Faster diameter computation in graphs of bounded Euler genus*

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Abstract

We show that for any fixed integer $k \geqslant 0$, there exists an algorithm that computes the diameter and the eccentricies of all vertices of an input unweighted, undirected n-vertex graph of Euler genus at most k in time

$$\mathcal{O}_k(n^{2-\frac{1}{25}}).$$

Furthermore, for the more general class of graphs that can be constructed by clique-sums from graphs that are of Euler genus at most k after deletion of at most k vertices, we show an algorithm for the same task that achieves the running time bound

$$\mathcal{O}_k(n^{2-\frac{1}{356}} \log^{6k} n).$$

Up to today, the only known subquadratic algorithms for computing the diameter in those graph classes are that of [Ducoffe, Habib, Viennot; SICOMP 2022], [Le, Wulff-Nilsen; SODA 2024], and [Duraj, Konieczny, Potępa; ESA 2024]. These algorithms work in the more general setting of K_h-minor-free graphs, but the running time bound is $\mathcal{O}_h(n^{2-c_h})$ for some constant $c_h > 0$ depending on h. That is, our savings in the exponent, as compared to the naive quadratic algorithm, are independent of the parameter k.

The main technical ingredient of our work is an improved bound on the number of distance profiles, as defined in [Le, Wulff-Nilsen; SODA 2024], in graphs of bounded Euler genus.

1 Introduction

Computing the diameter of an input (undirected, unweighted) graph G is a classic computational problem that can be trivially solved in $\mathcal{O}(nm)$ time[1]. In 2013, Roditty and Vassilevska-Williams showed that this running time bound cannot be significantly improved in general: any algorithm distinguishing graphs of diameter 2 and 3 running in time $\mathcal{O}(m^{2-\varepsilon})$, for any fixed $\varepsilon > 0$, would break the Strong Exponential Time Hypothesis [14]. This motivates the search for restrictions on G that would make the problem of computing the diameter more tractable.

As shown by Cabello and Knauer [3], sophisticated orthogonal range query data structures allow near-linear diameter computation in graphs of constant treewidth. A breakthrough result by Cabello [2] showed that the diameter of an n-vertex planar graph can be computed in $\widetilde{\mathcal{O}}(n^{11/6})$ time; this complexity has been later improved by Gawrychowski, Kaplan, Mozes, Sharir, and Weimann to $\widetilde{\mathcal{O}}(n^{5/3})$ [8][2]. A subsequent line of research [5, 6, 11] generalized this result to K_h-minor-free graphs: for every integer h, there exists a constant $c_h > 0$ such that the diameter problem in n-vertex K_h-minor-free graphs can

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[1] We follow the convention that the vertex and the edge count of the input graph are denoted by n and m, respectively.

[2] The $\widetilde{\mathcal{O}}(\cdot)$ notation hides factors polylogarithmic in n, and the $\mathcal{O}_k(\cdot)$ notation hides factors depending on a parameter k.

be solved in time $\mathcal{O}_h(n^{2-c_h})$. In the works [6, 11], it holds that $c_h = \Omega\left(\frac{1}{h}\right)$; so the savings tend to zero as the size of the excluded clique minor increases.

However, known lower bounds, including the one of [14], does not exclude the possibility that c_h can be made a universal constant. That is, no known lower bound refutes the following conjecture:

Conjecture 1.1. *There exists a constant $c > 0$ such that, for every integer $h > 1$, the diameter problem in (unweighted, undirected) n-vertex K_h-minor-free graphs can be solved in time $\mathcal{O}_h(n^{2-c})$.*

Graphs of bounded Euler genus. Our main result is the verification of Conjecture 1.1 for graphs of bounded Euler genus. Furthermore, our algorithm computes also the eccentricities of all the vertices of the input graph G. Recall here that the eccentricity of a vertex $v \in V(G)$ is defined as $\mathrm{ecc}(v) \coloneqq \max_{u \in V(G)} \mathrm{dist}_G(u, v)$, where $\mathrm{dist}_G(\cdot, \cdot)$ is the distance metric in G.

Theorem 1.2. *For every integers $k \geqslant 1$, there exists an algorithm that, given an (unweighted, undirected) n-vertex graph G of Euler genus at most k, runs in time $\mathcal{O}_k(n^{2-\frac{1}{25}})$ and computes the diameter of G and the eccentricity of every vertex of G.*

We remark that in [2, Section 9], Cabello briefly speculated that his approach could be also generalized to graphs embeddable on surfaces of bounded genus. However, as noted in [2], this would require significant effort, as the technique works closely on the embedding and in surfaces of higher genus, additional topological hurdles arise. In contrast, in our proof of Theorem 1.2 the main ingredient is an improved combinatorial bound on the number of so-called *distance profiles* [11] in graphs of bounded Euler genus. This proof uses topology only very lightly, while the rest of the argument is rather standard and topology-free. All in all, we obtain a robust methodology of approaching the problem, which, as we will see, can be also used to attack Conjecture 1.1 to some extent.

To explain our bound on distance profiles, we need to recall several relevant definitions.

Let G be a graph, $R \subseteq V(G)$ be a subset of vertices, and $s_R \in R$ be a vertex in R. The *distance profile* of a vertex $u \in V(G)$ to R (relative to s_R) is the function $\mathrm{prof}_{R,s_R}[u] \colon R \to \mathbb{Z}$ defined as follows:

$$\mathrm{prof}_{R,s_R}[u](s) = \mathrm{dist}_G(u, s) - \mathrm{dist}_G(u, s_R) \qquad \text{for all } s \in R.$$

Note that provided R is connected[3], we have $\mathrm{prof}_{R,s_R}[u](s) \in \{-|R|, -|R|+1, \ldots, |R|-1, |R|\}$. In [11], Le and Wulff-Nilsen proved that if R is connected and G is K_h-minor-free, then the set system

$$\big\{\{(s,i) \in R \times \{-|R|, \ldots, |R|\} \mid i \leqslant \mathrm{prof}_{R,s_R}[u](s)\} \;:\; u \in V(G)\big\}$$

has VC dimension at most $h-1$. Hence, by applying the Sauer-Shelah Lemma we obtain that

Theorem 1.3 ([11])**.** *For every integer $h \geqslant 1$, K_h-minor-free graph G, connected set $R \subseteq V(G)$, and $s_R \in R$, there are at most $\mathcal{O}_h(|R|^{2h-2})$ different distance profiles to R relative to s_R.*

The VC dimension argument applied above inevitably leads to a bound with the exponent depending on h. We show that for graphs of bounded Euler genus, the bound of Theorem 1.3 can be improved to a polynomial of degree independent of the parameter.

Theorem 1.4. *For every integer $k \geqslant 1$, (unweighted, undirected) graph G of Euler genus at most k, connected set $R \subseteq V(G)$, and $s_R \in R$, the number of distance profiles to R relative to s_R is at most $\mathcal{O}_k(|R|^{12})$.*

The main idea behind the proof of Theorem 1.4 is the following simple observation: if P is a shortest path from some $u \in V(G)$ to s_R, then, as one walks along P from u to s_R, the distance profile of the current vertex to R can only (point-wise) increase. A slightly more technical modification of this argument works for shortest paths from $u \in V(G)$ to R. This allows us to reduce the case of bounded Euler genus graphs to the planar case by cutting along a constant number of shortest-to-R paths, and analysing how the distance profiles change during such a process.

[3] A subset of vertices R of a graph G is *connected* if the induced subgraph $G[R]$ is connected.

One could ask whether an improvement similar to that of Theorem 1.4 would be possible even in the generality of K_h-minor-free graphs. Unfortunately, it seems that Theorem 1.4 is the limit of such improvements. More precisely, the following simple example shows that the linear dependency on h in the exponent of the bound on the number of profiles is inevitable even in graphs of treewidth h (which are $K_{(h+1)^2}$-minor-free).

Let $0 < k \ll \ell$ be positive integers. Let R be a path of length ℓ and $v_1, \ldots, v_k$ be k equidistant points on R (i.e., the distance between v_i and v_{i+1} is at least $p := \lfloor \ell/(k-1) \rfloor$). For every vector $\mathbf{a} = (a_1, \ldots, a_k) \in \{\ell, \ldots, \ell + p\}^k$, construct a vertex $u(\mathbf{a})$ and, for every $i \in \{1, \ldots, k\}$, connect it with v_i using a path of length a_i. This finishes the construction of the graph G; see Figure 1 for an illustration. Note that G has treewidth at most $k+1$, because $G - \{v_1, \ldots, v_k\}$ is a forest. Furthermore, since the distance between consecutive vertices v_i is at least p, we have that $\mathrm{dist}_G(u(\mathbf{a}), v_i) = a_i$ for every vector $\mathbf{a}$ and $i \in \{1, \ldots, k\}$. Consequently, if we restrict to vectors $\mathbf{a}$ with $a_1 = \ell$, every vertex $u(\mathbf{a})$ has a different distance profile to R relative to v_1. Finally, note that there are $(p+1)^{k-1} \geqslant (\ell/(k-1))^{k-1} = \Omega_k(\ell^k)$ different vectors $\mathbf{a}$ with $a_1 = \ell$, giving that many different profiles.

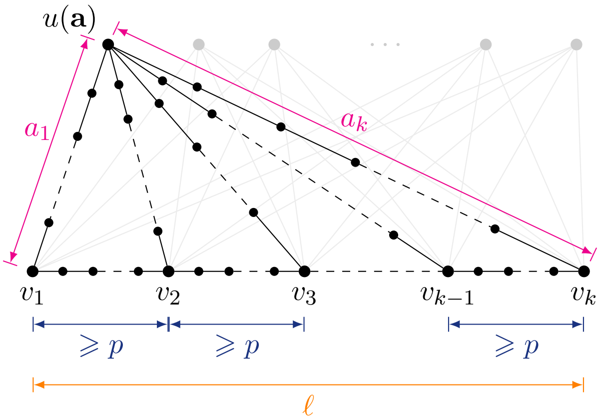

Figure 1: Illustration of a construction that shows that linear dependency on h in the exponent of the bound on the number of profiles is inevitable, even in graphs of treewidth h.

Our algorithm for Theorem 1.2 follows closely the approach of Le and Wulff-Nilsen [11] augmented by the bound provided by Theorem 1.4. Namely, we first compute an r-division of the input graph G into regions of size $r = n^\delta$, for some small $\delta > 0$. Then we use Theorem 1.4 for individual regions R to speed up the computation of distances between R and $V(G) - R$, by grouping vertices outside R according to their distance profiles to R. Each group is batch-processed in a single step.

Generalizations. Further, we show that our techniques combine well with the techniques for bounded treewidth graphs of Cabello and Knauer [3]. First, we show that Conjecture 1.1 holds for classes of graphs of bounded Euler genus with a constant number of *apices*, i.e., vertices that are arbitrarily connected to the rest of the graph.

Theorem 1.5. *For every integers $g, k \geqslant 1$, there exists an algorithm that, given an (unweighted, undirected) n-vertex graph G and a set $A \subseteq V(G)$ such that $|A| \leqslant k$ and $G - A$ is of Euler genus at most g, runs in time $\mathcal{O}_{g,k}(n^{2-\frac{1}{25}} \log^{k-1} n)$ and computes the diameter of G and the eccentricity of every vertex of G.*

Second, we show that Conjecture 1.1 holds for classes of graphs constructed by clique-sums of graphs as in Theorem 1.5. To state this result formally, we need some definitions. For a graph G, a *tree decompostion* of G is a pair (T, β) where T is a tree and β is a function that assigns to every $t \in V(T)$ a *bag* $\beta(t) \subseteq V(G)$ such that (1) for every $v \in V(G)$, the set $\{t \in V(T) \mid v \in \beta(t)\}$ is nonempty and connected in T, and (2) for every $uv \in E(G)$ there exists $t \in V(T)$ with $u, v \in \beta(t)$. The *torso* of the bag $\beta(t)$ is constructed from $G[\beta(t)]$ by adding, for every neighbor s of t in T, all edges between the vertices of $\beta(s) \cap \beta(t)$.

Theorem 1.6. *For every integer $k \geqslant 1$, there exists an algorithm with the following specification. The input consists of an (unweighted, undirected) n-vertex graph G together with a tree decomposition (T, β) of G and a set $A(t) \subseteq \beta(t)$ for every $t \in V(T)$ satisfying the following properties:*

- *For every node $t \in V(T)$, we have that $|A(t)| \leqslant k$ and the torso of $\beta(t)$ with the vertices of $A(t)$ deleted is a graph of Euler genus at most k.*
- *For every edge $st \in E(T)$, we have $|\beta(s) \cap \beta(t)| \leqslant k$.*

The algorithm runs in time $\mathcal{O}_k(n^{2-\frac{1}{356}} \log^{6k} n)$ and computes the diameter of G and the eccentricity of every vertex of G.

Note that the statements of Theorems 1.5 and 1.6 require the set A and the decomposition (T, β), respectively, to be provided explicitly on input; this should be compared with more general statements where the algorithm is given only G with a promise that such set A or decomposition (T, β) exist. At this point, we are not aware of any existing algorithm that would find in subquadratic time a set A as in Theorem 1.5, or the decomposition (T, β) with the sets A as in Theorem 1.6, even in the approximate sense. However, we were informed by Korhonen, Pilipczuk, Stamoulis, and Thilikos [10] that it seems likely that the techniques introduced in the recent almost linear-time algorithm for minor-testing [9] could be used to construct such an algorithm, with almost linear time complexity. With this result in place, the assumption about the decomposition and/or apex sets being provided on input could be lifted in Theorems 1.5 and 1.6; this is, however, left to future work.

Discussion. As one of the main outcomes of their theory of graph minors, Robertson and Seymour proved the following Structure Theorem [13]: every K_h-minor-free graph G admits a tree decomposition (T, β) such that

- for every pair s, t of adjacent nodes of T, the set $\beta(t) \cap \beta(s)$ has size $\mathcal{O}_h(1)$; and
- the torso of every bag $\beta(t)$ is "nearly embedable" into a surface of bounded (in terms of h) Euler genus.

The notion of being "nearly embeddable" encompasses adding a constant number of apices (which can be handled by Theorem 1.6) and a constant number of so-called vortices (which are not handled by Theorem 1.6). Thus, our methods fall short of verifying Conjecture 1.1 in full generality due to vortices.

We remark that recently, Thilikos and Wiederrecht [18] proved a variant of the Structure Theorem, where under the stronger assumption of excluding a minor of a *shallow vortex grid*, instead of a clique minor, they gave a decomposition as above, but with torsos devoid of vortices. Thus, the decomposition for shallow-vortex-grid-minor-free graphs provided by [18] can be directly plugged into Theorem 1.6, with the caveat that [18] does not provide a subquadratic algorithm to compute the decomposition.

Coming back to Conjecture 1.1, the simplest case that we are currently unable to solve is the setting when the input is a planar graph plus a single vortex. More formally, for a fixed integer k, let $\mathcal{G}_k$ be the class of graphs defined as follows. We have $G \in \mathcal{G}_k$ if there exist two subgraphs G_0, G_1 of G and a sequence of vertices $v_1, \ldots, v_b$ in $V(G_0) \cap V(G_1)$ such that:

- $V(G) = V(G_0) \cup V(G_1)$,
- $E(G) = E(G_0) \cup E(G_1)$,
- G_0 admits a planar embedding where the vertices $v_1, \ldots, v_b$ lie on one face in this order, and
- G_1 admits a tree decomposition (T_1, β_1), where T_1 is a path on nodes $t_1, \ldots, t_b$ and for every $i \in \{1, \ldots, b\}$, the bag $\beta_1(t_i)$ contains v_i and is of size at most k.

It is easy to see that graphs from $\mathcal{G}_k$ are $K_{k+\mathcal{O}(1)}$-minor-free. Do they satisfy Conjecture 1.1? That is, is there a constant $c > 0$ such that the diameter problem in $\mathcal{G}_k$ can be solved in time $\mathcal{O}_k(n^{2-c})$?

Organization. We prove Theorem 1.4 in Section 3. Theorem 1.5 is proven in Section 4; note that Theorem 1.2 follows from Theorem 1.5 for $k = 1$. Theorem 1.6 is proven in Section 5.

2 Preliminaries

Set systems and VC-dimension. A *set system* is a collection $\mathcal{F}$ of subsets of a given set A, which we call *ground set* of $\mathcal{F}$. We say that a subset $Y \subseteq A$ is *shattered by* $\mathcal{F}$ if $\{Y \cap S : S \in \mathcal{F}\} = 2^Y$, that is, the intersections of Y and the sets in $\mathcal{F}$ contain every subset of Y. The *VC-dimension* of a set system $\mathcal{F}$ with ground set A is the size of the largest subset $Y \subseteq A$ shattered by $\mathcal{F}$. The notion of VC-dimension was introduced by Vapnik and Chervonenkis [19].

We will use the following well-known Sauer-Shelah Lemma [15, 16], which gives a polynomial upper bound on the size of a set system of bounded VC-dimension.

Lemma 2.1 (Sauer-Shelah Lemma)**.** *Let $\mathcal{F}$ be a set system with ground set A. If the VC-dimension of $\mathcal{F}$ is at most k, then $|\mathcal{F}| = \mathcal{O}(|A|^k)$.*

Basic graph notation. All our graphs are undirected. For a graph G, the neighborhood of a vertex u is defined as $N_G(u) = \{v : uv \in E(G)\}$ and for $X \subseteq V(G)$ we have $N_G(X) = \bigcup_{u \in X} N_G(u) - X$.

The *length* of a path P, denoted $|P|$, is the number of edges of P. For two vertices u, v of a graph G, the *distance* between u and v, denoted $\mathrm{dist}_G(u, v)$, is defined as the minimum length of a path in G with endpoints u and v. For every $v \in V(G)$ and set $R \subseteq V(G)$, we set $\mathrm{dist}_G(v, R) \coloneqq \min\{\mathrm{dist}_G(v, y) \colon y \in R\}$. For vertices x, y appearing on a path P, by $P[x, y]$ we denote the subpath of P with endpoints x and y. The set of vertices traversed by a path P is denoted by $V(P)$. In all above notation, we sometimes drop the subscript if the graph is clear from the context.

For a nonnegative integer q, we use the shorthand $[q] \coloneqq \{1, \ldots, q\}$. For a vertex $v \in V(G)$ and a set $X \subseteq V(G)$, we define the *X-eccentricity* of v as $\mathrm{ecc}_X(v) \coloneqq \max_{x \in X} \mathrm{dist}(v, x)$. Thus, the eccentricity of v in G is the same as its $V(G)$-eccentricity.

The *Euler genus* of a graph G is the minimum Euler characteristic of a surface, where G is embeddable. We refer to the textbook of Mohar and Thomassen for more on surfaces and embedded graphs [12].

We will use the following result of Le and Wulff-Nilsen [11, Theorem 1.3] for planar graphs. Note that the set R is not necessarily connected.

Theorem 2.2. *Let $h \geqslant 1$ be an integer, G be a K_h-minor-free (unweighted, undirected) graph, R be a subset of $V(G)$, and $s_R \in R$. Then the set system*

$$\{\{(s, i) \in R \times \mathbb{Z} \mid i \leqslant \mathrm{dist}_G(u, s) - \mathrm{dist}_G(u, s_R)\} \;:\; u \in V(G)\}$$

has VC-dimension at most $h - 1$.

Algorithmic tools. All our algorithms assume the word RAM model.

To cope with apices, we will need the following classic data structure due to Willard [20].

Theorem 2.3 ([20])**.** *Let V be a set of n points in $\mathbb{R}^d$ and let $w \colon V \to \mathbb{R}$ be a weight function. By a* suffix range*, we mean any set of the form*

$$\mathsf{Range}(r_1, \ldots, r_d) \coloneqq \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i \geqslant r_i \text{ for all } i \in [d]\}$$

for some range parameters $r_1, \ldots, r_d \in \mathbb{R}$.

There is a data structure that uses $\mathcal{O}\left(n \log^{d-1} n\right)$ preprocessing time, $\mathcal{O}\left(n \log^{d-1} n\right)$ memory and answers the following suffix range queries in time $\mathcal{O}\left(\log^{d-1} n\right)$: given a tuple $(r_i)_{i \in [d]}$, find the maximum value of $w(v)$ over all $v \in V \cap \mathsf{Range}(r_1, \ldots, r_d)$.

We will also need the following standard statement about r-divisions.

Theorem 2.4 ([21])**.** *Let G be a K_t-minor-free graph on n vertices. For any fixed constant $\varepsilon > 0$, and for any parameter r with $Ct^2 \log n \leqslant r \leqslant n$, where C is some absolute constant, we can construct in time $\mathcal{O}\left(n^{1+\varepsilon}\sqrt{r}\right)$ an r-division of G, that is, a collection $\mathcal{R}$ of connected subsets of vertices of G such that:*

- *$\bigcup \mathcal{R} = V(G)$,*
- *$|R| \leqslant r$ for every $R \in \mathcal{R}$, and*
- *$\sum_{R \in \mathcal{R}} |\partial R| \leqslant \mathcal{O}(nt/\sqrt{r})$, where $\partial R = R \cap N_G(V(G) - R)$.*

3 Distance profiles in graphs of bounded Euler genus

In this section we prove Theorem 1.4. Our argument consists of a reduction to the planar case, where we can use the constant bound on the VC-dimension of the set system given by the distance profiles due to Le and Wulff-Nilsen [11]. The main idea behind the reduction is to consider certain notions of "extended" profiles, where the extension is built along a collections of shortest paths. These shortest paths can be chosen in such a way that by cutting the graph along these paths we obtain a plane graph. Then a bound on the number of the extended profiles in the obtained plane graph translates to a bound on the number of (standard) distance profiles in the original graph.

Preliminary definitions and results needed for defining profiles with respect to shortest paths are given in Section 3.1. These extended profiles are then defined in Section 3.2. There, we also prove that a fundamental lemma that equality of extended profiles entails equality of (standard) distance profiles. The main reduction providing the proof of Theorem 1.4 is given at the end of this section.

3.1 Milestones

Let G be a graph, R be a subset of $V(G)$, v_0 be a vertex in $V(G)$, and P be a shortest path from v_0 to R. Let x be the unique vertex in $V(P) \cap R$. Further, let $\leqslant_P$ be the linear ordering of the vertices traversed by P: for two vertices $v, u \in V(P)$, we have $v \leqslant_P u$ if u belongs to $P[v,x]$. We say that a vertex $v \in V(P)$ is a *milestone of* P if either $v = x$ or we have $\mathrm{prof}_{R,x}[v] \neq \mathrm{prof}_{R,x}[u]$, where u is the successor of v in $\leqslant_P$. We denote by $M_R(P)$ the set of all milestones of P. Given a milestone $v \in M_R(P)$, the *neutral prefix of* v *in* P is defined as the vertex set of the maximal subpath Q of $P[v_0, v]$ satisfying the following: v is the only milestone of P that belongs to Q.

The next lemma shows that minimum-length paths towards R that contain a vertex in the neutral prefix of a milestone can be assumed to pass through that milestone vertex.

Lemma 3.1. *Let G be a graph, R be a subset of $V(G)$, v_0 be a vertex in $V(G)$ and P be a shortest path from v_0 to R. Then for every $v \in M_R(P)$, every u in the neutral prefix of v, and every $y \in R$, it holds that $\mathrm{dist}(u,y) = |P[u,v]| + \mathrm{dist}(v,y)$.*

Proof. Let x be the unique vertex of $V(P) \cap R$. Note that, by definition, $\mathrm{prof}_{R,x}[v] = \mathrm{prof}_{R,x}[u]$. Also, $\mathrm{dist}(u,x) = \mathrm{dist}(u,v) + \mathrm{dist}(v,x)$ and $\mathrm{dist}(u,v) = |P[u,v]|$. Therefore, $\mathrm{dist}(u,y) = |P[u,v]| + \mathrm{dist}(v,y)$ for every $y \in R$. □

We also give an upper bound on the number of milestones.

Lemma 3.2. *Let G be a graph, R be a connected subset of $V(G)$, v_0 be a vertex of G, and P be a shortest path from v_0 to R. Then the number of milestones of P is at most $|R|^2 + 1$.*

Proof. Let x be the unique vertex of $V(P) \cap R$. First observe that since P is a shortest path from v_0 to R, we have $\mathrm{dist}(v,y) \geqslant \mathrm{dist}(v,x)$ for every $v \in V(P)$ and every $y \in R$; hence $\mathrm{prof}_{R,x}[v](y) \geqslant 0$. Also, since R is connected, for every $y \in R$ we have $\mathrm{prof}_{R,x}[x](y) \leqslant |R|$. To conclude the proof, it suffices to prove that for all $v_1, v_2 \in V(P)$ with $v_1 \leqslant_P v_2$, we have

$$\mathrm{prof}_{R,x}[v_1](y) \leqslant \mathrm{prof}_{R,x}[v_2](y) \qquad \text{for all } y \in R. \tag{1}$$

Indeed, (1) together with the previous observations shows that all the distinct distance profiles of the form $\mathrm{prof}_{R,x}[v]$ for $v \in V(P)$ can be treated as vectors of length $|R|$ with entries in $\{0, \ldots, |R|\}$, and they all have distinct sums $\sum_{y \in R} \mathrm{prof}_{R,x}[v](y)$. Since these sums range between 0 and $|R|^2$, the total number of distinct profiles is at most $|R|^2 + 1$, implying the same bound on the number of milestones.

To see why (1) holds, note that $\mathrm{dist}(v_1,y) \leqslant \mathrm{dist}(v_1,v_2) + \mathrm{dist}(v_2,y)$ implies that

$$\mathrm{dist}(v_1,y) \leqslant \mathrm{dist}(v_1,v_2) + \mathrm{prof}_{R,x}[v_2](y) + \mathrm{dist}(v_2,x) = \mathrm{prof}_{R,x}[v_2](y) + \mathrm{dist}(v_1,x);$$

the last equality follows from P being a shortest path containing v_1, v_2, and x (in this order). This in turn implies that $\mathrm{prof}_{R,x}[v_1](y) = \mathrm{dist}(v_1,y) - \mathrm{dist}(v_1,x) \leqslant \mathrm{prof}_{R,x}[v_2](y)$, as claimed. □

3.2 Anchor-distance profiles

Shortest path collections. Let G be a graph and R be a subset of vertices of G. We say that a collection $\mathcal{P}$ of paths in G is an *R-shortest path collection* if

- every $P \in \mathcal{P}$ is a shortest path from some $v^P \in V(G)$ to R, i.e., $|P| = \mathrm{dist}(v^P, R)$; and
- $R \subseteq \bigcup_{P \in \mathcal{P}} V(P)$.

For each $P \in \mathcal{P}$, we denote by x^P the endpoint of P in R. Note that the collection $\mathcal{P}$ obtained by taking, for every $y \in R$, the zero-length path from y to y, is an R-shortest path collection.

We say that an R-shortest path collection is *consistent* if, for every $P_1, P_2 \in \mathcal{P}$ and $v \in V(P_1) \cap V(P_2)$ the paths $P_1[v, x^{P_1}]$ and $P_2[v, x^{P_2}]$ are equal. That is, once two paths intersect, they continue together towards R.

The following statement is a direct consequence of the definition of an R-shortest path collection.

Observation 3.1. *Let G be a graph, R be a subset of vertices of G, and $\mathcal{P}$ be an R-shortest path collection. Then for every two paths $P_1, P_2 \in \mathcal{P}$ and every $v \in V(P_1) \cap V(P_2)$, we have $|P_1[v, x^{P_1}]| = |P_2[v, x^{P_2}]|$.*

Anchor vertices and their prefixes. Let G be a graph, R be a subset of $V(G)$, and $\mathcal{P}$ be an R-shortest path collection. We denote by $H_{\mathcal{P}}$ the union of the paths in $\mathcal{P}$, i.e., the graph $(\bigcup_{P \in \mathcal{P}} V(P), \bigcup_{P \in \mathcal{P}} E(P))$. We say that a vertex is an *anchor vertex* if either it has degree more than two in $H_{\mathcal{P}}$ or it is a milestone of a path $P \in \mathcal{P}$. We denote by $A_R(P)$ the set of all anchor vertices lying on a path $P \in \mathcal{P}$ and by $A_R(\mathcal{P})$ the set of all anchor vertices for $\mathcal{P}$. Given a path $P \in \mathcal{P}$ with endpoints v_0 and $y \in R$, and an anchor vertex $w \in A_R(P)$, the *prefix of w in P* is the vertex set of the maximal subpath Q of $P[v_0, v]$ satisfying the following: v is the only anchor vertex of P that belongs to Q. Note that for every anchor $w \in V(P)$ there is a milestone w' of P (possibly $w = w'$) such that the prefix of w in P is a subset of the neutral prefix of w' in P. Finally, for an anchor vertex w, the *tail* of w, denoted $\mathrm{tail}(w)$, is the subgraph of G consisting of the union of all prefixes of w in P over all paths $P \in \mathcal{P}$ that contain w.

Hat-distances. Let G be a graph, R be a subset of vertices of G, and $\mathcal{P}$ be an R-shortest path collection. We denote by

$$U_{\mathcal{P}} \coloneqq V(G) - \bigcup_{P \in \mathcal{P}} V(P).$$

For every $u \in U_{\mathcal{P}}$, and every anchor vertex $w \in A_R(\mathcal{P})$, we set

$$\widehat{\mathrm{dist}}(u, w) \coloneqq \min\{|Q_{u,z}| + |P[z, w]| \colon P \in \mathcal{P} \wedge w \in V(P) \wedge z \text{ is in the prefix of } w \text{ in } P\},$$

where $Q_{u,z}$ is a shortest path from u to z with all its internal vertices in $U_{\mathcal{P}}$. If such $Q_{u,z}$ does not exist for any $z \in V(\mathrm{tail}(w))$, we set $\widehat{\mathrm{dist}}(u, w) \coloneqq \infty$.

The following statement is a direct consequence of the definition of $\widehat{\mathrm{dist}}(\cdot, \cdot)$.

Observation 3.2. *Let G be a graph, R be a subset of vertices of G, and $\mathcal{P}$ be an R-shortest path collection. Then for every $u \in U_{\mathcal{P}}$, we have that*

$$\mathrm{dist}(u, R) = \min\left\{\widehat{\mathrm{dist}}(u, w) + \mathrm{dist}(w, R) \colon w \in A_R(\mathcal{P})\right\}.$$

Anchor-distance profiles. Let G be a graph, R be a subset of vertices of G, and $\mathcal{P}$ be an R-shortest path collection. The *anchor-distance profile* of a vertex $u \in U_{\mathcal{P}}$ to R with respect to $\mathcal{P}$ is a function $\mathrm{prof}^\star_{R,\mathcal{P}}[u]$ mapping each $w \in A_R(\mathcal{P})$ to

$$\mathrm{prof}^\star_{R,\mathcal{P}}[u](w) \coloneqq \widehat{\mathrm{dist}}(u, w) + \mathrm{dist}(w, R) - \mathrm{dist}(u, R).$$

Observation 3.2 implies that we always have $\mathrm{prof}^\star_{R,\mathcal{P}}[u](w) \geqslant 0$. We set

$$\widehat{\mathrm{prof}}_{R,\mathcal{P}}[u](w) \coloneqq \min\{\mathrm{prof}^\star_{R,\mathcal{P}}[u](w), |R| + 1\}.$$

Lemma 3.3. *Let G be a graph, let R be a connected subset of vertices of G, and $s_R \in R$. Also, let $\mathcal{P}$ be an R-shortest path collection. Then for all $u_1, u_2 \in U_{\mathcal{P}}$,*

$$\widehat{\mathrm{prof}}_{R,\mathcal{P}}[u_1] = \widehat{\mathrm{prof}}_{R,\mathcal{P}}[u_2] \qquad \textit{implies} \qquad \mathrm{prof}_{R,s_R}[u_1] = \mathrm{prof}_{R,s_R}[u_2].$$

Proof. Fix $u_1, u_2 \in U_{\mathcal{P}}$ with $\widehat{\mathrm{prof}}_{R,\mathcal{P}}[u_1] = \widehat{\mathrm{prof}}_{R,\mathcal{P}}[u_2]$. We start by proving the following.

Claim 3.1. *Let $u \in U_{\mathcal{P}}$ and $y \in R$. There is an anchor $w \in A_R(\mathcal{P})$ such that*

- $\widehat{\mathrm{dist}}(u, w) + \mathrm{dist}(w, y) = \mathrm{dist}(u, y)$ *and*
- $\widehat{\mathrm{prof}}_{R,\mathcal{P}}[u](w) \leqslant |R|$.

Proof. Let Q be a shortest path from u to y and let $P \in \mathcal{P}$ be the path which Q first intersects (if the first vertex of Q in $\bigcup_{P \in \mathcal{P}} V(P)$ belongs to more than one paths in $\mathcal{P}$, we choose P arbitrarily among these paths). Also, let u' be the first vertex of Q (when ordering from u to y) in $V(P)$ and w be the anchor of P that contains u' in its prefix (in P). Note that $u' \in V(\mathrm{tail}(w))$.

We first show that

$$\widehat{\mathrm{dist}}(u, w) + \mathrm{dist}(w, y) = \mathrm{dist}(u, y). \tag{2}$$

By Lemma 3.1 and the fact that $|Q[u', y]| = \mathrm{dist}(u', y)$, we have

$$\mathrm{dist}(w, y) = |Q[u', y]| - |P[u', w]|. \tag{3}$$

Also, by definition, we have

$$\widehat{\mathrm{dist}}(u, w) \leqslant |Q[u, u']| + |P[u', w]|. \tag{4}$$

By (3) and (4), we get that $\widehat{\mathrm{dist}}(u, w) + \mathrm{dist}(w, y) \leqslant |Q|$. Moreover, since Q is a shortest path from u to y and $\widehat{dist}(u, w) \geqslant \mathrm{dist}(u, w)$, we have

$$|Q| = \mathrm{dist}(u, y) \leqslant \mathrm{dist}(u, w) + \mathrm{dist}(w, y) \leqslant \widehat{dist}(u, w) + dist(w, y).$$

This proves (2).

Next, we show that $\widehat{\mathrm{prof}}_{R,\mathcal{P}}[u](w) \leqslant |R|$. Note that

$$\begin{aligned}\mathrm{prof}^{\star}_{R,\mathcal{P}}[u](w) + \mathrm{dist}(u, R) &= \widehat{\mathrm{dist}}(u, w) + \mathrm{dist}(w, R)\\ &\leqslant \widehat{\mathrm{dist}}(u, w) + \mathrm{dist}(w, y) = \mathrm{dist}(u, y).\end{aligned}$$

The connectivity of R implies that $\mathrm{dist}(u, y) \leqslant \mathrm{dist}(u, R) + |R|$, which gives $\mathrm{prof}^{\star}_{R,\mathcal{P}}[u](w) \leqslant |R|$, and the claim follows. ☐

We next show that there is an integer c such that for every $y \in R$, we have

$$\mathrm{dist}(u_1, y) = \mathrm{dist}(u_2, y) + c.$$

Note that this will immediately imply that $\mathrm{prof}_{R,s_R}[u_1] = \mathrm{prof}_{R,s_R}[u_2]$.

By Observation 3.2, for every $h \in \{1, 2\}$, there is an anchor $w_h \in A_R(\mathcal{P})$ such that $\mathrm{dist}(u_h, R) = \widehat{\mathrm{dist}}(u_h, w_h) + \mathrm{dist}(w_h, R)$, which is equivalent to $\mathrm{prof}^{\star}_{R,\mathcal{P}}[u_h](w_h) = 0$. If w_h lies on $P_h \in \mathcal{P}$, then $\mathrm{dist}(u_h, R) = \mathrm{dist}(u_h, x^{P_h})$. Therefore, as $\mathrm{prof}^{\star}_{R,\mathcal{P}}[u_1] = \mathrm{prof}^{\star}_{R,\mathcal{P}}[u_2]$, we can choose $w_1 = w_2$ and $P_1 = P_2$, hence $x^{P_1} = x^{P_2}$. In other words, there exists $x \in R$ such that $\mathrm{dist}(u_1, R) = \mathrm{dist}(u_1, x)$ and $\mathrm{dist}(u_2, R) = \mathrm{dist}(u_2, x)$. We set $c := \mathrm{dist}(u_1, x) - \mathrm{dist}(u_2, x) = \mathrm{dist}(u_1, R) - \mathrm{dist}(u_2, R)$.

Now, fix $y \in R$. Let $w_1 \in A_R(P)$ be the anchor from Claim 3.1 (applied for u_1 and y). As $\widehat{\mathrm{prof}}_{R,\mathcal{P}}[u_1] = \widehat{\mathrm{prof}}_{R,\mathcal{P}}[u_2]$ and $\mathrm{prof}^{\star}_{R,\mathcal{P}}[u_1](w_1) \leqslant |R|$, we have $\mathrm{prof}^{\star}_{R,\mathcal{P}}[u_1](w_1) = \mathrm{prof}^{\star}_{R,\mathcal{P}}[u_2](w_1)$, i.e.,

$$\widehat{\mathrm{dist}}(u_1, w_1) + \mathrm{dist}(w_1, R) - \mathrm{dist}(u_1, R) = \widehat{\mathrm{dist}}(u_2, w_1) + \mathrm{dist}(w_1, R) - \mathrm{dist}(u_2, R).$$

Therefore,

$$\begin{aligned}\mathrm{dist}(u_1,y) &= \widehat{\mathrm{dist}}(u_1,w_1)+\mathrm{dist}(w_1,y)\\ &= \widehat{\mathrm{dist}}(u_2,w_1)+\mathrm{dist}(w_1,y)+c \geqslant \mathrm{dist}(u_2,y)+c;\end{aligned}$$

the first equality follows from Claim 3.1. Thus $\mathrm{dist}(u_2,y)+c \leqslant \mathrm{dist}(u_1,y)$. A symmetric reasoning shows that also $\mathrm{dist}(u_1,y)-c \leqslant \mathrm{dist}(u_2,y)$. Therefore we get $\mathrm{dist}(u_1,y)=\mathrm{dist}(u_2,y)+c$, as required. □

3.3 Reduction from bounded genus graphs to planar graphs

We next recall several definitions related to embeddings of graphs on surfaces. Our basic terminology follows [12]. We say that a graph H embedded in a surface Σ is a *simple cut-graph* of Σ if H has a single face that is also homeomorphic to an open disk; equivalently, H has a single facial walk. Given a surface Σ and a simple cut-graph H on Σ, we denote by $\Sigma \mathbin{⋉} H$ the surface obtained by cutting Σ along H. Note that, provided H is a simple cut-graph, $\Sigma \mathbin{⋉} H$ is always a disk.

Suppose now that a graph G embedded on Σ and H is a subgraph of G that is a simple cut-graph of H. We define $G \mathbin{⋉} H$ to be the graph embedded on $\Sigma \mathbin{⋉} H$ obtained from G as follows. First, let σ be the (unique) facial walk of H and note that each edge e of H is contained exactly twice in σ and each vertex v of H is contained in σ as many times as the degree of v in H. To obtain $G \mathbin{⋉} H$, we replace H with a simple cycle C_σ whose vertex set is the set of copies of vertices of H and its edge set is the set of copies of edges of H in the obvious way. Notice that σ also prescribes for every edge uv of G between a vertex $u \in V(G)-V(H)$ and a vertex $v \in V(H)$, to which copy of v in $G \mathbin{⋉} H$ the vertex u should be adjacent to (in $G \mathbin{⋉} H$). See Figure 2 for an illustration.

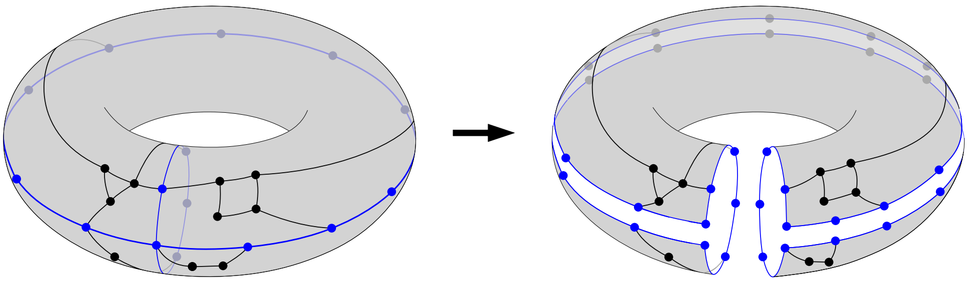

Figure 2: Left: A graph G embedded on a surface Σ and a subgraph H of G (in blue) that is a simple cut-graph of Σ. Right: The graph $G \mathbin{⋉} H$ embedded on the surface $\Sigma \mathbin{⋉} H$ (which is homeomorphic to a disk); the blue vertices/edges are copies of the vertices/edges of H.

We will use the following well-known result which appears in the literature under different formulations; see e.g. [1, 4, 7].

Lemma 3.4. *For every integer $k \geqslant 1$ and for every edge-weighted connected graph G embedded on a surface Σ of Euler genus at most k and every vertex $u \in V(G)$, there is a subgraph H of G with the following properties:*

- *H is a simple cut-graph of Σ, and*
- *$V(H)$ is the union of the vertex sets of $\mathcal{O}(k)$ shortest paths in G that have u as a common endpoint.*

We are now ready to proceed to the proof of Theorem 1.4. Employing Lemma 3.3, we aim at bouding the VC-dimension of the set system defined by the anchor-distance profiles. This is can be done by a suitable reduction to the planar setting using Lemma 3.4.

Proof of Theorem 1.4. We assume that G is connected – the distance profiles of all vertices that are not connected to R are equal. Let T_R be a spanning tree of $G[R]$ and let G_0 be the graph obtained from G after contracting T_R into a single vertex v_R. Consider an embedding of G_0 on a surface Σ of Euler genus at most k. By Lemma 3.4, there is a subgraph H_0 of G_0 that is a simple cut-graph of Σ and a family $\mathcal{P}_0$ of $\mathcal{O}(k)$ shortest paths in G_0, each with v_R as an endpoint, such that $V(H_0) = \bigcup_{P\in\mathcal{P}_0} V(P)$. Furthermore, as Lemma 3.4 handles edge weights, we can slightly perturb the weights so that shortest paths in G_0 are unique and, in particular, all shortest paths with one endpoint in v_R form a tree. Since H_0 is a simple cut-graph of Σ, $G_0 \mathbin{\text{✂}} H_0$ is disk-embedded. Uncontracting T_R, we get a subgraph H of G such that $G \mathbin{\text{✂}} H$ is disk-embedded. Let $\mathcal{P}$ be the family of projections of the paths of $\mathcal{P}_0$ onto G plus, for every $y \in R$, a zero-length path from y to y. Hence, $\mathcal{P}$ is an R-shortest paths collection of size $\mathcal{O}(k)$ with $V(\mathcal{P}) = V(H)$. Furthermore, since in G_0 the paths of $\mathcal{P}_0$ formed a tree rooted at v_R, $\mathcal{P}$ is consistent.

Note that due to Lemma 3.2 we have that $\sum_{P\in\mathcal{P}} |M_R(P)| \leqslant \mathcal{O}_k(|R|^2)$. Also, since $\mathcal{P}$ is consistent, if B are the vertices that are not in R (recall that vertices in R are milestones) and have degree more than two in the graph obtained by the union of the paths in $\mathcal{P}$, then $|B| \leqslant |\mathcal{P}| - 1$. Hence,

$$\sum_{P\in\mathcal{P}} |A_R(P)| \leqslant \mathcal{O}_k(|R|^2). \tag{5}$$

We set $\mathcal{T}$ be the set of all vertices of $G \mathbin{\text{✂}} H$ that are copies of the anchor vertices $A_R(\mathcal{P})$. Every anchor vertex has $\mathcal{O}_k(1)$ copies in $\mathcal{Q}$ and therefore, due to (5),

$$|\mathcal{T}| = \mathcal{O}_k(|R|^2). \tag{6}$$

For $s \in \mathcal{T}$, let $w(s) \in A_R(\mathcal{R})$ be the anchor vertex whose copy (in $G \mathbin{\text{✂}} H$) is s. In the other direction, for $w \in A_R(\mathcal{R})$, let $S(w)$ be the set of copies of w in $G \mathbin{\text{✂}} H$.

Let U be the set of vertices of $G \mathbin{\text{✂}} H$ that are *not* copies of vertices from H (i.e., $U = V(G) - V(H)$). We set E_{out} be the set of all edges uv of $G \mathbin{\text{✂}} H$ where $u \in U$ and v is a copy of a vertex from H, i.e., $v \in V(G \mathbin{\text{✂}} H) - U$. We also set E_{next} be the set of all edges uv of $G \mathbin{\text{✂}} H$ where u is a copy of an anchor vertex $w \in A_R(P)$ for some $P \in \mathcal{P}$ and v is a copy of the neighbor of w in P that *is not* in the prefix of w in P.

Let now $\widehat{G}$ be the graph obtained from $G \mathbin{\text{✂}} H$ after the following modifications:

- we subdivide $|V(G)|$-many times each edge in $E_{\mathsf{out}} \cup E_{\mathsf{next}}$,
- we introduce a new vertex t and add, for every $s \in \mathcal{T}$, a path between t and s of length

$$d_{w(s),t} \coloneqq |V(G)| + \mathrm{dist}_G(w(s), R).$$

See Figure 3. Observe that since $G \mathbin{\text{✂}} H$ is disk-embedded, $\widehat{G}$ is planar, because we may embed t together with all the added paths outside of the disk containing $G \mathbin{\text{✂}} H$.

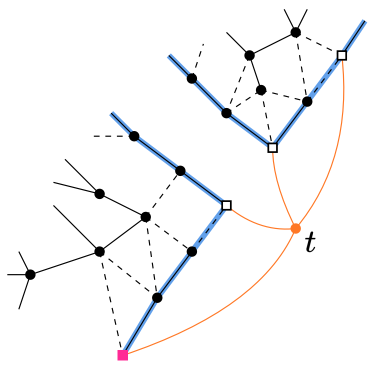

Figure 3: An illustration of (a part of) the construction of the graph $\widehat{G}$. The squared vertices are copies of anchor vertices. The marked squared vertex is also a copy of a vertex in R. The highlighted edges are copies of edges of H in $G \mathbin{\text{✂}} H$, while the paths obtained by subdividing the edges of $E_{\mathsf{out}} \cup E_{\mathsf{next}}$ are depicted with dashed edges. Edges adjacent to t correspond to paths of appropriate length.

For every $u \in U$, we define a function $\pi[u]$, mapping every $w \in A_R(\mathcal{P})$ to

$$\pi[u](w) := \min\{\mathrm{dist}_{\widehat{G}}(u,s) : s \in S(w)\} + d_{w,t} - \mathrm{dist}_{\widehat{G}}(u,t).$$

Also, we set $\widehat{\mathcal{X}} := \{\widehat{X}_u \mid u \in U\}$, where for $u \in U$,

$$\widehat{X}_u := \{(w,i) \in A_R(\mathcal{P}) \times \{0,\ldots,|R|+1\} \mid i \leqslant \pi[u](w)\}\,.$$

Claim 3.2. *The set system $\widehat{\mathcal{X}}$ has size $\mathcal{O}_k(|R|^{12})$.*

Proof. We set $\mathcal{T}^+ := \mathcal{T} \cup \{t\}$. We start with the set system $\mathcal{C}^1 := \left\{C_u^1 :\ u \in U\right\}$, where

$$C_u^1 := \left\{(s,i) \in \mathcal{T}^+ \times \mathbb{Z} \mid i \leqslant \mathrm{dist}_{\widehat{G}}(u,s) - \mathrm{dist}_{\widehat{G}}(u,t)\right\}.$$

As $\widehat{G}$ is planar, by Theorem 2.2 we infer that $\mathcal{C}$ has VC-dimension at most 4.

We now "shift columns" of $\mathcal{C}^1$. That is, define $\mathcal{C}^2 := \left\{C_u^2 :\ u \in U\right\}$, where

$$C_u^2 := \left\{(s,i) \in \mathcal{T}^+ \times \mathbb{Z} \mid i \leqslant \mathrm{dist}_{\widehat{G}}(u,s) + d_{w(s),t} - \mathrm{dist}_{\widehat{G}}(u,t)\right\}.$$

Clearly, the VC-dimension of $\mathcal{C}^1$ and $\mathcal{C}^2$ are equal: a set $Z \subseteq \mathcal{T}^+ \times \mathbb{Z}$ shatters $\mathcal{C}^1$ if and only if the set $\{(s, d_{w(s),t} + i) :\ (s,i) \in Z\}$ shatters $\mathcal{C}^2$.

Now, let $\mathcal{C}^3$ be "cropped" $\mathcal{C}^2$: $\mathcal{C}^3 := \left\{C_u^3 :\ u \in U\right\}$, where

$$C_u^3 := C_u^2 \cap \left(\mathcal{T}^+ \times \{0,\ldots,|R|+1\}\right).$$

Since restricting to a smaller universe cannot increase VC-dimension, $\mathcal{C}^3$ has VC-dimension at most 4. Since $|\mathcal{T}^+| = \mathcal{O}_k(|R|^2)$, by Sauer-Shelah lemma (Lemma 2.1) we have $|\mathcal{C}^3| = \mathcal{O}_k(|R|^{12})$.

Now observe that for every $u_1, u_2 \in U$

$$C_{u_1}^3 = C_{u_2}^3 \qquad \text{implies} \qquad \widehat{X}_{u_1} = \widehat{X}_{u_2}. \tag{7}$$

Indeed, the assumption $C_{u_1}^3 = C_{u_2}^3$ implies that for every $w \in A_R(\mathcal{P})$ and $s \in S(W)$ we have

$$\begin{aligned}&\max(0, \min(|R|+1, \mathrm{dist}_{\widehat{G}}(u_1,s) + d_{w,t} - \mathrm{dist}_{\widehat{G}}(u_1,t)))\\ &= \max(0, \min(|R|+1, \mathrm{dist}_{\widehat{G}}(u_2,s) + d_{w,t} - \mathrm{dist}_{\widehat{G}}(u_2,t))).\end{aligned}$$

For fixed $w \in A_R(\mathcal{P})$, we take a minimum of the above expression over all $s \in S(w)$, obtaining:

$$\begin{aligned}&\max(0, \min(|R|+1, \min\{\mathrm{dist}_{\widehat{G}}(u_1,s) :\ s \in S(w)\} + d_{w,t} - \mathrm{dist}_{\widehat{G}}(u_1,t)))\\ &= \max(0, \min(|R|+1, \min\{\mathrm{dist}_{\widehat{G}}(u_2,s) :\ s \in S(w)\} + d_{w,t} - \mathrm{dist}_{\widehat{G}}(u_2,t))).\end{aligned}$$

This proves (7). From (7), we infer $|\widehat{\mathcal{X}}| \leqslant |\mathcal{C}^3| = \mathcal{O}_k(|R|^{12})$, as desired. $\square$

We next relate the distance from a vertex $u \in U$ to R (in G) and to t (in $\widehat{G}$).

Claim 3.3. *For every $u \in U$, $\mathrm{dist}_G(u,R) = \mathrm{dist}_{\widehat{G}}(u,t) - 2|V(G)|$.*

Proof. Fix $u \in U$. We first show that $\mathrm{dist}_G(u,R) \leqslant \mathrm{dist}_{\widehat{G}}(u,t) - 2|V(G)|$. For this, consider a shortest path $\widehat{Q}$ in $\widehat{G}$ from u to t. Observe that there is a vertex $s \in \mathcal{T}$ that is a copy of an anchor vertex w, such that $\widehat{Q}[s,t]$ is the path from s to t of length $d_{w,t}$ added in the construction of $\widehat{G}$ from G✄H. Recall that $d_{w,t} = \mathrm{dist}_G(w,R) + |V(G)|$. Also, observe that $\widehat{Q}[u,s]$ contains at least one subdivided edge of E_{out}, as it starts in U and ends outside U, and otherwise corresponds to a walk from u to w in G. Therefore, we have

$$\begin{aligned}\mathrm{dist}_{\widehat{G}}(u,t) = |\widehat{Q}| &= |\widehat{Q}[u,s]| + |\widehat{Q}[s,t]|\\ &= |\widehat{Q}[u,s]| + \mathrm{dist}_G(w,R) + |V(G)|\\ &\geqslant |V(G)| + \mathrm{dist}_G(u,w) + \mathrm{dist}_G(w,R) + |V(G)|\\ &\geqslant \mathrm{dist}_G(u,R) + 2|V(G)|.\end{aligned}$$

We next show that $\mathrm{dist}_G(u,R) \geqslant \mathrm{dist}_{\widehat{G}}(u,t) - 2|V(G)|$. For this, consider a shortest path Q in G from u to R. Let $y \in R$ be the unique vertex in $R \cap V(Q)$. Also, let z be the first vertex of Q (when ordering from u to y) in $\bigcup_{P\in\mathcal{P}} V(P)$ and let $P \in \mathcal{P}$ be the path that z is contained (if z is contained to more than one paths, pick one of them arbitrarily). Also, let w be the first vertex of $P[z, x^P]$ (when ordering from z to x^P) that is an anchor vertex. Observe that $Q[u,z]$ corresponds to a path in $\widehat{G}$ from u to a copy s' of z that contains exactly one subdivided edge of E_{out} (and no edge of E_{next}) and there is a copy of $P[z,w]$ in $\widehat{G}$ from s' to a copy s of w that contains no edge of $E_{\mathsf{out}} \cup E_{\mathsf{next}}$. Therefore,

$$
\begin{aligned}
|Q| &= |Q[u,z]| + |Q[z,y]| \\
&= |Q[u,z]| + |P[z,x^P]| && (Q[z,y] \text{ and } P[z,x^P] \text{ being shortest paths from } z \text{ to } R) \\
&= |Q[u,z]| + |P[z,w]| + |P[w,x^P]| \\
&= |Q[u,z]| + |P[z,w]| + \mathrm{dist}_G(w,R) && (P \text{ being shortest path from a vertex } v^P \text{ to } R) \\
&\geqslant \mathrm{dist}_{\widehat{G}}(u,s) - |V(G)| + d_{w,t} - |V(G)| \\
&\geqslant \mathrm{dist}_{\widehat{G}}(u,t) - 2|V(G)|.
\end{aligned}
$$

Thus, we have $\mathrm{dist}_G(u,R) = |Q| \geqslant \mathrm{dist}_{\widehat{G}}(u,t) - 2|V(G)|$, as desired. □

Claim 3.4. *For every $u \in U$ and $w \in A_R(\mathcal{P})$, it holds that*

$$
\begin{aligned}
&\widehat{\mathrm{dist}}(u,w) < \infty \quad \text{if and only if} \quad \widehat{\mathrm{dist}}(u,w) = \min\left\{\mathrm{dist}_{\widehat{G}}(u,s) \,:\, s \in S(w)\right\} - |V(G)|, \text{ and} \\
&\widehat{\mathrm{dist}}(u,w) = \infty \quad \text{if and only if} \quad \min\left\{\mathrm{dist}_{\widehat{G}}(u,s) \,:\, s \in S(w)\right\} > 2|V(G)|.
\end{aligned}
$$

Proof. We first show that if $\widehat{\mathrm{dist}}(u,w) < \infty$, then there exists $s \in S(w)$ with $\mathrm{dist}_{\widehat{G}}(u,s) \leqslant |V(G)| + \widehat{\mathrm{dist}}(u,w)$. To this end, let Q be a path from u to w in G of length $\widehat{\mathrm{dist}}(u,w)$, as in the definition of $\widehat{\mathrm{dist}}(u,w)$. There exists $P \in \mathcal{P}$ with $w \in A_R(P)$ and a vertex $z \in V(P) \cap V(Q)$ such that Q decomposes into $Q[u,z]$ and $Q[z,w] = P[z,w]$, with all internal vertices of $Q[u,z]$ in U. Then, $\widehat{G}$ contains a copy s' of z such that $Q[u,z]$ projects to a path from u to s' with one subdivided edge of E_{out} (and no edge of E_{next}) and also a copy of $P[z,w]$ from s' to a copy s of w with no subdivided edge of $E_{\mathsf{out}} \cup E_{\mathsf{next}}$. The concatenation of these two paths witness that $\mathrm{dist}_{\widehat{G}}(u,s) \leqslant |V(G)| + \widehat{\mathrm{dist}}(u,w)$, as desired.

To finish the proof of the claim, it suffices to show that if there exists $s \in S(w)$ with $\mathrm{dist}_{\widehat{G}}(u,s) \leqslant 2|V(G)|$, then $\widehat{\mathrm{dist}}(u,w) \leqslant \mathrm{dist}_{\widehat{G}}(u,s) - |V(G)|$ (in particular, $\widehat{\mathrm{dist}}(u,w) \neq \infty$). To this end, let $\widehat{Q}$ be a path in $\widehat{G}$ from u to s of minimum length. Since $u \in U$ but $s \notin U$, $\widehat{Q}$ necessarily contains at least one subdivided edge of E_{out}. Since $|\widehat{Q}| \leqslant 2|V(G)|$, $\widehat{Q}$ contains exactly one edge of E_{out}, no edge of E_{next}, and no edge incident with t. Consequently, there exists a vertex s' on $\widehat{Q}$ which is a copy of a vertex z that lies in the prefix of w on some path $P \in \mathcal{P}$ such that $\widehat{Q}$ decomposes as $\widehat{Q}[u,s']$, which has all internal vertices in U, and $\widehat{Q}[s',s]$ going along a copy of $P[z,w]$ to $s \in S(w)$. Hence, $\widehat{Q}$ corresponds to a path Q in G from u to w that satisfies the requirements of the definition of $\widehat{\mathrm{dist}}(u,w)$ and witnesses $\widehat{\mathrm{dist}}(u,w) \leqslant |\widehat{Q}| - |V(G)|$, as desired.

This finishes the proof of the claim. □

Using the two previous claims, we infer that for every $u \in U$ and $w \in A_R(\mathcal{P})$ it holds that

$$
\widehat{\mathrm{prof}}_{R,\mathcal{P}}[u](w) = \min(|R|+1, \pi[u](w)). \tag{8}
$$

Indeed,

$$
\begin{aligned}
\min\left(|R|+1, \pi[u](w)\right) &= \min\left(|R|+1, \min\left\{\mathrm{dist}_{\widehat{G}}(u,s) \,:\, s \in S(w)\right\} + d_{w,t} - \mathrm{dist}_{\widehat{G}}(u,t)\right) \\
&= \min\left(|R|+1, \min\left\{\mathrm{dist}_{\widehat{G}}(u,s) \,:\, s \in S(w)\right\} - |V(G)| \right. \\
&\qquad\qquad \left. + \mathrm{dist}_G(w,R) - \mathrm{dist}_G(u,R)\right) && \text{by Claim 3.3} \\
&= \min\left(|R|+1, \widehat{\mathrm{dist}}(u,w) + \mathrm{dist}_G(w,R) - \mathrm{dist}_G(u,R)\right) && \text{by Claim 3.4} \\
&= \widehat{\mathrm{prof}}_{R,\mathcal{P}}[u](w).
\end{aligned}
$$

Here, in the third step we used the estimate $\mathrm{dist}_G(u,R) - \mathrm{dist}_G(w,R) \leqslant |U| \leqslant |V(G)| - |R|$, so if $\min\left\{\mathrm{dist}_{\widehat{G}}(u,s) \,:\, s \in S(w)\right\} > 2|V(G)|$ (which is equivalent to $\widehat{\mathrm{dist}}(u,w) = \infty$ by Claim 3.4), then the minimum is attained at value $|R|+1$.

For every $u \in U$, we set

$$B_u \coloneqq \left\{(w,i) \in A_R(\mathcal{P}) \times \mathbb{Z} \mid i \leqslant \widehat{\mathrm{prof}}_{G,R}[u](w)\right\}.$$

Claim 3.2 and (8) imply that the set system $\{B_u \,:\, u \in U\}$ has size $\mathcal{O}_k(|R|^{12})$.

Now, for every $v \in V(G)$, we set

$$S_v \coloneqq \left\{(s,i) \in R \times \{-|R|,\ldots,|R|\} \mid i \leqslant \mathrm{prof}_{R,s_R}[v](s)\right\}.$$

The bound on the size of the set system $\{B_u \,:\, u \in U\}$ and Lemma 3.3 imply that the size of $\{S_u \,:\, u \in U\}$ is bounded by $\mathcal{O}_k(|R|^{12})$. We conclude the proof of the lemma by bounding the size of $\{S_u \,:\, u \in V(G) - U\}$. For this, note that every vertex $v \in V(G) - U$ is either a milestone for some path $P \in \mathcal{P}$ or a vertex in the neutral prefix of a milestone. In the latter case, there is a path $P \in \mathcal{P}$ and a milestone $w \in M_R(P)$ such that $S_v = S_w$. Therefore, we have

$$|\{S_u \,:\, u \in V(G) - U\}| \leqslant \sum_{P \in \mathcal{P}} |M_R(P)| \leqslant \mathcal{O}_k(|R|^2),$$

where the second inequality follows from (5). Hence, the size of $\{S_v \,:\, v \in V(G)\}$ is at most

$$|\{S_u \,:\, u \in U\}| + |\{S_u \,:\, u \in V(G) - U\}| = \mathcal{O}_k(|R|^{12}).$$

This finishes the proof of Theorem 1.4. $\square$

4 Bounded Euler genus graphs with apices: proof of Theorem 1.5

In this section we prove Theorem 1.5. (Note that Theorem 1.2 is a special case of Theorem 1.5 for $k = 1$.) We start by deriving the following corollary from Theorem 2.3.

Corollary 4.1. *Let V be a set of n points in $\mathbb{R}^d$. There is a data structure that uses $\mathcal{O}\left(dn\log^{d-2} n\right)$ preprocessing time, $\mathcal{O}\left(dn\log^{d-2} n\right)$ memory and answers the following queries in time $\mathcal{O}\left(d\log^{d-2} n\right)$: given $r_1,\ldots,r_d \in \mathbb{R}$, find $\max_{v\in V}\min_{i\in[d]}(v_i + r_i)$, where v_i denotes the ith coordinate of v.*

Proof. Fix query parameters $r_1,\ldots,r_d \in \mathbb{R}$. Let $\lambda \coloneqq \max_{v\in V}\min_{i\in[d]}(v_i + r_i)$ denote the answer we want to find.

We say that a pair $(v,i) \in V \times [d]$ is *good* if for every $j \in [d]$, it holds that $v_j - v_i \geqslant r_i - r_j$. Let

$$\lambda' = \max\left\{v_i + r_i \colon i \in [d], v \in V, \text{ and } (v,i) \text{ is good}\right\}.$$

We claim that

$$\lambda = \lambda'. \tag{9}$$

Let $v' = \mathrm{argmax}_{v\in V}(\min_{i\in[d]} v_i + r_i)$ and let $i' = \mathrm{argmin}_{i\in[d]} v'_i + r_i$. By the choice of i', for each j we have $v'_j + r_j \geqslant v'_{i'} + r_{i'}$, implying $v'_j - v'_{i'} \geqslant r_{i'} - r_j$. Hence (v',i') is good, so $\lambda' \geqslant v'_{i'} + r_{i'} = \lambda$.

On the other hand, consider a good pair (v',i') maximizing $v'_{i'} + r_{i'}$. The goodness of (v',i') implies that $i' = \mathrm{argmin}_{i\in[d]} v'_i + r_i$, hence $\lambda \geqslant \min_{i\in[d]} v'_i + r_i = v'_{i'} + r_{i'} = \lambda'$. This proves (9).

For every $i \in [d]$, we set V_i to be the set

$$\{(v_1 - v_i, v_2 - v_i, \ldots, v_{i-1} - v_i, v_{i+1} - v_i, \ldots, v_d - v_i) : v \in V\} \subseteq \mathbb{R}^{d-1},$$

and set $w_i(v) \coloneqq v_i$. Let $\mathbb{D}_i$ be the data structure obtained by applying Theorem 2.3 to V_i and w_i. Consider the suffix range

$$R_i \coloneqq \mathsf{Range}(r_i - r_1, r_i - r_2, \ldots, r_i - r_{i-1}, r_i - r_{i+1}, \ldots, r_i - r_d) \subseteq \mathbb{R}^{d-1}.$$

Now, by (9) we have that

$$\lambda = \max\left\{r_i + \max\{w_i(v) : v \in V_i \cap R_i\} \colon i \in [d]\right\}.$$

This value can be computed by asking d queries to the data structures $\mathbb{D}_i$, for $i \in [d]$. This gives us a data structure satisfying the conditions given in the lemma statement. $\square$

The main work in the proof of Theorem 1.5 will be done in the following lemma, which provides a fast computation of eccentricities once a suitable division is provided on input. We adopt the notation for divisions introduced in the statement of Theorem 2.4.

Lemma 4.2. *Fix constants $0 < \alpha, \gamma, \rho < 1$ and $k \in \mathbb{N}$. Assume we are given a connected graph G on n vertices with $O(n)$ edges with positive integer weights, a subset of vertices X, a subset of apices $A \subseteq V(G)$ of size at most k, and a family $\mathcal{R}$ with $V(G) - A = \bigcup \mathcal{R}$ such that the following conditions are satisfied:*

- *$\sum_{R \in \mathcal{R}} |\partial R| \leqslant \mathcal{O}(n^\gamma)$;*
- *for every $R \in \mathcal{R}$, $|R| \leqslant \mathcal{O}(n^\rho)$ and $G[R]$ is connected and contains $O(|R|)$ edges; and*
- *for every $R \in \mathcal{R}$, the number of distance profiles in $G - A$ on ∂R is of $\mathcal{O}(n^\alpha)$.*

Then, we can compute X-eccentricity of every vertex of G in time $\mathcal{O}(n^{\gamma+2\rho}\log n + (n^{1+\gamma} + n^{1+\alpha})\log^{k-1} n)$.

Proof. Let $G' \coloneqq G - A$ and $X' \coloneqq X \cap V(G')$. Denote $A \coloneqq \{a_1, a_2, \ldots, a_k\}$. We first describe the procedure, and then discuss its time complexity.

For every $a \in A$ and $u \in V(G)$, we compute distance between a and u denoted $d_A(a, u)$.

Step 1. We start by precomputing the following information for every region $R \in \mathcal{R}$. For all $u, v \in R$, we compute the distance between u and v in $G'[R]$, denoted $d_R(u, v)$. For all $s \in \partial R$, $u \in V(G')$, we compute the distance between s and u in G', denoted $d_{\partial R}(u, s)$. We arbitrarily pick a pivot vertex $s_R \in \partial R$, and for brevity denote $p_R[u] \coloneqq \mathrm{prof}_{\partial R, s_R}[u]$, where the profile is considered in G'. That is, $p_R[u]$ is the (∂R)-profile of u with respect to s_R:

$$p_R[u](s) = d_{\partial R}(u, s) - d_{\partial R}(u, s_R), \quad \text{for all } u \in V(G') \text{ and } s \in \partial R.$$

We define $P_R \coloneqq \{p_R[u] \colon u \in V(G')\}$. By our assumption, we have $|P_R| \leqslant \mathcal{O}(n^\alpha)$. Finally, for every profile $p \in P_R$, we list all vertices $v \in X' - R$ such that $p_R[v] = p$ and set up the data structure of Corollary 4.1 for the points $(d_A(a_1, v), \ldots, d_A(a_k, v), d_{\partial R_u}(s_R, v))$; denote it by $\mathbb{D}_{R,p}$.

Step 2. For every $u \in V(G)$, we compute $\mathrm{ecc}_X(u)$ as follows. If $u \in A$, the answer is $\max_{v \in X} d_A(u, v)$. Hence, we may assume $u \notin A$. Let R_u denote any region of $\mathcal{R}$ containing u. For every $v \in R_u$, the shortest path from u to v in G either:

- goes through an apex, in which case its length is $\min_{a \in A} d_A(a, u) + d_A(a, v)$; or
- is disjoint from A and intersects ∂R_u, in which case its length is $\min_{s \in \partial R_u} d_{\partial R_u}(s, u) + d_{\partial R_u}(s, v)$; or
- is contained entirely in R_u, in which case its length is $d_{R_u}(u, v)$.

The length of this path is therefore the minimum among the three quantities. Using the above observation, we compute $\mathrm{dist}_G(u, v)$ explicitly for each $v \in R_u$, and define $\Delta_u^{R_u} \coloneqq \max_{v \in R_u \cap X} \mathrm{dist}_G(u, v)$.

For every $v \in V(G) - (A \cup R_u)$, the shortest path between u and v either crosses A or ∂R_u. The length of such path avoiding A is

$$\min_{s \in \partial R_u} d_{\partial R_u}(s, u) + d_{\partial R_u}(s, v) = d_{\partial R_u}(s_R, v) + \min_{s \in \partial R_u} \left(d_{\partial R_u}(s, u) + p_{R_u}[v](s)\right).$$

We partition the vertices v by their profile $p_{R_u}[v]$ and for every $p \in P_{R_u}$, we compute the maximum distance to a vertex with profile p separately. Let $V_p = \{v \in X' - R_u \mid p_{R_u}[v] = p\}$. For every $v \in V_p$, we have

$$\mathrm{dist}_G(u, v) = \min\left(\min_{a \in A} d_A(a, u) + d_A(a, v), d_{\partial R_u}(s_R, v) + \min_{s \in \partial R_u} \left(d_{\partial R_u}(s, u) + p(s)\right)\right).$$

We set $r_i \coloneqq d_A(a, u)$ for $i \in [k]$, and $r_{k+1} \coloneqq \min_{s \in \partial R_u} (d_{\partial R_u}(s, u) + p(s))$. Now,

$$\max_{v \in V_p} \operatorname{dist}_G(u, v) = \max_{v \in V_p} \min(r_1 + d_A(a_1, v), \ldots, r_k + d_A(a_k, v), r_{k+1} + d_{\partial R_u}(s_R, v)).$$

This value can be computed by querying $r_1, \ldots, r_{k+1}$ to the data structure $\mathbb{D}_{R_u,p}$. We define $\Delta_u^{V(G)-(A \cup R_u)}$ as the maximum of such values over all $p \in P_{R_u}$.

Finally, we set $\Delta_u^A \coloneqq \max_{a \in A \cap X} d_A(a, u)$, and report $\operatorname{ecc}_X(u) = \max\left(\Delta_u^A, \Delta_u^{R_u}, \Delta_u^{V(G)-(A \cup R_u)}\right)$.

It remains to argue that this algorithm can be implemented in the desired running time. For any source $u \in V(G)$, distance from u to all vertices of G can be calculated in time $\mathcal{O}((|V(G)| + |E(G)|) \log |V(G)|)$ using Dijkstra's algorithm. Therefore:

- computing $d_A(a, \cdot)$ for all a can be done in time $\mathcal{O}(n \log n)$,
- computing $d_{\partial R}(\cdot, \cdot)$ for all R can be done in time $\mathcal{O}(n \log n \cdot \sum_{R \in \mathcal{R}} |\partial R|) \leqslant \mathcal{O}(n^{1+\gamma} \log n)$,
- computing $d_R(\cdot, \cdot)$ for all $R \in \mathcal{R}$ can be done in time $\mathcal{O}(|\mathcal{R}| n^{2\rho} \log n) \leqslant \mathcal{O}(n^{\gamma+2\rho} \log n)$; constructing $G[R]$ takes $\mathcal{O}(|R|^2 \log n) = \mathcal{O}(n^{2\rho} \log n)$ time and calculating all pairs shortest paths can be done in time $\mathcal{O}(|R||E(G[R])| \log n) = \mathcal{O}(n^{2\rho} \log n)$.

Finally, the total size of the data structures $\mathbb{D}_{R,p}$ over all R, p is $\mathcal{O}(|\mathcal{R}|n) = \mathcal{O}(n^{1+\gamma})$, hence we can construct them in time $\mathcal{O}(n^{1+\gamma} \log^{k-1} n)$.

Consider $u \in V(G) - A$ fixed in step 2. Computing $\Delta_u^{R_u}$ takes $\mathcal{O}(|R| \cdot |\partial R_u|)$ time. Computing Δ_u^A can be done in constant time. Computing $\Delta_u^{V(G)-(A \cup R_u)}$ requires asking $|P_{R_u}|$ queries to some $\mathbb{D}_{R,p}$, which takes $\mathcal{O}(n^\alpha \log^{k-1} n)$ time in total. In total, step 2 for all vertices u can be done in time $\mathcal{O}(n^{1+\alpha} \log^{k-1} n + n^\rho \cdot \sum_{u \in V(G)-A} |\partial R_u|) = \mathcal{O}(n^{1+\alpha} \log^{k-1} n + n^{\gamma+2\rho})$.

We conclude that the total running time is $\mathcal{O}(n^{\gamma+2\rho} \log n + (n^{1+\gamma} + n^{1+\alpha}) \log^{k-1} n)$. □

The next statement is a reformulation of Theorem 1.5.

Theorem 4.3. *Fix constants $k, g \in \mathbb{N}$. Let $\mathcal{C}$ denote the class of all graphs that can be obtained by taking a graph G of Euler genus bounded by g, and adding k apices adjacent arbitrarily to the rest of G and to each other. Then there is an algorithm that given an unweighted graph G belonging to $\mathcal{C}$, together with its set of apices A, computes the eccentricity of every vertex in time $\mathcal{O}_{k,g}\left(n^{1+\frac{24}{25}} \log^{k-1} n\right)$.*

Proof. Let $A = \{a_1, \ldots, a_k\}$ denote the set of apices and let $G' = G - A$. Fix $\rho \coloneqq \frac{2}{25}$. Since graphs of bounded genus exclude some fixed clique as a minor, by Theorem 2.4 (with $\varepsilon = \rho/2$) we can find an $\mathcal{O}(n^\rho)$-division $\mathcal{R}$ of G' satisfying $\sum_{R \in \mathcal{R}} |\partial R| = \mathcal{O}(n^{1-\frac{\rho}{2}})$ in time $\mathcal{O}(n^{1+\rho})$. By Theorem 1.4, the graph G' has a degree 12 polynomial bound on the number of distance profiles. In particular, the number of profiles on every ∂R is of $\mathcal{O}(|R|^{12}) = \mathcal{O}(n^{12\rho})$. Let $X \coloneqq V(G)$, $\gamma \coloneqq 1 - \frac{\rho}{2} = \frac{24}{25}$ and $\alpha \coloneqq 12\rho = \frac{24}{25}$. Now, applying Lemma 4.2 gives us an algorithm computing all eccentricities in time $\mathcal{O}(n^{1+\frac{24}{25}} \log^{k-1} n)$. □

5 The general case: Proof of Theorem 1.6

First, we show that data structure of Corollary 4.1 can be used to compute distances witnessed by shortest paths that pass through a constant-size separator.

Lemma 5.1. *Fix a constant $k \in \mathbb{N}$. There exists an algorithm which as the input receives an edge-weighted graph G on n vertices and m edges together with a partition of its vertices into three sets A, B, C such that $|B| \leqslant k$ and there are no edges between A and C, and as the output computes $\max_{c \in C} \operatorname{dist}(a, c)$ for every $a \in A$. The running time is $\mathcal{O}(m \log n + n \log^{k-1} n)$.*

Proof. Let $B = \{b_1, \ldots, b_k\}$. For any $a \in A, c \in C$, we have $\operatorname{dist}(a, c) = \min_{i \in [k]} \operatorname{dist}(a, b_i) + \operatorname{dist}(c, b_i)$. First, we run Dijkstra's algorithm from every vertex in B to find $\operatorname{dist}(v, b_i)$ for every $v \in V(G)$ and $i \in [k]$. Next, we use Corollary 4.1 to construct a data structure $\mathbb{D}$ for the point set $\{(\operatorname{dist}(c, b_1), \ldots, \operatorname{dist}(c, b_k)) : c \in C\} \subseteq \mathbb{R}^k$. Now, the value $\max_{c \in C} \operatorname{dist}(a, c)$ for any given a is equal to the answer of $\mathbb{D}$ to the query with argument $(\operatorname{dist}(a, b_1), \ldots, \operatorname{dist}(a, b_k))$. □

After computing the distances over a constant-size separator, we will use the following observation to simplify one of the sides of the separation.

Lemma 5.2. *Let G be a edge-weighted connected graph and let A, B, C be a partition of its vertices such that there are no edges between A and C. For every pair of vertices $u, v \in B$, let $P_{u,v}$ be any shortest path from u to v with all internal vertices in C (assuming such a path exists).*

Let G' denote a graph obtained from $G[A \cup B]$ by adding an edge from u to v of weight equal to the length of $P_{u,v}$, for all $u, v \in B$ for which $P_{u,v}$ exists. Then,

$$\mathrm{dist}_G(s,t) = \mathrm{dist}_{G'}(s,t) \qquad \textit{for all } s,t \in A \cup B.$$

Proof. Let G'' be the graph obtained by adding new edges of G' to G. Fix any $s, t \in A \cup B$ and let P denote the shortest path from s to t in G'' which minimizes the number of vertices from C visited. Naturally, the weight of P is equal $\mathrm{dist}_G(s,t)$. Assume that such path visits at least one vertex of C. Then, the path P is of the form $s \xrightarrow{P_1} x \xrightarrow{P_2} y \xrightarrow{P_3} t$, where $x, y \in B$ and all the internal vertices of P_2 are in C. By the construction of G', P_2 can be replaced with a direct edge from x to y of the same weight. We obtain a same weight path with a smaller number of vertices of C visited, which is a contradiction. Therefore, P is entirely contained in $A \cup B$, hence it exists in G'. This shows that $\mathrm{dist}_G(s,t) = \mathrm{dist}_{G'}(s,t)$. $\square$

The next lemma encapsulates the main algorithmic content of the proof of Theorem 1.6. The algorithm will split the tree decomposition provided on input into smaller parts for which the eccentricities are easier to calculate. We use the following lemma to handle a single such part.

Lemma 5.3. *Fix constants $k, g \in \mathbb{N}, 0 < \delta < \frac{1}{54}$. Assume we are given $n \in \mathbb{N}$, an edge-weighted graph G on at most n vertices with a weight function $w\colon E(G) \to \mathbb{N}$, a vertex subset A and a collection of non-empty vertex subsets $V_0, V_1, \ldots, V_\ell$ satisfying the following conditions:*

- *The sum of weights of all the edges in G is bounded by $\mathcal{O}(n)$.*
- *$V(G) - A = V_0 \cup V_1 \cup \cdots \cup V_\ell$.*
- *$|A| \leqslant k$.*
- *For every $i \in [\ell]$, $G[V_i - V_0]$ is connected, $N_G(V_i - V_0) = V_i \cap V_0$, $|V_i| = \mathcal{O}(n^\delta)$, and $|V_0 \cap V_i| \leqslant 4$.*
- *For all $i, j \in [\ell], i \neq j$, $V_i - V_0$ and $V_j - V_0$ are disjoint and non-adjacent in G.*
- *Every edge $uv \in E(G)$ with $u, v \notin A$ is contained in $G[V_i]$ for some $i \in \{0, 1, \ldots, \ell\}$.*
- *The graph obtained by taking $G[V_0]$ and adding a clique on $V_0 \cap V_i$ for every $i \in [\ell]$ has Euler genus bounded by g.*

Then, we can compute the eccentricity of every vertex of G in time $\mathcal{O}\left(n^{1+\frac{150+54\delta}{151}} \log^k n\right)$.

Proof. Fix $\delta' = \frac{1+97\delta}{151}$; we have $\delta' - \delta = \frac{1-54\delta}{151} > 0$. Let E_i denote the set of edges with one endpoint in V_i and the other endpoint in $V_i - V_0$. For $i \in [\ell]$, we shall say that V_i is *heavy* if the sum of weights of E_i is larger than $n^{\delta'}$. Since the sets E_i are pairwise disjoint and the total sum of weights of all the edges is bounded by $\mathcal{O}(n)$, the number of heavy subsets is bounded by $\mathcal{O}(n^{1-\delta'})$. Without loss of generality, we may assume that $V_{\ell'+1}, \ldots, V_\ell$ are heavy and $V_1, \ldots, V_{\ell'}$ are not, for some $\ell' \in \{0, \ldots, \ell\}$.

For any source vertex s, we can calculate distances from s to every vertex of G using breadth first search in time $\mathcal{O}(\sum_{e \in E(G)} w(e)) = \mathcal{O}(n)$. In particular, for every $\ell' < i \leqslant \ell$, we can compute the distances from every vertex of V_i to every vertex of G in total time $\mathcal{O}(n^{2-\delta'+\delta})$, because

$$|V_{\ell'+1} \cup \ldots \cup V_\ell| \leqslant n^{1-\delta'} \cdot \mathcal{O}(n^\delta) = \mathcal{O}(n^{1-\delta'+\delta}).$$

Additionally, we calculate distances $\mathrm{dist}_G(a, v)$ for every $a \in A, v \in V(G)$ in time $O(n)$.

For every $i \in [\ell]$ and $u, v \in V_0 \cap V_i$, there exists a shortest path $P_{i,u,v}$ from u to v with all internal vertices belonging to $V_i - V_0$ due to the assumption that $G[V_i - V_0]$ is connected and $N_G(V_i - V_0) = V_i \cap V_0$. Therefore, the distance from u to v is bounded by the sum of weights of edges in E_i. In particular, for $i \in [\ell']$, $\mathrm{dist}_G(u,v) \leqslant n^{\delta'}$.

We define $\widetilde{G}$ to be the graph obtained by taking $G[A \cup V_0 \cup \cdots \cup V_{\ell'}]$ and applying the following operation for every $i \in \{\ell'+1, \ldots, \ell\}$: for each pair of vertices $u, v \in A \cup (V_0 \cap V_i)$, add an edge in $\widetilde{G}$ between u and v with weight equal to the total weight of $P_{i,u,v}$. For a fixed i, u, we can find $P_{i,u,v}$ for all v using breadth first search in time $\mathcal{O}(n)$. Taking a sum over all i, u, we get that $\tilde{G}$ can be computed in total time $\mathcal{O}(n^{2-\delta'})$.

Claim 5.1. *The sum of the edge weights in $\widetilde{G}$ is $\mathcal{O}(n)$. Moreover, for all $u, v \in V(\widetilde{G})$, we have $\mathrm{dist}_{\widetilde{G}}(u, v) = \mathrm{dist}_G(u, v)$.*

Proof. Consider $i \in \{\ell'+1, \ldots, \ell\}$ and any $u, v \in A \cup (V_0 \cap V_i)$ for which we added an edge. Its weight is bounded by the sum of weights of edges in E_i. Therefore, the total weight of all edges added is at most

$$\sum_{i \in \{\ell'+1,\ldots,\ell\}} \left(|A \cup (V_0 \cap V_i)|^2 \sum_{e \in E_i} w(e) \right) \leqslant (4+k)^2 \sum_{e \in E(G)} w(e) = \mathcal{O}(n).$$

This proves the first part of the claim.

For the second part of the claim, consider any $i \in \{\ell'+1, \ldots, \ell\}$ and observe that by our assumptions, $A \cup (V_0 \cap V_i)$ separates $(V_0 \cup \cdots \cup V_{\ell'} \cup V_{i+1} \cup \cdots \cup V_\ell) - V_i$ from $V_i - V_0$. Hence it suffices to repeatedly apply Lemma 5.2. □

For every $u \in V(\widetilde{G})$, we have $\mathrm{ecc}_G(u) = \max(\mathrm{ecc}_{\widetilde{G}}(v), \max_{v \in V(G)-V(\widetilde{G})} \mathrm{dist}_G(u, v))$. Note, that we already know all the distances $\mathrm{dist}_G(u, v)$ for $v \in V(G) - V(\widetilde{G})$. Similarly, we can already compute $\mathrm{ecc}_G(u)$ for every $u \in V(G) - V(\widetilde{G})$. Therefore, it remains to compute $\mathrm{ecc}_{\widetilde{G}}(v)$ for each $v \in V(\widetilde{G})$. Our goal is to show that this can be done efficiently using Lemma 4.2.

Now, let G' be the graph obtained from $\tilde{G}$ by replacing every edge e non-indicent to A with $w(e) \geqslant 2$ with a path of length $w(e)$ consisting of unit-weight edges. This operation again preserves the distances. Since the sum of edge weights in $\tilde{G}$ is of $\mathcal{O}(n)$, the total number of vertices in G' is of $\mathcal{O}(n)$. For $0 \leqslant i \leqslant \ell'$, we write V'_i to denote the set V_i together with all the vertices added as a part of a path between two endpoints in V_i. As V_i is not heavy for each $i \in [\ell']$, we have

$$|V'_i - V'_0| \leqslant |V_i| + \sum_{e \in E_i} w(e) = \mathcal{O}(n^{\delta'}) \qquad \text{for all } i \in [\ell'].$$

Let G_0 denote the graph $G'[V'_0]$ and let G_0^* denote the graph $G' - A$ with $V'_i - V'_0$ contracted to a single vertex v_i^*, for each $i \in [\ell']$; note that, all edges of G_0 and G_0^* have unit weight.

Claim 5.2. *The graph G_0^* is does not contain K_t as a minor, where $t = \mathcal{O}(\sqrt{g})$.*

Proof. Let $\bar{G}_0$ denote the graph obtained by taking G_0 and adding a clique on $V_0 \cap V_i$ for every $i \in [\ell']$. By lemma assumptions and the fact that subdividing edges does not increase the Euler genus, $\bar{G}_0$ has Euler genus at most g. In particular, $\bar{G}_0$ is $K_{t'}$-minor-free for some $t' = \mathcal{O}(\sqrt{g})$, because the Euler genus of $K_{t'}$ is $\Omega(t'^2)$.

Similarly, let $\bar{G}_0^*$ be the graph obtained by taking G_0^* and adding a clique on each $V_0 \cap V_i$. Note, that $\bar{G}_0^* - \{v_1^*, \ldots, v_{\ell'}^*\}$ is precisely $\bar{G}_0$. Let $t = \max(t', 6)$. Recall that a minor model of a clique K_t consists of t pairwise vertex-disjoint connected subgraphs, called branch sets, such that there is at least one edge between each pair of the branch sets. Consider a minor model φ of K_t in $\bar{G}_0^*$. Note that φ cannot contain any singleton branch set of the form $\{v_i^*\}$, for the degree of v_i^* in $\bar{G}_0^*$ is at most $4 < t - 1$. Furthermore, since $N_{\bar{G}_0^*}(v_i^*) = V_0 \cap V_i$, any branch set containing v_i^* and at least one other vertex contains some $u \in V_0 \cap V_i$, and $N_{\bar{G}_0^*}(v_i^*) \subseteq N_{\bar{G}_0^*}(u)$, hence removing v_i^* from this branch set preserves the model. Therefore, we can assume without loss of generality that all branch sets of φ are disjoint from $\{v_1^*, \ldots, v_{\ell'}^*\}$, hence φ is a minor model of K_t in $\bar{G}_0$. This is a contradiction, as $t \geqslant t'$ and $\bar{G}_0$ is $K_{t'}$-minor-free. Therefore, $\bar{G}_0^*$ is K_t-minor-free, hence G_0^* also. □

Let $\rho' = \frac{2-108\delta}{151} > 0$. The graph G_0^* is a unit-weight graph and is K_t-minor-free. Hence, by applying Theorem 2.4 to G_0^* (with $\varepsilon = \rho'/2$) we obtain an $n^{\rho'}$-division $\mathcal{R}_0$ in time $\mathcal{O}(n^{1+\rho'})$. We extend it to $G' - A$ by mapping every contracted vertex v_i^* to $N_{G'-A}[V_i' - V_0'] = (V_i' - V_0') \cup (V_0 \cap V_i)$. Formally, we put $V_i'' \coloneqq N_{G'-A}[V_i' - V_0']$ and

$$\mathcal{R} \coloneqq \left\{ (R_0 \cap V_0') \cup \bigcup_{i\colon v_i^* \in R_0} V_i'' \colon R_0 \in \mathcal{R}_0 \right\}.$$

Now, we argue that $\mathcal{R}$ is a reasonable division of $G' - A$. Clearly, all sets $R \in \mathcal{R}$ are connected in $G' - A$. Pick any $R \in \mathcal{R}$ and let R_0 be its corresponding set in $\mathcal{R}_0$. Every vertex v_i^* is mapped to a set of size $\mathcal{O}(n^{\delta'})$, therefore

$$|R| \leqslant |R_0| \cdot \mathcal{O}(n^{\delta'}) = \mathcal{O}(n^{\rho'+\delta'}).$$

By our construction, for every $i \in [\ell']$, R is either disjoint from $V_i' - V_0'$ or contains whole $N_{G'-A}[V_i' - V_0']$. This means that no vertex belonging to any $V_i' - V_0'$ can be in ∂R, hence $\partial R \subseteq V_0'$.

Pick any $u \in \partial R \cap R_0$. Assume that $u \notin \partial R_0$. Then every vertex of $N_{G_0^*}(u)$ must be in R_0, hence $N_{G-A'}(u) \subseteq R$, which is a contradiction. This means that $\partial R \cap R_0 \subseteq \partial R_0$.

Pick any $u \in \partial R - R_0$. Then, $u \in V_0 \cap V_i$ for some $i \in [\ell']$ such that $v_i^* \in R_0$. Moreover, $v_i^* \in \partial R_0$ and is adjacent to u in G_0^*. The number of such u is bounded by $4|\partial R_0 \cap \{v_1^*, \ldots, v_{\ell'}^*\}|$.

Putting two cases together, we obtain:

$$\sum_{R\in\mathcal{R}} |\partial R| = \sum_{R\in\mathcal{R}} (|\partial R \cap R_0| + |\partial R - R_0|) \leqslant \sum_{R_0\in\mathcal{R}_0} (|\partial R_0| + 4|\partial R_0 \cap \{v_1^*, \ldots, v_{\ell'}^*\}|) = \mathcal{O}(n^{1-\frac{1}{2}\rho'}).$$

It remains to show the following claim.

Claim 5.3. *Pick any $R \in \mathcal{R}$, $s_R \in R$. The number of different distance profiles on R relative to s_R in $G' - A$ is of $\mathcal{O}(n^{48\rho'+54\delta'})$.*

Proof. We look at every vertex $v \in V(G') - A$ and consider three cases: $v \in R$, $v \in V_0'$, and $v \in V_i' - (V_0' \cup R)$ for some $i \in [\ell']$. By our construction, $R \cap V_0'$ is non-empty, hence w.l.o.g. we can assume that $s_R \in V_0'$ as whether two vertices have the same profile on R is independent of the choice of the pivot vertex.

In the first case, there are at most $|R| = \mathcal{O}(n^{\rho'+\delta'})$ such vertices, hence they realise at most that many profiles.

In the second case, we want to observe that profile of any vertex $v \in V_0'$ on R depends only on its profile on $R \cap V_0'$ (relative to s_R). Pick any $t \in R - V_0'$. Then $t \in V_i' - V_0'$ for some $i \in [\ell']$, $V_i \cap V_0 \subseteq R \cap V_0'$, and every path from v to t intersects $V_i \cap V_0$. In particular, distances from v to vertices of $V_i \cap V_0$ determine its distance to t, which proves the observation.

Let $\tilde{G}_0$ denote the graph obtained by taking $G'[V_0']$ and for every $i \in [\ell'], u, v \in V_0 \cap V_i$ adding a disjoint path from u to v of length $\mathrm{dist}(u, v)$. Let P_i denote the vertex set of paths added between $V_0 \cap V_i$. For every $t \in V_0'$ we have $\mathrm{dist}_{G'-A}(v, t) = \mathrm{dist}_{\tilde{G}_0}(v, t)$, so it suffices to bound the number of profiles on $R \cap V_0'$ in $\tilde{G}_0$. By our assumptions, $\tilde{G}_0$ has Euler genus bounded by g and all P_i are of size $\mathcal{O}(n^{\delta'})$.

Let R_0 be the set of $\mathcal{R}_0$ corresponding to R. Let $\tilde{R}_0$ denote the set $(R \cap V_0') \cup \bigcup_{i:v_i^* \in R_0} P_i$. Such set is connected in $\tilde{G}_0$. Moreover, similarly to R, its size is $\mathcal{O}(n^{\rho'+\delta'})$. Applying Theorem 1.4, we get that the number of distance profiles on $\tilde{R}_0$ in $\tilde{G}_0$ is $\mathcal{O}(n^{12(\rho'+\delta')})$, which also bounds the number of profiles on R in $G' - A$ realised by V_0'.

For the third case, assume $v \in V_i' - (V_0' \cup R)$ for some $i \in [\ell']$. Every path from v to any vertex of R in $G' - A$ intersects $V_i \cap V_0$. Let $v_1, \ldots v_p$ be the vertices of $V_i \cap V_0$, where $p \leqslant 4$. The profile of v on R is then determined by the following:

(a) the profile of each v_j on R,

(b) $\mathrm{dist}_{G'-A}(v, v_j) - \mathrm{dist}_{G'-A}(v, v_1)$ for each $2 \leqslant j \leqslant p$, and
(c) $\mathrm{dist}_{G'-A}(s_R, v_j) - \mathrm{dist}_{G'-A}(s_R, v_1)$ for each $2 \leqslant j \leqslant p$ where s_R is some pivot vertex of R.

By the previous case, the number of distance profiles of each v_j is $\mathcal{O}(n^{12(\rho'+\delta')})$. The distances between v and v_j are bounded by $|V_i'|$, hence each quantity described in (b) can take $\mathcal{O}(n^{\delta'})$ different possible values. Similarly, since v_1 and v_j are connected via V_i', $|\mathrm{dist}_{G'-A}(s_R, v_j) - \mathrm{dist}_{G'-A}(s_R, v_1)| \leqslant \mathcal{O}(n^{\delta'})$. The number of different possible profiles of such v is therefore bounded by $\mathcal{O}(n^{48(\rho'+\delta')+6\delta'}) = \mathcal{O}(n^{48\rho'+54\delta'})$. This finishes the proof of the claim. $\square$

Now we can apply Lemma 4.2 to graph G' with apex set A, $X = V(\widetilde{G})$, and the following constants:

$$\rho = \rho' + \delta', \qquad \gamma = 1 - \frac{1}{2}\rho', \quad \text{and} \quad \alpha = 48\rho' + 54\delta'.$$

This allows us to calculate all $V(\widetilde{G})$-eccentricities in G' in time

$$\mathcal{O}\left(\left(n^{2-\frac{1}{2}\rho'} + n^{1+48\rho'+54\delta'}\right)\log^k n\right) = \mathcal{O}\left(n^{1+\frac{150+54\delta}{151}}\log^k n\right).$$

Since for each $v \in V(\widetilde{G})$ we have $\mathrm{ecc}_{\widetilde{G}}(v) = \max_{u\in V(\widetilde{G})} \mathrm{dist}_{\widetilde{G}}(v,u) = \max_{u\in V(\widetilde{G})} \mathrm{dist}_{G'}(v,u)$, this means that we have successfully computed all the eccentricities in $\widetilde{G}$; and as we argued, this is enough to compute all the eccentricities in G as well.

Finally, the total running time of the algorithm is

$$\mathcal{O}\left(n^{1+\frac{150+54\delta}{151}}\log^k n + n^{2-\delta'+\delta}\right) = \mathcal{O}\left(n^{1+\frac{150+54\delta}{151}}\log^k n\right).$$

$\square$

Lemma 5.4. *Fix constants $k, g \in \mathbb{N}$, $0 < \delta < \frac{1}{54}$. Assume we are given $n \in \mathbb{N}$, an edge-weighted graph G on at most n vertices with a weight function $w\colon E(G) \to \mathbb{N}$, a vertex subset A and a collection of non-empty vertex subsets $V_0, V_1, \ldots, V_\ell$ satisfying the same conditions as in Lemma 5.3 with the following differences:*

- *we don't require $G[V_i - V_0]$ to be connected and $V_i - V_0$ to be adjacent to whole $V_i \cap V_0$;*
- *instead of $|V_0 \cap V_i| \leqslant 4$, we require $|V_0 \cap V_i| \leqslant k$.*

Then, we can compute the eccentricity of every vertex of G in time $\mathcal{O}\left(n^{1+\frac{150+54\delta}{151}}\log^{k+5g} n\right)$.

Proof. We will reduce our input to one which will satisfy the conditions of Lemma 5.3. We start by addressing the adhesions $V_0 \cap V_i$ containing too many vertices.

Let G_0 denote the graph $G[V_0]$ with cliques placed at $V_0 \cap V_i$ for every $i \in [\ell]$. For every $i \in [\ell]$ we repeat the following procedure: while $|V_0 \cap V_i| > 4$, remove arbitrary 5 vertices from $V_0 \cap V_i$. Since $|V_0 \cap V_i| \leqslant k$ for each $i \in [\ell]$, this procedure can be implemented in total time $\mathcal{O}(n)$. As a result, at the end we have $|V_0 \cap V_i| \leqslant 4$ for all $i \in [\ell]$. Let M be the set of all the removed vertices. By our assumptions, G_0 has Euler genus bounded by g, hence it cannot contain $g+1$ pairwise disjoint copies of K_5 (as the Euler genus of a graph is the sum of the Euler genera of its 2-connected components [17] and K_5 is not planar). Each removed quintiple of vertices induces a K_5 in G_0, hence we have $|M| \leqslant 5g$. We set $A' = A \cup M$ and may thus assume that V_i is disjoint from A' for all $0 \leqslant i \leqslant \ell$.

Now, fix $i \in [\ell]$. Let $C_1^i, \ldots, C_{r_i}^i$ denote the connected components of $V_i - V_0$ in $G - A'$. We define $W_j^i := N_{G-A'}[C_j^i]$ for every $j \in [r_i]$. Clearly, all W_j^i induce a connected subgraph of G and satisfy $N_{G-A'}(W_j^i - V_0) = W_j^i \cap V_0$. We put $V_0' := V_0$ and enumerate

$$\{V_1', V_2', \ldots V_{\ell'}'\} := \{W_j^i \colon i \in [\ell], j \in [r_i]\}.$$

It is easy to verify that the sets A' and $V_0', V_1', \ldots, V_{\ell'}'$ satisfy the conditions of Lemma 5.3. We apply said lemma to calculate the eccentricity of every vertex of G in the desired time. $\square$

The next statement is a reformulation of Theorem 1.6.

Theorem 5.5. *Fix constants $k, g \in \mathbb{N}$. Assume we are given a graph G on n vertices together with its tree decomposition (T, β) and a set of private apices $A_t \subseteq \beta(t)$ for each node $t \in V(T)$ such that the following conditions hold:*

- *For every node $t \in V(T)$, we have $|A_t| \leqslant k$.*
- *For every edge $st \in E(T)$, we have $|\beta(v) \cap \beta(u)| \leqslant k$.*
- *For every node $t \in V(T)$, graph obtained by taking $G[\beta(t)] - A_t$ and turning $(\beta(t) \cap \beta(s)) - A_t$ into a clique for every edge $st \in E(T)$ has Euler genus bounded by g.*

Then, we can compute the eccentricity of every vertex of G in time $\mathcal{O}\left(n^{1+\frac{355}{356}} \log^{k+5g} n\right)$.

Proof. We may assume that $|V(T)| \leqslant n$, for every tree decomposition with no two bags comparable by inclusion has this property; and adjacent comparable bags can be merged by contracting the edge between them.

For a node $t \in V(T)$, by the *weight* of t we mean the size of the corresponding bag, that is, $|\beta(t)|$. For any subset of nodes $S \subseteq V(T)$, we define $\beta(S) := \bigcup_{t \in S} \beta(t)$ By the *weight* of S, we mean the total weight of the elements of S, that is, $\sum_{t \in S} |\beta(t)|$.

Claim 5.4. *The weight of $V(T)$ is of $\mathcal{O}(n)$.*

Proof. The sets $\beta'(t) := \beta(t) - \bigcup_{s \in N_T(t)} \beta(s)$ are pairwise disjoint. We have

$$\sum_{t \in V(T)} |\beta(t)| = \sum_{t \in V(T)} |\beta'(t)| + 2 \cdot \sum_{st \in E(T)} |\beta(s) \cap \beta(t)| \leqslant |V(T)| + 2k|E(T)| = \mathcal{O}(n).$$

□

Since every bag induces a graph of bounded Euler genus, the number of edges contained in a bag is linear in its size. In particular, this implies that the total number of edges of G is also bounded by $\mathcal{O}(n)$.

We set

$$\delta := \frac{1}{356} \qquad \text{and} \qquad \Delta := \frac{355}{356}.$$

Root the tree T in an arbitrarily chosen node; this naturally imposes an ancestor-descendant relation in T (for convenience, every node is considered its own ancestor and descendant).

We start by partitioning T into connected subtrees using the following procedure. We proceed bottom-up over T, processing nodes in any order so that a node is processed after all its strict descendants have been processed. Along the way, we mark some nodes and split the edges of T into heavy and light. Let $t \in V(T)$ be the currently processed non-root node of T and let $e \in E(T)$ be the edge connecting t with its parent. If the total weight of all the unmarked nodes that are descendants of t is at least n^δ (recall that this includes t itself as well), then we declare e heavy and mark all the descendants of t that were unmarked so far. Otherwise, the edge e is declared light and the procedure proceeds to further nodes of T.

Observe that removing all heavy edges splits T into connected subtrees, say $T'_1, \cdots T'_m$. All of the subtrees, except for possibly the subtree containing the root node, are of weight at least n^δ. In particular, the number of subtrees m, and therefore the number of heavy edges, is bounded by $\mathcal{O}(n^{1-\delta})$. Moreover, in every subtree T'_i, removing the node closest to the root splits T'_i into smaller components, each of weight less than n^δ.

Fix a heavy edge e and let T^e_1 and T^e_2 be the two subtrees into which T splits after removing e. Let $X^e_i = \beta(T^e_i)$ for $i \in \{1, 2\}$. Put $A_e = X^e_1 - X^e_2$, $C_e = X^e_2 - X^e_1$, and $B_e = X^e_1 \cap X^e_2$. By the properties of tree decompositions, such choice of A_e, B_e, C_e satisfies the conditions of Lemma 5.1, hence in time $\mathcal{O}(n \log^{k-1} n)$ we can compute $\max_{v \in X^e_2} \mathrm{dist}_G(u, v)$ for every $u \in X^e_1$, and $\max_{u \in X^e_1} \mathrm{dist}_G(u, v)$ for every $v \in X^e_2$. Computing this for every heavy edge e takes total time $\mathcal{O}(n^{2-\delta} \log^{k-1} n)$.

Fix any subtree $T' = T'_j$. Let $e_1 = t_1^{e_1}t_2^{e_1}, e_2 = t_1^{e_2}t_2^{e_2}, \ldots, e_\ell = t_1^{e_\ell}t_2^{e_\ell}$ denote the heavy edges incident to T', where $t_1^{e_i} \in V(T')$ and $V(T') \subseteq V(T_1^{e_i})$ for every $i \in [\ell]$. For a vertex $v \in \beta(T')$, let

$$d_0(v) = \max_{u \in \beta(T')} \mathrm{dist}_G(v,u) \qquad \text{and} \qquad d_i(v) = \max_{u \in X_2^{e_i}} \mathrm{dist}_G(v,u), \quad \text{for } i \in [\ell].$$

We have $\mathrm{ecc}(v) = \max\{d_i(v) \colon i \in \{0,1,\ldots,\ell\}\}$.The values of $d_i(v)$ are already calculated for all $i \in [\ell]$, hence it remains to compute $d_0(v)$.

For every $i \in [\ell]$ and every pair of vertices $u, v \in \beta(t_1^{e_i}) \cap \beta(t_2^{e_i})$ we find a shortest path between u and v with all internal vertices inside $X_2^{e_i}$ (or determine that it doesn't exist). For a fixed u, v this can be done in time $\mathcal{O}(n)$. Since in total we perform this step at most $2k^2$ times per heavy edge, it takes $\mathcal{O}(n^{2-\delta})$ time in total. Let $P_{i,u,v}$ denote such path, assuming it exists.

Let G' denote the graph obtained from $G[\beta(T')]$ by taking every i, u, v for which $P_{i,u,v}$ exists and adding an edge between u and v of weight equal to the total weight of $P_{i,u,v}$. The weight of every edge inserted in $\beta(t_1^{e_i}) \cap \beta(t_2^{e_i})$ is bounded by $|X_2^{e_i}| + 1$. The total weight of all edges inserted is therefore at most

$$\sum_{i \in [\ell]} |\beta(t_1^{e_i}) \cap \beta(t_2^{e_i})|^2 \cdot (|X_2^{e_i}| + 1) \leqslant k^2 \sum_{i \in [\ell]} (|X_2^{e_i}| + 1) = \mathcal{O}(n),$$

where the last equality follows from the fact that all the trees $T_2^{e_i}$ are pairwise disjoint. By Lemma 5.2, we have $\mathrm{dist}_{G'}(u,v) = \mathrm{dist}_G(u,v)$ for each $u, v \in \beta(T')$. Hence, computing $d_0(v)$ for every $v \in \beta(T')$ is equivalent to computing the eccentricity of every vertex in G'.

If the size of $\beta(T')$ is smaller than n^Δ, we compute the eccentricities naively in time $\mathcal{O}(|\beta(T')|^2)$, noting that G' has $\mathcal{O}(|\beta(T')|)$ edges (thanks to Claim 5.4 and bounded genus assumption of the last bullet of the theorem statement). Otherwise, we argue that we can use the algorithm in Lemma 5.3 as follows.

Let t be the node of T' closest to the root. Let $s_1, \ldots, s_p$ be the children of t in T and let T''_i denote the connected component of $T' - \{t\}$ containing s_i. Set $V_0 = \beta(t)$ and $V_i = \beta(T''_i)$ for $i \in [p]$.

It is now easy to verify that G' and sets $A, \{V_i \colon 0 \leqslant i \leqslant p\}$ selected this way satisfy the assumptions of Lemma 5.4. This allows us to use it to compute the eccentricities in G' in time

$$\mathcal{O}\left(n^{1+\frac{150+54\delta}{151}} \log^{k+5g} n\right) = \mathcal{O}\left(n^{1+\frac{354}{356}} \log^{k+5g} n\right).$$

As we argued, from these eccentricities, we may easily compute all the eccentricities in G.

Now, let us analyse the total running time of the whole algorithm. We invoke Lemma 5.3 $\mathcal{O}(n^{1-\Delta})$ times, since we apply it only to subtrees T'_i of size at least n^Δ. The total running time of those applications is hence

$$\mathcal{O}\left(n^{2+\frac{354}{356}-\Delta} \log^{k+5g} n\right) = \mathcal{O}\left(n^{1+\frac{355}{356}} \log^{k+5g} n\right).$$

We compute the eccentricities naively for subtrees smaller than n^Δ, hence the total running time of this computation is

$$\sum_{i \in [m] \colon |\beta(T'_i)| \leqslant n^\Delta} |\beta(T'_i)|^2 \leqslant n^\Delta \cdot \sum_{i \in m} |\beta(T'_i)| = \mathcal{O}(n^{1+\Delta}) = \mathcal{O}\left(n^{1+\frac{355}{356}}\right).$$

The rest of computation can be done in $\mathcal{O}(n^{2-\delta} \log^k n)$. Therefore, the whole algorithm runs in time $\mathcal{O}\left(n^{1+\frac{355}{356}} \log^{k+5g} n\right)$. □

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