{Bad}^{\text{lot}}| \underbrace{<}_{|\mathsf{Bad}^{\text{lot}}|<0.01\varepsilon k} 10^{-4}\cdot\varepsilon k^2. \tag{6.74}$$

Consequently, the number of points of ❒ colored in $\mathsf{Bad}^{\text{few}}$ is

$$\begin{aligned}\left|f^{-1}(\mathsf{Bad}^{\text{few}})\cap \text{❒}\right| &= |\text{❒}| - \left(N_{\mathsf{Good}} + N(\mathsf{Bad}^{\text{lot}}) + \sum_{\alpha\in\mathsf{Bad}^{\text{lot}}}(k-|D_\alpha|)\right)\\ &> 0.95\cdot\varepsilon k^2 - \left(0.01\cdot\varepsilon k^2 + 0.02\cdot\varepsilon k^2 + 10^{-4}\cdot\varepsilon k^2\right) > 0.91\cdot\varepsilon k^2,\end{aligned} \tag{6.75}$$

as desired. □

We further partition $\mathsf{Bad}^{\text{few}}$ into $\mathsf{Bad}^{\text{few}}_{\text{long}}$ and $\mathsf{Bad}^{\text{few}}_{\text{short}}$ defined as

$$\begin{aligned}\mathsf{Bad}^{\text{few}}_{\text{long}} &:= \left\{\alpha\in\mathsf{Bad}^{\text{few}} \;\middle|\; C_{x_\alpha,\alpha}\geqslant 0.01k\right\},\\ \mathsf{Bad}^{\text{few}}_{\text{short}} &:= \left\{\alpha\in\mathsf{Bad}^{\text{few}} \;\middle|\; C_{x_\alpha,\alpha} < 0.01k\right\}.\end{aligned} \tag{6.76}$$

Below, we will divide into two cases according to the size of $\mathsf{Bad}^{\text{few}}_{\text{long}}$.

(Case 2-2-2-1) $|\mathsf{Bad}^{\text{few}}_{\text{long}}| \geqslant 0.2\cdot|\mathsf{Bad}^{\text{few}}|$**.** Note that $\mathsf{Bad}^{\text{few}}_{\text{short}} \leqslant 0.8\cdot|\mathsf{Bad}^{\text{few}}|$ by assumption. We first show that a certain fraction of points of ❒ are colored in $\mathsf{Bad}^{\text{few}}_{\text{long}}$.

Claim 6.21. *It holds that*

$$\left|f^{-1}(\mathsf{Bad}^{\text{few}}_{\text{long}})\cap \text{❒}\right| > 0.07\cdot\varepsilon k^2, \tag{6.77}$$

namely, more than $0.07\cdot\varepsilon k^2$ *points of* ❒ *are colored in* $\mathsf{Bad}^{\text{few}}_{\text{long}}$.

Proof. The number of points colored in $\mathsf{Bad}^{\text{few}}_{\text{short}}$ can be bounded as follows:

- The number of points colored in $\mathsf{Bad}^{\text{few}}_{\text{short}}$ that *disagree* with f^* is

$$
\begin{aligned}
N(\mathsf{Bad}^{\text{few}}_{\text{short}}) &= \sum_{\alpha\in\mathsf{Bad}^{\text{few}}_{\text{short}}} N_\alpha \\
&\underbrace{<}_{N_\alpha<1.01k} 1.01k\cdot|\mathsf{Bad}^{\text{few}}_{\text{short}}| \\
&\underbrace{\leqslant}_{|\mathsf{Bad}^{\text{few}}_{\text{short}}|\leqslant 0.8\cdot|\mathsf{Bad}|} 1.01k\cdot 0.8\cdot|\mathsf{Bad}| \\
&\underbrace{<}_{|\mathsf{Bad}|<1.02\varepsilon k} 1.01k\cdot 0.8\cdot 1.02\varepsilon k < 0.83\varepsilon k^2.
\end{aligned}
\tag{6.78}
$$

- The number of points colored in $\mathsf{Bad}^{\text{few}}_{\text{short}}$ that *agree* with f^* is

$$
\sum_{\alpha\in\mathsf{Bad}^{\text{few}}_{\text{short}}}(k-|D_\alpha|) \underbrace{<}_{|D_\alpha|>0.99k} 0.01k\cdot|\mathsf{Bad}^{\text{few}}_{\text{short}}| \underbrace{<}_{|\mathsf{Bad}|<1.02\varepsilon k} 0.01k\cdot 0.8\cdot 1.02\varepsilon k < 0.01\cdot\varepsilon k^2.
\tag{6.79}
$$

Consequently, by Claim 6.20, the number of points of $\square$ colored in $\mathsf{Bad}^{\text{few}}_{\text{long}}$ is

$$
\begin{aligned}
\left|f^{-1}(\mathsf{Bad}^{\text{few}}_{\text{long}})\cap\square\right| &= \left|f^{-1}(\mathsf{Bad}^{\text{few}})\cap\square\right| - \left(N(\mathsf{Bad}^{\text{few}}_{\text{short}}) + \sum_{\alpha\in\mathsf{Bad}^{\text{few}}_{\text{short}}}(k-|D_\alpha|)\right) \\
&> 0.91\cdot\varepsilon k^2 - (0.83\cdot\varepsilon k^2 + 0.01\cdot\varepsilon k^2) = 0.07\cdot\varepsilon k^2,
\end{aligned}
\tag{6.80}
$$

as desired. $\square$

We show that $\mathcal{V}_{\text{stripe}}$'s rejection probability is $\Omega\left(\frac{|f^{-1}(\mathsf{Bad}^{\text{few}}_{\text{long}})\cap\square|}{k^3}\right)$. See Figure 9 for illustration of its proof.

Claim 6.22. *It holds that*

$$
\mathbb{P}\left[\mathcal{V}_{\text{stripe}} \textit{ rejects } f\right] \geqslant \frac{10^{-3}}{k^3}\cdot\left|f^{-1}(\mathsf{Bad}^{\text{few}}_{\text{long}})\cap\square\right| \geqslant \frac{10^{-5}\cdot\varepsilon}{k}.
\tag{6.81}
$$

Proof. For each color $\alpha\in\mathsf{Bad}^{\text{few}}_{\text{long}}$, $\mathcal{V}_{\text{stripe}}$ rejects f by α with probability

$$
\begin{aligned}
&\mathbb{P}\left[f(X_1,Y_1)=f(X_2,Y_2)=\alpha \;\middle|\; X_1\neq X_2 \text{ and } Y_1\neq Y_2\right] \\
&\geqslant \mathbb{P}_{\substack{X_1\neq X_2\\ Y_1\neq Y_2}}\left[f(X_1,Y_1)=f(X_2,Y_2)=\alpha \text{ and } X_1=x_\alpha \text{ and } (X_2,Y_2)\in f^{-1}(\alpha)\cap\square\right] \\
&= \underbrace{\mathbb{P}_{\substack{X_1\neq X_2\\ Y_1\neq Y_2}}\left[f(X_1,Y_1)=f(X_2,Y_2)=\alpha \;\middle|\; X_1=x_\alpha \text{ and } (X_2,Y_2)\in f^{-1}(\alpha)\cap\square\right]}_{\text{(first term)}} \\
&\quad\cdot \underbrace{\mathbb{P}_{\substack{X_1\neq X_2\\ Y_1\neq Y_2}}\left[X_1=x_\alpha \text{ and } (X_2,Y_2)\in f^{-1}(\alpha)\cap\square\right]}_{\text{(second term)}}.
\end{aligned}
\tag{6.82}
$$

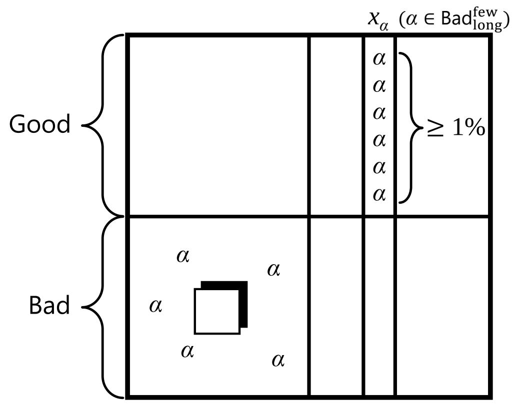


Figure 9: Illustration of the proof of Claim 6.22. Since each color α of $\mathsf{Bad}^{\mathsf{few}}_{\mathsf{long}}$ appears at least $0.01k$ times on the ${x_\alpha}^{\text{th}}$ column and $\mathsf{Bad}^{\mathsf{few}}_{\mathsf{long}}$ appears $\Omega(\varepsilon k^2)$ times in $\square$ due to Claim 6.21, we can bound the rejection probability from below.

Since $C_{x_\alpha,\alpha} \geqslant 0.01k$ by definition of $\mathsf{Bad}^{\mathsf{few}}_{\mathsf{long}}$ (i.e., there are $0.01k$ y's such that $f(x_\alpha, y) = \alpha$) and $\square$ does not contain ${x_\alpha}^{\text{th}}$ column, the first term can be bounded as follows:

$$\begin{aligned}
&\mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\left[f(X_1,Y_1) = f(X_2,Y_2) = \alpha \;\middle|\; X_1 = x_\alpha \text{ and } (X_2,Y_2) \in f^{-1}(\alpha) \cap \square\right] \\
&= \mathop{\mathbb{P}}_{\substack{X_2 \neq x_\alpha \\ Y_1 \neq Y_2}}\left[f(x_\alpha,Y_1) = \alpha \;\middle|\; (X_2,Y_2) \in f^{-1}(\alpha) \cap \square\right] \\
&\geqslant \frac{0.01k - 1}{k-1} \underbrace{>}_{k \geqslant k_0} 10^{-3}.
\end{aligned} \tag{6.83}$$

The second term can be bounded as follows:

$$\begin{aligned}
&\mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\left[X_1 = x_\alpha \text{ and } (X_2,Y_2) \in f^{-1}(\alpha) \cap \square\right] \\
&= \mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\left[X_1 = x_\alpha \;\middle|\; (X_2,Y_2) \in f^{-1}(\alpha) \cap \square\right] \cdot \mathop{\mathbb{P}}_{\substack{X_1 \neq X_2 \\ Y_1 \neq Y_2}}\left[(X_2,Y_2) \in f^{-1}(\alpha) \cap \square\right] \\
&\geqslant \frac{1}{k-1} \cdot \frac{|f^{-1}(\alpha) \cap \square|}{k^2} \geqslant \frac{|f^{-1}(\alpha) \cap \square|}{k^3}.
\end{aligned} \tag{6.84}$$

Therefore, $\mathcal{V}_{\text{stripe}}$'s rejection probability is at least

$$\begin{aligned}
\sum_{\alpha \in \mathsf{Bad}^{\mathsf{few}}_{\mathsf{long}}} \mathbb{P}\left[\mathcal{V}_{\text{stripe}} \text{ rejects } f \text{ by } \alpha\right] &\geqslant \sum_{\alpha \in \mathsf{Bad}^{\mathsf{few}}_{\mathsf{long}}} \frac{10^{-3}}{k^3} \cdot \left|f^{-1}(\alpha) \cap \square\right| \\
&= \frac{10^{-3}}{k^3} \cdot \left|f^{-1}(\mathsf{Bad}^{\mathsf{few}}_{\mathsf{long}}) \cap \square\right| \\
&\underbrace{>}_{\text{Claim 6.21}} \frac{10^{-3}}{k^3} \cdot 0.07 \cdot \varepsilon k^2 > \frac{10^{-5} \cdot \varepsilon}{k},
\end{aligned} \tag{6.85}$$

which completes the proof. $\square$

$\boxed{\textbf{(Case 2-2-2-2) } |\mathsf{Bad}^{\mathrm{few}}_{\mathrm{long}}| < 0.2 \cdot |\mathsf{Bad}^{\mathrm{few}}|\textbf{.}}$ We first show that a large majority of the points of ❐ are colored in $\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}$.

Claim 6.23. *The number of points of* ❐ *colored in* $\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}$ *that disagree with* f^* *is more than* $0.68 \cdot \varepsilon k^2$.

Proof. Observe the following:

- The number of points colored in $\mathsf{Bad}^{\mathrm{few}}_{\mathrm{long}}$ that disagree with f^* is

$$N(\mathsf{Bad}^{\mathrm{few}}_{\mathrm{long}}) \underbrace{<}_{N_\alpha < 1.01k} 1.01k \cdot |\mathsf{Bad}^{\mathrm{few}}_{\mathrm{long}}| \underbrace{<}_{|\mathsf{Bad}^{\mathrm{few}}_{\mathrm{long}}| < 0.2 |\mathsf{Bad}^{\mathrm{few}}|} 1.01k \cdot 0.2 \cdot 1.02\varepsilon k < 0.21 \cdot \varepsilon k^2. \tag{6.86}$$

- The number of points colored in $\mathsf{Bad}^{\mathrm{few}}$ that *agree* with f^* is

$$\sum_{\alpha \in \mathsf{Bad}^{\mathrm{few}}} (k - |D_\alpha|) \underbrace{<}_{|D_\alpha| > 0.99k} 0.01k \cdot |\mathsf{Bad}^{\mathrm{few}}| \underbrace{<}_{|\mathsf{Bad}^{\mathrm{few}}| \leqslant |\mathsf{Bad}|} 0.01k \cdot 1.02\varepsilon k < 0.02\varepsilon k^2. \tag{6.87}$$

Therefore, by Claim 6.20, the number of points of ❐ colored in $\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}$ that disagree with f^* is

$$\begin{aligned} &\left| f^{-1}(\mathsf{Bad}^{\mathrm{few}}) \cap \text{❐} \right| - \left(N(\mathsf{Bad}^{\mathrm{few}}_{\mathrm{long}}) + \sum_{\alpha \in \mathsf{Bad}^{\mathrm{few}}} (k - |D_\alpha|) \right) \\ &> 0.91 \cdot \varepsilon k^2 - (0.21 \cdot \varepsilon k^2 + 0.02 \cdot \varepsilon k^2) > 0.68 \cdot \varepsilon k^2, \end{aligned} \tag{6.88}$$

as desired. $\square$

Claim 6.23 implies $N(\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}) > 0.68 \cdot \varepsilon k^2$. Define

$$\widetilde{\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}} := \left\{ \alpha \in \mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}} \;\middle|\; N_\alpha \geqslant 0.01k \right\}. \tag{6.89}$$

Observe that

$$\begin{aligned} N\left(\widetilde{\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}}\right) &\geqslant N(\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}) - \sum_{\alpha \in \mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}} N_\alpha \cdot \left[\!\left[N_\alpha < 0.01k \right]\!\right] \\ &\geqslant N(\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}) - 0.01k \cdot |\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}| \\ &\geqslant N(\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}) - 0.01k \cdot |\mathsf{Bad}| \\ &\geqslant 0.68\varepsilon k^2 - 0.01k \cdot 1.02\varepsilon k \\ &> 0.66\varepsilon k^2. \end{aligned} \tag{6.90}$$

We will show that $\mathcal{V}_{\mathrm{stripe}}$'s rejection probability is $\Omega\left(\frac{1}{k^3} \cdot N\left(\widetilde{\mathsf{Bad}^{\mathrm{few}}_{\mathrm{short}}}\right)\right)$. See Figure 10 for illustration of its proof.

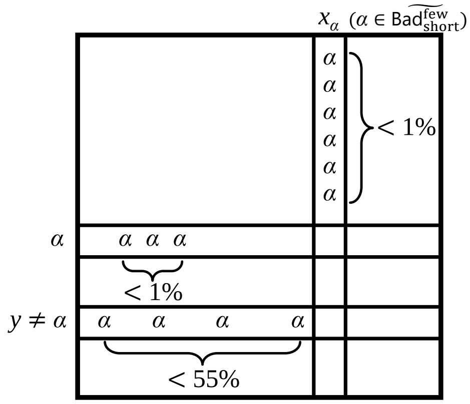


Figure 10: Illustration of the proof of Claim 6.24. Since $\alpha \in \widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}$ appears at most $0.55k$ times for any column and row and $N\left(\widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}\right) = \Omega(\varepsilon k^2)$ due to Claim 6.23, we can apply Lemma 6.16 to bound the rejection probability from below.

Claim 6.24.

$$\mathbb{P}\left[\mathcal{V}_{\text{stripe}} \textit{ rejects } f\right] \geqslant \frac{10^{-5}}{k^3} \cdot \left(N\left(\widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}\right) - 0.6k \cdot \left|\widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}\right|\right) \geqslant \frac{10^{-6} \cdot \varepsilon}{k}. \tag{6.91}$$

Proof. For each color $\alpha \in \widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}$, we claim that $R_{y,\alpha} < 0.55k$ for all $y \in [k]$. Suppose for contradiction that $R_{y^*,\alpha} \geqslant 0.55k$ for some $y^* \in [k]$. Note that $y^* \neq \alpha$ as $R_{\alpha,\alpha} = k - |D_\alpha| < 0.01k$ by definition of Bad. Since f^* is closest to f, we must have

$$\begin{aligned}
&R_{\alpha,\alpha} + R_{y^*,y^*} \geqslant R_{\alpha,y^*} + R_{y^*,\alpha} \\
&\implies R_{y^*,y^*} \geqslant R_{\alpha,y^*} + R_{y^*,\alpha} - R_{\alpha,\alpha} \geqslant R_{y^*,\alpha} - R_{\alpha,\alpha} > 0.54k \\
&\implies R_{y^*,\alpha} + R_{y^*,y^*} > k,
\end{aligned} \tag{6.92}$$

which is a contradiction.

Since $C_{x,\alpha} < 0.01k < 0.55k$ for all $x \in [k]$ by definition of $\mathsf{Bad}^{\text{few}}_{\text{short}}$, we apply Lemma 6.16 with $\vartheta = 0.55$ to every color $\alpha \in \widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}$ and derive

$$\mathbb{P}\left[\mathcal{V}_{\text{stripe}} \text{ rejects } f \text{ by } \alpha\right] \geqslant \frac{N_\alpha \cdot (N_\alpha - 0.55k)}{10^2 \cdot k^3} \geqslant \frac{0.01k \cdot (N_\alpha - 0.55k)}{10^2 \cdot k^3} = \frac{N_\alpha - 0.55k}{10^4 \cdot k^3}. \tag{6.93}$$

Therefore, $\mathcal{V}_{\text{stripe}}$'s rejection probability is at least

$$\begin{aligned}
\mathbb{P}\left[\mathcal{V}_{\text{stripe}} \text{ rejects } f\right] &\geqslant \sum_{\alpha \in \widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}} \frac{N_\alpha - 0.55k}{10^4 \cdot k^3} \\
&\geqslant \frac{10^{-4}}{k^3} \cdot \left(N\left(\widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}\right) - 0.55k \cdot \left|\widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}\right|\right) \\
&\geqslant \frac{10^{-4}}{k^3} \cdot \left(N\left(\widetilde{\mathsf{Bad}^{\text{few}}_{\text{short}}}\right) - 0.55k \cdot |\mathsf{Bad}|\right) \\
&\geqslant \frac{10^{-4}}{k^3} \cdot \left(0.66 \cdot \varepsilon k^2 - 0.55k \cdot 1.02\varepsilon k\right) > \frac{10^{-6} \cdot \varepsilon}{k},
\end{aligned} \tag{6.94}$$

which completes the proof. □

Using the claims shown so far, we eventually prove Lemma 6.7.

Proof of Lemma 6.7. By Claims 6.17 to 6.19, 6.22 and 6.24, the following hold:

- If f is ε-far from being striped for any $\varepsilon < \varepsilon_0 = 10^{-2}$, then $\mathcal{V}_{\text{stripe}}$ rejects with probability more than $\frac{10^{-6} \cdot \varepsilon}{k}$.
- If f is 10^{-2}-far from being striped, then $\mathcal{V}_{\text{stripe}}$ rejects with probability more than $\frac{10^{-6} \cdot 10^{-2}}{k}$.

Setting the rejection rate $\rho := 10^{-8}$ completes the proof. □

7 Deterministic $\left(1 - \frac{2}{k}\right)$-factor Approximation Algorithm for MAXMIN k-CUT RECONFIGURATION

In this section, we develop a deterministic $\left(1 - \frac{2}{k}\right)$-factor approximation algorithm for MAXMIN k-CUT RECONFIGURATION for every $k \geqslant 2$.

Theorem 7.1. *For every integer $k \geqslant 2$ and every real $\varepsilon > 0$, there exists a deterministic polynomial-time algorithm that given a simple graph G and a pair of its k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}}$, returns a reconfiguration sequence $\mathscr{F}$ from f_{start} to f_{end} such that*

$$\mathsf{val}_G(\mathscr{F}) \geqslant \left(1 - \frac{1}{k} - \varepsilon\right)^2 \cdot \min\left\{\mathsf{val}_G(f_{\mathsf{start}}), \mathsf{val}_G(f_{\mathsf{end}})\right\}. \tag{7.1}$$

In particular, letting $\varepsilon := \frac{1}{k^3}$, this algorithm approximates MAXMIN k-CUT RECONFIGURATION *on simple graphs within a factor of $\left(1 - \frac{1}{k} - \frac{1}{k^3}\right)^2 \geqslant 1 - \frac{2}{k}$.*

7.1 Outline of the Proof of Theorem 7.1

Our proof of Theorem 7.1 can be divided into the following three steps:

- In Section 7.2, we deal with the case that f_{start} or f_{end} has a low value, say $o(1)$. We show how to safely transform such a k-coloring into a $\frac{1}{2}$-value k-coloring.

- In Section 7.3, for a graph consisting only of "low-degree" vertices, we demonstrate that *a random reconfiguration sequence via a random k-coloring* makes $\approx \left(1-\frac{1}{k}\right)^2$-fraction of edges bichromatic with high probability, whose proof is based on the read-k Chernoff bound (Theorem 5.4).
- In Section 7.4, we handle "high-degree" vertices. Intuitively, k colors are distributed almost evenly over each high-degree vertex's neighbors with high probability.

7.2 Low-value Case

Here, we introduce the following lemma, which enables us to assume that $\mathsf{val}_G(f_{\mathsf{start}})$ and $\mathsf{val}_G(f_{\mathsf{end}})$ are at least $\frac{1}{2}$ without loss of generality.

Lemma 7.2. *For a graph $G=(V,E)$ and a k-coloring $f_{\mathsf{start}}\colon V\to[k]$ of G such that $\mathsf{val}_G(f_{\mathsf{start}}) < 1-\frac{1}{k}$, there exists a reconfiguration sequence $\mathscr{F}$ from f_{start} to another k-coloring $f'_{\mathsf{start}}\colon V\to[k]$ such that $\mathsf{val}_G(f'_{\mathsf{start}}) \geqslant 1-\frac{1}{k}$ and $\mathsf{val}_G(\mathscr{F}) = \mathsf{val}_G(f_{\mathsf{start}})$. Such $\mathscr{F}$ can be found in polynomial time.*

Proof. We first claim the following:

Claim 7.3. *For a k-coloring $f\colon V\to[k]$ with $\mathsf{val}_G(f) < 1-\frac{1}{k}$, there is another k-coloring $f'\colon V\to[k]$ such that $\mathsf{val}_G(f') > \mathsf{val}_G(f)$ and f and f' differ in a single vertex.*

Proof. We prove the contrapositive. Suppose that recoloring any single vertex does not decrease the number of monochromatic edges. Then, for every $v\in V$, $f(v)$ appears in at most $\frac{1}{k}$-fraction of v's neighbors; i.e., at least $\left(1-\frac{1}{k}\right)$-fraction of the incident edges to v must be bichromatic. This implies $\mathsf{val}_G(f) \geqslant 1-\frac{1}{k}$. □

By the above claim, until $\mathsf{val}_G(f) \geqslant 1-\frac{1}{k}$, one can find a pair of vertex $v\in V$ and color $\alpha\in[k]$ such that recoloring v to α strictly increases the number of bichromatic edges, as desired. □

7.3 Low-degree Case

We then show that if G contains only "low-degree" vertices, *a random reconfiguration sequence to a random k-coloring* makes $\approx \left(1-\frac{1}{k}\right)^2$-fraction of edges bichromatic with with high probability. For a graph $G=(V,E)$, we define $\Delta := |E|^{\frac{2}{3}}$, and we say that a vertex of G is *low degree* if its degree is less than Δ, and *high degree* otherwise.

Lemma 7.4. *Let $G=(V,E)$ be a graph of maximum degree at most $\Delta=|E|^{\frac{2}{3}}$ and $f_{\mathsf{start}}\colon V\to[k]$ be a proper k-coloring of G. Consider a uniformly random k-coloring $F\colon V\to[k]$ and a random irredundant reconfiguration sequence $\mathscr{F}$ uniformly chosen from $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow F)$. Then, it holds that*

$$\mathop{\mathbb{P}}_{\mathscr{F}}\left[\mathsf{val}_G(\mathscr{F}) < \left(1-\tfrac{1}{k}\right)^2 - |E|^{-\frac{1}{4}}\right] < \exp\left(-2\cdot|E|^{\frac{1}{12}}\right). \tag{7.2}$$

To prove Lemma 7.4, we first show that each edge consistently remains bichromatic through $\mathscr{F}$ with probability $\left(1-\frac{1}{k}\right)^2$.

Lemma 7.5. *Let $e=(v,w)$ be an edge and $f_{\mathsf{start}}\colon\{v,w\}\to[k]$ be a proper k-coloring of e. Consider a uniformly random k-coloring $F\colon\{v,w\}\to[k]$ and a random irredundant reconfiguration sequence $\mathscr{F}$*

uniformly chosen from $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow F)$. *Then,* $\mathscr{F}$ *keeps* e *bichromatic with probability at least* $\left(1-\frac{1}{k}\right)^2$*; namely,*

$$\mathbb{P}\Big[\forall f \in \mathscr{F},\, f(v) \neq f(w)\Big] \geqslant \left(1-\tfrac{1}{k}\right)^2. \tag{7.3}$$

Proof. Denote $\alpha_v := F(v)$, $\alpha_w := F(w)$, and $L := \{f_{\mathsf{start}}(v), f_{\mathsf{start}}(w)\}$. Consider the following case analysis on α_v and α_w:

(Case 1) If $\alpha_v \neq \alpha_w$ and $\{\alpha_v, \alpha_w\} \cap L = \emptyset$:
There are $(k-2)(k-3)$ such colorings in total. Observe easily that $\mathscr{F}$ succeeds with probability 1.

(Case 2) If $\alpha_v = f_{\mathsf{start}}(v)$ and $\alpha_w \notin L$:
There are $(k-2)$ such colorings. $\mathscr{F}$ succeeds with probability 1.

(Case 3) If $\alpha_w = f_{\mathsf{start}}(w)$ and $\alpha_v \notin L$:
There are $(k-2)$ such colorings. Similarly to (Case 3), $\mathscr{F}$ succeeds with probability 1.

(Case 4) If $\alpha_v = f_{\mathsf{start}}(v)$ and $\alpha_w = f_{\mathsf{start}}(w)$:
There is only one such coloring; $\mathscr{F}$ always succeeds.

(Case 5) If $\alpha_v = f_{\mathsf{start}}(w)$ and $\alpha_w \notin L$:
There are $(k-2)$ such colorings. If w is recolored at first, $\mathscr{F}$ fails; i.e., the success probability is $\frac{1}{2}$.

(Case 6) If $\alpha_w = f_{\mathsf{start}}(v)$ and $\alpha_v \notin L$:
There are $(k-2)$ such colorings. Similarly to the previous case, $\mathscr{F}$ succeeds with probability $\frac{1}{2}$.

(Case 7) If $\alpha_v = f_{\mathsf{start}}(w)$ and $\alpha_w = f_{\mathsf{start}}(v)$:
There is only one such coloring. $\mathscr{F}$ cannot succeed anyway.

(Case 8) Otherwise (i.e., $\alpha_v = \alpha_w$):
There are k such colorings; $\mathscr{F}$ never succeeds.

Summing the success probability over the preceding eight cases, we derive

$$\begin{aligned} &k^{-2} \cdot \Big((k-2)(k-3) \cdot 1 + (k-2) \cdot 1 + (k-2) \cdot 1 + 1 \cdot 1 + (k-2) \cdot \tfrac{1}{2} + (k-2) \cdot \tfrac{1}{2}\Big) \\ &= \left(1 - \frac{1}{k}\right)^2, \end{aligned} \tag{7.4}$$

completing the proof. $\square$

We now apply the read-k Chernoff bound to Lemma 7.5 and prove Lemma 7.4.

Proof of Lemma 7.4. Define $n := |V|$ and $m := |E|$. Let $\mathbf{I} = (I_v)_{v \in V}$ be a random integer sequence distributed uniformly over $[k]^V$ and $f_{\mathbf{I}} \colon V \to [k]$ denote a k-coloring of G such that $f_{\mathbf{I}}(v) := I_v$ for all $v \in V$. Let $\mathbf{R} = (R_v)_{v \in V}$ be a random real sequence distributed uniformly over $(0,1)^V$ and $\sigma_{\mathbf{R}} \colon [n] \to V$ denote an ordering of V such that $R_{\sigma(1)} > R_{\sigma(2)} > \cdots > R_{\sigma(n)}$. Let $\mathscr{F}(\mathbf{I}, \mathbf{R}; f_{\mathsf{start}})$ be a random irredundant reconfiguration sequence from f_{start} to $f_{\mathbf{I}}$ obtained by recoloring vertex $\sigma(i)$ from $f_{\mathsf{start}}(\sigma(i))$ to $f_{\mathbf{I}}(\sigma(i))$ (if $f_{\mathsf{start}}(\sigma(i)) \neq f_{\mathbf{I}}(\sigma(i))$) in the order of $\sigma(1), \ldots, \sigma(n)$. Observe easily that $\mathscr{F}(\mathbf{I}, \mathbf{R}; f_{\mathsf{start}})$ is uniformly distributed over $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow F)$.

For each edge $e = (v, w)$ of G, let Y_e be a random variable that takes 1 if e is bichromatic throughout $\mathscr{F}(\mathbf{I}, \mathbf{R}; f_{\mathsf{start}})$ and takes 0 otherwise; namely,

$$Y_e := \left[\!\!\left[\forall f \in \mathscr{F}(\mathbf{I}, \mathbf{R}; f_{\mathsf{start}}),\ f(v) \neq f(w) \right]\!\!\right]. \tag{7.5}$$

Note that each Y_e is Boolean and depends only on I_v, I_w, R_v, and R_w; thus, Y_e's are a read-Δ family. Let Y be the sum of Y_e over all edges e; namely,

$$Y := \sum_{e \in E} Y_e. \tag{7.6}$$

Since $\mathscr{F}(\mathbf{I}, \mathbf{R}; f_{\mathsf{start}})$ is distributed uniformly over $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow F)$, by Lemma 7.5, it holds that

$$\mathbb{E}[Y] \geqslant \left(1 - \tfrac{1}{k}\right)^2 \cdot m. \tag{7.7}$$

By applying the read-k Chernoff bound (Theorem 5.4) to Y_e's, we derive

$$\mathbb{P}\left[Y \leqslant \mathbb{E}[Y] - m^{\frac{3}{4}}\right] \leqslant \exp\left(-\frac{2 \cdot m^{\frac{3}{4}}}{\Delta}\right) \underbrace{<}_{\Delta = m^{\frac{2}{3}}} \exp\left(-2 \cdot m^{\frac{1}{12}}\right), \tag{7.8}$$

which completes the proof. □

7.4 Handling High-degree Vertices

We now handle high-degree vertices and show the following using Lemma 7.4.

Proposition 7.6. *Let $G = (V, E)$ be a simple graph such that $|E| \geqslant 10^6$, and $f_{\mathsf{start}} \colon V \to [k]$ be a k-coloring of G such that $\mathsf{val}_G(f_{\mathsf{start}}) \geqslant \frac{1}{2}$. Let $V_{\leqslant \Delta}$ and $V_{>\Delta}$ be the set of low-degree and high-degree vertices, respectively, where $\Delta = |E|^{\frac{2}{3}}$.*

Consider a uniformly random k-coloring $F \colon V \to [k]$ and a random reconfiguration sequence $\mathscr{F}$ from f_{start} to F uniformly chosen from $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow \check{f} \leftrightsquigarrow F)$, where $\check{f}$ agrees with f_{start} on $V_{>\Delta}$ and with F on $V_{\leqslant \Delta}$; namely,

$$\check{f}(v) := \begin{cases} f_{\mathsf{start}}(v) & \text{if } v \in V_{>\Delta}, \\ F(v) & \text{if } v \in V_{\leqslant \Delta}. \end{cases} \tag{7.9}$$

Then, it holds that

$$\mathbb{P}_{\mathscr{F}}\left[\mathsf{val}_G(\mathscr{F}) < \left(1 - \tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f_{\mathsf{start}}) - 5 \cdot |E|^{-\frac{1}{4}}\right] < \exp\left(-\Omega\left(k^{-5} \cdot |E|^{\frac{1}{24}}\right)\right). \tag{7.10}$$

Define $n := |V|$, $m := |E|$, and

$$\begin{aligned} V_{\leqslant \Delta} &:= \left\{ v \in V \;\middle|\; d_G(v) \leqslant \Delta \right\}, \\ V_{>\Delta} &:= \left\{ v \in V \;\middle|\; d_G(v) > \Delta \right\}. \end{aligned} \tag{7.11}$$

Partition G into the following three subgraphs:

$$\begin{aligned} G_1 &\coloneqq G[V_{\leqslant\Delta}], & m_1 &\coloneqq |E(G_1)|, \\ G_2 &\coloneqq G - (E(G[V_{\leqslant\Delta}]) \cup E(G[V_{>\Delta}])), & m_2 &\coloneqq |E(G_2)|, \\ G_3 &\coloneqq G[V_{>\Delta}], & m_3 &\coloneqq |E(G_3)|. \end{aligned} \tag{7.12}$$

Roughly speaking, G_1 is the subgraph of G induced by $V_{\leqslant\Delta}$, G_3 is the subgraph of G induced by $V_{>\Delta}$, and G_2 is the subgraph of G obtained by leaving only the edges connecting between G_1 and G_2. Note that the union of $E(G_1)$, $E(G_2)$, and $E(G_3)$ is equal to E. Since $|V_{>\Delta}| \cdot \Delta < 2m$, it holds that $|V_{>\Delta}| < 2 \cdot m^{\frac{1}{3}}$; thus, $m_3 \leqslant |V_{>\Delta}|^2 < 4 \cdot m^{\frac{2}{3}}$.

Let m'_1, m'_2, and m'_3 denote the number of bichromatic edges in G_1, G_2, and G_3, with respect to f_{start}, respectively; namely,

$$\begin{aligned} m'_1 &\coloneqq m_1 \cdot \mathsf{val}_{G_1}(f_{\mathsf{start}}), \\ m'_2 &\coloneqq m_2 \cdot \mathsf{val}_{G_2}(f_{\mathsf{start}}), \\ m'_3 &\coloneqq m_3 \cdot \mathsf{val}_{G_3}(f_{\mathsf{start}}). \end{aligned} \tag{7.13}$$

Note that $m'_1 + m'_2 + m'_3 = m \cdot \mathsf{val}_G(f_{\mathsf{start}}) \geqslant \frac{m}{2}$ by assumption.

We first demonstrate that the number of edges of G_2 that are bichromatic throughout $\mathscr{F}$, i.e., $m_2 \cdot \mathsf{val}_{G_2}(\mathscr{F})$, is at least $\left(1 - \frac{1}{k}\right)^2 \cdot m'_2$ with high probability.

Lemma 7.7. *It holds that*

$$\mathbb{P}\Big[m_2 \cdot \mathsf{val}_{G_2}(\mathscr{F}) < \left(1 - \tfrac{1}{k}\right)^2 \cdot m'_2\Big] < 4 \cdot m^{\frac{2}{3}} \cdot \exp\left(-\frac{m'_2}{48k^5 \cdot m^{\frac{1}{3}}}\right). \tag{7.14}$$

Proof. Fix a high-degree vertex $v \in V_{>\Delta}$. Let X_v denote a random variable for the maximum number of monochromatic edges between v and $V_{\leqslant\Delta}$, where the maximum is taken over all k-colorings of $\mathscr{F}$; namely,

$$X_v \coloneqq \max_{f \in \mathscr{F}} \left\{ \sum_{w \in \mathcal{N}_G(v) \cap V_{\leqslant\Delta}} \Big[\!\!\Big[f(v) = f(w) \Big]\!\!\Big] \right\}. \tag{7.15}$$

Observe easily that

$$m_2 \cdot \mathsf{val}_{G_2}(\mathscr{F}) \geqslant \sum_{v \in V_{>\Delta}} \Big(|\mathcal{N}_G(v) \cap V_{\leqslant\Delta}| - X_v \Big). \tag{7.16}$$

Let M_v and B_v be the set of vertices $w \in \mathcal{N}_G(v) \cap V_{\leqslant\Delta}$ such that (v,w) is monochromatic and bichromatic on f_{start}, respectively; namely,

$$M_v \coloneqq \Big\{ w \in \mathcal{N}_G(v) \cap V_{\leqslant\Delta} \;\Big|\; f_{\mathsf{start}}(v) = f_{\mathsf{start}}(w) \Big\}, \tag{7.17}$$

$$B_v \coloneqq \Big\{ w \in \mathcal{N}_G(v) \cap V_{\leqslant\Delta} \;\Big|\; f_{\mathsf{start}}(v) \neq f_{\mathsf{start}}(w) \Big\}. \tag{7.18}$$

Note that M_v and B_v form a partition of $\mathcal{N}_G(v) \cap V_{\leqslant\Delta}$, and

$$\sum_{v \in V_{\leqslant\Delta}} |B_v| = m'_2. \tag{7.19}$$

Since f_{start} makes all edges of B_v bichromatic, and $\mathscr{F}$ would recolor v after recoloring all vertices of B_v, we derive

$$\begin{aligned}
X_v &\leqslant |M_v| + \max\left\{ \sum_{w \in B_v} \left[\!\left[f_{\mathsf{start}}(v) = F(w) \right]\!\right], \sum_{w \in B_v} \left[\!\left[F(v) = F(w) \right]\!\right] \right\} \\
&\leqslant |M_v| + \max_{\alpha \in [k]} \left\{ \sum_{w \in B_v} \left[\!\left[\alpha = F(w) \right]\!\right] \right\} \\
&\leqslant |M_v| + \max_{\alpha \in [k]} \left| \left\{ w \in B_v \;\middle|\; F(w) = \alpha \right\} \right| \\
&= |M_v| + \max_{\alpha \in [k]} \{Y_\alpha\},
\end{aligned} \tag{7.20}$$

where $Y_1, \ldots, Y_k$ are random variables that follow the multinomial distribution with $|B_v|$ trials and event probabilities $\underbrace{\frac{1}{k}, \ldots, \frac{1}{k}}_{k \text{ times}}$. Here, the maximum of Y_i's may exceed $\left(\frac{1}{k} + \varepsilon\right)n$ with small probability, which will be proved later.

Claim 7.8. *For k random variables, $Y_1, \ldots, Y_k$, that follow the multinomial distribution with n trials and event probabilities $\underbrace{\frac{1}{k}, \ldots, \frac{1}{k}}_{k \text{ times}}$, it holds that for any $\varepsilon > 0$,*

$$\mathbb{P}\left[\max_{1 \leqslant i \leqslant k} \{Y_i\} > \left(\frac{1}{k} + \varepsilon\right) \cdot n \right] < n \cdot \exp\left(-\frac{\varepsilon^2 kn}{3} \right). \tag{7.21}$$

By taking a union bound of Claim 7.8 over all $v \in V_{>\Delta}$ *such that* $|B_v| \geqslant \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|}$, with probability at least $1 - |V_{>\Delta}|^2 \cdot \exp\left(-\frac{\varepsilon^2 k}{3} \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|} \right)$, it holds that

$$X_v \leqslant |M_v| + \left(\frac{1}{k} + \varepsilon\right) \cdot |B_v| \text{ for all } v \in V_{>\Delta} \text{ such that } |B_v| \geqslant \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|}. \tag{7.22}$$

In such a case, we derive

$$
\begin{aligned}
m_2 \cdot \mathsf{val}_{G_2}(\mathscr{F}) &\geqslant \sum_{v \in V_{>\Delta}} \Big(|\mathcal{N}_G(v) \cap V_{\leqslant \Delta}| - X_v \Big) \\
&\geqslant \sum_{v \in V_{>\Delta} : |B_v| \geqslant \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|}} \Big(|M_v| + |B_v| - X_v \Big) \\
&\geqslant \sum_{v \in V_{>\Delta} : |B_v| \geqslant \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|}} \Big(|M_v| + |B_v| - \Big(|M_v| + \big(\tfrac{1}{k} + \varepsilon\big) \cdot |B_v| \Big) \Big) \\
&= \big(1 - \tfrac{1}{k} - \varepsilon\big) \cdot \sum_{v \in V_{>\Delta} : |B_v| \geqslant \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|}} |B_v| \\
&= \big(1 - \tfrac{1}{k} - \varepsilon\big) \cdot \left(\sum_{v \in V_{>\Delta}} |B_v| - \sum_{v \in V_{>\Delta} : |B_v| < \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|}} |B_v| \right) \\
&\geqslant \big(1 - \tfrac{1}{k} - \varepsilon\big) \cdot (1 - \varepsilon) \cdot m_2'.
\end{aligned}
\tag{7.23}
$$

Setting finally $\varepsilon := \frac{1}{2k^2}$, we have

$$
m_2 \cdot \mathsf{val}_{G_2}(\mathscr{F}) \geqslant \big(1 - \tfrac{1}{k} - \varepsilon\big) \cdot (1 - \varepsilon) \cdot m_2' \geqslant \big(1 - \tfrac{1}{k}\big)^2 \cdot m_2' \tag{7.24}
$$

with probability at least

$$
1 - |V_{>\Delta}|^2 \cdot \exp\left(-\frac{\varepsilon^2 k}{3} \frac{\varepsilon \cdot m_2'}{|V_{>\Delta}|} \right) \geqslant 1 - 4 \cdot m^{\frac{2}{3}} \cdot \exp\left(-\frac{m_2'}{48k^5 \cdot m^{\frac{1}{3}}} \right), \tag{7.25}
$$

where we used the fact that $|V_{>\Delta}| < 2 \cdot m^{\frac{1}{3}}$, as desired. □

Proof of Claim 7.8. It is sufficient to bound $\mathbb{P}[Y_i > (\frac{1}{k} + \varepsilon)n]$. Since Y_i follows a binomial distribution with n trials and event probability $\frac{1}{k}$, which has mean $\mathbb{E}[Y_i] = \frac{n}{k}$, we apply the Chernoff bound (Theorem 5.2) to obtain

$$
\mathbb{P}\Big[Y_i \geqslant (1 + k\varepsilon)\, \mathbb{E}[Y_i] \Big] \leqslant \exp\left(-\frac{k^2 \varepsilon^2 \cdot \mathbb{E}[Y_i]}{3} \right) = \exp\left(-\frac{\varepsilon^2 k n}{3} \right). \tag{7.26}
$$

Taking a union bound over Y_i's accomplishes the proof. □

We are now ready to prove Proposition 7.6.

Proof of Proposition 7.6. Since $\mathscr{F}$ is uniformly distributed over $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow \check{f})$, we apply Lemma 7.4 to the subgraph of G_1 induced by bichromatic edges with respect to f_{start} and obtain

$$
\mathbb{P}\left[m_1 \cdot \mathsf{val}_{G_1}(\mathscr{F}) < \big(1 - \tfrac{1}{k}\big)^2 \cdot m_1' - m_1'^{\frac{3}{4}} \right] < \exp\left(-2 \cdot m_1'^{\frac{1}{12}} \right). \tag{7.27}
$$

By Lemma 7.7, we have

$$\mathbb{P}\left[m_2 \cdot \mathsf{val}_{G_2}(\mathscr{F}) < \left(1-\tfrac{1}{k}\right)^2 \cdot m_2'\right] < 4 \cdot m^{\frac{2}{3}} \cdot \exp\left(-\frac{m_2'}{48k^5 \cdot m^{\frac{1}{3}}}\right). \tag{7.28}$$

We proceed by a case analysis on m_1' and m_2':

(Case 1) if $m_1' \leqslant m^{\frac{1}{2}}$:
Since $m_1' + m_3' \leqslant m^{\frac{1}{2}} + 4 \cdot m^{\frac{2}{3}} \leqslant 5 \cdot m^{\frac{2}{3}}$, we have

$$\begin{aligned} m_2' &= (m_1' + m_2' + m_3') - (m_1' + m_3') \\ &\geqslant m \cdot \mathsf{val}_G(f) - 5 \cdot m^{\frac{2}{3}} \\ &\geqslant \tfrac{1}{2} \cdot m - 5 \cdot m^{\frac{2}{3}} \underbrace{\geqslant}_{m \geqslant 10^6} \tfrac{1}{4} \cdot m. \end{aligned} \tag{7.29}$$

Therefore, with probability at least

$$1 - \text{Eq. (7.28)} = 1 - 4 \cdot m^{\frac{2}{3}} \cdot \exp\left(-\frac{m_2'}{48k^5 \cdot m^{\frac{1}{3}}}\right) \geqslant 1 - 4 \cdot m^{\frac{2}{3}} \cdot \exp\left(-\frac{m^{\frac{2}{3}}}{192k^5}\right), \tag{7.30}$$

we derive

$$\begin{aligned} m \cdot \mathsf{val}_G(\mathscr{F}) &\geqslant m_2 \cdot \mathsf{val}_{G_2}(\mathscr{F}) \\ &\geqslant m_2' \cdot \left(1-\tfrac{1}{k}\right)^2 \\ &= (m_1' + m_2' + m_3') \cdot \left(1-\tfrac{1}{k}\right)^2 - (m_1' + m_3') \cdot \left(1-\tfrac{1}{k}\right)^2 \\ &\geqslant m \cdot \mathsf{val}_G(f_{\mathsf{start}}) \cdot \left(1-\tfrac{1}{k}\right)^2 - 5 \cdot m^{\frac{2}{3}} \end{aligned} \tag{7.31}$$

$$\implies \mathsf{val}_G(\mathscr{F}) \geqslant \left(1-\tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f_{\mathsf{start}}) - 5 \cdot m^{-\frac{1}{3}}. \tag{7.32}$$

(Case 2) if $m_2' \leqslant m^{\frac{1}{2}}$:
Since $m_2' + m_3' \leqslant m^{\frac{1}{2}} + 4 \cdot m^{\frac{2}{3}} \leqslant 5 \cdot m^{\frac{2}{3}}$, we have

$$\begin{aligned} m_1' &= (m_1' + m_2' + m_3') - (m_2' + m_3') \\ &\geqslant m \cdot \mathsf{val}_G(f) - 5 \cdot m^{\frac{2}{3}} \\ &\geqslant \tfrac{1}{2} \cdot m - 5 \cdot m^{\frac{2}{3}} \underbrace{\geqslant}_{m \geqslant 10^6} \frac{m}{4}. \end{aligned} \tag{7.33}$$

Thus, with probability at least

$$1 - \text{Eq. (7.27)} = 1 - \exp\left(-2 \cdot m_1'^{\frac{1}{12}}\right) \geqslant 1 - \exp\left(-m^{\frac{1}{12}}\right), \tag{7.34}$$

we get

$$\begin{aligned}
m \cdot \mathsf{val}_G(\mathscr{F}) &\geqslant m_1 \cdot \mathsf{val}_{G_1}(\mathscr{F}) \\
&\geqslant m_1' \cdot \left(1-\tfrac{1}{k}\right)^2 - m_1'^{\frac{3}{4}} \\
&= (m_1'+m_2'+m_3') \cdot \left(1-\tfrac{1}{k}\right)^2 - (m_2'+m_3') \cdot \left(1-\tfrac{1}{k}\right)^2 - m_1'^{\frac{3}{4}} \\
&\geqslant m \cdot \mathsf{val}_G(f_{\mathsf{start}}) \cdot \left(1-\tfrac{1}{k}\right)^2 - 6 \cdot m^{\frac{3}{4}}
\end{aligned} \tag{7.35}$$

$$\implies \mathsf{val}_G(\mathscr{F}) \geqslant \left(1-\tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f_{\mathsf{start}}) - 6 \cdot m^{-\frac{1}{4}}. \tag{7.36}$$

(Case 3) if $m_1' > m^{\frac{1}{2}}$ and $m_2' > m^{\frac{1}{2}}$:
Taking a union bound, with probability at least

$$\begin{aligned}
1 - \text{Eq. (7.27)} - \text{Eq. (7.28)} &\geqslant 1 - 4 \cdot m^{\frac{2}{3}} \cdot \exp\left(-\frac{m_2'}{48k^5 \cdot m^{\frac{1}{3}}}\right) - \exp\left(-2 \cdot m_1'^{\frac{1}{12}}\right) \\
&\geqslant 1 - 4 \cdot m^{\frac{2}{3}} \cdot \exp\left(-\frac{m^{\frac{1}{6}}}{48k^5}\right) - \exp\left(-2 \cdot m^{\frac{1}{24}}\right),
\end{aligned} \tag{7.37}$$

we have

$$\begin{aligned}
m \cdot \mathsf{val}_G(\mathscr{F}) &\geqslant m_1' \cdot \mathsf{val}_{G_1}(\mathscr{F}) + m_2' \cdot \mathsf{val}_{G_2}(\mathscr{F}) \\
&\geqslant m_1' \cdot \left(1-\tfrac{1}{k}\right)^2 - m_1'^{\frac{3}{4}} + m_2' \cdot \left(1-\tfrac{1}{k}\right)^2 \\
&\geqslant (m_1'+m_2'+m_3') \cdot \left(1-\tfrac{1}{k}\right)^2 - m_1'^{\frac{3}{4}} - m_3' \cdot \left(1-\tfrac{1}{k}\right)^2 \\
&\underbrace{\geqslant}_{m_3' \leqslant 4 \cdot m^{\frac{2}{3}}} m \cdot \mathsf{val}_G(f_{\mathsf{start}}) \cdot \left(1-\tfrac{1}{k}\right)^2 - m^{\frac{3}{4}} - 4 \cdot m^{\frac{2}{3}} \\
&\geqslant m \cdot \left(1-\tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f_{\mathsf{start}}) - 5 \cdot m^{\frac{3}{4}}
\end{aligned} \tag{7.38}$$

$$\implies \mathsf{val}_G(\mathscr{F}) \geqslant \left(1-\tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f_{\mathsf{start}}) - 5 \cdot m^{-\frac{1}{4}}. \tag{7.39}$$

Consequently, in either case, it holds that

$$\mathbb{P}\left[\mathsf{val}_G(\mathscr{F}) \geqslant \left(1-\tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f) - 6 \cdot m^{-\frac{1}{4}}\right] \geqslant 1 - \exp\left(-\Omega\left(k^{-5} \cdot m^{\frac{1}{24}}\right)\right), \tag{7.40}$$

which completes the proof. □

7.5 Putting Them Together: Proof of Theorem 7.1

We eventually conclude the proof of Theorem 7.1 using Proposition 7.6.

Proof of Theorem 7.1. Let $G = (V,E)$ be a simple graph on m edges and $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [k]$ be a pair of its k-colorings. Let $\varepsilon > 0$ be any small real. Let $m_0 \colon \mathbb{N} \to \mathbb{N}$ be a function such that $m_0(k) = \Theta\left(k^{240}\right)$. If $m < m_0(k)$, any optimal reconfiguration sequence from f_{start} to f_{start} can be found by running a brute-force search, which completes in $m^{\Theta(k)} = k^{\Theta(k)}$ time.

Hereafter, we assume $m > m_0(k)$. If $\mathsf{val}_G(f_{\mathsf{start}}) < \frac{1}{2}$, we use Lemma 7.2 to replace f_{start} by f'_{start} such that $\mathsf{val}_G(f'_{\mathsf{start}}) \geqslant \frac{1}{2}$ and $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) \geqslant \mathsf{val}_G(f_{\mathsf{start}})$. A similar proprocessing can be applied to f_{end} whenever $\mathsf{val}_G(f_{\mathsf{end}}) < \frac{1}{2}$. So, we can safely assume that $\mathsf{val}_G(f_{\mathsf{start}}) \geqslant \frac{1}{2}$ and $\mathsf{val}_G(f_{\mathsf{end}}) \geqslant \frac{1}{2}$.

Define $\Delta := m^{\frac{2}{3}}$, and define $V_{\leqslant\Delta}$ and $V_{>\Delta}$ by Eq. (7.11). Consider a uniformly random k-coloring $F : V \to [k]$ of G and a random reconfiguration sequence $\mathscr{F} := \mathscr{F}_1 \circ \mathscr{F}_2$ from f_{start} to f_{end} obtained by concatenating two reconfiguration sequences $\mathscr{F}_1 \sim \mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow \check{f}_{\mathsf{start}} \leftrightsquigarrow F)$ and $\mathscr{F}_2 \sim \mathbb{F}(F \leftrightsquigarrow \check{f}_{\mathsf{end}} \leftrightsquigarrow f_{\mathsf{end}})$. Here, $\check{f}_{\mathsf{start}}$ agrees with f_{start} on $V_{>\Delta}$ and with F on $V_{\leqslant\Delta}$, and $\check{f}_{\mathsf{end}}$ agrees with f_{end} on $V_{>\Delta}$ and with F on $V_{\leqslant\Delta}$; namely,

$$\check{f}_{\mathsf{start}}(v) := \begin{cases} f_{\mathsf{start}}(v) & \text{if } v \in V_{>\Delta}, \\ F(v) & \text{if } v \in V_{\leqslant\Delta}, \end{cases} \tag{7.41}$$

$$\check{f}_{\mathsf{end}}(v) := \begin{cases} f_{\mathsf{end}}(v) & \text{if } v \in V_{>\Delta}, \\ F(v) & \text{if } v \in V_{\leqslant\Delta}. \end{cases} \tag{7.42}$$

Such $\mathscr{F}$ can be generated by the following randomized procedure:

Generating a random reconfiguration sequence $\mathscr{F}$ from f_{start} to f_{end}.

Input: a simple graph $G = (V, E)$ and two k-colorings $f_{\mathsf{start}}, f_{\mathsf{end}} : V \to [k]$ of G.

let $\Delta := |E|^{\frac{2}{3}}$, $V_{\leqslant\Delta} := \{v \in V \mid d_G(v) \leqslant \Delta\}$, and $V_{>\Delta} := \{v \in V \mid d_G(v) > \Delta\}$.
sample a k-coloring $F : V \to [k]$ uniformly at random.
sample an ordering $\sigma_{\leqslant\Delta}$ over $V_{\leqslant\Delta}$ uniformly at random.
sample an ordering $\sigma_{>\Delta}$ over $V_{>\Delta}$ uniformly at random.
for each vertex $v \in V_{\leqslant\Delta}$ in order of $\sigma_{\leqslant\Delta}$ **do** ▷ *reconfiguration from f_{start} to $\check{f}_{\mathsf{start}}$*
  if $f_{\mathsf{start}}(v) \neq F(v)$ **then**
    recolor v from $f_{\mathsf{start}}(v)$ to $F(v)$.
for each vertex $v \in V_{>\Delta}$ in order of $\sigma_{>\Delta}$ **do** ▷ *reconfiguration from $\check{f}_{\mathsf{start}}$ to F*
  if $f_{\mathsf{start}}(v) \neq F(v)$ **then**
    recolor v from $f_{\mathsf{start}}(v)$ to $F(v)$.
for each vertex $v \in V_{>\Delta}$ in order of $\sigma_{>\Delta}$ **do** ▷ *reconfiguration from F to $\check{f}_{\mathsf{end}}$*
  if $F(v) \neq f_{\mathsf{end}}(v)$ **then**
    recolor v from $F(v)$ to $f_{\mathsf{end}}(v)$.
for each vertex $v \in V_{\leqslant\Delta}$ in order of $\sigma_{\leqslant\Delta}$ **do** ▷ *reconfiguration from $\check{f}_{\mathsf{end}}$ to f_{end}*
  if $F(v) \neq f_{\mathsf{end}}(v)$ **then**
    recolor v from $F(v)$ to $f_{\mathsf{end}}(v)$.

By applying Proposition 7.6 on $\mathscr{F}_1$ and $\mathscr{F}_2$ and taking a union bound, we have

$$\begin{aligned} &\mathbb{P}\Big[\mathsf{val}_G(\mathscr{F}) < \left(1 - \tfrac{1}{k}\right)^2 \cdot \min\Big\{\mathsf{val}_G(f_{\mathsf{start}}), \mathsf{val}_G(f_{\mathsf{end}})\Big\} - 5 \cdot m^{-\frac{1}{4}}\Big] \\ &\leqslant \mathbb{P}\Big[\mathsf{val}_G(\mathscr{F}_1) < \left(1 - \tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f_{\mathsf{start}}) - 5 \cdot m^{-\frac{1}{4}}\Big] \\ &+ \mathbb{P}\Big[\mathsf{val}_G(\mathscr{F}_2) < \left(1 - \tfrac{1}{k}\right)^2 \cdot \mathsf{val}_G(f_{\mathsf{end}}) - 5 \cdot m^{-\frac{1}{4}}\Big] \\ &< 2 \cdot \exp\left(-\Omega\left(k^{-5} \cdot m^{\frac{1}{24}}\right)\right) < \exp\left(-\Omega\left(m^{\frac{1}{48}}\right)\right) && \text{since } k^{-5} = \Omega\left(m^{-\frac{1}{48}}\right). \end{aligned} \tag{7.43}$$

In particular, we have

$$\begin{aligned}\mathbb{E}\Big[\mathsf{val}_G(\mathscr{F})\Big] &\geqslant \left(\left(1-\tfrac{1}{k}\right)^2 \cdot \min\Big\{\mathsf{val}_G(f_{\mathsf{start}}), \mathsf{val}_G(f_{\mathsf{end}})\Big\} - 5\cdot m^{-\frac{1}{4}}\right)\cdot\left(1-\exp\left(-\Omega\left(m^{\frac{1}{48}}\right)\right)\right) \\ &\geqslant \left(1-\tfrac{1}{k}\right)^2 \cdot \min\Big\{\mathsf{val}_G(f_{\mathsf{start}}), \mathsf{val}_G(f_{\mathsf{end}})\Big\} - O\left(m^{-\frac{1}{4}}\right) \\ &\geqslant \left(1-\tfrac{1}{k}-\varepsilon\right)^2 \cdot \min\Big\{\mathsf{val}_G(f_{\mathsf{start}}), \mathsf{val}_G(f_{\mathsf{end}})\Big\}.\end{aligned} \tag{7.44}$$

where the last inequality holds for all sufficiently large m. Hence, we can apply the method of conditional expectations [AS16] to the aforementioned randomized procedure to construct a reconfiguration sequence $\mathscr{F}^*$ such that

$$\mathsf{val}_G(\mathscr{F}^*) \geqslant \left(1-\tfrac{1}{k}-\varepsilon\right)^2 \cdot \min\Big\{\mathsf{val}_G(f_{\mathsf{start}}), \mathsf{val}_G(f_{\mathsf{end}})\Big\} \tag{7.45}$$

in deterministic polynomial time, which accomplishes the proof. □

7.6 A Simple $\left(1-\frac{9}{k}\right)$-factor Approximation Algorithm

For the sake of completeness, we give a simple $\left(1-\frac{9}{k}\right)$-factor approximation algorithm for MAXMIN k-CUT RECONFIGURATION.

Observation 7.9. *Let $G=(V,E)$ be a graph and $f_{\mathsf{start}}, f_{\mathsf{end}}\colon V\to[k]$ be a pair of its proper k-colorings. Consider a uniformly random k-coloring $F\colon V\to[k]$ and a random irredundant reconfiguration sequence $\mathscr{F}$ uniformly chosen from $\mathbb{F}(f_{\mathsf{start}} \leftrightsquigarrow F \leftrightsquigarrow f_{\mathsf{end}})$. Then, it holds that*

$$\mathbb{E}\Big[\mathsf{val}_G(\mathscr{F})\Big] \geqslant 1-\tfrac{9}{k}. \tag{7.46}$$

In particular, $\mathscr{F}$ is a $\left(1-\frac{9}{k}\right)$-factor approximation for MAXMIN k-CUT RECONFIGURATION *in expectation.*

Proof. It is sufficient to show that for each edge $e=(v,w)$ of G,

$$\mathbb{P}\Big[\forall f\in\mathscr{F},\ f(v)\neq f(w)\Big] \geqslant 1-\tfrac{9}{k}. \tag{7.47}$$

Define

$$L := \Big\{f_{\mathsf{start}}(v), f_{\mathsf{start}}(w), f_{\mathsf{end}}(v), f_{\mathsf{end}}(w)\Big\}. \tag{7.48}$$

Conditioned on the event that

- $\{F(v),F(w)\}\cap L=\emptyset$, and
- $F(v)\neq F(w)$,

$\mathscr{F}$ always keeps e bichromatic. Since there are k^2 possible k-colorings of (v,w), the desired event occurs with probability at least

$$\frac{(k-|L|)^2-(k-|L|)}{k^2} \geqslant 1-\frac{2|L|+1}{k} \geqslant 1-\tfrac{9}{k}, \tag{7.49}$$

which completes the proof. □

A Omitted Proofs in Section 6

A.1 Proof of Proposition 6.2

In this subsection, we prove Proposition 6.2, i.e., PSPACE-hardness of approximating MAXMIN 2-CUT RECONFIGURATION. Let us begin with PSPACE-hardness of approximating MAXMIN 6-CUT RECONFIGURATION, which is immediate from [BC09, HO24b, Ohs23] and Lemma 6.4.

Lemma A.1. *There exist a universal constant $\varepsilon_0 \in (0,1)$ such that* GAP$_{1,1-\varepsilon_0}$ 6-CUT RECONFIGURATION *is* PSPACE*-hard. Moreover, this same hardness result holds even if the maximum degree of input graphs is bounded by some constant $\Delta \in \mathbb{N}$.*

Proof. By the PCRP theorem of Hirahara and Ohsaka [HO24b] and a series of gap-preserving reductions of Ohsaka [Ohs23], GAP$_{1,1-\varepsilon}$ NONDETERMINISTIC CONSTRAINT LOGIC is PSPACE-hard for some constant $\varepsilon \in (0,1)$. Since a polynomial-time reduction from NONDETERMINISTIC CONSTRAINT LOGIC to 4-COLORING RECONFIGURATION due to Bonsma and Cereceda [BC09] is indeed gap-preserving, GAP$_{1,1-\Omega(\varepsilon)}$ 4-CUT RECONFIGURATION on graphs of maximum degree 5 is PSPACE-hard. Lastly, by Lemma 6.4, GAP$_{1,1-\Omega(\varepsilon)}$ 6-CUT RECONFIGURATION on graphs of maximum degree $O(1)$ is PSPACE-hard, completing the proof. □

Hereafter, we present a gap-preserving reduction from MAXMIN 6-CUT RECONFIGURATION to MAXMIN 2-CUT RECONFIGURATION, which along with Lemma A.1 implies Proposition 6.2.

Lemma A.2. *For every real $\varepsilon \in (0,1)$ and every integer $\Delta \in \mathbb{N}$, there exists a gap-preserving reduction from* GAP$_{1,1-\varepsilon}$ 6-CUT RECONFIGURATION *on graphs of maximum degree Δ to* GAP$_{1-\delta_c,1-\delta_s}$ 2-CUT RECONFIGURATION *on graphs of maximum degree $O(\Delta)$, where*

$$\delta_c := \frac{19+\frac{\varepsilon}{2}}{54} \text{ and } \delta_s := \frac{19+\varepsilon}{54}. \tag{A.1}$$

Reduction. Our reduction from GAP$_{1,1-\varepsilon}$ 6-CUT RECONFIGURATION to GAP$_{1-\delta_c,1-\delta_s}$ 2-CUT RECONFIGURATION is described below. Fix $\varepsilon \in (0,1)$ and $\Delta \in \mathbb{N}$. Let $(G, f_{\mathsf{start}}, f_{\mathsf{end}})$ be an instance of GAP$_{1,1-\varepsilon}$ 6-CUT RECONFIGURATION, where $G = (V,E)$ is a graph of maximum degree Δ, and $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [6]$ are a pair of its (proper) 6-colorings. We construct an instance $(H, f'_{\mathsf{start}}, f'_{\mathsf{end}})$ of MAXMIN 2-CUT RECONFIGURATION as follows. For each vertex v of G, create a set of four fresh vertices, denoted $Z_v := \{z_{v,1}, z_{v,2}, z_{v,3}, z_{v,4}\}$. Define

$$V(H) := \bigcup_{v \in V} Z_v. \tag{A.2}$$

Consider the following verifier $\mathcal{V}_{\text{A.2}}$, given oracle access to a 2-coloring $f' \colon V(H) \to [2]$:

Verifier $\mathcal{V}_{\text{A.2}}$.

Input: a graph $G = (V, E)$.
Oracle access: a 2-coloring $f' \colon V(H) \to [2]$.

select an edge (v, w) from E uniformly at random.
select $r \sim [0, 1]$.
if $0 \leqslant r < \frac{4}{9}$ **then** ▷ *with probability* $\frac{4}{9}$
  select a pair $z_{v,i} \neq z_{v,j}$ from Z_v uniformly at random.
  let $\alpha := f'(z_{v,i})$ and $\beta := f'(z_{v,j})$.
else if $\frac{4}{9} \leqslant r < \frac{8}{9}$ **then** ▷ *with probability* $\frac{4}{9}$
  select a pair $z_{w,i} \neq z_{w,j}$ from Z_w uniformly at random.
  let $\alpha := f'(z_{w,i})$ and $\beta := f'(z_{w,j})$.
else ▷ *with probability* $\frac{1}{9}$
  select i from $[4]$ uniformly at random.
  let $\alpha := f'(z_{v,i})$ and $\beta := f'(z_{w,i})$.
if $\alpha = \beta$ **then**
  declare reject.
else
  declare accept.

Create the set $E(H)$ of parallel edges so as to emulate $\mathcal{V}_{\text{A.2}}$ in a sense that for any 2-coloring $f' \colon V(H) \to 2$ of H,

$$\mathsf{val}_H(f') = \mathbb{P}\Big[\mathcal{V}_{\text{A.2}} \text{ accepts } f'\Big]. \tag{A.3}$$

Note that the maximum degree of H can be bounded by $\mathcal{O}(\Delta)$.

We define an encoding function $\mathsf{enc} \colon [6] \to [2]^4$ such that for any color $\alpha \in [6]$,

$$\mathsf{enc}(\alpha) := \begin{cases} (1,1,2,2) & \text{if } \alpha = 1, \\ (1,2,1,2) & \text{if } \alpha = 2, \\ (1,2,2,1) & \text{if } \alpha = 3, \\ (2,1,1,2) & \text{if } \alpha = 4, \\ (2,1,2,1) & \text{if } \alpha = 5, \\ (2,2,1,1) & \text{if } \alpha = 6. \end{cases} \tag{A.4}$$

For any 6-coloring $f \colon V \to [6]$ of G, consider a 2-coloring $f' \colon V(H) \to [2]$ of H such that $f'(z_{v,i}) := \mathsf{enc}(f(v))_i$ for all $z_{v,i} \in V(H)$. Construct finally two 2-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}}$ of H from $f_{\mathsf{start}}, f_{\mathsf{end}}$ by this procedure, respectively, which completes the description of the reduction.

Correctness. We first analyze the acceptance probability of $\mathcal{V}_{A.2}$. We define a decoding function $\mathsf{dec}\colon [2]^4 \to [6] \cup \{\perp\}$ such that for any 2-color vector $\boldsymbol{\alpha} \in [2]^4$,

$$\mathsf{dec}(\boldsymbol{\alpha}) := \begin{cases} 1 & \boldsymbol{\alpha} = (1,1,2,2), \\ 2 & \boldsymbol{\alpha} = (1,2,1,2), \\ 3 & \boldsymbol{\alpha} = (1,2,2,1), \\ 4 & \boldsymbol{\alpha} = (2,1,1,2), \\ 5 & \boldsymbol{\alpha} = (2,1,2,1), \\ 6 & \boldsymbol{\alpha} = (2,2,1,1), \\ \perp & \text{otherwise}, \end{cases} \tag{A.5}$$

For any 2-coloring f' of H, let $f'(Z_v) := (f'(z_{v,1}), f'(z_{v,2}), f'(z_{v,3}), f'(z_{v,4}))$ for each vertex $v \in V$.

Lemma A.3. *Conditioned on the edge $(v,w) \in E$ selected by $\mathcal{V}_{A.2}$, the following hold:*

- *if $\mathsf{dec}(f'(Z_v)) \neq \perp$, $\mathsf{dec}(f'(Z_w)) \neq \perp$, and $\mathsf{dec}(f'(Z_v)) \neq \mathsf{dec}(f'(Z_w))$, then $\mathcal{V}_{A.2}$ accepts with probability at least $\frac{35}{54}$;*
- *otherwise, $\mathcal{V}_{A.2}$ accepts with probability at most $\frac{34}{54}$.*

Proof. Conditioned on the selected edge (v,w), the acceptance probability of $\mathcal{V}_{A.2}$ is equal to

$$\frac{4}{9} \cdot \mathop{\mathbb{P}}_{i \neq j}\Big[f'(z_{v,i}) \neq f'(z_{v,j})\Big] + \frac{4}{9} \cdot \mathop{\mathbb{P}}_{i \neq j}\Big[f'(z_{w,i}) \neq f'(z_{w,j})\Big] + \frac{1}{9} \cdot \mathop{\mathbb{P}}_{i}\Big[f'(z_{v,i}) \neq f'(z_{w,i})\Big]. \tag{A.6}$$

- Suppose first $\mathsf{dec}(f'(Z_v)) \neq \perp$, $\mathsf{dec}(f'(Z_w)) \neq \perp$, and $\mathsf{dec}(f'(Z_v)) \neq \mathsf{dec}(f'(Z_w))$. Since 4 of 6 pairs $(z_{v,i}, z_{v,j})$ in $\binom{Z_v}{2}$ of are bichromatic, 4 of 6 pairs $(z_{w,i}, z_{w,j})$ in $\binom{Z_w}{2}$ are bichromatic, and at least 2 of 4 pairs $(z_{v,i}, z_{w,i})$ are bichromatic, we have

$$\text{Eq. (A.6)} = \frac{4}{9} \cdot \frac{4}{6} + \frac{4}{9} \cdot \frac{4}{6} + \frac{1}{9} \cdot \frac{2}{4} = \frac{35}{54}. \tag{A.7}$$

- Suppose next $\mathsf{dec}(f'(Z_v)) \neq \perp$, $\mathsf{dec}(f'(Z_w)) \neq \perp$, but $\mathsf{dec}(f'(Z_v)) = \mathsf{dec}(f'(Z_w))$. Since none of 4 pairs $(z_{v,i}, z_{w,i})$ are bichromatic, we have

$$\text{Eq. (A.6)} = \frac{4}{9} \cdot \frac{4}{6} + \frac{4}{9} \cdot \frac{4}{6} + \frac{1}{9} \cdot \frac{0}{4} = \frac{32}{54}. \tag{A.8}$$

- Suppose finally $\mathsf{dec}(f'(Z_v))$ or $\mathsf{dec}(f'(Z_w))$ is equal to $\perp$. Without loss of generality, we can assume $\mathsf{dec}(f'(Z_v)) = \perp$. Since at most 3 of 6 pairs $(z_{v,i}, z_{v,j})$ could be bichromatic by definition of dec, at most 4 of 6 pairs $(z_{w,i}, z_{w,j})$ are bichromatic, and at most 4 of 4 pairs $(z_{v,i}, z_{w,i})$ are bichromatic, we have

$$\text{Eq. (A.6)} = \frac{4}{9} \cdot \frac{3}{6} + \frac{4}{9} \cdot \frac{4}{6} + \frac{1}{9} \cdot \frac{3}{4} = \frac{34}{54}. \tag{A.9}$$

The above case analysis completes the proof. □

We are now ready to prove Lemma A.2.

Proof of Lemma A.2. We first prove the completeness; i.e.,

$$\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) = 1 \implies \mathsf{opt}_H(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}) \geqslant 1 - \delta_c. \tag{A.10}$$

It is sufficient to consider the case that f_{start} and f_{end} differ in a single vertex, say $v^\star$. Without loss of generality, we can assume that $|E|$ is sufficiently large so that $\frac{\Delta}{|E|} < \frac{\varepsilon}{70}$. Consider a reconfiguration sequence $\mathscr{F}'$ from f'_{start} to f'_{end} obtained by recoloring $z_{v^\star,i}$ from $f'_{\mathsf{start}}(z_{v^\star,i})$ to $f'_{\mathsf{end}}(z_{v^\star,i})$ for each $i \in [4]$. Conditioned on the selected edge not being incident to $v^\star$, $\mathcal{V}_{\mathrm{A.2}}$ accepts any intermediate coloring of $\mathscr{F}'$ with probability at least $\frac{35}{54}$ by Lemma A.3. Since at most Δ edges are incident to $v^\star$, $\mathcal{V}_{\mathrm{A.2}}$ accepts any intermediate coloring of $\mathscr{F}'$ with probability at least

$$\frac{35}{54} \cdot \left(1 - \frac{\Delta}{|E|}\right) > \frac{35}{54} \cdot \left(1 - \frac{\varepsilon}{70}\right) = 1 - \frac{19 + \frac{\varepsilon}{2}}{54} = 1 - \delta_c. \tag{A.11}$$

We then prove the soundness; i.e.,

$$\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) < 1 - \varepsilon \implies \mathsf{opt}_H(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}}) < 1 - \delta_s. \tag{A.12}$$

Let $\mathscr{F}' = (f'^{(1)}, \ldots, f'^{(T)})$ be any reconfiguration sequence from f'_{start} to f'_{end} such that $\mathsf{val}_H(\mathscr{F}') = \mathsf{opt}_H(f'_{\mathsf{start}} \leftrightsquigarrow f'_{\mathsf{end}})$. Construct then a sequence $\mathscr{F} = (f^{(1)}, \ldots, f^{(T)})$ from f_{start} to f_{end} such that $f^{(t)}$ is a 6-coloring of G defined as follows:

$$f^{(t)}(v) := \begin{cases} \mathsf{dec}(f'^{(t)}(Z_v)) & \text{if it is not } \bot \\ 1 & \text{otherwise} \end{cases} \text{ for all } v \in V. \tag{A.13}$$

Since $\mathscr{F}$ is a valid reconfiguration sequence, it includes $f^{(t)}$ such that $\mathsf{val}_G(f^{(t)}) < 1 - \varepsilon$; i.e., $f^{(t)}$ makes more than ε-fraction of edges of G monochromatic. For each of such monochromatic edges (v, w), it must hold that either $\mathsf{dec}(f'^{(t)}(Z_v)) = \bot$, $\mathsf{dec}(f'^{(t)}(Z_w)) = \bot$, or $\mathsf{dec}(f'^{(t)}(Z_v)) = \mathsf{dec}(f'^{(t)}(Z_w))$. Consequently, by Lemma A.3, $\mathcal{V}_{\mathrm{A.2}}$ accepts $f'^{(t)}$ with probability less than

$$\frac{35}{54} \cdot (1 - \varepsilon) + \frac{34}{54} \cdot \varepsilon = \frac{35}{54} \cdot \left(1 - \frac{\varepsilon}{35}\right) = 1 - \frac{19 + \varepsilon}{54} = 1 - \delta_s, \tag{A.14}$$

which completes the proof. □

A.2 Proof of Lemma 6.4

In this subsection, we prove Lemma 6.4; i.e., there is a gap-preserving reduction from Maxmin 2-Cut Reconfiguration to Maxmin k-Cut Reconfiguration for every $k \geqslant 3$.

Reduction. Our reduction from Gap$_{1-\varepsilon_c, 1-\varepsilon_s}$ 2-Cut Reconfiguration to Gap$_{1-\delta_c, 1-\delta_s}$ k-Cut Reconfiguration is described below. Fix $k \geqslant 3$, $\varepsilon_c, \varepsilon_s \in (0,1)$ with $\varepsilon_c < \varepsilon_s$, and $\Delta \in \mathbb{N}$. Let $(G, f_{\mathsf{start}}, f_{\mathsf{end}})$ be an instance of Gap$_{1-\varepsilon_c, 1-\varepsilon_s}$ 2-Cut Reconfiguration, where $G = (V, E)$ is a graph of maximum degree $\Delta \in \mathbb{N}$, and $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to [2]$ are a pair of its 2-colorings. We construct an instance $(H, f'_{\mathsf{start}}, f'_{\mathsf{end}})$ of Maxmin k-Cut Reconfiguration as follows. Create a copy of V, and a set of fresh k vertices for each vertex $v \in V$, denoted by $Z_v := \{z_{v,1}, \ldots, z_{v,k}\}$. Define

$$V(H) := V \cup \bigcup_{v \in V} Z_v. \tag{A.15}$$

Generate a 3-*regular expander graph* X on V whose edge expansion is a positive real $h > 0$; i.e.,

$$\min_{\emptyset \subsetneq S \subsetneq V} \frac{|\partial_X(S)|}{\min\{|S|, |V \setminus S|\}} \geqslant h, \\ \text{where } \partial_X(S) \coloneqq \left\{ (v,w) \in E(X) \;\middle|\; v \in S, w \notin S \right\}. \tag{A.16}$$

Such an expander graph can be constructed in polynomial time; see, e.g., [GG81, HLW06, RVW02]. Consider the following verifier $\mathcal{V}_{6.4}$, given oracle access to a k-coloring $f' \colon V(H) \to [k]$, parameterized by p_1 and p_2 with $p_1 + p_2 = 1$, whose values depend only on $k, \varepsilon_c, \varepsilon_s$ and will be determined later:

Verifier $\mathcal{V}_{6.4}$.

Input: a graph $G = (V,E)$, a 3-regular expander graph X, parameters $p_1, p_2 \in (0,1)$ with $p_1 + p_2 = 1$.
Oracle access: a k-coloring $f' \colon V(H) \to [k]$.

```
if with probability p_1 then
    ▷ first test ◁
    select an edge (v, w) of X uniformly at random.
    select a pair i ≠ j from [k] uniformly at random.
    let α := f'(z_{v,i}) and β := f'(z_{w,j}).
else                                            ▷ with probability p_2
    ▷ second test ◁
    select an edge (v, w) of G uniformly at random.
    select r ~ [0, 1].
    if 0 ⩽ r < 1/(2k−1) then                    ▷ with conditional probability 1/(2k−1)
        let α := f'(v) and β := f'(w).
    else if 1/(2k−1) ⩽ r < (k−1)/(2k−1) then    ▷ with conditional probability (k−2)/(2k−1)
        select i from {3, ..., k} uniformly at random.
        let α := f'(v) and β := f'(z_{v,i}).
    else                                        ▷ with conditional probability (k−2)/(2k−1)
        select i from {3, ..., k} uniformly at random.
        let α := f'(w) and β := f'(z_{w,i}).
if α = β then
    declare reject.
else
    declare accept.
```

Create the set $E(H)$ of parallel edges between $V(H)$ so as to emulate $\mathcal{V}_{6.4}$ in a sense that for any k-coloring f' of H,

$$\mathsf{val}_H(f') = \mathbb{P}\Big[\mathcal{V}_{6.4} \text{ accepts } f'\Big]. \tag{A.17}$$

Note that the maximum degree of H can be bounded by $\mathcal{O}(\Delta + \mathrm{poly}(k))$. Construct finally two k-colorings $f'_{\mathsf{start}}, f'_{\mathsf{end}}$ of H such that $f'_{\mathsf{start}}(v) = f_{\mathsf{start}}(v)$ and $f'_{\mathsf{end}}(v) = f_{\mathsf{end}}(v)$ for all $v \in V$, and $f'_{\mathsf{start}}(z_{v,i}) = f'_{\mathsf{end}}(z_{v,i}) = i$ for all $v \in V$ and $i \in [k]$. This completes the description of the reduction.

Correctness. We first investigate the (conditional) rejection probability of the first test. We say that Z_v is *good* regarding a k-coloring f' of H if $f'(z_{v,i}) = i$ for all $i \in [k]$, and *bad* otherwise.

Lemma A.4. *Suppose that more than δ-fraction and less than $\frac{1}{2}$-fraction of Z_v's are bad for $\delta \in \left(0, \frac{1}{2}\right)$. Conditioned on the first test executed, $\mathcal{V}_{6.4}$ rejects with probability more than $\frac{2h\cdot\delta}{3k(k-1)}$.*

Proof. Since for any good Z_v and bad Z_w, there must be a pair $i \neq j$ such that $z_{v,i} = z_{w,j}$, $\mathcal{V}_{6.4}$'s (conditional) rejection probability is

$$\begin{aligned}
&\mathop{\mathbb{P}}_{\substack{(v,w)\sim E(X)\\ (i,j)\sim\binom{[k]}{2}}}\Big[f'(z_{v,i}) = f'(z_{w,j})\Big] \\
&\geqslant \mathop{\mathbb{P}}_{\substack{(v,w)\sim E(X)\\ (i,j)\sim\binom{[k]}{2}}}\Big[f'(z_{v,i}) = f'(z_{w,j}) \;\Big|\; Z_v \text{ is good and } Z_w \text{ is bad}\Big] \cdot \mathop{\mathbb{P}}_{(v,w)\sim E(X)}\Big[Z_v \text{ is good and } Z_w \text{ is bad}\Big] \\
&\geqslant \frac{1}{k(k-1)} \cdot \mathop{\mathbb{P}}_{(v,w)\sim E(X)}\Big[Z_v \text{ is good and } Z_w \text{ is bad}\Big]
\end{aligned} \tag{A.18}$$

Letting S be the set of vertices $v \in V$ such that Z_v is bad, we have $\delta|V| < |S| < \frac{1}{2}|V|$, implying that

$$\mathop{\mathbb{P}}_{(v,w)\sim E(X)}\Big[Z_v \text{ is good and } Z_w \text{ is bad}\Big] = \frac{|\partial_X(S)|}{|E(X)|} > \frac{h\delta\cdot|V|}{|E(X)|} = \frac{2h}{3}\cdot\delta. \tag{A.19}$$

Consequently, $\mathcal{V}_{6.4}$'s rejection probability is more than $\frac{2h\cdot\delta}{3k(k-1)}$, as desired. □

We then examine the (conditional) rejection probability of the second test. We say that edge (v,w) is *legal* regarding a k-coloring f' of H if ($f'(v) \in [2]$, $f'(w) \in [2]$, and $f'(v) \neq f'(w)$), and *illegal* otherwise.

Lemma A.5. *Conditioned on the event that the second test is executed and both Z_v and Z_w are good for the selected edge $(v,w) \in E$, the following hold:*

- *if (v,w) is legal, $\mathcal{V}_{6.4}$ rejects with probability* 0;
- *if (v,w) is illegal, $\mathcal{V}_{6.4}$ rejects with probability at least $\frac{1}{2k-3}$*;
- *if $f'(v) \in [2]$, $f'(w) \in [2]$, and $f'(v) = f'(w)$, then $\mathcal{V}_{6.4}$ rejects with probability $\frac{1}{2k-3}$.*

Proof. Conditioned on the selected edge (v,w), the second test rejects with probability

$$\frac{k-2}{2k-3}\cdot\mathop{\mathbb{P}}_{i\sim\{3,\ldots,k\}}\Big[f'(v) = f'(z_{v,i})\Big] + \frac{k-2}{2k-3}\cdot\mathop{\mathbb{P}}_{i\sim\{3,\ldots,k\}}\Big[f'(w) = f'(z_{w,i})\Big] + \frac{1}{2k-3}\cdot\Big[\!\Big[f'(v) = f'(w)\Big]\!\Big] \tag{A.20}$$

Observe easily that Eq. (A.20) is equal to 0 whenever (v,w) is legal. Suppose then (v,w) is illegal; at least one of the following must hold:

(Case 1) $f'(v) \notin [2]$; thus, $\mathbb{P}_{i\sim\{3,\ldots,k\}}[f'(v) = f'(x_{v,i})] = \frac{1}{k-2}$.

(Case 2) $f'(w) \notin [2]$; thus, $\mathbb{P}_{i\sim\{3,\ldots,k\}}[f'(w) = f'(x_{w,i})] = \frac{1}{k-2}$.

(Case 3) $f'(v) = f'(w)$; thus, $[\![f'(v) = f'(w)]\!] = 1$.

Eq. (A.20) is thus at least

$$\min\left\{\frac{k-2}{2k-3}\cdot\frac{1}{k-2},\frac{k-2}{2k-3}\cdot\frac{1}{k-2},\frac{1}{2k-3}\cdot 1\right\}=\frac{1}{2k-3}. \tag{A.21}$$

Obviously, Eq. (A.20) is equal to $\frac{1}{2k-3}$ whenever $f'(v)\in[2]$, $f'(w)\in[2]$, and $f'(v)=f'(w)$, as desired. □

We are now ready to prove Lemma 6.4.

Proof of Lemma 6.4. We first prove the completeness; i.e.,

$$\begin{aligned}&\mathsf{opt}_G(f_{\mathsf{start}}\leftrightsquigarrow f_{\mathsf{end}})\geqslant 1-\varepsilon_c \implies \mathsf{opt}_H(f'_{\mathsf{start}}\leftrightsquigarrow f'_{\mathsf{end}})\geqslant 1-\delta_c,\\ &\text{where } \delta_c := p_2\cdot\frac{\varepsilon_c}{2k-3}.\end{aligned} \tag{A.22}$$

It is sufficient to consider the case that f_{start} and f_{end} differ in a single vertex, say $v^\star$. Note that $\mathscr{F}'=(f'_{\mathsf{start}},f'_{\mathsf{end}})$ is a reconfiguration sequence from f'_{start} to f'_{end}. Let f' be either of f'_{start} or f'_{end}. Since Z_v is good regarding f' for all $v\in V$, the first test never rejects f', and the second test's rejection probability is 0 if the selected edge is bichromatic and $\frac{1}{2k-3}$ if the selected edge is monochromatic by Lemma A.5. Therefore, $\mathcal{V}_{6.4}$ rejects f' with probability at most

$$p_1\cdot 0+p_2\cdot\left(0\cdot(1-\varepsilon_c)+\frac{1}{2k-3}\cdot\varepsilon_c\right)=p_2\cdot\frac{\varepsilon_c}{2k-3}=\delta_c, \tag{A.23}$$

as desired.

We then prove the soundness; i.e.,

$$\begin{aligned}&\mathsf{opt}_G(f_{\mathsf{start}}\leftrightsquigarrow f_{\mathsf{end}})< 1-\varepsilon_s \implies \mathsf{opt}_H(f'_{\mathsf{start}}\leftrightsquigarrow f'_{\mathsf{end}})< 1-\delta_s,\\ &\text{where } \delta_s := p_2\cdot\frac{\frac{\varepsilon_s+\varepsilon_c}{2}}{2k-3}.\end{aligned} \tag{A.24}$$

Let $\mathscr{F}'=(f'^{(1)},\ldots,f'^{(T)})$ be any reconfiguration sequence from f'_{start} to f'_{end} such that $\mathsf{val}_H(\mathscr{F}')=\mathsf{opt}_H(f'_{\mathsf{start}}\leftrightsquigarrow f'_{\mathsf{end}})$. Define $\overline{\varepsilon}$ as

$$\overline{\varepsilon}:=\frac{\varepsilon_s-\varepsilon_c}{4}\in\left(0,\tfrac{1}{4}\right). \tag{A.25}$$

Suppose first $\mathscr{F}'$ includes a k-coloring $f'^{(t)}$ of H such that more than $\overline{\varepsilon}$-fraction of Z_v's are bad. By Lemma A.4, $\mathcal{V}_{6.4}$ rejects $f'^{(t)}$ with probability more than

$$p_1\cdot\frac{2h\cdot\overline{\varepsilon}}{3k^2}. \tag{A.26}$$

Suppose next that for every k-coloring $f'^{(t)}$ of H in $\mathscr{F}'$, at most $\overline{\varepsilon}$-fraction of Z_v's are bad. Construct then a reconfiguration sequence $\mathscr{F}=(f^{(1)},\ldots,f^{(T)})$ from f_{start} to f_{end}, where each $f^{(t)}$ is a 2-coloring of G defined as follows:

$$f^{(t)}(v):=\begin{cases}1 & \text{if } f'^{(t)}(v)=1\\ 2 & \text{if } f'^{(t)}(v)=2\\ 1 & \text{otherwise}\end{cases} \text{ for all } v\in V. \tag{A.27}$$

By assumption, $\mathscr{F}$ includes a 2-coloring $f^{(t)}$ of G such that more than ε_s-fraction of edges of G are monochromatic, each of which must be illegal regarding $f'^{(t)}$ due to the construction of $f^{(t)}$. Since at most $\overline{\varepsilon}$-fraction of Z_v's are bad regarding $f'^{(t)}$, the fraction of edges of G incident to any bad Z_v can be bounded by

$$\frac{\overline{\varepsilon} \cdot |V| \cdot \Delta}{|E|} \leqslant 2\overline{\varepsilon}. \tag{A.28}$$

There are thus more than $(\varepsilon_s - 2\overline{\varepsilon})$-fraction of *illegal* edges $(v,w) \in E$ such that both Z_v and Z_w are good (regarding $f'^{(t)}$). By Lemma A.5, $\mathcal{V}_{6.4}$ rejects $f'^{(t)}$ with probability more than

$$p_2 \cdot \frac{\varepsilon_s - 2\overline{\varepsilon}}{2k-3}. \tag{A.29}$$

Setting the values of p_1 and p_2 so that

$$p_1 \cdot \frac{2h \cdot \overline{\varepsilon}}{3k^2} = p_2 \cdot \frac{\varepsilon_s - 2\overline{\varepsilon}}{2k-3} \text{ and } p_1 + p_2 = 1, \tag{A.30}$$

we find $\mathcal{V}_{6.4}$'s rejection probability to be more than

$$\min\left\{ p_1 \cdot \frac{2h \cdot \overline{\varepsilon}}{3k^2}, p_2 \cdot \frac{\varepsilon_s - 2\overline{\varepsilon}}{2k-3} \right\} = p_2 \cdot \frac{\frac{\varepsilon_s + \varepsilon_c}{2}}{2k-3} = \delta_s, \tag{A.31}$$

as desired. Observe finally that $\delta_c < \delta_s$, completing the proof. □

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