Sets of r-graphs that color all r-graphs

Yulai Ma[1,*], Davide Mattiolo[2,†], Eckhard Steffen[1], Isaak H. Wolf[1,‡]

[1] Department of Mathematics, Paderborn University, Warburger Str. 100, 33098 Paderborn, Germany.

[2] Department of Computer Science, KU Leuven Kulak, 8500 Kortrijk, Belgium.

yulai.ma@upb.de, davide.mattiolo@kuleuven.be, es@upb.de, isaak.wolf@upb.de

Abstract

An r-regular graph is an r-graph, if every odd set of vertices is connected to its complement by at least r edges. Let G and H be r-graphs. An *H-coloring* of G is a mapping $f\colon E(G) \to E(H)$ such that each r adjacent edges of G are mapped to r adjacent edges of H. For every $r \geq 3$, let $\mathcal{H}_r$ be an inclusion-wise minimal set of connected r-graphs, such that for every connected r-graph G there is an $H \in \mathcal{H}_r$ which colors G.

We show that $\mathcal{H}_r$ is unique and characterize $\mathcal{H}_r$ by showing that $G \in \mathcal{H}_r$ if and only if the only connected r-graph coloring G is G itself.

The Petersen Coloring Conjecture states that the Petersen graph P colors every bridgeless cubic graph. We show that if true, this is a very exclusive situation. Indeed, either $\mathcal{H}_3 = \{P\}$ or $\mathcal{H}_3$ is an infinite set and if $r \geq 4$, then $\mathcal{H}_r$ is an infinite set. Similar results hold for the restriction on simple r-graphs.

By definition, r-graphs of class 1 (i.e. those having edge-chromatic number equal to r) can be colored with any r-graph. Hence, our study will focus on those r-graphs whose edge-chromatic number is bigger than r, also called r-graphs of class 2. We determine the set of smallest r-graphs of class 2 and show that it is a subset of $\mathcal{H}_r$.

Keywords: perfect matchings, regular graphs, factors, r-graphs, edge-coloring, class 2 graphs, Petersen Coloring Conjecture, Berge-Fulkerson Conjecture.

1 Introduction

All graphs considered in this paper are finite and may have parallel edges but no loops. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. A graph is *r-regular* if

[*] Supported by Sino-German (CSC-DAAD) Postdoc Scholarship Program 2021 (57575640)

[†] Supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO), project number 1268323N

[‡] Funded by Deutsche Forschungsgemeinschaft (DFG) - 445863039

every vertex has degree r. An r-regular graph is an *r-graph*, if $|\partial_G(X)| \geq r$ for every $X \subseteq V(G)$ of odd cardinality, where $\partial_G(X)$ denotes the set of edges that have precisely one vertex in X.

Let G be a graph and S be a set. An *edge-coloring* of G is a mapping $f : E(G) \to S$. It is a *k-edge-coloring* if $|S| = k$, and it is *proper* if $f(e) \neq f(e')$ for any two adjacent edges e and e'. The smallest integer k for which G admits a proper k-edge-coloring is the *edge-chromatic number* of G, which is denoted by $\chi'(G)$. A *matching* is a set $M \subseteq E(G)$ such that no two edges of M are adjacent. Moreover, M is said to be *perfect* if every vertex of G is incident with an edge of M.

If $\chi'(G)$ equals the maximum degree of G, then G is said to be *class* 1; otherwise G is *class* 2. If $\chi'(G) = r$, then r is the minimum number such that $E(G)$ decomposes into r matchings, which are perfect matchings in case of r-regular graphs. For $r \geq 1$, let $\mathcal{T}_r$ be the set of the smallest r-graphs of class 2. For example, the only element of $\mathcal{T}_3$ is the Petersen graph, which is denoted by P throughout this paper.

The generalized Berge-Fulkerson Conjecture [**?**] states that every r-graph has $2r$ perfect matchings such that every edge is in precisely two of them. For $r = 3$ the conjecture was attributed to Berge and Fulkerson [**?**], who put it into print (cf. [**?**]). As a unifying approach to study some hard conjectures on cubic graphs, Jaeger [**?**] introduced colorings with edges of another graph. To be precise, let G and H be graphs. An *H-coloring* of G is a mapping $f : E(G) \to E(H)$ such that

- if $e_1, e_2 \in E(G)$ are adjacent, then $f(e_1) \neq f(e_2)$,
- for every $v \in V(G)$ there exists a vertex $u \in V(H)$ with $f(\partial_G(v)) = \partial_H(u)$.

If such a mapping exists, then we write $H \prec G$ and say H *colors* G. A set $\mathcal{A}$ of connected r-graphs such that for every connected r-graph G there is an element $H \in \mathcal{A}$ which colors G is said to be *r-complete*. For every $r \geq 3$, let $\mathcal{H}_r$ be an inclusion-wise minimal r-complete set.

For $r = 3$, Jaeger [**?**] conjectured that the Petersen graph colors every bridgeless cubic graph. If true, this conjecture would have far reaching consequences. For instance, it would imply that the Berge-Fulkerson Conjecture and the 5-Cycle Double Cover Conjecture (see [**?**]) are also true. The Petersen Coloring Conjecture is a starting point for research in several directions. Different aspects of it are studied and partial results are proved, see for instance [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

Analogously to the case $r = 3$, if all elements of $\mathcal{H}_r$ would satisfy the generalized Berge-Fulkerson Conjecture, then every r-graph would satisfy it. Mazzuoccolo et al. [**?**] asked whether there exists a connected r-graph H such that $H \prec G$ for every (simple) r-graph G, for all $r \geq 3$. We show that $\mathcal{H}_r$ is unique and that it is an infinite set when $r \geq 4$. Furthermore, if $r = 3$, then either $\mathcal{H}_3 = \{P\}$ (if the Petersen Coloring Conjecture is true) or $\mathcal{H}_3$ is an infinite set. More precisely, in Section **??** we characterize $\mathcal{H}_r$ and provide constructions for infinite subsets of $\mathcal{H}_r$. Similar results are proved for simple r-graphs.

By definition, any r-graph G of class 1 can be colored with any r-graph H. Indeed, let $M_1, \dots, M_r$ be r pairwise disjoint perfect matchings of G and v a vertex of H with $\partial_H(v) = \{e_1, \dots, e_r\}$. Every edge of M_i of G can be mapped to e_i in H. Hence, the aforementioned questions and conjectures reduce to r-graphs of class 2. In Section **??** we determine the set $\mathcal{T}_r$ of the smallest r-graphs of class 2 and prove that $|\mathcal{T}_r| \geq p'(r-3, 6)$, where $p'(r-3, 6)$ is the number of partitions of $r-3$ into at most 6 parts. Furthermore, we show that if $r \geq 4$, then $\mathcal{T}_r$ is a proper subset of $\mathcal{H}_r$.

The Petersen Coloring Conjecture has also been studied in the context of quasi-orders on the set of graphs, see [**?**, **?**]. In Section **??** we briefly put our results in this context. We conclude the paper with some open questions.

1.1 Definitions and basic results

Let G be a graph. For any subset X of $V(G)$, we use $G - X$ to denote the graph obtained from G by deleting all vertices of X and all incident edges. Similarly, for $F \subseteq E(G)$, denote by $G - F$ the graph obtained by deleting all edges of F from G. In particular, we simply write $G - x$ and $G - e$ for $G - X$ and $G - F$, respectively, when $X = \{x\}$ and $F = \{e\}$. The subgraph of G induced by the vertex set X is denoted by $G[X]$. Moreover, the graph obtained from G by identifying all vertices of X and deleting all resulting loops is denoted by G/X; we denote the new vertex by w_X. Let Y be a subset of $V(G)$ with $X \cap Y = \emptyset$. We use $[X, Y]_G$ to denote the set of all edges of G with one vertex in X and the other one in Y. Furthermore, if $Y = X^c = V(G) \setminus X$ and $[X, Y]_G$ is nonempty, then we call it an *edge-cut* of G and denote it by $\partial_G(X)$. If X or Y consists of one vertex, we skip the set-brackets notation. In addition, $|\partial_G(x)|$ is called the *degree* of $x \in V(G)$ and it is denoted by $d_G(x)$. If G is an r-graph, then $\partial_G(X)$ is *tight* if $|X|$ is odd and $|\partial_G(X)| = r$. A tight edge-cut is *trivial* if X or X^c consists of a single vertex. Moreover, for $v \in V(G)$ we denote by $N_G(v)$ the set of neighbors of v.

A *1-factor* of a graph G is a spanning 1-regular subgraph of G, and its edge set is a perfect matching. A connected 2-regular graph is called a *circuit*. A circuit of length k is called a *k-circuit* and it is denoted by C_k.

For two graphs G and H, if there are two bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $e = uv \in E(G)$ if and only if $\phi(e) = \theta(u)\theta(v) \in E(H)$, then we say that G and H are *isomorphic*, denoted by $G \cong H$, and call the pair of mappings (θ, ϕ) an *isomorphism* between G and H. In particular, an *automorphism* of a graph is an isomorphism of the graph to itself.

Let $H_1, \dots, H_t$ be a sequence of graphs such that $V(H_i) \subseteq V(H_1)$ for each $i \in \{2, \dots, t\}$. Denote by $H_1 + E(H_2) + \ldots + E(H_t)$ the graph obtained from H_1 by adding a copy of every edge of H_i for every $i \in \{2, \dots, t\}$. Let $\mathcal{M}$ be a finite multiset of perfect matchings of the Petersen graph P. The graph $P + \sum_{M \in \mathcal{M}} M$ is denoted by $P^{\mathcal{M}}$.

Lemma 1.1 ([?]). *For every finite multiset $\mathcal{M}$ of perfect matchings of the Petersen graph P, the graph $P^{\mathcal{M}}$ is class 2.*

The following observation will frequently be used without reference.

Observation 1.2. *Let $r \geq 3$, let G be an r-graph and let $X \subseteq V(G)$. If $|X|$ is even, then $|\partial_G(X)|$ is even. If $|X|$ is odd, then $|\partial_G(X)|$ has the same parity as r.*

One major fact that we use in this paper is that every r-graph can be decomposed into a k-graph which is class 1 and an $(r-k)$-regular graph, for a suitable $k \in \{1, \dots, r\}$. For every r-graph G let $\pi(G)$ be the largest integer t such that G has t pairwise disjoint perfect matchings. Let $r \geq 3$ and $k \in \{1, \dots, r\}$ be integers. Let $\mathcal{G}(r,k) = \{G \colon G \text{ is an } r\text{-graph with } \pi(G) = k\}$. Note that $\mathcal{G}(r, r-1) = \emptyset$, since every r-graph with $r-1$ pairwise disjoint perfect matchings is a class 1 graph and thus, it has r pairwise disjoint perfect matchings. If $k \leq r-2$, then the elements of $\mathcal{G}(r,k)$ are class 2 graphs and $\mathcal{G}(r,i) \cap \mathcal{G}(r,j) = \emptyset$, if $1 \leq i \neq j \leq r-2$. We are interested in the subset of $\mathcal{G}(r,k)$ consisting of all such graphs with the smallest order. This set is denoted by $\mathcal{T}(r,k)$. By definition, $\mathcal{T}_r \subseteq \bigcup_{i=1}^{r-2} \mathcal{T}(r,i)$.

2 Smallest r-graphs of class 2

2.1 Determination of $\mathcal{T}_r$

The following theorem extends Lemma **??** and characterizes the perfect matchings M on $V(P)$ such that $P + M$ is a class 2 graph.

Theorem 2.1. *Let P be the Petersen graph and H be a 1-regular graph on $V(P)$ with edge set M. Then $P + M$ is class 2 if and only if $M \subseteq E(P)$.*

Proof. Lemma **??** has shown that $M \subseteq E(P)$ is a sufficient condition for $P+M$ to be class 2. We establish its necessity by way of contradiction. Suppose that there exists an edge $e \in M \setminus E(P)$. Let $H_1 = P+M$. Since any two vertices of the Petersen graph are in a 5-circuit, the subgraph P of H_1 can be decomposed into two 5-circuits, C_5^1 and C_5^2, and a 1-factor H' such that e is a chord of C_5^1 in H_1. Without loss of generality, we assume $C_5^1 = u_1u_2u_3u_4u_5u_1$ with $e = u_2u_5$, as shown in Figure **??**. Let $H_2 = H_1 - E(H') = P+M-E(H')$. Note that H_2 is 3-regular and contains C_5^1 and C_5^2. If $|\partial_{H_2}(V(C_5^1))| \neq 1$, then H_2 is 2-edge-connected. This implies that H_2 is class 1 since it is not isomorphic to P, as it contains a 4-circuit $u_2u_3u_4u_5u_2$. So, $H_1 = H_2 + E(H')$ is also class 1, a contradiction. Therefore, we may assume $|\partial_{H_2}(V(C_5^1))| = 1$ and set $\partial_{H_2}(V(C_5^1)) = \{e'\}$. The remaining proof is split into two cases. First, if e' is incident with u_1, then M contains an edge incident with u_3 and u_4. Thus, $H_3 = H_1 - M_1$ contains a 3-circuit $u_1u_2u_5u_1$, a 2-circuit $u_3u_4u_3$ and a 5-circuit C_5^2, where $M_1 = (M \setminus \{u_2u_5, u_3u_4\}) \cup \{u_2u_3, u_4u_5\}$. Moreover, there are five edges between $V(C_5^1)$ and $V(C_5^2)$ in H_3, which implies that H_3 is 2-edge-connected. Thus,

H_3 is class 1 and so is H_1, a contradiction. Second, if e' is incident with u_3 or u_4, then, without loss of generality, we assume that e' is incident with u_3, and so M contains the edge u_1u_4. Let $M_2 = (M \setminus \{u_1u_4, u_2u_5\}) \cup \{u_1u_2, u_4u_5\}$ and let $H_4 = H_1 - M_2$.

There are two adjacent vertices v_1 and v_4 in P such that $v_i \in N_P(u_i) \setminus V(C_5^1)$ for each $i \in \{1, 4\}$. Then H_4 contains a 4-circuit $u_1u_4v_4v_1u_1$. Moreover, H_4 is 2-edge-connected since there are five edges between $V(C_5^1)$ and $V(C_5^2)$. This implies that H_4 is class 1 and therefore, H_1 is also class 1, a contradiction. □

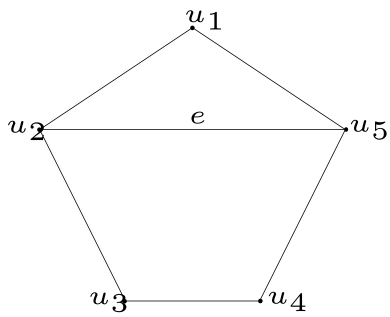


Figure 1: The 5-circuit C_5^1 with the edge e.

Theorem 2.2. *For all $r \geq 3$, $\mathcal{T}_r = \mathcal{T}(r, r-2) = \{P^{\mathcal{M}} \colon \mathcal{M}$ is a multiset of $r-3$ perfect matchings of the Petersen graph $P\}$.*

Proof. We will deduce the statement from the following three claims.

Claim 1. Let $r \geq 3$. If G is a smallest r-graph of class 2, then G has no non-trivial tight edge-cut.

Proof of Claim **??**. Suppose that there is an odd set $X \subseteq V(G)$ such that $|\partial_G(X)| = r$ and neither X nor X^c consists of a single vertex. By the minimality of $|V(G)|$, the r-graphs G/X and G/X^c are class 1. As a consequence, G is also class 1, a contradiction. ■

Claim 2. Let $r \geq 3$. If G is a smallest r-graph of class 2, then $|V(G)| = 10$ and G has $r-2$ pairwise disjoint perfect matchings.

Proof of Claim **??**. We prove the claim by induction on r. When $r = 3$, the statement follows from the fact that the smallest 3-graph of class 2 is the Petersen graph. Hence, let $r \geq 4$ and assume the statement is true for every $r' < r$.

Let G be a smallest r-graph of class 2. By Lemma **??**, $|V(G)| \leq 10$. Note that every r-graph has a perfect matching [**?**]. Thus, let M be a perfect matching of G.

If $H = G - M$ is an $(r-1)$-graph, then H is also class 2, since otherwise G would be class 1. Furthermore, we have $|V(G)| = |V(H)| \geq 10$ in this case, which implies $|V(G)| = |V(H)| = 10$. Thus, the statement follows by induction .

Therefore, we may assume that $H = G - M$ is not an $(r-1)$-graph. By the definition and Observation **??**, there is an odd set $X \subseteq V(G)$ such that $|\partial_G(X) \setminus M| \leq r-3$. Moreover, we have $|\partial_G(X)| \geq r+2$ by Claim **??**. Hence, $|\partial_G(X) \cap M| = |\partial_G(X)| - |\partial_G(X) \setminus M| \geq 5$. Since M is a perfect matching, we conclude that $|V(G)| = 10$. As a consequence, M has cardinality 5 and thus, $|\partial_G(X) \cap M| = 5$ and $|\partial_G(X)| = r+2$. Let x_1y_1 and x_2y_2 be two different edges of $\partial_G(X) \cap M$, where $x_1, x_2 \in X$. The graph $G' = G - \{x_1y_1, x_2y_2\} + \{x_1x_2, y_1y_2\}$ is still an r-graph. Indeed, for any odd set $Y \subseteq V(G')$ we have $|\partial_{G'}(Y)| \geq |\partial_G(Y)| - 2 \geq r$. Moreover, $|\partial_{G'}(X)| = r$ and hence, G' is class 1 by Claim **??**. Let $\mathcal{N}$ be a set of r pairwise disjoint perfect matchings of G' and let N_x and N_y be the perfect matchings containing x_1x_2 and y_1y_2 respectively (note that $N_x \neq N_y$ since otherwise G itself would be class 1). Then $\mathcal{N} \setminus \{N_x, N_y\}$ is a set of $r-2$ pairwise disjoint perfect matchings of G. ■

Claim 3. Let $r \geq 3$. If G is a smallest r-graph of class 2, then there is a set $\mathcal{M}$ of $r-3$ pairwise disjoint perfect matchings of G such that $G - \bigcup_{M \in \mathcal{M}} M \cong P$.

Proof of Claim **??**. We prove the claim by induction on r. When $r=3$, the statement is trivial since the smallest 3-graph of class 2 is the Petersen graph. Hence, let $r \geq 4$ and assume the statement is true for every $r' < r$.

Let G be a smallest r-graph of class 2. By Claim **??**, G is of order 10 and has a set $\mathcal{N}$ of $r-2$ pairwise disjoint perfect matchings. Let $M \in \mathcal{N}$. Then $G-M$ is class 2, since otherwise G would be class 1. If $G-M$ is an $(r-1)$-graph, then the statement follows by induction. Hence, there exists an odd set $X \subseteq V(G-M)$ with $|\partial_{G-M}(X)| \leq r-3$. Furthermore, $V(G-M) = V(G)$ and $|\partial_G(X) \setminus M| = |\partial_{G-M}(X)|$. By Claim **??** and Claim **??**, we have $|\partial_G(X)| \geq r+2$ and $|M| = 5$. As a consequence, we obtain $|\partial_G(X)| = r+2$ and $|\partial_G(X) \cap M| = 5$, which implies $|X| = 5$. Set $H = G - \cup_{N \in \mathcal{N}} N$ and note that H is a 2-factor of G, which contains at least two odd circuits, since otherwise G would be class 1. Every perfect matching of $\mathcal{N}$ contains at least one edge of $\partial_G(X)$ and hence, $|\partial_H(X)| = 0$. Thus, both $H[X]$ and $H[X^c]$ either consists of a 5-circuit or a 3-circuit and a 2-circuit. We consider the following two cases.

Case 1. $H+M$ is a 3-graph.

In this case $H+M \cong P$, since otherwise $H+M$ is class 1 which would imply that G is also class 1.

Case 2. $H+M$ is not a 3-graph.

Thus, $H+M$ has a bridge, which implies that both $H[X]$ and $H[X^c]$ consists of a 3-circuit and a 2-circuit and $|\partial_{H+M}(V(C) \cup V(C'))| = 1$, where C is the 3-circuit of $H[X]$ and C' is the 2-circuit of $H[X^c]$. As a consequence, there is only one possibility for the structure of $G+M$, which is depicted in Figure **??**. With respect to the vertex labels in Figure **??**, set $M' = (M \setminus \{v_1v_4, v_2v_3\}) \cup \{v_1v_2, v_3v_4\}$ and $\mathcal{N}' = (\mathcal{N} \setminus \{M\}) \cup \{M'\}$. Then, $\mathcal{N}'$ is a set of $r-2$ pairwise disjoint perfect matchings of G. Now, consider $\mathcal{N}'$ and M' instead of $\mathcal{N}$ and M, respectively, and repeat the same arguments as above. We deduce that $G-M'$ is an $(r-1)$-graph and the statement follows by induction. ■

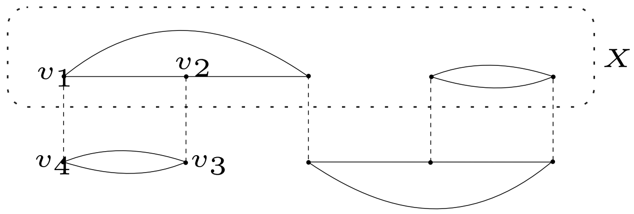


Figure 2: The graph $H + M$ in Case 2 of the proof of Claim **??** (Theorem **??**). The dashed edges belong to M.

By Claim **??**, we have $\mathcal{T}_r = \mathcal{T}(r, r-2)$. Moreover, by Theorem **??** and Claim **??**, for any multiset $\mathcal{M}$ of $r-3$ perfect matchings of P, the graph $P^{\mathcal{M}}$ is in $\mathcal{T}_r$. It remains to show that, if $G \in \mathcal{T}_r$, then $G \cong P^{\mathcal{M}}$ for a suitable multiset $\mathcal{M}$. By Claim **??**, there is a set $\mathcal{N}$ of $r-3$ pairwise disjoint perfect matchings of G such that the graph $H = G - \bigcup_{N \in \mathcal{N}} N$ is isomorphic to the Petersen graph. For every $N \in \mathcal{N}$, the graph $H + N$ is class 2, since otherwise G is class 1. Therefore, $G \cong P^{\mathcal{N}}$ by Theorem **??**. $\square$

2.2 Lower bounds for $|\mathcal{T}_r|$

The following lemma is a direct consequence of the fact that the Petersen graph is 3-arc-transitive, see e.g. Corollary 1.8 in [**?**]. That is, for any two paths of length 3 of P there is an automorphism of P which maps one to the other.

Lemma 2.3. *Let $M_1, \dots, M_6$ be the six perfect matchings of the Petersen graph P. Moreover, let $N_1, N_2, N_3 \in \{M_1, \dots, M_6\}$ and $g\colon \{N_1, N_2, N_3\} \to \{M_1, \dots, M_6\}$ be an injective function. There is an automorphism (θ, ϕ) of P such that, for all $i \in \{1, 2, 3\}$, $\phi(N_i) = g(N_i)$.*

Proof. Let N_1, N_2 and N_3 be pairwise different perfect matchings of P. If we prove the statement in this case then the proof is complete.

Note that the unique edge x_1x_2 in $N_1 \cap N_2$ and the unique edge x_3x_4 in $N_1 \cap N_3$ are at distance one, i.e. the subgraph $P[\{x_1, x_2, x_3, x_4\}]$ is a path T on four vertices. Up to changing names to such vertices, we may assume that $T = x_1x_2x_3x_4$. The same holds for the unique edge y_1y_2 in $g(N_1) \cap g(N_2)$ and the unique edge y_3y_4 in $g(N_1) \cap g(N_3)$. Without loss of generality, we can assume again that $y_1y_2y_3y_4$ is a path on four vertices.

Since P is 3-arc-transitive there is an automorphism (θ, ϕ) of P such that, for all $i \in \{1, \dots, 4\}$, $\theta(x_i) = y_i$. Since (θ, ϕ) is an automorphism, $\phi(N_1)$ must be a perfect matching. Moreover, since the only perfect matching of P containing both y_1y_2 and y_3y_4 is $g(N_1)$ we get $\phi(N_1) = g(N_1)$.

Similarly, $\phi(N_2)$ and $\phi(N_3)$ are perfect matchings of P different from $\phi(N_1)$, such that $y_1y_2 \in \phi(N_2)$ and $y_3y_4 \in \phi(N_3)$. Then, the only possibility is that $\phi(N_2) = g(N_2)$ and $\phi(N_3) = g(N_3)$. $\square$

We now consider partitions of integers, which are ways of writing an integer as a sum of positive integers, see e.g. [**?**]. We are interested in partitions of an integer into a fixed number of parts. We allow 0 to be a part of a partition. A *partition* of an integer n into k parts is a multiset of k integers $n_1, \dots, n_k$ with $n_i \geq 0$ for $i \in \{1, \dots, k\}$ such that $n = \sum_{i=1}^{k} n_i$. Two partitions of n are equal if they yield the same multiset, i.e. if they differ only in the order of their elements. For two positive integers $k \leq n$, let $p'(n,k)$ be the number of partitions of n into k parts. Set $p'(0,k) = 1$.

Theorem 2.4. *If $3 \leq r \leq 8$, then $|\mathcal{T}_r| = p'(r-3,6)$, and if $r \geq 9$, then $|\mathcal{T}_r| > p'(r-3,6)$.*

Proof. By Theorem **??**, any graph $G \in \mathcal{T}_r$ can be expressed as $G = P + \sum_{i=1}^{6} n_i M_i$, where $M_1, \dots, M_6$ are the six pairwise different perfect matchings of P. In this case, $n_1, \dots, n_6$ is a partition of $r-3$ into six parts. We say that G *induces* this partition of $r-3$.

Claim 1. Let $r \geq 3$ be an integer and $G, G' \in \mathcal{T}_r$. If $G \cong G'$, then G and G' induce the same partition of $r-3$.

Proof of Claim **??***.* We can assume that $G = P + \sum_{j=1}^{6} n_j M_j$ and $G' = P + \sum_{j=1}^{6} n'_j M_j$. For the subgraph P of G and G', we label an edge e of P by the set $\{p,q\}$ if $M_p \cap M_q = \{e\}$, $p \neq q$. Then all possible labels are used and no two edges receive the same label in P.

Since $G \cong G'$, there is an isomorphism between G and G' which maps the labeled edge $\{p,q\}$ of G to a labeled edge $\{i_p, i_q\}$ of G' for each $\{p,q\} \subseteq \{1, \dots, 6\}$. Furthermore, $n_p + n_q = n'_{i_p} + n'_{i_q}$. Thus, for $\{1,2\}, \dots, \{1,6\}$, we get that $4n_1 + \sum_{j=1}^{6} n_j = 4n'_{i_1} + \sum_{j=1}^{6} n'_{i_j}$. Since $\sum_{j=1}^{6} n_j = \sum_{j=1}^{6} n'_{i_j} = r-3$, it follows that $n_1 = n'_{i_1}$. With similar arguments, we further obtain that $n_j = n'_{i_j}$ for each $j \in \{1, \dots, 6\}$. ■

Claim 2. If $r \geq 9$, then there are non-isomorphic graphs in $\mathcal{T}_r$ which induce the same partition.

Proof of Claim **??***.* Let $N_1, \dots, N_4$ be four pairwise different perfect matchings of P such that the edge in $N_1 \cap N_2 = \{uv\}$ is adjacent to the edge in $N_3 \cap N_4 = \{uz\}$. There is a fifth perfect matching N_5 of P such that the unique edge in $N_3 \cap N_5$ is not adjacent to uv.

Let $t \geq 2$ be an integer and consider the $(t+7)$-graphs $G_t^1 = P + tN_1 + 2N_2 + N_3 + N_4$ and $G_t^2 = P + tN_1 + 2N_2 + N_3 + N_5$. Note that both G_t^1 and G_t^2 have exactly one pair of vertices connected by $t+3$ edges, i.e. $|[u,v]_{G_t^1}| = |[u,v]_{G_t^2}| = t+3$. On one hand, uv is adjacent to uz and $|[u,z]_{G_t^1}| = 3$. On the other hand, by the choice of N_5, uv is adjacent only to edges xy such that $|[x,y]_{G_t^2}| \leq 2$. We deduce that $G_t^1 \not\cong G_t^2$. ■

Claim 3. Let $r \leq 8$ and $G, G' \in \mathcal{T}_r$. If G and G' induce the same partition of $r-3$, then $G \cong G'$.

Proof of Claim **??***.* Assume that $G = P^{\mathcal{M}} = P + \sum_{j=1}^{6} n_j M_j$ and $G' = P^{\mathcal{M}'} = P + \sum_{j=1}^{6} n'_j M_j$ induce the same partition of $r-3$. Let $\mathcal{M}_0 = \{M_j \colon n_j \neq 0\}$ and $\mathcal{M}'_0 = \{M_j \colon n'_j \neq 0\}$. Then $|\mathcal{M}_0| = |\mathcal{M}'_0|$.

If $|\mathcal{M}_0| \le 3$, choose a bijection $g\colon \mathcal{M}_0 \to \mathcal{M}'_0$ such that $g(M_\alpha) = M_\beta$ if and only if $n_\alpha = n'_\beta$. By Lemma **??**, there is an automorphism (θ, ϕ) of P such that, for each perfect matching $N \in \mathcal{M}_0$, $\phi(N) = g(N)$. It follows that (θ, ϕ') is an isomorphism of $P^{\mathcal{M}}$ to $P^{\mathcal{M}'}$, where $\phi'(M_i) = \phi(M_i)$ for each $i \in \{1, \dots, 6\}$. The only other cases are the following.

- $r - 3 = 4$ with partition $1, 1, 1, 1, 0, 0$;
- $r - 3 = 5$ with partitions $2, 1, 1, 1, 0, 0$ or $1, 1, 1, 1, 1, 0$.

In such cases, we let $\mathcal{M}_1 = \{M_j : n_j = 1\}$ and $\mathcal{M}'_1 = \{M_j : n'_j = 1\}$. Let $\mathcal{N}_1$ be the set of perfect matchings of P different from those of $\mathcal{M}_1$ and $\mathcal{N}'_1$ be the set of perfect matchings of P different from those of $\mathcal{M}'_1$. Then, there is a bijection $g\colon \mathcal{N}_1 \to \mathcal{N}'_1$ such that $g(M_\alpha) = M_\beta$ if and only if $n_\alpha = n'_\beta$. The proof now, follows as above. Namely, since $|\mathcal{N}_1| = |\mathcal{N}'_1| \le 3$, by Lemma **??**, there is an automorphism (θ, ϕ) of P such that, for all $N \in \mathcal{N}_1$, $\phi(N) = g(N)$. Then, (θ, ϕ') is an isomorphism of $P^{\mathcal{M}}$ to $P^{\mathcal{M}'}$, where $\phi'(M_i) = \phi(M_i)$ for each $i \in \{1, \dots, 6\}$. ■

By Claims **??**, **??** and **??**, the theorem is proved. □

3 Complete sets

In this section we give the following characterization of $\mathcal{H}_r$: $G \in \mathcal{H}_r$ if and only if the only connected r-graph coloring G is G itself. Moreover, we show that $\mathcal{H}_r$ is an infinite set when $r \ge 4$. For $r = 3$ it turns out that, if the Petersen Coloring Conjecture is false, then $\mathcal{H}_3$ is an infinite set too. We prove similar results for the restriction on simple r-graphs.

We start with some preliminary technical results. In particular, we introduce a lifting operation for r-graphs.

3.1 Substructures and lifting

Let G be a graph and $F \subseteq E(G)$. We say that F *induces* a subgraph H of G if $E(H) = F$ and $V(H)$ contains all vertices of G which are incident with an edge of F. We denote such a subgraph H by $G[F]$. A spanning subgraph G' of G is a $\{K_{1,1}, C_m : m \ge 3\}$-factor if each component of G' is isomorphic to an element of $\{K_{1,1}, C_m : m \ge 3\}$, where $K_{s,t}$ is the complete bipartite graph with two partition sets of sizes s and t. Some of the following observations appear also in [**?**].

Observation 3.1. *Let H and G be graphs and let f be an H-coloring of G.*

(i) $\chi'(G) \le \chi'(H)$.

(ii) If $M_1, \dots, M_k$ are k pairwise disjoint perfect matchings in H, then $f^{-1}(M_1), \dots, f^{-1}(M_k)$ are k pairwise disjoint perfect matchings in G.

(iii) *If C is a 2-regular subgraph of H, then $f^{-1}(E(C))$ induces a 2-regular subgraph in G.*

(iv) *If H' is a $\{K_{1,1}, C_m \colon m \geq 3\}$-factor in H, then $f^{-1}(E(H'))$ induces a $\{K_{1,1}, C_m \colon m \geq 3\}$-factor in G.*

Proof. Let H' be a subgraph of H and G' be the subgraph of G induced by $f^{-1}(E(H'))$. By the definition of H-coloring, if H' is k-regular (spanning, respectively) then G' is k-regular (spanning, respectively). Then statements $(i), (ii)$ and (iii) can be obtained immediately. In order to show statement (iv), assume that H' is a $\{K_{1,1}, C_m \colon m \geq 3\}$-factor. We decompose H' into a 1-regular subgraph H_1 and a 2-regular subgraph H_2. The sets $f^{-1}(E(H_1))$ and $f^{-1}(E(H_2))$ induce a 1-regular subgraph G_1 and a 2-regular subgraph G_2 of G, respectively. By the definition of H-coloring, G_1 and G_2 are disjoint. This completes the proof. □

Let G be a graph and let $x \in V(G)$ with $|N_G(x)| \geq 2$. A *lifting* (of G) at x is the following operation: Choose two distinct neighbors y and z of x, delete an edge e_1 connecting x with y, delete an edge e_2 connecting x with z and add a new edge e connecting y with z; additionally, if e_1 and e_2 were the only two edges incident with x, then delete the vertex x in the new graph. We say e_1 and e_2 are *lifted to* e; the new graph is denoted by $G(e_1, e_2)$.

We will make use of the following fact. Let G be a graph, then $|\partial_G(X \cap Y)| + |\partial_G(X \cup Y)| \leq |\partial_G(X)| + |\partial_G(Y)|$ for every $X, Y \subseteq V(G)$.

Lemma 3.2. *Let $r \geq 2$ be an integer and let G be a connected graph of order at least 2 with a vertex $x \in V(G)$ such that*

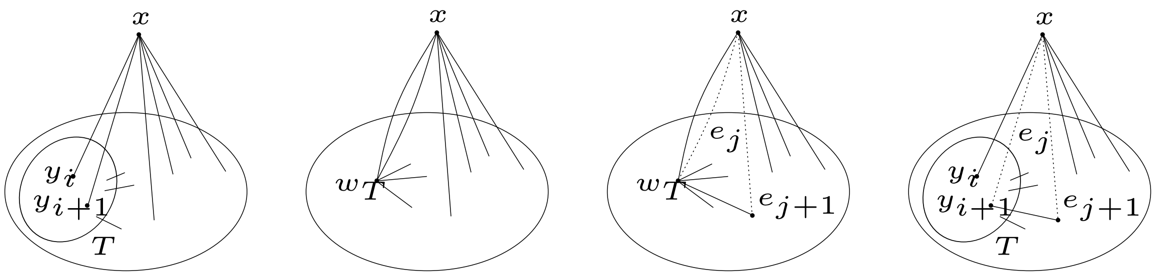
- *$d_G(v) = r$ for all $v \in V(G) \setminus \{x\}$, and*
- *if $|V(G)|$ is even, then $d_G(x) \neq r$, and*
- *$|\partial_G(S)| \geq r$ for every $S \subseteq V(G) \setminus \{x\}$ of odd cardinality.*

Then, for every labeling $\partial_G(x) = \{e_1, \dots, e_{d_G(x)}\}$ there exists an $i \in \mathbb{Z}_{d_G(x)}$ such that $G(e_i, e_{i+1})$ is a connected graph with $|\partial_{G(e_i,e_{i+1})}(S')| \geq r$ for every $S' \subseteq V(G(e_i, e_{i+1})) \setminus \{x\}$ of odd cardinality.

Proof. We argue by contradiction. Let G be a possible counterexample of smallest order, let $d = d_G(x)$, and let $e_i = xy_i$ for every $i \in \{1, \dots, d\}$.

First we show $|N_G(x)| \geq 2$. Suppose that x has just one neighbor x'. Note that $d_G(x') = r$ by our assumptions. If $|V(G)|$ is even, then $d_G(x) \neq r$. As a consequence, the set $S = V(G) \setminus \{x\}$ is a set of odd cardinality with $|\partial_G(S)| = d_G(x) < r$, a contradiction. If $|V(G)|$ is odd, then the set $S = V(G) \setminus \{x, x'\}$ is a set of odd cardinality with $|\partial_G(S)| = r - d_G(x) < r$, a contradiction again. Therefore, $|N_G(x)| \geq 2$.

Hence, we can choose an $i \in \mathbb{Z}_d$ such that $y_i \neq y_{i+1}$ and, if $G - x$ is not connected, then y_i and y_{i+1} belong to different components of $G - x$. Suppose that G has a bridge e. Then, for parity reasons, the component H of $G - e$ not containing x is of odd order, a contradiction since $|\partial_G(V(H))| = 1 < r$. Thus, G is bridgeless and hence, the graph $G(e_i, e_{i+1})$ is connected by the choice of i. As a consequence, there is a set $T \subseteq V(G(e_i, e_{i+1})) \setminus \{x\}$ of odd cardinality with $|\partial_{G(e_i,e_{i+1})}(T)| < r$, since G is a counterexample. Observe that $|\partial_G(T)|$ has the same parity as r, which implies $|\partial_G(T)| = r$ and $y_i, y_{i+1} \in T$. Set $G_1 = G/T$ and label the edges of $\partial_{G_1}(x)$ with the same labels as in G. Then, G_1 and x satisfy the conditions of the statement. Therefore, by the minimality of $|V(G)|$, there is an integer $j \in \mathbb{Z}_d$ such that the graph $G_2 = G_1(e_j, e_{j+1})$ satisfies $|\partial_{G_2}(S)| \geq r$ for every $S \subseteq V(G_2) \setminus \{x\}$ of odd cardinality. Set $G_3 = G(e_j, e_{j+1})$. The graphs G, G_1, G_2 and G_3 are depicted in Figure **??**.



(a) G (b) G_1 (c) G_2 (d) G_3

Figure 3: An example for the graphs G, G_1, G_2 and G_3.

Note that $V(G) = V(G_3)$ and $V(G_2) \setminus \{w_T\} = V(G_3) \setminus T$. Furthermore, we observe the following:

- for every $X \subseteq T$: $|\partial_G(X)| = |\partial_{G_3}(X)|$,
- for every $X \subseteq V(G_2) \setminus \{w_T\}$: $|\partial_{G_2}(X)| = |\partial_{G_3}(X)|$ and $|\partial_{G_2}(X \cup \{w_T\})| = |\partial_{G_3}(X \cup T)|$.

Now, let $S \subseteq V(G_3) \setminus \{x\}$ be a set of odd cardinality. Set $A = S \cap T$ and $B = S \setminus A$. We consider two cases.

Case 1. $|A|$ is even.

As a consequence, B and $T \setminus A$ are sets of odd cardinality. Therefore, by using the above observations we obtain the following:

$$\begin{aligned}
|\partial_{G_3}(S)| = |\partial_{G_3}(S^c)| &\geq |\partial_{G_3}(S^c \cap T)| + |\partial_{G_3}(S^c \cup T)| - |\partial_{G_3}(T)| \\
&= |\partial_{G_3}(T \setminus A)| + |\partial_{G_3}(B)| - |\partial_{G_3}(T)| \\
&= |\partial_G(T \setminus A)| + |\partial_{G_2}(B)| - |\partial_G(T)| \\
&\geq r + r - r \\
&= r.
\end{aligned}$$

Case 2. $|A|$ is odd.

Thus, B is a set of even cardinality, which implies

$$\begin{aligned}
|\partial_{G_3}(S)| &\geq |\partial_{G_3}(S \cap T)| + |\partial_{G_3}(S \cup T)| - |\partial_{G_3}(T)| \\
&= |\partial_{G_3}(A)| + |\partial_{G_3}(B \cup T)| - |\partial_{G_3}(T)| \\
&= |\partial_{G}(A)| + |\partial_{G_2}(B \cup \{w_T\})| - |\partial_{G}(T)| \\
&\geq r + r - r \\
&= r.
\end{aligned}$$

In any case, we have $|\partial_{G_3}(S)| \geq r$, which implies $|\partial_{G(e_j,e_{j+1})}(S')| \geq r$ for every $S' \subseteq V(G(e_j, e_{j+1})) \setminus \{x\}$ of odd cardinality. This is a contradiction to the assumption that G is a counterexample. $\square$

The previous lemma can be used in r-graphs as follows.

Theorem 3.3. *Let $r \geq 2$ be an integer, let G be a connected r-graph and let X be a non-empty proper subset of $V(G)$. If $|X|$ is even, then G/X can be transformed into a connected r-graph by applying $\frac{1}{2}|\partial_G(X)|$ lifting operations at w_X. If $|X|$ is odd, then G/X can be transformed into a connected r-graph by applying $\frac{1}{2}(|\partial_G(X)| - r)$ lifting operations at w_X.*

Proof. Consider any labeling of $\partial_{G/X}(w_X)$. The statement follows by applying repeatedly Lemma **??** to G/X at w_X. Note that w_X is removed in the last step when $|X|$ is even. $\square$

Note that the previous lifting operations can be applied so that they preserve embeddings of graphs in surfaces.

3.2 Characterization of $\mathcal{H}_r$

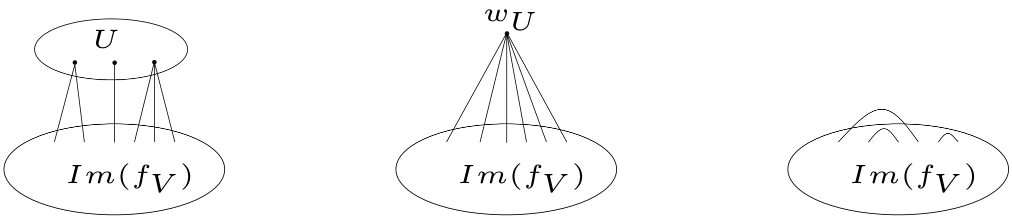
Let f be an H-coloring of G. The subgraph of H induced by the edge set $Im(f)$ is denoted by H_f. Observe that H_f also colors G. Furthermore, if H has no two vertices u_1, u_2 with $\partial_H(u_1) = \partial_H(u_2)$, then f induces a mapping $f_V : V(G) \to V(H)$, where every $v \in V(G)$ is mapped to the unique vertex $u \in V(H)$ with $f(\partial_G(v)) = \partial_H(u)$. Note that f_V is well defined if H is a connected graph with $|V(H)| > 2$. A vertex of $V(H) \setminus Im(f_V)$ is called *unused.*

Theorem 3.4. *Let $r \geq 3$ and let G be an r-graph of class 2 that cannot be colored by an r-graph of smaller order. If H is a connected r-graph and f is an H-coloring of G, then (f_V, f) is an isomorphism, i.e. $H \cong G$.*

Proof. Let $f : E(G) \to E(H)$ be an H-coloring of G. Note, that since G is class 2, H is also class 2 and therefore, f_V is well defined. We first prove three claims.

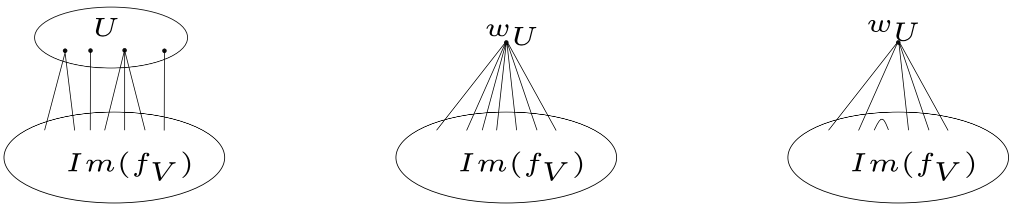
Claim 1. f is injective.

Proof of Claim **??**. Suppose to the contrary that f is not injective, which implies $|E(H_f)| < |E(G)|$. If H contains no unused vertices, then $|E(H)| = |E(H_f)| < |E(G)|$, which contradicts the assumption that G cannot be colored by an r-graph of smaller order. Thus, H contains unused vertices; let $U \subseteq V(H)$ be the set of them. Transform the graph H/U into a new r-graph H' as follows. If $|U|$ is even, then apply $\frac{1}{2}\,|\partial_H(U)|$ lifting operations at w_U (see Figure **??**). If $|U|$ is odd, then apply $\frac{1}{2}\,(|\partial_H(U)| - r)$ lifting operations at w_U (see Figure **??**). By Theorem **??**, this can be done in such a way that the resulting graph H' is indeed an r-graph.



(a) H (b) H/U (c) H'

Figure 4: An example for the graphs H, H/U and H' when $|U|$ is even.



(a) H (b) H/U (c) H'

Figure 5: An example for the graphs H, H/U and H' when $|U|$ is odd.

Note that every edge of $Im(f)$ is incident with at most one vertex of U. Thus, we can define a function $f' \colon E(G) \to E(H/U)$ as follows. For every $e \in E(G)$ let $f'(e)$ be the edge of H/U corresponding to the edge $f(e)$ of H. Observe that f' is an H/U-coloring of G, where w_U is the only unused vertex. Next, define a new mapping $f'' \colon E(G) \to E(H')$ as follows. For every $e \in E(G)$ set

$$f''(e) = \begin{cases} e' & \text{if } f'(e) \text{ is one of the two edges lifted to } e', \\ f'(e) & \text{if } f'(e) \in E(H'). \end{cases}$$

By construction, $f''(\partial_G(v)) = \partial_{H'}(f_V(v))$ for every $v \in V(G)$. Since G and H' are r-regular it follows that f'' is an H'-coloring. Therefore, $H' \prec G$ and hence $|V(H')| \geq |V(G)|$ by our assumptions. This is a contradiction, since

$$|E(H')| \leq |E(H/U)| = |E(H_f)| < |E(G)|.$$

■

Claim 2. f_V is surjective.

Proof of Claim ??. Suppose that H contains unused vertices. Then, there are $v_1, v_2 \in V(G)$ and $e \in [v_1, v_2]_G$ such that $f(e)$ is incident with exactly one unused vertex in H, since H is connected. Thus, $f(\partial_G(v_1)) = f(\partial_G(v_2))$, which contradicts Claim **??**. ■

Claim 3. $|V(H)| = |V(G)|$.

Proof of Claim ??. Since G cannot be colored by an r-graph of smaller order, we have $|V(H)| \geq |V(G)|$. On the other hand, $|V(H)| \leq |V(G)|$ by Claim **??**. ■

Claims **??**, **??** and **??** imply that f and f_V are bijections. Furthermore, we obtain that $e \in [v_1, v_2]_G$ if and only if $f(e) \in [f_V(v_1), f_V(v_2)]_H$. Therefore, (f_V, f) is an isomorphism between G and H, i.e. $H \cong G$. □

In [**?**], Mkrtchyan proves that if a connected 3-graph H colors the Petersen graph P, then $H \cong P$. The following result is implied by Theorem **??** together with Observation **??** (*ii*) and gives a generalization of Mkrtchyan's result in the r-regular case.

Corollary 3.5. *Let $r \geq 3$ and let G be an r-graph of class 2 such that $\pi(G') > \pi(G)$ for every r-graph G' with $|V(G')| < |V(G)|$. If H is a connected r-graph with $H \prec G$, then $H \cong G$.*

By Theorem **??**, $\mathcal{T}_r = \mathcal{T}(r, r-2) = \{P^{\mathcal{M}} \colon \mathcal{M}$ is a set of $r-3$ perfect matchings of the Petersen graph $P\}$. Hence, with Corollary **??** we obtain the following theorem.

Theorem 3.6. *Let $r \geq 3$, let H be a connected r-graph and let $G \in \mathcal{T}(r, r-2) \cup \mathcal{T}(r, 1)$. If $H \prec G$, then $H \cong G$.*

Theorem 3.7. *Let $r \geq 3$ and let G be a connected r-graph. The following statements are equivalent.*

1) $G \in \mathcal{H}_r$.

2) The only connected r-graph coloring G is G itself.

3) G cannot be colored by a smaller r-graph.

Proof. 2) $\implies$ 1) follows trivially.

1) $\implies$ 3). Assume by contradiction that 3) is not true. Then, let H be a smallest r-graph smaller than G such that $H \prec G$. Note that H cannot be colored by a smaller r-graph because otherwise, since the relation $\prec$ is transitive, G would be colored by an r-graph smaller than H. Hence, $H \in \mathcal{H}_r$ by Theorem **??**. Thus, $\mathcal{H}_r \setminus \{G\}$ is an r-complete set, in contradiction to the inclusion-wise minimality of $\mathcal{H}_r$.

3) $\implies$ 2) follows by Theorem **??**. □

Corollary 3.8. *For every $r \geq 3$, there exists only one inclusion-wise minimal r-complete set, i.e. $\mathcal{H}_r$ is unique.*

For $r = 3$, we have $\mathcal{T}(r, r-2) = \mathcal{T}(r, 1) = \{P\}$. The Petersen Coloring Conjecture states that $\mathcal{H}_3 = \{P\}$. This situation is very exclusive as we show in the following subsection.

3.3 Infinite subsets of $\mathcal{H}_r$

Lemma 3.9. *Let $r \geq 3$, let G and H be two connected r-graphs and let f be an H-coloring of G. For any 2-edge-cut $F = \{e_1, e_2\} \subseteq E(G)$, either $|f(F)| = 1$ or $f(F)$ is a 2-edge-cut of H.*

Proof. Let u and v be the endvertices of $f(e_1)$. Suppose by contradiction that $|f(F)| = 2$ but $f(F)$ is not a 2-edge-cut of H. Then, there is a uv-path T in H avoiding the edges of $f(F)$. Consider the circuit $C = T + f(e_1)$. By Observation **??** *(iii)*, $f^{-1}(E(C))$ is a union of circuits of G. This is a contradiction, since $f^{-1}(E(C))$ contains e_1 but not e_2. □

Let G, H be two graphs, let $f \colon E(G) \to E(H)$, $g \colon V(G) \to V(H)$ and let G' be a subgraph of G. The restriction of f to $E(G')$ is denoted by $f|_{G'}$; the restriction of g to $V(G')$ is denoted by $g|_{G'}$.

Lemma 3.10. *Let G and H be two r-graphs, where $r \geq 3$, and let f be an H-coloring of G. Let $\mathcal{M}$ be a multiset of $r-3$ perfect matchings of P and let $e_0 \in E(P^{\mathcal{M}})$. Let G' be an induced subgraph of G isomorphic to $P^{\mathcal{M}} - e_0$ and H' be the subgraph of H induced by $f(E(G'))$. Then, $(f_V|_{G'}, f|_{G'})$ is an isomorphism between G' and H', i.e. $H' \cong G'$.*

Proof. By the definition of G', we have $|\partial_G(V(G'))| = 2$. Assume that $\partial_G(V(G')) = \{e_1, e_2\}$ and $e_i = u_i v_i$ with $u_i \in V(G')$ for each $i \in \{1, 2\}$.

We first consider the case $f(e_1) = f(e_2)$. Let G^* be the r-graph obtained from G' by adding a new edge e_3 joining u_1 and u_2. Set $f^*(e) = f(e) = f|_{G'}(e)$ for each $e \in E(G^*) \setminus \{e_3\}$ and $f^*(e_3) = f(e_1) = f(e_2)$. Then f^* is an H-coloring of G^*. Since $G^* \cong P^{\mathcal{M}}$, we have that (f^*_V, f^*) is an isomorphism between G^* and H by Theorem **??**. Thus $(f_V|_{G'}, f|_{G'})$ is an isomorphism of G' to H' by the definition of f^*.

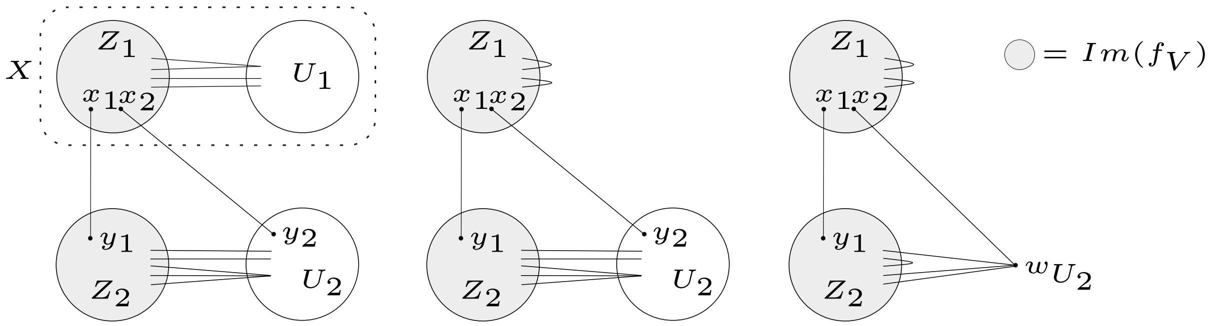
Now we assume that $f(e_1) \neq f(e_2)$. By Lemma **??**, $\{f(e_1), f(e_2)\}$ is a 2-edge-cut of H. Let X be a subset of $V(H)$ such that $\partial_H(X) = \{f(e_1), f(e_2)\}$. Denote $f(e_i) = x_i y_i$ with $x_i \in X$ for each $i \in \{1, 2\}$. We consider the following two cases.

Case 1. $f_V(V(G')) \subseteq X$ or $f_V(V(G')) \subseteq V(H) \setminus X$.

Without loss of generality, assume that $f_V(V(G')) \subseteq X$. Let G^* be the r-graph obtained from G' by adding a new edge e_3 joining u_1 and u_2, and H^* be the r-graph obtained from $H[X]$ by adding a new edge e_4 joining x_1 and x_2. Set $f^*(e) = f(e) = f|_{G'}(e)$ for each $e \in E(G^*) \setminus \{e_3\}$ and $f^*(e_3) = e_4$. Then f^* is an H^*-coloring of G^*. Since $G^* \cong P^{\mathcal{M}}$, we have that (f^*_V, f^*) is an isomorphism between G^* and H^* by Theorem **??**. Thus $(f_V|_{G'}, f|_{G'})$ is an isomorphism of G' to H' by the definition of f^* and the statement follows.

Case 2. $f_V(V(G')) \cap X \neq \emptyset$ and $f_V(V(G')) \cap (V(H) \setminus X) \neq \emptyset$.

We show that this case does not apply. Let $Z_1 = f_V(V(G')) \cap X$ and $Z_2 = f_V(V(G')) \cap (V(H) \setminus X)$. Observe that $\{f(e_1), f(e_2)\} \subseteq \partial_H(Z_1) \cup \partial_H(Z_2)$. Set $U_1 = X \setminus Z_1$ and $U_2 = (V(H) \setminus X) \setminus Z_2$. Note that U_1 and U_2 might be empty. We construct a new r-graph H_2 from H in two steps. First, if $U_1 = \emptyset$, set $H_1 = H$. Otherwise we can construct an r-graph H_1 starting from H/U_1 by taking suitable lifting operations at w_{U_1} as described in Theorem **??**, namely: if $|U_1|$ is even, then apply $\frac{1}{2}|\partial_H(U_1)|$ lifting operations at w_{U_1}; if $|U_1|$ is odd, then apply $\frac{1}{2}(|\partial_H(U_1)| - r)$ lifting operations at w_{U_1}. Observe that $U_2 \subset V(H_1)$. Next, if $U_2 = \emptyset$, set $H_2 = H_1$. Otherwise let H_2 be a graph obtained from H_1/U_2 by taking similar lifting operations as described above at the vertex w_{U_2}. An example for the construction of H_2 is given in Figure **??**.



(a) H (b) H_1 (c) H_2

Figure 6: An example for the graphs H, H_1 and H_2, when U_1, U_2 are non-empty, U_1 is of even cardinality and U_2 is of odd cardinality.

By Theorem **??**, this can be done such that H_2 is an r-graph. Furthermore, we have

$$|E(H_2)| \leq |E(H') \cup \{f(e_1), f(e_2)\}| \leq |E(G')| + 2.$$

As a consequence, $|V(H_2)| \leq 10$. Thus, H_2 is class 1 since it has a 2-edge-cut and hence, H_2 has r pairwise disjoint perfect matchings. By the construction of H_2, we deduce that H contains r pairwise disjoint sets of edges, denoted by $S_1, \dots, S_r$, such that $|\partial_H(y) \cap S_j| = 1$ for each $y \in f_V(V(G'))$ and each $j \in \{1, \dots, r\}$. Then $f^{-1}(S_1), \dots, f^{-1}(S_r)$ are r pairwise disjoint sets of edges of G such that $|\partial_G(u) \cap f^{-1}(S_j)| = 1$ for each $u \in V(G')$ and each $j \in \{1, \dots, r\}$. This is a contradiction since G' is class 2. $\square$

Let G and G' be two disjoint r-graphs of class 2 with $e \in E(G)$ and $e' \in E(G')$. Denote by $(G, e)|(G', e')$ the set of all graphs obtained from G by replacing the edge e of G by (G', e'), that is, deleting e from G and e' from G', and then adding two edges between $V(G)$ and $V(G')$ such that the resulting graph is regular (see Figure **??**).

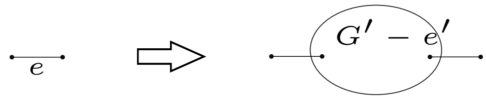


Figure 7: A replacement of the edge e by (G', e').

In fact, any graph in $(G, e)|(G', e')$ is an r-graph of class 2. Furthermore, we use $G|(G', e')$ to denote the set of all graphs obtained from G by replacing each edge of G by (G', e').

Theorem 3.11. *Let $\mathcal{M}$ be a multiset of $r-3$ perfect matchings of P, where $r \geq 3$, and let $e_0 \in E(P^{\mathcal{M}})$. Let G be an r-graph such that $G \ncong P^{\mathcal{M}}$. If $G \in \mathcal{H}_r$, then $G|(P^{\mathcal{M}}, e_0) \subset \mathcal{H}_r$.*

Proof. By Theorem **??**, it suffices to prove that any $G^* \in G|(P^{\mathcal{M}}, e_0)$ cannot be colored by a connected r-graph of smaller order. Let H be a connected r-graph such that G^* has an H-coloring, denoted by f. Label all subgraphs of G^* isomorphic to $P^{\mathcal{M}} - e_0$ as $G_1, \dots, G_\ell$, where $\ell = |E(G)|$, and denote by H_i the subgraph of H induced by $f_V(V(G_i))$. Note that $H_i \cong P^{\mathcal{M}} - e_0$ by Lemma **??**. For each $i \in \{1, \dots, \ell\}$, we label the two edges of $\partial_{G^*}(V(G_i))$ as e_i^1 and e_i^2, and let $e_i^t = u_i^t v_i^t$ with $v_i^t \notin V(G_i)$ for each $t \in \{1, 2\}$.

Claim 1. $f(\partial_{G^*}(V(G_i)))$ is a 2-edge-cut in H, for every $i \in \{1, \dots, \ell\}$.

Proof of Claim **??**. By Lemma **??**, we suppose to the contrary that there is $i \in \{1, \dots, \ell\}$ such that $f(e_i^1) = f(e_i^2)$. With $G_i \cong P^{\mathcal{M}} - e_0$, we have $H \prec P^{\mathcal{M}}$ by Lemma **??**, and so $H \cong P^{\mathcal{M}}$ by Theorem **??**. Then, $|f(F)| = 1$ for any 2-edge-cut $F \subset E(G^*)$ by Lemma **??** since $P^{\mathcal{M}}$ is 3-edge-connected. Thus, by the construction of G^*, we have $H \prec G$, which implies $H \cong G$ by Theorem **??**. This is a contradiction to the fact that $G \ncong P^{\mathcal{M}}$. ■

Claim 2. $V(H_i) = V(H_j)$ or $V(H_i) \cap V(H_j) = \emptyset$, for every $i, j \in \{1, \dots, \ell\}$.

Proof of Claim **??**. Assume $V(H_i) \cap V(H_j) \neq \emptyset$. To complete the proof, we shall show $V(H_i) \setminus V(H_j) = \emptyset$ and $V(H_j) \setminus V(H_i) = \emptyset$. Without loss of generality, suppose to the contrary that $V(H_j) \setminus V(H_i) \neq \emptyset$. Note that $f(\partial_{G^*}(V(G_i)))$ is a 2-edge-cut in H by Claim **??**. Observe that both H_i and H_j are isomorphic to $P^{\mathcal{M}} - e_0$ by Lemma **??**. Thus, at least one edge of $f(\partial_{G^*}(V(G_i)))$ is contained in $E(H_j)$, since H_j is connected. As a consequence, H_j either has a bridge or a 2-edge-cut consisting of two non-adjacent edges, since an r-graph has no cut-vertex. This is not possible. ■

Claim 3. $f_V(z) \notin \bigcup_{i=1}^{\ell} V(H_i)$, for every $z \in V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))$.

Proof of Claim **??**. Suppose to the contrary that there is a vertex $z \in V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))$ such that $f_V(z) \in V(H_j)$ for some $j \in \{1, \dots, \ell\}$. Let e be an edge incident with $f_V(z)$ in H_j. By the construction of G^*, the only edge of $f^{-1}(e) \cap \partial_{G^*}(z)$ is an element of $\partial_{G^*}(V(G_k))$ for some $k \in \{1, \ldots, \ell\}$. Thus, e is in a 2-edge-cut of H by Claim **??**, contradicting the fact that $H_j \cong P^{\mathcal{M}} - e_0$ by Lemma **??**. $\blacksquare$

By Claim **??**, $\partial_H(V(H_i)) = f(\partial_{G^*}(V(G_i))) = \{f(e_i^1), f(e_i^2)\}$. Let $f(e_i^t) = x_i^t y_i^t$ with $y_i^t \notin V(H_i)$ for each $t \in \{1, 2\}$.

Claim 4. $\{y_i^1, y_i^2\} \cap V(H_j) = \emptyset$, for every $i, j \in \{1, \ldots, \ell\}$.

Proof of Claim **??**. By contradiction, suppose $y_i^t \in V(H_j)$ for some $t \in \{1, 2\}$. Note that $f_V(v_i^t) \in \{y_i^t, x_i^t\}$ and $x_i^t \in V(H_i)$. Thus, $f_V(v_i^t) \in V(H_i) \cup V(H_j)$. This is a contradiction to Claim **??** since $v_i^t \in V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))$ by the construction of G^*. $\blacksquare$

Note that G can be obtained from G^* by deleting all vertices of G_i and adding a new edge edge e_i joining v_i^1 and v_i^2 for each $i \in \{1, \ldots, \ell\}$. By Claims **??** and **??**, $(V(H_i) \cup \{y_i^1, y_i^2\}) \cap V(H_j) = \emptyset$ if $V(H_i) \neq V(H_j)$ for each $i, j \in \{1, \ldots, \ell\}$. Thus, we can construct an r-graph H' from H by deleting all vertices of H_i and adding a new edge g_i joining y_i^1 and y_i^2 for each $i \in \{1, \ldots, \ell\}$. Note that, for some $i \neq j \in \{1, \ldots, \ell\}$, it might happen that $V(H_i) = V(H_j)$. In such a case, $g_i = g_j$. Define a mapping $f' \colon E(G) \to E(H')$ by letting $f'(e_i) = g_i$, for each $i \in \{1, \ldots, \ell\}$. By Claim **??**, $f'_V(z) \in V(H')$ for every vertex $z \in V(G) \subset V(G^*)$. Furthermore, we have $f'(\partial_G(z)) = \partial_{H'}(f'_V(z))$. Since both G and H' are r-graphs, f' is proper. Thus, f' is an H'-coloring of G. Then, (f'_V, f') is an isomorphism between G and H' by Theorem **??**. This implies that $|V(G^*) \setminus (\bigcup_{i=1}^{\ell} V(G_i))| = |V(H) \setminus (\bigcup_{i=1}^{\ell} V(H_i))|$, and $V(H_i) \neq V(H_j)$ for any distinct $i, j \in \{1, \ldots, \ell\}$ since $f'(e_i) \neq f'(e_j)$. Therefore, $|V(G^*)| = |V(H)|$ by Claims **??** and **??**, which completes the proof. $\square$

The following corollary answers the question of [**?**] whether for each $r \geq 4$, there exists a connected r-graph H with $H \prec G$ for every r-graph G.

Corollary 3.12. *Either $\mathcal{H}_3 = \{P\}$ or $\mathcal{H}_3$ is an infinite set. Moreover, if $r \geq 4$, then $\mathcal{H}_r$ is an infinite set.*

Proof. If $\mathcal{H}_3 \neq \{P\}$, then there is a smallest 3-graph G that cannot be colored by P. Note that G is class 2 and not isomorphic to P. Furthermore, if $H \prec G$ for a connected 3-graph H of smaller order, then $P \prec H$ by the choice of G and hence $P \prec G$, a contradiction. Thus, we can use Theorem **??** to inductively construct infinitely many graphs belonging to $\mathcal{H}_3$.

By Theorems **??** and **??**, $\mathcal{T}(r, 1) \subset \mathcal{H}_r$. Note that the set $\mathcal{T}(r, 1)$ is non-empty (see [**?**]), and for $r \geq 4$, it does not contain any graph isomorphic to $P^{\mathcal{M}}$, where $\mathcal{M}$ is any multiset of $r - 3$ perfect matchings of P. Hence, we can use Theorem **??** to inductively construct infinitely many graphs belonging to $\mathcal{H}_r$. $\square$

3.4 Simple r-graphs

In [?] the authors also asked whether for every $r \geq 4$, there is a connected r-graph coloring all simple r-graph. In this section we answer this question by showing that there is no finite set of connected r-graphs $\mathcal{H}'_r$ such that every connected simple r-graph can be colored by an element of $\mathcal{H}'_r$.

Lemma 3.13 ([?])**.** *Let r be a positive integer, G be an r-graph and $F \subseteq E(G)$. If $|F| \leq r-1$, then $G - F$ has a 1-factor.*

Recall that, for an r-graph G and an odd set $X \subseteq V(G)$, an edge-cut $\partial_G(X)$ is *tight* if it consists of exactly r edges.

Lemma 3.14. *Let $r \geq 3$, let G, H be connected r-graphs and let f be an H-coloring of G. If $F \subseteq E(G)$ is a tight edge-cut in G, then $f(F)$ is a tight edge-cut in H.*

Proof. Since F is a tight edge-cut, we have $|f(F)| \leq r$. Suppose that $|f(F)| < r$. By Lemma **??**, $H - f(F)$ has a perfect matching M. Thus, $f^{-1}(M)$ is a perfect matching of G such that $f^{-1}(M) \cap F = \emptyset$, a contradiction. Therefore, $|f(F)| = r$, and let $H_1, \dots, H_m$ be the components of $H - f(F)$.

We first claim that the two endvertices of each edge in $f(F)$ are in different components of $H - f(F)$. By contradiction, suppose that there is an edge $xy \in f(F)$ such that x and y are on the same component H' of $H - f(F)$. Let T be an xy-path contained in H'. Then, $f^{-1}(E(T) \cup \{xy\})$ induces a 2-regular subgraph in G (see Observation **??** (*iii*)) and intersects F exactly once, a contradiction.

The remaining proof is split into two cases as follows.

Case 1. $H - f(F)$ has a component of odd order.

If $m > 2$, then there is an odd component H' with $|\partial_G(V(H'))| < r$, since $H - f(F)$ has at least two components of odd order, a contradiction. Hence, $H - f(F)$ has exactly two components, which are of odd order and therefore, $f(F)$ is a tight edge-cut in H.

Case 2. Every component of $H - f(F)$ is of even order.

Let $\tilde{H}$ be the graph obtained from H by identifying all vertices in $V(H_i)$ to a new vertex for each $i \in \{1, \dots, m\}$. Since every component is of even order, $\tilde{H}$ is an eulerian graph on $|f(F)| = r$ edges.

Now, we shall prove that $\tilde{H}$ is bipartite. Suppose by contradiction that $\tilde{H}$ has an odd circuit of length $2t+1$. This means that there is an odd number of components $H_{i_1}, \dots, H_{i_{2t+1}}$ in $H - f(F)$ such that, for all $j \in \mathbb{Z}_{2t+1}$ there is an edge $x_j y_{j+1} \in f(F)$ such that $x_j \in V(H_{i_j})$ and $y_{j+1} \in V(H_{i_{j+1}})$. Moreover, for all $j \in \mathbb{Z}_{2t+1}$ there is an $x_j y_j$-path T_j contained in the component H_{i_j}, i.e. such that $E(T_j) \cap f(F) = \emptyset$. Consider the circuit C induced by $x_j y_{j+1}$ and all edges of T_j for all $j \in \mathbb{Z}_{2t+1}$. Then $|E(C) \cap f(F)| = 2t+1$ and $f^{-1}(E(C))$ induces a 2-regular subgraph in G such that $|F \cap f^{-1}(E(C))| = 2t+1$, a contradiction.

Since $\tilde{H}$ is a bipartite graph, we can assume without loss of generality that there is an $s \in \{1, \ldots, m-1\}$ such that $f(F) = \partial_H(W)$, where $W = V(H_1) \cup \cdots \cup V(H_s)$. Note that $|W|$ is even since every component of $H - f(F)$ has even order. Thus, a perfect matching M of H is such that $|M \cap \partial_H(W)| = |M \cap f(F)|$ is even. But then $|f^{-1}(M) \cap F|$ is even as well, a contradiction. $\square$

Lemma 3.15. *Let $r \geq 3$, let G and H be two r-graphs, and let X be a subset of $V(H)$ such that $\partial_H(X)$ is a tight cut and $\chi'(H/X^c) = r$. If $H \prec G$, then $H/X \prec G$.*

Proof. Assume that f is an H-coloring of G. Label the edges of $\partial_H(X)$ as $e_1, \ldots, e_r$. Since $\chi'(H/X^c) = r$, the subset $E(H[X]) \cup \partial_H(X)$ of $E(H)$ can be partitioned into r pairwise disjoint matchings, denoted by $M_1, \ldots, M_r$, such that each edge of $\partial_H(X)$ is contained in exactly one of them. Without loss of generality, we may assume $e_i \in M_i$ for each $i \in \{1, \ldots, r\}$. Note that $E(G) = f^{-1}(E(H)) = f^{-1}(E(H[X^c])) \cup f^{-1}(M_1) \cup \ldots \cup f^{-1}(M_r)$. Moreover, for convenience, every edge and every vertex of H/X is labeled as in H. We define a mapping $f' \colon E(G) \to E(H/X)$ as follows. For every $e \in E(G)$, set

$$f'(e) = \begin{cases} f(e) & \text{if } e \in f^{-1}(E(H[X^c])); \\ e_i & \text{if } e \in f^{-1}(M_i), \text{ for } i \in \{1, \ldots, r\}. \end{cases}$$

To conclude the proof, we shall show that f' is an H/X-coloring of G. Let v be a vertex of $V(G)$. If $f(\partial_G(v)) = \partial_H(u)$ for some vertex $u \in X^c \subset V(H)$, then $f'(\partial_G(v)) = f(\partial_G(v)) = \partial_H(u) = \partial_{H/X}(u)$ by the definition of f'. If $f(\partial_G(v)) = \partial_H(u)$ for some vertex $u \in X$, then the image under f of each edge of $\partial_G(v)$ is contained in one of $M_1, \ldots, M_r$. Hence, the image under f' of each edge of $\partial_G(v)$ appears once in $\partial_{H/X}(w_X)$. This implies $f'(\partial_G(v)) = \partial_{H/X}(w_X)$. Thus, f' is an H/X-coloring of G. $\square$

For any graph G, the number of isolated vertices of G is denoted by $iso(G)$. A simple graph H is *regularizable* if we can obtain a regular graph from H by replacing each edge of H by a nonempty set of parallel edges. We need the following lemma, which follows from two results of [**?**] and [**?**]. The equivalence of the first two statements is shown in [**?**]; the equivalence of the first and the third statement is shown in [**?**].

Lemma 3.16. *Let G be a simple connected graph which is not bipartite with two partition sets of the same cardinality. The following statements are equivalent:*

- *$iso(G - S) < |S|$, for all $S \subseteq V(G)$.*
- *G is regularizable [**?**].*

- *for every $v \in V(G)$, both $G - v$ and G have a $\{K_{1,1}, C_m \colon m \geq 3\}$-factor [?].*

Lemma 3.17. *Let $r \geq 3$, let G and H be r-graphs, where H is connected, and let $S \subseteq V(G)$ such that $\partial_G(S)$ is a tight cut and $G[S]$ has no $\{K_{1,1}, C_m : m \geq 3\}$-factor. If G has an H-coloring $f \colon E(G) \to E(H)$ and $\partial_H(X) = f(\partial_G(S))$ for an $X \subseteq V(H)$, then H/X or H/X^c is a bipartite graph with two partition sets of the same cardinality.*

Proof. Suppose to the contrary that both H/X and H/X^c are not bipartite graphs with two partition sets of the same cardinality. By Lemma **??**, the edge-cut $\partial_H(X)$ is tight and so both H/X and H/X^c are r-regular. Thus, the underling graphs of H/X and H/X^c are both regularizable and hence, both $H/X - w_X$ and $H/X^c - w_{X^c}$ have a $\{K_{1,1}, C_m : m \geq 3\}$-factor, by Lemma **??**. Let H' be the union of these two factors. Note that H' is a $\{K_{1,1}, C_m : m \geq 3\}$-factor of H, which contains no edge of $\partial_H(X)$. Since $\partial_H(X) = f(\partial_G(S))$ and by Observation **??** (*iv*), G has a $\{K_{1,1}, C_m : m \geq 3\}$-factor, which contains no edge of $\partial_G(S)$. This is a contradiction to the assumption that $G[S]$ has no $\{K_{1,1}, C_m : m \geq 3\}$-factor. $\square$

Let G be an r-regular graph with a vertex $v \in V(G)$. A *Meredith extension* of G at v is the following operation. Delete the vertex v from G and add a copy K of the complete bipartite graph $K_{r,r-1}$. Finally add r edges between $V(G - v)$ and $V(K)$ such that the resulting graph is r-regular.

Lemma 3.18 (Rizzi [?]). *Let G be a graph and $X \subseteq V(G)$ with $|X|$ odd. If G/X and G/X^c are both r-graphs, then G is an r-graph.*

Theorem 3.19. *Let $r \geq 3$ and let $\mathcal{H}$ be a set of connected r-graphs such that every $H \in \mathcal{H}$ does not contain a non-trivial tight edge-cut $\partial_H(X)$ such that H/X or H/X^c is class 1. If every connected simple r-graph can be colored by an element of $\mathcal{H}$, then every connected r-graph can be colored by an element of $\mathcal{H}$.*

Proof. Let G be an arbitrary r-graph. By applying a Meredith extension on every vertex of G, we obtain a simple r-regular graph G^e. From the fact that both G and $K_{r,r}$ are r-graphs, we know that G^e is also an r-graph by Lemma **??**. Hence, there is $H \in \mathcal{H}$ such that $H \prec G^e$. Let f be an H-coloring of G^e. Note that for any induced subgraph G' of G^e isomorphic to $K_{r,r-1}$, the edge-cut $\partial_{G^e}(V(G'))$ is tight, and so $f(\partial_{G^e}(V(G')))$ is also tight in H by Lemma **??**. Let $X \subset V(H)$ such that $\partial_H(X) = f(\partial_{G^e}(V(G')))$. Since $K_{r,r-1}$ contains no $\{K_{1,1}, C_m \colon m \geq 3\}$-factor, Lemma **??** implies that H/X or H/X^c is a bipartite graph with two partition sets of the same cardinality. In particular, H/X or H/X^c is class 1, which implies that X or X^c is a single vertex by the choice of $\mathcal{H}$. Therefore, the edge-cut $\partial_{G^e}(V(G'))$ is mapped to a trivial edge-cut of H under f. Since G' was chosen arbitrarily, we conclude that G also has an H-coloring, which completes the proof. $\square$

We obtain the main result of this section as a corollary.

Corollary 3.20. *Let $r \geq 3$ and let $\mathcal{H}'_r$ be a set of connected r-graphs such that every connected simple r-graph can be colored by an element of $\mathcal{H}'_r$.*

i) If the Petersen Coloring Conjecture is false, then $\mathcal{H}'_3$ is an infinite set.

ii) If $r \geq 4$, then $\mathcal{H}'_r$ is an infinite set.

Proof. By Lemma **??** we can contract suitable subsets of vertices of graphs in $\mathcal{H}'_r$ to obtain a set $\mathcal{H}''_r$ of connected r-graphs with the following properties.

- Every connected simple r-graph can be colored by an element of $\mathcal{H}''_r$.
- For every $H \in \mathcal{H}''_r$, there is no non-trivial tight edge-cut $\partial_H(X)$ such that H/X or H/X^c is class 1.

Hence, by Theorem **??**, every connected r-graph can be colored by an element of $\mathcal{H}''_r$. Thus, $\mathcal{H}_r \subset \mathcal{H}''_r$ and hence, $\mathcal{H}''_r$ is an infinite set by Corollary **??**. By the construction of $\mathcal{H}''_r$ we have $|\mathcal{H}'_r| \geq |\mathcal{H}''_r|$, and hence, $\mathcal{H}'_r$ is also an infinite set. □

4 Concluding remarks

4.1 Quasi-ordered sets

Jaeger [**?**] initiated the study of the Petersen Coloring Conjecture in terms of partial ordered sets. DeVos, Nešetřil and Raspaud [**?**] studied cycle-continuous mappings and asked whether there is an infinite set $\mathcal{G}$ of bridgeless graphs such that every two of them are cycle-continuous incomparable, i.e. there is no cycle-continuous map between any two graphs in $\mathcal{G}$. Šámal [**?**] gave an affirmative answer to the above question by constructing such an infinite set $\mathcal{G}$ of bridgeless cubic graphs. In fact, he also mentioned that this result can be considered in view of a quasi-order induced by cycle-continuous mappings on the set of bridgeless cubic graphs. That is, this quasi-ordered set contains infinite antichains.

For every integer $r \geq 1$, H-colorings of r-graphs induce a quasi-order on the set of r-graphs. Then, our result on r-graphs can be restated as follows: for any $r \geq 4$, there is an infinite set $\mathcal{H}_r$ of r-graphs such that each of them is incomparable to any other r-graph, and such infinite set exists for $r = 3$ if the Petersen Coloring Conjecture is false. In particular, the set $\mathcal{H}_r$ is an infinite antichain.

4.2 Open problems

The edge connectivity of an r-graph is equal to r or it is an even number. We have shown that $\mathcal{T}(r, r-2) \cup \mathcal{T}(r, 1) \subseteq \mathcal{H}_r$. Thus, for $r \neq 5$, for each possible edge-connectivity t there is a

t-edge-connected r-graph in $\mathcal{H}_r$. For $r = 5$, we do not know any 5-edge-connected 5-graph with this property, see [**?**] for a discussion of this topic. However, we know only a finite number of t-edge-connected r-graphs of $\mathcal{H}_r$ if $t \geq 3$.

Problem 4.1. *For $r, t \geq 3$, does $\mathcal{H}_r$ contain infinitely many t-edge-connected r-graphs?*

It is also not clear whether $\mathcal{H}_r$ contains elements of $\mathcal{T}(r, k)$ for $k \in \{2, \ldots, r-3\}$. So far, these sets are not determined for $k \in \{1, \ldots, r-3\}$. Indeed, we even do not know the order of their elements. Let $o(r, k)$ be the order of the graphs of $\mathcal{T}(r, k)$.

Problem 4.2. *For all $r \geq 3$ and $k \in \{1, \ldots, r-2\}$: Determine $o(r, k)$.*

By our results, $o(r, r-2) = 10$. By results of Rizzi [**?**], $o(r, 1) \leq 2 \times 5^{r-2}$. We conjecture the following to be true.

Conjecture 4.3. *For all $r \geq 3$ and $k \in \{2, \ldots, r-2\}$: $o(r, k-1) \geq o(r, k)$.*

If Conjecture **??** would be true, then it would follow with Corollary **??** that $\mathcal{T}(r, k) \subset \mathcal{H}_r$ for each $k \in \{1, \ldots, r-2\}$.

Similar problems arise for simple r-graphs. Let $o_s(r, k)$ be the smallest order of a simple r-graph G with $\pi(G) = k$. Small simple r-graphs of class 2 can be obtained as follows. Consider a perfect matching M of P and the graph $G = P + (r-3)M$. Let H be a simple r-graph of smallest order and $v \in V(H)$. Then, H is class 1 and $|V(H)| = r+1$ if r is odd and $|V(H)| = r+2$ if r is even. Now, replace appropriately five vertices of G by $H - v$ to obtain a simple r-graph G'. Since H is class 1 and $\pi(G) = r-2$, we have $\pi(G') = r-2$. Therefore, if r is odd, then $o_s(r, r-2) \leq 5(r+1)$ and if r is even, then $o_s(r, r-2) \leq 5(r+2)$. Furthermore, bounds for $o_s(r, k)$ can be obtained by using Meredith extensions, since if G' is a Meredith extension of an r-graph G, then $\pi(G') = \pi(G)$.

References

[1] L. Babai. Automorphism groups, isomorphism, reconstruction. In *Handbook of Combinatorics, Vol. 1, 2*, pages 1447–1540. Elsevier Sci. B. V., Amsterdam, 1995.

[2] C. Berge and M. Las Vergnas. On the existence of subgraphs with degree constrains. *Nederl. Akad. Wetensch. Indag. Math.*, 40:165–176, 1978.

[3] M. DeVos, J. Nešetřil, and A. Raspaud. On edge-maps whose inverse preserves flows or tensions. In *Graph theory in Paris*, Trends Math., pages 109–138. Birkhäuser, Basel, 2007.

[4] D. R. Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Mathematical programming*, 1(1):168–194, 1971.

[5] S. Grünewald and E. Steffen. Chromatic-index-critical graphs of even order. *J. Graph Theory*, 30:27–36, 1999.

[6] J. Hägglund and E. Steffen. Petersen-colorings and some families of snarks. *Ars Math. Contemp.*, 7(1):161–173, 2014.

[7] F. Jaeger. On graphic-minimal spaces. *Ann. Discrete Math.*, 8:123–126, 1980.

[8] F. Jaeger. On five-edge-colorings of cubic graphs and nowhere-zero flow problems. *Ars Combin.*, 20(B):229–244, 1985.

[9] F. Jaeger. Nowhere-zero flow problems. In *Selected topics in graph theory, 3*, pages 71–95. Academic Press, San Diego, CA, 1988.

[10] L. Jin. *Covers and Cores of r-graphs*. PhD thesis, Faculty of Computer Science, Electrical Engineering and Mathematics, Paderborn University, 2017.

[11] L. Jin and Y. Kang. Partially normal 5-edge-colorings of cubic graphs. *European J. Combin.*, 95:Paper No. 103327, 14, 2021.

[12] Y. Ma, D. Mattiolo, E. Steffen, and I. H. Wolf. Pairwise disjoint perfect matchings in r-edge-connected r-regular graphs. *available at* `https://arxiv.org/abs/2206.10975`, 2022.

[13] J. Matoušek and J. Nešetřil. *Invitation to Discrete Mathematics*. Oxford University Press, Oxford, second edition, 2009.

[14] G. Mazzuoccolo and V. V. Mkrtchyan. Normal edge-colorings of cubic graphs. *J. Graph Theory*, 94(1):75–91, 2020.

[15] G. Mazzuoccolo, G. Tabarelli, and J. P. Zerafa. On the existence of graphs which can colour every regular graph. *arXiv at* `https://arxiv.org/abs/2110.13684`, 2021.

[16] V. V. Mkrtchyan. A remark on the Petersen coloring conjecture of Jaeger. *Australas. J. Combin.*, 56:145–151, 2013.

[17] F. Pirot, J.-S. Sereni, and R. Škrekovski. Variations on the Petersen colouring conjecture. *Electron. J. Combin.*, 27(1):Paper No. 1.8, 14, 2020.

[18] W. R. Pulleyblank. Minimum node covers and 2-bicritical graphs. *Math. Programming*, 17(1):91–103, 1979.

[19] R. Rizzi. Indecomposable r-graphs and some other counterexamples. *J. Graph Theory*, 32(1):1–15, 1999.

[20] P. D. Seymour. On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc. (3)*, 38(3):423–460, 1979.

[21] R. Šámal. Cycle-continuous mappings—order structure. *J. Graph Theory*, 85(1):56–73, 2017.

[22] C.-Q. Zhang. *Integer flows and cycle covers of graphs*, volume 205 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1997.