Membership and Conjugacy in Inverse Semigroups

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Abstract

The membership problem for an algebraic structure asks whether a given element is contained in some substructure, which is usually given by generators. In this work we study the membership problem, as well as the conjugacy problem, for finite inverse semigroups. The closely related membership problem for finite semigroups has been shown to be PSPACE-complete in the transformation model by Kozen (1977) and NL-complete in the Cayley table model by Jones, Lien, and Laaser (1976). In the partial bijection model, the membership and the conjugacy problem for finite inverse semigroups were shown to be PSPACE-complete by Birget and Margolis (2008) and by Jack (2023).

Here we present a more detailed analysis of the complexity of the membership and conjugacy problems parametrized by varieties of finite inverse semigroups. We establish dichotomy theorems for the partial bijection model and for the Cayley table model. In the partial bijection model these problems are in NC (resp. NP for conjugacy) for strict inverse semigroups and PSPACE-complete otherwise. In the Cayley table model we obtain general L-algorithms as well as NPOLYLOGTIME upper bounds for Clifford semigroups and L-completeness otherwise.

Furthermore, by applying our findings, we show the following: the intersection non-emptiness problem for inverse automata is PSPACE-complete even for automata with only two states; the subpower membership problem is in NC for every strict inverse semigroup and PSPACE-complete otherwise; the minimum generating set and the equation satisfiability problems are in NP for varieties of finite strict inverse semigroups and PSPACE-complete otherwise.

2012 ACM Subject Classification Theory of computation → Algebraic language theory; Theory of computation → Problems, reductions and completeness; Theory of computation → Circuit complexity

Keywords and phrases inverse semigroups, membership, conjugacy, finite automata

Funding *Armin Weiß*: Supported by the German Research Foundation (DFG) grant WE 6835/1-2.

Contents

1 Introduction

In this work, we study the membership problem for inverse semigroups and some related problems such as the conjugacy problem. The *membership problem* for semigroups in the *transformation model* has first been studied by Kozen [62] in 1977. It receives as input a list $(u_1, \ldots, u_k)$ of functions $u_i \colon \Omega \to \Omega$ for some finite set Ω and a target function $t \colon \Omega \to \Omega$; the question is whether t can be written as composition of the u_i or, with other words, whether t is contained in the subsemigroup generated by $\{u_1, \ldots, u_k\}$. It is closely related to the DFA intersection non-emptiness problem, which receives as input a list of deterministic finite automata (DFAs) and asks whether there is a word accepted by *all* of the automata. Indeed, Kozen [62] showed that both problems are PSPACE-complete.

Inverse semigroups have been first studied by Wagner [104] and Preston [86] to describe partial symmetries. They constitute the arguably most natural class of algebraic structures containing groups and being contained in the semigroups. An *inverse semigroup* is a semigroup equipped with an additional unary operation $x \mapsto \overline{x}$ such that $x\overline{x}x = x$ and $\overline{x}x\overline{x} = \overline{x}$ for all x and $\overline{x}$ is unique with that property. This clearly generalizes the inverse operation in groups. Similar to groups being an algebraic abstraction of symmetries and semigroups being an algebraic abstraction of computation, inverse semigroups abstract *symmetric computation*. This notion of computation, where every computational step is invertible, was introduced by Lewis and Papadimitriou [66] in order to better describe the complexity of the accessibility problem in undirected graphs UGAP, which only much later was shown to be in L (deterministic logspace) by Reingold [89].

In the setting of inverse semigroups, it is natural to consider the *partial bijection model* for the membership problem, where the u_i and t are *partial* functions which are injective on their domain. The membership problem for inverse semigroups in the partial bijection model is also PSPACE-complete – and, thus, as difficult as for arbitrary semigroups – as observed by Birget and Margolis [17] and independently by Jack [51]. This observation is based on a result by Birget, Margolis, Meakin, and Weil [16] showing that the intersection non-emptiness problem for *inverse automata* is PSPACE-complete. Roughly speaking, an inverse automaton is a DFA with a partially defined transition function where every letter induces a partial bijection on the set of states and the action of each letter can be "inverted" by a sequence of letters. Thus, inverse automata can be seen as a generalization of permutation automata, for which the intersection non-emptiness problem is NP-complete [18]. Interestingly, the corresponding membership problem for permutation groups is even in NC as shown by Babai, Luks, and Seress [7] (for a series of preliminary results, see Section 1.3).

A different variant of the membership problem has been introduced by Jones, Lien, and Laaser [53] in 1976: for the membership problem in the *Cayley table model* the ambient semigroup is given as its full multiplication table (a.k.a. Cayley table), i.e., instead of the finite set Ω as above, the input includes a multiplication table of a semigroup and the elements are given as indices to rows/columns of that multiplication table. Clearly, this is a much less compressed form than the membership problem in the transformation or partial bijection model and, indeed, the membership problem in the Cayley table model is NL-complete [53].

Like in the transformation model, the case of groups appears to be easier than the general case. Indeed, in 1991, Barrington and McKenzie [10] observed that the membership problem for groups in the Cayley table model (which they denote by "GEN(groups)" and we by $\textsc{memb}_{\mathbf{CT}}(\mathbf{G})$) can be solved in L with an oracle for UGAP and speculated about it potentially being L^{UGAP}-complete. Indeed, they posed the following question.

> *Does* GEN(groups) *belong to* L*? We doubt that this is the case: we believe rather that* GEN(groups) *is complete for the* NC^1*-closure of* UGAP*, though we do not yet see how to apply the techniques in Cook and McKenzie (1987) to prove that* GEN(groups) *is even* L*-hard.*

This conjecture has been refuted by the first author of the present article [35, 36] by showing that $\textsc{memb}_{\mathbf{CT}}(\mathbf{G})$ can be solved in NPOLYLOGTIME (at least if we read completeness with respect to AC^0-reductions; when using NC^1-reductions, the results in [35, 36] give only a strong indication that the conjecture does not hold). Yet, in this work, we establish that the conjecture actually holds if we replace groups with inverse semigroups.

To find out in which cases the membership and conjugacy problem are easy and in which cases they are difficult, we study these problems restricted to certain varieties of finite inverse semigroups. A *variety of finite (inverse) semigroups*, frequently called a *pseudovariety*, is a class of (inverse) semigroups that is closed under finite direct products, (inverse) subsemigroups, and quotients. Important varieties of finite (inverse) semigroups are groups, semilattices, or aperiodic (inverse) semigroups. Varieties of finite semigroups are closely linked to varieties of formal languages (i.e., classes of languages enjoying natural closure properties) by Eilenberg's Correspondence Theorem [31].

Beaudry, McKenzie, and Thérien [14] classified the varieties of finite aperiodic monoids in terms of the complexity of their membership problem. They found the following five classes: AC^0, P-complete, NP-complete, NP-hard, and PSPACE-complete. Note that it might seem like a negligible difference whether semigroups or monoids are considered; however, the landscape of varieties of finite semigroup is much richer than the varieties of finite monoids. The aim of this work is to provide a similar classification for inverse semigroups.

1.1 Our Results

We consider the membership and conjugacy problems for inverse semigroups parametrized by a variety $\mathbf{V}$. For both problems we are given *inverse* semigroups $U \leqslant S$ where U is given by generators and $U \in \mathbf{V}$. Given an element $t \in S$, the membership problem asks whether $t \in U$. Given elements $s, t \in S$, the conjugacy problem asks whether $\bar{u}su = t$ and $s = ut\bar{u}$ for some $u \in U \cup \{1\}$. Both problems are examined with respect to two models of input – the Cayley table model and the partial bijection model. We write $\textsc{memb}_{\mathbf{CT}}(\mathbf{V})$, $\textsc{conj}_{\mathbf{CT}}(\mathbf{V})$, $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$, and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$, accordingly. Regarding further details on the definition we refer to Section 2.4.

Our main result regarding the Cayley table model is the following dichotomy. Herein $\mathbf{Cl}$ denotes the variety of finite Clifford semigroups, which is the smallest variety containing all finite groups and semilattices. The class NPOLYLOGTIME comprises all problems solvable by non-deterministic random access Turing machines in time $\log^{\mathcal{O}(1)} n$; see Section 2.5.

▶ **Theorem A** (Cayley Table Model)**.** *Let* $\mathbf{V}$ *be a variety of finite inverse semigroups.*
- *If* $\mathbf{V} \subseteq \mathbf{Cl}$*, then* $\textsc{memb}_{\mathbf{CT}}(\mathbf{V})$ *and* $\textsc{conj}_{\mathbf{CT}}(\mathbf{V})$ *are in* NPOLYLOGTIME *and in* L.
- *If* $\mathbf{V} \not\subseteq \mathbf{Cl}$*, then* $\textsc{memb}_{\mathbf{CT}}(\mathbf{V})$ *and* $\textsc{conj}_{\mathbf{CT}}(\mathbf{V})$ *are* L*-complete.*

In particular, both problems are in NPOLYLOGTIME $\subseteq \mathsf{qAC}^0$ if and only if $\mathbf{V} \subseteq \mathbf{Cl}$ as the class NPOLYLOGTIME contains no problem that is hard for L (with respect to AC^0 reductions). Furthermore, Theorem A establishes Barrington and McKenzie's conjecture [10] on $\mathsf{L}^{\textsc{ugap}}$-completeness[1] of the membership problem – however, for the larger class of inverse

[1] Recall that $\mathsf{L}^{\textsc{ugap}} = \mathsf{L}$ by Reingolds seminal result [89].

semigroups or, more specifically, any variety of finite inverse semigroups not contained in $\mathbf{Cl}$.

The condition in Theorem A can be equivalently formulated using the following fact. A variety of finite inverse semigroups is contained in $\mathbf{Cl}$ if and only if it does not contain the combinatorial Brandt semigroup B_2. The latter consists of elements $\{s, \bar{s}, s\bar{s}, \bar{s}s, 0\}$ where $s^2 = \bar{s}^2 = 0$ and all of the other products are as one would expect. Hence, by Theorem A, the problems $\textsc{memb}_{\mathbf{CT}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{CT}}(\mathbf{V})$ are L-complete if and only if $B_2 \in \mathbf{V}$.

We now turn to the partial bijection model. This input model is similar to the transformation model for semigroups considered above – however, the generators and the target elements are partial maps that need to be injective on their domain. Our main result for the partial bijection model is the following dichotomy.

▶ **Theorem B** (Partial Bijection Model)**.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups.*

- *If $\mathbf{V} \subseteq \mathbf{SIS}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ is in NC and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ is in NP.*
- *If $\mathbf{V} \not\subseteq \mathbf{SIS}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ are PSPACE-complete.*

Herein $\mathbf{SIS}$ denotes the variety of *strict inverse semigroups*, which is the smallest variety containing all groups and the combinatorial Brandt semigroup B_2; it contains $\mathbf{Cl}$, in particular, all semilattices (denoted as $\mathbf{Sl}$), and the variety generated by B_2 (denoted as $\mathbf{BS}$). As such, B_2 no longer serves as a key obstruction to an easy membership problem (as was the case in the Cayley table model). This rôle is now played by the combinatorial Brandt monoid B_2^1. Indeed, a variety of finite inverse semigroups $\mathbf{V}$ is contained in $\mathbf{SIS}$ if and only if $B_2^1 \notin \mathbf{V}$.

The case $\mathbf{V} \subseteq \mathbf{SIS}$ in Theorem B can be further refined as follows.

- If $\mathbf{V} \subseteq \mathbf{Sl}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ are in AC^0 (see [14]).
- If $\mathbf{V} = \mathbf{BS}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ are L-complete.
- If $\mathbf{V} \not\subseteq \mathbf{BS}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ is in NC and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ is in NP; both are hard for L.

Note that here AC^0 and NC refer to uniform circuit classes and, as such, the three levels of complexity $\mathsf{AC}^0 \subsetneq \mathsf{NC} \subsetneq \mathsf{PSPACE}$ are separated unconditionally. Therefore, in particular, each of the problems $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ is in AC^0 if and only if $\mathbf{V} \subseteq \mathbf{Sl}$, and the problem $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ is in NC if and only if $\mathbf{V} \subseteq \mathbf{SIS}$.

For $\mathbf{V} \subseteq \mathbf{SIS}$ we reduce the problems $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ to the corresponding problems for the variety of finite groups, matching their complexity. We build on the celebrated result of Babai, Luks, and Seress [7] which states that the membership problem for permutation groups is in NC, as well as the observation that, due to groups admitting polylogarithmic SLPs, the corresponding conjugacy problem is in NP.

As outlined above, membership problems are deeply intertwined with intersection non-emptiness problems for the corresponding classes of automata. In 2016, Bulatov, Kozik, Mayr, and Steindl [19] proved that the intersection non-emptiness problem for DFAs remains PSPACE-complete even if the input automata are restricted to at most three states exactly one of which is accepting. This corresponds to semigroups in the transformation model. Here we obtain a similar result for inverse automata, which corresponds to inverse semigroups in the partial bijection model, using the same reduction as for the hardness part of Theorem B.

▶ **Corollary C.** *The intersection non-emptiness problem for inverse automata is* PSPACE*-complete. This holds even if the automata have only two states, one of which is accepting.*

The reason that two states suffice to show PSPACE-hardness is grounded in the fact that inverse automata have partially defined transition functions, whereas the above-mentioned result concerns automata with total transition functions.

In the same work Bulatov, Kozik, Mayr, and Steindl also considered the *subpower membership problem* and showed that it is PSPACE-complete for arbitrary semigroups. Let S be a fixed (inverse) semigroup. The input for the subpower membership problem for S consists of a number m, elements $u_1, \ldots, u_k \in S^m$, and $t \in S^m$. The question is whether t is contained in the (inverse) subsemigroup generated by $\{u_1, \ldots, u_k\}$. As a consequence of Theorem B and Corollary C, we obtain the following dichotomy for the complexity of the subpower membership problem for inverse semigroups.

▶ **Corollary D.** *The subpower membership problem for an inverse semigroup S is in* NC *if and only if $S \in \mathbf{SIS}$. Otherwise, it is* PSPACE*-complete.*

Finally, we apply our results to the problems of determining the minimal size of a generating set (MGS) and deciding satisfiability of an equation (EQN) and obtain a similar dichotomy as above. The minimum generating set problems receives as input an inverse semigroup S and an integer k and asks whether S can be generated by at most k elements. For EQN the input is an inverse semigroup S and a single equation, and the question is whether there is a satisfying assignment of the variables to elements of S.

▶ **Corollary E.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups.*
- *If $\mathbf{V} \subseteq \mathbf{SIS}$, then $\mathrm{MGS}_{\mathbf{PB}}(\mathbf{V})$ and $\mathrm{EQN}_{\mathbf{PB}}(\mathbf{V})$ are in* NP*.*
- *If $\mathbf{V} \not\subseteq \mathbf{SIS}$, then $\mathrm{MGS}_{\mathbf{PB}}(\mathbf{V})$ and $\mathrm{EQN}_{\mathbf{PB}}(\mathbf{V})$ are* PSPACE*-complete.*

1.2 Technical Overview

A central role in our results is played by the combinatorial Brandt semigroup B_2 (defined above) and the Brandt monoid B_2^1, which is the Brandt semigroup with an adjoined identity. The former and the latter are the sole obstruction to inclusion in the variety $\mathbf{Cl}$ and $\mathbf{SIS}$, respectively. As such, both inverse semigroups are crucial obstructions preventing the membership and conjugacy problems from being "easy": for example in the Cayley table model, the Brandt semigroup B_2 is the obstruction from $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ being in NPOLYLOGTIME; in the partial bijection model the Brandt monoid B_2^1 makes the problem PSPACE-hard.

Outline of the Proof of Theorem A. The proof of Theorem A consists of three main steps: first, show that for Clifford semigroups both problems can be solved in NPOLYLOGTIME, second, show that in any case they can be solved in L, and finally, show that, if the variety under consideration contains the Brandt semigroup B_2, then the problems are hard for L.

For the first point there is not much left to do. Indeed, the first author [35, 36] showed that $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{Cl})$ is in NPOLYLOGTIME. Thus, it merely remains to apply this result also to the conjugacy problem. Yet, for the sake of completeness and because the proof allows us to describe some interesting consequences, we give a proof in Section 4.1. The crucial idea is to use compression via straight-line programs (SLPs) and the Reachability Lemma due to Babai and Szemerédi [8]: if G is a finite group and $g \in G$, then there is an SLP of length $\mathcal{O}(\log^2 |G|)$ that computes g. We also say that groups *admit polylogarithmic SLPs*. One can guess such an SLP and verify whether it actually computes g in NPOLYLOGTIME. In [35, 36] the first author extended the Reachability Lemma to Clifford semigroups.

Conversely and as a consequence of both parts of Theorem A we also obtain a complete characterization of when a variety of finite inverse semigroups $\mathbf{V}$ admits polylogarithmic SLPs, namely this is the case if and only if $\mathbf{V} \subseteq \mathbf{Cl}$ (see Corollary 49).

The proof that $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ and $\mathrm{CONJ}_{\mathbf{CT}}(\mathbf{V})$ are hard for L if $\mathbf{V} \not\subseteq \mathbf{Cl}$ follows via a reduction from undirected graph accessibility (UGAP), which is intimately related to the membership problem for the combinatorial Brandt semigroups B_n; for details, see Section 6.2.

Finally, to prove that $\textsc{memb}_{\mathbf{CT}}$ and $\textsc{conj}_{\mathbf{CT}}$ are in L, the crucial observation is that the strongly connected components of the Cayley graph of an inverse semigroup are actually undirected graphs. This allows to reduce the problem to $\textsc{ugap}$, which by Reingold's result [89] is in L. While for the conjugacy problem this yields a direct many-one reduction to $\textsc{ugap}$, for the membership problem we apply a greedy algorithm using oracle calls for $\textsc{ugap}$ in the associated Cayley graph. To be more precise, to decide whether $t \in U$, we start initializing a variable x to the neutral element, which we assume to exist and to be contained in U. We update x iteratively while maintaining the invariants that $x \in U$ and $x\overline{x}t = t$ (i.e., x is greater than or equal to t with respect to Green's relation $\mathcal{R}$). Using the $\textsc{ugap}$ oracle, in each iteration of the algorithm we greedily pick an element x_{new} to replace x that satisfies the invariants and is adjacent to the strongly connected component of x in the associated Cayley graph (so that x is strictly greater than x_{new} with respect to $\mathcal{R}$). While the invariants guarantee that, at any point, we still might multiply x on the right by another element of U to reach t, the way we choose x_{new} ensures that we actually make progress towards t.

Outline of the Proof of Theorem B. Our approach to the dichotomy result for the partial bijection model, i.e., to proving Theorem B, is similar to the above. In the group case, we build on the celebrated result of Babai, Luks, and Seress [7] which states that the membership problem for permutation groups is in NC, as well as the observation that, due to groups admitting polylogarithmic SLPs, the corresponding conjugacy problem is in NP.

Extending these bounds to Clifford semigroups is rather straight-forward: one can identify an appropriate subgroup (in fact, an $\mathcal{H}$-class[2]) to which the problem can be reduced to; for details, see Section 4.2. Interestingly, a similar reduction also works for strict inverse semigroups. However, the proof is much more involved as, in that case, identifying an appropriate subgroup is no longer possible in AC^0 but hard for L due to the presence of Brandt semigroups. We show that a L-reduction is nonetheless possible building on some special properties of representations of strict inverse semigroups which we now briefly describe.

Suppose that U is a strict inverse semigroup generated by a set Σ of partial bijections on some set Ω. We say that an element $u \in U$ is Δ-large for some U-invariant subset $\Delta \subseteq \Omega$ if its domain $\operatorname{dom}(u)$ or, equivalently, its range $\operatorname{ran}(u)$ intersects every U-orbit $x^U \subseteq \Delta$. Consider the graph $\mathrm{M}(\Delta; \Sigma)$ which, for each Δ-large $u \in \Sigma$, possesses an edge labeled u from a vertex associated with the set $\Delta \cap \operatorname{dom}(u) \subseteq \Omega$ to a vertex associated with the set $\Delta \cap \operatorname{ran}(u) \subseteq \Omega$. This graph, which we call the *Munn graph* $\mathrm{M}(\Delta; \Sigma)$ at Δ and with respect to Σ, is the basis of our reduction. As it turns out, every $\mathcal{D}$-class of the strict inverse semigroup U is generated (as a groupoid) by a connected component of the Munn graph $\mathrm{M}(\Delta; \Sigma)$ at some suitably chosen U-invariant set $\Delta \subseteq \Sigma$. Moreover, crucially, we can identify the set Δ suitable for the $\mathcal{D}$-class of U that contains a given element $t \in U$. This allows us to first reduce the membership problem for U to some $\mathcal{D}$-class of U and, ultimately, to some $\mathcal{H}$-class of U.

We refer the reader to Section 5.1 for details on the properties of Munn graphs and to Section 5.2 for details of the reduction outlined above as well as a corresponding reduction for the conjugacy problem based on the same ideas. We crucially rely on the graph accessibility problem ($\textsc{ugap}$) for the Munn graph. Even more, all relevant computations can be preformed in $\mathsf{L}^{\textsc{ugap}}$ and are, in fact, even easy to implement if one replaces oracle calls to $\textsc{ugap}$ with standard algorithms for this problem. On the other hand, using $\textsc{ugap} \in \mathsf{L}$ [89] yields $\textsc{memb}_{\mathbf{PB}}(\mathbf{SIS}) \leqslant_m^{\mathsf{L}} \textsc{memb}_{\mathbf{PB}}(\mathbf{G})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{SIS}) \leqslant_m^{\mathsf{L}} \textsc{conj}_{\mathbf{PB}}(\mathbf{G})$.

Finally, let us attend to the general case of the membership and conjugacy problems for

[2] Here $\mathcal{H}$ and $\mathcal{D}$ refer to Green's relations; for a definition, see Section 2.2.

inverse semigroups in the partial bijection model. On the one hand, it is well-known that both problems can be solved in PSPACE. On the other hand, both problems are PSPACE-hard in general [17, 51]. Here we show that the (idempotent) membership and conjugacy problems are PSPACE-hard for any variety of finite inverse semigroups $\mathbf{V}$ containing the combinatorial Brandt monoid B_2^1 or, equivalently, $\mathbf{V} \not\subseteq \mathbf{SIS}$. We do so via reduction from NCL, the configuration-to-configuration problem variant of non-deterministic constraint logic (NCL). This problem, introduced and shown to be PSPACE-complete by Hearn and Demaine [46], asks whether two given configurations of an NCL machine can be transformed into one another. Crucially, in this problem, configurations and transitions between these can be specified locally. We encode each local aspect of a problem instance into a (small) combinatorial Brandt monoid B_n^1 and the entire instance into the Cartesian product of all these B_n^1; for details, we refer the reader to Section 7.1. Here we use of the fact that B_n^1 divides the n-fold Cartesian power $(B_2^1)^n$ and, thus, $B_n^1 \in \mathbf{V}$ whenever $B_2^1 \in \mathbf{V}$.

The hardness proof for Corollary C closely follows the hardness proof of Theorem B, but using Cartesian powers of B_2^1 directly. Then we obtain Corollary D, the PSPACE-hardness of the subpower membership problem, as a corollary of Corollary C by simply replacing the combinatorial Brandt monoid B_2^1 with any inverse semigroup S divided by it.

Details on Further Results. Our results on the minimum generating set problem and on deciding satisfiability of equations in Corollary E are rather direct applications of our main results. In both cases, the upper bounds are established via an algorithm that guesses a witness or solution, which is then verified in polynomial time using access to an oracle for the membership problem.

The lower bounds, i.e., PSPACE-hardness, are obtained via reductions from (suitably restricted variants) of the membership and conjugacy problems. In the case of the minimum generating set problem, we reduce from (such a variant) $\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{V})$ to $\textsc{mgs}_{\mathbf{PB}}(\mathbf{V} \vee \mathbf{Sl})$ using the following idea. Given an instance $\Sigma \subseteq \mathcal{I}(\Omega)$ and $t \in \mathcal{I}(\Omega)$ of the former problem, we would like that the inverse subsemigroup $\langle \Sigma \cup \{t\} \rangle$ is generated by $|\Sigma|$ elements if and only if $t \in \langle \Sigma \rangle$. However, this is clearly not the case as Σ might contain redundant generators. Nevertheless, by adding extra elements to Ω for each element of Σ on which the generators behave as a semilattice, we can ensure that every single generator of Σ is needed; thus, in this modified instance the above wishful thinking actually applies.

To see PSPACE-hardness of deciding satisfiability of equations, observe first that conjugacy is represented by a system of equations. As such, the problem $\textsc{eqn}^{*}_{\mathbf{PB}}(\mathbf{V})$ of deciding whether a system of equations has a solution is PSPACE-hard whenever $\mathbf{V} \not\subseteq \mathbf{SIS}$ by Theorem B. Using a suitably restricted variant E-$\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{V})$ of the conjugacy problem, were a single equation suffices to represent conjugacy, we obtain that the corresponding problem $\textsc{eqn}_{\mathbf{PB}}(\mathbf{V})$ of deciding whether a single equation has a solution is also PSPACE-hard whenever $\mathbf{V} \not\subseteq \mathbf{SIS}$.

1.3 Related Work

Inverse semigroups have been introduced by Wagner [104] and Preston [86] to formalize partial symmetries. Implicitly, they had been studied even before for example in the context of so-called *pseudogroups* [40]. Inverse semigroup have been investigated extensively from geometric, combinatorial, and algorithmic viewpoints; see e.g. [29, 32, 42, 56, 58, 72, 75, 77, 80] for a rather random selection. For additional background on inverse semigroups, we refer to the standard books [65, 84] and the many references therein.

While membership problems in infinite semigroups recently have also gained a lot of attention (see [15, 21, 28, 30] for a few examples), let us in the following give an overview on related work on the membership problem with the input models we use in the present work.

Membership Problem (Cayley table model). The membership problem has been studied for many algebraic structures. Indeed [52] shows that the membership problem for magmas (i.e., having a binary operation with no additional axioms) is P-complete in the Cayley table model. In contrast, by [23] for quasigroups (magma with "inverses", a.k.a. latin squares) it is in NPOLYLOGTIME using similar techniques as we apply for the first part of Theorem A.

The membership problems for semigroups in the Cayley table model has been introduced by Jones, Lien, and Laaser [53]. Further studies by Barrington, Kadau, and Lange [9] showed that it can be solved in FOLL for nilpotent groups of constant class. This result has been further improved by Collins, Grochow, Levet, and the last author [23] showing that the problem can be solved in FOLL for all nilpotent groups (i.e., of arbitrary class) and served as a catalyst for the first author's work [35, 36] giving NPOLYLOGTIME algorithms for Clifford semigroups, on which we build in the present work.

Membership Problem (Transformation/Partial Bijection Model). The membership problem for semigroups in the transformation model has been shown to be PSPACE-complete by Kozen [62]. Beaudry [12] showed that the membership problem in commutative semigroups is NP-complete in the transformation model. This was later extended in [13, 14] to classify the complexity of the membership problem in aperiodic monoids as outlined above.

Based on Sims' work [95], Furst, Hopcroft, and Luks [37] showed that the membership problem for permutation groups is solvable in polynomial time, which after several partial results [69, 71, 74] was improved to NC by Babai, Luks, and Seress [7]. Interestingly, the problem of rational subset membership is NP-complete due to Luks [70] (see also [67]).

Turning our attention to the partial bijection model, it was observed by Birget and Margolis [17] and, recently, by Jack [51] that the membership problem for *inverse semigroups* given by partial bijections is PSPACE-complete. This follows from an earlier result by by Birget, Margolis, Meakin, and Weil [16] showing that the intersection non-emptiness problem for *inverse automata* is PSPACE-complete. Indeed, this problem remains PSPACE-complete over a two-letter alphabet [17].

Subpower Membership Problem. While there is no obvious generalization of the partial bijection or transformation semigroup model to non-associative structure such as magmas, the subpower membership problem still can be posed in this case. Indeed, the subpower membership problem initially has been studied within the context of universal algebra, see e.g. [20, 61, 73], and has turned out to be EXPTIME-complete [63] in general. For arbitrary semigroups the subpower membership problem has been shown to be PSPACE-complete by Bulatov, Kozik, Mayr, and Steindl [19].

Further results on the subpower membership problem in semigroups are due to Steindl giving a P vs. NP-completeness dichotomy for the special case of bands [97] and a P vs. NP-complete vs. PSPACE-complete trichotomy for combinatorial Rees matrix semigroups with adjoined identity [98]. Here it is interesting to note that by our results the NP-completeness case does not exists for inverse semigroups.

We now turn our attention to the intimately related intersection non-emptiness problem.

Intersection Non-Emptiness Problem. The DFA intersection non-emptiness problem has been introduced and shown to be PSPACE-complete by Kozen [62]. Further work studying the complexity (including parametrized and fine-grained complexity) of the DFA intersection non-emptiness problem can be found in [5, 26, 33, 47, 54, 64, 100, 105]. Two special cases are that the DFA intersection non-emptiness problem is NP-complete for DFA accepting finite languages [88] and for DFA over a unary alphabet [99] (see also [34]).

Another important special case are permutation automata [102] (a.k.a. group DFAs). This is closely linked to the membership problem in groups, which is in NC [7]. Thus, it comes rather as a surprise that the intersection non-emptiness problem is NP-complete as Brondin, Krebs, and McKenzie [18] showed; however, when restricting to permutation automata with a single accepting state it, indeed, is in NC [18]. Even more, intersection non-emptiness for permutation automata plus one context-free language is PSPACE-complete [67].

Note that every permutation automaton is an inverse automaton as studied in the present work and e.g. by Birget, Margolis, Meakin, and Weil [16]. Furthermore, inverse automata are a special case of *reversible* automata (or injective automata as they are called in [16]), which were studied e.g. by Pin [85] and Radionova and Okhotin [87].

Another problem related to membership is the conjugacy problem, which for infinite groups was introduced by Dehn in 1911 [27]. For generalizations to semigroups see [81].

Conjugacy Problem. The conjugacy problem for permutation groups is in NP and hard for graph isomorphism as shown by Luks [70]. Jack [51] showed that the conjugacy problem for inverse semigroups in the partial bijection model is PSPACE-complete. For a overview on different variants of conjugacy in (inverse) semigroups, we refer to [3].

The following problems, which are closely tied to the membership and conjugacy problems (see, e.g., Lemma 62 and Observation 66), have also attracted independent interest.

Minimum Generating Set Problem. The minimum generating set problem has first been considered by Papadimitriou and Yannahakis [83] and further studied in [6, 101] showing polylogarithmically space-bounded algorithms. For groups, it has been shown recently to be solvable in polynomial time by Lucchini and Thakkar [68]. This bound was further improved to NC by Collins, Grochow, Levet, and the last author [23]. Moreover, they also showed that the minimum generating set problem for magmas is NP-complete.

Equations. There is an extensive work on equations in algebraic structures. In particular, the case of groups has attracted a lot of attention after Goldmann and Russell [41] showed that deciding satisfiability of a system of equations is NP-complete for every fixed non-abelian group and in P for abelian groups. For more recent conditional lower bounds and algorithms for deciding satisfiability of (single) equations, see e.g. [39, 49, 50].

The case of semigroups has attracted much less attention. While here the closely related problem of checking identities has been investigated thoroughly, [2, 55, 59, 93, 94], there is relatively little work on deciding whether a (system of) equation(s) has a solution.

In [11] the problem of deciding whether a (single) equation in finite monoids is satisfiable has been investigated. Among other results it has been shown that in the Brandt monoid B_2^1, which also plays an important role in our work, this problem is NP-complete. Furthermore, in [60] systems of equations in semigroups were studied. They presented dichotomy results for the class of finite monoids and the class of finite regular semigroups. The result for finite regular semigroups is for a restricted variant of the problem, where one side of each equation contains no variable.

1.4 Outline

After fixing our notation in Section 2, we first consider the special cases of groups and Clifford semigroups in Section 3 and Section 4, respectively. After that, we turn our attention to strict inverse semigroups and prove that, like for Clifford semigroups, their membership and conjugacy problems can be reduced to the respective problems for groups.

In Section 6, we present and prove our results on the Cayley table model, i.e., the L-completeness statement in Theorem A. In Section 7 we give our PSPACE-hardness proof (part of Theorem B) and discuss the consequences for the intersection non-emptiness problem for inverse automata and the subpower membership problem. In Section 8 we apply our results to the minimum generating set problem and the problem of solving equations. Finally, in Section 9 we provide a short summary of our results and discuss interesting open questions.

2 Preliminaries and Notation

2.1 Inverse Semigroups

A *semigroup* is a non-empty set equipped with an associative binary operation denoted by xy. For a semigroup S we write $E(S)$ to denote its set of *idempotents*, i.e., elements $e \in S$ that satisfy $ee = e$. A monoid is a semigroup M with a *neutral element*, i.e., an element $1 \in M$ such that $1x = x = x1$ for all $x \in M$, which throughout is denoted as 1. A *zero element* z of S satisfies $zx = z = xz$ for all $x \in S$; if it exists, we denote it by 0. For further background on general semigroups we refer to [1, 22, 91] and for inverse semigroups to [65, 84].

An *inverse semigroup* is a semigroup S where every element $x \in S$ possesses a *unique* inverse $\overline{x} \in S$, i.e., $x\overline{x}x = x$ and $\overline{x}x\overline{x} = \overline{x}$ hold and $\overline{x}$ is the unique element with this property. In an inverse semigroup S all idempotents commute; in particular, $E(S)$ is always a subsemigroup of S and a semilattice. We denote the *natural order* on the elements of an inverse semigroup by $\leqslant$, i.e., $x \leqslant y$ if and only if $x = x\overline{x}y$ or, equivalently, $x = y\overline{x}x$.

For an inverse semigroup S and a subset Σ of S, we denote by $\langle \Sigma \rangle$ the inverse subsemigroup of S *generated by* Σ, i.e., the smallest inverse subsemigroup of S containing Σ. The elements of the set Σ are called *generators* for $\langle \Sigma \rangle$. Note that all elements of $\langle \Sigma \rangle$ can be written as words over $\Sigma \cup \overline{\Sigma}$. Therefore, we assume from now on that generating sets are closed under formation of inverses, i.e., $\overline{\Sigma} = \Sigma$. However, be aware that unlike in finite groups, arbitrary subsemigroups of an inverse semigroup need not be inverse semigroups again.

An inverse semigroup T is a *divisor* of S, written $T \preccurlyeq S$, if there exists a surjective homomorphism from an inverse subsemigroup of S onto T.

Symmetric Inverse Semigroups. The *symmetric inverse semigroup* $\mathcal{I}(\Omega)$ is the inverse semigroup of all partial bijections on a set Ω, i.e., partial maps $s \colon \Omega \to \Omega$ that induce a bijection from their *domain* $\operatorname{dom}(s)$ to their *range* $\operatorname{ran}(s)$. We write $\mathcal{I}_n = \mathcal{I}(\{1, \dots, n\})$.

For $s \in \mathcal{I}(\Omega)$ and $x \in \Omega$, we write x^s for the image of x under s, where $x^s = \perp$ means that x^s is undefined. We extend this notation to sets $\Delta \subseteq \Omega$, i.e., $\Delta^s = \{x^s \mid x \in \Delta\}$. Be aware that Δ^s might be empty even if Δ is not (this happens if $\Delta \cap \operatorname{dom}(s) = \emptyset$). For $\Delta \subseteq \Omega$ we write e_Δ for the idempotent associated to Δ, which is the partial identity on Δ. Every idempotent of $\mathcal{I}(\Omega)$ is of this form, and $e_{\Delta'} e_{\Delta''} = e_{\Delta' \cap \Delta''}$. We also write $e_{\Delta'} \vee e_{\Delta''} = e_{\Delta' \cup \Delta''}$.

The inverse subsemigroups of $\mathcal{I}(\Omega)$ are sometimes called partial bijection semigroups. An important result by Preston [86] and Wagner [104] states that every inverse semigroup S can be embedded into the symmetric inverse semigroup $\mathcal{I}(S)$ in a natural way.

Important Combinatorial Inverse Semigroups. We denote the two-element semilattice by Y_2, which consists of a zero 0 and a neutral element 1. The Cartesian power $(Y_2)^\Omega$ is naturally isomorphic to $E(\mathcal{I}(\Omega))$; in particular, every semilattice embeds into some power of Y_2.

The (combinatorial) *Brandt semigroup* $B(\Omega)$ on some set Ω is the inverse subsemigroup $\{s \in \mathcal{I}(\Omega) \mid |\mathrm{dom}(s)| \leqslant 1\}$ of $\mathcal{I}(\Omega)$. The (combinatorial) *Brandt monoid* on Ω is the inverse submonoid $B^1(\Omega) = B(\Omega) \cup \{1\}$ of $\mathcal{I}(\Omega)$. We write B_n and B_n^1 in case $\Omega = \{1, \ldots, n\}$.

The Brandt semigroup (or monoid) can be thought of as the complete directed graph on vertices Ω together with an additional zero element (and an identity) where the multiplication of edges (u, v) and (w, z) is (u, z) if $v = w$ and 0 otherwise. For example,

$B_2 = \{$  $\} \cup \{0\}$.

2.2 Green's Relations and Conjugacy

The following relative variants of *Green's relations* will be very useful. As usual, given an inverse semigroup S, we denote by S^1 the smallest inverse monoid containing S. However, in a relative context, i.e., for an inverse subsemigroup $U \leqslant S$, we denote by U^1 the inverse submonoid $U \cup \{1\} \leqslant S^1$. This abuse of notation is employed in the following definition.

► **Definition 1.** *Let $U \leqslant S$. Given elements $s, t \in S$, we write*

$$\begin{aligned} s \leqslant_{\mathcal{L}_U} t &\iff U^1 s \subseteq U^1 t, \\ s \leqslant_{\mathcal{R}_U} t &\iff s U^1 \subseteq t U^1, \\ s \leqslant_{\mathcal{J}_U} t &\iff U^1 s U^1 \subseteq U^1 t U^1. \end{aligned}$$

Furthermore, we write $s \mathrel{\mathcal{L}_U} t$ provided that $s \leqslant_{\mathcal{L}_U} t$ and $s \geqslant_{\mathcal{L}_U} t$; the Green's relations $\mathcal{R}_U$ and $\mathcal{J}_U$ are defined similarly. Finally, we let $\mathcal{H}_U = \mathcal{L}_U \cap \mathcal{R}_U$ and $\mathcal{D}_U = \mathcal{L}_U \vee \mathcal{R}_U$.[3]

We recover the usual definition of Green's relations [43] as $\leqslant_{\mathcal{X}} = \leqslant_{\mathcal{X}_S}$ and $\mathcal{X} = \mathcal{X}_S$, where $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$ or $\mathcal{X} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}\}$, respectively.

► **Lemma 2.** *Let S be a finite inverse semigroup. If $usv \geqslant_{\mathcal{J}} s$ for some $s \in S$ and $u, v \in S^1$, then $\bar{u}usv\bar{v} = s$. In particular, if $s \mathrel{\mathcal{X}} t$ for some $s, t \in S$ with $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$ and $U \leqslant S$, then $s \leqslant_{\mathcal{X}_U} t$ if and only if $s \geqslant_{\mathcal{X}_U} t$.*

Proof. If $usv \geqslant_{\mathcal{J}} s$, then $s = u'usvv'$ for some $u', v' \in S^1$. As such, we have

$$s = u'usvv' = (u'u)^\omega s(vv')^\omega = (u'u)^\omega \bar{u}usv\bar{v}(vv')^\omega = \bar{u}u(u'u)^\omega s(vv')^\omega v\bar{v} = \bar{u}usv\bar{v}.$$

Taking $u, v \in U^1$ in the above, shows that $s \geqslant_{\mathcal{J}_U} t \geqslant_{\mathcal{J}} s$ implies $t \geqslant_{\mathcal{J}_U} s$. The remaining cases $\mathcal{X} = \mathcal{L}$ and $\mathcal{X} = \mathcal{R}$ follow by setting $v = v' = 1$ and $u = u' = 1$, respectively. ◄

Similarly, we will use a relative variant of *conjugacy* in inverse semigroups defined as follows. Beware that there are other notions of conjugacy frequently considered in the context of semigroup theory; see also [3, 51].

► **Definition 3.** *Let $U \leqslant S$ be inverse semigroups. We call $s, t \in S$* conjugate *relative to U, written $s \sim_U t$, if there exists some $u \in U^1$ such that $\bar{u}su = t$ and $s = ut\bar{u}$.*

Conjugacy and $\mathcal{J}$-equivalence are closely related for idempotents of inverse semigroups.

► **Lemma 4.** *Let $U \leqslant S$ be inverse semigroups, and let $e, e' \in E(S)$. Then $e \geqslant_{\mathcal{J}_U} e'$ holds if and only if $e' = \bar{u}eu$ for some $u \in U^1$. If S is finite, then $e \mathrel{\mathcal{J}_U} e'$ if and only if $e \sim_U e'$.*

[3] The definition of $\mathcal{D}_U$ is provided here only for completeness, as $\mathcal{D}_U = \mathcal{J}_U$ whenever S is finite.

Proof. Suppose that $e \geqslant_{\mathcal{J}_U} e'$, i.e., there exist $v_1, v_2 \in U^1$ with $e' = v_1 e v_2$. Then

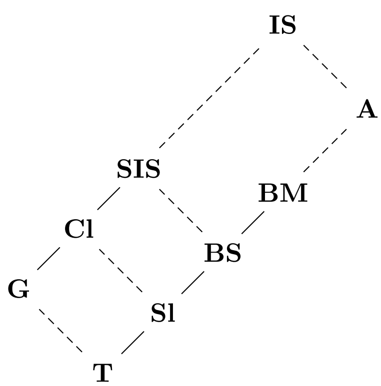
$$e' = e'e' = v_1 e v_2 \bar{v}_2 e \bar{v}_1 = v_1 e v_2 \bar{v}_2 v_2 \bar{v}_2 e \bar{v}_1 = v_1 v_2 \bar{v}_2 e e v_2 \bar{v}_2 \bar{v}_1 = v_1 v_2 \bar{v}_2 e v_2 \bar{v}_2 \bar{v}_1 = \bar{u} e u$$

where $u = v_2 \bar{v}_2 \bar{v}_1 \in U^1$. The converse is trivial. If S is finite, then $\bar{u} e u = e' \geqslant_{\mathcal{J}} e$ implies that $e = u \bar{u} e u \bar{u} = u e' \bar{u}$ by Lemma 2 and, hence, that $e \sim_U e'$. ◀

2.3 (Pseudo-)Varieties

A class $\mathcal{C}$ of finite inverse semigroups is called a *variety of finite inverse semigroups* (a.k.a. *pseudovariety*) if it is closed under formation of finite direct products and of divisors. By a theorem of Reiterman [90], such a class $\mathcal{C}$ consists of all finite inverse semigroups that satisfy some set of (pseudo-)identities. We also note that these classes are intimately related to certain classes of formal languages through Eilenberg's Correspondence Theorem [31].

We will use boldface roman letters to denote varieties of finite inverse semigroups and sometimes also give a set of defining inverse semigroup identities for them. For example, $\mathbf{G} = [\![x\bar{x} = y\bar{y}]\!] = [\![x\bar{x} = 1]\!]$ denotes the variety of finite groups, $\mathbf{Cl} = [\![x\bar{x} = \bar{x}x]\!]$ denotes the variety of finite *Clifford semigroups*, and $\mathbf{SIS} = [\![\bar{y}xy\bar{y}\bar{x}y = \bar{y}\bar{x}y\bar{y}xy]\!]$ denotes the variety of finite *strict inverse semigroups*. An overview of the varieties of finite inverse semigroups most relevant to our work is given in Figure 1.



Variety	Identities	Description
T	$x = y$	trivial
IS		inverse semigroups
G	$x\bar{x} = y\bar{y}$	groups
Cl	$x\bar{x} = \bar{x}x$	Clifford semigroups
SIS	$\bar{y}xy\,\bar{y}\bar{x}y = \bar{y}\bar{x}y\,\bar{y}xy$	strict inverse semigroups
Sl	$x^2 = x$	semilattices
BS	$(\bar{y}xy)^2 = \bar{y}xy$	generated by B_2
BM	not finitely based [58]	generated by B_2^1
A	$x^{\omega+1} = x^{\omega}$	aperiodic

Figure 1 Varieties of finite inverse semigroups, their relationships, and defining identities.

Varieties of finite inverse semigroups form a complete lattice under inclusion. (They are closed under arbitrary intersections and the variety of all finite inverse semigroups **IS** serves as a largest element.) The join of two varieties of finite inverse semigroups **U** and **V**, denoted by $\mathbf{U} \vee \mathbf{V}$, is the smallest variety of finite inverse semigroups containing both **U** and **V**.

The chain $\mathbf{T} \subseteq \mathbf{Sl} \subseteq \mathbf{BS} \subseteq \mathbf{BM}$ forms the bottom of the lattice of the varieties of finite combinatorial (i.e., aperiodic) inverse semigroups. Herein, $\mathbf{T} = [\![x = y]\!] = [\![x = 1]\!]$ denotes the variety of trivial inverse semigroups, $\mathbf{Sl} = [\![x^2 = x]\!]$ denotes the variety of all finite semilattices, and $\mathbf{BS} = [\![(\bar{y}xy)^2 = \bar{y}xy]\!]$ and **BM** denote the varieties of finite inverse semigroups generated by the (combinatorial) Brandt semigroup B_2 and monoid B_2^1, respectively.

▶ **Lemma 5** (Kleiman [57, Lemma 4]). *For every finite set Ω, $B(\Omega) \in \mathbf{BS}$ and $B^1(\Omega) \in \mathbf{BM}$.*

Proof. Let $S = (B_2)^{\Omega}$ be the Ω-fold Cartesian power of the Brandt semigroup B_2. The set $I = \{s \in S \mid \pi_x(s) = 0 \text{ for some } x \in \Omega\} \leqslant S$ is an ideal where $\pi_x \colon S \to B_2$ denotes projection onto the x coordinate. The Rees quotient $S/I \in \mathbf{BS}$ is isomorphic to $B(\Omega)$, and the Rees quotient $S^1/I \in \mathbf{BM}$ of $S^1 = S \cup \{1\} \leqslant (B_2^1)^\Omega$ is isomorphic to $B^1(\Omega)$. ◀

The bottom of the lattice of varieties of finite inverse semigroups is structured as follows. These results were originally proved for arbitrary varieties allowing also infinite direct products and then, by [44], transferred to pseudovarieties (i.e., varieties of finite inverse semigroups in our sense).

▶ **Proposition 6** (Djadchenko [29], Kleiman [56, 57]; see also [44]). *Let $\mathbf{V}$ be a variety of finite inverse semigroups. Then $\mathbf{V}$ is subject to each of the following alternatives.*

1. *Either $\mathbf{Sl} \subseteq \mathbf{V}$ or $\mathbf{V} \subseteq \mathbf{G} = \mathbf{G} \vee \mathbf{T}$.*
2. *Either $\mathbf{BS} \subseteq \mathbf{V}$ or $\mathbf{V} \subseteq \mathbf{Cl} = \mathbf{G} \vee \mathbf{Sl}$.*
3. *Either $\mathbf{BM} \subseteq \mathbf{V}$ or $\mathbf{V} \subseteq \mathbf{SIS} = \mathbf{G} \vee \mathbf{BS}$.*

Moreover, the intervals $[\mathbf{Sl}, \mathbf{Cl}]$ (or $[\mathbf{BS}, \mathbf{SIS}]$) and $[\mathbf{T}, \mathbf{G}]$ in the lattice of all varieties of finite inverse semigroups are isomorphic via $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{G}$ and $\mathbf{H} \mapsto \mathbf{H} \vee \mathbf{Sl}$ (or $\mathbf{H} \mapsto \mathbf{H} \vee \mathbf{BS}$).

2.4 Algorithmic Problems

The main focus of this work lies on analyzing the algorithmic complexity of two important decision problems for inverse semigroups, and several variants thereof. The first of these problems is the *membership problem* (MEMB); it is defined as follows.

Input. An inverse semigroup S, a subset $\Sigma \subseteq S$, and an element $t \in S$.
Question. Is $t \in U$ where $U = \langle \Sigma \rangle$?

The second main decision problem is the *conjugacy problem* (CONJ).

Input. An inverse semigroup S, a subset $\Sigma \subseteq S$, and elements $s, t \in S$.
Question. Is $s \sim_U t$ where $U = \langle \Sigma \rangle$?

Recall that we consider the relative variant of conjugacy (see Definition 3) meaning that we restrict the conjugating element to the inverse subsemigroup U, whereas the elements s and t are not restricted. Be aware that this might differ from other references in the literature, especially those concerned with infinite structures. Also, unlike what is common for infinite semigroups, we require that s and t are explicitly given as elements of S and not as words over some set of generators.

Input Models. As with any decision problem, its algorithmic complexity depends substantially on how its input is provided. We restrict our attention to two input models, the *Cayley table model* ($\mathbf{CT}$) and the *partial bijection model* ($\mathbf{PB}$). In the former model, the surrounding inverse semigroup is provided as a complete multiplication table, the so-called *Cayley table* of S, and all elements of S (in particular, Σ and t) are encoded as indices into this table.[4]

In the partial bijection model, the surrounding inverse semigroup is the symmetric inverse semigroup $\mathcal{I}_n$ on n elements with only n provided as part of the input. All elements of $S = \mathcal{I}_n$ (in particular, Σ and t) are given as partial bijections on $\Omega = \{1, \ldots, n\}$. More specifically, we assume that each partial bijection is encoded as a complete, ordered list of

[4] More precisely, a Cayley table of an n-element semigroup S is encoded as an array of size n^2 each of which entries is encoded using $\lceil \log n \rceil$ bits and where at position $i + jn$ we find the index of the element obtained by multiplying elements i and j (indices starting at 0).

its images (using a special symbol $\perp$ to denote undefined images)[5]. For example, $(2, \perp, 1)$ encodes the partial bijection on $\{1, 2, 3\}$ with $1 \mapsto 2$, $3 \mapsto 1$, and undefined on 2.

We denote by $\textsc{memb}_{\mathbf{CT}}$ and $\textsc{memb}_{\mathbf{PB}}$, and by $\textsc{conj}_{\mathbf{CT}}$ and $\textsc{conj}_{\mathbf{PB}}$ the membership and conjugacy problems in the respective input model. Intuitively, membership and conjugacy are easier to decide in the Cayley table model than in the partial bijection model. The fact that the Preston-Wagner representation [86, 104] of an inverse semigroup is efficiently computable allows us to make this intuition precise.

▶ **Lemma 7.** *On input of an inverse semigroup S given as a Cayley table, one can compute an embedding $S \to \mathcal{I}(S)$ in AC^0. Hence, $\textsc{memb}_{\mathbf{CT}} \leqslant_m^{\mathsf{AC}^0} \textsc{memb}_{\mathbf{PB}}$ and $\textsc{conj}_{\mathbf{CT}} \leqslant_m^{\mathsf{AC}^0} \textsc{conj}_{\mathbf{PB}}$.*

Proof. Every inverse semigroup S acts on itself via multiplication on the right. We can restrict this action to obtain a representation via partial bijections. Indeed, given $s \in S$, we define the partial map $\rho_s \colon S \to S$ via $t\rho_s = ts$ if $ts\bar{s} = t$ and $t\rho_s = \perp$ otherwise. The resulting map $\rho \colon S \to \mathcal{I}(S) \colon s \mapsto \rho_s$ is the desired embedding. Now note that encoding of the partial bijection ρ_s is simply the corresponding column of the Cayley table for S with some of its entries replaced by $\perp$. It thus remains to argue that we can decide the condition for such a replacement (i.e., whether $ts\bar{s} \neq t$) in AC^0. To see that this is indeed the case, note that we can compute the product of two elements of S in AC^0 and, given $s \in S$, we can compute $\bar{s}$ in AC^0 (for $\bar{s}$ is the unique element of S with $s\bar{s}s = s$ and $\bar{s}s\bar{s} = \bar{s}$). ◀

Idempotent Membership and Conjugacy. In order to obtain a detailed analysis of the algorithmic complexity, we impose certain restrictions on the allowed inputs. On the one hand, we consider the *idempotent membership* and *idempotent conjugacy* problems where we require that $s, t \in E(S)$. We denote these problem variants by E-$\textsc{memb}_{\mathbf{IM}}$ and E-$\textsc{conj}_{\mathbf{IM}}$ where $\mathbf{IM} \in \{\mathbf{CT}, \mathbf{PB}\}$. The latter, in particular, is closely tied to many other important problems regarding partial symmetries (e.g. the set transporter problem; see Section 3.2).

Restriction to Varieties. The other kind of restriction we impose is on the structure of the inverse subsemigroup $U = \langle \Sigma \rangle$ under consideration. More specifically, we consider the above problems with U confined to some fixed class $\mathbf{V}$ of finite inverse semigroups. We call these the (idempotent) *membership* and *conjugacy problem for* $\mathbf{V}$, and denote them by $\textsc{memb}_{\mathbf{IM}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{IM}}(\mathbf{V})$ where $\mathbf{IM} \in \{\mathbf{CT}, \mathbf{PB}\}$, respectively. Throughout, the class $\mathbf{V}$ will be some variety of finite inverse semigroups such as, e.g., the variety $\mathbf{G}$ of finite groups. Be aware that only U is restricted to the class $\mathbf{V}$, while S and the elements $s, t \in S$ can be arbitrary!

We also consider a more restricted variant of the problems, which we denote with a $\sharp$ superscript (e.g. $\textsc{memb}^{\sharp}_{\mathbf{PB}}$ and $\textsc{conj}^{\sharp}_{\mathbf{PB}}$). For these we require in the Cayley table model that $S \in \mathbf{V}$ and in the partial bijection model that there is some $S \leqslant \mathcal{I}(\Omega)$ with $S \in \mathbf{V}$ such that $\Sigma \subseteq S$ and $t \in S$ (resp. $s, t \in S$). For the conjugacy problem we also require that $s \sim_S t$. We use these restricted variants to show stronger statements for our hardness results.

2.5 Complexity

We assume that the reader is familiar with standard complexity classes such as PSPACE or NP; see any standard textbook [4, 82] on complexity theory. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are complexity classes, then we use the notation $\mathcal{C}^{\mathcal{D}}$ for the class of problems that can be solved in $\mathcal{C}$ with oracles for a finite set of problems from $\mathcal{D}$.

[5] Another different representation for permutations, which is also commonly used, is the cycle representation. While two permutations in our encoding (as a partial functions) can be multiplied in AC^0, in the cycle notation, multiplication is FL-complete [25].

Circuit Classes and Reductions. The circuit class AC^0 is defined as the class of problems decidable by polynomial-size, constant-depth Boolean circuits (where all gates may have arbitrary fan-in). Likewise AC^0-computable functions are defined. We say that a problem $K \subseteq \{0,1\}^*$ is AC^0-*(many-one-)reducible* to $L \subseteq \{0,1\}^*$ if there is an AC^0-computable function $f\colon \{0,1\}^* \to \{0,1\}^*$ such that $w \in K \iff f(w) \in L$. Throughout, we consider only uniform classes meaning that the circuits can be constructed (or verified) efficiently, for details see [103]. The classes qAC^0 and NC are defined analogously to AC^0 but allowing circuits of quasipolynomial (i.e., $2^{\log^{\mathcal{O}(1)} n}$) size (resp. polynomial size and depth $\log^{\mathcal{O}(1)} n$).

Logarithmic Space. We write L to denote logarithmic space. For many-one reductions computable in logarithmic space we write L-reductions. Recall that the composition of two L-reductions is again a L-reduction, that the class L is closed under L-reductions, and that the class L is low for itself, i.e., $\mathsf{L}^{\mathsf{L}} = \mathsf{L}$ (see e.g. [4, Lemma 4.17]). When talking about L-hard problems, throughout we refer to AC^0 many-one reductions.

Let UGAP denote the *undirected graph accessibility problem*, i.e., the input is an undirected graph and two vertices s and t and the question is to decide whether s and t lie in the same connected component. Note that UGAP is L-hard under AC^0 reductions [25] (even if the graphs are restricted to trees). The class L^{UGAP} is also denoted as SL. All our results on L will rely crucially on the following seminal result, which shows that $\mathsf{SL} = \mathsf{L}$.

▶ **Theorem 8** (Reingold [89]). *The problem* UGAP *is in* L.

▶ **Remark 9.** Using an UGAP oracle, we can compute a path between any two vertices of an undirected graph in L; in fact, we can even compute the vertices of a connected component, or a spanning tree of a connected component in L, for details see [78, Lemma 2.4].

Sublinear Time Classes. For sublinear time classes, we use random access Turing machines meaning that the Turing machine has a separate address tape and a query state; whenever the Turing machine goes into the query state and on the address tape the number i is written in binary, the i-th symbol of the input is read (the content of the address tape is *not* deleted after that). Apart from that, random access Turing machines work like regular Turing machines. The class $\mathsf{NPOLYLOGTIME}$ consists of the problems decidable by non-deterministic random access Turing machines in time $\log^{\mathcal{O}(1)} n$.

Overview. For an overview over the complexity classes we use in this paper and their relationships, see Figure 2. We also note that, by [92] and the space hierarchy theorem [96], we know that $\mathsf{NC} \subsetneq \mathsf{PSPACE}$. Moreover, by [23], $\mathsf{NPOLYLOGTIME}$ can be simulated by circuits of quasipolynomial size and depth two,[6] i.e., $\mathsf{NPOLYLOGTIME} \subseteq \mathsf{qAC}^0$. Clearly, AC^0 and L are not contained in $\mathsf{NPOLYLOGTIME}$ (as for example the conjunction of all input bits cannot be computed in $\mathsf{NPOLYLOGTIME}$). Even more, by [38, 45], qAC^0 does not contain any L-hard problem as it cannot compute, for instance, PARITY.

2.6 Straight-Line Programs

Let S be an inverse semigroup and $\Sigma \subseteq S$. A *straight-line program* (SLP) over Σ is a finite sequence $(s_1, \dots, s_k) \in S^k$ such that for all i either $s_i \in \Sigma$, or $s_i = s_j s_\ell$ for some $j, \ell < i$, or $s_i = \overline{s_j}$ for some $j < i$. An SLP as above *computes* an element $s \in S$ if $s \in \{s_1, \dots, s_k\}$.

[6] Thus, in terms of circuit depth, our corresponding results are optimal.

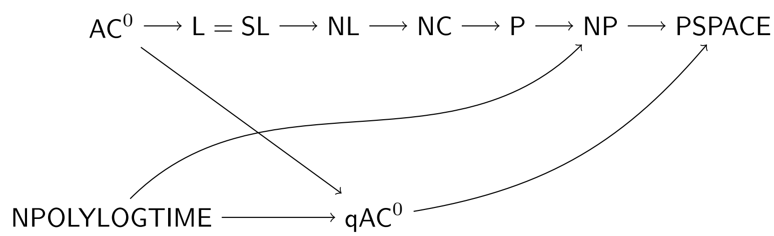


Figure 2 Complexity classes in this paper (all circuit classes are assumed to be uniform).

Note that for $\Sigma \subseteq S$ closed under formation of inverses (as is the case for generating sets by our convention) the rule allowing for $s_i = \overline{s_j}$ could have been omitted from the definition of a straight-line program over Σ. Note also that our definition of SLPs is according to Babai and Szemerédi [8] – other authors define them slightly differently via circuits (or equivalently context-free grammars). The difference is that in our definition the evaluation of the SLP in the semigroup S is already part of its definition.

▶ **Definition 10.** *We say that a class $\mathcal{C}$ of finite inverse semigroups* admits polylogarithmic SLPs *if there exists a polynomial P such that, for all $S \in \mathcal{C}$ and all generating sets $\Sigma \subseteq S$, every element $s \in S$ is computed by some SLP over Σ of length at most $P(\log|S|)$.*

The analogous property for semigroups and monoids was studied by the first author [35,36] under the name *polylogarithmic circuits property*. Using a straight-forward guess-and-check approach, we obtain the following result – for a proof see [35, Corollary 5.2].

▶ **Lemma 11** (Fleischer [35, Corollary 5.2])**.** *Let $\mathcal{C}$ be a class of finite (inverse) semigroups admitting polylogarithmic SLPs. Then the problem* $\textsc{Memb}_{\mathbf{CT}}(\mathcal{C})$ *is in* NPOLYLOGTIME*.*

▶ **Remark 12.** Notice that in [24] membership in quasigroups in the Cayley table model has been shown to be decidable in the class $\exists^{\log^2}\mathsf{DTISP}(\mathrm{polylog}, \log)$, meaning that, after non-deterministically guessing $\mathcal{O}(\log^2 n)$ bits, it can be verified deterministically in time $\log^{\mathcal{O}(1)} n$ with space restricted to $\mathcal{O}(\log n)$. Our results could be strengthened to the similar (yet slightly larger) class $\exists^{\log^{k+1}}\mathsf{DTISP}(\mathrm{polylog}, \log)$ where k is the degree of a polynomial P as in Definition 10. Note that $k \leqslant 2$ for the variety $\mathbf{Cl}$ of finite Clifford semigroups by Lemma 24, matching the bound for the variety $\mathbf{G}$ of finite groups obtained by Babai and Szemerédi [8]; see Lemma 14. Nevertheless, as $\mathsf{NPOLYLOGTIME} = \exists^{\log^{\mathcal{O}(1)}}\mathsf{DTISP}(\mathrm{polylog}, \log)$, this would give little additional insight and we refrain from doing so for the sake of a cleaner presentation.

3 Membership and Conjugacy in Groups

Groups are a primary example for inverse semigroups. Therefore, let us start exploring some known result and new observations about the membership and conjugacy problems in groups.

The following characterization of the variety of finite groups is well known (see, e.g. [44]).

▶ **Lemma 13.** *Let S be a finite inverse semigroup. Then the following are equivalent.*

1. *The inverse semigroup S is contained in $\mathbf{G} = [\![x\bar{x} = 1]\!]$.*
2. *The two-element semilattice Y_2 does not divide S.*

3.1 The Cayley Table Model

In the Cayley table model, deciding the membership and conjugacy problems for groups is comparatively easy. The best currently known approach [24] is based on the non-deterministic computation of a succinct representation of a target element as a product of generators.

We pursue a similar idea here and use SLPs for succinct representation. In the case of groups, this approach is afforded by the following Reachability Lemma.

▶ **Lemma 14** (Babai, Szemerédi [8, Theorem 3.1]). *The variety* $\mathbf{G}$ *of finite groups admits polylogarithmic SLPs. More precisely, for every group* G *and generating set* $\Sigma \subseteq G$*, every element of* G *is computed by an SLP over* Σ *of length* $\mathcal{O}(\log^2 |G|)$.

For the first part of the following result, see [35].

▶ **Proposition 15.** *The problems* $\textsc{memb}_{\mathbf{CT}}(\mathbf{G})$ *and* $\textsc{conj}_{\mathbf{CT}}(\mathbf{G})$ *are in* NPOLYLOGTIME.

Proof. The combination of Lemma 11 and Lemma 14 shows that $\textsc{memb}_{\mathbf{CT}}(\mathbf{G})$ is decidable in NPOLYLOGTIME. To see that this is also true for $\textsc{conj}_{\mathbf{CT}}(\mathbf{G})$, we observe that one can simply guess a conjugating element, thereby reducing the problem to $\textsc{memb}_{\mathbf{CT}}(\mathbf{G})$. ◀

3.2 The Partial Bijection Model

In the partial bijection model, the groups under consideration are permutation groups and it is in this latter setting that the membership and conjugacy problems have been widely studied. However, the problems $\textsc{memb}_{\mathbf{PB}}(\mathbf{G})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{G})$ are also subtly different from the corresponding problems for permutation groups simply because the former allow for more possible inputs (partial bijections instead of bijections). In case of the membership problem this distinction is mostly artificial (and can be resolved by appropriate AC^0 reductions).

▶ **Proposition 16** (Babai, Luks, and Seress [7]). *The problem* $\textsc{memb}_{\mathbf{PB}}(\mathbf{G})$ *is in* NC.

Proof. Let $\Sigma \subseteq \mathcal{I}(\Omega)$ such that $U = \langle \Sigma \rangle$ is a group and $t \in \mathcal{I}(\Omega)$ denote our input. First, observe that $\mathrm{dom}(u) = \mathrm{dom}(v)$ for all $u, v \in U$ as U is a group. Hence, to check membership, we first check whether $\mathrm{dom}(t) = \mathrm{ran}(t) = \mathrm{dom}(u)$ for some $u \in \Sigma$. If this is not the case, then $t \notin U$. Otherwise, we use the algorithm for permutation groups [7] to test whether $t \in \langle \Sigma \rangle$, where we interpret t and all elements of Σ as permutations on the set $\Omega^U = \mathrm{dom}(t) \subseteq \Omega$. ◀

We complement the above with the following hardness result.

▶ **Proposition 17.** *Let* $\mathbf{V}$ *be a variety of finite inverse semigroups containing a non-trivial group. Then* $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ *as well as its restricted variant* $\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{V})$ *are* L*-hard.*

Proof. Let us reduce UGAP to $\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{V})$. Let $G \in \mathbf{V}$ denote a non-trivial group with some non-trivial element $g \in G$. We can interpret G as a permutation group acting on itself. Given an undirected graph $\Gamma = (V, E)$, set $S = G^V \in \mathbf{V}$ (which can be interpreted as a permutation group acting on $\bigsqcup_{v \in V} G$). For each $v \in V$ we define $g_v \colon V \to G$ with $g_v(v) = g$ and $g_v(u) = 1$ otherwise, and for each pair $(u, v) \in V \times V$ we define $g_{uv} = \bar{g}_u g_v$. Clearly, $g_{st} \in \langle\, g_{uv} \mid \{u, v\} \in E \,\rangle$ if and only if $s \in V$ and $t \in V$ are in the same component of Γ. ◀

The complexity of the conjugacy problem for groups is more intricate. On the one hand, the problem is clearly in NP due to the existence of short SLPs (Lemma 14). This observation holds for permutation groups as well as groups in the partial bijection model.

On the other hand, the conjugacy problem is GI-hard for permutation groups, as was observed by Luks [70]. Luks also exhibited several other problems for permutation groups that are polynomial-time equivalent to conjugacy, among which is the *set transporter problem.*

Input. A group $G \leqslant \mathrm{Perm}(\Omega)$ given by generators, and sets $\Delta_s, \Delta_t \subseteq \Omega$.
Question. Does there exist an element $g \in G$ such that $\Delta_s^g = \Delta_t$?

The conjugacy problem for permutation groups is clearly a special case of the conjugacy problem for $\mathbf{G}$ in the partial bijection model, and so is the set transporter problem. In fact, the latter is precisely the idempotent conjugacy problem for $\mathbf{G}$ (recall that the idempotents of the symmetric inverse semigroup $\mathcal{I}(\Omega)$ are in canonical bijection with the subsets of Ω).

▶ **Lemma 18.** *The problem* $\textsc{conj}_{\mathbf{PB}}(\mathbf{G})$ *is* AC^0*-reducible to* E-$\textsc{conj}_{\mathbf{PB}}(\mathbf{G})$.

Proof. Let $U \leqslant \mathcal{I}(\Omega)$ be a group, and let $s, t \in \mathcal{I}(\Omega)$ be partial bijections. We may assume that U consists of bijections of Ω, for it consists of bijections on some $\Omega^U \subseteq \Omega$ and if $\mathrm{dom}(s) \cup \mathrm{dom}(t) \cup \mathrm{ran}(s) \cup \mathrm{ran}(t) \not\subseteq \Omega^U$, then $s \not\sim_U t$. Now let $\Delta_s, \Delta_t \subseteq \Omega \times \Omega$ be the graphs of s and t, respectively. We claim that $\Delta_s^u = \Delta_t$ for some $u \in U$ with respect to the diagonal action of U on $\Omega \times \Omega$ if and only if $\bar{u}su = t$ and $s = ut\bar{u}$. Indeed, a short calculation shows that $(x, x^s) \in \Delta_s$ and $(x, x^s)^u \in \Delta_t$ if and only if $x^s = x^{ut\bar{u}}$ with both sides defined.

Provided that the input is suitably encoded, transforming an instance of the conjugacy problem to its corresponding instance of the idempotent conjugacy problem, as above, (or to a default instance if $\mathrm{dom}(s) \cup \mathrm{dom}(t) \cup \mathrm{ran}(s) \cup \mathrm{ran}(t) \not\subseteq \Omega^G$) is possible with AC^0-circuits. ◀

The above discussion can be summarized as follows.

▶ **Proposition 19.** *Both of the problems* $\textsc{conj}_{\mathbf{PB}}(\mathbf{G})$ *and* E-$\textsc{conj}_{\mathbf{PB}}(\mathbf{G})$ *are polynomial-time equivalent to the conjugacy problem for permutation groups; hence,* GI*-hard and in* NP.

Notice that the situation is different for the restricted variants $\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{G})$ and E-$\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{G})$. As conjugacy in the symmetric group S_n can be tested in L, $\textsc{conj}_{\mathbf{PB}}(\mathbf{G})$ can be reduced to $\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{G})$; thus, the latter is as difficult as the general case. On the other hand, the restricted problem variant E-$\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{G})$ is trivial as every group only contains a single idempotent.

4 Membership and Conjugacy in Clifford Semigroups

We now turn to Clifford semigroups, which constitute the smallest variety of finite inverse semigroups $\mathbf{Cl}$ to properly contain the variety of finite groups $\mathbf{G}$. Our goal is to show that the membership and conjugacy problems for Clifford semigroups are essentially as complex as the corresponding problems for groups. To this end, let us first recall the following well known characterization of Clifford semigroups (see, e.g. [65, 84]).

▶ **Lemma 20.** *Let S be a finite inverse semigroup. Then the following are equivalent.*

1. *The inverse semigroup S is contained in $\mathbf{Cl} = [\![x\bar{x} = \bar{x}x]\!]$.*
2. *The Brandt semigroup B_2 does not divide S.*

The following simple observations regarding the structure of the idempotents $E(S)$ of a Clifford semigroup S are also of crucial importance for our discussion.

▶ **Lemma 21.** *Let $S \in \mathbf{Cl}$. If $s_1, \ldots, s_n \in S$, then*

$$s_1 s_2 \ldots s_n \bar{s}_n \ldots \bar{s}_2 \bar{s}_1 = s_1 \bar{s}_1 s_2 \bar{s}_2 \ldots s_n \bar{s}_n.$$

Proof. The identity holds for all $S \in \mathbf{G}$, since both sides are idempotent, and for all $S \in \mathbf{Sl}$, since $S \in \mathbf{Sl}$ implies $S = E(S)$ and idempotents in an inverse semigroup commute. Hence, the identity also holds for all $S \in \mathbf{G} \vee \mathbf{Sl} = \mathbf{Cl}$. ◀

▶ **Lemma 22.** *Let $S \in \mathbf{Cl}$. If $S = \langle s_1, \ldots, s_k \rangle$, then $E(S) = \langle s_1\bar{s}_1, \ldots, s_k\bar{s}_k \rangle$.*

Proof. Suppose that $S = \langle s_1, \ldots, s_k \rangle$ and let $e \in E(S)$. Then, by Lemma 21,

$$e = s_{i_1} \ldots s_{i_n} = s_{i_1} \ldots s_{i_n} \bar{s}_{i_n} \ldots \bar{s}_{i_1} = s_{i_1} \bar{s}_{i_1} \ldots s_{i_n} \bar{s}_{i_n} \in \langle s_1\bar{s}_1, \ldots, s_k\bar{s}_k \rangle.$$ ◀

4.1 The Cayley Table Model

As is the case for groups, Clifford semigroups afford succinct representations of their elements.

▶ **Lemma 23.** *Every finite semilattice E admits SLPs of length $\mathcal{O}(\log |E|)$.*

Proof. Let E be a finite semilattice generated by $\Sigma \subseteq E$. Given $e \in E$, let $e_1, \ldots, e_n \in \Sigma$ such that $e = e_1 \ldots e_n$ and n is minimal with this property. The elements $e_I = \prod_{i \in I} e_i \in E^1$ with $I \subseteq \{1, \ldots, n\}$ are pairwise distinct. Indeed, if $e_I = e_J$ for some $I \neq J$ and $i \in I \setminus J$, say, then $e = e_1 \ldots e_{i-1} e_{i+1} \ldots e_n$ which contradicts the minimality of n. Hence, $|E| \geqslant 2^n - 1$. ◀

The following result is also part of the first author's dissertation [35, Lemma 4.10].

▶ **Lemma 24.** *Every finite Clifford semigroup S admits SLPs of length $\mathcal{O}(\log^2 |S|)$.*

Proof. Let $S \in \mathbf{Cl}$ be generated by $\Sigma \subseteq S$ and let $t \in S$. The element $t\bar{t} \in E(S)$ can be computed by an SLP of length $\mathcal{O}(\log |E(S)|)$ over $\{s\bar{s} \mid s \in \Sigma\} \subseteq E(S)$ by Lemma 23. Moreover, t is contained in the subgroup $S' = \{s \in S \mid s\bar{s} = t\bar{t}\} \leqslant S$, which is generated by the set $\Sigma' = \{t\bar{t}s \in S \mid s \in \Sigma, s\bar{s} \geqslant t\bar{t}\} \subseteq S'$. Therefore, t can be computed by an SLP of length $\mathcal{O}(\log^2 |S'|)$ over Σ' by Lemma 14. Combining these two observations, we see that t can be computed by an SLP of length $\mathcal{O}(\log |E(S)| + \log^2 |S'|) \subseteq \mathcal{O}(\log^2 |S|)$ over Σ. ◀

The following is an immediate consequence (for membership see also [35]).

▶ **Proposition 25.** *The problems* $\textsc{memb}_{\mathbf{CT}}(\mathbf{Cl})$ *and* $\textsc{conj}_{\mathbf{CT}}(\mathbf{Cl})$ *are in* $\mathsf{NPOLYLOGTIME}$.

4.2 The Partial Bijection Model

Even in the partial bijection model, the complexity of the membership and conjugacy problems for Clifford semigroups is essentially equivalent to those for groups.

▶ **Lemma 26.** *Let $U \in \mathbf{Cl}$ be generated by $\Sigma \subseteq \mathcal{I}(\Omega)$. On input $e \in E(\mathcal{I}(\Omega))$ and Σ, the minimal idempotent $\hat{e} \in E(U) \cup \{1\}$ such that $\hat{e} \geqslant e$ can be computed in AC^0.*

Proof. This follows from the fact that $\hat{e} = \prod\{u\bar{u} \mid u \in \Sigma \cup \{1\}, u\bar{u} \geqslant e\}$ by Lemma 22. Moreover, note that $u\bar{u}$ is the idempotent associated with the set $\operatorname{dom}(u) \subseteq \Omega$ and, as such, the condition $u\bar{u} \geqslant e$ is equivalent to $\operatorname{dom}(u) \supseteq \operatorname{dom}(e)$ which can be verified in AC^0. The product in question and, in fact, the product $\prod_{f \in F} f$ of any set of idempotents $F \subseteq E(\mathcal{I}(\Omega))$ can also be computed in AC^0; it is the idempotent associated with the set $\bigcap_{f \in F} \operatorname{dom}(f)$. ◀

▶ **Proposition 27.** *If $\mathbf{H} \subseteq \mathbf{G}$ is a variety of finite groups, then the problems* $\textsc{memb}_{\mathbf{PB}}(\mathbf{H} \vee \mathbf{Sl})$ *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{H} \vee \mathbf{Sl})$ *are* AC^0*-reducible to* $\textsc{memb}_{\mathbf{PB}}(\mathbf{H})$ *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{H})$*, respectively.*

Proof. Let $U \leqslant \mathcal{I}(\Omega)$ with $U \in \mathbf{H} \vee \mathbf{Sl}$ generated by $\Sigma \subseteq U$. In the case of membership, we are given an additional element $t \in \mathcal{I}(\Omega)$. Using Lemma 26, we compute the minimal idempotent $\hat{e} \in E(U) \cup \{1\}$ with $\hat{e} \geqslant t\bar{t}$ and (as part of this computation) verify that $\hat{e} \in E(U)$. Next, we compute the generating set $\Sigma_{\hat{e}} = \{\hat{e}u \mid u \in \Sigma, \hat{e} \leqslant u\bar{u}\}$ of the $\mathcal{H}$-class $U_{\hat{e}} = \{u \in U \mid u\bar{u} = \hat{e}\} \leqslant U$ (which is a group). Then $t \in U$ if and only if $t \in U_{\hat{e}}$.

In case of conjugacy, on input $s, t \in \mathcal{I}(\Omega)$, we perform the reduction with $\hat{e} \in E(U) \cup \{1\}$ minimal such that $\hat{e} \geqslant s\bar{s} \vee \bar{s}s \vee t\bar{t} \vee \bar{t}t$ and the $\mathcal{H}$-class $U_{\hat{e}} \leqslant U^1$ of $\hat{e}$. Note that $\bar{u}su = t$ and $ut\bar{u} = s$ with $u \in U^1$ imply $u\bar{u} \geqslant \hat{e}$ and, therefore, $\bar{u}'su' = t$ and $u't\bar{u}' = s$ with $u' = \hat{e}u\hat{e} \in U_{\hat{e}}$. As such, $s \sim_U t$ if and only if $s \sim_{U_{\hat{e}}} t$. ◀

▶ **Corollary 28.** *The problems* $\textsc{memb}_{\mathbf{PB}}(\mathbf{Cl})$ *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{Cl})$ *are in* NC *and* NP*, respectively.*

▶ **Corollary 29** (Beaudry, McKenzie, Thérien [14]). *The problem* $\textsc{memb}_{\mathbf{PB}}(\mathbf{Sl})$ *is in* AC^0.

By Lemma 7, it follows from Corollary 29 that also $\textsc{memb}_{\mathbf{CT}}(\mathbf{Sl})$ is in AC^0. On the other hand, the following important question remains open.

▶ **Question 30.** *Are the problems* $\textsc{memb}_{\mathbf{CT}}(\mathbf{G})$ *and* $\textsc{memb}_{\mathbf{CT}}(\mathbf{Cl})$ *in* AC^0*?*

5 Membership and Conjugacy in Strict Inverse Semigroups

In this section we show that, in the partial bijection model, the membership and conjugacy problem for **SIS** are L-reducible to the corresponding problems for **G**. In the Cayley table model these problems are L-complete for **SIS** as we will show in Section 6.

Our reduction is explicit and based on special properties of the representation theory of the variety **SIS**. For now, let us recall the following characterization of **SIS** (see, e.g., [44]).

▶ **Lemma 31.** *Let S be a finite inverse semigroup. Then the following are equivalent.*

1. *The inverse semigroup S is contained in* **SIS**.
2. *If $e, f_1, f_2 \in E(S)$ with $e \geqslant f_1, f_2$ and $f_1 \mathcal{J} f_2$, then $f_1 = f_2$.*
3. *The Brandt monoid B_2^1 does not divide S.*

We will also need the following analogue of Lemma 21.

▶ **Lemma 32.** *Let $S \in \mathbf{SIS}$. If $s_1, \ldots, s_n \in S$ with $s_i\bar{s}_i = \bar{s}_i s_i$ for all $1 \leqslant i \leqslant n$ then*

$$s_1 s_2 \ldots s_n \bar{s}_n \ldots \bar{s}_2 \bar{s}_1 = s_1 \bar{s}_1 s_2 \bar{s}_2 \ldots s_n \bar{s}_n.$$

Proof. Note that $s\bar{s} = \bar{s}s$ if and only if $s = \bar{x}xx$ for some $x \in S$ (e.g., $x = s$). As such, the implication can be written as an identity. This identity holds for all $S \in \mathbf{G}$, since both sides are idempotent, and for B_2, since $s \in B_2$ with $s\bar{s} = \bar{s}s$ implies $s \in E(B_2)$ and idempotents in an inverse semigroup commute. Hence, the identity also holds for all $S \in \mathbf{G} \vee \mathbf{BS} = \mathbf{SIS}$. ◀

5.1 Representations of Strict Inverse Semigroups

Our goal is to develop an efficient description of the local structure of a strict inverse semigroup $U \leqslant \mathcal{I}(\Omega)$ based on its action on Ω and on U-invariant subsets $\Delta \subseteq \Omega$. To this end, we will show how to obtain a generating set for each $\mathcal{D}$-class of U (as a groupoid) from a given generating set $\Sigma \subseteq U$. As shown in the next subsection, this can be used to reduce the membership and conjugacy problems for U to an appropriate $\mathcal{D}$-class and, ultimately, to a single $\mathcal{H}$-class, i.e., to a subgroup of U. The attentive reader might notice that this strategy also underlies our approach for Clifford semigroups in Section 4 (and even the transition from groups of partial bijections to permutation groups in Section 3).

Given an inverse semigroup $U \leqslant \mathcal{I}(\Omega)$, we say that a set $\Delta \subseteq \Omega$ is *U-invariant* if $\Delta^s \subseteq \Delta$ for all $s \in U$. Equivalently, the set Δ is U-invariant if the idempotent $e_\Delta \in E(\mathcal{I}(\Omega))$ associated with it centralizes U, i.e., $se_\Delta = e_\Delta s$ holds for all $s \in U$. Clearly, each $\Delta \subseteq \Omega$ generates a U-invariant subset $\Delta^U := \bigcup_{s \in U} \Delta^s \subseteq \Omega$. If each point $x \in \Delta$ is in the domain of some $s \in U$, then Δ^U is the minimal U-invariant subset containing Δ. Conversely, the set of all points $x \in \Omega$ that are contained in the domain of some $s \in U$ is precisely Ω^U.

The following lemma describes the structure of the *orbit* $x^U := \{x\}^U$ of a point $x \in \Omega$ under an inverse semigroup $U \in \mathbf{SIS}$, i.e., of a minimal non-empty U-invariant subset. Throughout, it will be helpful to keep in mind that $y \in x^U$ if and only if $x \in y^U$.

▶ **Lemma 33.** *Let $U \leqslant \mathcal{I}(\Omega)$ with $U \in \mathbf{SIS}$, $\Delta = x^U$ for some $x \in \Omega$ and $s_1, s_2 \in U$. Then* $\mathrm{dom}(s_1) \cap \Delta = \mathrm{dom}(s_2) \cap \Delta$ *or* $\mathrm{dom}(s_1) \cap \mathrm{dom}(s_2) \cap \Delta = \emptyset$. *In other words, every orbit x^U is partitioned by the non-empty sets of the form* $\mathrm{dom}(s) \cap x^U$ *with $s \in U$.*

Proof. Suppose to the contrary that there exist $x_1, x_2 \in x^U$ with $x_1 \in \mathrm{dom}(s_1) \cap \mathrm{dom}(s_2)$ and $x_2 \in \mathrm{dom}(s_1) \setminus \mathrm{dom}(s_2)$. Since $x_1, x_2 \in x^U$, there exists some $t \in U$ with $x_1^t = x_2$. Observe that $e = s_1\bar{s}_1 \in U$ and $s = s_2\bar{s}_2 t \in U$ satisfy $x_1^e = x_1$, $x_2^e = x_2$, $x_1^s = x_2$, $x_2^s = \perp$. Hence, the restrictions of e and s to $\Omega' = \{x_1, x_2\} \subseteq \Omega$ generate the Brandt monoid $B^1(\Omega') \leqslant \mathcal{I}(\Omega')$. As such, B_2^1 divides U which contradicts $U \in \mathbf{SIS}$. ◀

Important to our cause are the elements of $U \leqslant \mathcal{I}(\Omega)$ that act on all orbits contained in some U-invariant set $\Delta \subseteq \Omega$. Formally, we say that $s \in U$ is Δ-*large* provided that $(\mathrm{dom}(s) \cap \Delta)^U = \Delta$ or, equivalently, $(\mathrm{ran}(s) \cap \Delta)^U = \Delta$. We claim that if $s \in U$ is Δ-large and $t \in U$ satisfies $s \leqslant_{\mathcal{J}} t$, then t is Δ-large itself. Indeed, if s is Δ-large and if $s \leqslant_{\mathcal{R}} t$ or $s \leqslant_{\mathcal{L}} t$, then t is Δ-large since $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$ or $\mathrm{ran}(s) \subseteq \mathrm{ran}(t)$, respectively; finally, if $s \leqslant_{\mathcal{J}} t$, then $s \leqslant_{\mathcal{R}} r \leqslant_{\mathcal{L}} t$ for some $r \in U$ from which we conclude that t is Δ-large.

▶ **Lemma 34.** *Let $U \leqslant \mathcal{I}(\Omega)$ with $U \in \mathbf{SIS}$ and $s, t \in U$. Further, let $\Delta \subseteq \Omega$ be U-invariant. If s is Δ-large and $e_\Delta s \leqslant e_\Delta t$, then t is Δ-large and $e_\Delta s = e_\Delta t$.*

Proof. Suppose that $s, t \in U$ are such that $e_\Delta s \leqslant e_\Delta t$. Then

$$\mathrm{dom}(s) \cap \Delta = \mathrm{dom}(e_\Delta s) \subseteq \mathrm{dom}(e_\Delta t) = \mathrm{dom}(t) \cap \Delta.$$

In particular, if s is Δ-large, then so is t. Let us show that in this case the above is an equality, i.e., $\mathrm{dom}(s) \cap \Delta = \mathrm{dom}(t) \cap \Delta$ and thus $e_\Delta s = e_\Delta t$. Consider some $x \in \mathrm{dom}(t) \cap \Delta$. Then $x \in x^U \subseteq \Delta = (\mathrm{dom}(s) \cap \Delta)^U$; hence, $\mathrm{dom}(s) \cap x^U \neq \emptyset$. As $\mathrm{dom}(s) \cap x^U \subseteq \mathrm{dom}(t) \cap x^U$, we conclude that $\mathrm{dom}(s) \cap x^U = \mathrm{dom}(t) \cap x^U$ by Lemma 33; hence, $x \in \mathrm{dom}(s)$. ◀

For the remainder of this section, we restrict our attention to the inverse semigroups contained in the variety of interest. We assume throughout that $U \in \mathbf{SIS}$ is generated by the set $\Sigma \subseteq \mathcal{I}(\Omega)$, which is closed under formation of inverses.

The construction we use is closely tied to the representation of an inverse semigroup via its action on idempotents by conjugation – the Munn representation (see e.g. [65, 84]). Indeed, the following graph can be obtained as (part of) the Schreier graph of such an action.

▶ **Definition 35.** *Let $\Delta \subseteq \Omega$ be U-invariant and let $\Sigma_\Delta := \{u \in \Sigma \mid u \text{ is } \Delta\text{-large}\}$. We then define the graph* $\mathrm{M}(\Delta; \Sigma)$*, which we call the* Munn graph *at Δ with respect to Σ, as follows. Its set of vertices is $E_\Delta := \{e_\Delta u\bar{u} \mid u \in \Sigma_\Delta\} \subseteq E(\mathcal{I}(\Omega))$ and its set of edges is Σ_Δ, where the edge $u \in \Sigma_\Delta$ connects its source vertex $e_\Delta u\bar{u}$ to its target vertex $e_\Delta \bar{u}u$.*

Recall that, as Δ is U-invariant, we have $e_\Delta u\bar{u} = ue_\Delta\bar{u} = u\bar{u}e_\Delta$ for all $u \in \Sigma$. The Munn graph is undirected in the sense that every edge $u \in \Sigma_\Delta$ has an inverse, viz. $\bar{u}$. As indicated above, paths in $\mathrm{M}(\Delta; \Sigma)$ encode the action of U on E_Δ by conjugation.

▶ **Lemma 36.** *Let $u_1, \ldots, u_n \in \Sigma$ and $e_s, e_t \in E_\Delta$ for some U-invariant subset $\Delta \subseteq \Omega$. Then the product $u = u_1 \cdots u_n \in U$ satisfies $\bar{u}e_s u = e_t$ if and only if the sequence $(u_1, \ldots, u_n)$ is a path from e_s to e_t in the Munn graph* $\mathrm{M}(\Delta; \Sigma)$ *(and thus, in particular, $u_1, \ldots, u_n \in \Sigma_\Delta$).*

Proof. Since the general case follows by a simple induction on the number n, we will only consider the case of a single generator $u = u_1 \in \Sigma$. If u is an edge from e_s to e_t in $\mathrm{M}(\Delta; \Sigma)$, then $u \in \Sigma_\Delta$ and $\bar{u}e_s u = \bar{u}e_\Delta u\bar{u}u = e_\Delta \bar{u}u\bar{u}u = e_\Delta \bar{u}u = e_t$.

Conversely, let us now assume that $\bar{u}e_s u = e_t$, and let $u_s, u_t \in \Sigma_\Delta$ with $e_s = e_\Delta u_s \bar{u}_s$ and $e_t = e_\Delta u_t \bar{u}_t$. Using the fact that $e_\Delta \geqslant e_s$, we obtain $e_\Delta \bar{u}u = \bar{u}e_\Delta u \geqslant \bar{u}e_s u = e_t = e_\Delta u_t \bar{u}_t$. By Lemma 34, $\bar{u}u$ is Δ-large (and thus so is $u\bar{u}$) and $e_\Delta \bar{u}u = e_t$. Applying Lemma 34 again, the inequality $e_\Delta u\bar{u} = ue_\Delta \bar{u}u\bar{u} = ue_t\bar{u} \leqslant e_s = e_\Delta u_s \bar{u}_s$ implies $e_\Delta u\bar{u} = e_s$. ◀

Next, we show that every $\mathcal{D}$-class of U can be recovered from the Munn graph $\mathrm{M}(\Delta;\Sigma)$ at an appropriately chosen $\Delta \subseteq \Omega$, beginning with the idempotents of such a class. Recall that the conditions $e \mathrel{\mathcal{D}} f$, $e \mathrel{\mathcal{J}} f$, and $e \sim f$ (i.e., e and f are conjugate) are equivalent for idempotents $e, f \in E(S)$ of a finite inverse semigroup S (see Lemma 4).

▶ **Lemma 37.** *Let $\Delta = (\mathrm{dom}(e))^U$ for some $e \in E(U)$. Then the set $\{f \in E(U) \mid e \sim_U f\}$ is the vertex set of a connected component of $\mathrm{M}(\Delta;\Sigma)$.*

Proof. Let $e, f \in E(U)$ with $e \sim_U f$, i.e., with $\bar{s}es = f$ and $sf\bar{s} = e$ for some $s \in U$. Then

$$(\mathrm{dom}(e))^U = (\mathrm{dom}(f))^{\bar{s}U} \subseteq (\mathrm{dom}(f))^U = (\mathrm{dom}(e))^{sU} \subseteq (\mathrm{dom}(e))^U$$

which shows that $(\mathrm{dom}(e))^U = (\mathrm{dom}(f))^U$. We now prove that e is a vertex of $\mathrm{M}(\Delta;\Sigma)$ which, by the preceding calculation, then also holds for f. Clearly, $\mathrm{dom}(e) \subseteq \Delta = (\mathrm{dom}(e))^U$. Hence, $e = e_\Delta e$ and e is Δ-large. Since $\Sigma \subseteq U$ is a generating set and $e \in E(U)$, we have $e \leqslant u\bar{u}$ for some $u \in \Sigma$. Therefore, we have $e = e_\Delta e \leqslant e_\Delta u\bar{u}$ which, by Lemma 34, implies that $e = e_\Delta u\bar{u}$ and $u \in \Sigma_\Delta$.

The fact that f and e are connected in $\mathrm{M}(\Delta;\Sigma)$ now follows from Lemma 36, as does the fact that every vertex of $\mathrm{M}(\Delta;\Sigma)$ connected to e is some $f \in E(U)$ with $e \sim_U f$. ◀

A $\mathcal{D}$-class D of an inverse semigroup S restricts to a groupoid with objects $D \cap E(S)$ and morphisms D where $s \in D$ is a morphism from $s\bar{s}$ to $\bar{s}s$ (see [76]). In the case at hand, we have already identified the objects as the vertices of a connected component of $\mathrm{M}(\Delta;\Sigma)$.

Given a vertex $e \in E_\Delta$ of the Munn graph $\mathrm{M}(\Delta;\Sigma)$ at some U-invariant $\Delta \subseteq \Omega$, we denote by $\mathrm{M}(\Delta, e;\Sigma) \subseteq \mathrm{M}(\Delta;\Sigma)$ the connected component of e and by $E_{\Delta,e}$ and $\Sigma_{\Delta,e}$ the set of its vertices and edges, respectively. It will become apparent from the arguments below, but not stated explicitly, that the elements $e_\Delta u = (e_\Delta u\bar{u})u(e_\Delta \bar{u}u)$ with $u \in \Sigma_{\Delta,e}$ generate the $\mathcal{D}$-class of $e \in E(U)$ as a groupoid when $\Delta = (\mathrm{dom}(e))^U$ is chosen as in Lemma 37.

▶ **Definition 38.** *Let $\Delta \subseteq \Omega$ be a U-invariant set and $e \in E_\Delta$. We call a map $\gamma\colon E_{\Delta,e} \to U$ a* basis *at (Δ, e) provided it satisfies the following conditions, wherein we write $\bar{\gamma}(f) = \overline{\gamma(f)}$.*

- *The element $\gamma(e)$ is idempotent (i.e., $\gamma(e) = \bar{\gamma}(e)$) and $\gamma(e) \geqslant \gamma(f)\bar{\gamma}(f)$ for all $f \in E_{\Delta,e}$.*
- *The element $\gamma(f)$ satisfies $\bar{\gamma}(f)e\gamma(f) = f$ and $\gamma(f)f\bar{\gamma}(f) = e$ for all $f \in E_{\Delta,e}$.*

To construct a basis γ at (Δ, e) we may proceed as follows. First, let $\tilde{e} \in E(U)$ be the product $\prod u\bar{u}$ extending over all $u \in \Sigma_{\Delta,e}$ with $u\bar{u} \geqslant e$. The idempotent $\tilde{e}$ will serve as $\gamma(e)$. Next, for each other vertex $f \in E_{\Delta,e}$, we choose a path $(u_1, \ldots, u_n)$ from e to f in $\mathrm{M}(\Delta, e;\Sigma)$ and set $\gamma(f) \coloneqq \tilde{e}u_1 \ldots u_n$. Using Lemma 36, it is easy to verify that γ is as claimed.

Given a basis γ at (Δ, e), we define $\lambda\colon \Sigma_{\Delta,e} \to U$ by $\lambda(u) = \gamma(e_\Delta u\bar{u})\, u\, \bar{\gamma}(e_\Delta \bar{u}u)$. Note that $\lambda(\bar{u})$ is the inverse of $\lambda(u)$. Moreover, we have $\lambda(u)\lambda(\bar{u}) = \lambda(\bar{u})\lambda(u)$ by Lemma 31 since, clearly, $\lambda(u)\lambda(\bar{u}) \mathrel{\mathcal{J}} \lambda(\bar{u})\lambda(u)$ and $\gamma(e) \geqslant \lambda(u)\lambda(\bar{u}), \lambda(\bar{u})\lambda(u)$.

▶ **Lemma 39.** *Let $e \in E(U)$, and let γ be a basis at (Δ, e) where $\Delta = (\mathrm{dom}(e))^U \subseteq \Omega$. Then the $\mathcal{H}$-class $U_e = \{s \in U \mid s\bar{s} = \bar{s}s = e\} \leqslant U$ is generated by $\Sigma_e \coloneqq \{e\lambda(u) \mid u \in \Sigma_{\Delta,e}\} \subseteq U$.*

Proof. It is easy to verify that each $s = e\lambda(u)$ satisfies $s\bar{s} = \bar{s}s = e$. Conversely, let $s \in U$ with $s\bar{s} = \bar{s}s = e$ and write $s = u_1 \ldots u_n$ with $u_1, \ldots, u_n \in \Sigma$. Then $(u_1, \ldots, u_n)$ is a path from e to e in $\mathrm{M}(\Delta, e;\Sigma)$ by Lemma 36 as $\bar{s}es = e$. In particular, $u_1, \ldots, u_n \in \Sigma_{\Delta,e}$.

Let $e = e_0, e_1, \ldots, e_n = e \in E_{\Delta,e}$ be the vertices along the path $(u_1, \ldots, u_n)$ and note that $e_0, e_1 \ldots, e_n \in E(U)$ by Lemma 37. We now compute

$$s = u_1 \ldots u_n \geqslant e\gamma(e_0)\, u_1\, \bar{\gamma}(e_1)\, e\gamma(e_1)\, u_2 \ldots u_n\, \bar{\gamma}(e_n) = e\lambda(u_1)\, e\lambda(u_2) \ldots e\lambda(u_n),$$

wherein the left side equals $e_\Delta s$ and the right side equals $e_\Delta \lambda(u_1)\lambda(u_2)\ldots\lambda(u_n)$. Therefore, we can then conclude that both sides of the inequality are equal by Lemma 34. ◀

▶ **Lemma 40.** *Let $e \in E(\mathcal{I}(\Omega))$. If $\hat{e} \in E(U)$ is minimal with $e \leqslant \hat{e}$, then there is a unique $e' \in E_\Delta$ with $e \leqslant e'$ where $\Delta = (\mathrm{dom}(e))^U$. Moreover, $\hat{e} = \prod \lambda(u)\lambda(\bar{u})$ where the product extends over all $u \in \Sigma_{\Delta,e'}$ and λ is obtained from some basis γ at (Δ, e').*

Note that $e \leqslant e' \leqslant \hat{e}$; hence, if $e = \hat{e}$, then also $e = e'$.

Proof. Suppose that $\hat{e} \in E(U)$. Then $\mathrm{dom}(e) \subseteq (\mathrm{dom}(e))^U = \Delta$ and thus $e \leqslant e_\Delta$. Let $u \in \Sigma$ with $\hat{e} \leqslant u\bar{u}$. Then $e = e_\Delta e \leqslant e_\Delta \hat{e} \leqslant e_\Delta u\bar{u}$ and $u\bar{u}$ is Δ-large; hence, $e' = e_\Delta u\bar{u} \in E_\Delta$. Conversely, if $u \in \Sigma_\Delta$ with $e \leqslant e_\Delta u\bar{u}$, then $e \leqslant u\bar{u}$ and, by minimality, $\hat{e} \leqslant u\bar{u}$. We obtain the inequality $e_\Delta \hat{e} \leqslant e_\Delta u\bar{u}$, which, by Lemma 34, is an equality. As such, $e' = e_\Delta \hat{e}$ is the unique vertex $e' \in E_\Delta$ with the property $e \leqslant e'$.

To see that $\hat{e}$ can be written as the product $\hat{e}_\lambda$ of all $\lambda(u)\lambda(\bar{u})$ with $u \in \Sigma_{\Delta,e'}$, we note that $\hat{e} = u_1 \ldots u_n \bar{u}_n \ldots \bar{u}_1$ for some $u_1, \ldots u_n \in \Sigma$. Then $(u_1, \ldots, u_n, \bar{u}_n, \ldots, \bar{u}_1)$ is a path from e' to e' in $\mathrm{M}(\Delta; \Sigma)$ by Lemma 36; so $u_1, \ldots, u_n \in \Sigma_{\Delta,e'}$. Let $e'_0, e'_1, \ldots, e'_n, \ldots, e'_1, e'_0 \in E_{\Delta,e'}$ be the vertices of $\mathrm{M}(\Delta; \Sigma)$ along this path. Inserting idempotents as in the proof of Lemma 39,

$$\begin{aligned} \hat{e} = u_1 \ldots u_n \bar{u}_n \ldots \bar{u}_1 &\geqslant \gamma(e'_0) u_1 \bar{\gamma}(e'_1) \ldots u_n \bar{\gamma}(e'_n) \gamma(e'_n) \bar{u}_n \ldots \gamma(e'_1) \bar{u}_1 \bar{\gamma}(e'_0) \\ &= \lambda(u_1) \ldots \lambda(u_n) \lambda(\bar{u}_n) \ldots \lambda(\bar{u}_1) = \lambda(u_1)\lambda(\bar{u}_1) \ldots \lambda(u_n)\lambda(\bar{u}_n) \geqslant \hat{e}_\lambda \end{aligned}$$

where the final equality follows from Lemma 32 as $\lambda(u)\lambda(\bar{u}) = \lambda(\bar{u})\lambda(u)$ for all $u \in \Sigma_{\Delta,e'}$. Since $e \leqslant e' \leqslant \hat{e}_\lambda$, we have $\hat{e} \leqslant \hat{e}_\lambda$ by minimality of $\hat{e}$. Hence, $\hat{e} = \hat{e}_\lambda$. ◀

5.2 The Membership and Conjugacy Problems in SIS

We now use the theoretical machinery developed in Section 5.1 to show how the membership problem (and also the conjugacy problem to a certain extend) for finite strict inverse semigroups can be solved efficiently. More precisely, we will show the following.

▶ **Theorem 41.** *Let $\mathbf{H} \subseteq \mathbf{G}$ be a variety of finite groups. The problems* $\textsc{memb}_{\mathbf{PB}}(\mathbf{H} \vee \mathbf{BS})$ *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{H} \vee \mathbf{BS})$ *are* L*-reducible to* $\textsc{memb}_{\mathbf{PB}}(\mathbf{H})$ *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{H})$ *for* $\mathbf{H}$*, respectively.*

Using the facts that $\textsc{memb}_{\mathbf{PB}}(\mathbf{G})$ is in NC (by [7], see Proposition 16) and $\textsc{conj}_{\mathbf{PB}}(\mathbf{G})$ is in NP (see Proposition 19), this implies the following upper bounds for $\mathbf{SIS} = \mathbf{G} \vee \mathbf{BS}$.

▶ **Corollary 42.** *The problem* $\textsc{memb}_{\mathbf{PB}}(\mathbf{SIS})$ *is in* NC *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{SIS})$ *is in* NP*.*

The other extreme, i.e., taking $\mathbf{H}$ to be the trivial variety $\mathbf{T}$ in Theorem 41, yields the following upper bounds. These are interesting because E-$\textsc{memb}_{\mathbf{CT}}(\mathbf{BS})$ and E-$\textsc{conj}_{\mathbf{CT}}(\mathbf{BS})$ are already complete for L as we will later show; see Proposition 54 and Proposition 53.

▶ **Corollary 43.** *The problems* $\textsc{memb}_{\mathbf{PB}}(\mathbf{BS})$ *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{BS})$ *are in* L*.*

We prove Theorem 41 using a sequence of short lemmas showing that the constructions from the previous subsection can actually be computed in L.

▶ **Lemma 44.** *Given $\Sigma \subseteq \mathcal{I}(\Omega)$ and $X \subseteq \Omega$ the $\langle\Sigma\rangle$-invariant set $X^{\langle\Sigma\rangle}$ can be computed in* L*.*

Proof. Define the (Schreier) graph Γ with vertex set Ω and an edge from x to $y \in \Omega$ whenever $y = x^u$ for some $u \in \Sigma$. Observe that this graph is undirected because $y = x^u$ if and only if $x = y^{\bar{u}}$. Now, $X^{\langle\Sigma\rangle}$ consists simply of the vertices with an incident edge and which are reachable from X. The latter can be checked using an oracle for UGAP. ◀

▶ **Lemma 45.** *Given $\Sigma \subseteq \mathcal{I}(\Omega)$ and Δ, the Munn graph $\mathrm{M}(\Delta;\Sigma)$ can be computed in* L.

Proof. To determine the edge set Σ_Δ of $\mathrm{M}(\Delta;\Sigma)$, we check whether $(\mathrm{dom}(u) \cap \Delta)^U = \Delta$ for each $u \in \Sigma$ using Lemma 44. Now, Definition 35 immediately gives us E_Δ and $\mathrm{M}(\Delta;\Sigma)$. ◀

▶ **Lemma 46.** *Given a Munn graph $\mathrm{M}(\Delta;\Sigma)$ and $e \in E_\Delta$, a basis $\gamma\colon E_{\Delta,e} \to U$ at (Δ, e) represented as a list $(f, \gamma(f))_{f \in E_{\Delta,e}}$, where $U = \langle\Sigma\rangle$, can be computed in* L.

Proof. The alphabet $\Sigma_{\Delta,e}$ needed for the definition of γ can be found by computing the connected component of $\mathrm{M}(\Delta;\Sigma)$ containing e, which can be done using an UGAP oracle. Next, to compute γ, we use [78, Lemma 2.4] to find a path $(u_1, \ldots, u_n)$ from e to some arbitrary $f \in E_{\Delta,e}$. Note that computing the product of a sequence of elements $u_1, \ldots, u_n \in \mathcal{I}(\Omega)$ can be done in L by evaluating $x^{u_1 \cdots u_n}$ for each $x \in \Omega$ separately. ◀

▶ **Lemma 47.** *Given $\Sigma \subseteq \mathcal{I}(\Omega)$ and $e \in E(\mathcal{I}(\Omega))$ the minimal $\hat{e} \in E(U) \cup \{1\}$ with $e \leqslant \hat{e}$ can be computed in L and, as part of the computation, one can decide whether $\hat{e} \in U = \langle\Sigma\rangle$.*

Proof. First, compute $\Delta = \mathrm{dom}(e)^U$ using Lemma 44 and the Munn graph $\mathrm{M}(\Delta;\Sigma)$ using Lemma 45. Next find some $e' \in E_\Delta$ with $e \leqslant e'$. If no such e' exists, then $\hat{e} = 1$ and $\hat{e} \notin U$. Otherwise, compute a basis $\gamma\colon E_{\Delta,e'} \to U$ at (Δ, e') by Lemma 46. Finally, to compute $\hat{e}$, we apply the formula from Lemma 40, which uses the already computed basis γ. ◀

Proof of Theorem 41. We first consider the membership problem. On the input of $\Sigma \subseteq \mathcal{I}(\Omega)$ with $U = \langle\Sigma\rangle \in \mathbf{H} \vee \mathbf{BS}$ and $t \in \mathcal{I}(\Omega)$, let $e = t\bar{t}$ and $f = \bar{t}t$. We then proceeds as follows.

Compute $\Delta = \mathrm{dom}(e)^U = \mathrm{dom}(f)^U$ using Lemma 44. If $\mathrm{dom}(e)^U \neq \mathrm{dom}(f)^U$, then $t \notin U$ and we can output a fixed negative instance. Next, we compute $\hat{e}$ and $\hat{f}$ (Lemma 47) and verify that $e = \hat{e} \in E_\Delta$ and $f = \hat{f} \in E_\Delta$ (if not, then $t \notin U$). Let $\gamma\colon E_{\Delta,e} \to U$ be a basis at (Δ, e), which we can compute by Lemma 46. Finally, using the basis γ we compute a generating set $\Sigma_e \subseteq \mathcal{I}(\Omega)$ of the $\mathcal{H}$-class $U_e = \{s \in U \mid s\bar{s} = \bar{s}s = e\}$ as in Lemma 39 and reduce to the question of whether $t' \coloneqq t\bar{\gamma}(f) \in \langle\Sigma_e\rangle$ – an instance of $\mathrm{MEMB}_{\mathbf{PB}}(\mathbf{H})$.

If $t' \in U_e$, then $t = t'\gamma(f) = t\bar{\gamma}(f)\gamma(f) \in U$. For the other direction, assume that $t \in U$. Then clearly $e, f \in U$ so $e = \hat{e}$ and $f = \hat{f}$. Moreover, e and f are vertices of a connected component of $\mathrm{M}(\Delta;\Sigma)$ by Lemma 37. Finally, $t' \in U_e$ as $t\bar{\gamma}(f)\gamma(f)\bar{t} = \gamma(f)\bar{t}t\bar{\gamma}(f) = e$.

We now turn our attention to the conjugacy problem. The input comprises $U = \langle\Sigma\rangle$ as above and two elements $s, t \in \mathcal{I}(\Omega)$. The question is whether there exists some $u \in U^1$ such that $\bar{u}su = t$ and $ut\bar{u} = s$. We assume throughout that $s \neq t$ (otherwise, we reduce to a fixed positive instance). Let $e = s\bar{s} \vee \bar{s}s \in E(\mathcal{I}(\Omega))$ and $f = t\bar{t} \vee \bar{t}t \in E(\mathcal{I}(\Omega))$.

Compute $\hat{e}$ and $\hat{f}$ (as above), as well as $\hat{\Delta} = \mathrm{dom}(\hat{e})^U = \mathrm{dom}(\hat{f})^U$. Note that we do not require that $e = \hat{e}$ or $f = \hat{f}$. If $\mathrm{dom}(\hat{e})^U \neq \mathrm{dom}(\hat{f})^U$ or $\hat{e}, \hat{f} \notin U$ or, similarly, if $\hat{e}$ and $\hat{f}$ are not connected in $\mathrm{M}(\hat{\Delta};\Sigma)$, then $s \not\sim_U t$. Next, compute a basis $\gamma\colon E_{\hat{\Delta},\hat{e}} \to U$ at $(\hat{\Delta}, \hat{e})$ and a corresponding generating set $\Sigma_{\hat{e}}$ of the $\mathcal{H}$-class $U_{\hat{e}}$ of $\hat{e}$ as in Lemma 39. Finally, reduce to the question of whether s is conjugate to $t' \coloneqq \gamma(\hat{f})t\bar{\gamma}(\hat{f})$ in $\langle\Sigma_{\hat{e}}\rangle$ – an instance of $\mathrm{CONJ}_{\mathbf{PB}}(\mathbf{H})$.

If $s \sim_{U_{\hat{e}}} t'$, then $s \sim_U t$ as $t \sim_U t'$. For the other direction, assume that s and t are conjugate by some element $u \in U$, i.e., $\bar{u}su = t$ and $ut\bar{u} = s$. Then $\hat{e}$ and $\hat{f}$ are also conjugate by the element u. To see this, note that $\bar{u}\hat{e}u \geqslant f$. Hence, $\bar{u}\hat{e}u \geqslant \hat{f}$ (by minimality of $\hat{f}$) and $u\hat{f}\bar{u} \geqslant \hat{e}$ (by symmetry). The chain of inequalities $\hat{f} = \bar{u}u\hat{f}\bar{u}u \geqslant \bar{u}\hat{e}u \geqslant \hat{f}$ then finally shows that $\bar{u}\hat{e}u = \hat{f}$ and $u\hat{f}\bar{u} = \hat{e}$ (by symmetry). It now follows from Lemma 37 that $\hat{e}$ and $\hat{f}$ are vertices of a connected component of $\mathrm{M}(\hat{\Delta}; \Sigma)$. Writing $v = \bar{\gamma}(\hat{f})$, we can easily verify that uv conjugates s to t', i.e., $\bar{v}\bar{u}suv = \bar{v}tv = t'$ and $uvt'\bar{v}\bar{u} = uv(\bar{v}tv)\bar{v}\bar{u} = s$. As clearly $\hat{e}s\hat{e} = s$, the element $\hat{e}uv$ also conjugates s to t'. Moreover, $\hat{e}uv \in U_{\hat{e}}$ and, as such, $s \sim_{U_{\hat{e}}} t'$. ◀

6 Inverse Semigroups in the Cayley Table Model

In this section we complete the proof of Theorem A, our main dichotomy theorem for Cayley table model, which is restated here for the readers convenience.

▶ **Theorem 48.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups.*
- *If $\mathbf{V} \subseteq \mathbf{Cl}$, then $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ and $\mathrm{CONJ}_{\mathbf{CT}}(\mathbf{V})$ are in* NPOLYLOGTIME *and in* L.
- *If $\mathbf{V} \not\subseteq \mathbf{Cl}$, then $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ and $\mathrm{CONJ}_{\mathbf{CT}}(\mathbf{V})$ are* L*-complete under* AC^0*-reductions.*

Recall that the variety $\mathbf{Cl}$ of finite Clifford semigroup is defined by the identity $\bar{x}x = x\bar{x}$; it is the smallest variety containing all groups and semilattices. In Section 4.1 we have seen that $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{Cl})$ and $\mathrm{CONJ}_{\mathbf{CT}}(\mathbf{Cl})$ are in NPOLYLOGTIME. Completing the proof of Theorem 48 consists of two steps: in Section 6.1 we establish L-algorithms for these problems, and in Section 6.2 we show that the problems are hard for L given that $\mathbf{V} \not\subseteq \mathbf{Cl}$. Before we go into the details of our proof, let us explore an interesting consequence of Theorem 48.

▶ **Corollary 49.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups. The following are equivalent.*
- *The class $\mathbf{V}$ comprises only Clifford semigroups, i.e., $\mathbf{V} \subseteq \mathbf{Cl}$.*
- *The class $\mathbf{V}$ admits polylogarithmic SLPs.*
- *The problem $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ is in* NPOLYLOGTIME.
- *The problem $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ is in* qAC^0.

Proof. By Lemma 24, any variety $\mathbf{V} \subseteq \mathbf{Cl}$ admits polylogarithmic SLPs. By Lemma 11, if $\mathbf{V}$ admits polylogarithmic SLPs, then $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ is in $\mathsf{NPOLYLOGTIME} \subseteq \mathsf{qAC}^0$.

Finally, if $\mathbf{V} \not\subseteq \mathbf{Cl}$, then the problem $\mathrm{MEMB}_{\mathbf{CT}}(\mathbf{V})$ is L-complete under AC^0-reductions by Theorem 48. Hence, this problem cannot be solved in qAC^0 as for example PARITY can be solved in L but not in qAC^0 [38, 45]. ◀

Note that the equivalence of the first two points of Corollary 49 can also be proved directly. Indeed, the first author determined in his dissertation [35] the maximal varieties of finite (not necessarily inverse) *monoids* admitting polylogarithmic SLPs, viz. the varieties of finite Clifford monoids and of finite commutative monoids. However, the situation for semigroups is considerably more intricate. Therefore, we point out the following open problem.

▶ **Question 50.** *Which varieties of arbitrary finite semigroups admit polylogarithmic SLPs?*

6.1 Membership and Conjugacy in L

It is known that the membership problem for semigroups in the Cayley table model belongs to NL [53]. This immediately carries over to inverse semigroups. In this section we go further and show that this can improved to L for inverse semigroups. Recall that NL is intricately related to *directed* graph accessibility, while SL = L by Reingold's result [89] corresponds to *undirected* graph accessibility. This difference in complexity is explained by the observation that strong connected components of the (right) Cayley graph of an inverse semigroup are actually undirected graphs. To formalize this, let us define the decision problem $\mathcal{R}$-$\mathrm{EQUIV}_{\mathbf{CT}}$.

Input. An inverse semigroup S as Cayley table, a subset $\Sigma \subseteq S$, and elements $s, t \in S$.
Question. Is $s \mathrel{\mathcal{R}_U} t$ where $U = \langle \Sigma \rangle$?

Recall that $x \mathrel{\mathcal{R}_U} y$ for $x, y \in S$ if and only if there are $r, s \in U$ with $xr = y$ and $ys = x$.

▶ **Proposition 51.** *The problems* $\mathcal{R}$-EQUIV$_{\mathbf{CT}}$ *and* CONJ$_{\mathbf{CT}}$ *are in* L.

Proof. Both problems are essentially reachability in an undirected graph. Indeed, define the undirected graph Γ with vertex set S and an edge between x and y for $x, y \in S$ whenever there is some $u \in \Sigma$ with $xu = y$ and $x = y\bar{u}$. Clearly, Γ can be computed in L.

By Lemma 2, if $xu \mathrel{\mathcal{R}_{\langle\Sigma\rangle}} x$ for $u \in \Sigma$ and $x \in S$, then x and xu are connected by an edge in Γ (and the converse is obviously true). In particular, the connected components of Γ are precisely the strongly connected components of the right Cayley graph of S with respect to the generating set Σ. Therefore, reachability in Γ is exactly the question whether $s \mathrel{\mathcal{R}_{\langle\Sigma\rangle}} t$.

To decide whether or not $s \sim_{\langle\Sigma\rangle} t$, we proceed in the same way but in the graph with edges $\{x, y\}$ whenever there is some $u \in \Sigma$ such that $\bar{u}xu = y$ and $x = uy\bar{u}$. ◀

▶ **Proposition 52.** *The problem* MEMB$_{\mathbf{CT}}$ *is in* L.

Proof. We are given an inverse semigroup S, a subset $\Sigma \subseteq S$ and an element $t \in S$ and we want to decide whether $t \in U = \langle\Sigma\rangle$. To this end, we describe a L-algorithm using oracle calls to UGAP based on Proposition 51.

Without loss of generality, we assume that both S and U contain a neutral element 1 (by simply adjoining such an element) and that Σ is closed under formation of inverses. Our algorithm then proceeds as follows.

1: $x \leftarrow 1$
2: **while** $\exists y \in S$, $u \in \Sigma$ **with** $x \mathrel{\mathcal{R}_U} y$ **and** $yu\bar{u} \neq y$ **and** $yu\bar{u}\bar{y}\,t = t$ **do**
3: $\quad x \leftarrow yu$
4: **if** $x \mathrel{\mathcal{R}_U} t$ **then**
5: $\quad$ **return true**
6: **return false**

The tests whether $x \mathrel{\mathcal{R}_U} y$ can be done in L by Proposition 51 (using oracle calls to UGAP). Moreover, the tests whether there exist $y \in S$, $u \in \Sigma$ meeting the conditions in line 2, can be done by iterating over all such elements checking whether the conditions are satisfied.

Throughout, we keep the invariant that $x \in U$. Therefore, if our algorithm outputs **true**, then, indeed, $t \in U$. On the other hand, let $t = u_1 \cdots u_n \in U$ with $u_i \in \Sigma$. The idea is that the algorithm finds this (or a slightly modified) sequence. Observe that besides $x \in U$ we maintain the invariant $x\bar{x}t = t$ (i.e., $x \geqslant_{\mathcal{R}} t$).

Next, observe that, by Lemma 2, $yu\bar{u} \neq y$ means that y is not $\mathcal{R}_U$-equivalent to yu or, more specifically, that $y >_{\mathcal{R}_U} yu$. As such, the while loop can be executed only finitely (indeed, at most $|U|$) many times. Therefore, we can proceed in the following by induction on the number of times that line 3, the body of the while loop, is being executed.

If $x \in U$ with $x\bar{x}t = t$ and $t \in U$, then either $x \mathrel{\mathcal{R}_U} t$ or there is some $j \in \{1, \ldots, n\}$ with $y = x\bar{x}u_1 \cdots u_j \mathrel{\mathcal{R}_U} x$ but $x\bar{x}u_1 \cdots u_{j+1} <_{\mathcal{R}_U} x$. In the former case, there are no y and u meeting the conditions in line 2; hence, the algorithm terminates and outputs **true**. In the latter case, we have $x \mathrel{\mathcal{R}_U} y$ and $yu_{j+1}\bar{u}_{j+1} \neq y$ and $yu_{j+1}\bar{u}_{j+1}\bar{y}t = t$. As such, the while loop will be executed at least one more time and, by induction, the algorithm will therefore answer correctly that $t \in U$. ◀

6.2 Hardness of Membership and Conjugacy for L

We now turn to hardness results for the (idempotent) membership and conjugacy problem for inverse semigroups in the Cayley table model. Recall that a variety of finite inverse semigroups $\mathbf{V}$ satisfies $\mathbf{V} \not\subseteq \mathbf{Cl}$ if and only if $\mathbf{BS} \subseteq \mathbf{V}$, i.e., $\mathbf{V}$ contains the combinatorial Brandt semigroup B_2; see the second item of Proposition 6.

▶ **Proposition 53.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups and suppose that $\mathbf{BS} \subseteq \mathbf{V}$. Then the problem E-$\textsc{conj}^{\sharp}_{\mathbf{CT}}(\mathbf{V})$ is L-hard under AC^0-reductions.*

Proof. To show hardness for $\mathsf{L} = \mathsf{SL}$, we reduce from the problem $\textsc{ugap}$. Given an undirected graph $G = (V, E)$ and vertices $s, t \in V$, we proceed as follows.

Let $S = B(V)$ be the Brandt semigroup on V (recall that $B_2 \in \mathbf{V}$ implies $B(V) \in \mathbf{V}$ by Lemma 5). Given $x, y \in V$, we denote by u_{xy} the unique element of S mapping x to y. Note that each non-zero element of S is of this form, and the non-zero idempotents are precisely the elements $e_x \coloneqq u_{xx}$. Crucially, for fixed idempotents e_x and e_y, the equation $\bar{u}e_x u = e_y$ has exactly one solution in S, namely the element $u = u_{xy}$. In particular, $e_x \sim_S e_y$.

Let $\Sigma = \{e_x \mid x \in V\} \cup \{u_{xy} \mid xy \in E\}$. An element u_{xy} is contained in $U = \langle \Sigma \rangle \leqslant S$ if and only if the vertices x and y are connected by a path in G. In particular, $e_s \sim_U e_t$ holds if and only if s and t are connected by a path in G. Since the Cayley table of S, the set $\Sigma \subseteq S$, and the idempotents $e_s, e_t \in E(S)$ can be computed in AC^0 from $G = (V, E)$ and $s, t \in V$, we conclude that the (restricted) idempotent conjugacy problem for $\mathbf{V}$ is L-hard. ◀

This argument also shows that the membership problem for $\mathbf{V}$ is L-hard in the Cayley table model as we can simply ask whether or not $u_{st} \in U$. Even more, the following holds.

▶ **Proposition 54.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups and suppose that $\mathbf{BS} \subseteq \mathbf{V}$. Then the idempotent membership problem E-$\textsc{memb}^{\sharp}_{\mathbf{CT}}(\mathbf{V})$ is L-hard.*

Together with Proposition 53, Proposition 54 establishes the hardness part of Theorem 48. Our proof of Proposition 54 is a slight variation of the proof of Proposition 53. It is based on the observation that certain instances of the idempotent conjugacy problem reduce to the idempotent membership problem for a slightly larger inverse semigroup.

▶ **Lemma 55.** *Let U be an inverse semigroup and $e_s, e_t \in E(U)$. Further, let $U' \leqslant U \times Y_2$ be generated by $(e_s, 0)$ and $U \times \{1\}$. Then $e_s \geqslant_{\mathcal{J}} e_t$ holds if and only if $(e_t, 0) \in U'$.*

Proof. If $e_s \geqslant_{\mathcal{J}} e_t$, then $e_t = ue_sv$ for some $u, v \in U^1$; hence, $(e_t, 0) = (u, 1)(e_s, 0)(v, 1) \in U'$. Conversely, if $(e_t, 0) \in U'$ holds, then $(e_t, 0) = u_1u_2 \ldots u_n$ for some $u_i \in \{(e_s, 0)\} \cup U \times \{1\}$. Clearly, at least one of the factors u_i must be equal to $(e_s, 0)$. By replacing $(e_s, 0)$ with $(e_s, 1)$ for all but one of the factors u_i, and using the fact that $U \times \{1\}$ is a subsemigroup of U', we obtain $(e_t, 0) = (u, 1)(e_s, 0)(v, 1)$ for some $u, v \in U^1$; hence, $e_s \geqslant_{\mathcal{J}} e_t$ as $e_t = ue_sv$. ◀

Proof of Proposition 54. Given an undirected graph $G = (V, E)$ and $s, t \in V$, we let $U \leqslant S$ and Σ be as in the proof of Proposition 53. We consider $S' = S \times Y_2$ and $U' = \langle \Sigma' \rangle \leqslant S'$ where $\Sigma' = \{(e_s, 0)\} \cup \Sigma \times \{1\}$. Note that $U', S' \in \mathbf{V}$ since $S \in \mathbf{V}$ and $Y_2 \in \mathbf{V}$. By Lemma 55, the idempotent $(e_t, 0) \in S'$ is contained in U' if and only if $e_s \geqslant_{\mathcal{J}_U} e_t$. By Lemma 4, this is equivalent to the existence of some $u \in U^1$ with $e_t = \bar{u}e_s u$. Since we already know that the latter holds if and only if s and t are connected by a path in G, this completes the reduction from the undirected graph reachability problem to the idempotent membership problem. ◀

7 Inverse Semigroups in the Partial Bijection Model

In this section we finally complete the proof of our dichotomy theorem regarding the membership and conjugacy problem for inverse semigroups in the partial bijection model.

▶ **Theorem 56** (Theorem B)**.** *Let* $\mathbf{V}$ *be a variety of finite inverse semigroups.*

- *If* $\mathbf{V} \subseteq \mathbf{SIS}$*, then* $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ *is in* NC *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ *is in* NP*.*
- *If* $\mathbf{V} \not\subseteq \mathbf{SIS}$*, then* $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ *and* $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ *are* PSPACE*-complete.*

As outlined in Section 1.1, the case $\mathbf{V} \subseteq \mathbf{SIS}$ can be further refined as follows implying actually an AC^0-vs.-NC-vs.-PSPACE-complete trichotomy for $\textsc{memb}$.

- If $\mathbf{V} \subseteq \mathbf{Sl}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ are in AC^0.
- If $\mathbf{V} = \mathbf{BS}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ are L-complete.
- If $\mathbf{V} \not\subseteq \mathbf{BS}$, then $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ is in NC and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ is in NP; both are hard for L.

Indeed, $\textsc{memb}_{\mathbf{PB}}(\mathbf{Sl})$ is considered in [14] and shown to be in AC^0. Note that the conjugacy problem for semilattices is trivially in AC^0 as in a semilattice S two elements are conjugate if and only if they are equal (recall that $S = E(S)$ and that S is $\mathcal{J}$-trivial).

If $\mathbf{V} \not\subseteq \mathbf{Sl}$, then either $\mathbf{V}$ contains the Brandt semigroup B_2 or some non-trivial group. In the first case, we get L-hardness by Theorem A as by the Preston-Wagner Theorem [86, 104] every inverse semigroup given as Cayley table can be interpreted as an inverse semigroup in the partial bijection model. In the second case, L-hardness of $\textsc{memb}_{\mathbf{PB}}(\mathbf{V})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{V})$ is established in Proposition 17. Thus, both problems are L-hard whenever $\mathbf{V} \not\subseteq \mathbf{Sl}$. The fact that $\textsc{memb}_{\mathbf{PB}}(\mathbf{BS})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{BS})$ are contained in L is the content of Corollary 43.

For proving Theorem 56, recall that either $\mathbf{V} \subseteq \mathbf{SIS}$ or $\mathbf{BM} \subseteq \mathbf{V}$ for each variety $\mathbf{V}$ of finite inverse semigroups by Proposition 6. With the first part of Theorem 56 covered by Corollary 42, it therefore suffices to show that the problems $\textsc{memb}_{\mathbf{PB}}(\mathbf{BM})$ and $\textsc{conj}_{\mathbf{PB}}(\mathbf{BM})$ are PSPACE-complete. Moreover, since these problems are clearly contained in PSPACE, it suffices to prove their hardness for it. We will do so by a suitable reduction in Section 7.1.

Slight variations of this reduction will then allow us to derive hardness results for the intersection non-emptiness problem as well as the subpower membership problem in Section 7.2 and Section 7.3, respectively.

7.1 Hardness of Membership and Conjugacy for PSPACE

As mentioned in the introduction to this section, our goal is to prove PSPACE-hardness of the membership and conjugacy problems for $\mathbf{BM}$ in the partial bijection model. In fact, we will show that this even holds for the idempotent variants of these problems.

▶ **Theorem 57.** *The problems* E-$\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$ *and* E-$\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$ *are* PSPACE*-complete.*

These problems are clearly contained in PSPACE. We prove their PSPACE-hardness by a reduction from the decision problem $\textsc{ncl}$ of *non-determinstic constraint logic* (*NCL*), which was introduced by Hearn and Demaine [46] and which we briefly describe in the following.

An *NCL machine* Γ is an edge-weighted simple undirected graph, every vertex of which has degree three, and every edge of which has weight one or two. A *configuration* of the NCL machine is an orientation of all edges such that the sum of the incoming edge weights (in-flow) at every vertex is at least two. We will denote the set of all configurations by $\mathcal{C}(\Gamma)$. Two configurations are related by a *transition* if they differ in the orientation of a single edge.

The decision problem $\textsc{ncl}$, in the *configuration-to-configuration* variant, is given as follows.

Input. An NCL machine Γ, and two configurations $C_s, C_t \in \mathcal{C}(\Gamma)$.
Question. Are C_s and C_t related by a sequence of transitions?

This problem is essentially a compressed version of the accessibility problem for undirected graphs, where accessibility is decided on the implicitly given graph of configurations and transitions between these. The problem $\textsc{ncl}$ is complete for PSPACE [46, Theorem 5].

Crucial to our cause is the observation that the validity of any configuration can be verified locally, as the minimum in-flow constraint has to be satisfied at each vertex individually. Moreover, transitions are also local in the sense that each transition affects only a single edge and, therefore, only the two vertices incident with that edge.

We call an orientation of all edges incident with a fixed vertex v a *local configuration* at v provided that the minimum in-flow constraint at v is satisfied. The set of all local configurations at v will be denoted by $\mathcal{C}(\Gamma, v)$. Note that $|\mathcal{C}(\Gamma, v)| \leqslant 7$ as v has degree three and at least one edge needs to be oriented towards v to satisfy the in-flow constraint.

Given an orientation O of all edges of Γ, we denote its restriction to an orientation of the edges incident with v by $O|_v$. Note that an orientation O is a configuration, i.e., $O \in \mathcal{C}(\Gamma)$, if and only if each restriction $O|_v$ is a local configuration, i.e., $O|_v \in \mathcal{C}(\Gamma, v)$ for all $v \in V(\Gamma)$.

Proof of Theorem 57. We first reduce the problem NCL to the problem E-$\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$. This shows that the latter problem is hard for PSPACE and, therefore, PSPACE-complete.

Given an NCL machine Γ, we associate to it the inverse semigroup $S_\Gamma = \prod_v B^1(\mathcal{C}(\Gamma, v))$ where the direct product extends over all vertices $v \in V(\Gamma)$. We identify S_Γ with an inverse subsemigroup of $\mathcal{I}(\Omega_\Gamma)$ where $\Omega_\Gamma = \bigsqcup_v \mathcal{C}(\Gamma, v)$. Note that $|\Omega_\Gamma| \in \mathcal{O}(|V(\Gamma)|)$.

Each configuration $C \in \mathcal{C}(\Gamma)$ has an idempotent $e(C) \in E(S_\Gamma)$ canonically associated to; the projection of $e(C)$ onto a factor $B^1(\mathcal{C}(\Gamma, v))$ is given by the idempotent at $C|_v \in \mathcal{C}(\Gamma, v)$. Note that any two such idempotents are conjugate in S_Γ. We now describe how to encode a transition by a corresponding element of S_Γ. Suppose given an oriented edge o from v_1 to v_2, say, and local configurations $c_i \in \mathcal{C}(\Gamma, v_i)$ with $o \in c_i$ for $i = 1, 2$. Let o' denote the reversal of o, and let c'_i denote the result of replacing o by o' in c_i. Provided that $c'_1 \in \mathcal{C}(\Gamma, v_1)$ and $c'_2 \in \mathcal{C}(\Gamma, v_2)$, we define $u(c_1, c'_2) \in S_\Gamma$ as follows. The projection of $u(c_1, c'_2)$ onto $B^1(\mathcal{C}(\Gamma, v))$ where $v = v_i$ is the unique partial bijection in $B^1(\mathcal{C}(\Gamma, v_i))$ with $c_i \mapsto c'_i$, and for $v \notin \{v_1, v_2\}$ the projection of $u(c_1, c'_2)$ onto the factor $B^1(\mathcal{C}(\Gamma, v))$ is the identity element. Note that we have $\overline{u(c_1, c'_2)} = u(c'_2, c_1)$ and that $u(c_1, c'_2)$ is defined if and only if $u(c'_2, c_1)$ is.

Let $U_\Gamma \leqslant S_\Gamma$ be the inverse subsemigroup generated by the set $\Sigma_\Gamma \subseteq S_\Gamma \leqslant \mathcal{I}(\Omega_\Gamma)$ of all elements $u(c_1, c'_2)$ as above. Note that $|\Sigma_\Gamma| \in \mathcal{O}(|V(\Gamma)|)$. We claim that $e(C_s) \sim_{U_\Gamma} e(C_t)$ for two configurations $C_s, C_t \in \mathcal{C}(\Gamma)$ if and only if these configurations can be transformed into one another by a sequence of transitions. To see this, consider an element of the form $\bar{u}eu$ where $e = e(C)$ for some $C \in \mathcal{C}(\Gamma)$ and with $u = u(c_1, c'_2) \in U_\Gamma$ as above. Then either $C|_{v_i} = c_i$ for $i = 1, 2$, in which case $\bar{u}eu = e(C')$ with $C' \in \mathcal{C}(\Gamma)$ obtained from C by reversal of o, or else $\bar{u}eu \in I_\Gamma$ where $I_\Gamma \subseteq S_\Gamma$ is the ideal comprising all elements with at least one projection equal to $0 \in B^1(\mathcal{C}(\Gamma, v))$. Since $e(C) \notin I_\Gamma$ for all $C \in \mathcal{C}(\Gamma)$, this proves the claim.

Given an instance of the problem NCL comprising an NCL machine Γ and $C_s, C_t \in \mathcal{C}(\Gamma)$, we associate with it the instance of the problem E-$\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$ comprising $\Sigma \coloneqq \Sigma_\Gamma \subseteq \mathcal{I}(\Omega_\Gamma)$ and idempotents $s \coloneqq e(C_s), t \coloneqq e(C_t) \in E(S_\Gamma) \subseteq E(\mathcal{I}(\Omega_\Gamma))$ as above. Clearly, the latter can be computed in polynomial time from the former. As the problem NCL is hard for PSPACE under polynomial-time many-one reductions, so is the problem E-$\textsc{conj}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$.

Finally, to see that the problem E-$\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$ is also hard for PSPACE, let $S'_\Gamma = S_\Gamma \times Y_2$ and U'_Γ be its inverse subsemigroup generated by $\Sigma_\Gamma \times \{1\}$ and the elements $(e(C_s), 0), (e(C_t), 1) \in S'_\Gamma$. Then $(e(C_t), 0) \in U'_\Gamma$ if and only if $e(C_s) \geqslant_{\mathcal{J}_{U_\Gamma}} e(C_t)$ by Lemma 55. Since $e(C_s) \mathrel{\mathcal{J}} e(C_t)$ holds in S_Γ, we then have $(e(C_t), 0) \in U'_\Gamma$ if and only if $e(C_s) \mathrel{\mathcal{J}_{U_\Gamma}} e(C_t)$ if and only if $e(C_s) \sim_{U_\Gamma} e(C_t)$ by Lemma 2 and by Lemma 4, respectively.

We finally note that $U'_\Gamma \leqslant S'_\Gamma$ can be realized as an inverse subsemigroup of $\mathcal{I}(\Omega_\Gamma \sqcup \{*\})$ and that $U'_\Gamma \leqslant S'_\Gamma \in \mathbf{BM}$. Hence, the problem NCL also admits a polynomial-time many-one reduction to the problem E-$\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$. As such, the latter is hard for PSPACE. ◀

7.2 The Intersection Non-Emptiness Problem

Next, let us explore a variant of the reduction from the proof of Theorem 57 to show further hardness results for the intersection non-emptiness problem for inverse automata.

For a definition of deterministic finite automata (DFA) we refer the reader to standard textbooks, e.g. [48]. We denote the accepted language of a DFA $\mathcal{A}$ by $\mathcal{L}(\mathcal{A})$.

An *inverse* automaton is a DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ with a *partially* defined transition function $\delta\colon Q \times \Sigma \to Q$ such that the following conditions are satisfied.

- For each $a \in \Sigma$, the partial map $\delta_a\colon Q \to Q$ with $q \mapsto \delta(q, a)$ is injective on its domain.
- For each $a \in \Sigma$, there is a word $w_a \in \Sigma^*$ such that $\delta_a \delta_{w_a}$ is the identity on $\mathrm{dom}(\delta_a)$, where $\delta_{w_a} = \delta_{a_1}\delta_{a_2}\ldots\delta_{a_n}$ is the partial map induced by $w_a = a_1a_2\ldots a_n$.[7]

▶ **Remark 58.** We will only encounter inverse automata over some alphabet endowed with an involution $a \mapsto \bar{a}$ where one can take $w_a = \bar{a}$ in the second condition above.

The *intersection non-emptiness problem* for a class $\mathcal{X}$ of automata (e.g., $\mathcal{X} = $ DFA or $\mathcal{X} = $ IA), which we denote by $\mathcal{X}$-INT-EMPTY, is the decision problem defined as follows.

Input. Automata $\mathcal{A}_1, \ldots, \mathcal{A}_k \in \mathcal{X}$.
Question. Is $L = \bigcap_{i=1}^{k} \mathcal{L}(\mathcal{A}_i)$ non-empty?

The problems DFA-INT-EMPTY and IA-INT-EMPTY are PSPACE-complete; see [62] and [16], respectively. We will give a short proof of these results based on the reduction from Section 7.1.

▶ **Theorem 59** (Corollary C)**.** *The problem* IA-INT-EMPTY *is complete for* PSPACE*. Moreover, this holds under the restriction that each automaton provided as part of the input has only two states, one of which is accepting.*

Proof. As in the proof of Theorem 57, we show PSPACE-hardness by polynomial-time many-one reduction from NCL to IA-INT-EMPTY. Let (Γ, C_s, C_t) be an instance of the former.

The idea is to encode a configuration $C \in \mathcal{C}(\Gamma)$ into the states of a collection of two-state inverse automata. To this end, we introduce one such automaton $\mathcal{A}(c)$ for each local configuration $c \in \mathcal{C}(\Gamma, v)$ at each vertex $v \in V(\Gamma)$. Its states $q_\top$ and $q_\bot$ indicate whether or not $C|_v = c$ holds. In particular, the starting state of $\mathcal{A}(c)$ is chosen to be $q_\top$ if $C_s|_v = c$ and $q_\bot$ if $C_s|_v \neq c$. Similarly, its accepting state is $q_\top$ if $C_t|_v = c$ and $q_\bot$ if $C_t|_v \neq c$.

As an alphabet Σ for our automata, we use the collection of all $u(c_1, c_2')$ as in the proof of Theorem 57 (with formation of inverses acting as involution). On input $u = u(c_1, c_2')$, with $c_i, c_i' \in \mathcal{C}(\Gamma, v_i)$ as in the proof of Theorem 57, the transitions are defined as follows. The automata $\mathcal{A}(c_i)$ with $i = 1, 2$ transitions from $q_\top$ to $q_\bot$ via u, and the automata $\mathcal{A}(c_i')$ with $i = 1, 2$ transitions from $q_\bot$ to $q_\top$ via u. For the automata $\mathcal{A}(c)$ with $c \in \{c_1, c_2, c_1', c_2'\}$ these are the only defined transitions via u. If $c \in \mathcal{C}(\Gamma, v_i)$ for some $i \in \{1, 2\}$ and $c \notin \{c_i, c_i'\}$, then $\mathcal{A}(c)$ has a transition from $q_\bot$ to $q_\bot$ via u and no other defined transitions via u. Finally, if $c \in \mathcal{C}(\Gamma, v)$ with $v \notin \{v_1, v_2\}$, then $\mathcal{A}(c)$ has a transition from $q_\top$ to $q_\top$ and from $q_\bot$ to $q_\bot$ via u, i.e., $\mathcal{A}(c)$ will remain in its current state upon reading u.

It is now easy to see that a word $u_1u_2\ldots u_n \in \Sigma^*$ is contained in $\bigcap_c \mathcal{L}(\mathcal{A}(c))$, where the intersection extends over all local configurations $c \in \mathcal{C}(\Gamma, v)$ at all vertices $v \in V(\Gamma)$, if and only if $u_1, u_2, \ldots, u_n$ describes a sequence of transitions of Γ from C_s to C_t. As, moreover, the collection of automata $\mathcal{A}(c)$ can be computed in polynomial time from Γ and C_s, C_t, we conclude that IA-INT-EMPTY is indeed hard for PSPACE. Since IA-INT-EMPTY is clearly contained in PSPACE, the problem is PSPACE-complete. ◀

[7] We view the maps δ_a as elements of $\mathcal{I}(Q)$ and, as such, compose them as operators acting on the right.

▶ **Corollary 60** (Bulatov, Kozik, Mayr, Steindl [19]). *The problem* DFA-INT-EMPTY *is complete for* PSPACE*. Moreover, this holds under the restriction that each automaton provided as part of the input has only three states, one of which is accepting.*

Note that herein the transition functions of the automata are *total* functions.

Proof. We can convert each inverse automaton into a deterministic finite automaton with a total transition function by introducing a failure state and appropriate transitions to it. ◀

7.3 The Subpower Membership Problem

From Theorem 59 we also derive a corresponding hardness result for the *subpower membership problem* of an inverse semigroup S. This problem is defined as follows.

Input. An integer k, a subset $\Sigma \subseteq S^k$, and an element $t \in S^k$.
Question. Is $t \in U$ where $U = \langle \Sigma \rangle$?

Be aware that here S is treated as a constant and *not* part of the input.

▶ **Corollary 61** (Corollary D). *The subpower membership problem of an inverse semigroup S is complete for* PSPACE *if and only if the Brandt monoid B_2^1 divides S; otherwise, it is in* NC.

Proof. The subpower membership problem clearly reduces to the problem MEMB$_{\mathbf{PB}}(\mathbf{V})$ where $\mathbf{V}$ is the variety generated by S. Hence, the subpower membership problem is in PSPACE and, if $S \in \mathbf{SIS}$ (i.e., $B_2^1 \not\prec S$ by Lemma 20), then it is in NC by Corollary 42.

From Theorem 59, it follows easily that the subpower membership for B_2^1 is PSPACE-complete (e.g. using the same argument as [17, Theorem 3.2] or [51, Theorem 4.10]). Moreover, note that the target element t obtained in the reduction does not project to the zero element of B_2^1 in any component of the direct product. Now, if B_2^1 divides S, then we can reduce the subpower membership problem of B_2^1 to the one of S in the following way.

As B_2^1 divides S, we can find some element $s \in S$ with $s\bar{s} \neq \bar{s}s$ and an idempotent $e \in E(S)$ with $e \geqslant s\bar{s} \vee \bar{s}s$. To see this, consider an inverse subsemigroup of S projecting onto $B_2^1 = \langle u, 1 \rangle$ and let e be an arbitrary idempotent in the preimage 1 and let $s = s'e$ where s' is any preimage of u. Note that we can multiply the elements $\{e, s, \bar{s}, s\bar{s}, \bar{s}s\}$ as in B_2^1 as long as their product does not project to 0 in B_2^1. Now, as the target element t does not project onto the zero element of B_2^1 in any component, we can safely replace each occurrence of 1 by e, of u by s, and so on in every component of t. Performing the same substitution on all elements of Σ completes our reduction. ◀

8 Further Related Problems

Finally, let us explore the consequences of our results to two further problems, namely the minimum generating set problem and the problem of solving equations.

8.1 The Minimum Generating Set Problem

As outlined in Section 1.3, the minimum generating set has been first considered by Papadimitriou and Yannahakis [83] and, in the Cayley table model, recently shown to be in P by Lucchini and Thakkar [68] and even NC [23]. Yet, the complexity for arbitrary semigroups and also for permutation groups remains wide open. In this section, we consider the minimum generating set problem for inverse semigroups in the partial bijection model. More formally, the minimum generating set problem (MGS) is defined as follows.

In this section, we consider the minimum generating set problem for inverse semigroups in the partial bijection model. More formally, the minimum generating set problem (MGS) is defined as follows.

Input. An inverse semigroup U and an integer k.
Question. Is there some $\Xi \subseteq U$ with $|\Xi| \leqslant k$ and $\langle \Xi \rangle = U$?

Be aware that, while in the above we assumed without loss of generality that generating sets are closed under formation of inverses, for MGS we drop this assumption. This is because adding inverses to Ξ, of course, changes $|\Xi|$. The hardness result below still applies to the variant of MGS where Ξ is required to be inverse-closed; however, our reduction then no longer is in AC^0 because we would need to count the number of self-inverse generators.

As for the membership problem, we write $\textsc{mgs}_{\mathbf{PB}}$ if U is given by generators of an inverse subsemigroup of $\mathcal{I}(\Omega)$ for some Ω and denote the restriction to a variety $\mathbf{V}$ by $\textsc{mgs}_{\mathbf{PB}}(\mathbf{V})$.

▶ **Lemma 62.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups. Then the restricted membership problem $\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{V})$ is AC^0-many-one-reducible to the problem $\textsc{mgs}_{\mathbf{PB}}(\mathbf{V} \vee \mathbf{Sl})$.*

Proof. Consider an instance $\Sigma \subseteq \mathcal{I}(\Omega)$ and $t \in \mathcal{I}(\Omega)$ for $\textsc{memb}^{\sharp}_{\mathbf{PB}}(\mathbf{V})$, i.e., the question is whether $t \in U = \langle \Sigma \rangle$. We define a subsemigroup $U' \leqslant \mathcal{I}(\Omega')$ where $\Omega' = \Omega \cup (\Sigma \times \{1,2\})$. To do so, we take $\Sigma' = \{t\} \cup (\Sigma \times \{1,2\})$ as a generating set, where t is viewed as a partial bijection on Ω' by leaving it undefined outside of Ω. For $(u,i) \in (\Sigma \times \{1,2\})$ and $x \in \Omega'$, we define $x^{(u,i)} = x^u$ if $x \in \Omega$ and $x^{(u,i)} = x$ if $x = (u,i)$ and otherwise $x^{(u,i)}$ is undefined. Thus, in particular, $U' = \langle \Sigma' \rangle$ is isomorphic to an inverse subsemigroup of $\langle \{t\} \cup \Sigma \rangle \times (Y_2 \times Y_2)^{\Sigma}$ and, as such, we have $U' \in \mathbf{V} \vee \mathbf{Sl}$. Clearly, the set $\Sigma' \subseteq \mathcal{I}(\Omega')$ can be computed in AC^0.

Let $k = |\Sigma|$. We claim that $U' = \langle \Sigma' \rangle$ is generated by $2k$ elements if and only if $t \in U$. To see this, first observe that the inverse subsemigroup $\langle \Sigma \times \{1,2\} \rangle$ of U' projects onto the semilattice $E(\mathcal{I}(\Sigma \times \{1,2\}))$; hence, it cannot be generated by less than $2k$ elements. Furthermore, we can find $U = \langle \Sigma \rangle$ as a subsemigroup of U' as $u = (u,1)\overline{(u,2)}(u,2)$ for $u \in \Sigma$. Thus, if $t \in U$, then U' is generated by exactly $2k$ elements. On the other hand, observe that the canonical projection $\mathcal{I}(\Omega') \to \mathcal{I}(\Omega)$ maps $\langle \Sigma \times \{1,2\} \rangle$ onto U and t to t; hence, if $t \notin U$, then $t \notin \langle \Sigma \times \{1,2\} \rangle$. This completes the proof of the claim. ◀

▶ **Remark 63.** This reduction applies to arbitrary transformation semigroups. There is only one minor adjustment to be aware of: U does not necessarily embed into U'; however, still all products of generators in U of length at least two embed into U'. This is enough for the reduction to be correct if one checks upfront whether any of the generators coincide.

▶ **Corollary 64** (First Part of Corollary E)**.** *Let $\mathbf{V}$ be a variety of finite inverse semigroups. Then the problem $\textsc{mgs}_{\mathbf{PB}}(\mathbf{V})$ is in NP if $\mathbf{V} \subseteq \mathbf{SIS}$ and PSPACE-complete otherwise.*

Proof. It is clear that $\textsc{mgs}_{\mathbf{PB}}(\mathbf{V})$ is in $\mathsf{NP}^{\textsc{memb}_{\mathbf{PB}}(\mathbf{V})}$ (just guess a generating set of an appropriate size and verify whether all of the original generators are in the inverse subsemigroup generated by the guessed generating set and vice versa). If $\mathbf{V} \not\subseteq \mathbf{SIS}$, then $\mathbf{V} \not\subseteq \mathbf{G}$ so $\mathbf{Sl} \subseteq \mathbf{V}$. Hence, the PSPACE-hardness of $\textsc{mgs}_{\mathbf{PB}}(\mathbf{V})$ follows from Lemma 62 and Theorem 57. ◀

8.2 Equations

In this section we explore the consequences of our hardness result for the conjugacy problem to the related problem of deciding satisfiability of equations. In particular, we are interested in the case, where the semigroup is part of the input. We will see that in the partial bijection model, this variant of the problem is harder than deciding whether an equation has a solution in a fixed inverse semigroup.

Let $\mathcal{X}$ be a set of variables. An equation $\ell = r$ in an inverse semigroup S is given as non-empty words $\ell, r \in (S \cup \mathcal{X} \cup \overline{\mathcal{X}})^+$. An assignment is a map $\sigma\colon \mathcal{X} \to S$, which naturally extends to a homomorphism from $\sigma\colon (S \cup \mathcal{X} \cup \overline{\mathcal{X}})^+ \to S$. The problem EQN^* of deciding whether a system of equations has a solution is defined as follows.

Input. An inverse semigroup S, and words $\ell_1, r_1, \ldots, \ell_k, r_k \in (S \cup \mathcal{X} \cup \overline{\mathcal{X}})^+$.
Question. Is there a $\sigma\colon \mathcal{X} \to S$ such that $\sigma(\ell_i) = \sigma(r_i)$ for all $1 \leqslant i \leqslant k$?

We denote by $\text{EQN}^*_{\mathbf{CT}}$ and $\text{EQN}^*_{\mathbf{PB}}$ this problem in the Cayley table model and in the partial bijection model, respectively. In the Cayley table model S is given as a multiplication table. In the partial bijection model the input is a set of generators $\Sigma \subseteq \mathcal{I}(\Omega)$ such that $\langle \Sigma \rangle = S$. The problem of deciding whether a single equation has a solution occurs as a special case when $k = 1$. We write $\text{EQN}_{\mathbf{CT}}$ and $\text{EQN}_{\mathbf{PB}}$, respectively.

We will show below, that the PSPACE-hardness of $E\text{-CONJ}^{\sharp}_{\mathbf{PB}}(\mathbf{BM})$ can be transferred to $\text{EQN}_{\mathbf{PB}}$. In addition, the problems of deciding whether a single equation or a system of equations have a solution are known to be NP-hard for many fixed inverse semigroups. We summarize these results here and give more details below. If $\mathbf{V}$ is a variety of finite inverse semigroups, then the following hold.

- The problem $\text{EQN}_{\mathbf{PB}}(\mathbf{V})$ is PSPACE-hard whenever $\mathbf{V} \not\subseteq \mathbf{SIS}$.
- The problems $\text{EQN}_{\mathbf{CT}}(\mathbf{V})$ and $\text{EQN}_{\mathbf{PB}}(\mathbf{V})$ are NP-hard whenever $\mathbf{V} \not\subseteq \mathbf{G}_{\text{sol}} \vee \mathbf{BS}$.
- The problems $\text{EQN}^*_{\mathbf{CT}}(\mathbf{V})$ and $\text{EQN}^*_{\mathbf{PB}}(\mathbf{V})$ are NP-hard whenever $\mathbf{V} \not\subseteq \mathbf{Com}$.

Herein $\mathbf{G}_{\text{sol}}$ denotes the variety of finite solvable groups and $\mathbf{Com} = \mathbf{Ab} \vee \mathbf{Sl}$ the variety of finite commutative inverse semigroups. Note that these hardness results are matched by a corresponding upped bound, which is presented in Observation 66 below. The second and third statement are a consequence of known NP-hardness results for fixed inverse semigroups. We will briefly describe them before proving the first statement in Lemma 65.

Whenever $\mathbf{V} \not\subseteq \mathbf{G}_{\text{sol}} \vee \mathbf{BS}$, then either $\mathbf{V}$ contains a non-solvable group or $B_2^1 \in \mathbf{V}$. According to Goldmann and Russell [41], the problem of determining whether a single equation over a (fixed) finite group G has a solution is NP-complete for every non-solvable group G. Likewise, the problem of determining whether a single equation over B_2^1 has a solution is also known to be NP-complete [11, Theorem 6]. We thus conclude that both of the problems $\text{EQN}_{\mathbf{CT}}(\mathbf{V})$ and $\text{EQN}_{\mathbf{PB}}(\mathbf{V})$ are NP-hard whenever $\mathbf{V} \not\subseteq \mathbf{G}_{\text{sol}} \vee \mathbf{BS}$.

The third statement is a special case of a dichotomy for regular semigroups: deciding whether a restricted system of equations, where the right-hand side is a constant, has a solution is in polynomial time if S is in the variety of finite semigroups generated by abelian groups and regular bands, and NP-complete otherwise [60]. The statement follows by restriction to inverse semigroups.

▶ **Lemma 65.** *Let $\mathbf{V}$ is a variety of finite inverse semigroups. Then the problem $\text{EQN}_{\mathbf{PB}}(\mathbf{V})$ is hard for* PSPACE *whenever $\mathbf{V} \not\subseteq \mathbf{SIS}$.*

Proof. We reduce from $E\text{-CONJ}^{\sharp}_{\mathbf{PB}}(\mathbf{V})$, which is PSPACE-hard by Theorem 57. We are given two idempotents e_s, e_t that are conjugate (hence, $\mathcal{J}$-equivalent) in $\mathcal{I}(\Omega)$. By definition $e_s \sim_U e_t$ if and only if there is some $X \in U$ with $\bar{X} e_s X = e_t$ and $X e_t \bar{X} = e_s$. As e_s and e_t are idempotent, this holds if and only if there is some $X \in \langle U \cup \{e_s, e_t\} \rangle \in \mathbf{V}$ with $\bar{X} e_s X = e_t$ and $X e_t \bar{X} = e_s$. Moreover, by Lemma 2 and Lemma 4, either one of the two equations has a solution if and only if the other one does. Hence, $\text{EQN}_{\mathbf{PB}}(\mathbf{V})$ is hard for PSPACE. ◀

▶ **Observation 66.** *Let* $\mathbf{V}$ *be a variety of finite inverse semigroups. The problem* $\mathrm{EQN}^*_{\mathbf{CT}}(\mathbf{V})$ *is in* NP *and* $\mathrm{EQN}^*_{\mathbf{PB}}(\mathbf{V})$ *is in* $\mathsf{NP}^{\mathrm{MEMB}_{\mathbf{PB}}(\mathbf{V})}$*, i.e., solvable in* NP *with an oracle for* $\mathrm{MEMB}_{\mathbf{PB}}(\mathbf{V})$*.*

Moreover, $\mathrm{EQN}^*_{\mathbf{PB}}(\mathbf{V})$ *remains in* $\mathsf{NP}^{\mathrm{MEMB}_{\mathbf{PB}}(\mathbf{V})}$ *if each variable can be constrained to some inverse subsemigroup and we allow arbitrary constants from* $\mathcal{I}(\Omega)$ *(not restricted to* $\mathbf{V}$*).*

In the partial bijection model we obtain that $\mathrm{EQN}^*_{\mathbf{PB}}$ is in PSPACE with Theorem 57, and that $\mathrm{EQN}^*_{\mathbf{PB}}(\mathbf{SIS})$ is in NP with Theorem 41.

Proof. The algorithm follows the guess-and-check pattern: we guess an assignment $\sigma\colon \mathcal{X} \to S$ and then verify that it satisfies the equations. In the Cayley table model this is straightforward and so it only remains to consider the partial bijection model. There, we guess a map $\sigma\colon \mathcal{X} \to \mathcal{I}(\Omega)$ and check whether $\sigma(X) \in \langle\Sigma\rangle$ for each $X \in \mathcal{X}$ using the $\mathrm{MEMB}_{\mathbf{PB}}(\mathbf{V})$ oracle. Then we verify that $\sigma(\ell_i) = \sigma(r_i)$ for all $1 \leqslant i \leqslant k$. The evaluations $\sigma(\ell_i)$ and $\sigma(r_i)$ can clearly be computed in polynomial time (in either input model).

Allowing inverse subsemigroup constraints does not increase the complexity since we already use the membership oracle anyway. Allowing arbitrary constants from $\mathcal{I}(\Omega)$ also does not increase difficulty, as evaluations are computed in $\mathcal{I}(\Omega)$ either way. ◀

Combining Lemma 65 with Observation 66, we obtain the following corollary, which constitutes the second part of Corollary E.

▶ **Corollary 67.** *Let* $\mathbf{V}$ *be a variety of finite inverse semigroups. Then the problems* $\mathrm{EQN}_{\mathbf{PB}}(\mathbf{V})$ *and* $\mathrm{EQN}^*_{\mathbf{PB}}(\mathbf{V})$ *are in* NP *if* $\mathbf{V} \subseteq \mathbf{SIS}$ *and* PSPACE*-complete otherwise.*

9 Discussion and Open Problems

By investigating the membership problem in inverse semigroups, we filled a gap between the rather restricted case of groups and the very general case of arbitrary semigroups. We gave a classification of the complexity of membership and conjugacy in inverse semigroups according to their combinatorial structure. Here the combinatorial Brandt semigroup and monoid are the critical obstructions for the membership problem being easy. Furthermore, by applying these results, we gained new insights on the complexity of closely related problems such as the intersection non-emptiness problem for inverse automata, the minimum generating set problem, and the equation satisfiability problem.

Applying Theorem A and Theorem B to the case of aperiodic inverse semigroups shows that the membership and conjugacy problems are either in AC^0, or L-complete, or PSPACE-complete (with PSPACE-complete only in the partial bijection model). Thus, in comparison to the classification for varieties of aperiodic monoids [14], we additionally have the L-complete case, whereas we do not have the P-complete, NP-complete, and NP-hard cases.

A corresponding classification of the complexity of the membership problem for varieties of arbitrary semigroups is still a major endeavor. While there is the classification for aperiodic monoids by Beaudry, McKenzie, and Thérien [14] mentioned above, the case for semigroups is considerably more involved as there are many more varieties of finite semigroups than of finite monoids. Moreover, it remains to integrate the case of groups into the consideration.

▶ **Open Problem 68.** *Give a classification of the varieties of finite semigroups in terms of their complexity for the membership problem.*

Another class of structures, situated between inverse semigroups and arbitrary semigroups, that we would like to draw attention to is the class of regular $*$-semigroups introduced by Nordahl and Scheiblich [79]. These are regular semigroups with distinguished inverses $x \mapsto x^*$

such that $x^{**} = x$, $(xy)^* = y^*x^*$, and $xx^*x = x$. The difference to inverse semigroups is that inverses need not be unique or, equivalently, that idempotents need not commute. As such, regular $*$-semigroups admit a far richer combinatorial structure.

▶ **Open Problem 69.** *Give a classification of the varieties of finite regular $*$-semigroups in terms of their complexity for the membership problem.*

Our algorithms for the NC (resp. NP) cases of our dichotomy result for the partial bijection model are very efficient in the sense that they provide reductions computable in L and also in linear or quadratic time to UGAP as well as to the respective problems for groups. Thus, the only open questions about the complexity of these problems come from the group case. For the membership problem this leads to the following rather far-reaching question.

▶ **Question 70.** *Is* $\textsc{memb}_{\mathbf{PB}}(\mathbf{G})$ *in* L*?*

We do not feel confident to make any guess about this question and want to use the present work to foster further research on this topic. On the other hand, we believe that the answer to the following question concerning the Cayley table model is likely to be negative.

▶ **Question 71.** *Is* $\textsc{memb}_{\mathbf{CT}}(\mathbf{G})$ *in* AC^0*?*

Another question is whether the $\mathcal{O}(\log^2 n)$ bound due to Babai and Szemerédi [8] on the length of straight-line programs in groups of order n is asymptotically optimal. In some special cases, like for Abelian groups, an (asymptotically optimal) $\mathcal{O}(\log n)$ bound can be obtained instead. Thus, the question here is whether this can be extended to *all* groups.

For inverse semigroups, we obtained a complete characterization of varieties admitting polylogarithmic SLPs in Corollary 49. A similar result for monoids was obtained by the first author [35]. Therefore, the natural question is to ask for a similar characterization for all semigroups. For some preliminary results in that direction, see also [35].

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