The excluded minors for embeddability into a compact surface

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September 15, 2025

Abstract

We determine the excluded minors characterising the class of countable graphs that embed into some compact surface.

Keywords: excluded minor, graphs in surfaces, outerplanar, star-comb lemma.
MSC 2020 Classification: 05C63, 05C83, 05C10, 05C75.

1 Introduction

The main aim of this paper is to provide the excluded minors characterising the class of countable graphs that embed into a compact surface, whereby we put no restriction on the genus. We will prove

Theorem 1.1. *A countable graph G embeds into a compact (orientable) surface if and only if it does not have one of the 8 graphs of Figure 1 as a minor.*[1]

It is an exercise to show that none of these graphs is a minor of another. Since none of these graphs embeds into a closed surface, orientable or not, our theorem remains valid if we remove the word 'orientable'.

All graphs in this paper are countable. In the locally finite case, only the first two obstructions $\Sigma_1 = \omega \cdot K_5, \Sigma_2 = \omega \cdot K_{3,3}$ are needed (see Corollary 4.6).

*Supported by EPSRC grants EP/V048821/1 and EP/V009044/1.

[1] Every non-trivial infinite graph has many 'minor-twins'; for example, for each pair x, y of vertices of infinite degree in $\Sigma_i, i \geq 5$, we could add or remove the x–y edge.

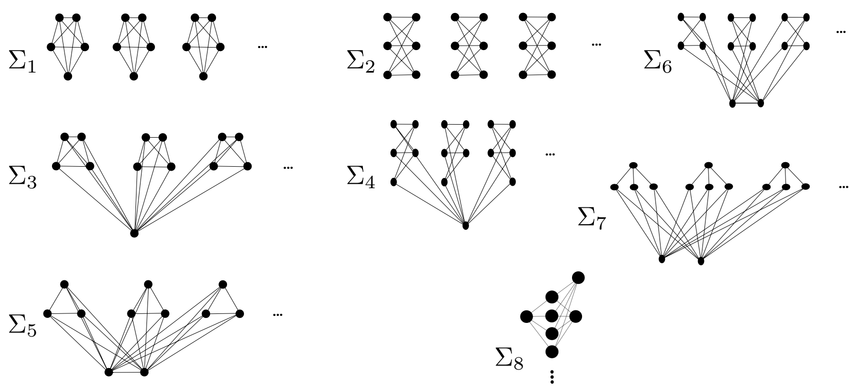

Figure 1: The excluded minors of Σ as provided by Theorem 1.1.
$\mathbf{\Sigma_1}$ (respectively $\mathbf{\Sigma_2}$): the disjoint union of infinitely many copies of K_5 (resp. $K_{3,3}$);
$\mathbf{\Sigma_3}$ (resp. $\mathbf{\Sigma_4}$): the graph arising from Σ_1 (resp. Σ_2) by picking one vertex from each component and identifying them;
$\mathbf{\Sigma_5}$ (resp. $\mathbf{\Sigma_6}$): the graph arising from Σ_1 (resp. Σ_2) by picking one edge from each component and identifying them;
$\mathbf{\Sigma_7}$: the graph arising from Σ_2 by picking one pair of non-adjacent vertices from each component and identifying these pairs; and $\mathbf{\Sigma_8} = K_{3,\omega}$.

An analogous statement for embeddings into a fixed surface is the following theorem of Robertson & Seymour [27] (previously announced in [5, 8, 13]). For every $n \in \mathbb{N}$, there is g, such that for every graph G of genus at least g, there is some Σ_i as in Figure 1, such that any subgraph of size n of Σ_i is a minor of G. I do not see a way to deduce this from Theorem 1.1 or vice-versa. An important difference between these two results is that if we restrict to the orientable case, then we need to allow the $n \times n$ projective grid as a further obstruction to embeddability into a fixed surface, but Theorem 1.1 proves that we have the same excluded minors with or without the orientability restriction when allowing arbitrarily high genus.

One of the tools for our proof of Theorem 1.1 is the following result of independent interest, saying that a graph embeds into a compact surface if and only if it can be decomposed into finitely many planar subgraphs with finite pairwise intersections.

Theorem 1.2. *A countable graph embeds into a compact (orientable) surface if and only if it admits a finitary decomposition into planar pieces.*

See Section 3 for the precise definitions. Theorem 1.2 supports the meta-conjecture that any result proved for infinite planar graphs generalises, rather easily, to graphs embeddable into a compact surface. Examples of such results include the main results of [4, 15, 17, 21, 23]. See Section 3.1 for more.

Part of the motivation for Theorem 1.1 comes from a well-known conjecture of Thomas [32] postulating that the countable graphs are well-quasi-ordered under the minor relation. The analogous statement for finite graphs is the celebrated Graph Minor Theorem of Robertson & Seymour [31]. A positive answer to Thomas's conjecture would imply that every minor-closed class $\mathcal{C}$ of countable graphs is characterised by forbidding a finite list $\mathrm{Ex}(\mathcal{C})$ of *excluded minors*. This is in general hard to show even for a concrete class of graphs like the class Σ of Theorem 1.1; indeed, it is a-priori not even clear that $\mathrm{Ex}(\Sigma)$ is finite. Apart from the fact that Σ is a natural class to consider, another reason why the finiteness of $\mathrm{Ex}(\Sigma)$ is a pressing question if one is interested in Thomas's conjecture is the important role played by classes of finite graphs embeddable in a fixed surface in the proof of the Graph Minor Theorem.

Many natural minor-closed graph classes $\mathcal{C}$, e.g. the graphs embeddable into a fixed surface, have the property that a graph is in $\mathcal{C}$ as soon as every finite subgraph is. This has the consequence that $\mathrm{Ex}(\mathcal{C})$ coincides with the list of excluded minors of the subclass of $\mathcal{C}$ comprising its finite elements. Apart from such classes, there are very few classes $\mathcal{C}$ of infinite graphs for which $\mathrm{Ex}(\mathcal{C})$ is explicitly known. The only example I am aware of are the graphs with accumulation-free embeddings in the plane [20].

Additional motivation for Theorem 1.1 comes from a question raised by Christian, Richter & Salazar [8], asking for a characterisation of the Peano continua that embed into a closed surface analogous to Claytor's [9] characterisation of the Peano continua embeddable into $\mathbb{S}^2$. The special case of graph-like continua was handled in [8], and the characterisation obtained is similar to Theorem 1.1. But the lack of compactness does not allow using that characterisation to deduce Theorem 1.1. Using Theorem 1.1 and the star-comb lemma (Lemma 2.2 below) it is not difficult to determine the excluded *topological* minors for embeddability into a compact surface, and this could be a first step towards answering the aforementioned question of Christian et al. [8].

Our proof of Theorem 1.1 is elementary (but involved), relying only on Kuratowski's theorem, and a classical result of Youngs about cellular embeddings of finite graphs. It is carried out mostly in Sections 3, 5 and 6. On the way to Theorem 1.1 we will develop techniques that allow us to find the excluded minors of families of infinite graphs that satisfy a property up to finitely many flaws: we will characterize the graphs that become forests after deleting, or contracting, finitely many edges (Sections 4 and 8), as well as the graphs that are outerplanar up to deleting finitely many edges (Section 7).

The star-comb lemma is one of the most useful tools in infinite graph theory. In Section 9 we obtain the following strengthening for 2-connected graphs:

Theorem 1.3. *Let G be a countable, 2-connected, graph, and $U \subseteq V(G)$ infinite. Then G contains a subdivision of an infinite ladder, or of an infinite fan, or of $K_{2,\infty}$, having infinitely many vertices in U.*

If G is locally finite, then this results in a ray in G containing an infinite subset of U.

The aforementioned conjecture of Thomas was studied by Robertson, Seymour & Thomas [28, 29], and they concluded that *there is not much chance of proving* it, as it would have implications about the ordering of finite graphs. It is therefore natural to try to extend the Graph Minor Theorem to an intermediate level covering all finite graphs but not necessarily all countable ones. A concrete approach for doing so is offered by Conjecture 10.2 and other questions in Section 10 arising from our results and methods.

2 Preliminaries

We follow the terminology of Diestel [10]. We use $V(G)$ to denote the set of vertices, and $E(G)$ the set of edges of a graph G. For $S \subseteq V(G)$, the subgraph $G[S]$ of G *induced* by S has vertex set S and contains all edges of G with both end-vertices in S.

The *degree* $d(v) = d_G(v)$ of a vertex v in a graph G, is the number of edges of G incident with v.

A *ray* is a one-way infinite path. We say that G is *locally finite*, if no vertex of G lies in infinitely many edges.

Let G, H be graphs. An H *minor* of G is a collection of disjoint connected subgraphs $B_v, v \in V(H)$ of G, called *branch sets*, and edges $E_{uv}, uv \in E(H)$ of G such that each E_{uv} has one end-vertex in B_u and one in B_v. We write $H < G$ to express that G has an H minor.

Given a set X of graphs, we write $\mathrm{Forb}(X)$ for the class of graphs H such that no element of X is a minor of H.

A *subdivision* of a graph G is a graph obtained by replacing some of the edges of G by paths with the same end-vertices.

A *surface* is a connected 2-manifold without boundary. An *embedding* of a countable graph G into a surface S is a map $f : G \to S$ from the 1-complex obtained from G when identifying each edge with the interval $[0, 1]$ to S such that the restriction of f to each finite subgraph of G is an embedding in the topological sense, i.e. a homeomorphism onto its image. (The reason why we restrict to finite subgraphs here is that the 1-complex topology of G is not metrizable when G is not locally finite, and so such G cannot have an embedding into a metrizable space S. For example, a star with infinitely many leaves admits an embedding into $\mathbb{R}^2$ in our sense but it does not admit a topological embedding. Let $\gamma(G)$ denote the minimum genus of an orientable surface into which a graph G embeds.

The following is perhaps folklore, but we sketch a proof for completeness. The locally finite case has been proved by Mohar [25, §5].

Lemma 2.1. *Let G be a countable graph, and S an orientable surface. Then G admits an embedding into S if each of its finite subgraphs does.*

When S is the sphere, Dirac & Shuster [11] provide a proof by an elementary compactness argument (which they atribute to Erdős). Our proof is a combination of this with Youngs' Theorem 3.4 below.

Proof. TOPROVE 2 □

We let Σ denote the class of countable graphs that embed into a compact orientable surface. (We will prove that every graph embeddable into a compact non-orientable surface also embeds into a compact orientable one.)

We let ω denote the smallest infinite ordinal.

2.1 The star-comb lemma

Given a graph G, and an infinite set $U \subseteq V(G)$, we define a *U-star* to be a subdivision of the infinite star $K_{1,\omega}$ in G all leaves of which lie in U. We define a *U-comb* to be the union of a ray R of G with infinitely many pairwise disjoint, possibly trivial, U–R paths. We call R the *spine* of C, and the U–R paths its *teeth*.

Lemma 2.2 (Star-comb lemma [10, Lemma 8.2.2])**.** *Let U be an infinite set of vertices in a connected graph G. Then G contains either a U-star or a U-comb.*

3 Decomposing into planar graphs

The aim of this section is to prove the orientable case of Theorem 1.2, which will be used as a tool for the proof of Theorem 1.1. (The non-orientable case will follow after we have proved Theorem 1.1.)

Definition 3.1. *A* decomposition *of a graph G is a family $(G_i)_{i\in\mathcal{I}}$ of subgraphs of G, called the* pieces*, such that $G = \bigcup_{i\in\mathcal{I}} G_i$. We say that a decomposition $(G_i)_{i\in\mathcal{I}}$ is* finitary*, if $\mathcal{I}$ is finite, and the intersection of any two distinct pieces is finite. Note that this means that at most finitely many vertices of G lie in more than one piece.*

Let us collect a few lemmas for the proof of Theorem 1.2, and for later use.

Lemma 3.2. *Let $G \in \Sigma$ and suppose G' is obtained from G by identifying two vertices $v, w \in V(G)$. Then $G' \in \Sigma$.*

Proof. TOPROVE 3 □

The power of Lemma 3.2 lies in our ability to apply it repeatedly. This way we obtain

Corollary 3.3. *Let G be a countable graph admitting a finitary decomposition $G_1, \dots, G_k$. If each G_i lies in Σ, then so does G.*

Proof. TOPROVE 5 □

Our last lemma is a classical result of Youngs about cellular embeddings. A *face* of an embedding $g : G \to \Gamma$ of a graph into a surface is a component of $\Gamma \backslash g(G)$.

Theorem 3.4 ([34]). *Let Γ be a closed orientable surface, and let G be a finite connected graph that embeds into Γ but does not embed into a closed orientable surface of smaller genus. Then for every embedding $g : G \to \Gamma$, each face of g is homeomorphic to an open disc.*

We are now ready for the proof of the main result of this section, which we restate for convenience:

Theorem 3.5. *A countable graph G embeds into a compact, orientable, surface if and only if it admits a finitary decomposition into planar pieces.*

Proof. TOPROVE 6 □

3.1 Implications of Theorem 3.5

We remark that Theorem 3.5 allows us to extend many results obtained for planar graphs, e.g. those of [4, 15, 17], to graphs in Σ. Motivated by the fact that some such results (e.g. [21, 23]) only apply to graphs with vertex-accumulation-free embeddings into the plane, we will now formulate and prove a refinement of Theorem 3.5 that takes accumulation points into account. This refinement is not needed for the proof of Theorem 1.1, and the reader may skip the rest of this section.

We let Σ^* denote the class of countable graphs G that embed into a compact orientable surface Γ so that there are at most finitely points of Γ that are accumulation points of vertices of G. We can always choose our embeddings so that no such accumulation point lies in the image of G. We define Planar* analogously, with Γ replaced by $\mathbb{S}^2$. Finally, we say that G is Vertex-Accumulation-Free, or *VAP-free* for short, if it admits an embedding in $\mathbb{R}^2$ with no accumulation point of vertices. We will prove

Corollary 3.6. *A countable graph G lies in Σ^* if and only if it admits a finitary decomposition into VAP-free pieces.*

Our proof will be a combination of Theorem 3.5 with the following basic fact about VAP-free graphs:

Lemma 3.7 ([33, LEMMA 7.1]). *A countable graph H is VAP-free, if and only if some embedding $g : H \to \mathbb{S}^2$ has the property that for every cycle C one of the two sides of $g(C)$ contains only finitely many vertices.*

Proof. TOPROVE 7 □

4 Graphs that have a property up to finitely many flaws

This section introduces classes of graphs that have a property up to finitely many 'flaws', and basic techniques for finding their excluded minors. This will suffice to prove the analogue of Theorem 1.1 for locally finite graphs.

Given a minor-closed family $\mathcal{C}$ of infinite graphs, one can define classes of graphs that are *almost* in $\mathcal{C}$ in the following sense.

Definition 4.1. *Let $\mathcal{C}_{\mathrm{V}}$ (respectively, $\mathcal{C}_{\mathrm{E}}$) denote the class of graphs G, such that by removing finitely many vertices (resp. edges) from G we obtain a graph belonging to $\mathcal{C}$. Similarly, we let $\mathcal{C}_{/\mathrm{E}}$ denote the graphs that belong to $\mathcal{C}$ after contracting finitely many edges.*

It is easy to see that $(\mathcal{C}_{\mathrm{E}})_{/\mathrm{E}} = (\mathcal{C}_{/\mathrm{E}})_{\mathrm{E}}$ for every $\mathcal{C}$, and we will simply write $\mathcal{C}_{\mathrm{E}/\mathrm{E}}$ instead.

The following examples show that neither of $\mathcal{C}_{/\mathrm{E}}, \mathcal{C}_{\mathrm{E}}$ is contained in the other in general.

Example 1: Let M denote the graph consisting of a ray emanating from the centre of an infinite star $K_{1,\omega}$ (we could call M the infinite mop), and let $\mathcal{C} := \mathrm{Forb}(M)$. Then $\mathcal{C}_{/\mathrm{E}} = \mathcal{C} \subsetneq \mathcal{C}_{\mathrm{E}}$, because $\mathcal{C}_{\mathrm{E}}$ contains M while $\mathcal{C}$ does not.

Example 2: It is easy to prove $\Sigma = \Sigma_{\mathrm{E}}$ similarly to Lemma 3.2. But $\Sigma \subsetneq \Sigma_{/\mathrm{E}}$, because $K_{3,\omega} \in \Sigma_{/\mathrm{E}}$.

Definition 4.2. *We write $\omega \cdot H$ for the disjoint union of countably infinitely many copies of a graph H. If H is vertex-transitive, we let $\bigvee H$ denote the graph obtained from $\omega \cdot H$ by picking one vertex from each copy of H and identifying them (by vertex-transitivity, it does not matter which vertices we pick).*

For example, $\bigvee K_3$ is a *bouquet of triangles*, i.e. an infinite union of triangles having exactly one vertex in common.

The next proposition provides the excluded minors of the class $\mathcal{F}_{\mathrm{E}}$ of 'almost forests' (the analogous result for $\mathcal{F}_{/\mathrm{E}}$ is given in Section 8). Although it is not formally needed, we include it here as a gentle introduction to the techniques we will later use to prove our main result (Theorem 1.1).

Proposition 4.3. *Let $\mathcal{F}$ denote the class of countable forests. Then* $\mathcal{F}_{\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_3, \bigvee K_3, K_{2,\omega})$.

Proof. TOPROVE 8 □

The class $\mathcal{F}_{\mathrm{V}}$ is easier to characterise in terms of excluded minors: we have $\mathcal{F}_{\mathrm{V}} = \mathrm{Forb}(\omega \cdot K_3)$. This is a special case of the following helpful fact. Its main argument is well-known in the context of Andreae's ubiquity conjecture [2].

Proposition 4.4. *Let $\mathcal{P} = \mathrm{Forb}(H_1, \dots, H_k)$ be a minor-closed class of countable graphs, where the H_i are finite. Then $\mathcal{P}_V = \mathrm{Forb}(\omega \cdot H_1, \dots, \omega \cdot H_k)$.*

Proof. TOPROVE 9 □

As an example application of Proposition 4.4, we deduce that $\mathrm{Planar}_\mathrm{V} = \mathrm{Forb}(\omega \cdot K_5, \omega \cdot K_{3,3})$, where we write Planar for the class of planar graphs. Since $\mathrm{Planar}_\mathrm{V} \subset \Sigma_V$, and $\omega \cdot K_5, \omega \cdot K_{3,3} \notin \Sigma_V$ ([3]; see also the proof of Theorem 1.1) this yields

Corollary 4.5. $\Sigma_V = \mathrm{Planar}_\mathrm{V} = \mathrm{Forb}(\omega \cdot K_5, \omega \cdot K_{3,3})$.

An alternative proof of Corollary 4.5 can be obtained by using Theorem 3.5: the latter implies that $\Sigma \subset \mathrm{Planar}_\mathrm{V}$, by removing the intersections of the pieces of any finitary planar decomposition of $G \in \Sigma$. This in turn implies $\Sigma_V \subset (\mathrm{Planar}_\mathrm{V})_V = \mathrm{Planar}_\mathrm{V}$, and so $\Sigma_V = \mathrm{Planar}_\mathrm{V}$ as the converse inclusion is trivial.

As another corollary of Proposition 4.4, we obtain an easy proof of the analogue of Theorem 1.1 for locally finite graphs:

Corollary 4.6. *Let G be a locally finite graph. Then $G \in \Sigma$ unless $\omega \cdot K_5 < G$ or $\omega \cdot K_{3,3} < G$.*

Proof. TOPROVE 10 □

Remark 1. The aforementioned fact that $\mathcal{F}_\mathrm{V} = \mathrm{Forb}(\omega \cdot K_3)$ can be thought of as the infinite version of the classical result of Erdős & Pósa [12] saying that every finite graph has either a $k \cdot K_3$ minor or a set of at most $f(k)$ vertices the removal of which results into a forest. I do not expect there to be an easy way to deduce the one from the other, as this Erdős-Pósa property fails for non-planar graphs in the finite case [30], while the infinite version holds for every finite graph by Proposition 4.4.

4.1 Almost planar graphs

The following is another consequence of Theorem 3.5 of independent interest. It is not needed for the proof of our other results, and the reader may skip to the next section.

Proposition 4.7. $\Sigma_{/\mathrm{E}} = \mathrm{Planar}_{/\mathrm{E}} = \mathrm{Planar}_{\mathrm{E}/\mathrm{E}}$.

Proof. TOPROVE 11 □

5 Outerplanar graphs and related classes

By Proposition 4.4 and Corollary 4.5, if a graph G does not lie in $\Sigma_\mathrm{V} = \mathrm{Planar}_\mathrm{V}$ then it has one of the desired minors, and so the most challenging part of our proof of Theorem 1.1 is to handle the case where G becomes planar after removing a finite vertex set W. When W is a single vertex, the latter is tantamount to saying that $G - W$ is planar but not outerplanar *relative to* the neighbourhood of W. The aim of this section is to characterise graphs that are outerplanar relative to a vertex set U, and the analogous class for embeddings

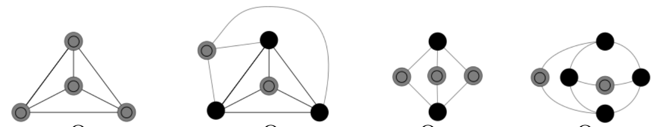

Figure 2: The excluded marked minors for relative outerplanarity. The sets of marked vertices U_i are shown in grey.

in any compact surface, in terms of forbidden minors that are *marked* by U. The precise definitions follow.

Let G be a graph and $U \subset V(G)$. Define the *U-cone* $C_U(G)$ of G to be the graph obtained from G by adding a new vertex u, the *cone vertex*, and joining it to each vertex in U with an edge. We say that G is *U-outerplanar*, if $C_U(G)$ is planar. If G is finite, it is easy to see that G is U-outerplanar if and only if it admits an embedding into $\mathbb{S}^2$ such that all vertices in U lie on a common face-boundary. Moreover, by letting $U = V(G)$ we recover the standard notion of outerplanarity: G is outerplanar if and only if it is $V(G)$-outerplanar.

A *marked graph* is a pair consisting of a graph G and a subset U of $V(G)$, called the *marked vertices*. Given two marked graphs $(G, U), (H, U')$, an H *marked minor* of G is defined just like an H minor of G (see Section 2), except that for each marked vertex v of H, we require that the corresponding branch set B_v contains at least one marked vertex of G. We write $(G, U) < (H, U')$ when this is possible.

Our next lemma adapts the well-known fact that the finite outerplanar graphs coincide with $\mathrm{Forb}(K_4, K_{2,3})$.

Lemma 5.1. *Suppose G is a countable planar graph, and let $U \subset V(G)$. Then G is U-outerplanar if and only if (G, U) does not contain one of the marked graphs $(\Theta_i, U_i), 1 \leq i \leq 4$ of Figure 2 as a marked minor.*

Proof. TOPROVE 12 □

Remark 2. *It follows from Lemma 5.1 that G is U-outerplanar as soon as each of its finite subgraphs is.*

We now introduce a generalisation of outerplanarity to arbitrary surfaces that will play an important role in the proof of Theorem 1.1:

Definition 5.2. *We say that a marked graph (G, U) lies in $\Sigma_\bullet$, and write $(G, U) \in \Sigma_\bullet$, if $C_U(G) \in \Sigma$.*

Lemma 5.3. *Suppose $G \in \Sigma$ is a countable graph, and (G, U) is not in $\Sigma_\bullet$ for some $U \subset V(G)$. Then (G, U) contains one of the following marked graphs as a marked minor:*
the graphs $\Phi_i, 1 \leq i \leq 5$, $\Phi'_i, 2 \leq i \leq 4$ of Figure 3, or
the graphs $\omega \cdot \Theta_i, 1 \leq i \leq 4$, with Θ_i as in Figure 2.

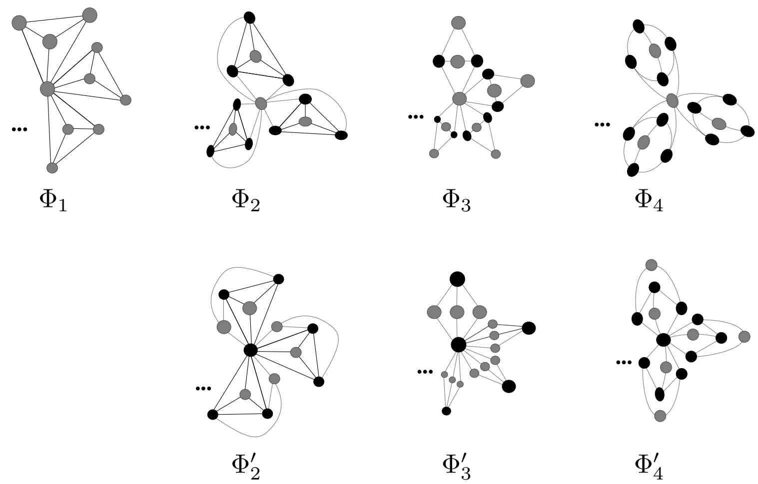

Figure 3: Some of the excluded marked minors of $\Sigma_\bullet$. The grey vertices represent the marked ones.

Lemma 5.3 is the technically most challenging part of the proof of Theorem 1.1. We prepare its proof with a number of lemmas. The first one is similar to Corollary 3.3.

Lemma 5.4. *Let G be a countable graph admitting a finitary decomposition $G_1, \ldots, G_k$. If each $(G_i, U \cap V(G_i))$ lies in $\Sigma_\bullet$ for some $U \subseteq V(G)$, then $G \in \Sigma_\bullet$.*

Proof. TOPROVE 13 □

Using this, we can now extend Lemma 5.1 from planar graphs to graphs in Σ:

Lemma 5.5. *Let G be a graph in Σ, and let $U \subset V(G)$. Then G lies in $\Sigma_\bullet$ if and only if it does not contain one of the marked graphs $(\Theta_i, U_i), 1 \le i \le 4$ of Figure 2 as a marked minor.*

Proof. TOPROVE 14 □

Given a marked graph (G, U) with $G \in \Sigma$, and a vertex $x \in V(G)$, we say that x is *$\Sigma_\bullet$-critical*, if (G, U) is not in $\Sigma_\bullet$ but $(G - x, U - x)$ is. The most difficult part of the proof of Lemma 5.3 lies in finding a Φ_5 minor in the case where G contains at least two $\Sigma_\bullet$-critical vertices. This is achieved (and refined) by the following two lemmas. Recall the definition of a U-star from Section 2.1.

Lemma 5.6. *Suppose $G \in \Sigma$, and $x \in V(G)$ is $\Sigma_\bullet$-critical for some $U \subseteq V(G)$. Then G contains a U-star with x as the infinite-degree vertex.*

Proof. TOPROVE 15 □

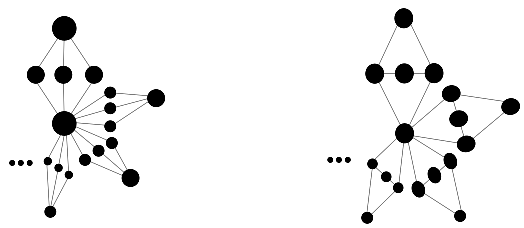

Figure 4: Two of the excluded minors of Proposition 7.1, arising by combining infinitely many copies of $K_{2,3}$.

We use Lemma 5.6 in order to prove

Lemma 5.7. *Suppose $G \in \Sigma$, and G has two $\Sigma_\bullet$-critical vertices x, y for some $U \subseteq V(G)$. Then G contains the marked double-star Φ_5 as a marked minor.*

Proof. TOPROVE 16 □

We can now prove the main result of this section:

Proof. TOPROVE 17 □

Remark 3. The converse of Lemma 5.3 holds too, that is, if G has one of the U-marked minors as in the statement, then G is not in $\Sigma_\bullet$. Indeed, the U-cone of each of these graphs is not in Σ, because such a cone contains one of the forbidden structures of Theorem 1.1 as we will see in the proof of Theorem 1.1.

6 The excluded minors of Σ

We can now prove our main result.

Proof. TOPROVE 18 □

7 Almost outerplanar graphs

The aim of this section is to prove the analogue of Proposition 4.3 for outerplanar graphs. This is included as a result of independent interest, proved using some of the techniques developed above.

Let OP denote the class of countable outerplanar graphs. Let G_1 (respectively, G_2) be the graph obtained from $\omega \cdot K_{2,3}$ by choosing a vertex of degree 3 (resp. degree 2) from each copy of $K_{2,3}$ and identifying them (Figure 4).

Proposition 7.1. $OP_{\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_4, \omega \cdot K_{2,3}, \bigvee K_4, G_1, G_2, K_{2,\omega})$.

Proof. TOPROVE 19 □

8 Almost forests revisited

The aim of this section is to prove the following result, which complements Proposition 4.3.

Proposition 8.1. *Let $\mathcal{F}$ denote the class of countable forests. Then* $\mathcal{F}_{/\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_3, \bigvee K_3) = \mathcal{F}_{\mathrm{E}/\mathrm{E}}$.

Combined with Proposition 4.3, it follows that $\mathcal{F}_{\mathrm{E}} \subsetneq \mathcal{F}_{/\mathrm{E}}$. For our proof we will need the following extension of the star-comb lemma.

A *2-star* is a graph obtained from the star $K_{1,\omega}$ by subdividing each edge at least once. In other words, a 2-star is obtained from the disjoint union of infinitely many paths of length at least 2 by identifying their first vertices.

We say that a vertex-set D *dominates* another vertex-set U of a graph, if $\overline{N}(D) \supseteq U$, where $\overline{N}(D)$ consists of D and all vertices sending an edge to D.

Lemma 8.2. *Let G be a connected graph, and $U \subseteq V(G)$. Then G contains at least one of the following:*

(i) A U-comb;

(ii) a 2-star with leaves in U;

(iii) a finite vertex-set dominating U.

Proof. TOPROVE 20 □

Proof. TOPROVE 21 □

9 A star-comb lemma for 2-connected graphs

While trying to prove Theorem 1.1 I came up with the following strengthening of the star-comb lemma for 2-connected graphs. Although it is not used for any of our proofs, I decided to include as it might become useful elsewhere. The star-comb lemma is one of the most useful tools in infinite graph theory. Some other strengthenings were obtained in a recent series of 4 papers by Bürger & Kurkofka [6]–[7]. A related result determining unavoidable induced subgraphs for infinite 2-connected graphs is obtained by Allred, Ding & Oporowski [1].

In analogy with U-stars and U-combs as in the statement of the star-comb lemma, we introduce the following structures. A *double-star* is a subdivision of $K_{2,\omega}$. A *ladder* consists of two disjoint rays R, L and an infinite collection of pairwise disjoint R–L paths. A *fan* consists of a ray R, a vertex $d \notin V(R)$, and an infinite collection of d–R paths having only d in common. For each of these three terms, adding the prefix *U-* means that the structure has infinitely many of its vertices in U. With this terminology, Theorem 1.3 from the introduction can be formulated as follows.

Theorem 9.1. *Let G be a 2-connected graph, and $U \subseteq V(G)$ infinite. Then G contains a U-double-star, or a U-ladder, or a U-fan.*

The following follows from the statement of Theorem 9.1, but we need to prove it first as a first step towards the proof of the latter.

Lemma 9.2. *Let G be a 2-connected, locally finite graph, and $U \subseteq V(G)$ infinite. Then G has a ray containing an infinite subset of U.*

In this section we assume that the reader is familiar with the basics about the end-compactification of a graph, and normal spanning trees; we refer to [10] therefor.

Proof. TOPROVE 22 □

Proof. TOPROVE 23 □

Problem 9.1. *Is it possible to generalise Theorem 9.1 to k-connected graphs, obtaining a finite list of subdivisions of k-connected graphs as unavoidable structures?*

Results of similar flavour have been obtained by Gollin & Heuer [18].

10 Final remarks

It would be interesting to find the excluded minors for the classes $\Sigma_{/\mathrm{E}}$, $\mathrm{Planar}_{\mathrm{E}}$, $\mathrm{Planar}_{/\mathrm{E}}$ and $\mathrm{Planar}_{\mathrm{E}/\mathrm{E}}$, and this should be within reach with the above methods and a little bit more work. I suspect that

$$\Sigma_{/\mathrm{E}} = \mathrm{Planar}_{/\mathrm{E}} = \mathrm{Planar}_{\mathrm{E}/\mathrm{E}} = \mathrm{Forb}(\omega \cdot K_5, \omega \cdot K_{3,3}, \bigvee K_5, \bigvee K_{3,3}).$$

The first two equalities have been proved in Proposition 4.7. I also suspect that $\mathrm{Planar}_{\mathrm{E}} = \mathrm{Forb}(\mathrm{Ex}(\Sigma) \cup \{K_5^{\oslash}, K_{3,3}^{\oslash}\})$, where $K^{\oslash}$ is obtained from a graph K by replacing each edge uv by infinitely many u–v paths of length 2.

Let us say that a minor-closed class $\mathcal{C}$ of graphs is *good*, if $\mathcal{C} = \mathrm{Forb}(X)$ for a finite set X of (possibly infinite) graphs. A well-know conjecture of Thomas [32] postulates that the countable graphs are well-quasi-ordered under the minor relation. A positive answer would imply that all minor-closed classes of countable graphs are good, but as mentioned in the introduction, this seems out of reach at the moment. Still, we could seek to extend the Graph Minor Theorem [31] by finding sufficient conditions for classes of infinite graphs to be good. The following questions suggest a possible direction, and the methods of this paper could be helpful. For further questions in a similar vein see [16].

Question 10.1. *Suppose $\mathcal{C}$ is a good minor-closed class of countable graphs. Must each of $\mathcal{C}_{\mathrm{V}}, \mathcal{C}_{\mathrm{E}}, \mathcal{C}_{/\mathrm{E}}, \mathcal{C}_{\mathrm{E}/\mathrm{E}}$ be good?*

We say that a class $\mathcal{C}$ of graphs is *co-finite*, if $\mathcal{C} = \mathrm{Forb}(S)$ for a set S of finite graphs (which set can be chosen to be finite by the Graph Minor Theorem [31]). Note that a graph G belongs to such a class $\mathcal{C}$ if and only if every finite minor of G does. Question 10.1 is open in general even if $\mathcal{C}$ is co-finite, except that $\mathcal{C}_{\mathrm{V}}$ is covered by Proposition 4.4 in this case. This papers provides some techniques for attacking it. In a similar spirit, one can ask whether the class of graphs admitting a finitary decomposition into graphs in $\mathcal{C}$ is good whenever $\mathcal{C}$ is good/co-finite.

We say that a class $\mathcal{C}$ of graphs is *UNCOF* (Union of Nested Co-finite classes), if there is a sequence $(C_n)_{n\in\mathbb{N}}$ of co-finite classes such that $\mathcal{C} = \bigcup_{n\in\mathbb{N}} C_n$ and $C_n \subseteq C_{n+1}$ holds for every $n \in \mathbb{N}$. The classes studied in this paper $(\Sigma, \mathcal{F}_{\mathrm{E}}, \mathcal{F}_{/\mathrm{E}}, OP_{\mathrm{E}}$, etc.) are easily seen to be UNCOF. Our results support

Conjecture 10.2. *Every UNCOF class of countable graphs is good.*

Another interesting example of an UNCOF class $\mathcal{C}$ comprises the graphs G of finite Colin de Verdière invariant $\mu(G)$, whereby for infinite G we define $\mu(G)$ to be the supremal m such that every finite subgraph $H \subset G$ satisfies $\mu(H) \leq m$. Is this $\mathcal{C}$ good? Can we determine $\mathrm{Ex}(\mathcal{C})$?

Not every proper minor-closed class is UNCOF. For example, $\mathrm{Forb}(K_\omega)$ is not, because it contains the disjoint union of $K_n, n \in \mathbb{N}$, which no proper co-finite class contains. Thus Conjecture 10.2 is weaker than Thomas' conjecture. Beware however that Conjecture 10.2 implies the Graph Minor Theorem: any minor-closed class of finite graphs is shown to be UNCOF by letting C_n be its sub-class comprising the elements with at most n vertices.

Acknowledgement

I thank Nathan Bowler and Max Pitz for spotting a mistake in an earlier version of the paper. I thank the anonymous referees for proposing several substantial improvements.

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