Graph minors and metric spaces

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Abstract

We present problems and results that combine graph-minors and coarse geometry. For example, we ask whether every geodesic metric space (or graph) without a fat H minor is quasi-isometric to a graph with no H minor, for an arbitrary finite graph H. We answer this affirmatively for a few small H. We also present a metric analogue of Menger's theorem and König's ray theorem. We conjecture metric analogues of the Erdős–Pósa Theorem and Halin's grid theorem.

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1 Introduction

Gromov's [22] coarse geometry perspective had a groundbreaking impact on geometric group theory, also spreading onto nearby areas. As an example highlight, it has been proved that (any Cayley graph of) every finitely generated group with finite asymptotic dimension embeds coarsely into Hilbert space [38], and therefore it satisfies the Novikov conjecture [22].

Several recent papers study the interplay between Gromov's asymptotic dimension and graph-theoretic notions such as graph minors [5, 7, 17, 18, 25, 34, 35]. With these in the background, we present some results and questions with a strong interplay between geometry and graph theory. We hope that this will evolve into a coherent theory that could be called *'Coarse Graph Theory'* or *'Graph-Theoretic Geometry'*.

We would like to view graphs 'from far away' to see their large-scale geometry and its implications. More generally, the same idea can be applied to Riemannian manifolds, or more generally, arbitrary length spaces. In what follows X

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can stand for a length space, or an infinite graph, or a family of finite graphs, endowed with their graph-distance to turn them into metric spaces; none of our statements are affected by this choice, the reader may choose their preferred setting. Our favourite such problem is

Conjecture 1.1. *Let X be a graph or a length space, and let H be a finite graph. Then X has no K-fat H minor for some $K \in \mathbb{N}$ if and only if X is $f(K)$-quasi-isometric to a graph with no H minor, where $f : \mathbb{N} \to \mathbb{N}$ depends on H only.*[1]

A quasi-isometry is a generalisation of a bi-Lipschitz map that allows for an additive error; see Section 2.1. This notion lies at the heart of Gromov's programme [22]. A fat minor is a coarse version of the notion of a graph minor defined as follows. Recall that X has a (usually finite) graph H as a *minor*, if we can find connected *branch sets* and *branch paths* in X, such that after contracting each branch set to a point, and each branch path to an edge, we obtain a copy of H. We say that X has H as a *K-fat minor*, if we can choose the above sets to be pairwise at distance $\geq K$, except that we do not require this for incident branch set–branch edge pairs —which must be at distance 0 by definition; see also Definition 1 below. This notion of fat minor also features in [7], and a slight variation features in [8].

If X has a K-fat H minor for every K, then we say that X has H as an *asymptotic minor*, and write $H \prec^\infty X$. An important feature of asymptotic minors is that they are preserved under quasi-isometries, and in particular they are stable under changing the generating set in a Cayley graph of a group (Observation 2.4).

For example, let L be the square lattice on $\mathbb{Z}^2$, and let $L' := L \times K_2$ be its cartesian product with an edge. Then one can easily show that L' has every finite graph as a minor, but it does not have K_5 or $K_{3,3}$ as a 2-fat minor. Thus this notion correctly captures the intuition that L' looks like the plane from a coarse point of view. As another example, the cubic lattice on $\mathbb{Z}^3$ has every finite graph H as an asymptotic minor; to see this, embed H in $\mathbb{R}^3$, fatten it slightly, then zoom $\mathbb{Z}^3$ out while keeping your H fixed.

We know that Conjecture 1.1 holds for some very small graphs H. If $H = K_2$ (a pair of vertices with an edge), then Conjecture 1.1 becomes the obvious fact that if X has finite diameter then it is quasi-isometric to a point. The case $H = K_3$ is already non trivial and it follows from Manning's characterization of quasi-trees [31]. We give a self contained exposition of this which entails a shorter proof of Manning's result (Theorem 3.1). Further alternative characterisations of quasi-trees have been recently obtained by Berger & Seymour [6].

Chepoi, Dragan, Newman, Rabinovich & Vaxès [8] settled the case $H = K_{2,3}$ of Conjecture 1.1 using slightly different terminology. The similar case $H = K_4^-$, i.e. K_4 with an edge removed, has been settled by Fujiwara and the second author [17], along with a stronger result characterising quasi-cacti. In Section 4 we settle the case where H is a star $K_{1,m}, m \geq 1$. This is, as far as we know, a new result.

Call two graphs H_1, H_2 *asymptotically equivalent*, if $H_1 \prec^\infty X$ implies $H_2 \prec^\infty X$ and vice-versa for every space X. In Section 5 we show that if two

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[1] The backward direction is straightforward.

finite graphs H_1, H_2 are homeomorphic as 1-complexes, then they are asymptotically equivalent (the converse is true as well if we allow X to be disconnected). This immediately settles Conjecture 1.1 for paths and cycles H. Moreover, combined with the aforementioned result of [17], it implies that a graph is quasi-isometric to a cactus if and only if it is quasi-isometric to an outerplanar graph (Corollary 5.2), a fact previously proved by Chepoi et al. [8].

In Conjecture 1.1 we are mostly interested in the case where H is finite, but we point out some infinite cases of interest: in Section 6 we prove a coarse analogue of König's theorem, and in Section 7 we discuss coarse analogues of Halin's grid theorem.

We observe below (Observation 2.2) that when H is 2-connected in Conjecture 1.1, then instead of a quasi-isometry we can equivalently ask for a bi-Lipschitz map. Bi-Lipschitz embeddings, aka. embeddings of bounded multiplicative distortion, into various targets, most notably tree metrics and normed spaces, are an actively researched topic, see e.g. [8, 15, 32, 36] and references therein. Embeddings of bounded additive distortion are also of interest [8]. This raises:

Question 1.2. *For which finite graphs H is it true that if a graph X has no K-fat H minor for some $K \in \mathbb{N}$, then X admits a map of bounded additive distortion onto a graph with no H minor?*

We know that this is the case when H is a cycle [26] (see also claim (5) in Section 3) or $H = K_{1,3}$.

Ostrovskii & Rosenthal [35] proved that (infinite, locally finite) graphs with a forbidden minor have finite asymptotic dimension. Bonamy et al. [7] improve this by proving that such graphs have asymptotic dimension at most 2 (as was conjectured in [17]). They ask whether forbidding a fat minor implies bounded asymptotic dimension. Note that a positive answer to Conjecture 1.1 would imply this. The following question is similar in spirit.

Conjecture 1.3 (Coarse Hadwiger conjecture)**.** *For every $n \in \mathbb{N}$, if X has no asymptotic K_{n+1} minor, then X has Assouad-Nagata dimension $\mathrm{ANdim}(X)$ at most $n - 1$.*

Since $\mathrm{ANdim}(X)$ can be defined via colouring (Section 2.2)—and $\mathrm{ANdim}(X) = n - 1$ means that n colours suffice— Conjecture 1.3 can be thought of as the coarse version of the Hadwiger conjecture, perhaps the oldest well-known open problem of graph theory, which asserts that every graph with no K_{n+1} minor is n–colourable. If we drop the word 'asymptotic' in the statement of Conjecture 1.3, then a better bound $\mathrm{ANdim}(X) \leq 2$ was recently obtained by Distel [12] and by Liu [29]. In fact 2 might be the right upper bound in Conjecture 1.3 as well. A somewhat related result relaxes the Hadwiger conjecture by allowing monochromatic components of bounded size [13].

Apart from its theoretic interest due to the connection with the Hadwiger conjecture, Conjecture 1.3 is also interesting from an algorithmic perspective. Given the theoretical and algorithmic importance of minor-closed graph classes, one hopes that the graphs we encounter in practice come from such a class. This is however rarely the case, because e.g. a little bit of randomness will result in arbitrarily large cliques. Thus it is natural to seek extensions of the

theory of graph minors to a larger setting. This has been a mainstream trend in computer science, see e.g. [21] and references therein. Conjecture 1.3 pursues something similar in a different direction. The kind of practical problems we have in mind are those typically encountered in computational geometry, e.g. the Traveling Salesman Problem (TSP). Computer scientists have developed a notion called *sparse partition schemes*, and have shown how these can be used to solve important practical problems such as the Universal Steiner Tree and the Universal TSP problem. The definition of sparse partition schemes is a slightly more quantitative version of the colourings used to define the Assouad-Nagata dimension (Section 2.2); see [7] and references therein for the precise relationship and further applications). Recall that the presence of large cliques in real-world graphs is an obstruction to using algorithms coming from graph minor theory. Our coarse approach overcomes such obstructions: a clique of any size has diameter 1, and so it looks just like a point when seen from far away.

In the hope to develop more basic tools towards Conjecture 1.1 and Conjecture 1.3, we considered the following coarse version of Menger's theorem:

Conjecture 1.4 (Coarse Menger Theorem). *For every $n \in \mathbb{N}$, there is a constant C_n, such that the following holds for every graph G and every two subsets A, Z of its vertex set. For every $r \in \mathbb{Z}$, either there is a set of n A–Z paths in G at distance at least r from each other, or there is a set S of less than n vertices of G, such that removing the ball of radius $C_n r$ around S separates A from Z in G.*

In Section 8 we will prove the special case $n = 2$ of this conjecture[2] We warn the reader that Conjecture 1.4 seems very difficult even for $n = 3$. In Section 9 we discuss related problems that might be more accessible.

2 Preliminaries

2.1 Metric spaces

Let (X, d) be a metric space. Given $x \in X$ and $R \in \mathbb{R}_+$, we let $\mathrm{Ball}_x(R)$ denote the ball of radius R around x.

For $Y, Z \subseteq X$ we define $d(Y, Z) := \inf_{y \in Y, z \in Z} d(y, z)$.

A *length space* is a metric space (X, d) such that for every $x, y \in X$ and $\epsilon > 0$ there is an x–y arc of length at most $d(x, y) + \epsilon$. Thus every geodesic metric space, in particular every graph endowed with its standard metric, is a length space.

An (M, A)-*quasi-isometry* between graphs $G = (V, E)$ and $H = (V', E')$ is a map
$f : V \to V'$ such that the following hold for fixed constants $M \geq 1, A \geq 0$:

(i) $M^{-1}d(x, y) - A \leq d(f(x), f(y)) \leq Md(x, y) + A$ for every $x, y \in V$, and

(ii) for every $z \in V'$ there is $x \in V$ such that $d(z, f(x)) \leq A$.

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[2] An alternative proof was independently obtained by Albrechtsen et al. [1], who also state Conjecture 1.4.

Here $d(\cdot, \cdot)$ stands for the graph distance in the corresponding graph G or H. We say that G and H are *quasi-isometric*, if such a map f exists. Quasi-isometries between arbitrary metric spaces are defined analogously. The following is well-known, see e.g. [9, Proposition 3.B.7(6)]:

Observation 2.1. *Every length space is quasi-isometric to a connected graph.*

Proof (sketch). Given a length space (X, d), fix $\epsilon > 0$ and pick a subset U such that $d(x, y) > \epsilon$ for every $x, y \in U$ and $d(z, U) \leq \epsilon$ for every $z \in X$. Define a graph on U by joining any two points $x, y \in U$ satisfying $d(x, y) < 3\epsilon$ with an edge. $\square$

When $f : X \to Y$ between metric spaces satisfies (i) with $A = 0$, then we say that f has bounded *multiplicative distortion*. When $M = 1$, we say that f has bounded *additive distortion*. We observe that in many cases where Y is not fixed, but allowed to vary inside a class of graphs, then a quasi-isometry can be turned into a map of bounded multiplicative distortion:

Observation 2.2. *Let $\mathcal{H}$ be a set of 2-connected graphs, and let $\mathcal{C} = \mathrm{Forb}(\mathcal{H})$ be the class of graphs that do not have any element of $\mathcal{H}$ as a minor. Then a graph X is quasi-isometric to an element of $\mathcal{C}$, if and only if X admits a map of bounded multiplicative distortion into an element of $\mathcal{C}$.*

Proof. The converse direction is trivial. For the forward direction, let $f : X \to G$ be a (M,A)-quasi-isometry with $G \in \mathcal{C}$. For every $v \in V(G)$, we attach the centre of a star S_v with $|V(X)|$ leaves to v, and subdivide each edge of S_v into a path of length $\lceil A \rceil$. Let G' be the resulting graph, and notice that $G' \in \mathcal{C}$ because the elements of $\mathcal{H}$ are 2-connected.

Define a map $f' : V(X) \to V(G')$ as follows. For each $v \in V(G)$, let f' map each vertex in $f^{-1}(v)$ to a distinct leaf of S_v; since S_v has $|V(X)|$ leaves, this is possible. It is straightforward to check that f' is a $((M + 3\lceil A \rceil), 0)$-quasi-isometry. $\square$

2.2 Asymptotic dimension

We recall one of the standard definitions of asymptotic dimension $\mathrm{asdim}(X)$ of a metric space (X, d) from e.g. [25]. The reader can think of X as being a graph endowed with its graph distance d. See [4] for a survey of this rich topic.

Let $\mathcal{U}$ be a collection of subsets of (X, d) —usually a cover. For $s > 0$, we say that $\mathcal{U}$ is *s-disjoint*, if $d(U, U') := \inf \{d(x, x') \mid x \in U, x' \in U'\} \geq s$ whenever $U, U' \in \mathcal{U}$ are distinct. More generally, we say that $\mathcal{U}$ is $(n + 1, s)$-*disjoint*, if $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ for subcollections $\mathcal{U}_i$ each of which is s-disjoint. The indices $1, \ldots, n + 1$ are the *colours* of $\mathcal{U}$. We define the *asymptotic dimension* $\mathrm{asdim}(X)$ as the smallest n such that for every $s \in \mathbb{R}_+$ there is an $(n + 1, s)$-disjoint cover $\mathcal{U}$ of X with $\sup_{U \in \mathcal{U}} \mathrm{diam}(U) < \infty$. Here, the diameter $\mathrm{diam}(U)$ is measured with respect to the metric d of X. We say that $\mathcal{U}$ is *D-bounded* for any $D > \sup_{U \in \mathcal{U}} \mathrm{diam}(U)$.

The *Assouad-Nagata dimension* $\mathrm{ANdim}(X)$ is defined by putting a stronger restriction on the diameters: we define $\mathrm{ANdim}(X)$ to be the smallest n for which there is $c \in \mathbb{R}$ such that for every $s \in \mathbb{R}_+$ there is an $(n + 1, s)$-disjoint and cs-bounded cover $\mathcal{U}$ of X.

2.3 Fat minors

Definition 1 (Fat minors)**.** *Let (X, d) be a metric space and Δ a graph. A K-fat Δ-minor of X is a collection of disjoint connected subspaces $B_v, v \in V(\Delta)$ of X (called* branch-sets*), and disjoint arcs $P_e, e \in E(\Delta)$ (called* branch-paths*) such that:*

   (i) Each P_{uw} intersects $\bigcup_{v \in V(\Delta)} B_v$ exactly at its two endpoints, which lie in B_u and B_w.

   (ii) For any $Y, Z \in \{B_v \mid v \in V(\Delta)\} \cup \{P_e \mid e \in E(\Delta)\}$, we have $d(Y, Z) \geq K$, unless $Y = Z$, or $Y = P_{vw}$ and $Z = B_v$ for some $v, w \in V(\Delta)$, or vice versa.

Definition 2.3 (Asymptotic minors)**.** *Let Δ be a finite graph. We say that X has Δ as an* asymptotic minor*, and write $\Delta \prec^\infty X$, if X has Δ as a K-fat minor for all $K > 0$.*

Observation 2.4. *The property of having a fixed finite graph H as an asymptotic minor is preserved under quasi-isometries between length spaces.*

In fact, asymptotic minors are preserved under the more general *coarse embeddings* as defined in [22, 36], but we will not use this fact here.

3 Manning's theorem — quasi-trees

Manning's theorem [31] says that a graph is quasi-isometric to a tree if and only if it has the following δ-Bottleneck Property for some $\delta \in \mathbb{R}_+$:

Definition 2. *We say that a metric space X has the δ-Bottleneck Property, if for all $x, y \in X$ there is a 'midpoint' $m = m(x, y)$ with $d(x, m) = d(y, m) = d(x, y)/2$ such that any path from x to y is at distance less than δ from m.*

Our next result contains Manning's theorem, and at the same time proves the case $H = K_3$ of Conjecture 1.1:

Theorem 3.1. *The following are equivalent for every graph G*

   (i) G satisfies the δ-Bottleneck Property for some $\delta > 0$;

   (ii) G has no K-fat K_3-minor for some $K > 0$; and

   (iii) G is quasi-isometric to a tree.

Remark 1: In this formulation our statement only makes sense for graphs G of infinite diameter, but our proof yields explicit relations between the constants involved that make it meaningful for finite G as well.

Remark 2: Krön & Möller [28] provide an alternative characterisation of quasi-trees (for graphs G of infinite diameter). Kerr [26] provides another proof of Manning's theorem, with multiplicative constant 1 for the quasi-isometry. Similar results had previously been obtained by Chepoi et al. [8]. Our proof of Theorem 3.1 in fact also gives Kerr's theorem (with a different proof), as we will obtain a $(1, 10K)$-quasi-isometry to a tree (5). In other words, (i)–(iii) of

Theorem 3.1 are equivalent to admitting an embedding into a tree with bounded additive distortion.

We will use the following terminology. Let (X, d) be a metric space. We say that $C \subseteq X$ is M-*near-connected* for $M \in \mathbb{R}_+$, if for every $x, y \in C$, there is a sequence $x = x_0, x_1, \ldots, x_k = y$ of points of C such that $d(x_i, x_{i+1}) \leq M$ for every $i < k$. Such a sequence $P = \{x_i\}$ will be called an M-*near path* from x to y. An M-*near-component* is a maximal subset of X that is M-near-connected.

Our method below bears some similarity with [8], but our use of M-near-components instead of components in a layered partition is an essential difference.

Proof of Theorem 3.1. ● (i) → (ii):
Suppose that G has a 2δ-fat K_3-minor M with branch sets B_1, B_2, B_3 and paths P_{12}, P_{23}, P_{31}. Let $x \in B_1, y \in B_2$ be the two endvertices of P_{12}. We may assume without loss of generality that

$$d(x, B_2) = 2\delta. \tag{1}$$

Indeed, if (1) fails, then let x' be the last vertex along P_{12} as we move from x to y such that $d(xP_{12}x', B_2) = 2\delta$, where $xP_{12}x'$ denotes the subpath of P_{12} from x to x'. Such an x' always exists since $d(x, B_2) \geq 2\delta$ and $y \in B_2$. Easily, $d(x', B_2) = 2\delta$. We can modify M by adding $xP_{12}x'$ to B_1 and replacing P_{12} by $x'P_{12}y$, to obtain a new K_3-minor M' of G which satisfies (1) by the choice of x'. It is straightforward to check that M' is still 2δ-fat.

Next, we claim that we may assume each B_i to be a path with end-vertices in $\bigcup_{j \neq i} P_{ij}$. Indeed, each B_i contains such a path P_i by the definitions, and discarding $B_i \backslash P_i$ cannot reduce the fatness of M. This means that the vertices and edges of G in the branch sets and paths of M form a cycle C of G.

Let γ be a x–B_2 geodesic, let m be its midpoint, and let $b \in B_2$ be its final vertex. Thus $d(m, B_2) = \delta$ by (1). Let C_1 be the subpath of C from x to b containing P_{12}, and let C_2 be the other subpath of C from x to b.

As we are assuming (i), there is a point z_1 in C_1 such that $d(z_1, m) < \delta$, and a point z_2 in C_2 such that $d(z_2, m) < \delta$. By the triangle inequality we thus have

$$d(z_1, z_2) < 2\delta. \tag{2}$$

Likewise, since $d(m, B_2) = \delta$, we also obtain

$$d(z_i, B_2) < 2\delta \text{ and } d(z_i, B_2) > 2\delta - \delta = \delta \text{ for } i = 1, 2. \tag{3}$$

In particular, neither of z_1, z_2 lies in B_2. Since all of B_1, B_3 and P_{13} are at distance at least 2δ from B_2, we must then have $z_1 \in P_{12}$ and $z_2 \in P_{23}$. But combined with (2) this implies $d(P_{12}, P_{23}) < 2\delta$, contradicting that M is 2δ-fat.

● (ii) → (iii):
Fix a 'root' $o \in V(G)$, and let $S_n := \{v \in V(G) \mid d(v, o) = n\}$ be the sphere at distance $n \in \mathbb{N}$.

Let $\mathcal{C}_n$ denote the set of $5K$-near-components of S_n, where we endow S_n with the metric $d = d_G$ inherited from G.

Claim 1: Each $C \in \mathcal{C}_n$ has diameter at most $D = D(K) := 10K$ for every $n \in \mathbb{N}$.

If $n \leq D/2$ this is obvious, so assume $n > D/2$ from now on. Suppose, to the contrary, there is $C \in \mathcal{C}_n$ and $x_1, x_2 \in C$ with $d(x_1, x_2) > D$. Let P be a x_1–x_2 path in G such that $d(P, S_n) \leq 5K/2$, which exists by the definition of a $5K$-near-component and the fact that $C \subseteq S_n$. Let $\pi_{x_i o}$ be a geodesic from x_i to o for $i = 1, 2$, and let B_i be the component of x_i in $(\pi_{x_i o} \cup P) \cap B(x_i, 7K/2)$, where $B(x, R)$ denotes the ball of radius R with center x in G. Let $P_{x_i o}$ be the subarc of $\pi_{x_i o}$ of length K starting at the last point of B_i (which point is an interior point of an edge unless K is an even integer). Let $P_{x_1 x_2}$ be a subpath of P joining B_1 to B_2. Note that $d(P_{x_i o}, S_n) = 7K/2$, and therefore $d(P_{x_1 x_2}, P_{x_i o}) \geq K$ by the triangle inequality because $d(P, S_n) \leq 5K/2$. Let $B_o = B(o, n - 9K/2)$, and note that this implies $d(B_o, P) \geq 2K$ and $d(B_o, B_i) \geq K$ for similar reasons. We claim that $d(P_{x_1 o}, P_{x_2 o}) \geq K$. Indeed, if not, then let Q be a $P_{x_1 o}$–$P_{x_2 o}$ path of length less than K. Then by concatenating subpaths of the five paths $(\pi_{x_1 o} \cap B_1), P_{x_1 o}, Q, P_{x_2 o}, (\pi_{x_2 o} \cap B_2)$ we obtain an x–y *path* of length less than $10K$, contradicting our assumption.

It is now straightforward to check, using the above inequalities, that the above paths form a K-fat K_3-minor of G with branch sets B_1, B_2, B_o. This contradicts our assumption thus proving Claim 1.

Define a tree T as follows. The vertex set $V(T) := \bigcup_n \bigcup \mathcal{C}_n$ is the set of all $5K$-near-components of all S_n. We put an edge between $C \in \mathcal{C}_i$ and $C' \in \mathcal{C}_j$ in T whenever there is an edge of G between C and C'. We can think of $\mathcal{C}_0$, which is just a singleton $\{\{o\}\}$, as the root of T. Notice that there is no edge of T between distinct elements of $\mathcal{C}_i$ by the definition of M-near-component. The fact that T is a tree thus follows from

Claim 2: For every $n \in \mathbb{N}$ and every $C \in \mathcal{C}_{n+1}$, there is exactly one $C' \in \mathcal{C}_n$ such that $CC' \in E(T)$.

Indeed, suppose there are $C' \neq C'' \in \mathcal{C}_n$ connected to C by an edge. In this case, there is a path A in G such that the endvertices a, z of A lie in distinct elements of $\mathcal{C}_n$ (which elements are not necessarily C' and C''), and the interior of A is disjoint from $\bigcup_{i \leq n} \bigcup \mathcal{C}_i$. To see this, let e be an edge of G joining C' to C, let f be an edge of G joining C'' to C, and let A' be a path in G joining the endvertices of e and f lying in C such that A' is at distance at most $5K$ from C; such an A' exists since C is $5K$-near-connected. It is easy to see that the concatenation $eA'f$ contains a subpath A with the desired properties. Note that

$$d(a, z) > 5K \tag{4}$$

since a, z lie in distinct $5K$-near-components. Let π_{ao} and π_{zo} be a geodesic from a to o and a geodesic from z to o respectively. Let Z be an a–z path contained in their union $\pi_{ao} \cup \pi_{zo}$.

We construct a K-fat K_3-minor in G as follows. Let B_a be the initial subpath of π_{ao} of length $2K$ and let B_z be the initial subpath of π_{zo} of length $2K$. Notice that $d(B_a, B_z) > K$ by (4). Let $P_{az} = Z \setminus \{B_a \cup B_z\}$. Let p be the last point

along A, as we move from a to z, such that $d(p, B_a) \leq K$ holds, and let q be the first point along A after p such that $d(q, B_z) \leq K$ holds. Let B_w be the subpath of A between p and q. Easily, $d(B_w, B_a), d(B_w, B_z) \geq K$ hold. Let P_{aw} be a shortest p–B_a path in G, and let P_{wz} be a shortest q–B_z path in G. Note that $|P_{aw}| = |P_{wz}| = K$.

We claim that $d(P_{aw}, P_{wz}) \geq K$. To see this, suppose there is a P_{aw}–P_{wz} path Q with $|Q| < K$. Let Q_a be the subpath of π_{ao} from a to P_{aw}, and notice that $|Q_a| \leq K$ because P_{aw} starts at S_{n+1} and has length K, and therefore cannot meet a point of π_{ao} at distance K or more from $a \in S_n$. We define Q_z similarly, as a subpath of π_{zo}. Thus by concatenating subpaths of the five paths $Q_a, P_{aw}, Q, P_{wz}, Q_z$ we obtain an a–z path of length less than $5K$, contradicting (4). This proves $d(P_{aw}, P_{wz}) \geq K$. Similarly, we have $d(P_{aw}, B_z) \geq K$ and $d(P_{wz}, B_a) \geq K$.

Finally, note that both P_{aw}, P_{wz} start at distance at least $n+1$ from o, while all of P_{az} is within distance $n - 2K$ from o. Combining the last two inequalities we deduce $d(P_{aw}, P_{az}), d(P_{wz}, P_{az}) > K$. Similarly, we have $d(B_w, P_{az}) > K$.

Thus these sets form a K-fat K_3-minor. This contradicts our assumption (ii), hence proving Claim 2.

Notice that T is obtained from G by contracting disjoint sets (the above $5K$-near-components) into points. Since these sets have diameter bounded by $10K$ by Claim 1, the map $f : V(G) \to V(T)$ mapping each vertex to the $5K$-near-component it belongs to defines a quasi-isometry from G to T. More precisely, we claim that

$$f \text{ is a } (1, 10K)\text{-quasi-isometry from } G \text{ to } T. \tag{5}$$

Indeed, given $x, y \in V(G)$, let P be an x–y path in G with length $d_G(x, y)$, and let Q be the unique $f(x)$–$f(y)$ path in T. It is easy to see that $d_G(x, y) \geq d_T(f(x), f(y))$, because $f(P)$ is connected, hence it must contain all edges of Q.

For the converse, let Z be the vertex of Q that is closest to the root $f(o)$ of T. Let Q_1 be the (possibly trivial) subpath of Q from $f(x)$ to Z, and Q_2 the subpath from Z to $f(y)$. Let Π_x (respectively, Π_y) be the x–o (resp. y–o) geodesic in G. Let z_x be the unique vertex of $\Pi_x \cap Z$, and z_y the unique vertex of $\Pi_y \cap Z$. Let S be a shortest z_x–z_y path in G, and let $||S||$ denote its length. Then $||S|| \leq 10K$ by Claim 1. We can form an x–y path by concatenating $x\Pi_x z_x$ with S and then $z_y \Pi_y y$. Notice that $||x\Pi_x z_x|| = ||Q_1||$, and $||z_y \Pi_y y|| = ||Q_2||$ (where $x\Pi_x z_x$ stands for the x–z_x subpath of Π_x). It follows that $d(x, y) \leq 10K + ||Q|| = 10K + d_T(f(x), f(y))$ as desired.

- The implication (iii) → (i) is obvious. □

4   Conjecture 1.1 for stars

In this section we establish the special case of Conjecture 1.1 where H is a *star*, i.e. $H = K_{1,m}$:

Theorem 4.1. *Let X be a length space, and $m \in \mathbb{N}_{>0}$. Then X has no asymptotic $K_{1,m}$ minor if and only if X is quasi-isometric to a graph with no $K_{1,m}$ minor.*

Proof. By Observation 2.1 we may assume that X is a graph. The backward implication is immediate from Observation 2.4. We will prove the forward implication: we assume that X does not have a K-*fat* $K_{1,m}$ minor, and will construct a $K_{1,m}$-minor-free graph quasi-isometric to X.

We follow the layered partition technique of the proof of Theorem 3.1, however, instead of the metric *spheres* S_n we will now decompose X into its metric *annuli* of width $M := 4K + 1$. More precisely, fix a 'root' $o \in V(X)$, and let $A_n := \{v \in V(X) \mid d(v, o) \in [nM, (n+1)M)\}, n \in \mathbb{N}$ be the nth annulus.

Let $\mathcal{C}_n$ denote the set of K-near-components of A_n, where we endow A_n with the metric $d = d_G$ inherited from G. An element of $\mathcal{C}_n, n \in \mathbb{N}$, is called a *box*. Similarly to Claim 1 from the proof of Theorem 3.1, we have

Claim a: Each box $C \in \mathcal{C}_n$ has diameter less than $D := cK^2$ for every $n \in \mathbb{N}$, where $c = c_m$ is a universal constant.

Indeed, this follows by combining the argument of Claim 1 with a pigeonhole argument: given $C \in \mathcal{C}_n$, and $x, y \in C$ with $d(x, y) = \operatorname{diam}(C)$, let P be an x–y path contained in the ball B of radius $K/2$ around A_n, which exists by the definition of a K-near-component. If $d(x, y)$ is large enough, then we can pick points $x_0, \dots, x_{m(M+K)}$ along P with pairwise distance at least $3K$. Since B has bounded width, i.e. $B \subset \{v \in V(X) \mid d(v, o) \in [nM - K/2, (n+1)M) + K/2\}$, the pigeonhole principle implies that some subset $\{x'_1, \dots x'_m\}$ of the x_i have equal distance to o. Letting π_i be a geodesic from each x'_i to o, we can easily find a K-fat $K_{1,m}$ minor of X inside $\bigcup \pi_i$, letting the singletons $\{x'_i\}$ be the branch sets corresponding to the leaves of $K_{1,m}$. This contradiction bounds $d(x, y)$ and therefore proves Claim a. (The resulting constant c_m is at most $15m$.)

Call a box $C \in \mathcal{C}_n$ *short*, if it send no edge to $\mathcal{C}_{n+1}$, and call it *tall* otherwise. As in the proof of Theorem 3.1, we are going to form a graph G by contracting each box into a vertex. Our hope that G will be $K_{1,m}$-minor-free will not quite come true, because a $K_{1,m}$ minor in G that involves some short components will not give rise to a fat $K_{1,m}$ minor in X. Therefore, we want to suppress the short boxes, and we do so as follows. Call two tall boxes $C_1, C_2 \in \mathcal{C}_n$ *siblings*, if they send an edge to a common short box (of $\mathcal{C}_{n+1}$). Let $\sim$ denote the transitive reflexive closure of the sibling relation. For each $\sim$-equivalence class $[C]$, define a *super-box* C' comprising the union of all elements of $[C]$ as well as all short boxes in $\mathcal{C}_{n+1}$ sending an edge to an element of $[C]$. Finally, let $\mathcal{C}'_n$ denote the set of super-boxes C' with $C \in \mathcal{C}_n$. Notice that distinct super-boxes are disjoint, and therefore $\bigcup_{n \in \mathbb{N}} \mathcal{C}'_n$ is a partition of $V(X)$. Moreover, for each $C' \in \mathcal{C}'_n$, we have $C' \subseteq \mathcal{C}_n \cup \mathcal{C}_{n+1}$ by definition, and therefore

$$\operatorname{diam}(C') < 2D. \tag{6}$$

Indeed, we can repeat the proof of Claim a with M replaced by $2M$.

Each super-box $C' \in \mathcal{C}_n$ contains at least one tall box C, which by definition has a non-empty intersection C_M with the *middle-layer* of A_n, i.e. the set $A_n^{1/2} := \{v \in V(X) \mid d(v, o) = nM + 2K + 1\}$. We call the vertices in $C' \cap A_n^{1/2}$ the *mid-points* of C' —we could have alternatively defined a box to be *tall* if it has at least one mid-point. Mid-points are helpful because they are far from

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any other super-box:

Let $C_1' \neq C_2'$ be two super-boxes, and x_i a mid-point of C_i'. Then $d(x_1, C_2') > 2K$, and $d(x_1, x_2) > 4K$. $\qquad(7)$

Indeed, every path from x_i to the complement of C_i' has length greater than $2K$ —because the width of A_n is $4K + 1$— from which both inequalities easily follow.

We now define the graph $G := X/\bigcup_{n \in \mathbb{N}} \mathcal{C}_n'$ by contracting each super-box of X into a vertex. In other words, we let $V(G) := \bigcup_{n \in \mathbb{N}} \mathcal{C}_n'$, and we put an edge between C_1', C_2' in G whenever there is an edge between these vertex-sets in X.

As in the proof of Theorem 3.1, (6) implies that G is $(2D, 2D)$-quasi-isometric to X. Therefore, it only remains to prove that

G does not have a $K_{1,m}$ minor. $\qquad(8)$

Suppose to the contrary that such a minor Δ exists, and let B be the branch set corresponding to the degree-m vertex and $v_1, \ldots v_m$ the branch set corresponding to the leaves of $K_{1,m}$. Easily, we may assume that each v_i is a vertex of G. We will turn Δ into a K-fat $K_{1,m}$ minor of X, a contradiction to our assumption.

Recall that each v_i is a super-box of X, and so we can pick a mid-point $x_i \in v_i$. Let $\bar{B}$ be a connected subgraph of the ball of radius $K/2$ around $\bigcup B$ in X containing $\bigcup B$, which exists because $\bigcup B$ is K-near-connected being a union of adjacent K-near-components. Notice that (7) implies that $d(x_i, \bar{B}) > 3K/2$, and $d(x_i, x_j) > 4K$ for $i \neq j$.

Let $\pi_i, 1 \leq i \leq m$, be geodesic from x_i to $\bar{B}$, and let P_i be the initial subpath of π_i of length K. We can now form a K-fat $K_{1,m}$ minor of X as follows. We let the singletons $\{x_i\}$ be the branch sets corresponding to the m leaves of $K_{1,m}$. We let P_i be the branch paths corresponding to the edges, and we let $\bar{B} \cup \bigcup_{1 \leq i \leq m} \pi_i \backslash Q_i$ be the branch set corresponding to the centre. These sets form the desired K-fat $K_{1,m}$ minor of X. This contradiction proves (8), completing our proof. $\qquad\square$

Remark: In the special case $m = 3$ of Theorem 4.1, the resulting graph G becomes a path or a cycle. In this case, after subdividing each edge of G M times, the quasi-isometry from X to G —obtained by mapping each vertex to its super-box— becomes a map of bounded additive distortion, i.e. a $(1, K')$-quasi-isometry. This can be proved similarly to (5). We do not see how to extend this to the case $m > 3$.

5 More on Conjecture 1.1

Despite our Conjecture 1.1, we remark that the correspondence between minor-closed families of graphs and quasi-isometry classes is not 1–1. Theorem 5.1 below makes this point clear, but it also provides a tool for studying Conjecture 1.1. Here we think of graphs as topological spaces, by considering the corresponding 1-complex.

Theorem 5.1. *Let Γ, Δ be finite graphs that are homeomorphic as topological spaces, and let X be a length space. Then $\Gamma \prec^\infty X$ if and only if $\Delta \prec^\infty X$.*

Let Θ be the graph obtained from K_4 by removing one edge (or from $K_{2,3}$ by contracting one). Recall that outerplanar graphs can be characterised as $\mathrm{Forb}(K_4, K_{2,3})$. Combining this with Theorem 5.1, and a result of Fujiwara & Papasoglou [17] stating that a graph is quasi-isometric to a cactus if and only if it has no asymptotic Θ minor, we obtain

Corollary 5.2. *The following are equivalent for a length space X:*

  (i) X is quasi-isometric to a cactus;

  (ii) X is quasi-isometric to an outerplanar graph;

 (iii) X has no asymptotic K_4 or $K_{2,3}$ minor, and

 (iv) there is a subdivision Θ' of Θ such that X has no asymptotic Θ' minor.

Here, we say that a graph Δ' is a *subdivision* of a graph Δ, if Δ' is obtained by replacing some of the edges of Δ by independent paths with the same end-vertices.

Proof. The implication (i) $\to$ (ii) is obvious because every cactus is outerplanar.

The implication (ii) $\to$ (iii) is straightforward since quasi-isometric spaces have the same asymptotic minors and an outerplanar graph has no K_4 or $K_{2,3}$ minor, let alone an asymptotic one.

The implication (iii) $\to$ (iv) is obvious because $K_{2,3}$ is a subdivision of Θ.

To prove (iv) $\to$ (i), notice first that X is quasi-isometric to a graph G, which by the above remark has no asymptotic Θ' minor. Then G has no asymptotic Θ minor by Theorem 5.1. Finally, G is quasi-isometric to a cactus by the aforementioned result of Fujiwara & Papasoglu [17], hence so is X. $\qquad\square$

Note that two graphs are homeomorphic (as topological spaces) if and only if they are subdivisions of the same graph (which can be obtained by suppressing all vertices of degree 2). Therefore Theorem 5.1 follows by applying the following lemma twice:

Lemma 5.3. *Let Δ' be a subdivision of a finite graph Δ, and let X be a length space. Then $\Delta \prec^\infty X$ if and only if $\Delta' \prec^\infty X$.*

Proof. For simplicity, let us assume that X is a geodesic metric space, it is straightforward to adapt the following arguments to more general length spaces.

By the definitions, $\Delta' \prec^\infty X$ implies $\Delta \prec^\infty X$ because $\Delta \prec \Delta'$. To show the converse, it suffices to consider the case where Δ' is obtained from Δ by subdividing an edge $e \in E(\Delta)$ once, since we can then repeatedly subdivide edges to obtain any subdivision Δ' of Δ.

So given $e = xy \in E(\Delta)$, a geodesic space X with $\Delta \prec^\infty X$, and $K \in \mathbb{N}$, let $M = (\{V_x, x \in V(\Delta)\}, \{P_f, f \in E(\Delta)\})$ be a $3K$-fat Δ-minor in X. We construct a K-fat Δ'-minor M' in X as follows. For every $x \in V(\Delta)$, we just keep $V'_x := V_x$ as a branch set. We need to provide the branch set V'_o of the vertex o of Δ' arising from the subdivision of e. For this, let p be the last point along P_e, as we move from V_x to V_y, such that $d(p, V_x) \le K$ holds, and let q be the first point along P_e after p such that $d(q, V_y) \le K$ holds. Let V'_o be the subpath of P_e between p and q. Easily, $d(V'_o, V_z) \ge K$ holds for every $z \in V(\Delta)$. Let P'_{xo} be a shortest p–V_x path in X, and let

P'_{yo} be a shortest q–V_y path in X, and notice that $|P'_{xo}| = |P'_{yo}| = K$. For every other edge f of Δ' we have $f \in E(\Delta)$ and we just let $P'_f := P_f$ to complete the definition of $M' = (\{V'_x, x \in V(\Delta')\}, \{P'_f, f \in E(\Delta')\})$. We have $d(P'_{xo}, P'_f) > K$ for every $f \in E(\Delta)$ because $d(p, P_f) \geq 3K$ and $P'_{xo} \subseteq B_p(K)$. Moreover, $d(P'_{xo}, P'_{yo}) \geq K$ because P'_{xo}, P'_{yo} are within distance K from V_x, V_y, respectively, and $d(V_x, V_y) \geq 3K$.

Thus M' is a K-fat Δ'-minor in X, establishing $\Delta' \prec^\infty X$.

$\square$

The converse of Theorem 5.1 is false in general: for example, we can let Γ be K_2, and Δ be its complement. But if we insist that Γ, Δ are both connected, and allow X to be disconnected, i.e. we allow for infinite distances between pairs of points of X, then the converse becomes true and not hard to prove[3]. If we insist that X be connected however, then the converse fails: if Γ is a cycle, and Δ the union of two cycles with exactly one common vertex, then Γ, Δ are asymptotically equivalent though non-homeomorphic. Still, we can obtain a converse to Theorem 5.1 (for connected X) by restricting to 2-connected graphs. We leave the details to the interested reader.

6   Coarse König Theorem

König's classical theorem states that every infinite connected graph contains a countably infinite star or a ray as a subgraph [11, Proposition 8.2.1.]. It is a cornerstone of infinite graph theory as it facilitates compactness arguments. In this section we prove a metric analogue.

Let P_n denote the path (in the graph theoretic sense) of length n. Let $\bigvee P_n$ be the graph obtained from the disjoint union of $(P_n)_{n \in \mathbb{N}}$ by identifying the first vertex of each P_n. A *ray* is a 1-way infinite path.

Theorem 6.1. *(Coarse König Theorem) Let X be a (connected) length space of infinite diameter. Then X has either the ray or $\bigvee P_n$ as an asymptotic minor.*

We prepare the proof of this with a couple of lemmas of independent interest. A ray P in a graph G is called *divergent*, if $P \cap S$ is finite for every subset S of $V(G)$ with bounded diameter.

Lemma 6.2. *Let G be a graph that has a divergent ray. Then G has the ray as an asymptotic minor.*

Proof. Let P be a divergent ray of G, and let o be its starting vertex. Given $R \in \mathbb{N}$, let $r(n)$ denote the last vertex v of P with $d(o, v) \leq n$. We form an R-fat ray minor of G as follows. Let $B_0 := \{o\}$. Having defined B_{i-1}, we proceed to define $B_i, i \geq 1$ inductively by letting B_i be the subpath of P from the last vertex of B_{i-1} to $r(R + s_n)$, where $s_n := \max\{d(o, v) \mid v \in \bigcup_{j<i} V(B_j)\}$. This choice guarantees that $d(B_i, B_j) \geq R$ holds whenever $|i - j| \geq 2$ (Figure 1).

Thus we can let the B_i with even i be the branch sets, and the B_i with odd i the branch paths, of a ray-minor of G, which minor is R-fat by the last remark. $\square$

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[3]For this, given non-homeomorphic Γ_1, Γ_2, let $X_i, i = 1, 2$ be the disjoint union of ω copies of Γ_i, with each edge of the nth copy subdivided n times. Then $\Gamma_i \prec^\infty X_i$, but $\Gamma_i \not\prec^\infty X_{3-i}$.
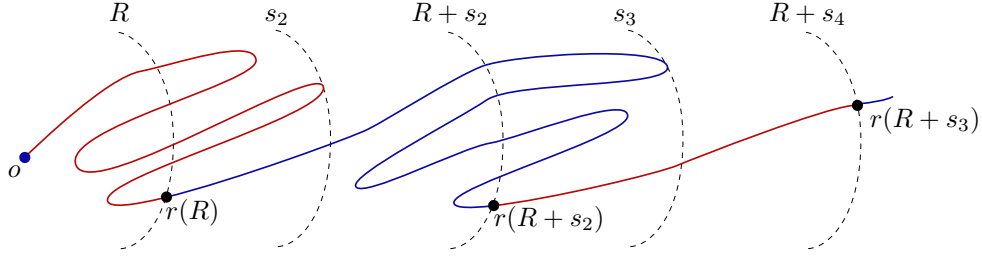
Figure 1: Turning a diverging ray into a fat ray minor. Points meeting the same dashed line have the same distance from o, indicated at the top. Branch sets are depicted in blue, if colour is shown, and branch paths in red.

By the Infinite Ramsey theorem, every infinite graph contains either an infinite clique or an infinite independent set. We will use the following corollary of this.

Lemma 6.3. *Let $G_1 \supseteq G_2 \supseteq G_3 \ldots$ be graphs with a common infinite vertex set V. Then either some G_i contains an infinite independent set, or there are infinite subsets $S_1 \supseteq S_2 \supseteq S_3 \ldots$ of V such that $G_i[S_i]$ is complete for every i.*

Proof. Apply the Infinite Ramsey theorem to G_1. Either it returns an independent set and we are done, or it returns a clique $G_1[S_1]$. Apply the Infinite Ramsey theorem to the subgraph $G_2[S_1]$ induced on G_2 by the vertices of that clique, and proceed recursively. In each step i, either we find an infinite independent set in G_i, or a clique $G_i[S_i]$ where $S_i \subseteq S_{i-1} \ldots \subseteq S_1$. □

We can now prove the main result of this section.

Proof of Theorem 6.1. For convenience we will assume X to be a graph, which we can because every length space is quasi-isometric to a graph (Observation 2.1).

Since X has infinite diameter, there is a sequence $(P_n)_{n \in \mathbb{N}}$ of geodesics $P_n : [0, n] \to X$ parametrised by arc length with $P_n(0) = o \in X$ a fixed point. Call this set of geodesic paths S. Fix $R > 0, R \in \mathbb{N}$. We will show that X contains either an R-fat $\bigvee P_n$ or an R-fat ray.

We say that $\{P_i, P_j\}$ is a (R, n)-*bad pair*, if there are vertices $v_i \in V(P_i), v_j \in V(P_j)$ satisfying $d(v_i, v_j) \leq R$ and $d(v_i, o), d(v_j, o) > n$.

We define a sequence of auxiliary graphs on S as follows. Let $G_n = G_n(R)$ be the graph on S whose edges are the (R, n)-bad pairs. Apply Lemma 6.3 to this sequence $(G_n)_{n \in \mathbb{N}}$. There are two cases depending on the outcome that Lemma 6.3 returns.

If it returns some G_n with an independent set S_n, then we can find a $\bigvee P_n$ R-fat minor in $\bigcup S_n$ as follows. Choose a sequence $\{P'_k\}_{k \in \mathbb{N}}$ of elements of S_n such that the length of P'_k is greater than $2kR + n$. We set

$$B_{v_0} = \bigcup_{k \in \mathbb{N}} P'_k([0, n]),$$

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which will be the branch set corresponding to the infinite-degree vertex of $\bigvee P_n$. For the other branch sets, we decompose each $P'_k \backslash P'_k([0, n])$ into $2k$ intervals of length at least R, and let every other such interval be a branch path and every other one a branch set. This defines an $\bigvee P_n$ minor in X, which is R-fat because no pair $\{P'_k, P'_j\}$ is (R, n)-bad, and each P'_k is a geodesic.

Otherwise, our application of Lemma 6.3 returns a sequence $S_1 \supseteq S_2 \supseteq S_3 \ldots$ with each $G_i[S_i]$ complete. In this case, we will find a divergent ray D in X, and therefore an asymptotic ray minor by Lemma 6.2. We construct D recursively as follows. Let D_0 be the trivial path $\{o\}$, and pick an element P^1 of S_1. Then for $i = 1, 2, \ldots$, assume we have defined a connected subgraph D_{i-1} of X and some $P^i \in S_i$ such that $D_{i-1} \cap P^i$ contains a vertex h_i with $d(o, h_i) \geq i - 1 - R$. Pick $P^{i+1} \in S_{i+1}$, and note that since $G_i[S_i]$ is complete, and $P^{i+1} \in S_{i+1} \subseteq S_i$, the pair $\{P^i, P^{i+1}\}$ is (R, i)-bad. This means that there is a P^i–P^{i+1} path X_i of length at most R, and at distance at least $i - R$ from o. Note that $\lim_{j \to \infty} d(o, X_j) = \infty$ as R is fixed. Let Z_i be the subpath of P^i from X_i to h_i, and let $D_i := D_{i-1} \cup X_i \cup Z_i$. Letting $h_{i+1} := P^{i+1} \cap X_i$, our inductive hypothesis is preserved. Note that $\lim_{j \to \infty} d(o, Z_j)$ is infinite as well.

Let $D' := \bigcup_{i \in \mathbb{N}} D_i$. Clearly, D' is connected since each D_i is. We claim that D' is locally finite. Indeed, notice that each D_i is finite, and it meets at most finitely many of the $X_j \cup Z_j$, $j \in \mathbb{N}$, because $\lim_{j \to \infty} d(o, X_j \cup Z_j) = \infty$.

Let D be a ray in D', which exists by König's theorem mentioned at the beginning of this section. Then D is divergent by the last argument. $\qquad\square$

7 Coarse Halin Theorems

For a cardinal k, and a graph H, let $k \cdot H$ denote the disjoint union of k copies of H. Let R denote the ray, i.e. the 1-way infinite path. Halin [23] proved that for every graph G, if $n \cdot R \subset G$ holds for every $n \in \mathbb{N}$, then $\omega \cdot R \subset G$. Does the coarse version hold?

Conjecture 7.1. *(Coarse Halin Ray Theorem) Let X be a length space. Suppose $n \cdot R \prec^\infty X$ holds for every $n \in \mathbb{N}$. Then $\omega \cdot R \prec^\infty X$.*

Let $\mathcal{H}$ denote the 'half' square grid on $\mathbb{N} \times \mathbb{N}$, and let $\mathcal{F}$ denote the 'full' square grid on $\mathbb{Z} \times \mathbb{Z}$. Halin's grid theorem [23] says that every 1-ended graph satisfying $n \cdot R \subset G$ for every $n \in \mathbb{N}$ contains a subdivision of $\mathcal{H}$.

Problem 7.2. *(Coarse Halin Grid Theorem) Let X be an 1-ended length space. Suppose $\omega \cdot R \prec^\infty X$. Then $\mathcal{H} \prec^\infty X$.*

Problem 7.3. *Let G be a Cayley graph of an 1-ended finitely generated group. Must $\mathcal{H} \prec^\infty G$ hold? Must $\mathcal{F} \prec^\infty G$ hold?*

8 Coarse Menger Theorem

In this section we prove the case $n = 2$ of Conjecture 1.4. In Section 8.3 below we present a corollary for diverging rays in infinite graphs. We restate our special case for convenience:

Theorem 8.1. *Let X be a graph or a geodesic metric space, and A, Z two subsets of X. For every $K > 0$, there is either*

    (i) a set $S \subset X$ with $\mathrm{diam}(S) \leq K$ such that $X \setminus S$ contains no path joining A to Z, or

    (ii) two paths joining A to Z at distance at least $a = K/16 \cdot 17 = K/272$ from each other.

Moreover, for every $x \in A, z \in Z$, we can choose the paths of (ii) so that both x, z are among their four endpoints.

We prepare the proof with some preliminaries. We will assume that (i) does not hold and we will show that (ii) holds.

We fix throughout the proof a shortest path $\gamma : [0, \ell] \to X$ joining A to Z.

Definition 8.2 (Bridges)**.** *A bridge B is a union of three successive paths $c_1 \cup b \cup c_2$ such that:*

    • *for any $x \in b, y \in \gamma$, we have $d(x, y) \geq K/8$.*

    • *If $b \cap A \neq \emptyset$ then c_1 is an endpoint of b and similarly if $b \cap Z \neq \emptyset$ then c_2 is an endpoint of b; otherwise c_1, c_2 are (geodetic) paths of length $K/8$ joining the endpoints of b to γ.*

We call c_1, c_2 the *legs* of B, and b its *spine*. The two endpoints of B on γ, if they exist, are denoted by B^0, B^1 so that $B^0 < B^1$. The *interval* of B is the interval $[B^0, B^1]$ of γ. If c_1 is a point (of A) then the interval of B is $[\gamma(0), v]$ where v is the endpoint of c_2, and we define similarly the interval of B when c_2 is a point of Z.

We say that a bridge is *maximal* if its interval is not contained properly in the interval of another bridge.

Proposition 8.3. *Let B be a maximal bridge with spine b. For any point $x \in b$, let x' be a point of γ minimizing $d(x, x')$. Then $x' \in [B^0, B^1]$.*

Proof. If $x' \notin [B^0, B^1]$, say $x' > B^1$, then we can modify B into a bridge B' by adding an x–x' geodesic and removing the subpath of B from x to B^1. The interval of B' is $[B^0, x']$ which strictly contains $[B^0, B^1]$, contradicting the maximality of B. $\square$

We assume that γ is parametrized by arc-length. If $a \in \gamma$ and $r > 0$, we denote by $a - r$ the point at distance r from a on γ which is before a. We define $a + r$ similarly.

Definition 8.4 (Surrounding)**.** *We say that a bridge B d-surrounds a point $x = \gamma(t) \in \gamma$ if the interval $[t_1, t_2]$ of B contains $x - d$ and $x + d$. Here if $t < d$ or $t > \ell - d$ we set $x - d = \gamma(0)$ and $x + d = \gamma(\ell)$ respectively.*

We restrict our attention now to maximal bridges.

Definition 8.5 (Order)**.** *Let B_1, B_2 be maximal bridges. If $B_2^0 > B_1^0$ (in which case $B_2^1 > B_1^1$ holds too by maximality), we write $B_2 > B_1$.*

For $R \in \mathbb{R}$, we say that B_2 *R-crosses* B_1 if B_2 R-surrounds B_1^1.

Lemma 8.6. *Assume condition (i) of Theorem 8.1 fails. Then for any $x \in \gamma$ there is some bridge B that $3K/8$-surrounds x.*

Proof. The ball $\mathrm{Ball}_x(3K/8)$ separates γ into two components γ_A and γ_Z. Let $A' := A \cup \{y \in X \mid d(y, \gamma_A) \leq K/8\}$, and define Z' analogously. Let α be a shortest path in $X \backslash \mathrm{Ball}_x(K/2)$ joining $A \cup \gamma_A$ to $Z \cup \gamma_Z$, which exists by our assumption. Note that α contains at least one subpath b from a point $p \in A'$ to a point $q \in Z'$ the interior of which is disjoint from $A' \cup Z'$. If $p \in A$ we let c_1 be the trivial path containing just p, otherwise we let c_1 be a geodesic from p to γ_A. Define c_2 analogously.

We claim that $c_1 \cup b \cup c_2$ is a bridge that $3K/8$-surrounds x. Indeed, notice that each c_i is either a singleton or has length $K/8$ and it starts outside $\mathrm{Ball}_x(K/2)$, therefore it avoids $\mathrm{Ball}_x(3K/8)$. Moreover, for every $z \in b$, we have $d(z, \gamma_A \cup \gamma_Z) \geq K/8$ and $d(z, \mathrm{Ball}_x(3K/8)) \geq K/8$. Since $\gamma \subseteq \gamma_A \cup \gamma_Z \cup \mathrm{Ball}_x(3K/8)$, it follows that $d(z, \gamma) \geq K/8$, and our claim is proved. $\square$

Lemma 8.7. *Let $B < C$ be maximal bridges with $C^0 > B^1$. Assume that there are $x \in B, y \in C$ such that $d(x, y) \leq K/4$. Then $d(B^1, C^0) < 3K/4$.*

Proof. Consider a shortest path α joining x, y, and let $|\alpha| \leq K/4$ denote its length. Let $x' \in [B^0, B^1]$, $y' \in [C^0, C^1]$ be closest points to x, y respectively on γ, where we used Proposition 8.3. By the maximality of the bridges, there is a point z on α such that $d(z, z') < K/8$ for some $z' \in \gamma$. Notice that $d(x, x') \leq d(x, z') < |x\alpha z| + K/8$, where $x\alpha z$ denotes the subpath of α from x to z. Similarly, we have $d(y, y') < |z\alpha y| + K/8$. Then

$$d(x', y') \leq d(x, x') + d(x, y) + d(y, y') < 2K/8 + 2|\alpha| \leq K/4 + 2K/4.$$

It follows that $d(B^1, C^0) < 3K/4$. $\square$

We choose a smallest set $\mathcal{U}$ of maximal bridges such that any $x \in \gamma$ is $K/4$-surrounded by some bridge in $\mathcal{U}$, and

$$\text{every } B \in \mathcal{U} \text{ has an interval of length at least } 3K/4. \tag{9}$$

Such a $\mathcal{U}$ exists by Lemma 8.6. We remark that $\mathcal{U}$ is finite because γ is compact, and any bridge that $3K/4$-surrounds $x \in \gamma$ also $K/4$-surrounds any point in an open interval of x on γ.

The minimality of $\mathcal{U}$ implies that the initial points of its bridges are not too dense on γ:

Proposition 8.8. *Let B, C, D be bridges in $\mathcal{U}$. Suppose that C r-crosses B, and D r-crosses C for some $r \leq K/4$. Then $d(B^0, D^0) \geq r$.*

Proof. Suppose $d(B^0, D^0) < r$. This implies $D^0 \in [B^0, B^1]$ by (9). We claim that any point that is $K/4$-surrounded by C is also $K/4$-surrounded by either B or D, contradicting the minimality of $\mathcal{U}$. So suppose p is $K/4$-surrounded by C. Then $d(D^1, p) > K/4$ as $D^1 > C^1$. Moreover, we have $p > D^0$ because $d(C^0, p) \geq K/4$ and $d(C^0, D^0) < r$ by our assumptions.

We consider two cases. If $p > B^1$, then either $d(D^0, p) \geq K/4$, in which case p is $K/4$-surrounded by D and we are done, or $d(D^0, B^1) < d(D^0, p) < K/4$. In

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the latter case, however, the triangle inequality yields $d(B^0, B^1) \leq d(B^0, D^0) + d(D^0, B^1) < K/2$, contradicting (9).

It remains to consider the case where $p \leq B^1$. As above, we obtain $d(D^0, p) < K/4$ unless p is $K/4$-surrounded by D. Similarly, we have $d(p, B^1) < K/4$ unless p is $K/4$-surrounded by B. But then the triangle inequality yields $d(B^0, B^1) \leq d(B^0, D^0) + d(D^0, p) + d(p, B^1) < 3K/4$, again contradicting (9). (This last inequality is the reason why we chose $\mathcal{U}$ so that any $x \in \gamma$ is $K/4$-surrounded, sacrificing some of the $3K/8$ provided by Lemma 8.6.) □

A sequence $\mathcal{B} = B_1, \ldots B_n$ of bridges is said to be *R-crossing* for some $R \in \mathbb{R}$, if B_1 R-surrounds $\gamma(0)$, each $B_i, i \geq 2$ R-crosses B_{i-1}, and $B_n^1 \in Z$. Note that B_n must R-surround $\gamma(1)$ as $\gamma(1) > B_{n-1}^1$. Such a sequence is *perfect*, if in addition its initial points are R-apart, i.e. $d(B_i^0, B_j^0) \geq R$ holds for every $1 \leq i, j \leq n$.

Remark: *A perfect R-crossing sequence $\mathcal{B}$ remains a perfect R-crossing sequence if we exchange the roles of A and Z and reverse the direction of γ and the ordering of $\mathcal{B}$. But we will not use this fact.*

It is straightforward to check that if we order the elements of $\mathcal{U}$ by $<$ then the resulting sequence S is a $K/4$-crossing sequence. Our next lemma says that we can obtain a perfect subsequence by sacrificing half of our crossing constant.

Lemma 8.9. *Let $S = \{B_i\}_{1 \leq i \leq n}$ be a $K/4$-crossing sequence of bridges. Then there is a perfect $K/8$-crossing subsequence S' of S.*

Proof. We claim that

> there are no three initial points $x < y < z$ of elements of S such that $d(x, y), d(y, z) < K/8$. (10)

Indeed, if such a triple exists, then we may assume that no other initial point of an element of S lies in $[x, z]$ by choosing x, y, z as close to each other as possible. It follows that there is $i \leq n$ such that $x = B_i^0, y = B_{i+1}^0, z = B_{i+2}^0$, and so B_{i+1} $K/4$-crosses B_i and B_{i+2} $K/4$-crosses B_{i+1}. This contradicts Proposition 8.8, and so our claim (10) is proved.

Thus the pairs $\{x, y\}$ of initial points of elements of S with $d(x, y) < K/8$ form a matching, i.e. no point participates in more than one such pair. For every such pair with $x < y$, remove the bridge B_x with initial point x from S, and let S' be the remaining subsequence of S. Clearly, we cannot have a pair $\{x, y\}$ as above in S'. To see that S' is $(R - K/8)$-crossing, notice that whenever we removed a bridge B_x from S, we left a bridge B_y in S' such that $B_y^0 < B_x^0 + K/8$ and $B_y^1 > B_x^1$. Thus any $p \in \gamma$ that was R-surrounded by B_x is $(R - K/8)$-surrounded by B_y. □

Proposition 8.10. *Suppose that B_1, B_2, B_3, B_4, B_5 are bridges such that each $B_i, i \geq 2$, $K/8$-crosses B_{i-1}, and B_i^0 lies in the interval of B_1 for every $2 \leq i \leq 5$. Then at least one B_i does not lie in $\mathcal{U}$.*

Proof. We will show that any point $K/8$-surrounded by B_4 is $K/8$-surrounded by either B_2, B_3, or B_5, so the result follows by the minimality of $\mathcal{U}$.

For this, notice that our assumptions impose the ordering

$$B_1^0 < B_2^0 < B_3^0 < B_4^0 < B_5^0 < B_1^1 < B_2^1 < B_3^1 < B_4^1 < B_5^1.$$

Figure 2: One of the possibilities in the proof of Proposition 8.10.

The definition of crossing implies that $d(B_{i-1}^1, B_i^j) \geq K/8$ for every $i \in \{2, 5\}$ and $j \in \{0, 1\}$. Moreover, applying Proposition 8.8 twice, once on B_2, B_3, B_4, and once on B_3, B_4, B_5, we also obtain $d(B_2^0, B_4^0), d(B_3^0, B_5^0) \geq K/8$.

Suppose now that a point p is $K/8$-surrounded by B_4, and so $p \in [B_4^0, B_4^1]$. If $p \in [B_4^0, B_1^1]$ (Figure 2), then it is $K/8$-surrounded by B_2, because $d(B_2^0, B_4^0), d(B_1^1, B_2^1) \geq K/8$. If $p \in [B_1^1, B_2^1]$, then it is $K/8$-surrounded by B_3, because $d(B_2^1, B_3^1) \geq K/8$ and $d(B_3^0, B_1^1) > d(B_3^0, B_5^0) \geq K/8$. Finally, if $p \in [B_2^1, B_4^1]$, then it is $K/8$-surrounded by B_5, because $d(B_5^0, B_2^1) > d(B_0^1, B_2^1) \geq K/8$ and $d(B_4^1, B_5^1) \geq K/8$. This proves our claim that p is always $K/8$-surrounded by either B_2, B_3, or B_5, contradicting the minimality of $\mathcal{U}$. $\square$

Lemma 8.11. *Suppose that $B_i : i = 1, \ldots, n$ are bridges in $\mathcal{U}$ such that B_i crosses B_{i-1} for $i = 2, \ldots, n$, and that there are $x \in B_1, y \in B_n$ such that $d(x, y) < K/4$. Then $n \leq 8$.*

Proof. Suppose $n \geq 9$. By Proposition 8.10, the intervals of B_1, B_5 are disjoint. By the same argument, the intervals of B_5, B_n are disjoint too. By lemma 8.7, we have $d(B_1^1, B_9^0) < 3K/4$. These facts combined imply that the interval of B_5 has length less than $3K/4$, contradicting (9). $\square$

Our plan is to construct the two paths α_1, α_2 joining A to Z required by Theorem 8.1 by combining subpaths of γ with bridges from a sequence S' as in Lemma 8.9. However, we can have bridges B, C in S' arbitrarily close to each other (but not intersecting, as this would contradict maximality), in which case we have to avoid putting B in α_1 and C in α_2. Therefore, if $d(B, C) < a$, we will use a B–C path α of length less than a to join B, C, and produce a new path called a *meta-bridge*. We may need to iterate the process, joining two nearby meta-bridges into a new meta-bridge. But the following lemma will exploit Lemma 8.11 to upper bound the complexity of meta-bridges.

Formally, we can simply define a *meta-bridge* B to be a path with endpoints in $\gamma \cup A \cup Z$ whose interior is disjoint from $\gamma \cup A \cup Z$. But the meta-bridges we will actually work with will obey further restrictions, being constructed recursively starting from bridges as explained above. Much of the notation we introduced for bridges can be applied verbatim to meta-bridges, and we will do so without further comment.

A *join* of a meta-bridge D is a maximal subpath of D consisting of points that lie on no bridge of S'.

Choose $a = K/(16 \cdot 17)$.

Lemma 8.12. *There is a perfect, $K/8$-crossing, sequence M of meta-bridges such that*

(i) *each join J of an element of M has length $\ell(J) < 15a$;*

(ii) *we have $d(J, \gamma) \geq a$ for each join J of an element of M, and*

(iii) *if x, y are points of distinct meta-bridges B, C of M, then $d(x, y) \geq a$ unless there are endpoints $B^i, C^{1-i}, i \in \{0, 1\}$ with $d(B^i, C^{1-i}) < K/8$.*

Proof. We start with a sequence S' of maximal bridges as in Lemma 8.9, and construct M recursively by joining (meta-)bridges together as long as (iii) is not satisfied. Since S' is finite, the process will terminate and (iii) will then be satisfied. The fact that (i) and (ii) hold will then be deduced from Lemma 8.11.

To make this precise, set $M_0 := S'$, and proceed inductively as follows. At step $i = 1, 2, \ldots$, we have defined a sequence M_{i-1} of meta-bridges. If M_{i-1} satisfies (iii), we set $M := M_{i-1}$ and stop. Otherwise we can find points x, y in distinct elements B, C of M_{i-1}, such that $d(x, y) < a$ and $d(B^i, C^j) \geq K/8$ for every $i, j \in \{0, 1\}$. We pick a shortest x–y path α, and combine it with a subpath of each of B, C to form a meta-bridge D as follows. We notice that α separates $B \cup C$ into 4 subpaths, and choose two of them B_α, C_α so as to maximize the interval of the meta-bridge $D := B_\alpha \cup \alpha \cup C_\alpha$. Notice that D now $K/8$-surrounds any point of γ previously $K/8$-surrounded by either B or C. Thus the sequence M_i obtained by replacing the subsequence of M_{i-1} between B and C by D is a perfect $K/8$-crossing sequence. For later use, we colour the path α *brown*.

The final sequence M of this process satisfies (iii) by definition. To prove that it satisfies (i), we will introduce some terminology.

A *join-tree* is any maximal (with respect to inclusion) connected subspace of X all points of which had been coloured brown in some step i of the above process. It is easy to see, by induction on i, that every join-tree is indeed homeomorphic to a finite simplicial tree, and that every join of an element of M is contained in some join-tree. The *length* $\ell(T)$ of a join-tree T is its 1-dimensional Hausdorff measure, and it equals the sum of the lengths of its constituent paths. We will prove (i) by showing that $\ell(T) < 15a$ holds for every join-tree T.

It is easy to see —by induction on i— that every leaf of a join-tree T lies on a bridge of S'. The *rank* $\mathrm{Rank}(T)$ of T is the number of leaves of T. Note that every two leaves of T lie in distinct elements of S', because when a new brown path is introduced it joins distinct meta-bridges. (Such a leaf P is a (trivial) join-tree itself, and we have $\mathrm{Rank}(P) = 1$.)

We claim that for every join-tree T with $\mathrm{Rank}(T) \geq 2$, we have

$$\ell(T) < a(\mathrm{Rank}(T) - 1). \tag{11}$$

We prove this by induction on $\mathrm{Rank}(T)$. Let α be an edge of T, and recall that $\ell(\alpha) < a$. The case $\mathrm{Rank}(T) = 2$ occurs exactly when $T = \alpha$, and so our claim is corroborated. For the general case, notice that $T \backslash \alpha$ has two components T_1, T_2. We have $\mathrm{Rank}(T) = \mathrm{Rank}(T_1) + \mathrm{Rank}(T_2)$, and $\ell(T) < \ell(T_1) + \ell(\alpha) + \ell(T_2)$ and

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so our inductive hypothesis yields $\ell(T) < a(\mathrm{Rank}(T_1) + \mathrm{Rank}(T_2) - 2) + a = a(\mathrm{Rank}(T) - 1)$ as claimed.

Next we claim that

$$\text{there is no join-tree } T \text{ with } \mathrm{Rank}(T) \geq 17. \tag{12}$$

Indeed, if such a T exists, then by recursively removing its last edge coloured brown, we can obtain a join-subtree T' with $16 \geq \mathrm{Rank}(T') \geq 9$ by the pigeonhole principle. Let B_i, B_j be bridges of S' incident with T', such that $j \geq i + 8$, which exist since $\mathrm{Rank}(T') \geq 9$. By (11) we have $\ell(T') < 16a$, and so $d(B_i, B_j) < 16a$. This contradicts Lemma 8.11 because $16a \leq K/4$, and so (12) is proved. Combining (12) with (11) yields (i).

To prove (ii), we will check that

$$\text{every join-tree } T \text{ has a leaf } p \text{ such that } d(p, \gamma) \geq K/16 - a. \tag{13}$$

For this, let α be the first subpath of T coloured brown, let x, y be its two endpoints, and let B_x, B_y be the bridges in S' containing them. Thus both x, y are leaves of T. Since the spines of B_x, B_y are at distance at least $K/8$ from γ, and $\ell(\alpha) < a$, the only interesting case is where both x, y lie on legs. Let x', y' be the endpoints of those legs on γ, and recall that we must have $d(x', y') \geq K/8$ for otherwise we would not have coloured α brown. Thus the triangle inequality yields $d(x, \gamma) + d(y, \gamma) + a \geq K/8$. Moreover, we have $d(y, \gamma) \leq d(x, \gamma) + a$. Combining these inequalities we obtain $2d(x, \gamma) + 2a \geq K/8$, and letting $p := x$ proves (13).

Combining (i) and (13) we deduce that every join-tree lies at distance at least $K/16 - a - 15a = K/16 - 16a$ from γ. Since $17a \leq K/16$, and every join is contained in a join-tree, this implies (ii). $\qquad\square$

Remark: The last inequality is the bottleneck for maximizing the distance a between the two paths α_i of Theorem 8.1.

We have collected all the necessary ingredients for the proof of our Theorem 8.1.

Proof of Theorem 8.1. Assuming condition (i) fails, we will now define the two paths α_1, α_2 of (ii), joining A to Z, and satisfying $d(\alpha_1, \alpha_2) \geq a$.

Each α_j will consist of intervals of γ and meta-bridges in a family M as in Lemma 8.12. It will be convenient to colour the paths contained in α_1 red and the paths of α_2 green. We will define these paths inductively.

Since M is a perfect sequence, it starts with a meta-bridge M_1 surrounding $t_0 = \gamma(0)$. We colour M_1 red, and set $\beta_1^1 := M_1$. We colour the initial subpath of γ from t_0 to $t_1 := M_1^1$ green, and set $\beta_2^1 := [t_0, t_1]$. This completes step 1 of our induction.

For step n, assume that we have already defined a red path $\beta_1 = \beta_1^{n-1}$ and a green path $\beta_2 = \beta_2^{n-1}$ with a common endpoint $t = t_{n-1} \in \gamma$. Let $C \in M$ be a meta-bridge that $K/8$-surrounds t. Let s be a point of $\gamma \cap (\beta_1 \cup \beta_2)$ closest to C^0 (it is possible that $s = C^0$). Suppose $s \in \beta_i$. Remove the subpath of β_i from s to t, and replace it by $C \cup I$ where I is the subpath of γ between s and C^0. Finally, extend the other path β_{3-i} by the subpath of γ from t to C^1, if

$C^1 \in \gamma$, or from t to $\gamma(1)$ if $C^1 \in Z$. In the latter case we terminate the process and set $\alpha_j := \beta_j, j \in \{1, 2\}$. This completes the construction of α_1, α_2.

It remains to check that $d(\alpha_1, \alpha_2) \geq a$, and to do so we first point out some properties of our construction.

Firstly, every meta-bridge $C \in M$ contained in an α_1 is traversed from C^0 to C^1 as α_1 moves from A to Z.

Secondly, any meta-bridge B removed from $\beta_1 \cup \beta_2$ in some step n cannot return to $\beta_1 \cup \beta_2$ in a later step $n' > n$, because $B^1 \leq t_{n-1}$.

Finally, notice that if $d(C^0, B^1) < K/8$ holds for some $meta-bridges$ $B, C \in M$, then either $s = B^j$, in which case B and C become part of the same β_i, or $s \neq B^j$ and B is not part of $\beta_1 \cup \beta_2$ at step n, hence nor at any later step. To summarize, we have proved that

if $d(C^i, B^{i'}) < K/8$ holds for some meta-bridges $B, C \in M$, then B, C cannot lie in distinct α_j. $\qquad$ (14)

Define a γ–edge to be a shortest interval of γ whose endpoints are both endpoints of meta-bridges of M. Thus (14) implies

if two γ–edges e, f are at distance $< K/8$, then e, f cannot lie in distinct α_j. $\qquad$ (15)

We now check that $d(x, y) \geq a$ holds for every $x \in \alpha_1, y \in \alpha_2$, distinguishing three cases according to whether these points lie on meta-bridges or on γ.

If x, y lie on meta-bridges B, C, then $B \neq C$ by construction. Then (14) says that the endpoints of B, C are at distance at least $K/8$ from each other, and so property (iii) of Lemma 8.12 yields $d(x, y) \geq a$.

If $x, y \in \gamma$, then (15) implies again $d(x, y) \geq a$.

Finally, if $x \in B \in M$, and $y \in \gamma$, then property (ii) of Lemma 8.12 yields $d(x, y) \geq a$ unless x lies on a leg of B. Here, a *leg* of a meta-bridge is a maximal initial or final subpath contained in a bridge in S'. If x lies on a leg c of B, we recall that if c lies in α_j, then one of the two γ–edges e incident with its endpoint $x' \in \gamma$ also lies in α_j. Since y cannot lie in e, (15) yields $d(y, x') \geq K/8$. We also know that $d(x, y) \geq d(x, x')$ by the definition of a leg, and so $d(x, y) \geq a$ easily follows.

This completes the proof of Theorem 8.1. $\qquad \square$

8.1 Alternative construction of the paths — reducing to the classical Menger theorem

This section offers an alternative proof of Theorem 8.1, which is simpler but gives a weaker lower bound a on $d(\alpha_1, \alpha_2)$. The main difference is that instead of constructing α_1, α_2 explicitly, we apply Menger's theorem to an auxiliary graph, consisting of γ–edges and meta-bridges in M.

For this, we first adapt (iii) of Lemma 8.12 by lowering the constant $K/8$ to $K/20$, say, and lowering a proportionately; the proof can be repeated verbatim after this change. This change allows us to deduce the following corollary of Lemma 8.12:

Corollary 8.13. *If B, C are meta-bridges in M or γ–edges, then $d(B, C) \geq a$ unless B, C are incident or as in (iii) of Lemma 8.12 (i.e. they have $K/20$-close end-points).*

Proof. If $B, C \in M$, then this is just Lemma 8.12 (iii). If B, C are both γ–edges, then it is obvious because γ is a geodesic. Otherwise, we have $B \in M, C \subseteq \gamma$ or vice-versa. In this case we use Lemma 8.12 (ii), that fact that spines of meta-bridges are far from γ, and that legs of meta-bridges are geodetics to γ. $\qquad\square$

Alternative proof of Theorem 8.1. Let G be the graph whose edge set consists of M (as in Lemma 8.12) and the γ–edges that M defines. For every $x, y \in V(G)$ joined by a γ–edge e of length less than $K/20$ (the new constant of (iii) of Lemma 8.12), contract e. Let G' be the graph obtained from G after all these contractions. Notice that the contracted edges of G are disjoint, and therefore G' has no cut-vertex x since M contains a meta-bridge surrounding x. Let α_1, α_2 be two disjoint A–Z paths, which exist by Menger's theorem. Then $d(\alpha_1, \alpha_2) > a$ by Corollary 8.13. $\qquad\square$

8.2 Choosing endpoints

We remark that $\gamma(0), \gamma(1)$ are among the four endpoints of $\alpha_1, y \in \alpha_2$ in the above construction. This can be seen directly from the construction, or by noticing that $\mathcal{U}$, and hence S' and M, contains at most one bridge with an endpoint in each of A and Z by maximality. Recall that γ was chosen as an arbitrary shortest A–Z path. By modifying X we can strengthen the above conclusion:

Corollary 8.14. *For every $x \in A$ and $z \in Z$, we can choose the paths of Theorem 8.1 so that both x, z are among their four endpoints.*

Proof. By the above remark, the statement holds if $d(x, z) = d(A, Z)$. We will extend X into a geodesic metric space X' and define $A' \ni x, Z' \ni z$ so that $d(x, z) = d(A', Z')$ and X' has no subset S with $\mathrm{diam}(S) \le K$ separating A' from Z'.

To do so, for every point $p \in A \backslash \{x\}$, attach a geodetic arc A_p of length larger than $d(x, z)$ to p, and let p' denote the other endpoint of A_p. Let $A' := \{x\} \cup \{p' \mid p \in A, p \ne x\}$. Let $Z' := Z$. There is a natural way to extend the metric of X to X', and this is also the unique metric d that makes X' a geodesic metric space. Note that $d(x, z) = d(A', Z')$ holds.

Let S be a subset of X' separating A' from Z' with $\mathrm{diam}(S) \le K$. We modify S into a subset S' of X by 'projecting' $S \backslash X$ onto A. To make this precise, let $S_A \subset A$ be the set of those $p \in A$ such that S intersects A_p, and let $S' := S_A \cup (S \cap X)$. It is straightforward to check that $\mathrm{diam}(S') \le \mathrm{diam}(S)$, and that S' separates A from Z in X. It follows that no such $S \subset X'$ can exist by our assumption on X. Applying Theorem 8.1, and the above remark, to X', A', Z' we obtain two A'–Z' paths that are a-far apart and contain x, z among their four endpoints. Removing a subpath of the form A_p from one of them completes our proof. $\qquad\square$

8.3 An application — diverging rays

It is not hard to extend Theorem 8.1 to the situation where Z is replaced by ∞; that is, we have

Corollary 8.15. *Let X be a locally finite metric graph, and $A \subset X$. For every $K > 0$, there is either*
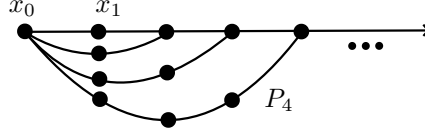
Figure 3: A non-locally-finite graph failing the conclusion of Corollary 8.15.

(i) a set $S \subset X$ with $\mathrm{diam}(S) \leq K$ such that $X \setminus S$ contains no path of infinite length starting at A, or

(ii) two paths of infinite length starting at A at distance at least $a = K/16 \cdot 17$ from each other.

Proof. Apply Theorem 8.1 to a sequence of spaces $(X_n)_{n \in \mathbb{N}}$, namely the balls of radius $n \in \mathbb{N}$ around A in X, with $Z = Z_n$ being the set of points at distance exactly n from A. This yields pairs of A–Z_n paths at distance at least a from each other, and we can use a compactness argument to obtain A–∞ paths at least a from each other. $\qquad\square$

Remark: the local-finiteness condition is necessary in Corollary 8.15. A non-locally-finite counterexample can be obtained as follows. Let $R = x_0 x_1 \ldots$ be a ray, and for each $i > 1$, join x_0 to x_i by a path P_i of length i, so that $P_i \cap P_j = \{x_0\}$ for every $i \neq j$, to obtain a graph X (in which x_0 is the only vertex of infinite degree, see Figure 3). Let A be any infinite set of vertices of X. Easily, no set of finite diameter separates A from infinity in X, i.e. (i) fails. Moreover, if X contains two infinite paths, then so does $X' := X - x_0$. But X' is an 1-ended tree. Thus (ii) fails as well.

We say that two rays R, L in a graph G *diverge*, if for every $n \in \mathbb{N}$ they have tails $R' \subseteq R, L' \subseteq L$ satisfying $d(R', L') > n$.

Theorem 8.16. *Let G be a graph with bounded degrees which has an infinite set of pairwise disjoint rays. Then G has a pair of diverging rays.*

The idea is to apply Corollary 8.15, letting the scale K increase as we progress towards infinity. Instead of increasing K, we will use the equivalent but more convenient method of scaling down the edge-lengths as the distance from a fixed origin tends to infinity. This is the role of the following lemma:

Lemma 8.17. *Let $G = (V, E)$ be a graph with bounded degrees which has an infinite set of pairwise disjoint rays. Then there is a finite vertex-set $A \subset V$ and an assignment of edge-lengths $\ell : E \to \mathbb{R}_{>0}$ with the following properties:*

(i) *no ball of radius 1 in the corresponding metric d_ℓ separates A from ∞, and*

(ii) $\lim_{e \in E} \ell(e) = 0$.[4]

Let us first see how this implies Theorem 8.16.

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[4]This means that for every $\epsilon > 0$ all but finitely many $e \in E$ satisfy $\ell(e) < \epsilon$.

Proof of Theorem 8.16. Applying Corollary 8.15 to G endowed with d_ℓ as in Lemma 8.17, we obtain a pair of rays R, L starting at A such that $d_\ell(R, L) > b$ for some (universal) constant $b > 0$. Let $E_n := \{e \in E \mid \ell(e) \geq 1/n\}$. Thus by (ii), E_n is finite. Let E'_n be the set of edges of G at distance at most n from E_n with respect to d_G, which is finite too since G is locally finite. To see that R, L diverge, choose tails R', L' avoiding E'_n. Notice that any R'–L' path P either avoids E_n or contains at least n edges in $E'_n \backslash E_n$. The fact that $\ell(P) > b$ implies that P contains at least $\min\{n, bn\}$ edges, proving that R, L diverge. $\square$

It remains to prove Lemma 8.17.

Proof of Lemma 8.17. Let $(R_n)_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint rays in G. Let k be an upper bound on the vertex-degrees of G. Fix a vertex $o \in V$. For $n = 1, 2, \ldots$, choose $r_n \in \mathbb{N}$ large enough that

(i) $B_o(r_n)$ meets at least $k^{n+3} + 2$ of the R_i, and

(ii) $r_n > r_{n-1} + n$,

where we set $r_0 := 1$, say. Let $S_1 := B_o(r_1)$, and for $n = 2, 3, \ldots$ let $S_n := B_o(r_n) \backslash S_{n-1}$. Assign length $\ell(e) = 1/n$ for each edge e in $S_n, n \geq 1$.

Clearly, this assignment satisfies property (ii) of the statement as each S_n is finite. We claim that ℓ also satisfies (i) with $A := \bigcup_{1 \leq i \leq 4} S_i$. To see this, pick any $x \in S_n$, and let B denote its ball of radius 1 with respect to d_ℓ. Thus we have to show that there is a ray in $G \backslash B$ starting in A.

It follows easily from (ii) that

$$B \subseteq S_{n-1} \cup S_n \quad \text{or} \quad B \subseteq S_n \cup S_{n+1}. \tag{16}$$

Thus for any $y \in B$ we have $d_G(x, y) \leq n+1$ because $d_\ell(x, y) \leq d_G(x, y)/(n+1)$ by the previous remark. This means that B has radius at most $n+1$ in d_G, and so it contains at most $k^{n+1} + 1$ vertices since G has maximum degree k. We consider two cases: if $n \geq 3$, then by (i), B avoids at least one ray R_i meeting S_{n-2}. Joining that ray to A by a shortest possible path —which avoids B by (16)— we obtain a ray starting in A and avoiding B. If $n < 3$, then again by (16) we can find a ray starting in $S_4 \subset A$ and avoiding B. This proves that (i) is satisfied. $\square$

9 Further problems

As we saw in the introduction, Conjecture 1.1 can be very difficult even for a simple graph H. The following special case is of particular interest (Conjecture 1.1 can be formulated with H replaced by a finite set of finite graphs):

Conjecture 9.1 (Coarse Kuratowski–Wagner theorem)**.** *A length space X is quasi-isometric to a planar graph if and only if it has no asymptotic K_5 or $K_{3,3}$ minor.*

Here, and more generally in Conjecture 1.1, we may have to allow the resulting planar graph to be edge-weighted; we do not know whether every planar graph with arbitrary real-valued edge-lengths (used to induce a length space) is quasi-isometric to a planar graph with all edge-lengths equal to 1.

This special case is well-motivated in itself due to widespread interest in planar graph metrics, and due to the interest in groups acting on planar surfaces [20].

Another interesting class of graphs is that of graphs of finite tree width.

Conjecture 9.2 (Coarse grid theorem). *A length space X is quasi-isometric to a graph of finite tree width if and only if it does not have the $n \times n$ grid as an asymptotic minor for some $n \in \mathbb{N}$.*

One can also ask whether under the same hypothesis $asdim(X) \leq 1$ (as this is weaker).

Our Conjecture 1.1 would imply that the fundamental Robertson-Seymour graph minor theory has a coarse geometric generalisation. One can of course formulate weaker versions of our conjecture in the same direction. For example, one could ask whether excluding an infinite family of asymptotic minors is equivalent to excluding a finite such family. In geometry and topology one often asks for the 'nicest' metric among all metrics in a homeomorphism class. This is the classical topic of uniformisation of surfaces, and this point of view was crucial in the recent classification of 3-manifolds after Thurston and Perelman. Conjecture 1.1 can be seen as a coarse analog to uniformisation: we ask for a 'nice' representative in a quasi-isometry class of graphs.

Our next conjecture is a coarse version of a result of Thomassen [37] saying that an 1-ended vertex transitive graph is either planar or has an infinite clique as a minor.

Conjecture 9.3. *Let G be a locally finite, vertex-transitive graph. Then either G is quasi-isometric to a planar graph, or it contains every finite graph as an asymptotic minor.*

Recall that a graph G is *k-planar*, if it can be drawn in $\mathbb{R}^2$ so that each of its edges crosses at most k other edges.

Problem 9.4. *Let G be a locally finite, vertex-transitive graph. Is it true that G is quasi-isometric to a planar graph if and only if it is k-planar for some $k \in \mathbb{N}$?*

Vertex-transitivity is essential here, for otherwise we could let G be a sequence of subdivisions of K_5.

Our next problem is motivated by the following theorem of Khukhro:

Theorem 9.1 ([27]). *A finitely generated, infinite group Γ is virtually free, if and only if for every finitely generated Cayley graph G of Γ, there is some finite graph that is not a minor of G.*

Conjecture 9.5. *A finitely generated group Γ is virtually planar, if and only if for every finitely generated Cayley graph G of Γ, there is some finite graph that is not an asymptotic minor of G.*

Virtually planar here means that Γ has a finite index subgroup H which has a planar Cayley graph. The forward implication of this is easy: by the Svarc–Milnor lemma, any finitely generated Cayley graphs of H and Γ are quasi-isometric, and so the statement follows from Observation 2.4. (The analogous

implication of Theorem 9.1 was proved 'by hand' by Ostrovskii & Rosenthal [35].)

As already mentioned, we expect Conjecture 1.4 to be very difficult for $n \geq 3$. The following is a related algorithmic question. Let MM3 denote the algorithmic problem that takes as input a graph G on n vertices and two subsets A, Z of its vertex set, and asks whether there is a triple of A–Z paths that are pairwise at distance at least 3.

Conjecture 9.6. *MM3 is NP-complete.*

A classical result of Erdős & Pósa [14] says that every finite graph has either a $k \cdot K_3$ minor or a set of at most $f(k)$ vertices the removal of which results into a forest, where $f(k) \sim k \log k$ is universal. The following proposes a coarse analogue, motivated by both Conjecture 1.1 and Conjecture 1.4:

Conjecture 9.7 (Coarse Erdős–Pósa Theorem)**.** *There is a function $f : \mathbb{N} \to \mathbb{N}$ and a universal constant C, such that the following holds for every length space X. If $n \cdot K_3$ is not a K-fat minor of G, then there is a set S of at most $f(n)$ points of X such that the ball of radius CK around S meets all K-fat K_3 minors of X. (Alternatively, X is a graph, and removing $B_{CK}(S)$ results into a quasi-forest.)*

10   Latest progress

Many of the questions of this paper have been answered between the original submission and the revision. The aim of this section is to provide an update.

Nguyen, Scott & Seymour [33] found a counterexample to Conjecture 1.4. This was used by Davies, Hickingbotham, Illingworth & McCarty [10] to disprove Conjecture 1.1. Conjecture 9.6 has been proved by Baligács & MacManus [3].

The case $H = K_4$ of Conjecture 1.1 has been proved by Albrechtsen, Jacobs, Knappe, & Wollan [2]. Special cases of Conjecture 1.4, for $r = 2$, and sparse graphs, have been proved by Gartland, Korhonen & Lokshtanov [19] and by Hendrey, Norin, Steiner & Turcotte [24]. The special case of Conjecture 9.3 for finitely presented Cayley graphs has been proved by MacManus [30]. The "only if" direction of Proposition 9.4 has been proved by Esperet & Giocanti [16].

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