A Dichotomy for Maximum PCSPs on Graphs*

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Fix two non-empty loopless graphs G and H such that G maps homomorphically to H. The *Maximum Promise Constraint Satisfaction Problem* parameterised by G and H is the following computational problem, denoted by $\mathrm{MaxPCSP}(G, H)$: Given an input (multi)graph X that admits a map to G preserving a ρ-fraction of the edges, find a map from X to H that preserves a ρ-fraction of the edges. As our main result, we give a complete classification of this problem under Khot's Unique Games Conjecture: The only tractable cases are when G is bipartite and H contains a triangle.

Along the way, we establish several results, including an efficient approximation algorithm for the following problem: Given a (multi)graph X which contains a bipartite subgraph with ρ edges, what is the largest triangle-free subgraph of X that can be found efficiently? We present an SDP-based algorithm that finds one with at least 0.8823ρ edges, thus improving on the subgraph with 0.878ρ edges obtained by the classic Max-Cut algorithm of Goemans and Williamson.

1 Introduction

Given two undirected graphs[1] G and H, a *homomorphism* from G to H is an edge preserving map h from $V(G)$ to $V(H)$; that is, if $(u, v) \in E(G)$ then $(h(u), h(v)) \in E(H)$. A classic result of Hell and Nešetřil established a computational *dichotomy* for the so-called H-colouring problem [HN90], for a fixed graph H: if H is bipartite then deciding whether an input graph G is homomorphic to H is solvable in polynomial time, and for every other H this problem is NP-complete. Going beyond graphs, Feder and Vardi conjectured that a similar dichotomy holds for all finite relational structures [FV98], not only for graphs and for relational structures over the Boolean domain [Sch78].[2] Bulatov [Bul17] and, independently, Zhuk [Zhu20], confirmed the tractability part of the dichotomy, which together with the NP-hardness part [BJK05], answered the Feder-Vardi conjecture in the affirmative. The homomorphism problem is also known as the *constraint satisfaction problem* (CSP) [Jea98]. CSPs can be equivalently defined as problems seeking an assignment of values to the given variables subject to

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  All data is provided in full in the results section of this paper.

[1] All graphs in this article are loopless and non-empty, meaning having at least one edge (and thus at least two vertices).

[2] We will not need relational structures, but, intuitively, one should think of them as generalisations of (hyper)graphs in which one is given a ground set and a collection of relations on the ground set.

the given constraints. The fixed target structure in the homomorphism problem corresponds to the set of allowed (domain) values and the set of allowed relations in the constraints. Concrete examples of CSPs include solvability of linear equations over finite fields and variants of (hyper)graph colourings.

A well-studied line of work focuses on *approximability* of CSPs [KSTW00, TSSW00]. A classic example here is the Max-Cut problem. In Max-Cut, the variables correspond to the vertices of the input graph, the values are just 0 and 1 (corresponding to the two sides of a cut), and the constraints are binary disequalities associated with the edges of the graph. Given a CSP, the computational task could be to find a solution maximising the number of satisfied constraints as in Max-Cut, or finding a (perfect) solution satisfying all constraints as discussed in the previous paragraph.

A *promise* CSP (PCSP) is a CSP in which each constraint comes in two forms, a strong one and a weak one. The promise is that a solution exists using the strong versions of the constraints, while the (possibly easier) task is to find a solution using the weak constraints. A recent line of work by Austrin, Guruswami, and Håstad [AGH17], Brakensiek and Guruswami [BG21], and Barto, Bulín, Krokhin, and Opršal [BBKO21] initiated a systematic study of PCSPs with perfect completeness, i.e., finding a solution satisfying all weak constraints given the promise that a solution satisfying all strong constraints exists. Canonical examples include approximate graph [GJ76] and hypergraph [DRS05, ABP20, BBB21] colouring problems, e.g., finding a 5-colouring of a given 3-colourable graph [KOWŽ23]. PCSPs are a vast generalisation of CSPs and their complexity is not well understood, not even on the Boolean domain [FKOS19, BGS23a] or for graphs. In particular, Brakensiek and Guruswami conjectured that only bipartite graphs lead to tractable PCSPs on graphs [BG21] (cf. Conjecture 3 in Section 3 for a precise statement). Resolving their conjecture would in particular include resolving the notoriously difficult approximate graph colouring problem, cf. [AFO+25] for exciting recent progress.

In this work, we will focus on the *approximability of maximisation PCSPs*. The ultimate goal is to understand the precise approximation factor for all MaxPCSPs, and thus identify where the transition from tractability to intractability occurs. This is an ambitious, long-term goal that would encompass many existing fundamental results.

An example of a MaxPCSP is the following problem. Given a graph G that admits a 2-colouring of the vertices with a ρ-fraction of the edges coloured properly, find a 3-colouring of G with an $\alpha\rho$-fraction of the edges coloured properly, where $0 < \alpha \leq 1$ is the approximation factor. As one of the results in the present paper, we will show that there is a 1-approximation algorithm; i.e., given a graph with a 2-colouring with ρ-fraction of non-monochromatic edges, one can efficiently find a 3-colouring of the graph with the same fraction of non-monochromatic edges.

As our main result, we will establish a dichotomy for 1-approximation of graph PCSPs under Khot's Unique Games Conjecture (UGC) [Kho02].

Theorem 1. *Let G and H be two fixed graphs such that there is a homomorphism from G to H. If G is bipartite and H contains a triangle then* $\mathrm{MaxPCSP}(G, H)$ *is 1-approximable. Otherwise, 1-approximation of* $\mathrm{MaxPCSP}(G, H)$ *is* NP*-hard assuming the* UGC*.*

Along the way to prove Theorem 1, we will design two efficient approximation algorithms. We shall discuss one of them briefly here, with an overview of both algorithms and all results in Section 3.

Given an undirected (multi)graph G, what is the bipartite subgraph of G with the most edges? This is nothing but the already mentioned Max-Cut problem, one of the most fundamental problems in computer science. Max-Cut was among the 21 problems shown to be NP-hard by Karp [Kar72]. Papadimitriou and Yannakakis showed that Max-Cut is APX-hard [PY91] and thus does not admit a polynomial-time approximation scheme, unless P = NP. However, there are several simple 0.5-approximation algorithms. Goemans and Williamson used semidefinite programming and randomised rounding to design a 0.878-approximation algorithm [GW95]. Khot, Kindler, Mossel, O'Donnell, and Oleszkiewicz established the optimality of this algorithm [KKMO07, MOO10] under the UGC.

What if the task is merely finding a large triangle-free subgraph (rather than a bipartite one)?

While the Goemans-Williamson algorithm can still be used, as one of our results we design an algorithm with a better approximation guarantee: If G contains a bipartite subgraph with ρ edges, our algorithm efficiently finds a triangle-free subgraph of G with 0.8823ρ edges.[3] Our algorithm is a randomised combination of the Goemans-Williamson original "random hyperplane algorithm", and an algorithm that first selects "long edges" (meaning edges for which the angle between the corresponding vectors from the SDP solution is above a certain threshold) and then applies a random hyperplane rounding, selecting "shorter edges" (still longer than some other threshold). The probability of the biased coin that selects one of the two algorithms depends on certain geometric quantities which guarantee that the resulting subgraph is indeed triangle-free. We complement our tractability result for this problem by showing that it is NP-hard to find a triangle-free subgraph with $(25/26+\varepsilon)\rho \approx (0.961+\varepsilon)\rho$ edges. This result is obtained by a reduction from Håstad's 3-bit PCP [Hås01].

Related work The notion of MaxPCSPs is a natural generalisation of the well-studied notion of MaxCSPs. For finite-domain MaxCSPs, it is known that a certain rounding of the basic SDP relaxation gives, up to some ε, the UGC-optimal approximation ratio (in time doubly exponential in $1/\varepsilon$) [Rag08, RS09]. However, the Raghavendra-Steurer algorithm does not immediately give a 1-approximation algorithm due to the above-mentioned ε, even for MaxCSPs. Moreover, that result is established only for finite-domain MaxCSPs. On other hand, our results include a 1-approximation for MaxPCSP(K_2, K_3), and an algorithm for infinite-domain structures, namely for MaxPCSP$(K_2, \mathfrak{G}_3)$, which captures the bipartite vs. triangle-free subgraph discussed above (cf. Section 2 for a precise definition).

Approximation of concrete MaxPCSPs has been studied for decades, include several papers on almost approximate graph colouring [EH08, DKPS10, KS12, HMS23], approximate colouring [NŽ23], and promise Max-3-LIN [BLŽ25]. Our work initiates a systematic investigation, giving a complete classification for 1-approximation of the graph case.

Recent work of Brakensiek, Guruswami, and Sandeep [BGS23b] studied robust approximation of MaxPCSPs; in particular, they state that Raghavendra's above-mentioned theorem on approximate MaxCSPs [Rag08] applies verbatim to MaxPCSPs. This in combination with the work of Brown-Cohen and Raghavendra [BR15] gives a framework for studying approximation of MaxPCSPs. An alternative framework for studying approximation of MaxPCSPs has recently been put forward by Barto, Butti, Kazda, Viola, and Živný [BBK+24].

Paper organisation After defining MaxPCSPs formally and few other basic concepts in Section 2, we will state all our results precisely in Section 3. The rest of the paper is then split in different parts of the proofs, including two tractability results in Section 4 and Section 5 and hardness results in Section 6 and Section 7

[3] We note that our algorithm is easy to extend to the case where edges have positive weights.

2 Preliminaries

For two nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, we denote by $\angle(\mathbf{x}, \mathbf{y})$ the angle between $\mathbf{x}$ and $\mathbf{y}$ in radians; i.e., $\angle(\mathbf{x}, \mathbf{y}) = \arccos\left(\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}\right)$. The following useful fact is well-known, cf. [Euc26, Book XI, Proposition 21].

Lemma 2. *For any three nonzero vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^N$, we have $\angle(\mathbf{x}_1, \mathbf{x}_2) + \angle(\mathbf{x}_2, \mathbf{x}_3) + \angle(\mathbf{x}_3, \mathbf{x}_1) \leq 2\pi$.*

Graphs and (partial) homomorphisms. All graphs will be nonempty, undirected and loopless but with possibly multiple edges. Fix two graphs $G = (V, E)$, $H = (U, F)$, and $\rho \in \mathbb{N}$. We say that there exists a *partial homomorphism* of weight ρ from G to H, and write $G \xrightarrow{\rho} H$, if there exists a mapping $h : V \to U$ such that for ρ edges $(x, y) \in E$ we have $(h(x), h(y)) \in F$. If $G \xrightarrow{|E|} H$, we say that there exists a homomorphism from G to H and write $G \to H$. (Note that for any G, H, I, if $G \xrightarrow{\rho} H \to I$ then $G \xrightarrow{\rho} I$.)

We denote by K_2 a clique on two vertices. A partial homomorphism $h : G \xrightarrow{\rho} K_2$ represents a cut of weight ρ, namely the edges (x, y) with $h(x) \neq h(y)$. Equivalently, it represents a bipartite subgraph of G with weight ρ. We now introduce a graph that similarly captures triangle-free subgraphs. Let $\mathfrak{G}_3$ be the direct sum of all finite triangle-free graphs. In other words, for every finite triangle-free graph $G = (V, E)$, the graph $\mathfrak{G}_3$ contains vertices x_G for $x \in V$, and edges (x_G, y_G) for $(x, y) \in E$. Then, for finite G, a partial homomorphism $h : G \xrightarrow{\rho} \mathfrak{G}_3$ represents a triangle-free subgraph of G with weight ρ: all the edges that connect vertices that are mapped by h to neighbouring vertices in $\mathfrak{G}_3$ form a triangle-free subgraph of G.

Maximum PCSPs. Fix two (possibly infinite) graphs $G \to H$. Then the *maximum promise constraint satisfaction problem* (MaxPCSP) for undirected graphs, denoted by $\mathrm{MaxPCSP}(G, H)$, is defined as follows. In the search version of the problem, we are given a (multi)graph X such that $X \xrightarrow{\rho} G$, and must find $h : X \xrightarrow{\rho} H$; this problem can be approximated with the approximation ratio α if we can find $h : X \xrightarrow{\lceil\alpha\rho\rceil} H$. In the decision version, we are given a (multi)graph X and a number $\rho \in \mathbb{N}$ and must output YES if $X \xrightarrow{\rho} G$, and NO if not even $X \xrightarrow{\rho} H$. This problem can be approximated with approximation ratio α if we can decide between $X \xrightarrow{\rho} G$ and not even $X \xrightarrow{\lceil\alpha\rho\rceil} H$. (In all cases, ρ is *not* part of the input.)

In particular, approximating the problem $\mathrm{MaxPCSP}(K_2, \mathfrak{G}_3)$ with approximation ratio α means the following. In the search version: given a graph G that contains a cut of weight ρ, find a triangle-free subgraph of weight $\alpha\rho$. In the decision version: given a graph G and a number $\rho \in \mathbb{N}$, output YES if it has a cut of weight ρ, and NO if it has no triangle-free subgraph of weight $\alpha\rho$.

We define the problem $\mathrm{PCSP}(G, H)$ identically to $\mathrm{MaxPCSP}(G, H)$, except that it is guaranteed that ρ is the number of edges of G. Thus observe that $\mathrm{PCSP}(G, H)$ reduces to $\mathrm{MaxPCSP}(G, H)$ trivially, in the sense that there is a polynomial-time reduction from $\mathrm{PCSP}(G, H)$ to $\mathrm{MaxPCSP}(G, H)$ that does not change the input.

Suppose $G \to G' \to H' \to H$. Then, it follows that $\mathrm{PCSP}(G, H)$ polynomial-time reduces to $\mathrm{PCSP}(G', H')$ and $\mathrm{MaxPCSP}(G, H)$ polynomial-time reduces to $\mathrm{MaxPCSP}(G', H')$ (and the same holds for α-approximation). Furthermore, the decision version of $\mathrm{PCSP}(G, H)$ and $\mathrm{MaxPCSP}(G, H)$ polynomial-time reduces to the search version of $\mathrm{PCSP}(G, H)$ and $\mathrm{MaxPCSP}(G, H)$, respectively. In other words, the decision version is no harder than the search version. Hence by proving our tractability results for the search version, and our hardness results for the decision version, we prove them for both versions of the problems.

SDP. For the Max-Cut problem, which is just $\mathrm{MaxPCSP}(K_2, K_2)$, the *basic SDP relaxation* for a graph $G = (V, E)$ with n vertices, which can be solved within additive error ε in polynomial time with respect to the size of G and $\log(1/\varepsilon)$,[4] is as follows:

$$\begin{aligned} \max \quad & \sum_{(u,v)\in E} \frac{1 - \mathbf{x}_u \cdot \mathbf{x}_v}{2} \\ \text{s.t.} \quad & ||\mathbf{x}_u||^2 = 1, \\ & \mathbf{x}_u \in \mathbb{R}^n. \end{aligned} \tag{1}$$

Goemans and Williamson [GW95] gave a rounding algorithm for the SDP (1) with approximation ratio

$$\alpha_{GW} = \left(\max_{0\le\tau\le\pi} \frac{\pi}{2} \frac{1 - \cos\tau}{\tau} \right)^{-1} = 0.878\cdots,$$

thus beating the trivial approximation ratio of $1/2$ obtained by, e.g., a random cut. Their algorithm solves the SDP (1), selects a uniformly random hyperplane in $\mathbb{R}^N$, and returns the cut induced by the hyperplane.

3 Results

Our main result is the following.

Theorem 1. *Let G and H be two fixed graphs such that there is a homomorphism from G to H. If G is bipartite and H contains a triangle then* $\mathrm{MaxPCSP}(G, H)$ *is 1-approximable. Otherwise, 1-approximation of* $\mathrm{MaxPCSP}(G, H)$ *is* NP*-hard assuming the* UGC.

This is an optimisation variant of a conjecture by Brakensiek and Guruswami on the tractability boundary of promise CSPs on undirected graphs.

Conjecture 3 ([BG21]). *Let G and H be two fixed graphs such that there is a homomorphism from G to H. If G is bipartite then* $\mathrm{PCSP}(G, H)$ *is tractable. Otherwise,* $\mathrm{PCSP}(G, H)$ *is* NP*-hard.*

The currently known cases supporting Conjecture 3 are NP-hardness of $\mathrm{PCSP}(K_3, K_5)$ [BBKO21], $\mathrm{PCSP}(K_k, K_{\binom{k}{\lfloor k/2 \rfloor}} - 1)$ for $k \geq 4$ [KOWŽ23], and $\mathrm{PCSP}(C_{2k+1}, K_4)$ for $k \geq 1$ [AFO+25], where C_{2k+1} denotes a cycle on $2k+1$ vertices. As $\mathrm{PCSP}(G, H)$ reduces to $\mathrm{MaxPCSP}(G, H)$, Conjecture 3 implies that $\mathrm{MaxPCSP}(G, H)$ is NP-hard whenever G is non-bipartite. We establish this result under the UGC but *not* relying on Conjecture 3. It follows from our Theorem 1 that not all cases of $\mathrm{MaxPCSP}(G, H)$ with bipartite G are 1-approximable, and thus the tractability boundary lies elsewhere for 1-approximation.

We note that establishing Theorem 1 appears easier than resolving Conjecture 3, similarly to how the complexity of exact solvability of MaxCSPs [TŽ16] was resolved before the complexity of decision CSPs [Bul17, Zhu20].

An important part in proving Theorem 1 is the NP-hardness of finding a triangle-free subgraph. For this problem, we also establish a non-trivial approximation result.

[4] Throughout we will ignore issues of real precision.

Theorem 4. $\mathrm{MaxPCSP}(K_2, \mathfrak{G}_3)$ *is* 0.8823*-approximable (in the search version) in polynomial time, and it is* NP*-hard to* $(25/26 + \varepsilon)$*-approximate (even in the decision version) for any fixed* $\varepsilon > 0$.

Note that crucially $0.8823 > 0.878\ldots$, thus our algorithm beats the Goemans-Williamson algorithm. Theorem 4 is proved in two parts: the tractability side in Section 4 and the hardness side in Section 6. We quickly give an intuitive explanation of why such an algorithm is possible. Define

$$\tau_{GW} = \arg\max_{0 \leq \tau \leq \pi} \frac{\pi}{2} \frac{1 - \cos \tau}{\tau} \approx 0.742\pi.$$

The function $\tau \mapsto (\pi/2)(1 - \cos \tau)/\tau$, depicted in Figure 1, is increasing up to τ_{GW}, then decreasing. Why would $\mathrm{MaxPCSP}(K_2, \mathfrak{G}_3)$ be easier to approximate than $\mathrm{MaxPCSP}(K_2, K_2)$? Consider the

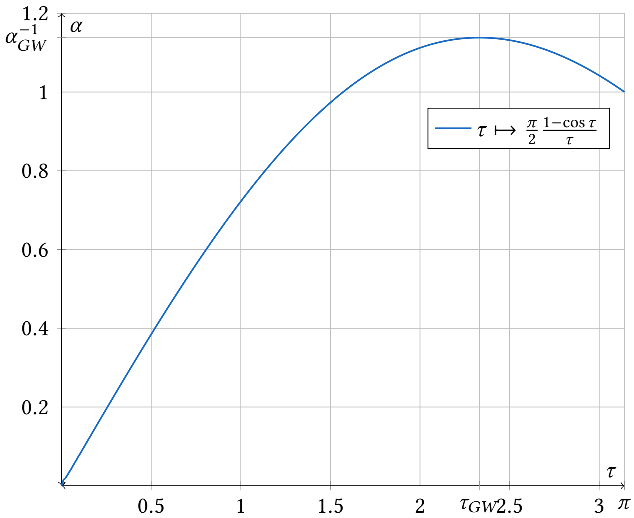

Figure 1: Function giving rise to α_{GW}, τ_{GW}.

Goemans-Williamson algorithm for $\mathrm{MaxPCSP}(K_2, K_2)$; the worst-case performance of this algorithm appears in a graph where the embedding into $\mathbb{R}^N$ given by solving the SDP (1) gives all edges an angle of approximately τ_{GW}. Observe that $\tau_{GW} > 2\pi/3$ — so immediately (cf. Lemma 2) this instance is triangle-free. So, for this instance an algorithm for $\mathrm{MaxPCSP}(K_2, \mathfrak{G}_3)$ could just return the entire graph! Indeed, in order to create an instance that contains triangles one needs to introduce shorter edges. This suggests that a hybrid algorithm, that either selects "long edges" or some appropriate selection of "shorter edges" (still longer than some threshold), should have better performance. The details can be found in Section 4, while the NP-hardness part of Theorem 4 is proved in Section 6.

Our next result is a combination of the Goemans-Williamson SDP for Max-Cut [GW95] and a rounding scheme due to Frieze and Jerrum for Max-3-Cut [FJ97].

Theorem 5. $\mathrm{MaxPCSP}(K_2, K_3)$ *is 1-approximable in polynomial time (in the search version).*

This algorithm is somewhat similar to the Goemans-Williams algorithm, except that rather than selecting a uniformly random hyperplane, it selects three normally distributed vectors, and partitions the vertices according to which vector they are closest to (where closeness is measured in terms of inner products). Thus, while this algorithm solves the same SDP as the algorithm from Theorem 4, it rounds the solution differently. This rounding scheme is the same as the rounding scheme of [FJ97]; it is also very similar to one of the rounding schemes of [KMS98]. The details can be found in Section 5.

The key in proving NP-hardness in Theorem 1 is the following result, established by generalising the proof of Dinur, Mossel, and Regev[DMR09] with a multilayered unique games conjecture, in the style of [BWŽ21]. The details can be found in Section 7.

Theorem 6. *For every $k \geq 1$ and $\ell \geq 3$, 1-approximation of* MaxPCSP(C_{2k+1}, K_ℓ) *is* NP*-hard assuming the* UGC.

We now have all tools to prove Theorem 1.

Proof. TOPROVE 0 □

We note that our hardness result depends in an essential way on the fact that the input graph can have multiple edges; we equivalently could have allowed non-negative integer weights on the edges. This variant of the problem is most natural when looking at it as a constraint satisfaction problem. It is interesting to ask what the complexity of MaxPCSP$(K_2, \mathfrak{G}_3)$ is if the input graph is both weightless and without multiple edges.

4 Approximation of MaxPCSP$(K_2, \mathfrak{G}_3)$

In this section, we will prove the tractability part of the following result.

Theorem 4. MaxPCSP$(K_2, \mathfrak{G}_3)$ *is* 0.8823*-approximable (in the search version) in polynomial time, and it is* NP*-hard to* $(25/26 + \varepsilon)$*-approximate (even in the decision version) for any fixed* $\varepsilon > 0$.

We will need the following technical lemma.

Lemma 7. *There exist $\alpha, P, Q, \tau \in \mathbb{R}$ with $P + Q = 1, P \geq 0, Q \geq 0$, $\tau \in [2\pi/3, \tau_{GW}]$, such that the following hold*

$$P\frac{\theta}{\pi} + Q \geq \alpha\frac{1-\cos\theta}{2} \qquad \theta \in [\tau, \pi] \tag{2}$$

$$P\frac{\varphi}{\pi} + Q\frac{\varphi}{\pi} \geq \alpha\frac{1-\cos\varphi}{2} \qquad \varphi \in [\pi - \tau/2, \tau] \tag{3}$$

$$P\frac{\psi}{\pi} \geq \alpha\frac{1-\cos\psi}{2} \qquad \psi \in [0, \pi - \tau/2]. \tag{4}$$

In particular, we can take $\alpha \geq 0.88232, \tau = 2.18746$, $Q = 1 - P$ and

$$P = \frac{\alpha\pi}{2}\left(\frac{1-\cos(\pi-\tau/2)}{\pi-\tau/2}\right) \approx 0.987535.$$

Proof. TOPROVE 1 □

We now prove the desired tractability result.[5]

Theorem 8. MaxPCSP$(K_2, \mathfrak{G}_3)$ *can be* 0.8823*-approximated in polynomial time.*

[5]We remark in passing that *no* SDP-based algorithm can have performance greater than $8/9 = 0.888\ldots$, since for the triangle K_3 the SDP value is $9/4$, yet the largest triangle-free subgraph has weight 2 (and $2/(9/4) = 8/9$).

Proof. TOPROVE 2 □

It is also interesting to consider what the power of our approximation algorithm is in the *almost satisfiable regime*, i.e. if an input graph that has a cut of value $1-\varepsilon$. It turns out that in this case we output a triangle-free subgraph with $1-O(\varepsilon)$ edges, significantly more than the $1-O(\sqrt{\varepsilon})$ edges outputted by the Goemans-Williamson algorithm [GW95]. This is not very hard to see, it follows immediately from the fact that our algorithm can choose all edges of angle $>\tau$ immediately.

Theorem 9. *The derandomised algorithm from Theorem 8, if run on an input graph G with a cut with a $(1-\varepsilon)$-fraction of edges, produces a triangle-free subgraph with $(1-O(\varepsilon))$-fraction of edges.*

Indeed, the (extremely loose) analysis below gives us that it returns a triangle-free subgraph with a $(1-15\varepsilon)$-fraction of edges at least.

Proof. TOPROVE 3 □

Interestingly, the non-derandomised algorithm has worse performance! Since it chooses at random between selecting all long edges deterministically and cutting according to the Goemans-Williamson algorithm, the performance degrades to $1-O(\sqrt{\varepsilon})$.

5 Approximation of MaxPCSP(K_2, K_3)

We first introduce some useful notation. For any predicate φ, we let $[\varphi] = 1$ if φ is true, and 0 otherwise.

For an event φ we let $\Pr[\varphi]$ be the probability that φ is true. For a random variable X, we let $\mathbb{E}[X]$ denote its expected value. Note that $\mathbb{E}[[\varphi]] = \Pr[\varphi]$. For any two distributions $\mathcal{D}, \mathcal{D}'$ with domains A, A', we let $\mathcal{D} \times \mathcal{D}'$ denote the product distribution, whose domain is $A \times A'$. For any distribution $\mathcal{D}$ over $\mathbb{R}$ and $a, b \in \mathbb{R}$, the distribution $a\mathcal{D} + b$ is the distribution of $aX + b$ when $X \sim \mathcal{D}$. We use the standard probability theory abbreviations i.i.d. (independent and identically distributed) and p.m.f. (probability mass function).

We introduce a few classic distributions we will need. The uniform distribution $\mathcal{U}(D)$ over a discrete set D is the distribution with p.m.f. $f : D \to [0,1]$ given by $f(x) = 1/|D|$. Note that $\mathcal{U}(D^n)$ is the same as $\mathcal{U}(D)^n$, a fact which we will use implicitly. We let $\mathrm{NBin}(n)$ denote a normalised binomial distribution: it is the distribution of $X_1 + \cdots + X_n$, where $X_i \sim \mathcal{U}(\{-1/\sqrt{n}, 1/\sqrt{n}\})$. The domain of this distribution is $\{(-n+2k)/\sqrt{n} \mid 0 \le k \le n\}$, the probability mass function is $(-n+2k)/\sqrt{n} \mapsto \binom{n}{k}/2^n$, the expectation is 0, and the variance is 1. If $\mu, \sigma \in \mathbb{R}$, then we let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2. Fixing d, if $\mu \in \mathbb{R}^d, \Sigma \in \mathbb{R}^{d\times d}$, then we let $\mathcal{N}(\mu, \Sigma)$ denote the multivariate normal distribution with mean μ and covariance matrix Σ. We let $\mathbf{I}_d$ denote the $d \times d$ identity matrix. Observe that if $\mathbf{x} \sim \mathcal{N}(\mu, \Sigma)$, where $\mathbf{x} \in \mathbb{R}^d$, then for any matrix $A \in \mathbb{R}^{d'\times d}$ we have that $A\mathbf{x} \sim \mathcal{N}(A\mu, A\Sigma A^T)$. Furthermore if $\mathbf{x} \sim \mathcal{N}(\mu, \Sigma)$ with Σ positive semidefinite, then by finding the Cholesky decomposition $\Sigma = \mathbf{A}\mathbf{A}^T$, where $\mathbf{A} \in \mathbb{R}^{d\times d}$, we find that $\mathbf{x}$ is identically distributed to $\mathbf{A}\mathbf{x}' + \mu$, where $\mathbf{x}' \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$.

Our goal is to prove the following result.

Theorem 5. MaxPCSP(K_2, K_3) *is 1-approximable in polynomial time (in the search version).*

Our proof will be split into three parts: First we prove some technical bounds which we will need. Next, we provide a randomised algorithm. Finally, we derandomise the algorithm.

Technical bounds. For the proof, we will need a technical lemma, stated as Lemma 11 below. The proof of Lemma 11 is an application of the following result of Cheng.

Theorem 10 ([Che68][Che69, Equation (2.18)]). *Suppose* $\mathbf{u} = (u_1, u_2, u_3, u_4) \sim \mathcal{N}(\mathbf{0}, \Sigma)$ *are drawn from a quadrivariate normal distribution with mean zero and covariance matrix*

$$\Sigma = \begin{pmatrix} 1 & a & b & ab \\ a & 1 & ab & b \\ b & ab & 1 & a \\ ab & b & a & 1 \end{pmatrix},$$

where $a, b \in [-1, 1]$. *Then* $\Pr_{\mathbf{u}}[u_1 \geq 0, u_2 \geq 0, u_3 \geq 0, u_4 \geq 0]$ *is*

$$\frac{1}{16} + \frac{\arcsin a + \arcsin b + \arcsin ab}{4\pi} + \frac{(\arcsin a)^2 + (\arcsin b)^2 - (\arcsin ab)^2}{4\pi^2}.$$

Lemma 11. *Fix* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha^2 + \beta^2 = 1$. *Suppose* $x_1, x_2, x_3, y_1, y_2, y_3 \sim \mathcal{N}(0, 1)$ *i.e. they are i.i.d. standard normal variables. The probability that*

$$\begin{aligned} x_1 &\geq x_2 \\ x_1 &\geq x_3 \\ \alpha x_1 + \beta y_1 &\geq \alpha x_2 + \beta y_2 \\ \alpha x_1 + \beta y_1 &\geq \alpha x_3 + \beta y_3 \end{aligned}$$

is precisely

$$P(\alpha) = \frac{1}{9} + \frac{\arcsin \alpha + \arcsin \frac{\alpha}{2}}{4\pi} + \frac{(\arcsin \alpha)^2 - (\arcsin \frac{\alpha}{2})^2}{4\pi^2}.$$

Proof. TOPROVE 4 □

We will need a bound on the $P(\alpha)$ function from Lemma 11.

Lemma 12. *For* $-1 \leq \alpha \leq -1$, $1 - 3P(\alpha) \geq \frac{1-\alpha}{2}$.

The functions involved are shown in Figure 2.

Proof. TOPROVE 5 □

Randomised algorithm. We first provide a randomised version of our algorithm, accurate up to some ε.

Theorem 13 (Randomised version of Theorem 5). *There exists a randomised algorithm which, given a graph* $G = (V, E)$ *that has a cut with* ρ *edges and an accuracy parameter* ε, *finds a 3-colouring of* G *that satisfies* $\rho - \varepsilon$ *edges in expectation, in polynomial time with respect to the size of* G *and* $\log(1/\varepsilon)$.

Proof. TOPROVE 6 □

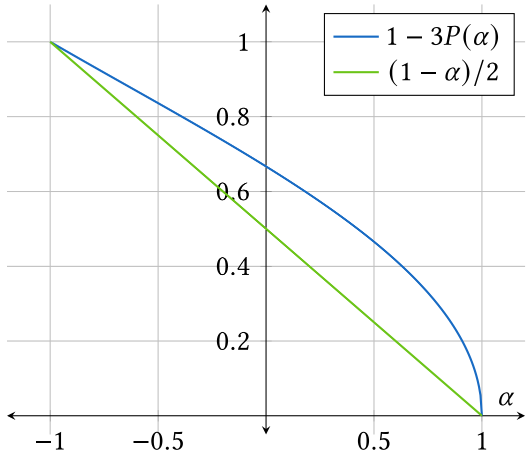

Figure 2: Plot of expressions from Lemma 12.

Derandomised algorithm. We will now show how to derandomise our algorithm, using the method of conditional expectations, which was also used by Mahajan and Ramesh [MR99]. The approach of Bhargava and Kosaraju [BK05] derandomises conditional probabilities by an approximation of normal distributions via polynomials; we approximate simply just with a scaled bionomial distribution. We believe that the results of the literature are sufficient to prove the derandomisation theorem we need (which crucially needs to work for our 1-approximation setting); however we propose a simpler derandomisation method that we believe will be easier to apply in general. (Indeed, our method avoids integration alltogether.) There is an interesting duality between our approach and that of Mahajan and Ramesh: we discretise the normal distribution, whereas they discretise the SDP vectors.

Our goal will be the following general derandomisation theorem.

Theorem 14. *Fix a constant d. There exists an algorithm that does the following. Suppose we are given $n, m \in \mathbb{N}$, $\mathbf{x}_{ij} \in \mathbb{R}^n$ and $y_{ij}, \varepsilon \in \mathbb{R}$ for all $i \in [m], j \in [d]$. Suppose $\mathbf{a} = (a_1, \dots, a_n) \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ and that*

$$\sum_{i=1}^{m} \Pr_{\mathbf{a}}\left[\bigwedge_{j=1}^{d} \mathbf{x}_{ij} \cdot \mathbf{a} > y_{ij}\right] \geq \rho$$

for some $\rho \in \mathbb{R}$. Then the algorithm computes some particular $\mathbf{a}^* = (a_1^*, \dots, a_n^*) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{m} \left[\bigwedge_{j=1}^{d} \mathbf{x}_{ij} \cdot \mathbf{a}^* > y_{ij}\right] \geq \rho - \varepsilon,$$

in polynomial time with respect to $n, m, 1/\varepsilon$.

To facilitate the proof of Theorem 14, we will need a multidimensional version of the Berry-Esseen theorem. We will use the following version with explicit constants, due to Raič [Rai19].

Theorem 15 ([Rai19, Theorem 1.1]). *Suppose* $\mathbf{t}_1, \ldots, \mathbf{t}_N \in \mathbb{R}^d$ *are independent random variables with mean zero, such that the sum of their covariance matrices is* $\mathbf{I}_d$. *Let* $\mathbf{s} = \mathbf{t}_1 + \cdots + \mathbf{t}_N$. *Suppose* $\mathbf{a} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$, *and let* $C \subseteq \mathbb{R}^d$ *be convex and measurable. Then*

$$|\Pr_{\mathbf{s}}[\mathbf{s} \in C] - \Pr_{\mathbf{a}}[\mathbf{a} \in C]| \leq \left(42\sqrt[4]{d} + 16\right) \sum_{i=1}^{N} \mathbf{E}\left[||\mathbf{t}_i||^3\right].$$

The following is an easy and well-known corollary of Theorem 15: We can approximate a multivariate normal distribution with binomial distributions. For completeness, we provide a proof.

Corollary 16. *Let* $d \in \mathbb{N}$ *be a constant and take* $\varepsilon \in (0,1)$. *Take*

$$N = N_\varepsilon \geq \left(\frac{42d^{7/4} + 16d^{3/2}}{\varepsilon}\right)^2 = \frac{\xi_d}{\varepsilon^2}, \tag{5}$$

where $\xi_d = O(d^{7/2})$ *depends only on* d. *Suppose* $s_1, \ldots, s_N \sim \mathrm{NBin}(N)$ *are i.i.d., and let* $\mathbf{s} = (s_1, \ldots, s_d)$. *Let* $\mathbf{a} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$. *Then for all convex measurable sets* $C \subseteq \mathbb{R}^d$ *we have*

$$|\Pr_{\mathbf{s}}[\mathbf{s} \in C] - \Pr_{\mathbf{a}}[\mathbf{a} \in C]| \leq \varepsilon.$$

Proof. TOPROVE 7 □

Theorem 17. *Fix a constant* d, *and take* $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{R}^n$, $y_1, \ldots, y_d \in \mathbb{R}$, $z_1, \ldots, z_d \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$. *Consider the function*

$$p(t) = \Pr_{\mathbf{a} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)}\left[\bigwedge_{i=1}^{d} \mathbf{x}_i \cdot \mathbf{a} + z_i t > y_i\right].$$

There exists a step function $\widehat{p}$ *with* $\mathrm{poly}(1/\varepsilon)$ *steps, where the steps and the values at those steps are computable in polynomial time with respect to* $1/\varepsilon$ *and* n, *such that* $|\widehat{p}(t) - p(t)| \leq \varepsilon$ *for all* $t \in \mathbb{R}$.

Proof. TOPROVE 8 □

Proof. TOPROVE 9 □

This is enough to derandomise our algorithm.

Proof. TOPROVE 10 □

6 Hardness of MaxPCSP$(K_2, \mathfrak{G}_3)$

In this section, we will prove the hardness part of the following result.

Theorem 4. MaxPCSP$(K_2, \mathfrak{G}_3)$ *is* 0.8823*-approximable (in the search version) in polynomial time, and it is* NP*-hard to* $(25/26 + \varepsilon)$*-approximate (even in the decision version) for any fixed* $\varepsilon > 0$.

Our general strategy will be to gadget reduce from the 3-bit PCP of Håstad [Hås01], similarly to [TSSW00] or [BGS98]. The main difficulty comes in from the fact that it is not possible to "negate" variables in an obvious way, since "negation" is not globally preserved by the property of being triangle-free, as opposed to that of being bipartite. Some mild complications will be forced by this. Recall first the definition of exactly-3 linear equations.

Definition 18. In the problem $\mathsf{E3Lin}_\delta$, one is given a system of mod-2 linear equations with exactly 3 variables per equation; i.e. $x+y+z \equiv 0 \bmod 2$ or $x+y+z \equiv 1 \bmod 2$. If it is possible to simultaneously solve a $1-\delta$ fraction of all the equations, one must answer YES; otherwise, if it is not even possible to simultaneously solve a $\frac{1}{2}+\delta$ fraction of the equations, one must answer NO.

Theorem 19 ([Hås01]). *For every small enough δ, the problem $\mathsf{E3Lin}_\delta$ is NP-hard.*

To deal with our negation problems, we will need a "balanced" version of this problem.

Definition 20. In the problem $\mathsf{BalancedE3Lin}_\delta$, one is given a system of mod-2 linear equations with exactly 3 variables per equation; i.e. $x+y+z \equiv 0 \bmod 2$ or $x+y+z \equiv 1 \bmod 2$. Furthermore, the number of equations of the two types is equal. A *balanced solution* to such a system of equations is one that satisfies exactly as many equations of form $x+y+z \equiv 0 \bmod 2$ as those of form $x+y+z \equiv 1 \bmod 2$.

If it is possible to find a balanced solution that satisfies a $1-\delta$ fraction of all the equations, one must answer YES; otherwise, it if is not even possible to find any (*possibly even unbalanced*) solution that satisfies a $\frac{1}{2}+\delta$ fraction of the equations, one must answer NO.

We believe that [Hås01] proves, without being explicit about it, NP-hardness of $\mathsf{BalancedE3Lin}_\delta$, although it is not straightforward to see it from the proof in [Hås01]. For completeness, we provide a simple, self-contained reduction.

Lemma 21. *For every small enough δ, the problem $\mathsf{BalancedE3Lin}_\delta$ is NP-hard.*

Proof. TOPROVE 11 □

We now define the notion of "gadget" that we will need for this particular reduction. This is along the same lines as [BGS98, TSSW00], but (i) generalised to deal with promise problems and (ii) specialised to our particular promise problem.

For the following, if $G = (V, E)$ is any bipartite graph, and $V' \subseteq V$, then we say that a function $c : V' \to \{0, 1\}$ is compatible with G if it is possible to extend c into a 2-colouring of G.

Definition 22. A gadget with performance $\alpha \in \mathbb{N}$ and parity $p \in \{0, 1\}$ is a graph $G = (V, E)$, with $0, x, y, z \in V$, where the following hold.

1. For any function $c : \{0, x, y, z\} \to \{0, 1\}$ such that $c(0) + c(x) + c(y) + c(z) \equiv p \bmod 2$, there exists a bipartite subgraph H of G with α edges, such that c is compatible with H.

2. Any triangle-free subgraph of G has at most α edges.

3. Every triangle-free subgraph H of G with strictly more than $\alpha - 1$ edges puts $0, x, y, z$ in the same connected component C, and the distance from 0 to x, y, z respectively is at most 2. Furthermore C is bipartite, and for any $c : \{0, x, y, z\} \to \{0, 1\}$ that is compatible with C, we have that $c(0) + c(x) + c(y) + c(z) \equiv p \bmod 2$.

Lemma 23. *Suppose we have n containers with capacities $c_1, \ldots, c_n \geq 0$. Suppose we distribute a volume of $c_1 + \cdots + c_n - n + a$ among the containers, distributing $v_i \leq c_i$ volume to container i. Then the number of containers i for which $v_i > c_i - 1$ is at least a.*

Proof. TOPROVE 12 □

The next theorem encodes our reduction from the 3-bit PCP of [Hås01] to $\mathrm{MaxPCSP}(K_2, \mathfrak{G}_3)$. This reduction is standard, needing only some care to deal with the fact that the triangle-free graph selected in the soundness case must be "connected enough".

Theorem 24. *Suppose that for $i \in \{0, 1\}$ there exist gadgets G_i with performance α_i and parity i. Then it is NP-hard to approximate $\mathrm{MaxPCSP}(K_2, \mathfrak{G}_3)$ with approximation ratio $1 - 1/(\alpha_0 + \alpha_1) + \varepsilon$.*

Proof. TOPROVE 13 □

We now exhibit the gadgets. The first gadget is identical to a gadget of Bellare, Goldreich and Sudan [BGS98] (although our analysis is slightly more complicated). In [BGS98], this gadget is called "PC-CUT", defined immediately before [BGS98, Claim 4.17]. The second gadget is a generalisation of the first. Recall that the gadgets of [BGS98] were improved in [TSSW00], and indeed the results of [TSSW00] indicate a generic method to find optimal gadgets for finite-domain CSPs. We do not believe this approach directly applies to our case because the property of being triangle-free is not captured by any finite CSP template (indeed, $\mathfrak{G}_3$ is infinite, and any homomorphism-equivalent structure must also be).

We will write our gadgets as graphs with non-negative integer weights for simplicity of presentation. These gadgets can then be implemented by adding edges multiple times.

Lemma 25. *There exists a gadget with performance 9 and parity 1.*

Proof. TOPROVE 14 □

Lemma 26. *There exists a gadget with performance 17 and parity 0.*

Proof. TOPROVE 15 □

7 Hardness of MaxPCSP(C_{2k+1}, K_ℓ)

In this section, we will prove the following result.

Theorem 6. *For every $k \geq 1$ and $\ell \geq 3$, 1-approximation of* MaxPCSP(C_{2k+1}, K_ℓ) *is* NP*-hard assuming the* UGC*.*

The proof is a generalisation of the proof in [DMR09] that establishes the NP-hardness of *almost 3-colouring* (i.e. given an n vertex graph G, output Yes if G has a 3-colourable $(1-\varepsilon)n$ vertex-induced subgraph, and No if G does not even have an independent set with more than εn vertices), assuming the UGC. We will need, unlike [DMR09], a *multilayered unique games conjecture*, in the style of [BWŽ21]. We first set up the necessary ingredients. The proof is based on *Markov-chain noise operators* — the one we will use is intimately related to C_{2k+1}.

Definition 27. Define M_n to be the matrix which has $\frac{1}{2}$ at position (i, j) if and only if $|i - j| \equiv 1 \bmod n$. I.e. M_n is the *circulant matrix* given by the vector $(0, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2})$, of length n. Note that M_n is a Markov chain on $[n]$, and (i, j) has nonzero transition probability if and only if (i, j) is an edge of C_n.

Lemma 28. *The uniform distribution is the stationary distribution of M_n.*

Proof. TOPROVE 16 □

Lemma 29. *Let ω be the n-th root of unity. The eigenvalues of M_n are $\cos(2k\pi/n)$ for $k = 0, \ldots, n-1$. In particular, if n is odd then exactly one eigenvalue has absolute value 1 and the rest have absolute value at most $\cos(1 - \pi/n) < 1$.*[6]

Proof. TOPROVE 17 □

These are the key properties needed to apply the theory of [DMR09]. We will need a multi-layered Unique Games Conjecture, which we now state.

Definition 30. An ℓ-layered unique label-cover instance consists of a set of variables $X_1, \ldots, X_\ell$, a domain $[D]$, and a multiset of constraints. Each constraint consists of ℓ variables $(x_1, \ldots, x_\ell) \in X_1 \times \cdots \times X_\ell$, together with a family of permutations π_{ij} on $[D]$ for $1 \leq i < j \leq \ell$, such that $\pi_{ik} = \pi_{jk} \circ \pi_{ij}$. A solution is an assignment $c : (X_1 \cup \ldots \cup X_\ell) \to [D]$. The assignment c *strongly satisfies* a constraint given by $(x_1, \ldots, x_\ell)$ and $(\pi_{ij})_{1 \leq i < j \leq \ell}$ if $\pi_{ij}(c(x_i)) = c(x_j)$ *for all* $1 \leq i < j \leq \ell$. The assignment c *weakly satisfies* this same constraint if $\pi_{ij}(c(x_i)) = c(x_j)$ *for at least one pair* $1 \leq i < j \leq \ell$. The strong value of an instance is the maximum fraction of constraints that can be simultaneously strongly satisfied by some assignment; the weak value is given by the maximum fraction of constraints that can be simultaneously weakly satisfied by some assignment.

Note that for $\ell = 2$, weak satisfaction and strong satisfaction (and hence weak and strong values) coincide. Hence for $\ell = 2$ we drop the weak/strong distinction.

Conjecture 31 (UGC [Kho02]). *For every ε there exists D such that, given a 2-layered unique label-cover instance with domain $[D]$, it is* NP*-hard to distinguish if the value is at least $1 - \varepsilon$ or not even ε.*

[6] This would *not* be true if n were even — there would be two eigenvalues with absolute value 1, namely ± 1, taking $k = 0$ and $k = n/2$.

We will show that Conjecture 31 implies the following conjecture. Our proof closely follows [BWŽ21], which in turn builds on [DGKR05], but is generalised to deal with imperfect completeness.

Conjecture 32 (Multilayered UGC). *For every $\varepsilon, \ell \geq 2$ there exists some D such that, given an ℓ-layered unique label-cover instance with domain $[D]$, it is* NP*-hard to distinguish if the strong value is at least $1-\varepsilon$, or if the weak value is not even ε.*

Proof. TOPROVE 18 □

The proof of Theorem 6 will be based on the *long code construction* [BGS98]. We will now describe the building blocks of our reduction.

Definition 33. Fix k, ℓ, D. A *cloud* of vertices, denoted by $\vec{f}$, is a set of vertices $f(a_1, \dots, a_D)$ for $a_1, \dots, a_D \in [2k+1]$. For ℓ clouds of variables $\vec{f_1}, \dots, \vec{f_\ell}$ and a family of permutations $\pi_{ij} : [D] \to [D]$ for $1 \leq i < j \leq \ell$ as in Definition 30, we define the set of edges $E_\pi(\vec{f_1}, \dots, \vec{f_\ell})$ as follows: for every $1 \leq i < j \leq \ell$, $a_1, \dots, a_D, b_1, \dots, b_D \in [2k+1]$ and for which a_t, b_t differ by ± 1 modulo $2k+1$, we include the edge $f_i(a_{\pi_{ij}(1)}, \dots, a_{\pi_{ij}(D)}) - f_j(b_1, \dots, b_D)$.

The following is the key theorem from [DMR09] that we will use. We will define some notions, however we refer to [DMR09] for a full treatment.

Definition 34. For a symmetric Markov operator T on $[q]$, and letting $f, g : [q]^n \to \mathbb{R}$, the value $\langle f, T^{\otimes n} g\rangle$ has the following interpretation. Let $x \in [q]^n$ be distributed uniformly at random, and let $y \in [q]^n$ be such that y_i is distributed according to the transition probabilities in T starting at x_i. Then $\langle f, T^{\otimes n} g\rangle$ is the expected value of $f(x)g(y)$.

The quantity $\langle F_\mu, U_\rho(1-F_{1-\nu})\rangle_\gamma$, which we also denote by $\Gamma_\rho(\mu, \nu)$, has the following interpretation. Let x, y be two normally distributed variables with mean 0, variance 1 and covariance ρ. Then this value is the probability that $x \leq \Phi^{-1}(\mu), y \geq \Phi^{-1}(1-\nu)$, where Φ is the cumulative distribution function of the normal distribution. Essentially, this value is nondecreasing in both μ and ν.

For a function $f : [q]^n \to \mathbb{R}$, the value $I_i^{\leq t}(f)$ is the *low-degree influence* of coordinate i in f. In particular, if $f(x) \in [0,1]$, it can be shown that $\sum_{i=1}^n I_i^{\leq t}(f) \leq t$ and $I_i^{\leq t}(f) \geq 0$. Furthermore, the influence of a coordinate is defined compatibly with permuting coordinates i.e. if the influence of coordinate i is x, and we permute the coordinates of f so as to move i to position j, yielding a function g, then the influence of j in g is still x.

Theorem 35 ([DMR09]). *Fix q and let T be a symmetric Markov operator on $[q]$ with spectral radius $\rho < 1$ (by spectral radius we mean the second largest eigenvalue of T in absolute value). Then for any $\varepsilon > 0$ there exist $\delta > 0$ and $t \in \mathbb{N}$ so that if $f, g : [q]^n \to [0,1]$ are two functions with*

$$\min(I_i^{\leq t}(f), I_i^{\leq t}(g)) < \delta,$$

for all i, then

$$\langle f, T^{\otimes n} g\rangle \geq \langle F_\mu, U_\rho(1-F_{1-\nu})\rangle_\gamma - \varepsilon,$$

where $\mu = E[f], \nu = E[g]$.

Similarly to [DMR09], we will use this in the contrapositive, in particular in the following form.

Corollary 36. *Fix q and let T be a symmetric Markov operator on $[q]$ with spectral radius $\rho < 1$. For every ε there exists $\delta > 0$ and $t \in \mathbb{N}$ so that, if $f, g : [q]^n \to [0,1]$ are two functions with $E[f] \geq \varepsilon, E[g] \geq \varepsilon$ and $\langle f, T^{\otimes n} g\rangle \leq \delta$, then there exists $i \in [n]$ so that $I_i^{\leq t}(f) \geq \delta$ and $I_i^{\leq t}(g) \geq \delta$.*

Proof. TOPROVE 19 □

Lemma 37. *There exists s, δ which depend only on k, ℓ so that the following holds for any D.*

Consider an ℓ-colouring $\vec{f} \to [\ell]$ of the cloud of vertices $\vec{f}$. We denote the vertex $f(a_1, \ldots, a_D)$ receiving colour c by $f(a_1, \ldots, a_D) = c$. There exists a way to assign any such cloud a subset $I(\vec{f})$ of $[D]$ of size s such that the following holds.

Consider any $\ell + 1$ clouds $\vec{f_1}, \ldots, \vec{f_{\ell+1}}$ and a family of permutations π_{ij} as in Definition 30. Suppose that these vertices are ℓ-coloured, and that the ℓ-colouring satisfies a $(1-\delta)$-fraction of the edges in $E_\pi(\vec{f_1}, \ldots, \vec{f_{\ell+1}})$. Then there exists $1 \leq i < j \leq \ell + 1$ such that $\pi_{ij}(I(\vec{f_i})) \cap I(\vec{f_j}) \neq \emptyset$.

Proof. TOPROVE 20 □

Theorem 38. *Conjecture 32 implies Theorem 6.*

Proof. TOPROVE 21 □

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References

[AB09] Sanjeev Arora and Boaz Barak. *Computational complexity: a modern approach.* Cambridge University Press, 2009.

[ABP20] Per Austrin, Amey Bhangale, and Aditya Potukuchi. Improved inapproximability of rainbow coloring. In *Proc. 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'20)*, pages 1479–1495, 2020. arXiv:1810.02784, doi:10.1137/1.9781611975994.90.

[AFO+25] Sergey Avvakumov, Marek Filakovský, Jakub Opršal, Gianluca Tasinato, and Uli Wagner. Hardness of 4-Colourings G-Colourable Graphs. In *Proc. 57th Annual ACM Symposium on Theory of Computing (STOC'25)*. ACM, 2025. arXiv:2504.07592.

[AGH17] Per Austrin, Venkatesan Guruswami, and Johan Håstad. $(2+\varepsilon)$-Sat is NP-hard. *SIAM J. Comput.*, 46(5):1554–1573, 2017. doi:10.1137/15M1006507.

[BBB21] Libor Barto, Diego Battistelli, and Kevin M. Berg. Symmetric Promise Constraint Satisfaction Problems: Beyond the Boolean Case. In *Proc. 38th International Symposium on Theoretical Aspects of Computer Science (STACS'21)*, volume 187 of *LIPIcs*, pages 10:1–10:16, 2021. arXiv:2010.04623, doi:10.4230/LIPIcs.STACS.2021.10.

[BBK+24] Libor Barto, Silvia Butti, Alexandr Kazda, Caterina Viola, and Stanislav Živný. Algebraic approach to approximation. In *39th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'24)*, pages 10:1–10:14. ACM, 2024. arXiv:2401.15186, doi:10.1145/3661814.3662076.

[BBKO21] Libor Barto, Jakub Bulín, Andrei A. Krokhin, and Jakub Opršal. Algebraic approach to promise constraint satisfaction. *J. ACM*, 68(4):28:1–28:66, 2021. arXiv:1811.00970, doi:10.1145/3457606.

[BG21] Joshua Brakensiek and Venkatesan Guruswami. Promise Constraint Satisfaction: Algebraic Structure and a Symmetric Boolean Dichotomy. *SIAM J. Comput.*, 50(6):1663–1700, 2021. arXiv:1704.01937, doi:10.1137/19M128212X.

[BGS98] Mihir Bellare, Oded Goldreich, and Madhu Sudan. Free Bits, PCPs, and Nonapproximability — Towards Tight Results. *SIAM J. Comput.*, 27(3):804–915, 1998. doi:10.1137/S0097539796302531.

[BGS23a] Joshua Brakensiek, Venkatesan Guruswami, and Sai Sandeep. Conditional dichotomy of Boolean ordered promise CSPs. *TheoretiCS*, 2, 2023. arXiv:2102.11854, doi:10.46298/theoretics.23.2.

[BGS23b] Joshua Brakensiek, Venkatesan Guruswami, and Sai Sandeep. SDPs and Robust Satisfiability of Promise CSP. In *Proc. Annual 55th ACM Symposium on Theory of Computing (STOC'23)*, pages 609–622. ACM, 2023. arXiv:2211.08373, doi:10.1145/3564246.3585180.

[BJK05] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM J. Comput.*, 34(3):720–742, 2005. doi:10.1137/S0097539700376676.

[BK05] Ankur Bhargava and S. Rao Kosaraju. Derandomization of dimensionality reduction and SDP based algorithms. In *Proc. 8th International Workshop on Algorithms and Data Structures (WADS'05)*, pages 396–408. Springer Berlin Heidelberg, 2005. doi:10.1007/11534273_35.

[BLŽ25] Silvia Butti, Alberto Larrauri, and Stanislav Živný. Optimal inapproximability of promise equations over finite groups. In *Proc. 52nd International Colloquium on Automata, Languages, and Programming (ICALP'25)*, 2025. arXiv:2411.01630.

[BR15] Jonah Brown-Cohen and Prasad Raghavendra. Combinatorial optimization algorithms via polymorphisms. Technical report, 2015. arXiv:1501.01598.

[Bul17] Andrei A. Bulatov. A dichotomy theorem for nonuniform CSPs. In *Proc. 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS'17)*, pages 319–330, USA, 2017. IEEE. arXiv:1703.03021, doi:10.1109/FOCS.2017.37.

[BWŽ21] Alex Brandts, Marcin Wrochna, and Stanislav Živný. The complexity of promise SAT on non-Boolean domains. *ACM Trans. Comput. Theory*, 13(4):26:1–26:20, 2021. arXiv:1911.09065, doi:10.1145/3470867.

[Che68] M. Cheng. The clipping loss in correlation detectors for arbitrary input signal-to-noise ratios. *IEEE Trans. Inf. Theory*, 14(3):382–389, 1968. doi:10.1109/TIT.1968.1054159.

[Che69] M. C. Cheng. The orthant probabilities of four gaussian variates. *Ann. Math. Stat.*, 40(1):152–161, 1969. doi:10.1214/aoms/1177697812.

[DGKR05] Irit Dinur, Venkatesan Guruswami, Subhash Khot, and Oded Regev. A new multilayered PCP and the hardness of hypergraph vertex cover. *SIAM J. Comput.*, 34(5):1129–1146, 2005. doi:10.1137/S0097539704443057.

[DKPS10] Irit Dinur, Subhash Khot, Will Perkins, and Muli Safra. Hardness of finding independent sets in almost 3-colorable graphs. In *Proc. 51th Annual IEEE Symposium on Foundations of Computer Science (FOCS'10)*, pages 212–221. IEEE Computer Society, 2010. doi:10.1109/FOCS.2010.84.

[DMR09] Irit Dinur, Elchanan Mossel, and Oded Regev. Conditional Hardness for Approximate Coloring. *SIAM J. Comput.*, 39(3):843–873, 2009. doi:10.1137/07068062X.

[DRS05] Irit Dinur, Oded Regev, and Clifford Smyth. The hardness of 3-uniform hypergraph coloring. *Comb.*, 25(5):519–535, September 2005. doi:10.1007/s00493-005-0032-4.

[EH08] Lars Engebretsen and Jonas Holmerin. More efficient queries in PCPs for NP and improved approximation hardness of maximum CSP. *Random Struct. Algorithms*, 33(4):497–514, 2008. doi:10.1002/RSA.20226.

[Euc26] Euclid. *The Thirteen Books of Euclid's Elements*. Cambridge University Press, Cambridge, second edition, 1926. Trans. Sir Thomas Little Heath.

[FJ97] Alan M. Frieze and Mark Jerrum. Improved Approximation Algorithms for MAX k-CUT and MAX BISECTION. *Algorithmica*, 18(1):67–81, 1997. doi:10.1007/BF02523688.

[FKOS19] Miron Ficak, Marcin Kozik, Miroslav Olšák, and Szymon Stankiewicz. Dichotomy for Symmetric Boolean PCSPs. In *Proc. 46th International Colloquium on Automata, Languages, and Programming (ICALP'19)*, volume 132, pages 57:1–57:12, Dagstuhl, Germany, 2019. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. arXiv:1904.12424, doi:10.4230/LIPIcs.ICALP.2019.57.

[FV98] Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. *SIAM J. Comput.*, 28(1), 1998. doi:10.1137/S0097539794266766.

[GJ76] M. R. Garey and D. S. Johnson. The complexity of near-optimal graph coloring. *J. ACM*, 23(1):43–49, 1976. doi:10.1145/321921.321926.

[GW95] Michel X. Goemans and David P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42(6):1115–1145, 1995. doi:10.1145/227683.227684.

[Hås01] Johan Håstad. Some optimal inapproximability results. *J. ACM*, 48(4):798–859, 7 2001. doi:10.1145/502090.502098.

[HMS23] Yahli Hecht, Dor Minzer, and Muli Safra. NP-Hardness of Almost Coloring Almost 3-Colorable Graphs. In *Proc. Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM'23)*, volume 275 of *LIPIcs*, pages 51:1–51:12. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPIcs.APPROX/RANDOM.2023.51.

[HN90] Pavol Hell and Jaroslav Nešetřil. On the Complexity of H-coloring. *J. Comb. Theory, Ser. B*, 48(1):92–110, 1990. doi:10.1016/0095-8956(90)90132-J.

[Jea98] Peter G. Jeavons. On the Algebraic Structure of Combinatorial Problems. *Theor. Comput. Sci.*, 200(1-2):185–204, 1998. doi:10.1016/S0304-3975(97)00230-2.

[Kar72] Richard M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations: Proceedings of a symposium on the Complexity of Computer Computations*, pages 85–103. Springer US, 1972. doi:10.1007/978-1-4684-2001-2_9.

[Kho02] Subhash Khot. On the power of unique 2-prover 1-round games. In *Proc. 34th Annual ACM Symposium on Theory of Computing (STOC'02)*, pages 767–775. ACM, 2002. doi:10.1145/509907.510017.

[Kho10] Subhash Khot. On the unique games conjecture (invited survey). In *Proc. 25th Annual IEEE Conference on Computational Complexity (CCC'10)*, pages 99–121. IEEE Computer Society, 2010. doi:10.1109/CCC.2010.19.

[KKMO07] Subhash Khot, Guy Kindler, Elchanan Mossel, and Ryan O'Donnell. Optimal Inapproximability Results for MAX-CUT and Other 2-Variable CSPs? *SIAM J. Comput.*, 37(1):319–357, 2007. doi:10.1137/S0097539705447372.

[KMS98] David R. Karger, Rajeev Motwani, and Madhu Sudan. Approximate graph coloring by semidefinite programming. *J. ACM*, 45(2):246–265, 1998. doi:10.1145/274787.274791.

[KOWŽ23] Andrei A. Krokhin, Jakub Opršal, Marcin Wrochna, and Stanislav Živný. Topology and adjunction in promise constraint satisfaction. *SIAM J. Computing*, 52(1):37–79, 2023. arXiv:2003.11351, doi:10.1137/20M1378223.

[KS12] Subhash Khot and Rishi Saket. Hardness of finding independent sets in almost q-colorable graphs. In *Proc. 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS'12)*, pages 380–389. IEEE Computer Society, 2012. doi:10.1109/FOCS.2012.75.

[KSTW00] Sanjeev Khanna, Madhu Sudan, Luca Trevisan, and David P. Williamson. The approximability of constraint satisfaction problems. *SIAM J. Comput.*, 30(6):1863–1920, 2000. doi:10.1137/S0097539799349948.

[MOO10] Elchanan Mossel, Ryan O'Donnell, and Krzysztof Oleszkiewicz. Noise stability of functions with low influences: invariance and optimality. *Ann. of Math. (2)*, 171(1):295–341, 2010. doi:10.4007/annals.2010.171.295.

[MR99] Sanjeev Mahajan and H. Ramesh. Derandomizing approximation algorithms based on semidefinite programming. *SIAM J. Comput.*, 28(5):1641–1663, 1999. doi:10.1137/S0097539796309326.

[NŽ23] Tamio-Vesa Nakajima and Stanislav Živný. Maximum k- vs. ℓ-colourings of graphs, 2023. arXiv:2311.00440.

[PY91] Christos H. Papadimitriou and Mihalis Yannakakis. Optimization, approximation, and complexity classes. *J. Comput. Syst. Sci.*, 43(3):425–440, 1991. doi:10.1016/0022-0000(91)90023-X.

[Rag08] Prasad Raghavendra. Optimal algorithms and inapproximability results for every CSP? In *Proc. 40th Annual ACM Symposium on Theory of Computing (STOC'08)*, pages 245–254, 2008. doi:10.1145/1374376.1374414.

[Rai19] Martin Raič. A multivariate Berry–Esseen theorem with explicit constants. *Bernoulli*, 25(4A):2824 – 2853, 2019. arXiv:1802.06475, doi:10.3150/18-BEJ1072.

[RS09] Prasad Raghavendra and David Steurer. How to round any CSP. In *Proc. 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS'09)*, pages 586–594. IEEE Computer Society, 2009. doi:10.1109/FOCS.2009.74.

[Sch78] Thomas Schaefer. The complexity of satisfiability problems. In *Proc. 10th Annual ACM Symposium on the Theory of Computing (STOC'78)*, pages 216–226, USA, 1978. ACM. doi:10.1145/800133.804350.

[TSSW00] Luca Trevisan, Gregory B. Sorkin, Madhu Sudan, and David P. Williamson. Gadgets, approximation, and linear programming. *SIAM J. Comput.*, 29(6):2074–2097, 2000. doi:10.1137/S0097539797328847.

[TŽ16] Johan Thapper and Stanislav Živný. The Complexity of Finite-Valued CSPs. *J. ACM*, 63(4):37:1–37:33, 2016. doi:10.1145/2974019.

[Zhu20] Dmitriy Zhuk. A proof of the CSP dichotomy conjecture. *J. ACM*, 67(5):30:1–30:78, 2020. arXiv:1704.01914, doi:10.1145/3402029.