Three-cuts are a charm: acyclicity in 3-connected cubic graphs*

František Kardoš

LaBRI, CNRS, University of Bordeaux, Talence, F-33405, France;
Department of Computer Science, Faculty of Mathematics, Physics and Informatics
Comenius University, Mlynská Dolina, 842 48 Bratislava, Slovakia

Edita Máčajová

Department of Computer Science, Faculty of Mathematics, Physics and Informatics
Comenius University, Mlynská Dolina, 842 48 Bratislava, Slovakia

Jean Paul Zerafa

St. Edward's College, Triq San Dwardu
Birgu (Città Vittoriosa), BRG 9039, Cottonera, Malta;
Department of Computer Science, Faculty of Mathematics, Physics and Informatics
Comenius University, Mlynská Dolina, 842 48 Bratislava, Slovakia;
Faculty of Economics, Management & Accountancy, and Faculty of Education
L-Università ta' Malta, Msida, MSD 2080, Malta

Abstract

Let G be a bridgeless cubic graph. In 2023, the three authors solved a conjecture (also known as the S_4-Conjecture) made by Mazzuoccolo in 2013: there exist two perfect matchings of G such that the complement of their union is a bipartite subgraph of G. They actually show that given any 1^+-factor F (a spanning subgraph of G such that its vertices have degree at least 1) and an arbitrary edge e of G, there exists a perfect matching M of G containing e such that $G \setminus (F \cup M)$ is bipartite. This is a step closer to comprehend better the Fan–Raspaud Conjecture and eventually the Berge–Fulkerson Conjecture. The S_4-Conjecture, now a theorem, is also the weakest assertion in a series of three conjectures made by Mazzuoccolo in 2013, with the next stronger statement being: there exist two perfect matchings of G such that the complement of their union is an acyclic subgraph of G. Unfortunately, this conjecture is not true: Jin, Steffen, and Mazzuoccolo later showed that there exists a counterexample admitting 2-cuts. Here we show that, despite of this, every cyclically 3-edge-connected cubic graph satisfies this second conjecture.

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E-mail addresses: frantisek.kardos@u-bordeaux.fr (František Kardoš), macajova@dcs.fmph.uniba.sk (Edita Máčajová), zerafa.jp@gmail.com (Jean Paul Zerafa)

1 Introduction

In 2013, Giuseppe Mazzuoccolo [?] proposed three beguiling conjectures about bridgeless cubic graphs. His first conjecture, implied by the Berge–Fulkerson Conjecture [?], is the following.

Conjecture 1.1 (Mazzuoccolo, 2013 [?]). *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is a bipartite graph.*

This conjecture, which is no longer open, has been solved by the three authors. More precisely they prove the following stronger statement.

Theorem 1.2 (Kardoš, Máčajová & Zerafa, 2023 [?]). *Let G be a bridgeless cubic graph. Let F be a 1^+-factor of G and let $e \in E(G)$. Then, there exists a perfect matching M of G such that $e \in M$, and $G \setminus (F \cup M)$ is bipartite.*

We note that a 1^+-*factor* of G is the edge set of a spanning subgraph of G such that its vertices have degree 1, 2 or 3. Theorem **??** not only shows the existence of two perfect matchings of G whose deletion leaves a bipartite subgraph of G, but that for every perfect matching of G there exists a second one such that the deletion of the two leaves a bipartite subgraph of G. In particular, Theorem **??** also implies that for every collection of disjoint odd circuits of G, there exists a perfect matching which intersects at least one edge from each odd circuit (this was posed as an open problem by Mazzuoccolo and the last author in [?], see also [?]).

Mazzuoccolo moved on to propose two stronger conjectures, with Conjecture **??** being the strongest of all three.

Conjecture 1.3 (Mazzuoccolo, 2013 [?]). *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is an acyclic graph.*

Conjecture 1.4 (Mazzuoccolo, 2013 [?]). *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is an acyclic graph, whose components are of order 2 or 3.*

Clearly, these last two conjectures are true for 3-edge-colourable cubic graphs, and Janos Hägglund verified the strongest of these conjectures (Conjecture **??**) by computer for all non-trivial snarks (non 3-edge-colourable cubic graphs) of order at most 34 [?]. However, 5 years later, Jin, Steffen, and Mazzuoccolo [?] gave a counterexample to Conjecture **??**. Their counterexample contains a lot of 2-edge-cuts and the authors state that the conjecture "could hold true for 3-connected or cyclically 4-edge-connected cubic graphs". In fact, as in real life, being more connected has its own benefits, and in this paper we show the following stronger statement.

Theorem 1.5. *Let G be a cyclically 3-edge-connected cubic graph, which is not a Klee-graph. Then, for any $e \in E(G)$ and any 1^+-factor F of G, there exists a perfect matching M of G containing e such that $G \setminus (F \cup M)$ is acyclic.*

We remark that Klee-graphs (see Definition **??**), which are to be discussed further in Section **??**, are 3-edge-colourable cubic graphs and so are not a counterexample to Conjecture **??**. However, the stronger statement given in Theorem **??** does not hold for this class of graphs, and this is the reason why we exclude them.

Although Theorem **??** is not a direct consequence of the Berge–Fulkerson Conjecture, we believe that the results presented here and in [**?**] are valuable steps towards trying to decipher long-standing conjectures such as the Fan–Raspaud Conjecture [**?**], and the Berge–Fulkerson Conjecture itself.

In fact, we will prove the following statement, which is equivalent to Theorem **??**.

Theorem 1.6. *Let G be a cyclically 3-edge-connected cubic graph, which is not a Klee-graph. Then, for any $e \in E(G)$ and any collection of disjoint circuits $\mathcal{C}$, there exists a perfect matching M of G containing e such that every circuit in $\mathcal{C}$ contains an edge from M.*

Indeed, given a collection of disjoint circuits $\mathcal{C}$, its complement is a 1^+-factor, say $F_{\mathcal{C}}$. A perfect matching M containing e such that $G \setminus (F_{\mathcal{C}} \cup M)$ is acyclic must contain an edge from every circuit in $\mathcal{C}$. On the other hand, given a 1^+-factor F, its complement is a collection of disjoint paths and circuits, and so it suffices to consider the collection $\mathcal{C}_F$ of circuits disjoint from F. A perfect matching M containing e such that every circuit in $\mathcal{C}_F$ contains an edge from M, clearly makes $G \setminus (F \cup M)$ acyclic.

1.1 Important definitions and notation

Graphs considered in this paper are simple, that is, they cannot contain parallel edges and loops, unless otherwise stated.

Let G be a graph and (V_1, V_2) be a partition of its vertex set, that is, $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$. Then, by $E(V_1, V_2)$ we denote the set of edges having one endvertex in V_1 and one in V_2; we call such a set an *edge-cut*. An edge which itself is an edge-cut of size one is a *bridge*. A graph which does not contain any bridges is said to be *bridgeless*.

An edge-cut $X = E(V_1, V_2)$ is called *cyclic* if both graphs $G[V_1]$ and $G[V_2]$, obtained from G after deleting X, contain a *circuit* (a 2-regular connected subgraph). The *cyclic edge-connectivity* of a graph G is defined as the smallest size of a cyclic edge-cut in G if G admits one; it is defined as $|E(G)| - |V(G)| + 1$, otherwise. For cubic graphs, the latter only concerns K_4, $K_{3,3}$, and the graph consisting of two vertices joined by three parallel edges, whose cyclic edge-connectivity is thus 3, 4, and 2, respectively. An *acyclic* graph is a graph which does not contain any circuits.

Let G be a bridgeless cubic graph. A 1^+*-factor* of G is the edge set of a spanning subgraph of G such that its vertices have degree 1, 2 or 3. In particular, a *perfect matching* and a 2*-factor* of G are 1^+-factors whose vertices have exactly degree 1 and 2, respectively.

2 Klee-graphs

Definition 2.1 ([**?**]). A graph G is a Klee-graph if G is the complete graph on 4 vertices K_4 or there exists a Klee-graph G_0 such that G can be obtained from G_0 by replacing a vertex by a triangle (see Figure **??**).

Figure 1: Examples of Klee-graphs on 4 upto 12 vertices, left to right.

For simplicity, if a graph G is a Klee-graph, we shall sometimes say that G is Klee. We note that there is a unique Klee-graph on 6 vertices (the graph of a 3-sided prism), and a unique Klee-graph on 8 vertices. As we will see in Section **??**, these two graphs are Klee ladders, and shall be respectively denoted as KL_6 and KL_8.

Lemma 2.2 ([**?**]). *The edge set of any Klee-graph can be uniquely partitioned into three pairwise disjoint perfect matchings. In other words, any Klee-graph is 3-edge-colourable, and the colouring is unique up to a permutation of the colours.*

Since Klee-graphs are 3-edge-colourable, they easily satisfy the statement of Conjecture **??**.

Proposition 2.3. *Let G be a Klee-graph. Then, G admits two perfect matchings M_1 and M_2 such that $G \setminus (M_1 \cup M_2)$ is acyclic.*

The new graph obtained after expanding a vertex of a Hamiltonian graph (not necessarily Klee) into a triangle is still Hamiltonian, and so, since K_4 is Hamiltonian, all Klee-graphs are Hamiltonian. Hamiltonian cubic graphs have the following distinctive property.

Proposition 2.4. *Let G be a Hamiltonian cubic graph. Then, for any collection of disjoint circuits $\mathcal{C}$ of G there exists a perfect matching M of G which intersects at least one edge of every circuit in $\mathcal{C}$.*

Proof. TOPROVE 0 $\square$

Corollary 2.5. *For any collection of disjoint circuits $\mathcal{C}$ of a Klee-graph G there exists a perfect matching M of G which intersects at least one edge of every circuit in $\mathcal{C}$.*

On the other hand, we have to exclude Klee-graphs from Theorem **??** (and Theorem **??**) since for some Klee-graphs there are edges contained in a unique perfect matching, as we will see in the following subsection.

2.1 Other results about Klee-graphs

Lemma 2.6 ([**?**]). *Let G be a Klee-graph on at least 6 vertices. Then, G has at least two triangles and all its triangles are vertex-disjoint.*

Indeed, expanding a vertex into a triangle can only destroy triangles containing the vertex to be expanded.

We will now define a series of particular Klee-graphs, which we will call *Klee ladders*. Let KL_4 be the complete graph on 4 vertices, and let u_4v_4 be an edge of KL_4. For any even $n \geq 4$, let KL_{n+2} be the Klee-graph obtained from KL_n by expanding the vertex u_n into a triangle. In the resulting graph KL_{n+2}, we denote the vertex corresponding to v_n by v_{n+2}, and denote the vertex of the new triangle adjacent to v_{n+2} by u_{n+2}.

In other words, the graph KL_{2k+2} consists of the Cartesian product $P_2 \square P_k$ (where P_t denotes a path on t vertices) with two additional vertices u_{2k+2} and v_{2k+2} adjacent to each other, such that u_{2k+2} (v_{2k+2}) is adjacent to the two vertices in the first (last, respectively) copy of P_2 in $P_2 \Box P_k$ (see Figure **??**).

Klee ladders can be used to illustrate why we have to exclude Klee-graphs from our main result. For a given Klee ladder G there exists an edge e such that e is contained in a unique perfect matching of G, and therefore there is no hope for a statement like Theorem **??** to be true.

Figure 2: An example of a Klee ladder, KL_{12}. There is a unique perfect matching (here depicted using dotted lines) containing the edge e. The complement of this perfect matching is a Hamiltonian circuit.

We will frequently use the following structural property of certain Klee-graphs.

Lemma 2.7. *Let G be a Klee-graph on at least 8 vertices having exactly two (disjoint) triangles. Then,*

(i) exactly one edge of each triangle lies on a 4-circuit; and

(ii) if G admits an edge joining the two triangles, then G is a Klee ladder.

Proof. TOPROVE 1 □

3 Proof of Theorem ??

Proof. TOPROVE 2 □

Here are some consequences of Theorem **??**. Corollary **??** follows by the above result and Corollary **??**.

Corollary 3.1. *Let G be a cyclically 3-edge-connected cubic graph and let $\mathcal{C}$ be a collection of disjoint circuits of G. Then, there exists a perfect matching M such that $M \cap E(C) \neq \emptyset$, for every $C \in \mathcal{C}$.*

Corollary 3.2. *Let G be a cyclically 3-edge-connected cubic graph. For every perfect matching M_1 of G, there exists a perfect matching M_2 of G such that $G \setminus (M_1 \cup M_2)$ is acyclic.*

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