The Long Arm of Nashian Allocation in Online p-Mean Welfare Maximization

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Abstract

We study the online allocation of divisible items to n agents with additive valuations for p-mean welfare maximization, a problem introduced by Barman, Khan, and Maiti (2022). Our algorithmic and hardness results characterize the optimal competitive ratios for the entire spectrum of $-\infty \le p \le 1$. Surprisingly, our improved algorithms for all $p \le 1/\log n$ are simply the greedy algorithm for the Nash welfare, supplemented with two auxiliary components to ensure all agents have non-zero utilities and to help a small number of agents with low utilities. In this sense, the long arm of Nashian allocation achieves near-optimal competitive ratios not only for Nash welfare but also all the way to egalitarian welfare.

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1 Introduction

Resource allocation is a fundamental challenge in many domains of Computer Science, Economics, and beyond, such as cloud computing (e.g., Ghodsi et al., 2011) and food banks (e.g., Aleksandrov et al., 2015). The primary objectives in these scenarios are efficiency and fairness. For $-\infty \le p \le 1$, the p-means of the agents' utilities, known as the p-mean welfare, form an axiomatically justified family of objectives (Moulin, 2004, Chapter 3) with different tradeoffs between these two factors. On one extreme when $p = 1$, this is the utilitarian welfare, i.e., the sum of the agents' utilities. On the other extreme when $p = -\infty$, this is the egalitarian welfare, i.e., the minimum utility among the agents. In the middle when $p = 0$ is the Nash welfare, which reconciles the two extremes and satisfies several notions of fairness including envy-freeness and proportionality (Varian, 1974).

This paper studies online resource allocation algorithms for maximizing the p-mean welfare among n agents when the items arrive sequentially and need to be allocated immediately upon arrival. This *online p-mean welfare maximization* problem was proposed by Barman, Khan, and Maiti (2022), assuming that 1) the items are divisible, 2) the agents' utilities are additive, and 3) each agent's utility for receiving all items, a.k.a., its monopolist utility, is equal to 1. For the whole spectrum of $-\infty \le p \le 1$, Barman et al. (2022) gave competitive online algorithms and hardness results. However, there are gaps between their algorithmic and hardness results. For any negative constant p, the gap is polynomially large in the number of agents.

By contrast, the non-negative regime is better understood. For any positive p, the problem is equivalent to a special case of the *online matching with concave returns* problem introduced by Devanur and Jain (2012). Their online primal dual approach gave a $1/p$ competitive algorithm even without the third assumption above about unit monopolist utilities. We include an exposition of this algorithm in Appendix A to be self-contained, and also in Appendix B an improvement for $0 < p \le 1/\log n$ using the third assumption. For Nash welfare ($p = 0$), Banerjee, Gkatzelis, Gorokh, and Jin (2022) gave an $O(\log n)$-competitive algorithm,[1] which is the best possible.

1.1 Our Contributions

Optimal Competitive Ratios (up to Lower-Order Terms). We improve the upper and lower bounds for the competitive ratios of online p-mean welfare maximization. These bounds characterize the optimal competitive ratios for all $-\infty \le p \le 1$. See Table 1 for a summary.

For $p \ge 1/\log n$, we prove that the $1/p$ competitive ratio by Devanur and Jain (2012) is optimal up to a $\log 1/p$ term even if the instance satisfies the assumption of unit monopolist utilities.

For the Nashian regime, i.e., when $-1/\log n \le p \le 1/\log n$, we improve the $O(\log^3 n)$ competitive ratio by Barman et al. (2022) to $O(\log n)$, and give a corresponding hardness result matching it up to a doubly logarithmic factor. Note that for Nash welfare ($p = 0$), our result matches the optimal ratio of $O(\log n)$ obtained by Banerjee et al. (2022).

Between the Nashian regime and the harmonic mean, i.e., when $-1 \le p \le -1/\log n$, we give an online algorithm that is $n^{|p|/(|p|+1)}$-competitive, omitting lower-order terms. This is better than the state-of-the-art by Barman et al. (2022) by a polynomial factor for $p \ge -1/4$, and a poly-logarithmic factor for other values of p. We complement the algorithmic result with a hard instance, showing that no online algorithm can do better up to a logarithmic term. This hardness result is a polynomial improvement compared to the existing $n^{|p|/(2|p|+1)}$ for any negative constant p.

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[1] More precisely, their model assumes having predictions of the agents' monopolist utilities, which can be used to normalize the agents' utilities to restore the assumption of unit monopolist utilities. See also Huang, Li, Shu, and Wei (2023) for online algorithms that relaxed this assumption to allow the absence of predictions of agents' monopolist utilities but still require the max-min ratio of monopolist utilities to be bounded.

Table 1: Summary of Results. We omit constant and lower-order terms for brevity, and defer the precise bounds to the subsequent sections. The bounds on some rows were known for a special value of p: Banerjee et al. (2022) gave $\log n$ upper and lower bounds for $p = 0$, and Barman et al. (2022) showed a $\sqrt{n}$ lower bound for $p = -\infty$.

	Algorithmic Results			Hardness Results	
$1/\log n \leq p \leq 1$	$1/p$	Devanur and Jain (2012)		$1/p$	Theorem 13
$-1/\log n \leq p \leq 1/\log n$	$\log n$	Corollary 4		$\log n$	Theorem 13
$-o(1) \leq p \leq -1/\log n^2$	$n^{|p|}$	Theorem 3		$n^{|p|}$	Theorem 20
$-1 \leq p \leq -\Omega(1)$	$n^{\frac{|p|}{|p|+1}}$	Theorem 6		$n^{\frac{|p|}{|p|+1}}$	Theorem 21
$-\infty \leq p \leq -1$	$\sqrt{n}$	Theorem 9; Barman et al. (2022)		$\sqrt{n}$	Theorem 22

Last but not least, from harmonic mean to egalitarian welfare, i.e., when $-\infty \leq p \leq -1$, we show that the optimal competitive ratio is $\sqrt{n}$ up to a lower-order term. Our hardness result is a polynomial improvement on the existing $n^{|p|/(2|p|+1)}$ bound by Barman et al. (2022) for any finite p; our algorithmic result is slightly better than theirs by a $\sqrt{\log n}$ factor.

Long Arm of Nashian Allocation. Besides the improved competitive ratios, we find it conceptually interesting that our algorithmic results for the whole spectrum of $-\infty \leq p \leq 1/\log n$ are obtained by just two algorithms, both based on the greedy algorithm for the Nash welfare. *A priori*, it is surprising that the greedy algorithm for $p = 0$ also achieves nearly optimal competitive ratios for other values of p, even those that are far from zero. This can be viewed as further evidence for the effectiveness of Nash welfare maximization for balancing efficiency and fairness in the context of online optimization, echoing the unreasonable fairness from offline Nash welfare maximization showed by Caragiannis et al. (2019).

More precisely, the improvements for $-1 \leq p \leq 1/\log n$ are obtained by combining the greedy algorithm for the Nash welfare with the common idea of distributing a constant fraction of each item uniformly to all agents to ensure an $\Omega(1/n)$ base utility for the agents (see, e.g., Banerjee et al., 2022; Huang et al., 2023, for previous analyses of this algorithm for $p = 0$). The base utilities allow us to avoid the irregularity of p-mean welfare when an agent has zero utility, for any $p \leq 0$.

We further address the cases of $-\infty \leq p \leq -1$ by introducing an auxiliary component that considers the other extreme of the spectrum, greedily maximizing the egalitarian welfare with respect to the agents' *regularized utilities*. An agent's regularized utility is the sum of its utility and a regularization term that is linear in the agent's monopolist utility for the remaining items. We stress that the greedy algorithm for the Nash welfare still plays the main role in this regime. In particular, it ensures that at most $\tilde{O}(\sqrt{n})$ agents may have low regularized utilities; the auxiliary component is designed to help these agents by making regularized egalitarian allocation.

Our Techniques. Next, we sketch the competitive analysis of the greedy algorithm for the Nash welfare. Consider the algorithm's allocation and any alternative allocation. Denote each agent a's utilities for these allocations as U_a and $\tilde{U}_a$ respectively. We will prove that the ratio of these utilities, i.e., $\tilde{U}_a/U_a$, is at most $O(\log n)$ averaging over all agents a. This is easy to prove in hindsight, based on 1) the greedy criteria with respect to the Nash welfare, and the fact that 2) the agents'

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[2]The omitted terms are of lower order for $p \leq -\omega(\log \log n/\log n)$. For $-O(\log \log n/\log n) \leq p \leq -1/\log n$, our results show that the optimal competitive ratio is poly-logarithmic, but do not characterize the degree of the poly-logarithms.

maximum and minimum utilities differ by at most an $O(n)$ multiplicative factor due to the $\Omega(1/n)$ base utilities from distributing a constant fraction of each item uniformly to all agents. Nonetheless, this is powerful enough to derive all subsequent lemmas and competitive ratios in this paper; we call it the *Fundamental Lemma of Nashian Allocation* (Lemma 5).

In particular, we will use this fundamental lemma to derive upper bounds for the number of agents whose utilities are smaller than a threshold (Lemma 8), and the number of agents whose regularized utilities are smaller than a threshold (Lemma 10). Intuitively, these bounds are useful because the p-mean welfare for any negative $p < 0$ may be seen as a softmin function over the agents' utilities. Hence, lower bounding the p-mean welfare of the algorithm's allocation reduces to upper bounding the number of agents whose (regularized) utilities are too small.

Finally, our hardness results consider upper triangular instances that are prevalent in online resource allocation and online matching (e.g., Karp, Vazirani, and Vazirani, 1990; Kalyanasundaram and Pruhs, 2000). We supplement the upper triangular instances with items arriving at the end to satisfy the assumption of unit monopolist utilities.

1.2 Further Related Work

Online resource allocation problems have been studied extensively, including online packing (Alon et al., 2006; Buchbinder and Naor, 2009) and online matching (Karp et al., 1990; Mehta, 2013; Huang et al., 2024). Walsh (2011) and Aleksandrov et al. (2015) started the study of online fair division with agent and item arrivals respectively. See Aleksandrov and Walsh (2020) for a survey.

Closest to this paper is the work by Barman et al. (2022), which we have already discussed and compared against. Better results have been achieved in relaxed models of online allocation. Cohen and Panigrahi (2023) studied the problem in a learning-augmented setting, i.e., where the algorithm has access to some extra (machine-learned) information. Hajiaghayi et al. (2022) considered the case of egalitarian welfare and indivisible items, assuming that the items arrive by a random order and the instance satisfies a large-market assumption.

Another line of work considered online resource allocation for other notions of fairness. Benadè et al. (2018) achieved envy-freeness in online allocation in the sense of no-regret learning. Zeng and Psomas (2020) explored the fairness-efficiency tradeoffs in online fair division with respect to envy-freeness and Pareto optimality. Gkatzelis et al. (2021) considered the online allocation of divisible items (to two agents) to satisfy envy-freeness and at the same time approximately maximize utilitarian welfare. Banerjee et al. (2023) studied online allocation of public goods for proportional fairness, with predictions for the agents' monopolist utilities.

Finally, offline maximization for p-mean welfare has also been studied in a long line of work. Since the case of additive valuations and divisible items reduces to standard convex optimization, researchers focused on the harder case with indivisible items and non-additive valuations. For general $-\infty \le p \le 1$, Barman et al. (2020) and Chaudhury et al. (2021) independently gave $O(n)$-approximation algorithms for maximizing the p-mean welfare when the valuations are subadditive.

For $p = 0$, maximizing the Nash welfare is APX-hard (Lee, 2017). The state-of-the-art for additive valuations is an $e^{1/e}$-approximation algorithm by Barman et al. (2018), which has been generalized to the weighted case by Feng and Li (2024). Li and Vondrák (2022) obtained a constant approximation for submodular valuations while Garg et al. (2020) showed an $e/(e-1)$ lower bound. Recently, Dobzinski et al. (2024) gave a constant approximation for subadditive valuations.

For $p = -\infty$, maximizing the egalitarian welfare for additive valuations and indivisible items, a.k.a., the Santa Claus problem (e.g., Bansal and Sviridenko, 2006), is also APX-hard (Bezáková and Dani, 2005). Chakrabarty et al. (2009) gave an $\tilde{O}(n^\varepsilon)$-approximate algorithm that runs in $n^{O(1/\varepsilon)}$ time, for any $\varepsilon = \Omega(\log\log n/\log n)$.

2 Preliminaries

We write log for the natural logarithm in this paper.

2.1 Model

Discrete Time. Consider the problem of online allocation of m *divisible* items I to n agents A. Agent a has value $v_{ai} \geq 0$ for item i. The items arrive one by one. The agents' values for an item are unknown initially and revealed when the item arrives. Upon an item's arrival, the algorithm must allocate it to some agent(s) immediately and irrevocably. Denote the algorithm's allocation as $x = (x_{ai})_{a \in A, i \in I}$, where $x_{ai} \geq 0$ is the portion of item i allocated to agent a. Agent a's *utility* for allocation x is:

$$U_a = \sum_{i \in I} v_{ai} \cdot x_{ai} .$$

For some $p \leq 1$, we want to maximize the *p-mean welfare*:

$$\left(\frac{1}{n} \sum_{a \in A} U_a^p \right)^{\frac{1}{p}} . \tag{1}$$

When $p = 1$, this is the *utilitarian welfare* (up to a $1/n$ factor). When $p = 0$, Equation (1) is defined as the geometric mean of the agents' utilities, and is known as the *Nash welfare*. When $p = -1$, this becomes the harmonic mean of the agents' utilities; we therefore call it the *harmonic welfare*. Last but not least, when $p = -\infty$, Equation (1) is the minimum utility among the agents, and is called the *egalitarian welfare*.

Continuous Time. It will be more convenient to present our algorithms and analyses in the more general continuous-time model. Next, we present the model and then explain the reduction from the discrete-time model to the continuous-time model. Consider a continuum of infinitesimal items arriving in time horizon $[0, T)$ with unit arrival rate. We will refer to the item arriving at time t as item t, and thus, let $I = [0, T)$ denote the set of items. Agent a has unit value $v_a(t)$ for item t; we assume that $v_a(t)$ is piecewise constant, changing its value only a finite number of times. Denote the algorithm's allocation as $x = (x_a(t))_{a \in A, t \in I}$ where $x_a(t)$ is the portion of item t allocated to agent a. Correspondingly, agent a's utility for allocation x is:

$$U_a = \int_0^T v_a(t) x_a(t) \, \mathrm{d}t .$$

Given any instance in the discrete-time model where we without loss of generality denote the set of m divisible items as $\{1, 2, \ldots, m\}$, we can reinterpret it in the continuous-time model with $T = m$, such that agent a's value for items $i - 1 \leq t < i$ is $v_a(t) = v_{ai}$, for any $1 \leq i \leq m$.

Assumption of Unit Monopolist Utilities. Following Barman et al. (2022), we assume that the agents have unit monopolist utilities. That is, for any agent $a \in A$, we have:

$$\int_0^T v_a(t) \, \mathrm{d}t = 1 .$$

Trivial hardness results exist if the algorithm has no information on the agents' monopolist utilities. Banerjee et al. (2022) provided a hard instance that prevents any algorithm from performing

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better than $\Omega(n)$-competitive in online Nash welfare maximization ($p = 0$). The proof of $\Omega(n)$ hardness can be generalized to any $p \leq 0$ using the same hard instance.

Our results still hold under a weaker assumption that the monopolist utilities are known and within a constant factor of each other (see Appendix C). This corresponds to having predictions for the agents' monopolist utilities, which could be derived from historical data and machine learning models. Moreover, the agents are of similar importance in the market. Further relaxing this assumption is an interesting research direction. See Banerjee et al. (2022) and Huang et al. (2023) for some related results for the Nash welfare.

2.2 Competitive Analysis

Offline Optimal Allocation. We will compare the p-mean welfare of the algorithm's allocation to the offline optimal benchmark, obtained from optimizing the allocation based on full knowledge of the instance. We can compute the offline optimal benchmark by solving a convex program:

$$\text{maximize} \quad \left(\frac{1}{n} \sum_{a \in A} U_a^p \right)^{\frac{1}{p}}$$

$$\text{subject to} \quad \sum_{a \in A} x_a(t) \leq 1 \qquad \forall t \in I$$

$$U_a = \int_0^T v_a(t) x_a(t) \, \mathrm{d}t \qquad \forall a \in A$$

$$x_a(t) \geq 0 \qquad \forall a \in A, \forall t \in I$$

We denote the optimal allocation as $x^* = \left(x_a^*(t) \right)_{a \in A, t \in I}$ and the corresponding agents' utilities and p-mean welfare as $\{U_a^*\}_{a \in A}$ and OPT respectively. If there are multiple allocations that achieve the optimal p-mean welfare, we will pick an arbitrary one as x^*.

Competitive Online Algorithms. Denote the p-mean welfare for the online algorithm's allocation as ALG. An online algorithm is Γ-competitive if for every instance of online p-mean welfare maximization, we have:

$$\text{OPT} \leq \Gamma \cdot \text{ALG} .$$

2.3 Relaxation of Online p-Mean Welfare Maximization

We can guarantee a simple lower bound for all agents' utilities with the Uniform Allocation below.

Uniform Allocation

For each item $t \in I$, allocate it uniformly to all agents, i.e., $x_a(t) = 1/n$.

Lemma 1. *Every agent $a \in A$ gets utility $1/n$ from Uniform Allocation.*

As a result of this simple bound, we can consider a relaxation of online p-mean welfare maximization, in which each agent starts with $1/n$ base utility for the algorithm's allocation. In other words, an agent a's utility for an allocation x is:

$$U_a \overset{\text{def}}{=} \frac{1}{n} + \int_0^T v_a(t) x_a(t) \, \mathrm{d}t .$$

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We further write $U_a(t)$ for agent a's utility for the items it received before time t, i.e.:

$$U_a(t) \stackrel{\text{def}}{=} \frac{1}{n} + \int_0^t v_a(s)x_a(s)\,\mathrm{d}s \ .$$

As for the benchmark, we will still compare against the original optimal p-mean welfare without the $1/n$ base utility.

Lemma 2. *If an online algorithm A is Γ-competitive for the relaxed online p-mean welfare maximization problem, then allocating half of each item by Uniform Allocation and the other half by algorithm A is 2Γ-competitive for the original problem.*

3 Nashian Regime: $-o(1) \leq p \leq 1/\log n$

This section considers the Nashian regime where p is close to zero. We will analyze the greedy algorithm for maximizing the Nash welfare and prove that it is competitive for all values of p in this regime simultaneously.

Recall that the Nash welfare is the case of $p = 0$, defined as the geometric mean of the agents' utilities. Equivalently, we may consider maximizing its logarithm (up to a factor n):

$$\sum_{a \in A} \log U_a \ .$$

For each item t, allocating it to agent a would increase this objective by:

$$\frac{v_a(t)}{U_a(t)} \ .$$

Hence, the Nashian Greedy algorithm allocates each item to an agent to maximize the above increment. We present below a formal definition of the algorithm.

Nashian Greedy

Initialization: Let $x_a(t) = 0$, $U_a(0) = \frac{1}{n}$ for any agent $a \in A$ and any time $t \in I$.

Online Decisions: Allocate each item $t \in I$ to an agent a that maximizes:

$$\frac{v_a(t)}{U_a(t)} \ ,$$

to greedily maximize the (logarithm of) Nash welfare:

$$\sum_{a \in A} \log U_a(t) \ .$$

The main results of this section are the following theorem and its corollary when $|p| \leq 1/\log n$.

Theorem 3. *For any $|p| = o(1)$, Nashian Greedy is $O\big(n^{|p|} \cdot \log n\big)$-competitive for the relaxed online p-mean welfare maximization problem.*

Corollary 4. *For any p such that $|p| \leq 1/\log n$, Nashian Greedy is $O(\log n)$-competitive for the relaxed online p-mean welfare maximization problem.*

Next, we present the most important lemma for the analysis of Nashian Greedy. Despite its simple form and proof, it is the foundation of all subsequent lemmas and competitive analyses in this paper.

Lemma 5 (Fundamental Lemma of Nashian Allocation). *Consider any allocation of the items $\tilde{x} = (\tilde{x}_{ai})_{a \in A, i \in I}$. For any time $t \in I$ and the agents' utilities for allocation $\tilde{x}$ up to time t, denoted as $\tilde{U}(t) = (\tilde{U}_a(t))_{a \in A}$, we have:*

$$\frac{1}{n} \sum_{a \in A} \frac{\tilde{U}_a(t)}{U_a(t)} \le \log(n+1) .$$

Proof. TOPROVE 0 □

Proof. TOPROVE 1 □

4 From Nash to Harmonic Welfare: $-1 \le p \le -\Omega(1)$

In this section, we continue to analyze the Nashian Greedy algorithm. Surprisingly, it remains competitive even when p is bounded away from zero. The resulting competitive ratios are nearly optimal for all values of p from Nash welfare to harmonic welfare.

Theorem 6. *For any $-1 \le p \le -\Omega(1)$, Nashian Greedy is:*

$$O\left(n^{\frac{|p|}{|p|+1}} (\log n)^{\frac{1}{|p|+1}}\right)$$

competitive for the relaxed online p-mean welfare maximization problem.

4.1 Further Properties of Nashian Allocation

Definition 1 (Bad Agents). For any time $t \in I$ and any $\beta > 0$, let $B_\beta(t)$ be the set of β-*bad* agents whose utilities at time t are at most a β fraction of the optimal p-mean welfare, i.e.:

$$B_\beta(t) \stackrel{\text{def}}{=} \left\{ a \in A : U_a(t) \le \beta \cdot \mathrm{OPT} \right\} .$$

Further, we write B_β for $B_\beta(T)$, the set of β-bad agents at the end.

The next lemma considers a subset of β-bad agents, and upper bounds the sum of their utilities for the optimal allocation by a linear combination of $\beta \cdot \mathrm{OPT}$, the upper bound of these agents' utilities for the algorithm's allocation, and the sum of these agents' remaining monopolist utilities.

Lemma 7. *For any time $t \in I$, any $\beta > 0$, and any subset of β-bad agents $S \subseteq B_\beta(t)$, we have:*

$$\frac{1}{n} \sum_{a \in S} U_a^* \le \beta \log(n+1) \cdot \mathrm{OPT} + \frac{1}{n} \sum_{a \in S} \left(1 - \int_0^t v_a(s) \, \mathrm{d}s \right) .$$

Proof. TOPROVE 2 □

Lemma 8. *For any $\beta > 0$, the fraction of β-bad agents at the end is at most:*

$$\frac{|B_\beta|}{n} \le \left(\beta \log(n+1) \right)^{\frac{|p|}{|p|+1}} .$$

Proof. TOPROVE 3 □

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4.2   Proof of Theorem 6

We compare the p-mean welfare of Nashian Greedy to the optimal benchmark by:

$$\left(\frac{\text{ALG}}{\text{OPT}}\right)^p = \frac{1}{n}\sum_{a\in A}\left(\frac{U_a}{\text{OPT}}\right)^p .$$

By $U_a \geq 1/n$ and $\text{OPT} \leq 1$, and recalling that $p < 0$, we have :

$$\left(\frac{U_a}{\text{OPT}}\right)^p \leq n^{|p|} .$$

Therefore, the above ratio can be written as:

$$\left(\frac{\text{ALG}}{\text{OPT}}\right)^p = \int_0^{n^{|p|}} \big(\text{fraction of agents with } \left(U_a/\text{OPT}\right)^p \geq \alpha\big)\,\mathrm{d}\alpha$$

$$\leq \int_0^{n^{|p|}} \big(\alpha^{\frac{1}{p}}\log(n+1)\big)^{\frac{|p|}{|p|+1}}\,\mathrm{d}\alpha \qquad\qquad \text{(Lemma 8)}$$

$$= \int_0^{n^{|p|}} \big(\log(n+1)\big)^{\frac{|p|}{|p|+1}}\alpha^{-\frac{1}{|p|+1}}\,\mathrm{d}\alpha$$

$$= \frac{|p|+1}{|p|}\cdot n^{\frac{|p|^2}{|p|+1}}\big(\log(n+1)\big)^{\frac{|p|}{|p|+1}} .$$

Taking p-th root on both sides gives:

$$\frac{\text{ALG}}{\text{OPT}} \geq \underbrace{\left(\frac{|p|}{|p|+1}\right)^{\frac{1}{|p|}}}_{\Omega(1)\ \text{for}\ |p| = \Omega(1)} \cdot n^{-\frac{|p|}{|p|+1}}\big(\log(n+1)\big)^{-\frac{1}{|p|+1}} .$$

5   From Harmonic to Egalitarian Welfare: $-\infty \leq p \leq -1$

This section considers the regime between harmonic welfare and egalitarian welfare. In this case, we need to introduce an auxiliary component to obtain a new lower bound of the agents' utilities that is better than the $1/n$ base utility given by Uniform Allocation.

Although Nashian Greedy on its own fails to provide a better bound, we observe that the number of agents who are in trouble cannot exceed $\sqrt{n\log(n+1)}$ (see Corollary 11). Hence, we may reserve a constant fraction of each item to help these agents.

How can we identify the agents in trouble? Naïvely, it is natural to consider an egalitarian greedy algorithm that allocates the item to the agents who currently have the lowest utilities. However, an agent may have low utility just because its valuable items are yet to arrive. It would be a mistake to allocate items to such an agent purely based on egalitarian consideration. Instead, we introduce for each agent $a \in A$ a regularization term R_a, which equals the agent's remaining monopolist utility, i.e., its total utility for the items yet to arrived, scaled by a $\sqrt{n\log(n+1)}$ factor that is driven by the analysis. Our new component is the egalitarian greedy allocation with respect to the regularized utilities $U_a + R_a$.

We remark that Barman et al. (2022) also introduced an algorithmic sub-routine to identify the agents in trouble, and already observed the necessity to consider both the agents' utilities and their remaining monopolist utilities. Our regularized egalitarian allocation gives an alternative, and in

our opinion, more natural way to implement this idea. The resulting competitive ratios are also asymptotically better than those from Barman et al. (2022).

Below we present the formal definition of this Mixed Greedy algorithm that combines Nashian Greedy and Regularized Egalitarian Greedy. For cleaner constants in the analysis, we further relax the problem by letting there be two copies of each item. This relaxation only affects the competitive ratio by a constant factor.

Mixed Greedy (Nashian Greedy and Regularized Egalitarian Greedy)

Initialization: Let $x_a(t) = 0$, $U_a(0) = \frac{1}{n}$ for any agent $a \in A$ and any time $t \in I$.

Regularization: Define the regularizer of any agent $a \in A$ as:

$$R_a(t) = \frac{1}{\Phi}\left(1 - \int_0^t v_a(s)\,\mathrm{d}s\right),$$

where $\Phi = \sqrt{n\log(n+1)}$.

Online Decisions: For each item $t \in I$, which comes in two copies:

1. Allocate the *Nashian copy* using Nashian Greedy.

2. Allocate the *egalitarian copy* to the agents with the smallest regularized utility:

$$U_a(t) + R_a(t),$$

to greedily maximize the regularized egalitarian welfare:

$$\min_{a \in A}\left(U_a(t) + R_a(t)\right).$$

This algorithm achieves the same and nearly optimal competitive ratio simultaneously for all values of p from harmonic welfare to egalitarian welfare.

Theorem 9. *For any $-\infty \leq p \leq -1$, Mixed Greedy is $O\!\left(\sqrt{n\log n}\right)$-competitive for the relaxed online p-mean welfare maximization problem.*

5.1 Properties of the Regularized Utilities

Definition 2 (Critical Agents)**.** For any time $t \in I$, and for any $\beta \geq 0$, let $C_\beta(t)$ be the set of β-*critical* agents at time t, whose regularized utilities are at most a β fraction of the optimal p-mean welfare, i.e.:

$$C_\beta(t) = \left\{a \in A : U_a(t) + R_a(t) \leq \beta \cdot \mathrm{OPT}\right\}.$$

Lemma 10. *For any time $t \in I$ and any $\beta > 0$, the fraction of β-critical agents at time t is upper bounded by:*

$$\frac{|C_\beta(t)|}{n} \leq \max\left\{\left(2\beta\log(n+1)\right)^{\frac{|p|}{|p|+1}}, \left(2\Phi\beta\right)^{|p|}\right\}.$$

Proof. TOPROVE 4 □

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Corollary 11. *For any time $t \in I$, and:*

$$\beta^* = \frac{1}{2} \cdot n^{-\frac{1}{2} - \frac{1}{2|p|}} \left(\log(n+1)\right)^{-\frac{1}{2} + \frac{1}{2|p|}} ,$$ (2)

we have:

$$\left| C_{\beta^*}(t) \right| \leq \sqrt{n \log(n+1)} .$$

Using Corollary 11, we derive a universal lower bound for all agents' utilities.

Lemma 12. *For the choice of β^* in Equation* (2), *and any agent $a \in A$, we have:*

$$U_a \geq \beta^* \cdot \text{OPT} .$$

Proof. TOPROVE 5 □

5.2 Proof of Theorem 9

The proof is almost verbatim to that of Theorem 6, except that we will use the newly developed Lemma 12 to lower bound the agents' utilities, replacing the basic bound $U_a \geq 1/n$.

Consider the p-th power of the Mixed Greedy algorithm's p-mean welfare, normalized by the p-th power of OPT:

$$\left(\frac{\text{ALG}}{\text{OPT}}\right)^p = \frac{1}{n} \sum_{a \in A} \left(\frac{U_a}{\text{OPT}}\right)^p .$$

By Lemma 12 and $p < 0$, we have:

$$\left(\frac{U_a}{\text{OPT}}\right)^p \leq (\beta^*)^p .$$

The above ratio can therefore be written as:

$$
\begin{aligned}
\left(\frac{\text{ALG}}{\text{OPT}}\right)^p &= \int_0^{(\beta^*)^p} \left(\text{fraction of agents with } \left(U_a/\text{OPT}\right)^p \geq \alpha\right) d\alpha \\
&\leq \int_0^{(\beta^*)^p} \left(\alpha^{\frac{1}{p}} \log(n+1)\right)^{\frac{|p|}{|p|+1}} d\alpha && \text{(Lemma 8)} \\
&= \int_0^{(\beta^*)^p} \left(\log(n+1)\right)^{\frac{|p|}{|p|+1}} \alpha^{-\frac{1}{|p|+1}} d\alpha \\
&= \frac{|p|+1}{|p|} (\beta^*)^{-\frac{|p|^2}{|p|+1}} \left(\log(n+1)\right)^{\frac{|p|}{|p|+1}} \\
&= \frac{|p|+1}{|p|} 2^{\frac{|p|^2}{|p|+1}} \left(n \log(n+1)\right)^{\frac{|p|}{2}} .
\end{aligned}
$$

Taking p-th root on both sides gives:

$$
\frac{\text{ALG}}{\text{OPT}} \geq \underbrace{\left(\frac{|p|}{|p|+1}\right)^{\frac{1}{|p|}} 2^{-\frac{|p|}{|p|+1}}}_{\Omega(1) \text{ for } p \leq -1} \cdot \left(\sqrt{n \log(n+1)}\right)^{-1} .
$$

6 Hardness for the Nashian and Positive Regimes: $p \geq -1/\log n$

Recall that the online primal dual algorithm by Devanur and Jain (2012) is $1/p$-competitive for $1/\log n \leq p \leq 1$ (Theorem 24), and Nashian Greedy is $O(\log n)$-competitive for $-1/\log n \leq p \leq 1/\log n$ (Corollary 4). This section will complement these competitive ratios with hardness results that match them up to a lower-order term. Formally, we will show that:

Theorem 13. *For any $p \leq 1$ and any online algorithm for online p-mean welfare maximization, the competitive ratio is no smaller than:*

$$\begin{cases} \Omega\left(\frac{1}{p}\right) & \text{if } p \geq \frac{\log \log n}{\log n}; \\ \Omega\left(\frac{\log n}{\log \log n}\right) & \text{otherwise,} \end{cases}$$

even when the agents have binary valuations (and unit monopolist utilities).

This asymptotic lower bound holds for a sufficiently large number of agents. It implicitly covers three cases. If p is a positive constant independent of the number of agents n, it characterizes how the competitive ratio changes as p tends to 0. If $p(n)$ is a function of the number of agents n that converges to zero as n goes to infinity, e.g., $p(n) = 1/\log n$, the theorem asserts that for the competitive ratio is no smaller than $\Omega(1/p(n))$ if $p(n)$ converges to zero slower than function $\log \log n/\log n$, or no smaller than $\Omega(\log n/\log \log n)$ otherwise. The latter bound also applies when p is a negative constant or a negative function $p(n)$ that converges to zero.

Since the items are divisible, we may without loss of generality focus on deterministic online algorithms. Given any randomized online algorithm, we may instead consider a corresponding deterministic algorithm that allocates each item $i \in I$ to each agent $a \in A$ by the expected amount allocated by the randomized algorithm.

6.1 Hard Instance

It suffices to consider $p \leq 1/16$ and a sufficiently large n to prove the asymptotic bounds in Theorem 13. The hard instance depends on two parameters:

$$M = \begin{cases} \frac{1}{4p} & \text{if } p \geq \frac{\log \log n}{\log n}; \\ \frac{\log n}{4 \log \log n} & \text{otherwise.} \end{cases} \quad , \quad L = \frac{\log n}{2} \geq 2M \log \log n \ .$$

By $p \leq 1/16$ and for a sufficiently large n, we have $M \geq 4$.

The items arrive in two stages: an *Upper Triangular Stage* and a *Makeup Stage*. See Figure 1 for an illustration of this hard instance.

Upper Triangular Stage. This stage proceeds in L rounds, during which we will adaptively partition the agents into $L + 1$ groups. After each round, we will determine one group of agents in the partition. We will refer to the agents in these L groups as the *bad agents* because their utilities for the algorithm's allocation will be at most an $O(1/M)$ fraction of their utilities for the optimal allocation. After these L rounds, we will put all remaining agents into the last group, and call them the *good agents* because their utilities for the algorithm's allocation will be greater than or equal to their utilities for the optimal allocation. However, the fraction of good agents will be small.

We now describe the Upper Triangular Stage in detail. In the first round, morally speaking, we would like to have one item with supply:

$$v_1 \cdot \frac{n}{M} \quad \text{where} \quad v_1 = e^{-L} = \frac{1}{\sqrt{n}} \ ,$$

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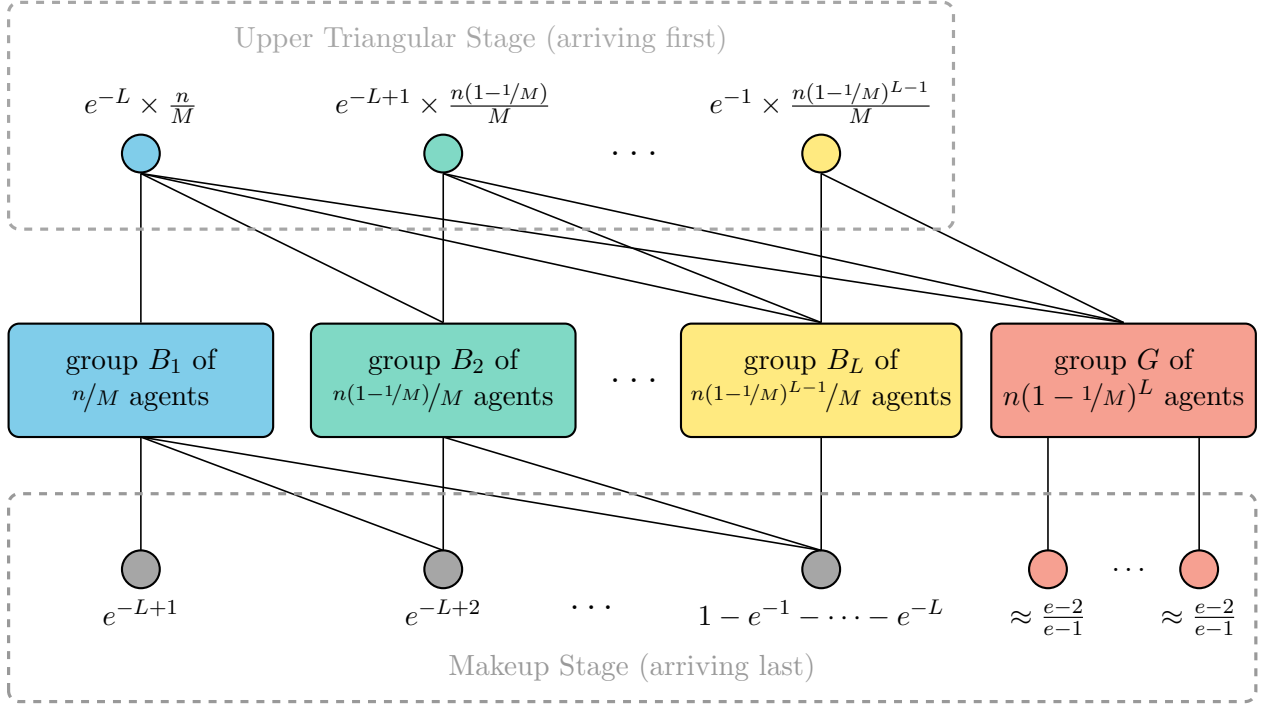
(a) Structure of the hard instance. Rounded rectangles represent groups of agents. Circles represent items, labeled by their supplies. Having an edge between a group of agents and an item means that the agents have value 1 for the item; the parallel edges between the good agents in group G and the corresponding items in the Makeup Stage indicate having a separate item for every agent in the group. The items in the Upper Triangular Stage arrive from left to right; each item in the picture corresponds to multiple items in the gadget illustrated in part (b). The items' arrival order in the Makeup Stage does not affect the argument. The groups of agents and items are colored by the offline optimal allocation; some items are colored gray because their allocations do not affect the argument.
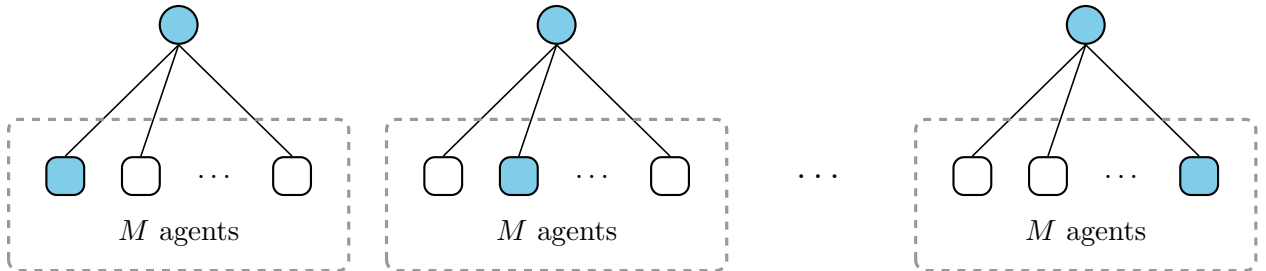
(b) Gadget for a single round in the Upper Triangular Stage. The (currently) ungrouped agents are partitioned into subsets of size M, represented by dashed rounded rectangles. For each subset in the partition, there is an item of value 1 only to the agents in the subset; the item's supply is $v_\ell = e^{-L-1+\ell}$ in round ℓ. The colored agent in each subset is an agent with the lowest utility after the allocation of this item. These colored agents will form the group of bad agents from this round.

Figure 1: Illustration of the hard instance for Nashian and positive regimes, i.e., when $p \geq -1/\log n$.

for which all agents have value 1. However, this would violate the assumption of unit monopolist utilities. Instead, we use the following gadget from the hard instance by Banerjee et al. (2022). We partition the agents into subsets of M agents arbitrarily. For each subset S in the partition, let there be an item with supply v_1 that is of value 1 to the agents in S, and 0 to the others. Note that the items' total value remains $v_1 \cdot n/M$, as long as we allocate every item to an agent with value 1 for it. Meanwhile, the contribution to each agent's monopolist utility reduces to v_1.

Since the instance so far is symmetric for all agents, intuitively the online algorithm should simply distribute every item equally to the agents who have value 1 for it. In that case, the agents' utilities would be v_1/M and we may let B_1 be an arbitrary subset of n/M agents. In general, we let B_1 be comprised of an agent with the *lowest* utility from each subset in the partition.

We repeat this process L times to adaptively define L groups of bad agents $B_1, B_2, \ldots, B_L$. At the beginning of round $1 \le \ell \le L$, there are:

$$n_\ell \overset{\text{def}}{=} n \left(1 - \frac{1}{M} \right)^{\ell-1}$$

agents not in groups $B_1, \ldots, B_{\ell-1}$. Morally speaking, we would like to have one item with supply:

$$v_\ell \cdot \frac{n_\ell}{M} \quad \text{where} \quad v_\ell \overset{\text{def}}{=} e^{-L-1+\ell} ,$$

that is of value 1 to these n_ℓ ungrouped agents, and 0 to the others. To reduce the contribution to the agents' monopolist utilities, we partition the ungrouped agents into subsets of M agents arbitrarily. For each subset S in the partition, let there be an item with supply v_ℓ that is of value 1 to the agents in S, and 0 to the others. After the allocation, we let B_ℓ be comprised of an agent with the lowest utility from each subset in the partition.

After all L rounds of the Upper Triangular Stage, there are still:

$$n \left(1 - \frac{1}{M} \right)^L$$

ungrouped agents. Let G denote this set of agents, which we will refer to as the good agents.

Makeup Stage. Observe that the agents' monopolist utilities are still smaller than 1 after the Upper Triangular Stage. Next, we let additional items arrive in the Makeup Stage to achieve unit monopolist utilities. For the good agents, the algorithm's allocation yields higher utilities than the offline benchmark. Hence, we will help the offline benchmark by letting each good agent have its own item in the Makeup Stage, so that its utility for the offline benchmark is at least $\Omega(1)$. For the bad agents, by contrast, we will let them compete for the same items, in order to limit the online algorithm's ability to catch up with the offline benchmark in terms of the bad agents' utilities.

We now describe the Makeup Stage in detail. For each good agent $a \in G$, the Makeup Stage has an item with supply:

$$1 - \sum_{\ell=1}^{L} v_\ell \;=\; 1 - \frac{1}{e} - \frac{1}{e^2} - \cdots - \frac{1}{e^L} \;\ge\; \frac{e-2}{e-1} \tag{3}$$

and is of value 1 only to agent a.

Finally, we consider the bad agents. For each group B_ℓ, the agents therein have monopolist utilities $\sum_{i=1}^{\ell} v_i$ after the Upper Triangular Stage. Let there be an item with supply $1 - \sum_{\ell=1}^{L} v_\ell$ that is of value 1 to all bad agents, and 0 to the good agents. Further, for any $1 < \ell \le L$, let there be an item with supply v_ℓ that is of value 1 to the agents in B_1 to $B_{\ell-1}$, and 0 to the others. By the construction of these items, all bad agents have unit monopolist utilities at the end.

6.2 Properties of the Hard Instance

First, we describe an offline allocation for the above hard instance, and lower bound the agents' utilities for it. These bounds for the agents' utilities will provide a lower bound for the optimal p-mean welfare in our proof of Theorem 13.

Lemma 14. *There is an allocation such that for any $1 \leq \ell \leq L$, the agents in group B_ℓ have utilities at least v_ℓ, and the agents in group G have utilities at least $(e-2)/(e-1)$.*

Proof. TOPROVE 6 □

Further, we characterize the limitations of the allocation by any deterministic online algorithm. For any $1 \leq \ell \leq L$, let $\bar{U}_\ell$ denote the average utility of the agents in group B_ℓ. We show that these bad agents on average only receive an $O(1/M)$ fraction of what they get in Lemma 14.

Lemma 15. *For any $1 \leq \ell \leq L$, we have $\bar{U}_\ell \leq 3v_\ell/M$.*

Proof. TOPROVE 7 □

6.3 Proof of Theorem 13

By Lemma 14, we have:

$$
\begin{aligned}
\text{OPT} &\geq \left(\left(1 - \frac{1}{M}\right)^L \cdot \left(\frac{e-2}{e-1}\right)^p + \sum_{\ell=1}^{L} \frac{1}{M}\left(1 - \frac{1}{M}\right)^{\ell-1} \cdot v_\ell^p \right)^{\frac{1}{p}} \\
&\geq \frac{e-2}{e-1}\left(\left(1 - \frac{1}{M}\right)^L + \sum_{\ell=1}^{L} \frac{1}{M}\left(1 - \frac{1}{M}\right)^{\ell-1} \cdot v_\ell^p \right)^{\frac{1}{p}} .
\end{aligned}
$$

By the concavity of p-mean welfare in the agents' utilities and Lemma 15, we have:

$$
\begin{aligned}
\text{ALG} &\leq \left(\left(1 - \frac{1}{M}\right)^L \cdot 1^p + \sum_{\ell=1}^{L} \frac{1}{M}\left(1 - \frac{1}{M}\right)^{\ell-1} \cdot \bar{U}_\ell^p \right)^{\frac{1}{p}} \\
&\leq \left(\left(1 - \frac{1}{M}\right)^L + \sum_{\ell=1}^{L} \frac{1}{M}\left(1 - \frac{1}{M}\right)^{\ell-1} \cdot \left(\frac{3v_\ell}{M}\right)^p \right)^{\frac{1}{p}} .
\end{aligned}
$$

To simplify notations, let:

$$
A = \left(1 - \frac{1}{M}\right)^L \quad , \quad B(p) = \sum_{\ell=1}^{L} \frac{1}{M}\left(1 - \frac{1}{M}\right)^{\ell-1} v_\ell^p .
$$

We get that:

$$
\frac{\text{ALG}}{\text{OPT}} \leq \frac{e-1}{e-2}\left(\frac{A + B(p) \cdot \left(\frac{3}{M}\right)^p}{A + B(p)} \right)^{\frac{1}{p}}
$$

Next, we bound the ratio between A and $B(p)$. Recall that $v_\ell = e^{-L-1+\ell}$. We have:

$$
\frac{B(p)}{A} = \frac{1}{M} \sum_{\ell=1}^{L} \left(\left(1 - \frac{1}{M}\right)e^p \right)^{-(L+1-\ell)} .
$$

By $1 - x \leq e^{-x}$ and $p \leq 1/2M$, we have:

$$\left(1 - \frac{1}{M}\right)e^p \leq e^{-\frac{1}{M}+p} \leq e^{-\frac{1}{2M}} .$$

Hence, we get that:

$$\frac{B(p)}{A} \geq \frac{1}{M}\sum_{\ell=1}^{L} e^{\frac{L+1-\ell}{2M}} = \frac{e^{\frac{L}{2M}} - 1}{M(1 - e^{-\frac{1}{2M}})} .$$

On one hand, we have:

$$1 - e^{-\frac{1}{2M}} \leq \frac{1}{2M} .$$

On the other hand, recall that $L \geq 2M \log \log n$. We get that:

$$e^{\frac{L}{2M}} \geq \log n$$

Therefore, for sufficiently large n, we conclude that:

$$\frac{B(p)}{A} \geq 2\big(\log n - 1\big) \geq \log n .$$

It remains to bound:

$$\left(\frac{A}{A+B(p)} + \frac{B(p)}{A+B(p)}\left(\frac{3}{M}\right)^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{1 + \log n} + \frac{\log n}{1 + \log n}\left(\frac{3}{M}\right)^p\right)^{\frac{1}{p}} . \tag{4}$$

Consider any $p \geq \log\log n/\log n$. The right-hand-side of Equation (4) equals:

$$\frac{3}{M}\left(1 + \frac{1}{1 + \log n}\left(\left(\frac{M}{3}\right)^p - 1\right)\right)^{\frac{1}{p}} .$$

Further, we have:

$$\left(\frac{M}{3}\right)^p = e^{p \log \frac{M}{3}}$$
$$\leq e^{p \log \frac{1}{p}} \qquad\qquad (M = 1/4p)$$
$$\leq 1 + 2p \log \frac{1}{p} . \qquad\qquad (e^x \leq 1 + 2x \text{ for } 0 \leq x \leq 1)$$

Hence, Equation (4) is at most:

$$\frac{3}{M}\left(1 + \frac{1}{1 + \log n}2p \log \frac{1}{p}\right)^{\frac{1}{p}} \leq \frac{3}{M}(1 + p)^{\frac{1}{p}} \qquad\qquad (p \geq \frac{1}{\log n})$$
$$\leq \frac{3e}{M} .$$

For $p < \log\log n/\log n$, M does not depend on p. Hence, the right-hand-side of Equation (4) can be seen as a weighted p-mean of 1 and $3/M$. This is increasing in p. Hence, it reduces to the case of $p = \log\log n/\log n$.

7 Hardness for the Negative Regime: $p \leq -1/\log n$

Recall that Nashian Greedy is $n^{|p|}$-competitive for $-o(1) \leq p \leq -1/\log n$ and $n^{|p|/(|p|+1)}$-competitive for $-1 \leq p \leq -\Omega(1)$, up to lower-order and logarithmic factors (Theorem 3). Moreover, Mixed Greedy is $O(\sqrt{n \log n})$-competitive for $-\infty \leq p \leq -1$ (Theorem 9). This section complements these algorithmic results with almost matching lower bounds.

7.1 Family of Hard Instances

We consider a family of hard instances $I(L, \alpha)$ parameterized by a positive integer L and a real number $0 \leq \alpha < |p|$. The items arrive in two stages: an Upper Triangular Stage and a Makeup Stage. See Figure 2 for an illustration of this family of hard instances.
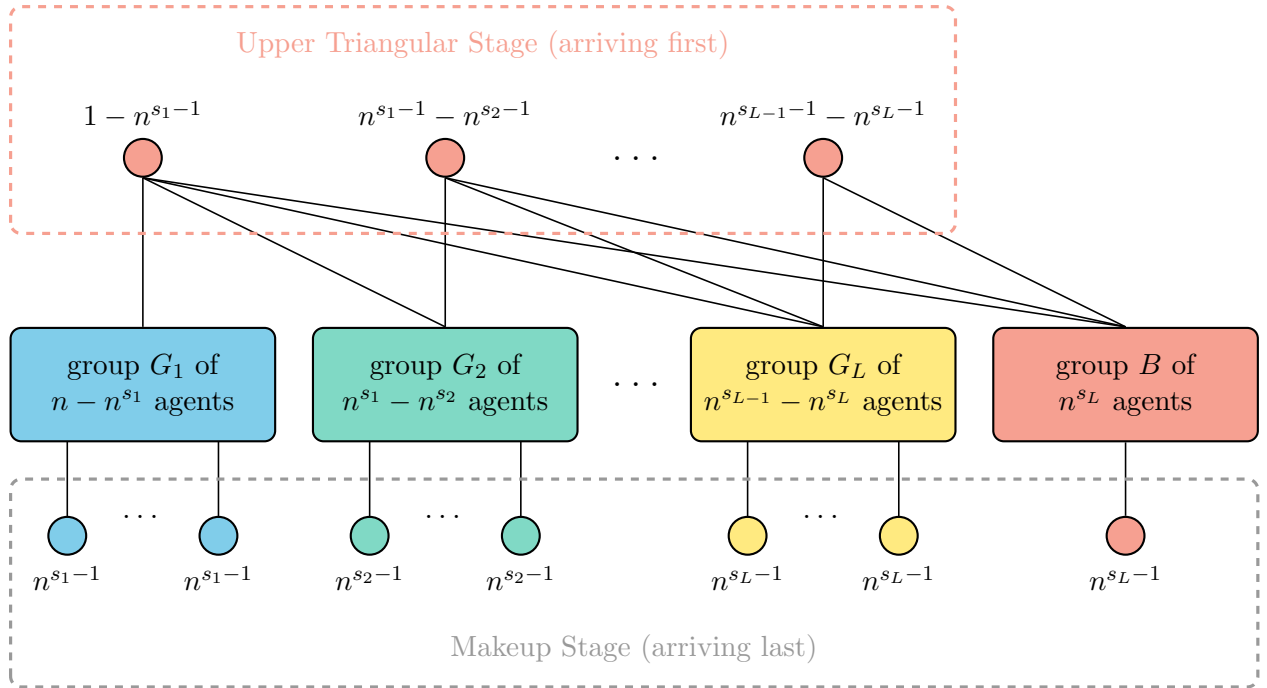

Figure 2: Illustration of the family of hard instances for the negative regime. Here, $L \geq 1$ is the number of groups of good agents. A decreasing sequence $1 = s_0 > s_1 > \cdots > s_L \geq 0$ determines the group sizes. Rounded rectangles represent groups of agents, labeled by the groups' indices and sizes. Circles represent items, labeled by the items' supplies. Having an edge between a group of agents and an item means that the agents have value 1 for the item; the parallel edges between the good agents and the corresponding items in the Makeup Stage indicate having a separate item for every such agent. The items in the Upper Triangular Stage arrive first from left to right; then, those in the Makeup Stage arrive by an arbitrary order. The groups of agents and items are colored according to the offline optimal allocation. For example, the items in the Upper Triangular Stage are colored red because the offline benchmark allocates them to group B which is also colored red.

Upper Triangular Stage. This stage proceeds in L rounds, during which we will adaptively partition the agents into $L + 1$ groups. After each round, we will determine one group of agents in the partition. We will refer to the agents in these L groups as the *good agents* because their utilities

for the algorithm's allocation might be greater than their utilities for the optimal allocation. After these L rounds, we will put all remaining agents into the last group, and call them the *bad agents* because their utilities for the algorithm's allocation will be low.

We now describe the Upper Triangular Stage in detail. Based on parameters L and α, we will choose a decreasing sequence:

$$1 = s_0 > s_1 > \cdots > s_L \geq 0 \,,$$

such that the ℓ-th group of good agents G_ℓ has size $n^{s_{\ell-1}} - n^{s_\ell}$. Correspondingly, the number of bad agents in B equals n^{s_L}. We will first present the structure of the hard instance, deferring the choice of this decreasing sequence to the end of the subsection.

In the first round, let an item arrive with supply $n^{s_0-1} - n^{s_1-1}$ and value 1 to all agents. Given the online algorithm's allocation of this item, let G_1 be comprised of $n^{s_0} - n^{s_1}$ agents with the *highest* utilities (breaking ties arbitrarily), so that the remaining agents' average utility is at most:

$$\frac{1}{n} \cdot \left(n^{s_0-1} - n^{s_1-1}\right) < \frac{1}{n} \,.$$

Here, recall that we may consider deterministic online algorithms without loss of generality.

We repeat this process to adaptively define L groups of good agents $G_1, G_2, \ldots, G_L$. In each round $1 \leq \ell \leq L$, let an item arrive with supply $n^{s_{\ell-1}-1} - n^{s_\ell-1}$ and value 1 to the ungrouped agents, i.e., those not in G_1 to $G_{\ell-1}$. Given the online algorithm's allocation of this item, let the G_ℓ be comprised of the $n^{s_{\ell-1}} - n^{s_\ell}$ ungrouped agents with the highest utilities.

After all L rounds of the Upper Triangular Stage, there are still n^{s_L} ungrouped agents. Let B denote this set of agents, which we will refer to as the bad agents.

Makeup Stage. We next let additional items arrive such that the agents have unit monopolist utilities at the end. Since the good agents might have high utilities for the algorithm's allocation, we will help the offline benchmark by letting each good agent have its own item in the Makeup Stage, so that the agent's utility for the offline benchmark is at least $\Omega(1)$. By contrast, the bad agents' utilities for the algorithm's allocation are low. We will therefore let them compete for the same item in the Makeup Stage, limiting the algorithm's ability to catch up with the offline benchmark in terms of the bad agents' utilities.

More precisely, for each good agent $a \in G_\ell$, let there be an item with supply $n^{s_\ell-1}$ and value 1 only to agent a. Let there be another item for the bad agents, where the item has supply n^{s_L-1}, and value 1 only to the bad agents.

Choice of Group Sizes. Finally, we define the decreasing sequence $1 = s_0 > s_1 > \cdots > s_L \geq 0$, and correspondingly, the group sizes. Recall that $s_0 = 1$. We define the rest of the sequence to satisfy a set of linear equations:

$$s_{\ell-1} + |p|(1 - s_\ell) = s_L(1 + |p|) - \alpha \,, \qquad \forall 1 \leq \ell \leq L \,. \tag{5}$$

The closed-form solution is:

$$s_\ell = 1 - \frac{|p| - \alpha}{2\sum_{i=1}^{L} |p|^i + 1} \sum_{i=L-\ell}^{L-1} |p|^i \,, \qquad \forall 0 \leq \ell \leq L \,. \tag{6}$$

By this closed-form, the sequence is indeed decreasing when $\alpha < |p|$.

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7.2 Properties of the Hard Instances

We first show that the bad agents have low utilities for the online algorithm's allocation, because most of the items in the Upper Triangular Stage have been allocated to the good agents.

Lemma 16. *The average utility of the agents in B for the algorithm's allocation is less than $(L+1)/n$.*

Proof. TOPROVE 8 □

Next, we upper bound the p-mean welfare of the online algorithm's allocation, using the fact that the average utility of the bad agents in group B is low.

Lemma 17. *For hard instance $I(L, \alpha)$, we have:*

$$\text{ALG} \leq (L+1) \cdot n^{\frac{1-s_L-|p|}{|p|}} .$$

Proof. TOPROVE 9 □

Finally, we lower bound the p-mean welfare of the offline optimal allocation.

Lemma 18. *For hard instance $I(L, \alpha)$, we have:*

$$\text{OPT} \geq \left(1 + \frac{L}{n^\alpha}\right)^{-\frac{1}{|p|}} n^{\frac{1-s_L(|p|+1)}{|p|}}$$

Proof. TOPROVE 10 □

Corollary 19. *For hard instance $I(L, \alpha)$, we have:*

$$\frac{\text{OPT}}{\text{ALG}} \geq \frac{\left(1 + \frac{L}{n^\alpha}\right)^{-\frac{1}{|p|}} \cdot n^{1-s_L}}{L+1} .$$

7.3 Lower Bounds for Competitive Ratios

In the range of $-O(\log\log n/\log n) \leq p \leq -1/\log n$, we resort to the $\Omega(\log n/\log\log n)$ lower bound by Theorem 13. Comparing this lower bound with the $O(n^{|p|} \log n)$ upper bound by Nashian Greedy (Theorem 3), we know that the optimal competitive ratio is poly-logarithmic in the number of agents, transitioning toward the polynomial dependence when p is a negative constant.

We next present the lower bounds for $p \leq -\omega(\log\log n/\log n)$.

Theorem 20. *For any $-o(1) \leq p \leq -\omega(\log\log n/\log n)$ and any online algorithm for online p-mean welfare maximization, the competitive ratio is no smaller than:*

$$n^{|p|-o(|p|)} ,$$

even when the agents have binary valuations (and unit monopolist utilities).

Proof. TOPROVE 11 □

Theorem 21. *For any $-1 \leq p \leq \Omega(1)$ and any online algorithm for online p-mean welfare maximization, the competitive ratio is no smaller than:*

$$n^{\frac{|p|}{|p|+1}-o(1)},$$

even when the agents have binary valuations (and unit monopolist utilities).

Proof. TOPROVE 12 □

Theorem 22. *For any* $-\infty \leq p \leq -1$ *and any online algorithm for online p-mean welfare maximization, the competitive ratio is no smaller than:*

$$n^{\frac{1}{2}-o(1)}.$$

Proof. TOPROVE 13 □

References

M. Aleksandrov and T. Walsh. Online fair division: A survey. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence*, pages 13557–13562, 2020.

M. Aleksandrov, H. Aziz, S. Gaspers, and T. Walsh. Online fair division: analysing a food bank problem. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence*, pages 2540–2546, 2015.

N. Alon, B. Awerbuch, Y. Azar, N. Buchbinder, and J. Naor. A general approach to online network optimization problems. *ACM Transactions on Algorithms*, 2(4):640–660, 2006.

S. Banerjee, V. Gkatzelis, A. Gorokh, and B. Jin. Online Nash social welfare maximization with predictions. In *Proceedings of the 33rd Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1–19. SIAM, 2022.

S. Banerjee, V. Gkatzelis, S. Hossain, B. Jin, E. Micha, and N. Shah. Proportionally fair online allocation of public goods with predictions. In *Proceedings of the 32nd International Joint Conference on Artificial Intelligence*, pages 20–28, 2023.

N. Bansal and M. Sviridenko. The santa claus problem. In *Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, page 31–40, 2006.

S. Barman, S. K. Krishnamurthy, and R. Vaish. Finding fair and efficient allocations. In *Proceedings of the 19th ACM Conference on Economics and Computation*, pages 557–574, 2018.

S. Barman, U. Bhaskar, A. Krishna, and R. G. Sundaram. Tight approximation algorithms for p-mean welfare under subadditive valuations. In *Proceedings of the 28th Annual European Symposium on Algorithms*, pages 11:1–11:17, 2020.

S. Barman, A. Khan, and A. Maiti. Universal and tight online algorithms for generalized-mean welfare. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence*, pages 4793–4800, 2022.

G. Benadè, A. M. Kazachkov, A. D. Procaccia, and C.-A. Psomas. How to make envy vanish over time. In *Proceedings of the 19th ACM Conference on Economics and Computation*, pages 593–610, 2018.

I. Bezáková and V. Dani. Allocating indivisible goods. *ACM SIGecom Exchanges*, 5(3):11–18, 2005.

N. Buchbinder and J. Naor. Online primal-dual algorithms for covering and packing. *Mathematics of Operations Research*, 34(2):270–286, 2009.

I. Caragiannis, D. Kurokawa, H. Moulin, A. Procaccia, N. Shah, and J. Wang. The unreasonable fairness of maximum nash welfare. *ACM Transactions on Economics and Computation*, 7(3): 1–32, 2019.

D. Chakrabarty, J. Chuzhoy, and S. Khanna. On allocating goods to maximize fairness. In *Proceedings of the 50th Annual IEEE Symposium on Foundations of Computer Science*, pages 107–116, 2009.

B. R. Chaudhury, J. Garg, and R. Mehta. Fair and efficient allocations under subadditive valuations. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence*, pages 5269–5276, 2021.

I. R. Cohen and D. Panigrahi. A general framework for learning-augmented online allocation. In *Proceedings of the 50th International Colloquium on Automata, Languages, and Programming*, pages 43:1–43:21, 2023.

N. R. Devanur and K. Jain. Online matching with concave returns. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing*, pages 137–144, 2012.

S. Dobzinski, W. Li, A. Rubinstein, and J. Vondrák. A constant-factor approximation for nash social welfare with subadditive valuations. In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*, pages 467–478, 2024.

Y. Feng and S. Li. A note on approximating weighted nash social welfare with additive valuations. In *Proceedings of the 51st EATCS International Colloquium on Automata, Languages and Programming*, 2024.

J. Garg, P. Kulkarni, and R. Kulkarni. Approximating Nash social welfare under submodular valuations through (un)matchings. In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2673–2687, 2020.

A. Ghodsi, M. Zaharia, B. Hindman, A. Konwinski, S. Shenker, and I. Stoica. Dominant resource fairness: Fair allocation of multiple resource types. In *Proceedings of the 8th USENIX Symposium on Networked Systems Design and Implementation*, 2011.

V. Gkatzelis, A. Psomas, and X. Tan. Fair and efficient online allocations with normalized valuations. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence*, pages 5440–5447, 2021.

M. Hajiaghayi, D. Panigrahi, M. Khani, and M. Springer. Online algorithms for the Santa Claus problem. In *Proceedings of the 36th Annual Conference on Advances in Neural Information Processing Systems*, pages 30732–30743, 2022.

Z. Huang, M. Li, X. Shu, and T. Wei. Online nash welfare maximization without predictions. In *Proceedings of the 19th International Conference on Web and Internet Economics*, pages 402–419. Springer, 2023.

Z. Huang, Z. G. Tang, and D. Wajc. Online matching: A brief survey. *ACM SIGecom Exchanges*, 22:135–158, 2024.

B. Kalyanasundaram and K. R. Pruhs. An optimal deterministic algorithm for online b-matching. *Theoretical Computer Science*, 233(1-2):319–325, 2000.

R. Karp, U. Vazirani, and V. Vazirani. An optimal algorithm for on-line bipartite matching. In *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing*, pages 352–358, 1990.

E. Lee. APX-hardness of maximizing Nash social welfare with indivisible items. *Information Processing Letters*, 122:17–20, 2017.

W. Li and J. Vondrák. A constant-factor approximation algorithm for nash social welfare with submodular valuations. In *Proceedings of the 62nd Annual IEEE Symposium on Foundations of Computer Science*, pages 25–36, 2022.

A. Mehta. Online matching and ad allocation. *Foundations and Trends in Theoretical Computer Science*, 8(4):265–368, 2013.

H. Moulin. *Fair division and collective welfare*. MIT press, 2004.

H. R. Varian. Equity, envy, and efficiency. *Journal of Economic Theory*, 9(1):63–91, 1974.

T. Walsh. Online cake cutting. In *Proceedings of the 2nd International Conference on Algorithmic Decision Theory*, pages 292–305. Springer, 2011.

D. Zeng and A. Psomas. Fairness-efficiency tradeoffs in dynamic fair division. In *Proceedings of the 21st ACM Conference on Economics and Computation*, pages 911–912, 2020.

A   Sufficiently Positive Regime: $p \geq 1/\log n$

A.1   Online Primal Dual Framework

In the positive regime, we follow the online primal dual framework under Fenchel duality. First, we present an equivalent primal program of the offline optimal benchmark for any $p \in (0, 1]$, where we transform the p-mean welfare by raising to the p-th power:

$$
\begin{aligned}
\text{maximize} \quad & P \overset{\text{def}}{=} \frac{1}{p} \sum_{a \in A} U_a^p \\
\text{subject to} \quad & \sum_{a \in A} x_a(t) \leq 1 && \forall t \in I \\
& U_a = \int_0^T x_a(t) v_a(t) \, \mathrm{d}t && \forall a \in A \\
& x_a(t) \geq 0 && \forall a \in A, \ \forall t \in I
\end{aligned}
\tag{\textbf{Primal}}
$$

Consequently, an online algorithm is Γ-competitive for online p-mean welfare maximization if and only if it is Γ^p-competitive for **Primal**.

The corresponding Fenchel dual convex program of **Primal** is:

$$
\begin{aligned}
\text{minimize} \quad & \int_0^T \alpha(t) \, \mathrm{d}t + \left(\frac{1}{p} - 1 \right) \sum_{a \in A} \beta_a^{-\frac{p}{1-p}} \\
\text{subject to} \quad & \alpha(t) \geq v_a(t) \beta_a && \forall a \in A, \ \forall t \in I \\
& \beta_a \geq 0 && \forall a \in A, \ \forall t \in I
\end{aligned}
$$

It is more convenient in the subsequent analysis to change variables with $\gamma_a = \beta_a^{-1/(1-p)}$, to make the (online) relation between U_a and γ_a "almost homogeneous". Hence, we rewrite the dual program as follows:

$$\text{minimize} \quad D \stackrel{\text{def}}{=} \int_0^T \alpha(t)\, \mathrm{d}t + \left(\frac{1}{p} - 1\right)\sum_{a \in A} \gamma_a^p$$

$$\text{subject to} \quad \alpha(t) \geq v_a(t)\gamma_a^{-(1-p)} \qquad \forall a \in A,\ \forall t \in I \qquad \textbf{(Dual)}$$

$$\gamma_a \geq 0 \qquad \forall a \in A,\ \forall t \in I$$

An online primal dual algorithm will maintain not only an online allocation x and the agents' corresponding utilities U_a's as the primal assignment, but also a dual assignment α and γ. The next lemma summarizes the sufficient conditions under which an online primal-dual algorithm is Γ-competitive. Recall that P and D denote the primal and dual objective values.

Lemma 23. *Suppose that an online primal dual algorithm $\mathcal{A}$ satisfies the following conditions:*

(a) $x = \big(x_a(t)\big)_{a \in A, t \in I}$ forms a feasible assignment of **Primal***;*

(b) $\alpha = \big(\alpha(t)\big)_{t \in I}$ and $\gamma = \big(\gamma_a\big)_{a \in A}$ form a feasible assignment of **Dual***; and*

(c) $\Gamma^p \cdot P \geq D$.

Then, the algorithm is Γ-competitive for online p-mean welfare maximization.

Proof. TOPROVE 14 □

The design and analysis of online primal dual algorithms are driven by the primal and dual programs. The Karush-Kuhn-Tucker (KKT) conditions assert that the optimal primal solution x^*, U^* and the optimal dual solution α^*, β^*, and the corresponding γ^* satisfy that:

$$\beta_a^* = \big(U_a^*\big)^{-(1-p)}$$

and, thus, by $\gamma_a^* = (\beta_a^*)^{-1/(1-p)}$ we have:

$$\gamma_a^* = U_a^* \ .$$

Further, if an item t is (partially) allocated to an agent a, i.e., if $x_a(t) > 0$, then the corresponding dual constraint is tight:

$$\alpha^*(t) = v_a(t)\big(\gamma_a^*\big)^{-(1-p)} = v_a(t)\big(U_a^*\big)^{-(1-p)} \ .$$

Since the partial derivative of U_a in $x_a(t)$ equals $v_a(t)U_a^{-(1-p)}$, the above condition says that every item is allocated to the agents that yield the maximum marginal increase in the objective of **Primal**, conditioned on the allocation of the other items.

When an online algorithm decides how to allocate item t, it only knows an agent a's current utility U_a but not the final one. Hence, the algorithm has insufficient information to decide which agent yields the maximum increase in the objective with respect to the final utility. The online primal dual algorithms in this section will project the agents' final utilities based on the current ones, and choose the primal and dual assignments based on the projected utilities.

A.2 Greedy

Primal as an online algorithm problem is a special case of online matching with concave returns studied by Devanur and Jain (2012). Applying their results to degree-p polynomial functions, we have a p^{-p} competitive ratio for **Primal**, and correspondingly a $1/p$ competitive ratio for p-mean welfare maximization. We summarize this result below as a theorem, and present their algorithm and analysis to be self-contained.

Theorem 24 (c.f. Devanur and Jain (2012)). *For any $p \in (0, 1]$, there is a $1/p$-competitive primal dual algorithm for online p-mean welfare maximization.*

Greedy as an Online Primal Dual Algorithm. Naïvely, we may project the final utility to be C times larger than the current utility for some factor $C > 1$, and assign values to the primal and dual variables based on the projected utilities. The online primal dual algorithm of Devanur and Jain (2012) chooses $C = 1/p$. Hence, we let $\gamma_a = U_a/p$, $\alpha(t) = \max_{a \in A} v_a(t)\gamma_a^{-(1-p)}$, and allocate each item t to an agent that achieves this maximum.

Interestingly, the projected utilities of the agents merely scale their current utilities by the same factor. Hence, the primal allocation rules with and without the scaling are identical. In other words, the algorithm of Devanur and Jain (2012) is identical to the greedy algorithm that allocates each item to maximize the immediate increase of the p-mean welfare; the projected utilities mainly play their parts in the dual assignment and the analysis.

Greedy (as an Online Primal Dual Algorithm)

Initialization: Let $x_a(t) = U_a = \alpha(t) = \gamma_a = 0$ for any agent $a \in A$ and any time $t \in I$.

Invariants: Maintain at all time that for any agent $a \in A$:

$$U_a = \int_0^T v_a(t)x_a(t)\,\mathrm{d}t \quad , \quad \gamma_a = \gamma(U_a) \overset{\text{def}}{=} \frac{U_a}{p} . \tag{7}$$

Online Decisions: For each item $t \in I$:

1. Let $\alpha(t) = \max_{a \in A} v_a(t)\gamma_a^{-(1-p)}$.

2. Allocate the item to an agent that achieves the above maximum.

Proof. TOPROVE 15 □

B Online Primal Dual in the Positive Nashian Regime

Our result is an $\Theta(\log n)$-competitive primal dual algorithm for any $p \in [0, 1]$. Recall that it suffices to prove this competitive ratio for the relaxed problem in which agents start with $1/n$ base utilities. We modify the online primal dual algorithm of Devanur and Jain (2012) by designing a suitable regularized utility function.

<div style="border: 1px solid black; padding: 10px;">

Regularized Greedy (as an Online Primal Dual Algorithm)

Initialization: For any agent $a \in A$ and any time $t \in I$, let:

$$x_a(t) = 0 \ , \quad U_a = \frac{1}{n} \ , \quad \alpha(t) = 0 \ , \quad \gamma_a = \frac{\log(n+1)+1}{n} \ .$$

Invariants: Maintain at all time that for any agent $a \in A$:

$$U_a = \int_0^T v_a(t) x_a(t) \, \mathrm{d}t \quad , \quad \gamma_a = \gamma(U_a) \overset{\text{def}}{=} U_a \cdot \left(1 + \log\left(1 + \frac{1}{n}\right) - \log U_a \right) . \qquad (8)$$

Online Decisions: For each item $t \in I$:

1. Allocate the item to an agent a that with the maximum $v_a(t) \cdot \gamma_a^{-(1-p)}$.

2. Let $\alpha(t) = v_a(t) \cdot \gamma_a^{-(1-p)}$.

</div>

Lemma 25. *The regularized utility $\gamma(U)$ is increasing in $U \in [1/n, 1 + 1/n]$.*

Theorem 26. *For any $0 < p \le 1/\log n$, Regularized Greedy is $O(\log n)$-competitive for online p-mean welfare maximization.*

Proof. TOPROVE 16 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

C Generalizing to Non-Unit Monopolist Utilities

This paper introduces algorithms designed to maximize p-mean welfare for various ranges of p. To facilitate a clearer comparison with previous work (e.g., Barman et al. (2022)), we initially focused on a simplified model where all agents have unit monopolist utilities. This approach allowed us to isolate the complexities introduced by non-unit monopolist utilities.

In this section, we extend our discussion to generalize these algorithms for scenarios where agents possess non-unit monopolist utilities. Assume that each agent a's monopolist utility is accurately predicted to be $V_a \ge 0$, and the max-min ratio of the monopolist utilities is a constant, i.e.,

$$K \overset{\text{def}}{=} \frac{\max_{a \in A} V_a}{\min_{a \in A} V_a} = O(1).$$

Our results in this general setting are summarized as follows:

1. For $1/\log n \le p \le 1$, the online primal dual algorithm by Devanur and Jain (2012), remains $1/p$-competitive. Indeed, the result naturally extends to the more general setting where the algorithm has zero information about the monopolist utilities of the agents. The details can be found in Appendix A.

2. For $-1 \le p \le 1/\log n$, the competitive results (Theorem 3, Corollary 4, and Theorem 6) are preserved with a slight adjustment to Nashian Greedy.

3. For $-\infty \le p \le -1$, the competitive result (Theorem 9) is preserved with a slight adjustment to Mixed Greedy.

In the remainder of this section, we discuss how to maintain the competitive results in this general setting for $-\infty \le p \le 1/\log n$.

C.1 Preliminaries

We first restate the definition of Uniform Allocation.

> **Uniform Allocation**
>
> For each item $t \in I$, allocate it uniformly to all agents, i.e., $x_a(t) = 1/n$.

The base utility guaranteed by Uniform Allocation now varies for each individual agent based on their own monopolistic utilities.

Lemma 27 (c.f. Lemma 1). *Every agent $a \in A$ gets utility V_a/n from Uniform Allocation.*

As before, we consider a relaxation of the online p-mean welfare maximization, in which each agent a starts with the base utility $U_a(0) = V_a/n$ for the algorithm's allocation, and the benchmark is the original optimal p-mean welfare without the V_a/n base utility.

Lemma 28 (c.f. Lemma 2). *If an online algorithm A is Γ-competitive for the relaxed online p-mean welfare maximization problem with accurate predictions of monopolist utilities, then allocating half of each item by Uniform Allocation and the other half by algorithm A is 2Γ-competitive for the original problem with accurate predictions of monopolist utilities.*

To apply the reduction in Lemma 28, the initialization of the base utilities of the agents in Nashian Greedy and Mixed Greedy should be modified in accordance with Lemma 27.

In alignment with the earlier organization, we first re-define Nashian Greedy and defer the re-definition of Mixed Greedy to Appendix C.4.

> **Nashian Greedy (Extended Version)**
>
> *Initialization:* Let $x_a(t) = 0$, $U_a(0) = \frac{V_a}{n}$ for any agent $a \in A$ and any time $t \in I$.
>
> *Online Decisions:* Allocate each item $t \in I$ to an agent a that maximizes:
>
> $$\frac{v_a(t)}{U_a(t)} \, ,$$
>
> to greedily maximize the (logarithm of) Nash welfare:
>
> $$\sum_{a \in A} \log U_a(t) \, .$$

A key observation is that the Fundamental Lemma of Nashian Allocation (Lemma 5) remains valid under this extended definition of the Nashian Greedy algorithm.

Lemma 29 (Fundamental Lemma of Nashian Allocation, c.f. Lemma 5). *Consider any allocation of the items $\tilde{x} = (\tilde{x}_{ai})_{a \in A, i \in I}$. For any time $t \in I$ and the agents' utilities for allocation $\tilde{x}$ up to time t, denoted as $\tilde{U}(t) = (\tilde{U}_a(t))_{a \in A}$, we have:*

$$\frac{1}{n} \sum_{a \in A} \frac{\tilde{U}_a(t)}{U_a(t)} \le \log(n+1) \, .$$

Proof. TOPROVE 17 □

C.2   Nashian Regime: $-o(1) \leq p \leq 1/\log n$

The same results for $-o(1) \leq p \leq 1/\log n$ are established using the extended version of Nashian Greedy.

Theorem 30 (c.f. Theorem 3). *For any $|p| = o(1)$, Nashian Greedy is $O\big(n^{|p|} \cdot \log n\big)$-competitive for the relaxed online p-mean welfare maximization problem with accurate predictions of monopolist utilities.*

Corollary 31 (c.f. Corollary 4). *For any p such that $|p| \leq 1/\log n$, Nashian Greedy is $O(\log n)$-competitive for the relaxed online p-mean welfare maximization problem with accurate predictions of monopolist utilities.*

Proof. TOPROVE 18 □

C.3   From Nash to Harmonic Welfare: $-1 \leq p \leq -\Omega(1)$

The same result for $-1 \leq p \leq -\Omega(1)$ are also established using the extended version of Nashain Greedy.

Theorem 32 (c.f. Theorem 6). *For any $-1 \leq p \leq -\Omega(1)$, Nashian Greedy is:*

$$O\Big(n^{\frac{|p|}{|p|+1}}(\log n)^{\frac{1}{|p|+1}}\Big)$$

competitive for the relaxed online p-mean welfare maximization problem with accurate predictions of monopolist utilities.

C.3.1   Further Properties of Nashian Allocation

Definition 3 (Bad Agents, Definition 1 restated). *For any time $t \in I$ and any $\beta > 0$, let $B_\beta(t)$ be the set of β-bad agents whose regularized utilities at time t are at most a β fraction of the optimal p-mean welfare, i.e.:*

$$B_\beta(t) \stackrel{\text{def}}{=} \Big\{a \in A : U_a(t) \leq \beta \cdot \text{OPT}\Big\} .$$

Further, we write B_β for $B_\beta(T)$, the set of β-bad agents at the end.

Lemma 33 (c.f. Lemma 7). *For any time $t \in I$, any $\beta > 0$, and any subset of β-bad agents $S \subseteq B_\beta(t)$, we have:*

$$\frac{1}{n}\sum_{a \in S} U_a^* \leq \beta \log(n+1) \cdot \text{OPT} + \frac{1}{n}\sum_{a \in S}\Big(V_a - \int_0^t v_a(s)\,\mathrm{d}s\Big) .$$

Proof. TOPROVE 19 □

Lemma 34 (c.f. Lemma 8). *For any $\beta > 0$, the fraction of β-bad agents at the end is at most:*

$$\frac{|B_\beta|}{n} \leq \big(\beta \log(n+1)\big)^{\frac{|p|}{|p|+1}} .$$

Proof. TOPROVE 20 □

C.3.2 Proof of Theorem 32

We compare the p-mean welfare of Nashian Greedy to the optimal benchmark by:

$$\left(\frac{\text{ALG}}{\text{OPT}}\right)^p = \frac{1}{n}\sum_{a \in A}\left(\frac{U_a}{\text{OPT}}\right)^p.$$

By $U_a \geq V_a/n \geq \min_{a \in A} V_a/n$ and $\text{OPT} \leq \max_{a \in A} V_a$, and recalling that $p < 0$, we have :

$$\left(\frac{U_a}{\text{OPT}}\right)^p \leq (Kn)^{|p|}.$$

Therefore, the above ratio can be written as:

$$\left(\frac{\text{ALG}}{\text{OPT}}\right)^p = \int_0^{(Kn)^{|p|}}\left(\text{fraction of agents with }\left(U_a/\text{OPT}\right)^p \geq \alpha\right)\mathrm{d}\alpha$$

$$\leq \int_0^{(Kn)^{|p|}}\left(\alpha^{\frac{1}{p}}\log(n+1)\right)^{\frac{|p|}{|p|+1}}\mathrm{d}\alpha \qquad \text{(Lemma 34)}$$

$$= \int_0^{(Kn)^{|p|}}\left(\log(n+1)\right)^{\frac{|p|}{|p|+1}}\alpha^{-\frac{1}{|p|+1}}\mathrm{d}\alpha$$

$$= \frac{|p|+1}{|p|}\cdot (Kn)^{\frac{|p|^2}{|p|+1}}\left(\log(n+1)\right)^{\frac{|p|}{|p|+1}}.$$

Taking p-th root on both sides gives:

$$\frac{\text{ALG}}{\text{OPT}} \geq \underbrace{\left(\frac{|p|}{|p|+1}\right)^{\frac{1}{|p|}}}_{\Omega(1)\text{ for }|p|=\Omega(1)}\cdot (Kn)^{-\frac{|p|}{|p|+1}}\left(\log(n+1)\right)^{-\frac{1}{|p|+1}}.$$

C.4 From Harmonic to Egalitarian Welfare: $-\infty \leq p \leq -1$

We now present the extended definition of this Mixed Greedy algorithm that combines Nashian Greedy and Regularized Egalitarian Greedy. In line with the earlier definition, we further relax the problem by letting there be two copies of each item. This relaxation only affects the competitive ratio by a constant factor.

> **Mixed Greedy (Extended Version)**
>
> *Initialization:* Let $x_a(t) = 0$, $U_a(0) = \frac{V_a}{n}$ for any agent $a \in A$ and any time $t \in I$.
>
> *Regularization:* Define the regularizer of any agent $a \in A$ as:
>
> $$R_a(t) = \frac{1}{\Phi}\left(V_a - \int_0^t v_a(s)\,\mathrm{d}s\right),$$
>
> where $\Phi = \sqrt{n \log(n+1)}$.
>
> *Online Decisions:* For each item $t \in I$, which comes in two copies:
>
> 1. Allocate the *Nashian copy* using Nashian Greedy.
>
> 2. Allocate the *egalitarian copy* to the agents with the smallest regularized utility:
>
> $$U_a(t) + R_a(t),$$
>
> to greedily maximize the regularized egalitarian welfare:
>
> $$\min_{a \in A}\left(U_a(t) + R_a(t)\right).$$

The same result for $-\infty \leq p \leq -1$ is established using the extended version of Mixed Greedy.

Theorem 35 (c.f. Theorem 9). *For any $-\infty \leq p \leq -1$, Mixed Greedy is $O\left(\sqrt{n \log n}\right)$-competitive for the relaxed online p-mean welfare maximization problem.*

C.4.1 Properties of the Regularized Utilities

Definition 4 (Critical Agents, Definition 2 restated). *For any time $t \in I$, and for any $\beta \geq 0$, let $C_\beta(t)$ be the set of β-critical agents at time t, whose regularized utilities are at most a β fraction of the optimal p-mean welfare, i.e.:*

$$C_\beta(t) = \left\{a \in A: U_a(t) + R_a(t) \leq \beta \cdot \mathrm{OPT}\right\}.$$

Lemma 36 (c.f. Lemma 10). *For any time $t \in I$ and any $\beta > 0$, the fraction of β-critical agents at time t is upper bounded by:*

$$\frac{|C_\beta(t)|}{n} \leq \max\left\{(2\beta \log(n+1))^{\frac{|p|}{|p|+1}}, (2\Phi\beta)^{|p|}\right\}.$$

Proof. TOPROVE 21 □

Corollary 37 (c.f. Corollary 11). *For any time $t \in I$, and:*

$$\beta^* = \frac{1}{2} \cdot n^{-\frac{1}{2} - \frac{1}{2|p|}}(\log(n+1))^{-\frac{1}{2} + \frac{1}{2|p|}}, \tag{9}$$

we have:

$$\left|C_{\beta^*}(t)\right| \leq \sqrt{n \log(n+1)}.$$

Using Corollary 37, we derive a universal lower bound for all agents' utilities.

Lemma 38 (c.f. Lemma 12). *For the choice of β^* in Equation (9), and any agent $a \in A$, we have:*

$$U_a \geq \frac{\beta^*}{K} \cdot \text{OPT} .$$

Proof. TOPROVE 22 □

C.4.2 Proof of Theorem 35

The proof is almost verbatim to that of Theorem 32, except that we will use the newly developed Lemma 38 to lower bound the agents' utilities, replacing the basic bound $U_a \geq 1/n$.

Consider the p-th power of the Mixed Greedy algorithm's p-mean welfare, normalized by the p-th power of OPT:

$$\left(\frac{\text{ALG}}{\text{OPT}}\right)^p = \frac{1}{n} \sum_{a \in A} \left(\frac{U_a}{\text{OPT}}\right)^p .$$

By Lemma 38 and $p < 0$, we have:

$$\left(\frac{U_a}{\text{OPT}}\right)^p \leq \left(\frac{\beta^*}{K}\right)^p .$$

The above ratio can therefore be written as:

$$
\begin{aligned}
\left(\frac{\text{ALG}}{\text{OPT}}\right)^p &= \int_0^{(\beta^*/K)^p} \left(\text{fraction of agents with } (U_a/\text{OPT})^p \geq \alpha\right) d\alpha \\
&\leq \int_0^{(\beta^*/K)^p} \left(\alpha^{\frac{1}{p}} \log(n+1)\right)^{\frac{|p|}{|p|+1}} d\alpha \qquad\qquad (\text{Lemma 34}) \\
&= \int_0^{(\beta^*/K)^p} \left(\log(n+1)\right)^{\frac{|p|}{|p|+1}} \alpha^{-\frac{1}{|p|+1}} d\alpha \\
&= \frac{|p|+1}{|p|} \left(\frac{\beta^*}{K}\right)^{-\frac{|p|^2}{|p|+1}} \left(\log(n+1)\right)^{\frac{|p|}{|p|+1}} \\
&= \frac{|p|+1}{|p|} (2K)^{\frac{|p|^2}{|p|+1}} \left(n \log(n+1)\right)^{\frac{|p|}{2}} .
\end{aligned}
$$

Taking p-th root on both sides gives:

$$\frac{\text{ALG}}{\text{OPT}} \geq \underbrace{\left(\frac{|p|}{|p|+1}\right)^{\frac{1}{|p|}} (2K)^{-\frac{|p|}{|p|+1}}}_{\Omega(1) \text{ for } p \leq -1} \cdot \left(\sqrt{n \log(n+1)}\right)^{-1} .$$