

# Faster, Deterministic and Space Efficient Subtrajectory Clustering

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## Abstract

Given a trajectory  $T$  and a distance  $\Delta$ , we wish to find a set  $C$  of curves of complexity at most  $\ell$ , such that we can cover  $T$  with subcurves that each are within Fréchet distance  $\Delta$  to at least one curve in  $C$ . We call  $C$  an  $(\ell, \Delta)$ -clustering and aim to find an  $(\ell, \Delta)$ -clustering of minimum cardinality. This problem variant was introduced by Akitaya *et al.* (2021) and shown to be NP-complete. The main focus has therefore been on bicriteria approximation algorithms, allowing for the clustering to be an  $(\ell, \Theta(\Delta))$ -clustering of roughly optimal size.

We present algorithms that construct  $(\ell, 4\Delta)$ -clusterings of  $\mathcal{O}(k \log n)$  size, where  $k$  is the size of the optimal  $(\ell, \Delta)$ -clustering. We use  $\mathcal{O}(n^3)$  space and  $\mathcal{O}(kn^3 \log^4 n)$  time. Our algorithms significantly improve upon the clustering quality (improving the approximation factor in  $\Delta$ ) and size (whenever  $\ell \in \Omega(\log n / \log k)$ ). We offer deterministic running times improving known expected bounds by a factor near-linear in  $\ell$ . Additionally, we match the space usage of prior work, and improve it substantially, by a factor super-linear in  $n\ell$ , when compared to deterministic results.

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## 1 Introduction

In subtrajectory clustering, the goal is to partition an input trajectory  $T$  with  $n$  vertices into subtrajectories and group them into *clusters* such that all subtrajectories within a cluster have low Fréchet distance to one another. Clustering under the Fréchet distance is a natural application of the Fréchet distance and a well-studied topic [9, 10, 11, 15, 16] with applications in, for example, map reconstruction [6, 7]. In recent years, several variants of this algorithmic problem have been proposed [1, 3, 5, 8]. Regardless of the variant, the subtrajectory clustering problem has been shown to be NP-complete [1, 3, 8].

We focus on the problem variant proposed by Akitaya, Brüning, Chambers, and Driemel [3]. Given a trajectory  $T$  and a distance  $\Delta$ , and some  $\ell$ , they compute what we call an  $(\ell, \Delta)$ -clustering  $C$  of  $T$ . Each cluster  $Z \in C$  is a set of subtrajectories together with a center curve

# Clusters	$\Delta' =$	Time	Space	Source
$\mathcal{O}(k\ell^2 \log(k\ell))$	$19\Delta$	$\tilde{\mathcal{O}}(k\ell^4 \lambda^2 + n\lambda)$	$\mathcal{O}(n + \lambda)$	[3]
$\mathcal{O}(k\ell \log k)$	$11\Delta$	$\tilde{\mathcal{O}}(kn^3 \ell)$	$\tilde{\mathcal{O}}(n^3)$	[5]
$\mathcal{O}(k \log n)$	$11\Delta$	$\tilde{\mathcal{O}}(kn^4 \ell + n^4 \ell^2)$	$\tilde{\mathcal{O}}(n^4 \ell)$	[13]
$\mathcal{O}(k \log n)$	$4\Delta$	$\mathcal{O}(kn^3 \log^4 n)$	$\mathcal{O}(n^3)$	Thm. 16

■ **Table 1** Prior work and our result. The first two (red) rows indicate randomized results.  $k$  denotes the smallest  $(\ell, \Delta)$ -clustering size of  $T$ .  $\lambda$  denotes the arc length of  $T$  relative to  $\Delta$ .

(the ‘reference curve’  $P_Z$ ) of complexity at most  $\ell$ . Each curve in a cluster must have Fréchet distance at most  $\Delta$  to the center and each point on  $T$  must be present in at least one cluster. The goal is to compute an  $(\ell, \Delta)$ -clustering of minimum cardinality. Note that the parameter  $\ell$  is necessary to not trivialize the problem. Indeed, if  $P_Z$  may have arbitrary complexity, then a trivial  $(n, 0)$ -clustering exists consisting of a single cluster  $Z$  where  $Z = \{T\}$ .

Akitaya *et al.* [3] propose a bicriteria approximation scheme: Given  $\ell$  and  $\Delta$ , let  $k$  be the minimum size of an  $(\ell, \Delta)$ -clustering of  $T$ . The goal is to compute an  $(\ell, \Theta(\Delta))$ -clustering of size  $\mathcal{O}(f(k))$ . This paradigm was studied in [3, 5, 13] and previous results are summarised in Table 1. [3] computes an  $(\ell, 19\Delta)$ -clustering of  $\mathcal{O}(k\ell^2 \log(k\ell))$  size. The running time and space bounds depend on the *arc length* of  $T$  relative to  $\Delta$ . Brüning, Conradi and Driemel [5] compute an  $(\ell, 11\Delta)$ -clustering of  $\mathcal{O}(k\ell \log k)$  size (where the hidden constant is exceptionally large). Their algorithm uses  $\tilde{\mathcal{O}}(n^3)$  space and has  $\tilde{\mathcal{O}}(kn^3)$  expected running time. Recently, Conradi and Driemel [13] improve both the size and the quality of the clustering. They compute an  $(\ell, 11\Delta)$ -clustering of  $\mathcal{O}(k \log n)$  size in  $\tilde{\mathcal{O}}(n^4 \ell)$  space and  $\tilde{\mathcal{O}}(kn^4 \ell + n^4 \ell^2)$  time.

**Results.** We present a bicriteria approximation algorithm that uses  $\mathcal{O}(kn^3 \log^4 n)$  time and  $\mathcal{O}(n^3)$  space, and computes an  $(\ell, 4\Delta)$ -clustering of size  $\mathcal{O}(k \log n)$ . When compared to previous works [3, 5, 13] our results:

- obtain deterministic results and improve the running time by a factor near-linear in  $\ell$ ,
- match the space usage,
- improve the approximation in  $\Delta$  from a factor 11 to 4,
- asymptotically match the clustering size (whenever  $\ell \in \Omega(\log n / \log k)$ ).

Compared exclusively to deterministic results [13], we instead improve time by a factor near-linear in  $n\ell$ , space by a factor super-linear in  $n\ell$ , and obtain asymptotically equal clustering size for all  $\ell$  (see also Table 1).

**Methodology and contribution.** Our algorithm constructs a clustering iteratively by greedily adding a cluster that covers an approximately-maximum set of uncovered points on  $T$ . The challenge is to compute such a cluster. Previous work [3, 5] presented randomized algorithms for constructing a cluster based on  $\varepsilon$ -net sampling over the set of all candidate clusters. They shatter the set of candidate clusters and show that it has bounded VC dimension, which leads to their asymptotic approximation of  $k$  — the minimum size of an  $(\ell, \Delta)$ -clustering. The algorithm of Conradi and Driemel [13] is more similar to ours. They also simplify the input and iteratively select the cluster with the (exact) maximum coverage to obtain an  $(\ell, \Delta)$ -clustering of size  $\mathcal{O}(k \log n)$ . The key difference lies in finding the next cluster. Conradi and Driemel [13] explicitly consider a set of  $\mathcal{O}(n^3 \ell)$  candidate clusters, which requires  $\mathcal{O}(n^4 \ell)$  time and space to construct.

We make two key contributions that distinguish us from prior works: First, we present a novel simplification algorithm that computes a curve  $S$  such that we may restrict potential

reference curves of clusters to be subcurves of  $S$ . This new curve simplification technique allows us to create a clustering where clusters have radius at most  $4\Delta$  as opposed to  $11\Delta$ . Second, we prove that we may restrict the reference curves to be one of two types:

- vertex-subcurves of  $S$ , which are subcurves that start and end at a vertex of  $S$ ,  
(we may furthermore only consider subcurves whose complexity is a power of 2)
- and subedges of  $S$ , which are subcurves that are a subsegment of a single edge of  $S$ .

We prove that a greedy algorithm that exclusively adds maximal clusters where the reference curve is of one of these two types creates a clustering of size  $\mathcal{O}(k \log n)$ . This characterization reduces the set of candidate clusters from  $\tilde{\mathcal{O}}(n^3 \ell)$  to  $\tilde{\mathcal{O}}(n^2)$  which significantly reduces the time spent compared to [13]. The geometric characterization of these subcurves allow us to compute candidate clusters on the fly, significantly reducing space usage.

## 2 Preliminaries

A (*polygonal*) curve with  $n$  vertices is a piecewise-linear map  $P: [1, n] \rightarrow \mathbb{R}^d$  whose breakpoints (called *vertices*) are at each integer parameter, and whose pieces are called *edges*. We denote by  $P[a, b]$  the subcurve of  $P$  that starts at  $P(a)$  and ends at  $P(b)$ . If  $a$  and  $b$  are integers, we call  $P[a, b]$  a *vertex subcurve* of  $P$ . Let  $|P|$  denote the number of vertices of  $P$ .

**Fréchet distance.** A *reparameterization* of  $[1, n]$  is a non-decreasing surjection  $f: [0, 1] \rightarrow [1, n]$ . Two reparameterizations  $f$  and  $g$  of  $[1, m]$  and  $[1, n]$ , respectively, describe a *matching*  $(f, g)$  between two curves  $P$  and  $Q$  with  $n$  and  $m$  vertices, where for any  $t \in [0, 1]$ , point  $P(f(t))$  is matched to  $Q(g(t))$ . A matching  $(f, g)$  is said to have *cost*  $\max_t \|P(f(t)) - Q(g(t))\|$ , where  $\|\cdot\|$  denotes the Euclidean norm. A matching with cost at most  $\Delta$  is called a  $\Delta$ -*matching*. The (continuous) *Fréchet distance*  $d_F(P, Q)$  between  $P$  and  $Q$  is the minimum cost over all matchings.

**Free space diagram.** The *parameter space* of curves  $P$  and  $Q$  with  $m$  and  $n$  vertices, respectively, is given by the orthogonal rectangle  $[1, m] \times [1, n]$ . This parameter space is associated with a regular grid whose cells are the squares  $[i, i+1] \times [j, j+1]$  for integers  $i$  and  $j$ . A point  $(x, y)$  in the parameter space corresponds to the pair of points  $P(x)$  and  $Q(y)$ . We say that  $(x, y)$  is  $\Delta$ -*free* if  $\|P(x) - Q(y)\| \leq \Delta$ . The  $\Delta$ -*free space diagram*  $\Delta\text{-FSD}(P, Q)$  of  $P$  and  $Q$  is the set of  $\Delta$ -free points in the parameter space of  $P$  and  $Q$ . The *obstacles* of  $\Delta\text{-FSD}(P, Q)$  are the connected components of  $([1, m] \times [1, n]) \setminus \Delta\text{-FSD}(P, Q)$ .

Alt and Godau [4] observe that the Fréchet distance between  $P[x_1, x_2]$  and  $Q[y_1, y_2]$  is at most  $\Delta$  if and only if there is a bimonotone path in  $\Delta\text{-FSD}(P, Q)$  from  $(x_1, y_1)$  to  $(x_2, y_2)$  (and  $x_1 \leq x_2$  and  $y_1 \leq y_2$ ).

**Input and output.** Our input is a curve  $T$  with  $n$  vertices, which we will call the *trajectory*, some integer parameter  $\ell \geq 2$ , and some distance parameter  $\Delta \geq 0$ . We consider *clustering* subtrajectories of  $T$  using *pathlets*:

► **Definition 1 (Pathlet).** An  $(\ell, \Delta)$ -pathlet is a tuple  $(P, \mathcal{I})$  where  $P$  is a curve with  $|P| \leq \ell$  and  $\mathcal{I}$  is a set of intervals in  $[1, n]$ , where  $d_F(P, T[a, b]) \leq \Delta$  for all  $[a, b] \in \mathcal{I}$ . We call  $P$  the reference curve of  $(P, \mathcal{I})$ .

We can see a pathlet  $(P, \mathcal{I})$  as a cluster, where the center is  $P$  and all subtrajectories induced by  $\mathcal{I}$  get mapped to  $P$ . See Figure 1. An  $(\ell, \Delta)$ -clustering of  $T$  is defined as follows:

► **Definition 2.** An  $(\ell, \Delta)$ -clustering  $C$  is a set of  $(\ell, \Delta)$ -pathlets with  $\bigcup_{(P, \mathcal{I}) \in C} \bigcup_{I \in \mathcal{I}} I = [1, n]$ .

Throughout this paper, we let  $k_\ell(\Delta)$  denote the smallest integer for which there exists an  $(\ell, \Delta)$ -clustering of size  $k_\ell(\Delta)$ . The goal is to find an  $(\ell, \Delta')$ -clustering  $C$  where  $|C|$  is not too large compared to  $k_\ell(\Delta)$ , and  $\Delta' \in \mathcal{O}(\Delta)$ .

**Weighting a cluster.** We define a *universe*  $\mathcal{U}$  as any set of interior-disjoint closed intervals that together cover  $[1, n]$ . Given a fixed universe  $\mathcal{U}$ , we can weigh each pathlet by what we call its *coverage*:

► **Definition 3.** The coverage over  $\mathcal{U}$  of a pathlet  $(P, \mathcal{I})$  is  $\text{Cov}_{\mathcal{U}}(P, \mathcal{I}) = \{I \in \mathcal{U} \mid I \subseteq \text{Cov}(P, \mathcal{I})\}$ . The coverage of a set  $C$  of pathlets is  $\text{Cov}_{\mathcal{U}}(C) = \sum_{(P, \mathcal{I}) \in C} \text{Cov}_{\mathcal{U}}(P, \mathcal{I})$ .

Whenever  $\mathcal{U}$  is clear from context we omit the subscript  $\mathcal{U}$ .

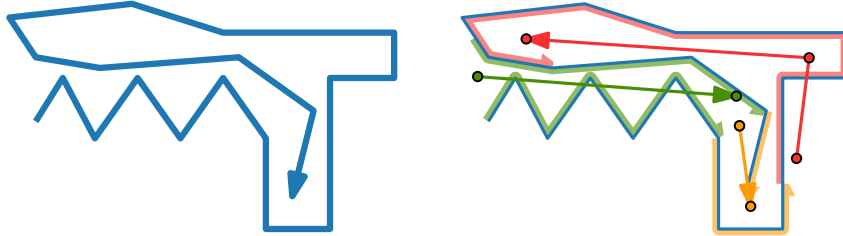
► **Definition 4** (Reference optimal). Let the universe  $\mathcal{U}$  be fixed and let  $C$  be a set of  $(\ell, \Delta)$ -pathlets. An  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$  is reference-optimal if its coverage over  $\mathcal{U} \setminus \text{Cov}(C)$ , i.e.,  $|\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)|$ , is maximum over all  $(\ell, \Delta)$ -pathlets with the same reference curve.

► **Definition 5.** Let the universe  $\mathcal{U}$  be fixed and let  $C$  be a set of  $(\ell, \Delta)$ -pathlets. An  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$  is optimal whenever  $|\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)|$  is maximum over all  $(\ell, \Delta)$ -pathlets.

### 3 Algorithmic outline

Our algorithmic input is a trajectory  $T$ , an integer  $\ell \geq 2$ , and value  $\Delta \geq 0$ . We provide a high-level overview of our algorithm here. Our approach can be decomposed as follows:

1. Reference curves may lie anywhere in the ambient space. Our first step is to restrict where these reference curves may lie. In Section 4 we construct a  $2\Delta$ -simplification  $S$  of  $T$ , and prove that for any  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$ , there exists a subcurve  $S[a, d]$  of  $S$  for which  $(S[a, d], \mathcal{I})$  is an  $(\ell + 2 - |\mathbb{N} \cap \{a, d\}|, \Delta')$ -pathlet, where  $\Delta' = 4\Delta$ . Hence we may restrict our attention to pathlets where the reference curve is a subcurve of  $S$ , if we allow for a slightly higher complexity. This higher complexity is circumvented later on, to still give an  $(\ell, \Delta')$ -clustering.
2. In Section 5, Given  $S$  and  $T$ , we smartly create some universe  $\mathcal{U}$ . We prove, by adapting the argument for greedy set cover, that any algorithm that iteratively computes an optimal  $(\ell, \Delta)$ -pathlet outputs a clustering of size  $\mathcal{O}(k_\ell(\Delta) \log n)$ .



■ **Figure 1** The trajectory  $T$  (blue, left) is covered by three pathlets. Each pathlet is defined by a reference curve (green, red, yellow) and the subcurve(s) of  $T$  the curve covers.

3. In Section 6 we give the general algorithm. We choose some  $\Delta' \in \Theta(\Delta)$ . We iteratively construct an  $(\ell, \Delta')$ -clustering of size  $\mathcal{O}(k_\ell(\Delta) \log n)$ . Our greedy iterative algorithm maintains a set  $C$  of pathlets and adds an  $(\ell, \Delta')$ -pathlet  $(P, \mathcal{I})$  to  $C$  at every iteration. Consider having a set of pathlets  $C = \{(P_i, \mathcal{I}_i)\}$ . We greedily select a pathlet  $(P, \mathcal{I})$  that covers as much of  $\mathcal{U} \setminus \text{Cov}(C)$  as possible, and add it to  $C$ . Formally, we select a  $(\Delta, \frac{1}{17})$ -maximal  $(\ell, \Delta')$ -pathlet: an  $(\ell, \Delta')$ -pathlet  $(P, \mathcal{I})$  such that

$$|\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)| \geq \frac{1}{17} |\text{Cov}(P', \mathcal{I}') \setminus \text{Cov}(C)|$$

for all  $(\ell, \Delta)$ -pathlets  $(P', \mathcal{I}')$ .

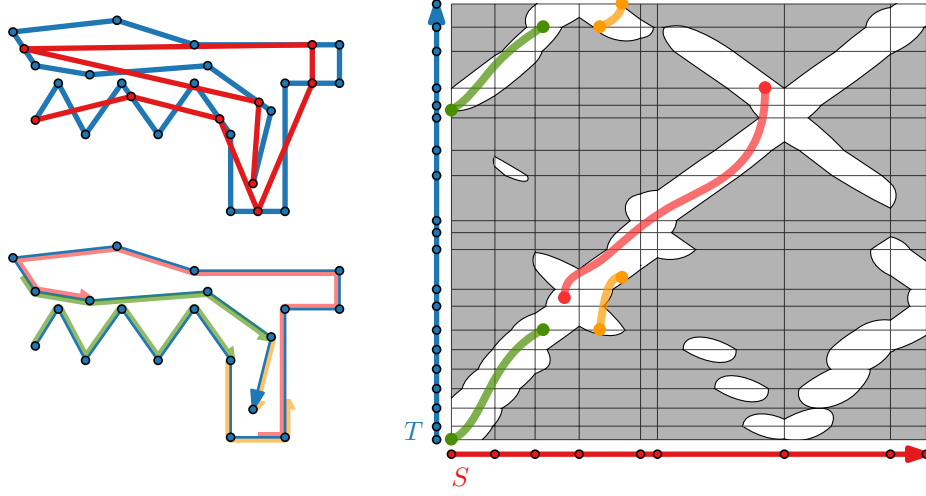
4. The subsequent goal is to compute  $(\Delta, \frac{1}{17})$ -maximal pathlets. We restrict pathlets to two types: those where the reference curve is 1) a vertex subcurve of  $S$ , or 2) a subsegment of an edge of  $S$ . Then we give algorithms for constructing pathlets of these types with a certain quality guarantee, i.e., pathlets that cover at least a constant fraction of what the optimal pathlet of that type covers. These algorithms are given in Sections 8 and 9.

**Reachability graph.** We introduce the *reachability graph* in Section 7. This graph is defined on a subcurve  $W$  of  $S$  and a set  $Z$  of points in  $\Delta'$ -FSD( $W, T$ ). The reachability graph  $G(W, T, Z)$  is a directed acyclic graph whose vertices are the set of points  $Z$ , together with certain boundary points of the free space  $\Delta'$ -FSD( $W, T$ ) and a collection of *steiner points*. Given two points  $(x, y)$  and  $(x', y')$  in  $Z$ , the graph contains a directed path from  $(x, y)$  to  $(x', y')$  if and only if  $d_F(W[x, x'], T[y, y']) \leq \Delta'$ .

We treat the free space diagram as a rectilinear polygon  $\mathcal{R}$  with rectilinear holes, obtained by reducing all obstacles of  $\Delta'$ -FSD( $W, T$ ) to their intersections with the parameter space grid. We show that a bimonotone path between two points  $p$  and  $q$  exists in  $\Delta'$ -FSD( $W, T$ ) if and only if a rectilinear shortest path between  $p$  and  $q$  in  $\mathcal{R}$  has length  $\|p - q\|_1$ , the  $L_1$ -distance between  $p$  and  $q$ . The reachability graph  $G(W, T, Z)$  is defined as the *shortest paths preserving graph* [20] for the set  $Z$  with respect to  $\mathcal{R}$ , made into a directed graph by directing edges, which are all horizontal or vertical, to the right or top. This graph has  $\mathcal{O}((|W|n + |Z|) \log(n|Z|))$  complexity, and a shortest path in the graph between points in  $Z$  is also a rectilinear shortest path between the corresponding points in  $\mathcal{R}$ .

**Vertex-to-vertex pathlets.** In Section 8 we construct a pathlet where the reference curve is a vertex subcurve of  $S$ . For a given vertex  $S(i)$  of  $S$ , we construct reference-optimal  $(\ell, \Delta')$ -pathlets  $(S[i, i + j], \mathcal{I}_j)$  for all  $j \in [\ell]$ . We first identify a set  $Z$  of  $\mathcal{O}(n\ell)$  *critical points* in  $\Delta'$ -FSD( $S[i, i + \ell], T$ ). We show that for every reference curve  $S[i, i + j]$ , there is a reference-optimal  $(\ell, \Delta')$ -pathlet  $(S[i, i + j], \mathcal{I}_j)$  where for each interval  $[y, y'] \in \mathcal{I}_j$ , the points  $(i, y)$  and  $(i + j, y')$  are critical points. We construct the intervals  $\mathcal{I}_j$  through a sweepline algorithm over the reachability graph  $G(S[i, i + \ell], T, Z)$ , which has  $\mathcal{O}(n\ell \log n)$  complexity. Our sweepline computes, for all  $j \in [\ell]$ , a reference-optimal  $(j, \Delta')$ -pathlet  $(S[i, i + j], \mathcal{I}_j)$  by iterating over all in-edges to critical points  $(i + j, y)$  in  $G(S[i, i + \ell], T, Z)$ . Doing this for all  $i$  (and remembering the optimum) thereby takes  $\mathcal{O}(n^2 \ell \log^2 n)$  time and  $\mathcal{O}(n\ell \log n)$  space.

**Subedge pathlets.** In Section 9 we construct a pathlet where the reference curve is a subsegment of an edge of  $S$ . For a given edge  $e$  of  $S$ , we again first identify a set  $Z$  of  $\mathcal{O}(n^2)$  *critical points* in  $\Delta'$ -FSD( $e, T$ ). However, rather than restricting the intervals in pathlets based on these critical points, we restrict the reference curves based on these critical points. Specifically, there are  $m = \mathcal{O}(n)$  unique  $x$ -coordinates of points in  $Z$ , which we order as



■ **Figure 2** Top left: A simplification  $S$  (red) of the trajectory  $T$  (blue). Right: The diagram  $\Delta'$ -FSD( $S, T$ ) in white. The obstacles of the diagram are colored in gray. The clustering (bottom left) corresponds to a set of colored bimonotone paths, where paths of a given color are horizontally aligned, and the paths together span the entire vertical axis.

$x_1, \dots, x_m$ . We show that by allowing for pathlets to use subsegments of the reversal  $\overleftarrow{e}$  of  $e$  as reference curves, we may restrict reference curves to be of the form  $e[x_i, x_{i'}]$  or  $\overleftarrow{e}[x_i, x_{i'}]$  to not lose much coverage. That is, the optimal  $(2, \Delta')$ -pathlet with such a reference curve covers at least one-fourth of what any other  $(2, \Delta')$ -pathlet using a subsegment of  $e$  as a reference curve covers.

The remainder of our subedge pathlet construction algorithm follows the same procedure as for vertex-to-vertex pathlets, though with the following optimization. We consider every  $x_i$  separately, for  $i \in [m]$ . However, rather than considering all reference curves  $e[x_i, x_{i'}]$ , of which there are  $m - i$ , we consider only  $\mathcal{O}(\log(m - i))$  reference curves. The main observation is that we may split a pathlet  $(e[x_i, x_{i'}], \mathcal{I})$  into two:  $(e[x_i, x_{i+2^j}], \mathcal{I}_1)$  and  $(e[x_{i+2^j}, x_{i'}], \mathcal{I}_2)$ , for some  $j \leq \log(m - i)$ . One of the two pathlets covers at least half of what  $(e[x_i, x_{i'}], \mathcal{I})$  covers, so an optimal  $(2, \Delta')$ -pathlet  $(e[x_i, x_{i+2^j}], \mathcal{I})$  that is defined by critical points covers at least one-eighth of any other subedge  $(2, \Delta')$ -pathlet  $(e[x, x'], \mathcal{I}')$ .

For every  $i \in [m]$ , we let  $Z_i \subseteq Z$  be the subset of critical points with  $x$ -coordinate equal to  $x_i$  or  $x_{i+2^j}$  for some  $j \leq \log(m - i)$ . We construct the reachability graph  $G(e, T, Z_i)$ , which has  $\mathcal{O}(n \log^2 n)$  complexity. We then proceed as with the vertex-to-vertex pathlets, using a sweepline through the reachability graph. Doing this for all  $i$  (and remembering the optimal pathlet) thereby takes  $\mathcal{O}(n^2 \log^3 n)$  total time and  $\mathcal{O}(n \log^2 n)$  space. Taken over all edges of  $S$ , we obtain a subedge pathlet in  $\mathcal{O}(n^3 \log^3 n)$  time and  $\mathcal{O}(n \log^2 n)$  space.

#### 4 Pathlet-preserving simplifications

We first aim to limit our attention to  $(\ell, 4\Delta)$ -pathlets  $(P, \mathcal{I})$  whose reference curves  $P$  are subcurves of some universal curve  $S$ . This way, we may design an algorithm that considers all subcurves of  $S$ , rather than all curves in  $\mathbb{R}^d$ . This has the additional benefit of allowing the use of the free space diagram  $4\Delta$ -FSD( $S, T$ ) to construct pathlets, as seen in Figure 2.

For any  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$  there exists an  $(n, 2\Delta)$ -pathlet  $(P', \mathcal{I})$  where  $P'$  is a subcurve of  $T$ . Indeed, consider any interval  $[a, b] \in \mathcal{I}$  and choose  $P' = T[a, b]$ . However, restricting



the subcurves of  $T$  to have complexity at most  $\ell$  may significantly reduce the maximum coverage, see for example Figure 3. Instead of restricting pathlets to be subcurves of  $T$ , we restrict them to be subcurves of a different curve  $S$ . We enforce the following property:

► **Definition 6.** For a trajectory  $T$  and value  $\Delta \geq 0$ , a *pathlet-preserving simplification* is a curve  $S$  together with a  $2\Delta$ -matching  $(f, g)$ , where for any subtrajectory  $T[a, b]$  of  $T$  and all curves  $P$  with  $d_F(P, T[a, b]) \leq \Delta$ , the subcurve  $S[s, t]$  matched to  $T[a, b]$  by  $(f, g)$  has complexity  $|S[s, t]| \leq |P| + 2 - |\mathbb{N} \cap \{s, t\}|$ .

► **Theorem 7.** Let  $(S, f, g)$  be a pathlet-preserving simplification of  $T$ . For any  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$ , there exists a subcurve  $S[s, t]$  such that  $(S[s, t], \mathcal{I})$  is an  $(\ell + 2 - |\mathbb{N} \cap \{s, t\}|, 4\Delta)$ -pathlet.

**Proof.** TOPROVE 0 ◀

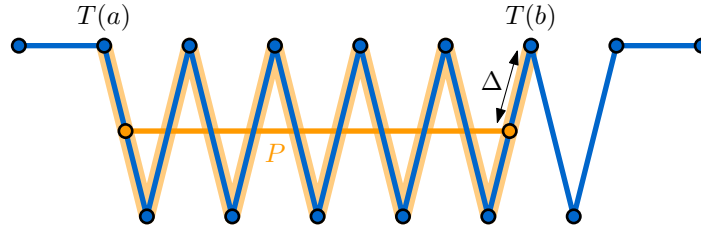
**Prior simplifications.** The curve  $S$  that we construct is a *curve-restricted  $\alpha\Delta$ -simplification* of  $T$ ; a curve whose vertices lie on  $T$ , where for every edge  $s = \overline{T(a)T(b)}$  of  $S$  we have  $d_F(s, T[a, b]) \leq \alpha\Delta$ . Various  $\alpha\Delta$ -simplification algorithms have been proposed [2, 14, 17, 19].

If  $T$  is a curve in  $\mathbb{R}^2$ , Guibas *et al.* [17] provide an  $\mathcal{O}(n \log n)$  time algorithm that constructs a  $2\Delta$ -simplification  $S$  for which there is no  $\Delta$ -simplification  $S'$  with  $|S'| < |S|$ . Their algorithm is not efficient in higher dimensions however.

Agarwal *et al.* [2] also construct a  $2\Delta$ -simplification  $S$  of  $T$  in  $\mathcal{O}(n \log n)$  time. This was applied by Akitaya *et al.* [3] for their subtrajectory clustering algorithm under the discrete Fréchet distance. The simplification  $S$  has a similar guarantee as the simplification of [17]: there exists no *vertex-restricted*  $\Delta$ -simplification  $S'$  with  $|S'| < |S|$ . This guarantee is weaker than that of [17], as vertex-restricted simplifications are simplifications formed by taking a subsequence of vertices of  $T$  as the vertices of the simplification. It can, however, be constructed efficiently in higher dimensions.

As we show in Figure 3, the complexity of a vertex-restricted  $\Delta$ -simplification can be arbitrarily bad compared to the (unrestricted)  $\Delta$ -simplification with minimum complexity. Brüning *et al.* [5] note that for the subtrajectory problem under the continuous Fréchet distance, one requires an  $\alpha\Delta$ -simplification whose complexity has guarantees with respect to the optimal (unrestricted) simplification. They present a  $3\Delta$ -simplification  $S$  (whose definition was inspired by de Berg, Gudmundsson and Cook [14]) with the following property: for any subcurve  $T[a, b]$  of  $T$  within Fréchet distance  $\Delta$  of some line segment, there exists a subcurve  $S[s, t]$  of  $S$  with complexity at most 4 that has Fréchet distance at most  $3\Delta$  to  $T[a, b]$ . Thus, there exists no  $\Delta$ -simplification  $S'$  with  $|S'| < |S|/2$ .

**Our new curve simplification.** In Definition 6 we presented yet another curve simplification under the Fréchet distance for curves in  $\mathbb{R}^d$ . Our simplification has a stronger property than



■ **Figure 3** There exists a segment  $P$  where  $d_F(P, T[a, b]) \leq \Delta$ . In contrast, for any vertex-restricted  $S$  with  $d_F(T[a, b], S) \leq \Delta$ , the complexity of  $S$  is  $\Theta(|T[a, b]|)$ .

the one that is realized by Brüning *et al.* [5]: for any subcurve  $T[a, b]$  and any curve  $P$  with  $d_F(P, T[a, b]) \leq \Delta$ , we require that there exists a subcurve  $S[s, t]$  with  $d_F(S[s, t], T[a, b]) \leq 2\Delta$  that has at most two more vertices than  $P$ . This implies both the property of Brüning *et al.* [5] and ensures that no  $\Delta$ -simplification  $S'$  exists with  $|S'| < |S| - 2$ .

In Appendix B, we provide an efficient algorithm for constructing pathlet-preserving simplifications. We relegate this section to the appendix to not distract from the main storyline. The algorithm is an extension of the vertex-restricted simplification of Agarwal *et al.* [2] to construct a curve-restricted simplification instead. For this, we use the techniques of Guibas *et al.* [17] to quickly identify if an edge of  $T$  is suitable to place a simplification vertex on. We combine this check with the algorithm of [2] and obtain:

► **Theorem 8.** *For any trajectory  $T$  with  $n$  vertices and any  $\Delta \geq 0$ , we can construct a pathlet-preserving simplification  $S$  in  $\mathcal{O}(n \log n)$  time.*

## 5 The universe $\mathcal{U}$ and greedy set cover

Subtrajectory clustering is closely related to the *set cover* problem. In this problem, we have a discrete universe  $\mathcal{U}$  and a family of sets  $\mathcal{S}$  in this universe, and the goal is to pick a minimum number of sets in  $\mathcal{S}$  such that their union is the whole universe. The decision variant of set cover is NP-complete [18]. However, the following greedy strategy gives an  $\mathcal{O}(\log |\mathcal{U}|)$  approximation of the minimal set cover size [12]. Suppose we have picked a set  $\hat{\mathcal{S}} \subseteq \mathcal{S}$  that does not yet cover all of  $\mathcal{U}$ . The idea is then to add a set  $S \in \mathcal{S}$  that maximizes  $|S \cap (\mathcal{U} \setminus \bigcup \hat{\mathcal{S}})|$ , and to repeat the procedure until  $\mathcal{U}$  is fully covered.

**Defining the universe  $\mathcal{U}$ .** We apply this greedy strategy to subtrajectory clustering, putting the focus on constructing a pathlet that covers the most of some universe  $\mathcal{U}$ . For subtrajectory clustering, the universe is, in principle, infinite. We therefore first define a discrete universe  $\mathcal{U}$  consisting of  $\mathcal{O}(n^3)$  intervals that together cover  $[1, n]$ . We choose this universe carefully, as an optimal covering of  $\mathcal{U}$  with pathlets must have roughly the same size as an optimal covering of  $[1, n]$ . We define  $\mathcal{U}$  using the following set of *critical points* in  $\Delta'$ -FSD( $S, T$ ):

► **Definition 9.** *For  $i \in [|S| - 1]$  and  $j \in [n - 1]$ , consider their corresponding cell (the area  $[i, i + 1] \times [j, j + 1]$ ) and the following six extreme points:*

- *A leftmost point of  $\Delta'$ -FSD( $S, T$ )  $\cap ([i, i + 1] \times [j, j + 1])$ ,*
- *A rightmost point of  $\Delta'$ -FSD( $S, T$ )  $\cap ([i, i + 1] \times [j, j + 1])$ ,*
- *The leftmost and rightmost points of  $\Delta'$ -FSD( $S, T$ )  $\cap ([i, i + 1] \times \{j\})$ , and*
- *The leftmost and rightmost points of  $\Delta'$ -FSD( $S, T$ )  $\cap ([i, i + 1] \times \{j + 1\})$ .*

*Let  $X_{i,j}$  be the set of corresponding  $x$ -coordinates and  $X := \bigcup_{i,j} X_{i,j}$ . For each  $x \in X$ , we call every point  $(x, y)$  that is an endpoint of a connected component (vertical segment) of  $\Delta'$ -FSD( $S, T$ )  $\cap (\{x\} \times [1, n])$  a critical point.*

► **Definition 10.** *Let  $Y^*$  be the set of critical points, sorted by their  $y$ -coordinate. We define the set  $\mathcal{U}$  as the set of intervals in  $[1, n]$  between two consecutive  $y$ -coordinates in  $X$ . Since there are at most  $6n$  critical points in  $[i, i + 1] \times [j, j + 1]$  for each  $i \in [|S| - 1]$  and  $j \in [n - 1]$ , it follows that  $|\mathcal{U}| \leq 6n^3 - 1 = \mathcal{O}(n^3)$ .*

► **Lemma 11.** *We can construct  $\mathcal{U}$  in  $\mathcal{O}(n^3)$  time.*

**Proof.** TOPROVE 1

◀



**Applying greedy set cover.** In the remainder of this paper, we let  $\mathcal{U}$  denote this discrete universe. We generalize the analysis of the greedy set cover argument to pathlets that cover a (constant) fraction of what the optimal pathlet covers. This relaxes the requirements on the pathlets and helps reduce complexity of the problem. For this, we introduce the following:

► **Definition 12** (Maximal pathlets). *Given a set  $C$  of pathlets, a  $(\Delta, \frac{1}{c})$ -maximal  $(\ell, \Delta')$ -pathlet  $(P', \mathcal{I}')$  is a pathlet such that there exists no  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$  with*

$$\frac{1}{c} |\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)| \geq |\text{Cov}(P', \mathcal{I}') \setminus \text{Cov}(C)|.$$

In Lemma 13, we show that if we keep greedily selecting  $(\Delta, \frac{1}{c})$ -maximal pathlets for our clustering, the size of the clustering stays relatively small compared to the optimum size. The bound closely resembles the bound obtained by the argument for greedy set cover.

► **Lemma 13.** *Iteratively adding  $(\Delta, \frac{1}{c})$ -maximal pathlets yields a clustering of size at most  $3c \cdot k_\ell(\Delta) \ln(6n) + 1$ .*

**Proof.** TOPROVE 2 ◀

## 6 Subtrajectory clustering

In this section we present our algorithm for subtrajectory clustering. We first restrict our attention to reference curves of two types.

Recall that using the pathlet-preserving simplification  $S$  of  $T$ , we may already restrict our attention to reference curves that are subcurves of  $S$ . Still, the space of possible reference curves remains infinite. We wish to discretize this space by identifying certain finite classes of reference curves that contain a “good enough” reference curve, i.e., one with which we can construct a pathlet that is  $(\Delta, \frac{1}{c})$ -maximal for some small constant  $c$ .

We distinguish between two types of pathlets, based on their reference curves (note that not all pathlets fit into a class, and that some may fit into both classes):

1. Vertex-to-vertex pathlets: pathlets  $(P, \mathcal{I})$  where  $P$  is a vertex subcurve of  $S$ .
2. Subedge pathlets: pathlets  $(P, \mathcal{I})$  where  $P$  is a subsegment of an edge of  $S$ .

We construct pathlets of the above types, ensuring that they all cover at least some constant fraction of the optimal coverage for pathlets of the same type. Let  $(P_{\text{ver}}, \mathcal{I}_{\text{ver}})$  and  $(P_{\text{sub}}, \mathcal{I}_{\text{sub}})$  respectively be a vertex-to-vertex and subedge  $(\ell, \Delta')$ -pathlet, that respectively cover at least a factor  $\frac{1}{c_{\text{ver}}}$  and  $\frac{1}{c_{\text{sub}}}$  of an optimal pathlet of the same type. We show that one of these two pathlets is a  $(\Delta, \frac{1}{c})$ -maximal pathlet, for  $c = c_{\text{ver}} + 2c_{\text{sub}}$ . For intuition, refer to Figure 4.

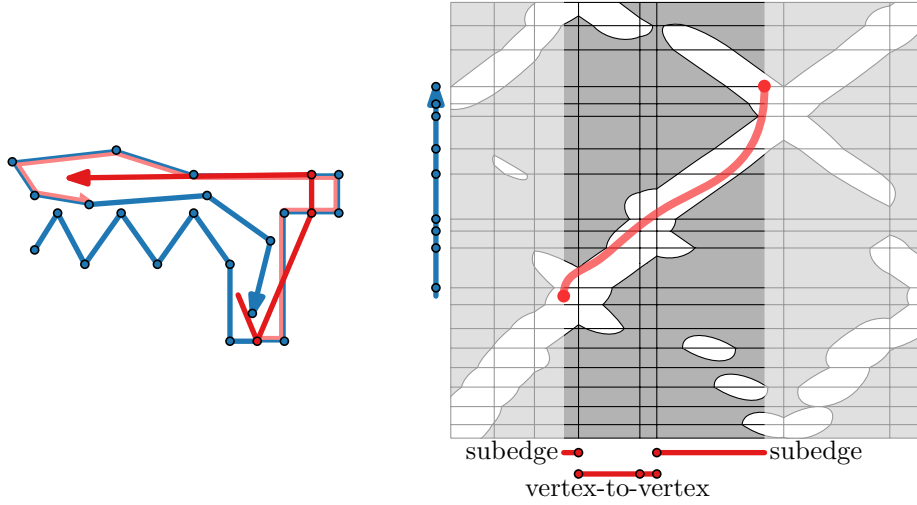
► **Lemma 14.** *Given a collection  $C$  of pathlets, let*

$$(P, \mathcal{I}) \in \{(P_{\text{ver}}, \mathcal{I}_{\text{ver}}), (P_{\text{sub}}, \mathcal{I}_{\text{sub}})\}$$

*be a pathlet with maximal coverage among the uncovered points. Then  $(P, \mathcal{I})$  is  $(\Delta, \frac{1}{c})$ -maximal with respect to  $C$ , for  $c = c_{\text{ver}} + 2c_{\text{sub}}$ .*

**Proof.** TOPROVE 3 ◀

Next we combine the previous ideas on simplification and greedy algorithms and present our algorithm for subtrajectory clustering. The algorithm uses subroutines for constructing the two types of pathlets described above, as well as a data structure for comparing their coverages to select the best pathlet for the clustering.



■ **Figure 4** A pathlet (left), corresponding to the red  $\Delta'$ -matching (right), gets split into a vertex-to-vertex and two subedge pathlets. The new pathlets correspond to the parts of the red matching that are vertically above the part of the  $x$ -axis corresponding to the new reference curve.

Our pathlet construction algorithms guarantee that  $c_{\text{ver}} = 1$  and  $c_{\text{sub}} = 8$ . By Lemma 14, the pathlet with the most coverage is therefore  $(\Delta, \frac{1}{c})$ -maximal with respect to the uncovered points, for  $c = 1 + 2 \cdot 8 = 17$ . By Lemma 13, the resulting  $(\ell, \Delta')$ -clustering has a size of at most  $17k_\ell(\Delta) \ln(|\mathcal{U}| - 1) + 1$ . Since our universe  $\mathcal{U}$  has size at most  $6n^3 - 1$ , the clustering has a size of at most  $51k_\ell(\Delta) \ln(6n) + 1$ .

**A data structure for comparing pathlets.** Recall that we fixed some discrete universe  $\mathcal{U}$  of  $\mathcal{O}(n^3)$  intervals, and that we denote  $\text{Cov}(P, \mathcal{I}) = \text{Cov}_{\mathcal{U}}(P, \mathcal{I})$ . In each iteration of our greedy algorithm, we select one of two pathlets whose coverage is the maximum over  $\mathcal{U} \setminus \text{Cov}(C)$ , given the current set of picked pathlets  $C$ . To compare the coverages of pathlets, we make use of binary search trees built on  $\mathcal{U}$  and  $\text{Cov}(C)$ :

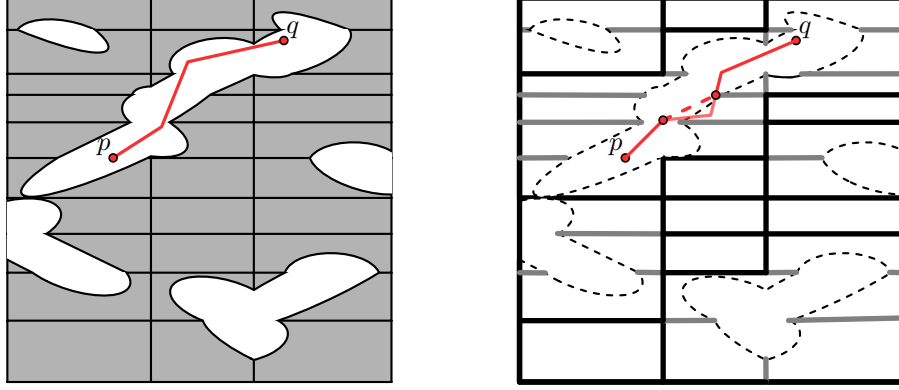
► **Lemma 15.** *In  $\mathcal{O}(n^3 \log n)$  time, we can preprocess  $\mathcal{U}$  and  $\text{Cov}(C)$  into a data structure of  $\mathcal{O}(n^3)$  size, such that given a pathlet  $(P, \mathcal{I})$ , the value  $|\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)|$  can be computed in  $\mathcal{O}(|\mathcal{I}| \log n)$  time.*

**Proof.** TOPROVE 4 ◀

**Asymptotic complexities.** Our algorithm iteratively constructs a set  $C$  of  $\mathcal{O}(k_\ell(\Delta) \log n)$  pathlets. Before we start constructing pathlets, we compute the universe  $\mathcal{U}$  of  $\mathcal{O}(n^3)$  intervals. This takes  $\mathcal{O}(n^3)$  time (Lemma 11).

In each iteration, we construct the data structure of Lemma 15 on the universe  $\mathcal{U}$  and current set of pathlets  $C$ . This takes  $\mathcal{O}(n^3 \log n)$  time and uses  $\mathcal{O}(n^3)$  space. Constructing the vertex-to-vertex pathlet then takes  $\mathcal{O}(n^2 \ell \log^2 n)$  time and uses  $\mathcal{O}(n \ell \log n)$  space (Theorem 20). The subedge pathlet takes  $\mathcal{O}(n^3 \log^3 n)$  time and  $\mathcal{O}(n \log^2 n)$  space to construct (Theorem 23).

To decide which pathlet to use in the clustering, we make further use of the data structure of Lemma 15. All constructed pathlets  $(P, \mathcal{I})$  have  $|\mathcal{I}| \leq n$ , and so we compute the coverages of the two pathlets in  $\mathcal{O}(n \log n)$  time. By summing up all complexities, we derive our main theorem:



■ **Figure 5** (left) The  $\Delta'$ -free space diagram of  $W$  and  $T$  with points  $p$  and  $q$  connected by a bimonotone path. (right) The obstacles of  $\mathcal{R}$  are made up of all grid edges that are entirely contained in the obstacles of  $\Delta'$ -FSD( $W, T$ ) (shown in black) plus the gray segments. We may transform any bimonotone path between  $p$  and  $q$  into one that lies in  $\Delta'$ -FSD( $W, T$ ).

► **Theorem 16.** *Given a trajectory  $T$  with  $n$  vertices, an integer  $\ell \geq 2$ , and a value  $\Delta \geq 0$ , we can construct an  $(\ell, 4\Delta)$ -clustering of size at most  $51k_\ell(\Delta) \ln(6n) + 1$  in  $\mathcal{O}(k_\ell(\Delta)n^3 \log^4 n)$  time and using  $\mathcal{O}(n^3)$  space.*

## 7 The reachability graph

Let  $\Delta' = 4\Delta$ . For any subcurve  $W$  of  $S$  and a set of points  $Z$  in  $\Delta'$ -FSD( $W, T$ ) we define the *reachability graph*  $G(W, T, Z)$ . The vertices of this graph are the set of points  $Z$ , together with some Steiner points in  $[1, |W|] \times [1, |T|]$ . The reachability graph  $G(W, T, Z)$  is a directed graph where, for any two  $\mu_1, \mu_2 \in Z$ , there exists a directed path from  $\mu_1$  to  $\mu_2$  if and only if  $\mu_2$  is reachable from  $\mu_1$  in the free space  $\Delta'$ -FSD( $W, T$ ).

We define a reachability graph with  $\mathcal{O}((n|W| + |Z|) \log(n|Z|))$  vertices and edges, and can be constructed in  $\mathcal{O}((n|W| + |Z|) \log(n|Z|))$  time.

► **Theorem 17.** *Let  $W$  be a subcurve of  $S$  and  $Z$  a set of points in  $\Delta'$ -FSD( $W, T$ ). The reachability graph  $G(W, T, Z)$  has  $\mathcal{O}((n|W| + |Z|) \log(n|Z|))$  vertices and edges, and can be constructed in  $\mathcal{O}((n|W| + |Z|) \log(n|Z|))$  time.*

**Constructing the graph.** Lemma 18 shows that when focusing on reachability between points in  $\Delta'$ -FSD( $W, T$ ), we can simplify obstacles of the free space diagram to the parameter space grid, minus the free space. See Figure 5.

These simplified obstacles can be represented in  $\mathcal{O}(|W|n)$  time as a set of horizontal and vertical line segments (whose endpoints are not included, except possibly some that meet the boundary of  $[1, |W|] \times [1, |T|]$ ). The complement of these segments in the parameter space  $[1, |W|] \times [1, |T|]$  gives a rectilinear polygon with rectilinear holes  $\mathcal{R}$ .

► **Lemma 18.** *Let  $p$  and  $q$  be two points in  $\Delta'$ -FSD( $W, T$ ). There is a bimonotone path from  $p$  to  $q$  in  $\Delta'$ -FSD( $W, T$ ) if and only if there is a bimonotone path from  $p$  to  $q$  in  $\mathcal{R}$ .*

**Proof.** TOPROVE 5 ◀

To obtain  $G(W, T, Z)$  we first construct an undirected graph  $G(Z)$ . This graph is the *shortest paths preserving graph* by Widmayer [20]. The vertices of  $G(Z)$  are the points in  $Z$ , together

with the vertices of  $\mathcal{R}$  and some Steiner points. By weighting each edge by its length, the graph perfectly encodes rectilinear distances between points in  $Z$ . That is, the rectilinear distance in  $\mathcal{R}$  between two points in  $Z$  is equal to their distance in  $G(Z)$ .

The number of vertices of  $\mathcal{R}$  is  $\mathcal{O}(n|W|)$ , giving the graph a complexity of  $\mathcal{O}((n|W| + |Z|)\log(n|Z|))$ . The graph can be constructed in  $\mathcal{O}((n|W| + |Z|)\log(n|Z|))$  time [20].

The edges of  $G(Z)$  are all horizontal or vertical line segments. We set  $G(W, T, Z)$  to be the graph  $G(Z)$ , but with each edge directed towards the right (if horizontal) or top (if vertical). Observe that  $G(W, T, Z)$  perfectly encodes reachability: for two points  $p = (x, y)$  and  $q = (x', y')$  in  $Z$ , if there is a bimonotone rectilinear path from  $p$  to  $q$  in  $\mathcal{R}$ , then any rectilinear shortest path from  $p$  to  $q$  must be bimonotone, and hence there must be a (bimonotone) path between them in  $G(W, T, Z)$ . Conversely, any path in  $G(W, T, Z)$  is also a path in  $\mathcal{R}$ . Thus  $d_F(P[x, x'], T[y, y']) \leq \Delta'$  if and only if there is a (bimonotone) path from  $(x, y)$  to  $(x', y')$  in  $G(W, T, Z)$ .

► **Theorem 17.** *Let  $W$  be a subcurve of  $S$  and  $Z$  a set of points in  $\Delta'$ -FSD( $W, T$ ). The reachability graph  $G(W, T, Z)$  has  $\mathcal{O}((n|W| + |Z|)\log(n|Z|))$  vertices and edges, and can be constructed in  $\mathcal{O}((n|W| + |Z|)\log(n|Z|))$  time.*

## 8 Vertex-to-vertex pathlets

Let  $\Delta' = 4\Delta$ , and let  $C$  be a set of pathlets. Recall that we can compute  $|\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)|$ , for a given pathlet  $(P, \mathcal{I})$ , in  $\mathcal{O}(|\mathcal{I}| \log n)$  time (Lemma 15). We fix some integer  $i$ . We then give an algorithm for constructing a vertex-to-vertex  $(\ell, \Delta')$ -pathlet  $(P, \mathcal{I})$  where  $P$  starts at the  $i$ 'th vertex of  $S$ , and its coverage over  $\mathcal{U} \setminus \text{Cov}(C)$  is maximum.

We find for each subcurve  $S'$  of  $S$  of length at most  $\ell$  a reference-optimal  $(\ell, \Delta')$ -pathlet. To this end, we consider each vertex  $S(i)$  of  $S$  separately. We construct a set of reference-optimal pathlets  $(S[i, i+1], \mathcal{I}_1), \dots, (S[i, i+j], \mathcal{I}_j), \dots, (S[i, i+\ell], \mathcal{I}_\ell)$ . We let each interval  $\mathcal{I}_j$  contain all maximal intervals  $[y, y']$  for which  $d_F(S[i, i+j], T[y, y']) \leq \Delta'$ , and thus all maximal intervals for which  $(i, y)$  can reach  $(i+j, y')$  by a bimonotone path in  $\Delta'$ -FSD( $S, T$ ).

Recall that in Definition 9 we defined a set of *critical points*. Let  $Z$  denote all critical points in  $\Delta'$ -FSD( $S[i, i+\ell], T$ ) of the form  $(i+j, y)$ , for integers  $j \in [\ell]$ . That is,  $Z$  contains for all  $j \in [\ell]$  the endpoints of all connected components (vertical line segments) of  $\Delta'$ -FSD( $S, T$ )  $\cap (\{i+j\} \times [1, n])$ . Since each cell has at most  $\mathcal{O}(1)$  such critical points, it follows that  $|Z| \in \mathcal{O}(n\ell)$ . Observe that for any  $\Delta'$ -matching between  $T$  and a vertex-to-vertex subcurve, we can always extend each curve in the matching to start and end at a point in  $Z$ :

► **Observation 19.** *Let  $(P, \mathcal{I})$  be a vertex-to-vertex  $(\ell, \Delta')$ -pathlet where  $P$  starts at the  $i$ 'th vertex. Then there exists an  $(\ell, \Delta')$ -pathlet  $(P, \mathcal{I}')$  with  $\text{Cov}(P, \mathcal{I}) \subseteq \text{Cov}(P, \mathcal{I}')$  such that for each interval in  $\mathcal{I}'$ , the corresponding bimonotone path in  $\Delta'$ -FSD( $S, T$ ) starts and ends at a point in  $Z$ .*

We create a sweepline algorithm that, for each  $j \in [\ell]$ , constructs a reference-optimal  $(\ell, \Delta')$ -pathlet  $(S[i, j], \mathcal{I}_j)$ . We let each interval  $\mathcal{I}_j$  contain all maximal intervals  $[y, y']$  for which  $d_F(S[i, i+j], T[y, y']) \leq \Delta'$ , and thus all maximal intervals for which  $(i, y)$  can reach  $(i+j, y')$  by a bimonotone path in  $\Delta'$ -FSD( $S, T$ ). Note that both  $(i, y)$  and  $(i+j, y')$  are critical points. Thus we aim to find all maximal intervals  $[y, y']$  for which  $\Delta'$ -FSD( $S, T$ ) contains a bimonotone path between critical points  $(i, y)$  and  $(i+j, y')$ .

To this end, we construct, for each  $i \in [n]$ , the reachability graph  $G(S[i, i+\ell], T, Z)$  from Section 7, which encodes reachability between all critical points. This graph takes  $\mathcal{O}((n\ell + |Z|)\log(n|Z|)) = \mathcal{O}(n\ell \log n)$  time to construct and has complexity  $\mathcal{O}(n\ell \log n)$ .

(see Theorem 17). We aim to annotate each vertex  $\mu$  (which does not necessarily have to be a critical point) in  $G(S[i, i + \ell], T, Z)$  with the minimum  $y$ , such that there exists a critical point  $(i, y)$  that can reach  $\mu$ . We annotate  $\mu$  with  $\infty$  if no such value  $y$  exists.

**Annotating vertices.** We begin by annotating the vertices  $(i, y)$  in  $\mathcal{O}(n)$  time, by scanning over them in order of increasing  $y$ -coordinate. We go over the remaining vertices in  $yx$ -lexicographical order, where we go over the vertices based on increasing  $y$ -coordinate, and increasing  $x$ -coordinate when ties arise. Each vertex  $\mu$  that we examine has only incoming arcs originating from vertices below and left of  $\mu$ . By our lexicographical ordering, each of these vertices are already annotated. The minimal  $y$  for which there exists a path from  $(i, y)$  to  $\mu$ , must be the minimum over all its incoming arcs which we compute in time proportional to the in-degree of  $\mu$ . If  $\mu$  has no incoming arcs, we annotate it with  $\infty$ .

Let  $V$  and  $A$  be the sets of  $\mathcal{O}(n\ell \log n)$  vertices and arcs of  $G(S[i, i + \ell], T, Z)$ . For the above annotation procedure, we first compute the  $yx$ -lexicographical ordering of the vertices, based on their corresponding points in the parameter space. This takes  $\mathcal{O}(|V| \log |V|)$  time. Afterwards, we go over each vertex and each incoming arc exactly once, which take an additional  $\mathcal{O}(|V| + |A|)$  time. In total, we annotate all vertices in  $\mathcal{O}(n\ell \log^2 n)$  time.

**Constructing the pathlets.** With the annotations, constructing the pathlets becomes straightforward. For each  $j \in [\ell]$ , we construct  $\mathcal{I}_j$  as follows. We iterate over all critical point  $(i + j, y')$  in the graph  $G(S[i, i + \ell], T, Z)$ . For each critical point  $(i + j, y')$  with a finite annotation  $y$ , we add the interval  $[y, y']$  to  $\mathcal{I}_j$ . This procedure ensures that  $\mathcal{I}_j$  contains all maximal intervals  $[y, y']$  for which  $d_F(S[i, i + j], T[y, y']) \leq \Delta'$ , creating an optimal pathlet  $(S[i, i + j], \mathcal{I}_j)$  with respect to its reference curve. Since there are  $\mathcal{O}(n)$  critical points per  $j$ , this algorithm uses  $\mathcal{O}(n\ell)$  time. Storing the pathlets takes  $\mathcal{O}(n\ell)$  space. We conclude:

► **Theorem 20.** *Suppose  $\text{Cov}(C)$  is preprocessed by Lemma 15. In  $\mathcal{O}(n^2 \ell \log^2 n)$  time and using  $\mathcal{O}(n\ell \log n)$  space, we can construct an optimal vertex-to-vertex  $(\ell, \Delta')$ -pathlet  $(P, \mathcal{I})$ .*

**Proof.** TOPROVE 6 ◀

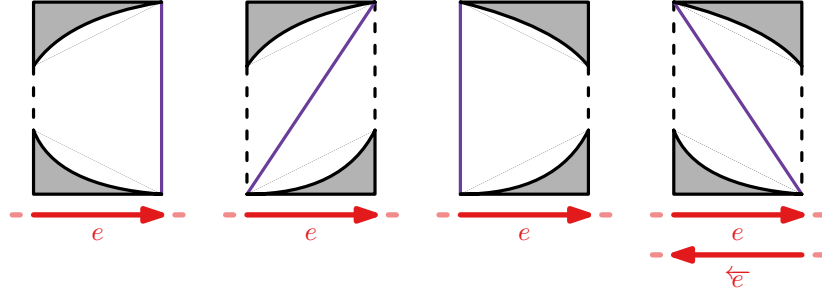
## 9 Subedge pathlets

Let  $\Delta' = 4\Delta$ , and let  $C$  be a set of pathlets. We assume that  $\text{Cov}(C)$  has at most  $\mathcal{O}(n^2 \log n)$  connected components. We provide an algorithm for constructing a subedge  $(2, \Delta')$ -pathlet  $(P, \mathcal{I})$  whose coverage – (the sum of lengths in  $\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)$ ) – is at least one-eighth the optimum.

Recall that a subedge pathlet  $(P, \mathcal{I})$  is a pathlet where  $P = e[x, x']$  is a subsegment of some edge  $e$  of  $S$ . We construct a subedge pathlet given the edge  $e$ . We first discretize the problem, identifying a set of  $\mathcal{O}(n^2)$  *critical points* in  $\Delta'$ -FSD( $e, T$ ). This set ensures that there exists a subedge pathlet  $(e[x, x'], \mathcal{I})$  with at least one-fourth the coverage of any pathlet using a subedge of  $e$  as a reference curve, where for all  $[y, y'] \in \mathcal{I}$ , the points  $(x, y)$  and  $(x', y')$  are both critical points.

For  $j \in [n - 1]$ , consider the following six extreme points of  $\Delta'$ -FSD( $e, T$ )  $\cap ([1, 2] \times [j, j + 1])$  (where some points may not exist):

- A leftmost point of  $\Delta'$ -FSD( $e, T$ )  $\cap ([1, 2] \times [j, j + 1])$ ,
- A rightmost point of  $\Delta'$ -FSD( $e, T$ )  $\cap ([1, 2] \times [j, j + 1])$ ,
- The leftmost and rightmost points of  $\Delta'$ -FSD( $e, T$ )  $\cap ([1, 2] \times \{j\})$ , and
- The leftmost and rightmost points of  $\Delta'$ -FSD( $e, T$ )  $\cap ([1, 2] \times \{j + 1\})$ .



■ **Figure 6** The connected components of  $\Delta'$ -FSD( $e', T$ ) fall into these four cases, based on where the minima and maxima of the bottom and top parabolic arcs lie. In the first three cases, there is a clear matching with optimal coverage (purple). In the fourth case, the matching is only valid when mirroring the free space, achieved by using  $\overleftarrow{e}$  instead of  $e$ .

Let  $X_j$  be the set of  $x$ -coordinates of these points, and let  $X = \bigcup X_j$  be the set of all these coordinates. Let  $x_1, \dots, x_m$  be the set of values in  $X$ , sorted in increasing order. We call every point  $(x_i, y)$  that is an endpoint of a connected component (vertical segment) of  $\Delta'$ -FSD( $e, T$ )  $\cap (\{x_i\} \times [1, n])$  a critical point. Let  $Z$  be the set of at most  $4n^2 = \mathcal{O}(n^2)$  critical points.

Before we restrict pathlets to be defined by these critical points, we first allow a broader range of pathlets. We consider the edge  $\overleftarrow{e}$ , obtained by reversing the direction of  $e$ , and look at constructing a pathlet that is a subedge of either  $e$  or  $\overleftarrow{e}$ . We show that by restricting pathlets to be defined by  $Z$ , while allowing for reference curves that are subcurves of  $\overleftarrow{e}$ , results in losing only a factor four in the maximum coverage.

► **Lemma 21.** *Let  $C$  be a set of pathlets. For any subedge  $(2, \Delta')$ -pathlet  $(e[x, x'], \mathcal{I})$ , there exists a subedge  $(2, \Delta')$ -pathlet  $(P, \mathcal{I}')$  with*

$$|\text{Cov}(P, \mathcal{I}') \setminus \text{Cov}(C)| \geq \frac{1}{4} |\text{Cov}(e[x, x'], \mathcal{I}) \setminus \text{Cov}(C)|,$$

where  $P$  is equal to  $e[x_i, x_j]$  or  $\overleftarrow{e}[x_i, x_j]$  for some  $i$  and  $j$ , and for every interval  $[y, y'] \in \mathcal{I}'$ , the points  $(x_i, y)$  and  $(x_j, y')$  are contained in  $Z$ .

**Proof.** TOPROVE 7 ◀

We find for each point  $e(x_i)$  of  $e$  corresponding to a critical point a subedge pathlet whose reference curve starts at  $e(x_i)$  and ends at some point  $e(x_j)$  that also corresponds to a critical point. To this end, we consider each point  $e(x_i)$  separately. We proceed akin to the construction for vertex-to-vertex pathlets Section 8, with some optimization steps.

It proves too costly to consider each reference curve  $e[x_i, x_{i'}]$  for every  $x_i$  we consider. By sacrificing the quality of the pathlet slightly, settling for a pathlet with at least one-eighth the coverage of any subedge pathlet rather than one-fourth, we can reduce the number of reference curves we have to consider from  $\Theta(m^2) = \mathcal{O}(n^2)$  to  $\mathcal{O}(m \log m)$ . Let  $(e[x_i, x_{i'}], \mathcal{I})$  be a subedge pathlet. We can split  $e[x_i, x_{i'}]$  into two subedges  $e[x_i, x_{i+2j}]$  and  $e[x_{i'-2j}, x_{i'}]$ . The matchings corresponding to  $\mathcal{I}$  naturally decompose into two sets of matchings (whose matched subcurves may overlap), giving rise to two pathlets  $(e[x_i, x_{i+2j}], \mathcal{I}_1)$  and  $(e[x_{i'-2j}, x_{i'}], \mathcal{I}_2)$  with  $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$ . Thus at least one of these pathlets has at least half the coverage that  $(e[x_i, x_{i'}], \mathcal{I})$  has. By Lemma 21, a pathlet  $(e[x_i, x_{i+2j}], \mathcal{I})$  that has maximum coverage out of all such pathlets then covers at least one-eighth of what any other subedge pathlet  $(e[x, x'], \mathcal{I}')$  covers.



We create a sweepline algorithm that, for each  $e[x_i, x_{i+2^j}]$  (with  $j \leq \log(m-i)$ ), constructs a reference-optimal  $(\ell, \Delta')$ -pathlet  $(e[x_i, x_{i+2^j}], \mathcal{I}_j)$ . We let each interval  $\mathcal{I}_j$  contain all maximal intervals  $[y, y']$  for which  $d_F(e[x_i, x_{i+2^j}], T[y, y']) \leq \Delta'$ , and thus all maximal intervals for which  $(x_i, y)$  can reach  $(x_{i+2^j}, y')$  by a bimonotone path in  $\Delta'$ -FSD( $S, T$ ). Note that both  $(x_i, y)$  and  $(x_{i+2^j}, y')$  are critical points. Thus we aim to find all maximal intervals  $[y, y']$  for which the critical point  $(x_i, y)$  can reach the critical point  $(x_{i+2^j}, y')$  by a bimonotone path in  $\Delta'$ -FSD( $S, T$ ).

Let  $Z_i$  be the subset of  $\mathcal{O}(n \log n)$  critical points with  $x$ -coordinate equal to  $x_i$  or  $x_{i+2^j}$  for some  $j \leq \log(m-i)$ . We construct, for each  $i \in [n]$ , the reachability graph  $G(e, T, Z_i)$  from Section 7, which encodes reachability between the critical points in  $Z_i$ . This graph takes  $\mathcal{O}((n + |Z_i|) \log(n|Z_i|)) = \mathcal{O}(n \log^2 n)$  time to construct and has complexity  $\mathcal{O}(n \log^2 n)$  (see Theorem 17). We aim to annotate each vertex  $\mu$  (note that  $\mu$  does not have to be a critical point) in  $G(e, T, Z_i)$  with the minimum  $y$ , such that there exists a critical point  $(x_i, y)$  that can reach  $\mu$ . We annotate  $\mu$  with  $\infty$  if no such  $y$  exists.

**Annotating vertices and asymptotic analysis.** We first annotate the vertices  $(x_i, y)$  in  $\mathcal{O}(n)$  time by scanning over them in order of increasing  $y$ -coordinate. We process the remaining vertices in  $yx$ -lexicographical order, first by increasing  $y$ -coordinate, and by increasing  $x$ -coordinate when ties arise. Each vertex  $\mu$  that we consider has only incoming arcs that originate from vertices below and left of  $\mu$ . By our lexicographical ordering, each of these vertices are already annotated. The minimal  $y$  for which there exists a path from  $(x_i, y)$  to  $\mu$ , must be the minimum over all its incoming arcs which we compute in time proportional to the in-degree of  $\mu$ . If  $\mu$  has no incoming arcs, we annotate it with  $\infty$ .

Let  $V$  and  $A$  be the sets of  $\mathcal{O}(n \log^2 n)$  vertices and arcs of  $G(e, T, Z_i)$ . For the above annotation procedure, we first compute the  $yx$ -lexicographical ordering of the vertices, based on their corresponding points in the parameter space. This takes  $\mathcal{O}(|V| \log |V|)$  time. Afterwards, we go over each vertex and each incoming arc exactly once, which take an additional  $\mathcal{O}(|V| + |A|)$  time. In total, we annotate all vertices in  $\mathcal{O}(n \log^3 n)$  time.

**Constructing the pathlets.** Using the annotations, constructing the pathlets becomes straightforward. For each  $j \in [\log(m-i)]$ , we construct  $\mathcal{I}_j$  as follows. We iterate over all critical point  $(x_{i+2^j}, y')$  in the graph  $G(e, T, Z_i)$ . For each critical point  $(x_{i+2^j}, y')$  with a finite annotation  $y$ , we add the interval  $[y, y']$  to  $\mathcal{I}_j$ . This procedure ensures that  $\mathcal{I}_j$  contains all maximal intervals  $[y, y']$  for which  $d_F(e[x_i, x_{i+2^j}], T[y, y']) \leq \Delta'$ , making an optimal pathlet  $(e[x_i, x_{i+2^j}], \mathcal{I}_j)$  with respect to its reference curve. As there are  $\mathcal{O}(n)$  critical points per  $j$ , this algorithm uses  $\mathcal{O}(n \log n)$  time. Storing the pathlets takes  $\mathcal{O}(n \log n)$  space. Thus, we conclude the following:

► **Lemma 22.** *Let  $C$  be a set of pathlets where  $\text{Cov}(C)$  has  $\mathcal{O}(n^2 \log n)$  connected components. Suppose  $\text{Cov}(C)$  is preprocessed into the data structure of Lemma 15. In  $\mathcal{O}(n^2 \log^3 n)$  time and using  $\mathcal{O}(n \log^2 n)$  space, we can construct a  $(2, \Delta')$ -pathlet  $(P, \mathcal{I})$  with*

$$\|\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)\| \geq \frac{1}{8} \|\text{Cov}(P', \mathcal{I}') \setminus \text{Cov}(C)\|$$

for any  $(2, \Delta')$ -pathlet  $(P', \mathcal{I}')$  where  $P'$  is a subsegment of a given directed line segment  $e$ . The intervals in  $\mathcal{I}$  all have endpoints that come from a set of at most  $4n^2$  values.

**Proof.** TOPROVE 8 ◀

► **Theorem 23.** *Suppose that the universe  $\mathcal{U}$  and the coverage  $\text{Cov}(C)$  is preprocessed into the data structure of Lemma 15. In  $\mathcal{O}(n^3 \log^3 n)$  time and using  $\mathcal{O}(n \log^2 n)$  space, we can construct a  $(2, \Delta')$ -pathlet  $(P, \mathcal{I})$  with*

$$|\text{Cov}(P, \mathcal{I}) \setminus \text{Cov}(C)| \geq \frac{1}{8} |\text{Cov}(P', \mathcal{I}') \setminus \text{Cov}(C)|$$

for any subedge  $(2, \Delta')$ -pathlet  $(P', \mathcal{I}')$ .

## 10 Conclusion

In this work, we presented an improved approximation algorithm for subtrajectory clustering. We discuss our technical contribution, and how it differs from previous works, our asymptotic improvements and finally interesting directions for future work.

**Technical contribution.** Our technical contributions are threefold:

First, we introduced a new type of curve simplification in Section 4. This simplification allows us to construct a curve  $S$ , such that our clustering needs to consider only pathlets whose reference curve is a subcurve of  $S$ . Although numerous similar curve-simplification algorithms exist, our method distinguishes itself by lying significantly closer to the input curve  $T$ . Consequently, our approximation algorithm is a 4-approximation in  $\Delta$ , compared to the 11-approximations of prior works. We consider this simplification to be of independent interest, as future works may immediately use our simplification method to obtain 4-approximations in  $\Delta$  also.

Secondly, we considered in Section 6 an extension to the greedy set cover algorithm wherein each iteration adds an approximately-maximum covering element, rather than a maximum one. Observe that  $P$  can always be divided into at most three subcurves, where at most one of them starts and ends at a vertex of  $S$  (a vertex-subcurve) and at most two of them are subcurves of an edge of  $S$  (a subedge of  $S$ ). We design a greedy meta-algorithm, that in each iteration computes an  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$  with approximately-maximum coverage, whose reference curve is a vertex-subcurve or subedge of  $S$ . Our approximately greedy set cover analysis shows that our meta-algorithm computes a clustering of size  $\mathcal{O}(k \log n)$ . A key takeaway from our construction is that by restricting our attention to vertex-subcurves and subedges of  $S$ , we significantly reduce the set of candidate pathlets from  $\tilde{\mathcal{O}}(n^3 \ell)$  to  $\tilde{\mathcal{O}}(n^2)$ . We consider this fact to also be of independent interest. Indeed, our subsequent algorithm spends near-linear time per candidate pathlet but future works may discover more efficient algorithms over the same smaller candidate set.

Finally, we presented algorithms in Sections 8 and 9 that compute the corresponding candidate pathlet for a candidate reference curve in near-linear time and near-linear space. The key observation to this contribution is that we show that it suffices to compute all candidate pathlets on the fly, significantly reducing the space.

**Asymptotic improvements.** Compared to the best prior deterministic work [13], our algorithm improves the running time by a factor near-linear in  $n\ell$ , improves the space by a factor near-linear in  $n^2\ell$ , and improves the approximation in  $\Delta$  from a factor 11 to 4, all whilst asymptotically matching the size of the clustering. We consider this a significant improvement over the state-of-the-art.

When we compare to previous randomized work [3, 5] we improve the running time by a factor near-linear in  $\ell$ , improve space by a factor  $n$ , and improve the approximation in  $\Delta$  from a factor 11 to 4. A downside of our approach is that, compared to randomised

works, we only asymptotically match the clustering size whenever  $\ell$  is relatively large (i.e.,  $\ell \in \Omega(\log n / \log k)$ ). However, we note that on all other algorithmic quality measures, we still offer a substantial improvement whilst also being deterministic. In addition, when considering algorithmic performance in practice, we note that these previous randomized results [3, 5] use  $\varepsilon$ -net sampling. Such a sampling procedure leads to very high hidden constants in the asymptotic clustering size which makes such an approach impractical.

**Future work.** We think it remains an interesting open problem whether one can obtain a clustering size of  $\mathcal{O}(k\ell \log k)$  in a deterministic manner. We also note that, currently, our algorithm considers a set of  $\tilde{\mathcal{O}}(n^2)$  reference curves  $P$ , and computes an  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$  with approximately-maximum coverage for each reference curve independently, in near-linear time. We think it is an interesting open problem whether one can present an algorithm that is overall more efficient whenever these maximum pathlets are considered simultaneously rather than independently.

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## A

 The interior-disjoint setting

Previous definitions of subtrajectory clustering have imposed various restrictions on the pathlets in the clustering. For example, in [6, 7, 8, 16] the pathlets must be *interior-disjoint*. A pathlet  $(P, \mathcal{I})$  is interior-disjoint whenever the intervals in  $\mathcal{I}$  are pairwise interior-disjoint. While we do not give dedicated algorithms for the interior-disjoint setting, we show in Lemma 25 that we can efficiently convert any pathlet into two interior-disjoint pathlets with the same coverage. This gives a post-processing algorithm for converting a clustering  $C$  into an interior-disjoint clustering  $C'$  with at most twice the number of pathlets. We first show the following auxiliary lemma.

► **Lemma 24.** *Given a set of intervals  $\mathcal{I}$ , we can compute a subset  $\mathcal{I}' \subseteq \mathcal{I}$  with  $\text{ply}^1$  at most two and with  $\bigcup \mathcal{I}' = \bigcup \mathcal{I}$  in  $\mathcal{O}(|\mathcal{I}| \log |\mathcal{I}|)$  time.*

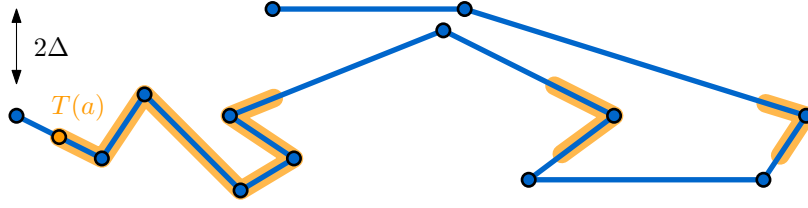
**Proof.** TOPROVE 9 ◀

► **Lemma 25.** *Given an  $(\ell, \Delta)$ -pathlet  $(P, \mathcal{I})$ , we can construct two interior-disjoint  $(\ell, \Delta)$ -pathlets  $(P_1, \mathcal{I}_1)$  and  $(P_2, \mathcal{I}_2)$  with  $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$  in  $\mathcal{O}(|\mathcal{I}| \log |\mathcal{I}|)$  time.*

**Proof.** TOPROVE 10 ◀

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<sup>1</sup> The ply of a set of intervals is the maximum number of intervals with a common intersection.



■ **Figure 7** Consider the trajectory  $T$  and some point  $T(a)$ . For some  $\Delta$ , we can indicate all  $T(b) \in T$  with  $a \leq b$  where for line  $s = \overline{T(a)T(b)}$ ,  $d_F(s, T[a, b]) \leq 2\Delta$ . Note that this set  $\mathbb{B}(a)$  is not a connected set of intervals on  $[1, n]$ .

## B Constructing a pathlet-preserving simplification

### B.1 Defining our $2\Delta$ -simplification $S$

We consider the vertex-restricted simplification defined by Agarwal *et al.* [2] and generalize their  $2\Delta$ -simplification definition, allowing vertices to lie anywhere on  $T$  (whilst still appearing in order along  $T$ ). This way, we obtain a simplification with at most as many vertices as the optimal unrestricted  $\Delta$ -simplification (see Figure 7).

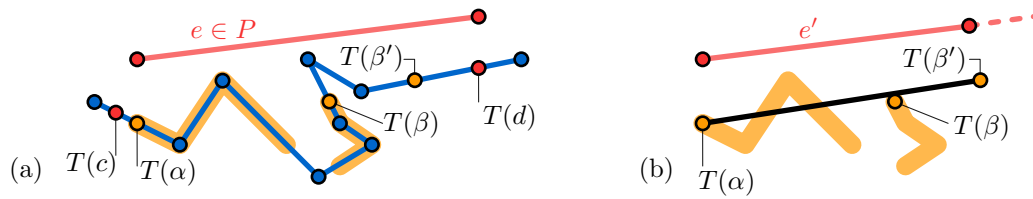
► **Definition 26.** Let  $T$  be a trajectory with  $n$  vertices,  $\Delta \geq 0$  and  $a \in [1, n]$ . We define the set  $\mathbb{B}(a) = \{b \geq a \mid d_F(\overline{T(a)T(b)}, T[a, b]) \leq 2\Delta\}$ .

► **Definition 27.** Let  $T$  be a trajectory with  $n$  vertices and  $\Delta \geq 0$ . We define our  $2\Delta$ -simplified curve  $S$  as follows: the first vertex of  $S$  is  $T(1)$ . The second vertex of  $S$  may be any point  $T(a)$  with  $a$  as a rightmost endpoint of an interval in  $\mathbb{B}(1)$ . The third vertex of  $S$  may be any point  $T(b)$  with  $b$  as a rightmost endpoint of an interval in  $\mathbb{B}(a)$ , and so forth.

Per definition of the set  $\mathbb{B}(a)$ , the resulting curve  $S$  is a  $2\Delta$ -simplified curve. Let  $(f, g)$  be any  $2\Delta$ -matching between  $S$  and  $T$  that matches the vertices of  $S$  to the points on  $T$  that define them. We prove that  $(S, f, g)$  is a pathlet-preserving simplification.

► **Lemma 28.** The curve  $S$  as defined above, together with the matching  $(f, g)$ , forms a pathlet-preserving simplification.

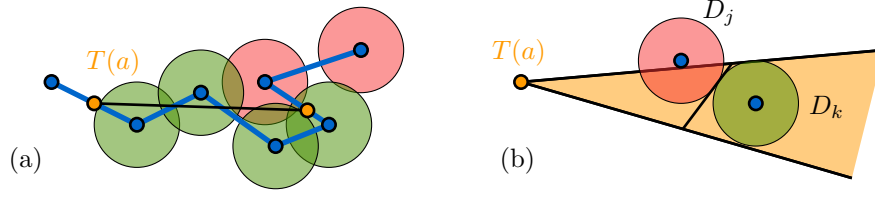
**Proof.** TOPROVE 11



■ **Figure 8** (a) The construction in the proof of Lemma 28. We have an edge  $e$  with  $d_F(e, T[c, d]) \leq \Delta$ . Moreover, for some  $\alpha \in [c, d]$  we show  $\mathbb{B}(\alpha)$  in orange where  $\beta$  is the last value in some connected component of  $\mathbb{B}(\alpha)$ . (b) For any  $\beta' \in [\alpha, d]$ , there exists a subedge  $e'$  of  $e$  with  $d_F(e', T[a, \beta']) \leq \Delta$ .

### B.2 Constructing the simplification

We give an  $\mathcal{O}(n \log n)$  time algorithm for constructing our greedy simplification  $S$  (Definition 27). Given any point  $T(a)$  on  $T$ , we decompose the problem of finding a  $b \in \mathbb{B}(a)$  over



■ **Figure 9** (a) For two points  $T(a)$  and  $T(b)$  on  $T$ , we consider all  $j \in [a, b] \cap N$  and draw a ball with radius  $2\Delta$  around them in green. (b) For all  $k \in [a, j-1] \cap \mathbb{N}$ , the area  $SW_{j-1}$  must be contained in the orange cone.

all edges of  $T$ . That is, given an  $a \in [1, n]$ , we consider an individual edge  $T[i, i+1]$  of  $T$ . We show how to report the maximal  $b \in [i, i+1] \cap \mathbb{B}(a)$ . Our procedure is based off of the work by Guibas *et al.* [17] on ordered stabbing of disks in  $\mathbb{R}^2$ , and takes  $\mathcal{O}(i-a)$  time. We fix a plane  $H$  in  $\mathbb{R}^d$  that contains  $T(a)$  and  $T[i, i+1]$ . On a high-level, we apply the argument by Guibas *et al.* in  $\mathbb{R}^d$  by restricting the disks to their intersection with  $H$ :

► **Definition 29.** Let  $H$  be some fixed two-dimensional plane in  $\mathbb{R}^d$ . For any  $x \in [1, n]$  denote by  $B_x$  be the ball in  $H$  that is obtained by intersecting a ball of radius  $2\Delta$  centered at  $T(x)$  with  $H$ . For  $a \leq b$  we say that a directed line segment  $e$  in  $H$  stabs all balls in  $[a, b]$  in order if for all  $k \in \{a\} \cup ([a, b] \cap \mathbb{N}) \cup \{b\}$  there are points  $p_k \in e \cap B_k$  such that  $p_k$  comes before  $p_{k'}$  on  $e$  whenever  $k \leq k'$  (see Figure 9 (a)).

► **Lemma 30** ([17, Theorem 14]). A line segment  $e$  is within Fréchet distance  $2\Delta$  of a subcurve  $T[a, b]$  of  $T$  if and only if the following conditions are met:

1.  $e$  starts within distance  $2\Delta$  of  $T(a)$ ,
2.  $e$  ends within distance  $2\Delta$  of  $T(b)$ , and
3.  $e$  stabs all balls in  $[a, b]$  in order.

**Computing the maximal  $b \in [i, i+1] \cap \mathbb{B}(a)$ .** For any edge  $e = \overline{T(a)T(b)}$  of  $Z$ , the endpoints lie on  $T$  and thus  $e$  trivially satisfies the first two criteria. It follows that if we fix some  $T(a)$  on  $S$  and some edge  $T[i, i+1]$ , then the maximal  $b \in [i, i+1]$  (with  $b \geq a$ ) for which  $\overline{T(a)T(b)}$  stabs balls  $[a, b]$  in order is also the maximal  $b \in [i, i+1] \cap \mathbb{B}(a)$ . We consider the following (slightly reformulated) lemma by Guibas *et al.* [17]:

► **Lemma 31** ([17, Lemma 8]). Let  $[a_j, b_j]$  be a sequence of intervals. There exist  $p_j \in [a_j, b_j]$  with  $p_j \leq p_k$  for all  $j \leq k$ , if and only if there is no pair  $j \leq k$  with  $b_k < a_j$ .

The above lemma is applicable to segments in  $H$  stabbing balls in  $H$ . Indeed, consider all integers  $j \in [a, i]$ . We may view any directed line segment  $e$  in  $H$  as (part of) the real number line, and view the intersections between  $e$  and the disks  $D_j$  as intervals. Lemma 31 then implies that  $e$  stabs  $[a, i]$  in order, if and only if no integers  $j, k \in [a, i]$  exist with  $j \leq k$  such that  $e$  leaves  $D_k$  before it enters  $D_j$  (assuming  $e$  intersects all disks).

For all integers  $j \in [a, i]$ , let  $W_j := \{p \in H \mid \overline{T(a)p} \text{ intersects } D_j\}$ . We define the *stabbing wedge*  $SW_j := \{p \in H \mid \overline{T(a)p} \text{ intersects } [a, j] \text{ in order}\}$ . We prove the following:

► **Lemma 32.** Either  $SW_j = \bigcap_{k \in [a, j] \cap \mathbb{N}} W_k$ , or  $SW_j = \emptyset$ .

**Proof.** TOPROVE 12 ◀

► **Lemma 33.** Given  $a \in [1, n]$  and edge  $T[i, i+1]$  of  $T$ , we can compute the maximum  $b \in [i, i+1] \cap \mathbb{B}(a)$ , or report that no such  $b$  exists, in  $\mathcal{O}(1+i-a)$  time.



**Proof.** TOPROVE 13 ◀

► **Lemma 34.** *Given  $a \in [1, n]$ , we can compute a value  $b^*$  that is the maximum of some connected component of  $\mathbb{B}(a)$  in  $\mathcal{O}((1 + b^* - a) \log n)$  time.*

**Proof.** TOPROVE 14 ◀

We now iteratively apply Lemma 34 to construct our curve  $S$ . We obtain a  $2\Delta$ -matching  $(f, g)$  by constructing separate matchings between the edges  $\overline{T(a)T(b)}$  of  $S$  and the subcurves  $T[a, b]$  that they simplify. By Lemma 28 this gives a pathlet-preserving simplification  $(S, f, g)$ .

► **Theorem 8.** *For any trajectory  $T$  with  $n$  vertices and any  $\Delta \geq 0$ , we can construct a pathlet-preserving simplification  $S$  in  $\mathcal{O}(n \log n)$  time.*