Complexity of approximate conflict-free, linearly-ordered, and nonmonochromatic hypergraph colourings*

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Abstract

Using the algebraic approach to promise constraint satisfaction problems, we establish complexity classifications of three natural variants of hypergraph colourings: standard nonmonochromatic colourings, conflict-free colourings, and linearly-ordered colourings.

Firstly, we show that finding an ℓ-colouring of a k-colourable r-uniform hypergraph is NP-hard for all constant $2 \leq k \leq \ell$ and $r \geq 3$. This provides a shorter proof of a celebrated result by Dinur et al. [FOCS'02/Combinatorica'05].

Secondly, we show that finding an ℓ-conflict-free colouring of an r-uniform hypergraph that admits a k-conflict-free colouring is NP-hard for all constant $2 \leq k \leq \ell$ and $r \geq 4$, except for $r = 4$ and $k = 2$ (and any ℓ); this case is solvable in polynomial time. The case of $r = 3$ is the standard nonmonochromatic colouring, and the case of $r = 2$ is the notoriously difficult open problem of approximate graph colouring.

Thirdly, we show that finding an ℓ-linearly-ordered colouring of an r-uniform hypergraph that admits a k-linearly-ordered colouring is NP-hard for all constant $3 \leq k \leq \ell$ and $r \geq 4$, thus improving on the results of Nakajima and Živný [ICALP'22/ACM TocT'23].

1 Introduction

Graph colouring Graph colouring is one of the most studied computational problems: Given a graph G and an integer k, is there a k-colouring, i.e., an assignment of one of k colours to the vertices of the graph so that adjacent vertices are assigned different colours?

Deciding the existence of a 3-colouring is one of Karp's 21 NP-complete problems [Kar72]. Since finding a graph colouring with the smallest number of colours is NP-hard, there has been much interest in the *approximate graph colouring* (AGC) problem: Given a graph G that admits a colouring with k colours, find a colouring with ℓ colours for some $k \leq \ell$. It

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is believed that for every constant $3 \leq k \leq \ell$, this problem remains NP-hard [GJ76]. While some conditional results are known (i.e. AGC is NP-hard if we assume the *d-to-1 conjecture with perfect completeness* [GS20]), proving unconditional results seems elusive. The strongest results known so far are for $\ell = 2k - 1$ [BBKO21] and $\ell = \binom{k}{\lfloor k/2 \rfloor} - 1$ [KOWŽ23] (the first result is stronger for $k = 3, 4$, and equal to the second for $k = 5$). As progress on proving the hardness of AGC seems to have hit a barrier, it is natural to try to attack variants of AGC, to see if any of the ideas and insights from those problems could apply to the AGC. In this paper, we will focus on hypergraph generalisations of graph colourings.

Recall that a *hypergraph* is a pair (V, E), where V is the vertex set and $E \subseteq 2^V$ the edge set. A hypergraph is called *r-uniform* if all the edges have size r. Thus, a 2-uniform hypergraph is a graph. A colouring of a graph (V, E) is an assignment of colours $c(v)$ to the vertices $v \in V$ such that for every edge $\{u, v\} \in E$ we have $c(u) \neq c(v)$. Put differently in three different but equivalent ways, for every edge $\{u, v\} \in E$ we have that (i) the set $\{c(u), c(v)\}$ contains at least 2 elements, or (ii) some colour in the multiset $\{\{c(u), c(v)\}\}$ appears exactly once, or (iii) the largest colour in the multiset $\{\{c(u), c(v)\}\}$ appears exactly once.[1] When these three definitions are applied to hypergraphs, we get three different notions, namely nonmonochromatic (NAE) colourings, conflict-free (CF) colourings, and linearly-ordered (LO) colourings. Note that any LO colouring is a CF colouring, and any CF colouring is an NAE colouring. The three notions of colourings are different already for 4-uniform[2] hypergraphs.[3]

Promise CSPs We will now review the most relevant literature on the three variants of hypergraph colourings. It will be convenient to present the existing results in the framework of so-called *promise constraint satisfaction problems* (PCSPs) [BG21], as we shall use the tools developed for understanding the computational complexity of PCSPs [BBKO21]. Constraint satisfaction problems (CSPs) are problems that can be cast as homomorphisms between relational structures. We will only need a special case of relational structures that contain only one relation. Formally, a *relational structure* $\mathbf{A} = (A, R^{\mathbf{A}})$ is a pair, where A is the universe of $\mathbf{A}$ and $R^{\mathbf{A}} \subseteq A^r$ is an r-ary relation. By abuse of language, we call r the *arity* of $\mathbf{A}$.

An example of a relational structure is a graph or an r-uniform hypergraph, where the universe is the vertex set and the relation is the edge set — the arity of the relation is 2 for graphs and r for r-uniform hypergraphs. A *homomorphism* from one relational structure of arity r, say $\mathbf{A} = (A, R^{\mathbf{A}})$, to another of the same arity, say $\mathbf{B} = (B, R^{\mathbf{B}})$, is a map $h : A \to B$ that preserves the relation: if $(a_1, \ldots, a_r) \in R^{\mathbf{A}}$ then $(h(a_1), \ldots, h(a_r)) \in R^{\mathbf{B}}$. We denote the existence of a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ by writing $\mathbf{A} \to \mathbf{B}$.

Given two relational structures $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A} \to \mathbf{B}$, the promise constraint satisfaction problem with template $(\mathbf{A}, \mathbf{B})$, denoted by PCSP$(\mathbf{A}, \mathbf{B})$, is the following computational problem. Given a relational structure $\mathbf{X}$ with $\mathbf{X} \to \mathbf{A}$, find a homomorphism from $\mathbf{X}$ to $\mathbf{B}$. This is the search version of the problem. In the decision version, which reduces to the search version, one is given a relational structure $\mathbf{X}$ with the same arity as $\mathbf{A}$ and the task is to output YES if $\mathbf{X} \to \mathbf{A}$ and NO if $X \not\to \mathbf{B}$.[4]

[1] This notion assumes that the colours are taken from a totally ordered set, e.g., the natural numbers.

[2] For 3-uniform hypergraphs, NAE and CF colourings coincide, but LO colourings are different.

[3] Indeed, the edge $\{a, b, c, d\}$ could be assigned colours $\{\{1, 1, 2, 2\}\}$ in an NAE colouring, but not in the other two; whereas the edge could be assigned colours $\{\{1, 2, 2, 2\}\}$ in a CF colouring, but not in an LO colouring.

[4] Since the decision version reduces to the search version, solving the decision version is no harder than solving the search version. All of our results will hold for both versions — hardness results will hold even for the decision version, and tractability results will hold even for the search version.

In order to cast approximate hypergraph NAE/CF/LO-colourings as PCSPs, we will need to encode the NAE/CF/LO-colourability of a hypergraph by a homomorphism to a suitable relational structure. We will thus describe three families of relational structures capturing the three types of hypergraph colourings mentioned above (and therefore implicitly graph colouring). For any arity $r \geq 2$ and domain size k, we define:[5]

$$\begin{aligned}
\mathbf{NAE}_k^r &= ([k], \{(x_1, \dots, x_r) \mid \exists i, j \in [r] \cdot x_i \neq x_j\}), \\
\mathbf{CF}_k^r &= ([k], \{(x_1, \dots, x_r) \mid \exists i \in [r] \cdot \forall j \in [r] \cdot i = j \vee x_i \neq x_j\}), \\
\mathbf{LO}_k^r &= ([k], \{(x_1, \dots, x_r) \mid \exists i \in [r] \cdot \forall j \in [r] \cdot i = j \vee x_i > x_j\}).
\end{aligned}$$

Observe that an r-uniform hypergraph $\mathbf{X}$ has an NAE k-colouring if and only if $\mathbf{X} \to \mathbf{NAE}_k^r$. The analogous statement holds for CF and LO colourings. Since NAE, LO and CF colourings are all identical to graph colouring on uniformity 2, we see that k vs. ℓ AGC is the same as $\text{PCSP}(\mathbf{NAE}_k^2, \mathbf{NAE}_\ell^2)$ — or equivalently $\text{PCSP}(\mathbf{CF}_k^2, \mathbf{CF}_\ell^2)$ or $\text{PCSP}(\mathbf{LO}_k^2, \mathbf{LO}_\ell^2)$.

Nonmonochromatic colourings The most studied hypergraph colourings are nonmonochromatic colourings, also known as weak hypergraph colourings. This is the weakest non-trivial restriction one can impose when colouring the vertices of a hypergraph, i.e., any type of hypergraph colouring (that excludes constant colourings) is also a nonmonochromatic colouring. As mentioned before, nonmonochromatic k-colourings of an r-uniform hypergraph correspond to homomorphisms from the hypergraph to $\mathbf{NAE}_k^r$. Since nonmonochromatic colouring is NP-hard for any uniformity $r \geq 3$ and number of colours $k \geq 2$, Dinur, Regev, and Smith investigated the approximate version, establishing the following result [DRS05].

Theorem 1. $\text{PCSP}(\mathbf{NAE}_k^r, \mathbf{NAE}_\ell^r)$ *is* NP*-hard for all constant* $2 \leq k \leq \ell$ *and* $r \geq 3$.

In this paper we will show a simpler proof of this result. The proof in [DRS05] relies on constructing a somewhat ad-hoc reduction and analysing its completeness and soundness. We recast this proof in the recent algebraic framework for PCSPs [BBKO21]. We also replace the use of Schrijver graphs with the simpler Kneser graphs, plus a (correct and very easy) case of Hedetniemi's conjecture (cf. Lemma 9). We believe that our simplification is of interest since it replaces a more quantitative analysis of the polymorphisms with one that only deals with constants everywhere. In particular, this means that our proof does not require the full strength of the PCP theorem — only the "baby PCP" of Barto and Kozik [BK22].

Conflict-free colourings A conflict-free hypergraph colouring is a colouring of the vertices in a hypergraph such that every hyperedge has at least one uniquely coloured vertex [ELRS03, Smo13]. As mentioned before, conflict-free k-colourings of an r-uniform hypergraph correspond to homomorphisms from the hypergraph to $\mathbf{CF}_k^r$. We shall determine the complexity of $\text{PCSP}(\mathbf{CF}_k^r, \mathbf{CF}_\ell^r)$ for all constants $2 \leq k \leq \ell$ and $r \geq 3$ (the case of $r = 3$ corresponding to nonmonochromatic colourings, i.e., $\mathbf{CF}_k^3 = \mathbf{NAE}_k^3$ for every k).

After the easy observation that $\text{PCSP}(\mathbf{CF}_k^r, \mathbf{CF}_\ell^r)$ reduces to $\text{PCSP}(\mathbf{CF}_k^{r+t}, \mathbf{CF}_\ell^{r+t})$ for $t \geq 2$ (cf. Lemma 6 in Section 2), Theorem 1 directly implies NP-hardness for promise conflict-free colouring for uniformity $r \geq 5$. The crux of the result is to deal with the case of uniformity $r = 4$. Note that finding a conflict-free colouring of a 4-uniform hypergraph using 2 colours is identical to solving systems of equations of the form $x + y + z + t \equiv 1 \pmod 2$ over $\mathbb{Z}_2$, and

[5] For any integer k, we write $[k]$ for the set $\{1, 2, \dots, k\}$.

is hence in P and consequently so is PCSP($\mathbf{CF}_2^4, \mathbf{CF}_\ell^4$) for every $\ell \geq 2$. We resolve the only remaining case, showing in Section 3.3 that PCSP($\mathbf{CF}_k^4, \mathbf{CF}_\ell^4$) is NP-hard for all $3 \leq k \leq \ell$. Summarising, we have

Theorem 2. PCSP($\mathbf{CF}_k^r, \mathbf{CF}_\ell^r$) *is* NP*-hard for all constant* $2 \leq k \leq \ell$ *and* $r \geq 3$*, except for* $k = 2$ *and* $r = 4$*, which is in* P*.*

This also immediately implies the following (much weaker) corollary, which does not appear to have been known in the CF-colouring literature.

Corollary 3. *It is* NP*-hard to approximate the conflict-free chromatic number*[6] *of a hypergraph to within any constant factor, even if it is* r*-uniform for some constant* $r \geq 3$*.*

Linearly-ordered colourings A linearly-ordered [BBB21] (or unique-maximum [CKP13]) hypergraph colouring is a colouring of the vertices in a hypergraph with linearly-ordered colours such that the maximum colour in every hyperedge is unique. As mentioned before, linearly-ordered k-colourings of an r-uniform hypergraph correspond to homomorphisms from the hypergraph to $\mathbf{LO}_k^r$.

Barto at al. [BBB21] conjectured that PCSP($\mathbf{LO}_k^3, \mathbf{LO}_\ell^3$) is NP-hard for all constant $2 \leq k \leq \ell$, but even the case PCSP($\mathbf{LO}_2^3, \mathbf{LO}_3^3$) is still open. A recent result of Filakovský et al. [FNO+24] established NP-hardness of PCSP($\mathbf{LO}_3^3, \mathbf{LO}_4^3$) by generalising the topological methods of Krokhin et al. [KOWŽ23]. Nakajima and Živný [NŽ23] also showed NP-hardness of PCSP($\mathbf{LO}_k^r, \mathbf{LO}_\ell^r$) for every $2 \leq k \leq \ell$ and $r \geq \ell - k + 4$. We strengthen this result, showing NP-hardness of PCSP($\mathbf{LO}_k^r, \mathbf{LO}_\ell^r$) when $3 \leq k \leq \ell$ and $r \geq 4$.

Theorem 4. PCSP($\mathbf{LO}_k^r, \mathbf{LO}_\ell^r$) *is* NP*-hard for all constant* $3 \leq k \leq \ell$ *and* $r \geq 4$*.*

Observe that this theorem covers nearly all the cases from the result of [NŽ23]: the only case not covered is $k = 2$ and $r \geq \ell + 2$. In particular, Theorem 4 has no requirement on r in terms of ℓ, unlike the result in [NŽ23]. Indeed, Theorem 4 covers the full range of parameters except for the cases $r = 3$ or $k = 2$ (and thus the conjecture of Barto et al. remains open).

It is worth digressing somewhat to discuss the appearance of topological methods within these proofs. Our proof uses the chromatic number of the Kneser graph as an essential ingredient — this is a topological fact, and thus our proof is in some sense topological. (This is similar to the appearance of topology within hardness proofs for rainbow colourings [ABP20].) On the other hand, the topological approach of [FNO+24], which proved that PCSP($\mathbf{LO}_3^3, \mathbf{LO}_4^3$) is NP-hard, is rather different. It assigns each relational structure an equivariant simplicial complex in a “nice enough” way so that the topological properties of these simplicial complexes imply the hardness of the original template. It would be interesting to see if these two approaches can be merged, or combined to strengthen both.

2 Preliminaries

Let $(\mathbf{A}, \mathbf{B})$ be a PCSP template with $\mathbf{A}$ and $\mathbf{B}$ of arity r. A *polymorphism* of arity $n = \mathrm{ar}(f)$ of $(\mathbf{A}, \mathbf{B})$ is a function $f : A^n \to B$ such that if f is applied component-wise to any n-tuple of elements of $R^{\mathbf{A}}$ it gives an element of $R^{\mathbf{B}}$. In more detail, whenever (a_{ij}) is an $r \times n$ matrix such that every column is in $R^{\mathbf{A}}$, then f applied to the rows gives an r-tuple which is in

[6] That is, the minimum number of colours needed to CF-colour a given hypergraph.

$R^{\mathbf{B}}$. We say that the rows of such a matrix are *compatible*. We denote by $\mathrm{Pol}^{(n)}(\mathbf{A},\mathbf{B})$ the collection of n-ary polymorphisms of $(\mathbf{A},\mathbf{B})$, and we let $\mathrm{Pol}(\mathbf{A},\mathbf{B}) = \bigcup_n \mathrm{Pol}^{(n)}(\mathbf{A},\mathbf{B})$.

For an n-ary function $f : A^n \to B$ and a map $\pi : [n] \to [m]$, we say that an m-ary function $g : A^m \to B$ is the *minor of f given by π* if $g(x_1,\dots,x_m) = f(x_{\pi(1)},\dots,x_{\pi(n)})$. We write $f \xrightarrow{\pi} g$ if g is the minor of f given by π. Note that $\mathrm{Pol}(\mathbf{A},\mathbf{B})$ is closed under minors.

We use $\leq_p$ to denote a polynomial-time many-one reduction.

Theorem 5 ([BG21])**.** *If* $\mathrm{Pol}(\mathbf{A},\mathbf{B}) \subseteq \mathrm{Pol}(\mathbf{A}',\mathbf{B}')$ *then* $\mathrm{PCSP}(\mathbf{A}',\mathbf{B}') \leq_p \mathrm{PCSP}(\mathbf{A},\mathbf{B})$.

Lemma 6. *For any $t \geq 2$,* $\mathrm{PCSP}(\mathbf{CF}_k^r, \mathbf{CF}_\ell^r) \leq_p \mathrm{PCSP}(\mathbf{CF}_k^{r+t}, \mathbf{CF}_\ell^{r+t})$.

Proof. TOPROVE 0 □

An *ℓ-chain of minors* is a sequence of the form $f_0 \xrightarrow{\pi_{0,1}} f_1 \xrightarrow{\pi_{1,2}} \cdots \xrightarrow{\pi_{\ell-1,\ell}} f_\ell$. We shall then write $\pi_{i,j} : [\mathrm{ar}(f_i)] \to [\mathrm{ar}(f_j)]$ for the composition of $\pi_{i,i+1},\dots,\pi_{j-1,j}$, for any $0 \leq i < j \leq \ell$. Note that $f_i \xrightarrow{\pi_{i,j}} f_j$. We shall use the following NP-hardness criterion for PCSPs.

Theorem 7 ([BWŽ21])**.** *Suppose there are constants k,ℓ and an assignment* sel *which, for every $f \in \mathrm{Pol}(\mathbf{A},\mathbf{B})$, outputs a set $\mathrm{sel}(f) \subseteq [n]$ of size at most k, where n is the arity of f. Suppose furthermore that for every ℓ-chain of minors there are i,j such that $\pi_{i,j}(\mathrm{sel}(f_i)) \cap \mathrm{sel}(f_j) \neq \varnothing$. Then,* $\mathrm{PCSP}(\mathbf{A},\mathbf{B})$ *is* NP*-hard.*

For a graph G, the *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that $G \to K_k$, where K_k is the clique on k vertices. We will rely on Lovász's result for the chromatic number of Kneser graphs [Lov78]. For $1 \leq h \leq |A|$, write $A^{(h)}$ for the family of subsets of A of size h. The *Kneser graph* is defined as $\mathrm{KG}(A,h) = (A^{(h)}, E)$, where $\{S,T\} \in E$ if and only if $S \cap T = \varnothing$. For the special case $A = [n]$, we use the notation $\mathrm{KG}(n,h) = \mathrm{KG}([n],h)$.

Theorem 8 ([Lov78])**.** $\chi(\mathrm{KG}(n,h)) = n - 2h + 2$ *for any $n,h \geq 1$.*

Lemma 9. *Let $\chi(G) > n$. Then $\chi(G \times K_{n+1}) > n$.*

Proof. TOPROVE 1 □

If $\mathbf{A} \to \mathbf{A}' \to \mathbf{B}' \to \mathbf{B}$ then $(\mathbf{A},\mathbf{B})$ is a *homomorphic relaxation* of $(\mathbf{A}',\mathbf{B}')$. In this case it follows from the definitions that $\mathrm{PCSP}(\mathbf{A},\mathbf{B}) \leq_p \mathrm{PCSP}(\mathbf{A}',\mathbf{B}')$ [BBKO21].

3 Proofs of hardness

3.1 Avoiding sets imply hardness

Our hardness proofs will revolve around the notion of *avoiding sets* for polymorphisms [BBKO21], defined below. We denote by $\mathbf{1}_X$ the *indicator vector* of X: $(\mathbf{1}_X)_i = 1$ when $i \in X$ and $(\mathbf{1}_X)_i = 0$ otherwise. The overall length of the vector will be clear from context.

Definition 10. Take A so that $\{0,1\} \subseteq A$. Let $f : A^n \to B$ and $T \subseteq B$. A *T-avoiding set* for f is a set $S \subseteq [n]$ such that for any $R \supseteq S$, we have $f(\mathbf{1}_R) \notin T$. For $t \in \mathbb{N}$, we call a set S t-avoiding for f if it is T-avoiding for f for some subset $T \subseteq B$ of size t.

We will first collect some simple properties of avoiding sets.

Lemma 11. *Let $f : A^n \to B$ and $\ell = |B|$.*

(i) There are no ℓ-avoiding sets for f.

(ii) $[n]$ is an $(\ell - 1)$-avoiding set for f.

(iii) If U is T-avoiding for f then so is every $V \supseteq U$.

(iv) Take $\pi : [n] \to [m]$ and suppose $f \xrightarrow{\pi} g$ for some $g : A^m \to B$. Suppose $S \subseteq [n]$ is T-avoiding for f. Then $\pi(S)$ is T-avoiding for g.

Proof. TOPROVE 2 □

To apply Theorem 7, we want to build $\mathrm{sel}(f)$ out of (small) avoiding sets for f. This is a good idea because avoiding sets are preserved by minors, as shown in Lemma 11 (iv). The issue is that we might have too many avoiding sets. For the polymorphisms in this paper, many (small) t-avoiding sets which are pairwise disjoint imply the existence of a (small) $(t+1)$-avoiding set. Thus, since there can be no sets that avoid every output in the range, as shown in Lemma 11 (i), there must be some maximal t for which a (small) avoiding set exists. By maximality, there cannot be too many disjoint t-avoiding sets. Thus, we can build $\mathrm{sel}(f)$ out of these disjoint "maximally avoiding" sets.

Theorem 12. *Let $(\mathbf{A}, \mathbf{B})$ be a PCSP template with $\{0, 1\} \subseteq A$ and $\ell = |B|$. Suppose that there exist constants $N, \{\alpha_t\}_{t=1}^{\ell}, \{\beta_t\}_{t=1}^{\ell}$ such that every $f \in \mathrm{Pol}(\mathbf{A}, \mathbf{B})$ has the following properties:*

1. *f has a 1-avoiding set of size $\leq \beta_1$.*
2. *If f is of arity $\geq N$ and has a disjoint family of $> \alpha_t$ many t-avoiding sets, all of size $\leq \beta_t$, then f has a $(t+1)$-avoiding set of size $\leq \beta_{t+1}$.*

Then, $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.

Proof. TOPROVE 3 □

The rest of this papers will show hardness of certain PCSP templates by showing that they have the two properties from Theorem 12.

3.2 Hardness of promise nonmonochromatic colouring

In this section we prove the advertised hardness results for NAE colouring. The most important case is $\mathrm{PCSP}(\mathbf{NAE}_2^3, \mathbf{NAE}_\ell^3)$ for $\ell \geq 2$; all other cases derive from this one by either gadget reductions or homomorphic relaxations.

Lemma 13. *Let $\ell \geq 2$ and $n \in \mathbb{N}$. Any $f \in \mathrm{Pol}^{(n)}(\mathbf{NAE}_2^3, \mathbf{NAE}_\ell^3)$ has a 1-avoiding set of size $\leq \ell$.*

Proof. TOPROVE 4 □

Lemma 14. *Let $1 \leq t < \ell$ and $n \geq (\ell+1)\ell^t + \ell + 1$. Suppose $f \in \mathrm{Pol}^{(n)}(\mathbf{NAE}_2^3, \mathbf{NAE}_\ell^3)$ has $> \binom{\ell}{t} \cdot \ell$ disjoint t-avoiding sets of size $\leq \ell^t$. Then f has a $(t+1)$-avoiding set of size $\leq \ell^{t+1}$.*

Proof. TOPROVE 5 □

Theorem 1. $\mathrm{PCSP}(\mathbf{NAE}_k^r, \mathbf{NAE}_\ell^r)$ *is NP-hard for all constant $2 \leq k \leq \ell$ and $r \geq 3$.*

Proof. TOPROVE 6 □

3.3 Hardness of promise conflict-free and linearly-ordered colouring

In this section we prove the advertised hardness results for both LO and CF colourings. For the CF colourings, by Lemma 6 it suffices to establish hardness for $r = 4$. However, our proof is the same for any $r \geq 4$ and thus we will present it that way. Since $\mathbf{LO}_k^r \to \mathbf{CF}_k^r$, we can then do both LO and CF colourings "in one go" by proving the hardness of $\mathrm{PCSP}(\mathbf{LO}_3^r, \mathbf{CF}_\ell^r)$ for all $\ell \geq 3$ and $r \geq 4$. In the following proofs, we let $\mathbf{0}$ denote the vector whose elements are all 0, and we let $\mathbf{2}_X$ denote a "scaled indicator vector": $\mathbf{2}_X = 2 \cdot \mathbf{1}_X$.

Lemma 15. *Let $\ell \geq 3$ and $r \geq 4$. Then any $f \in \mathrm{Pol}^{(n)}(\mathbf{LO}_3^r, \mathbf{CF}_\ell^r)$ has a 1-avoiding set of size $\leq \ell$.*

Proof. TOPROVE 7 □

Lemma 16. *Let $\ell \geq 3$ and $r \geq 4$. Let $1 \leq t < \ell$, and suppose that $f \in \mathrm{Pol}^{(n)}(\mathbf{LO}_3^r, \mathbf{CF}_\ell^r)$ has a disjoint family of $> \binom{\ell}{t}\ell$ many t-avoiding sets, all of size $\leq t\ell$. Then f has a $(t+1)$-avoiding set of size $\leq (t+1)\ell$.*

Proof. TOPROVE 8 □

Theorem 17. $\mathrm{PCSP}(\mathbf{LO}_3^r, \mathbf{CF}_\ell^r)$ *is* NP*-hard for all constant $\ell \geq 3$ and $r \geq 4$.*

Proof. TOPROVE 9 □

Theorem 2 and Theorem 4 follow immediately:

Theorem 2. $\mathrm{PCSP}(\mathbf{CF}_k^r, \mathbf{CF}_\ell^r)$ *is* NP*-hard for all constant $2 \leq k \leq \ell$ and $r \geq 3$, except for $k = 2$ and $r = 4$, which is in* P*.*

Proof. TOPROVE 10 □

Theorem 4. $\mathrm{PCSP}(\mathbf{LO}_k^r, \mathbf{LO}_\ell^r)$ *is* NP*-hard for all constant $3 \leq k \leq \ell$ and $r \geq 4$.*

Proof. TOPROVE 11 □

Remark 18. In the above proofs of Lemma 15 and Lemma 16, the only required property of $\mathbf{CF}_\ell^r$ is that if a tuple in the relation has two entries which are equal, then the remaining $r - 2$ entries in the tuple cannot all be equal. Thus, the same proof also shows a stronger result: Define $\mathbf{BNAE}_\ell^{s,r-s}$ (Block-NAE) as the template on domain $[\ell]$, with a single relation which contains exactly the tuples for which if any s coordinates have the same value, then the remaining $r - s$ coordinates cannot all have the same value. (This relation with $s = 2$ is strictly larger than the relation corresponding to $\mathbf{CF}_\ell^r$ when $r \geq 6$). Then we have shown that $\mathrm{PCSP}(\mathbf{LO}_3^r, \mathbf{BNAE}_\ell^{2,r-2})$ is NP-hard for all constant $\ell \geq 3, r \geq 4$. In fact, using the same proof technique, one can also show NP-hardness of $\mathrm{PCSP}(\mathbf{LO}_3^r, \mathbf{BNAE}_\ell^{s,r-s})$ for all constant $\ell \geq 3, r \geq 4, 2 \leq s \leq r-2$ by considering the s-uniform Kneser hypergraph $\mathrm{KG}^{(s)}(n,a)$, whose chromatic number is known to be $\left\lceil \frac{n-s(a-1)}{s-1} \right\rceil$ [AFL86].

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