Forbidden Induced Subgraphs for Bounded Shrub-Depth and the Expressive Power of MSO*

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Abstract

The graph parameter shrub-depth is a dense analog of tree-depth. We characterize classes of bounded shrub-depth by forbidden induced subgraphs. The obstructions are well-controlled flips of large half-graphs and of disjoint unions of many long paths. Applying this characterization, we show that on every hereditary class of unbounded shrub-depth, MSO is more expressive than FO. This confirms a conjecture of [Gajarský and Hliněný; LMCS 2015] who proved that on classes of bounded shrub-depth FO and MSO have the same expressive power. Combined, the two results fully characterize the hereditary classes on which FO and MSO coincide, answering an open question by [Elberfeld, Grohe, and Tantau; LICS 2012].

Our work is inspired by the notion of stability from model theory. A graph class $\mathcal{C}$ is MSO-stable, if no MSO-formula can define arbitrarily long linear orders in graphs from $\mathcal{C}$. We show that a hereditary graph class is MSO-stable if and only if it has bounded shrub-depth. As a key ingredient, we prove that every hereditary class of unbounded shrub-depth FO-interprets the class of all paths. This improves upon a result of [Ossona de Mendez, Pilipczuk, and Siebertz; Eur. J. Comb. 2025] who showed the same statement for FO-transductions instead of FO-interpretations.

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Contents

1 Introduction

The main result of this paper is the following Theorem 1.1 which yields various characterizations for hereditary graph classes of bounded shrub-depth, in terms of forbidden induced subgraphs, monadic second-order logic (MSO), counting monadic second-order logic (CMSO), and first-order logic (FO). All notions appearing in Theorem 1.1 will be motivated, defined, and explained in the remainder of this introduction.

Theorem 1.1. *For every hereditary graph class $\mathcal{C}$, the following are equivalent.*

1. *$\mathcal{C}$ has bounded shrub-depth.*
2. *There is a $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped half-graphs of order t and all flipped tP_t.*
3. *There is a $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped half-graphs of order t and all flipped $3P_t$.*
4. *$\mathcal{C}$ is MSO-stable.*
5. *$\mathcal{C}$ is monadically MSO-stable.*
6. *$\mathcal{C}$ is CMSO-stable.*
7. *$\mathcal{C}$ is monadically CMSO-stable.*
8. *$\mathcal{C}$ does not 1-dimensionally FO-interpret the class of all paths.*
9. *FO and MSO have the same expressive power on $\mathcal{C}$.*

Stability

We start motivating Theorem 1.1 by introducing the model-theoretic notion of *stability*, a context in which the graph parameter shrub-depth will naturally arise. Originating in the 70s and pioneered by Shelah, stability theory is a prolific branch of model theory which seeks to classify the complexity of theories (or in our case: graph classes). There, stability is the most important dividing line, which separates the well-behaved *stable* classes from the complex *unstable* ones. Intuitively, a graph class $\mathcal{C}$ is stable, if one cannot define arbitrarily large linear orders in $\mathcal{C}$ using logical formulas. More precisely, for a logic $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}, \mathrm{CMSO}\}$, an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, and a graph class $\mathcal{C}$, we say φ has the *order-property* on $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is a graph $G \in \mathcal{C}$ and a sequence $\bar{a}_i, \dots, \bar{a}_\ell$ of tuples of vertices of G, such that for all $i, j \in [\ell]$

$$G \models \varphi(\bar{a}_i, \bar{a}_j) \quad \Leftrightarrow \quad i \leqslant j.$$

For example the FO-formula $\varphi(x, y)$ expressing "the neighborhood of x is a superset of the neighborhood of y" has the order-property on the class of all half-graphs, as it orders the sequence $a_1, \dots, a_t$ in the half-graph H_t of order t for every $t \in \mathbb{N}$, as depicted in Figure 1.

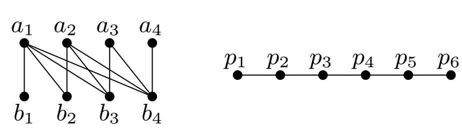

Figure 1: On the left: the half-graph of order 4 (denoted as H_4) with vertices $\{a_1, \dots, a_4, b_1, \dots, b_4\}$ and edges between a_i and b_j if $i \leqslant j$. On the right: the 6-vertex path (denoted as P_6).

Similarly, the MSO-formula $\psi(x_1x_2, y_1y_2)$ expressing "x_1 and x_2 are not connected after deleting y_1" has the order-property on the class of all paths, as it orders the sequence of 2-tuples $p_1p_t, p_2p_t, \dots, p_tp_t$ in the t-vertex path P_t for every $t \in \mathbb{N}$. A graph class $\mathcal{C}$ is *$\mathcal{L}$-stable* if no $\mathcal{L}$-formula has the order-property on $\mathcal{C}$, and *$\mathcal{L}$-unstable* otherwise. The class of all half-graphs (all paths) is the arguably simplest example of an FO-unstable (MSO-unstable) graph class.

Apart from its extensive study on infinite structures in model theory, FO-stability has recently gained a lot of attention in finite model theory, in particular in structural and algorithmic graph theory. Podewski and Ziegler [33] and Adler and Adler [1] observed that on *monotone*[1] graph classes, FO-stability coincides with the combinatorial property of being *nowhere dense*. Nowhere dense classes are very general classes of sparse graphs [30], enjoying strong combinatorial and algorithmic properties, such as fixed-parameter tractable (fpt) FO model checking [22]. The equivalence of FO-stability and nowhere denseness in monotone classes elegantly bridges the fields of model theory and structural graph theory. It has prompted the question whether also *hereditary*[2] FO-stable graph classes are combinatorially well-behaved. In the hereditary setting, FO-stability significantly generalizes nowhere denseness: for example the class of all cliques and the class of all map graphs are both FO-stable but not nowhere dense. A current research program has uncovered multiple natural combinatorial characterizations of hereditary FO-stable classes [11, 13, 19, 8] and shown that they also admit fpt FO model checking [19, 11, 12].

In contrast to the rich literature on FO-stability, we are not aware of any previous works studying its natural restriction MSO-stability. We attribute this to the fact that the compactness theorem, which is the main tool for working on *infinite* structures where stability originates, fails for MSO. Now, the recent successes in the study of FO-stable classes of *finite* graphs raise the question whether also MSO-stability can be understood through the lens of structural graph theory. In this work we show that this is indeed the case, by proving the following.

Theorem 1.2. *A hereditary graph class is MSO-stable if and only if it has bounded shrub-depth.*

The folklore fact that every monotone class of bounded shrub-depth also has bounded tree-depth yields the following corollary.

Corollary 1.3. *A monotone graph class is MSO-stable if and only if it has bounded tree-depth.*

Shrub-Depth

Shrub-depth is a parameter for graph classes introduced in [20]. It generalizes tree-depth to dense classes and can be seen as a bounded depth analog of clique- and rank-width, similar to how tree-depth is a bounded depth analog of tree-width. Classes of bounded shrub-depth are extremely well-behaved from algorithmic, combinatorial, and logical points of view. For instance, they admit quadratic time isomorphism testing [31] and fpt MSO model checking with an elementary dependence on the size of the input formula [18, 20]; they can be characterized by vertex minors and FO-transductions [32]; and FO and MSO have the same expressive power on these classes [18].

The above examples show that the structure side of bounded shrub-depth is well charted. Indeed, we prove the direction "bounded shrub-depth implies MSO-stability" of Theorem 1.2 using mainly existing tools. In contrast, proving the "hereditary and unbounded shrub-depth implies MSO-instability" direction requires insights about the non-structure side of shrub-depth (i.e., the properties shared by all classes that have unbounded shrub-depth). Up until now, the picture here was much more vague: In [21] it is proved that for every $d \in \mathbb{N}$ there exists a finite set of graphs $\mathcal{F}_d$ such that a graph has shrub-depth at most d if and only if it excludes all of $\mathcal{F}_d$ as induced subgraphs. This result is non-constructive and does not reveal the concrete sets $\mathcal{F}_d$. In [25] and [32] it is shown that the class of all paths can be produced from any class $\mathcal{C}$ of unbounded shrub-depth by

[1] A graph class is *monotone*, if it is closed under taking subgraphs.

[2] A graph class is *hereditary*, if it is closed under taking induced subgraphs.

taking vertex-minors and also by an FO-transduction, respectively. The high expressive power and irreversibility of vertex-minors and transductions limits our ability to draw conclusions about the structure of the original class $\mathcal{C}$ from these two results. In this paper, we improve the non-structure situation by providing a characterization of classes of bounded shrub-depth by explicitly listing forbidden induced subgraphs.

Forbidden Induced Subgraphs

We will briefly introduce the notions needed to state the obstructions. For two graphs G and H on the same vertex set and a partition $\mathcal{P}$ of $V(G)$, we say H is a *$\mathcal{P}$-flip* of G if it can be obtained by complementing the edge relation between pairs of parts of $\mathcal{P}$ (see Figure 2). We refer to the preliminaries for formal definitions. Our obstructions for bounded shrub-depth will be well-controlled flips of disjoint unions of long paths and of large half-graphs, as made precise in the following two definitions and their accompanying Figures 2 and 3.

Definition 1.4. P_t is the t-vertex path and mP_t is the disjoint union of m many P_t, with vertices $[m] \times [t]$, where (i, j) is the jth vertex on the ith path. A *flipped mP_t* is an $\mathcal{L}$-flip of mP_t for the partition $\mathcal{L} = \{L_1, \dots, L_t\}$ of the paths into *layers*, where $L_j = \{(1, j), \dots, (m, j)\}$ contains the jth vertices of all the paths for $j \in [t]$.

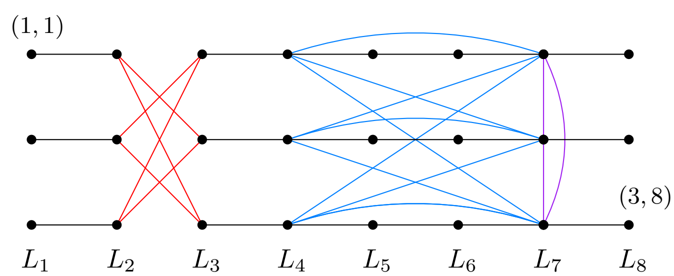

Figure 2: A flipped $3P_8$. More precisely the depicted graph is an $\mathcal{L}$-flip of $3P_8$ with $\mathcal{L} = \{L_1, \dots, L_8\}$, where the following parts were flipped: L_2 with L_3 (red), L_4 with L_7 (blue), L_7 with L_7 (purple).

Definition 1.5. The *half-graph of order t* (denoted as H_t) is the graph on vertices $a_1, \dots, a_t$ and $b_1, \dots, b_t$ where a_i and b_j are adjacent if and only if $i \leqslant j$. A *flipped H_t* is an $\{A, B\}$-flip of H_t, where the flip-partition has parts $A = \{a_1, \dots, a_t\}$ and $B = \{b_1, \dots, b_t\}$.

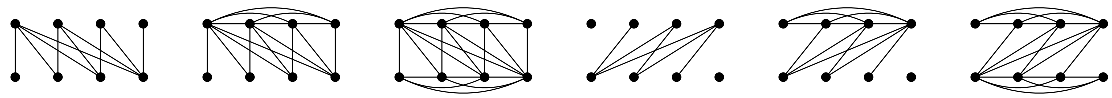

Figure 3: All flipped H_4s (up to isomorphism). Figure replicated with permission from [28, Fig. 2.5].

We are now ready to state our characterization.

Theorem 1.6. *A graph class $\mathcal{C}$ has bounded shrub-depth if and only if there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.*

Moreover, our proofs show that in Theorem 1.6, one can replace tP_t with $3P_t$. While the stated tP_t variant of the theorem is more useful for hardness proofs, the $3P_t$ variant is a step towards finding the simplest obstructions that cause unbounded shrub-depth. It remains open whether the theorem also holds for $2P_t$, but we know it fails for $1P_t$, as every graph on t vertices is a flipped $1P_t$.

Interpretations and Monadic Stability

In order to prove the stability characterization (Theorem 1.2) from the induced subgraph characterization (Theorem 1.6), we visit another model-theoretic concept called *interpretations*. For a logic $\mathcal{L}$ a 1-dimensional[3] $\mathcal{L}$-interpretation I is defined by two $\mathcal{L}$-formulas: a domain formula $\delta(x)$ and an irreflexive, symmetric edge formula $\varphi(x, y)$. It maps each graph G to the graph $H := I(G)$ with vertex set $V(H) := \{v \in V(G) : G \models \delta(v)\}$ and edges $E(H) := \{(u, v) \in V(H)^2 : G \models \varphi(u, v)\}$. We say a graph class $\mathcal{C}$ *interprets* a graph class $\mathcal{D}$, if there is an interpretation I such that $\mathcal{D} \subseteq \{I(G) : G \in \mathcal{C}\}$. Building on our forbidden induced subgraph characterization, we show the following.

Theorem 1.7. *A hereditary graph class has unbounded shrub-depth if and only if it* 1*-dimensionally FO-interprets the class of all paths.*

As we have already seen, the class of all paths is MSO-unstable. Theorem 1.7 can be used to lift instability to all hereditary classes of unbounded shrub-depth. This is a strengthening of a recent result by Ossona de Mendez, Pilipczuk, and Siebertz [32], which states that every class of unbounded shrub-depth *FO-transduces*[4] the class of all paths. Their result implies that every class of unbounded shrub-depth is *monadically MSO-unstable*: a graph class $\mathcal{C}$ is monadically $\mathcal{L}$-stable if every coloring of $\mathcal{C}$ is $\mathcal{L}$-stable. In general, monadic MSO-stability is more restrictive than MSO-stability. For instance the class $\mathcal{K}$ of all 1-subdivided cliques is MSO-stable, but monadically MSO-unstable. When considering only hereditary classes, this example fails: the hereditary closure of $\mathcal{K}$ contains every 1-subdivided graph and is FO-unstable (so in particular also monadically FO-unstable, MSO-unstable, and monadically MSO-unstable). Braunfeld and Laskowski showed that FO-stability and monadic FO-stability coincide in all hereditary classes [7]. We show that the same collapse happens for MSO and CMSO.

Theorem 1.8. *For hereditary graph classes, the notions MSO-stability, monadic MSO-stability, CMSO-stability, and monadic CMSO-stability are all equivalent.*

The Expressive Power of MSO

For a graph class $\mathcal{C}$, we say FO and MSO *have the same expressive power on* $\mathcal{C}$ if for every MSO-sentence φ, there exists an FO-sentence ψ such that for every graph $G \in \mathcal{C}$ we have $G \models \varphi \Leftrightarrow G \models \psi$. Otherwise, we say MSO is *more expressive* than FO on $\mathcal{C}$.

It was shown by Grohe, Elberfeld, and Tantau that for all monotone classes $\mathcal{C}$, FO and MSO have the same expressive power if and only if $\mathcal{C}$ has bounded tree-depth [15]. As an open question, they asked for a characterization of the hereditary classes where FO and MSO coincide. As a dense analog of bounded tree-depth, bounded shrub-depth is a natural candidate here. Indeed, Gajarský and Hliněný showed that FO and MSO have the same expressive power on every class of bounded shrub-depth [18, Thm. 5.14]. However, they could not prove the reverse direction, which they attributed to a lack of known obstructions for classes of bounded shrub-depth. As an application of our forbidden induces subgraph characterization, we provide the missing part of the puzzle by showing that MSO is more expressive than FO on every hereditary class of unbounded shrub-depth. Together with the result by Gajarský and Hliněný, this completely characterizes the hereditary classes on which FO and MSO coincide.

[3]We will later also define and work with higher dimensional interpretations, whose formulas have tuples as free variables.

[4]A transduction first colors the input graph non-deterministically, and then applies a fixed interpretation that can access these colors. For a single input graph G, it produces multiple (possibly isomorphic) output graphs: one for each coloring of G. Due to the access to an arbitrary coloring, transductions are more expressive than interpretations.

Theorem 1.9. *For every hereditary graph class $\mathcal{C}$, FO and MSO have the same expressive power on $\mathcal{C}$ if and only if $\mathcal{C}$ has bounded shrub-depth.*

Overview of the Paper

Figure 4 shows the implications which comprise Theorem 1.1. The only notion in the figure that has not been discussed so far is ∞-*flip-flatness*. This is another characterization of shrub-depth with strong ties to stability theory, recently proved by Dreier, Mählmann, and Toruńczyk [14], which we define in the upcoming preliminaries (Section 2). The proofs of the individual implications are presented in Sections 3 to 6. We conclude with an outlook in Section 7, where we discuss potential strengthenings of our induced subgraph characterization and a second major dividing line from model theory, named *dependence*.

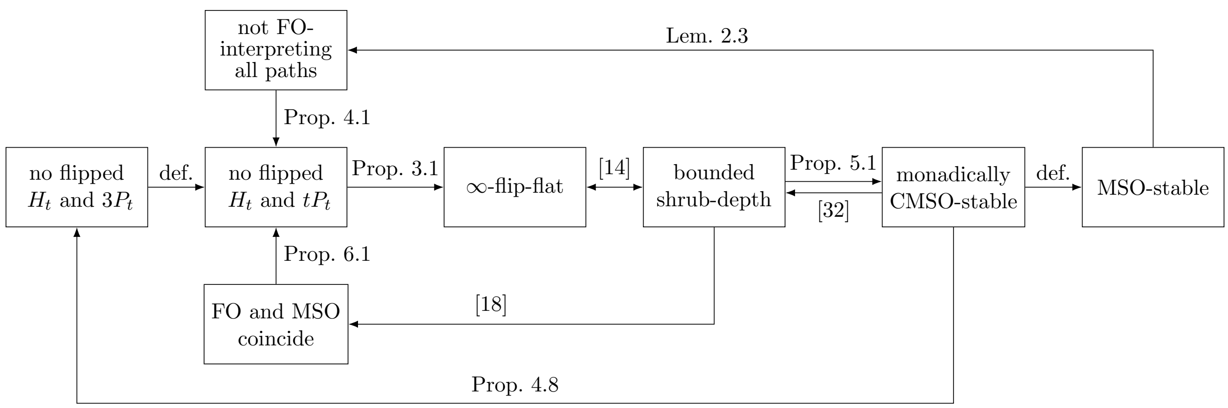

Figure 4: A map of Theorem 1.1: combinatorial and logical characterizations of hereditary classes of bounded shrub-depth.

2 Preliminaries

We use $[n]$ to refer to the set $\{1, \ldots, n\}$. For an s-tuple $\bar{a} = a_1 \ldots a_s$, we use $|\bar{a}|$ to refer to its length s and $\bar{a}[i]$ to refer to its ith component a_i.

2.1 Colored Structures and Graphs

A *(relational) signature* Σ is a set of relation symbols, each with an implicitly assigned arity. A structure G over Σ is called a Σ-*structure*. We denote by $\Sigma^{(k)} := \Sigma \cup \{C_1, \ldots, C_k\}$ the *expansion* of Σ by k many new unary predicates, which we can think of as colors. A single element is allowed to have multiple colors, but this will not be important. A $\Sigma^{(k)}$-structure G^+ is a k-*coloring* of G, if G is the Σ-structure obtained by forgetting the color predicates in G^+. For a class of Σ-structures $\mathcal{C}$, we denote by $\mathcal{C}^{(k)}$ the class of $\Sigma^{(k)}$-structures consisting of all the k-colorings of structures from $\mathcal{C}$.

A *graph* is a Γ-structure for the signature $\Gamma := \{E\}$ consisting of a single binary relation (called *edge relation*) that is interpreted symmetrically and irreflexive. We call $\Gamma^{(k)}$-structures k-*colored graphs*. For any structure G, we use $V(G)$ to refer to its universe: if G is a graph, then $V(G)$ is its vertex set.

2.2 Logic and Types

We assume familiarity with first-order logic (FO) and monadic second-order logic (MSO). As we model graphs as structures with a binary edge relation, MSO on graphs allows for quantification of vertex sets, but not of edge sets. This is also known as MSO_1 in the literature. *Counting* monadic second-order logic (CMSO) extends MSO for all set variables X and $0 \leqslant r < k$ by the cardinality constraint $\mathrm{card}_{r,k}(X)$ that holds true if and only if $|X| \equiv r \pmod{k}$.

Fix a logic $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}, \mathrm{CMSO}\}$ and a signature Σ. We use $\mathcal{L}[\Sigma]$ to refer to the set of $\mathcal{L}$-formulas over Σ and $\mathcal{L}[\Sigma]_q$ for the set of $\mathcal{L}$-formulas over Σ with quantifier rank at most q. We often drop the signature in the notation if it is clear from the context. For a Σ-structure G, a tuple $\bar{a}$, and a set of Σ-formulas Φ, the *type* $\mathrm{tp}(\bar{a}, G, \Phi)$ is the set of formulas $\varphi(\bar{x}) \in \Phi$, with $|\bar{x}| = |\bar{a}|$ such that $G \models \varphi(\bar{a})$. In particular $\mathrm{tp}(G, \Phi)$ refers to the *sentences* (i.e., formulas without free variables) from Φ that hold in G. For example $\mathrm{tp}(G, \mathrm{FO}_q)$ are the FO-sentences of quantifier rank at most q that hold in G.

2.3 Stability

Fix a logic $\mathcal{L}$ and a signature Σ. An $\mathcal{L}[\Sigma]$-formula $\varphi(\bar{x}, \bar{y})$ has the *ℓ-order-property* on a Σ-structure G, if there exists a sequence $(\bar{a}_i)_{i \in [\ell]}$ of tuples of elements of G, such that for all $i, j \in [\ell]$

$$G \models \varphi(\bar{a}_i, \bar{a}_j) \quad \Leftrightarrow \quad i \leqslant j.$$

The formula φ has the *order-property* on a class of Σ-structures $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is $G \in \mathcal{C}$ such that φ has the ℓ-order-property on G. We call a class of Σ-structures *$\mathcal{L}$-stable*, if no $\mathcal{L}[\Sigma]$-formula has the order-property on $\mathcal{C}$. Moreover, $\mathcal{C}$ is *monadically $\mathcal{L}$-stable* if for every $k \in \mathbb{N}$, the class $\mathcal{C}^{(k)}$ of k-colorings from $\mathcal{C}$ is $\mathcal{L}$-stable.

2.4 Interpretations

For a logic $\mathcal{L}$, $d \in \mathbb{N}$, and signatures Σ_1 and Σ_2, a *d-dimensional $\mathcal{L}$-interpretation I from Σ_1 to Σ_2* consists of

- an $\mathcal{L}[\Sigma_1]$-formula $\delta_I(\bar{x})$ called the *domain formula*,
- an $\mathcal{L}[\Sigma_1]$-formula $\varphi_{I,R}(\bar{x}_1, \ldots, \bar{x}_k)$ for every k-ary relation R in Σ_2,

where $\bar{x}$ and all $\bar{x}_i$ are d-tuples. Given a Σ_1-structure G, we define the $H := I(G)$ to be the Σ_2-structure with the universe $V(H) := \{\bar{a} \in V(G)^d : G \models \delta_I(\bar{a})\}$ and relations $R(H) := \{\bar{a}_1 \ldots \bar{a}_k \in V(H)^k : G \models \varphi_{I,R}(\bar{a}_1, \ldots, \bar{a}_k)\}$ for every k-ary relation R in Σ_2.

We will use interpretations as a reduction mechanism. The following lemma is crucial for this purpose. It is easy to prove by an inductive formula rewriting procedure. See, e.g., [24, Thm. 4.3.1].

Lemma 2.1. *For every d-dimensional FO-interpretation I from Σ_1 to Σ_2, and $\mathrm{FO}[\Sigma_2]$-formula $\varphi(\bar{x})$ there exists an $\mathrm{FO}[\Sigma_1]$-formula $\varphi_I(\bar{x})$ such that for every Σ_1-structure G and tuple $\bar{a} \in V(I(G))^{|\bar{x}|}$ we have $I(G) \models \varphi(\bar{a})$ if and only if $G \models \varphi_I(\bar{a})$.*

The above lemma also holds for MSO- and CMSO-interpretations, when restricted to a single dimension. See, e.g., [9, Thm. 7.10]. Higher dimensions are problematic because MSO and CMSO cannot quantify over sets of tuples.

Lemma 2.2. *Fix $\mathcal{L} \in \{\mathrm{MSO}, \mathrm{CMSO}\}$. For every 1-dimensional $\mathcal{L}$-interpretation I from Σ_1 to Σ_2, and $\mathcal{L}[\Sigma_2]$-formula $\varphi(\bar{x})$ there exists an $\mathcal{L}[\Sigma_1]$-formula $\varphi_I(\bar{x})$ such that for every Σ_1-structure G and tuple $\bar{a} \in V(I(G))^{|\bar{x}|}$ we have $I(G) \models \varphi(\bar{a})$ if and only if $G \models \varphi_I(\bar{a})$.*

For an interpretation I and a class $\mathcal{C}$, we write $I(\mathcal{C})$ for the class $\{I(G) : G \in \mathcal{C}\}$. We say $\mathcal{C}$ *d-dimensionally $\mathcal{L}$-interprets* a class $\mathcal{D}$, if there exists a d-dimensional $\mathcal{L}$-interpretation I such that $\mathcal{D} \subseteq I(\mathcal{C})$. We have already seen in the introduction, that the class of all paths is MSO-unstable. Lemma 2.2 lifts instability to all classes that 1-dimensionally MSO-interpret it.

Lemma 2.3. *Fix $\mathcal{L} \in \{\mathrm{MSO}, \mathrm{CMSO}\}$. Every class that 1-dimensionally $\mathcal{L}$-interprets the class of all paths is $\mathcal{L}$-unstable.*

2.5 Transductions

Intuitively, a transduction is the composition of a coloring step and an interpretation. We will make this more precise now. Fix a logic $\mathcal{L}$ and signatures Σ_1, Σ_2. An *$\mathcal{L}$-transduction T from Σ_1 to Σ_2* is defined by a 1-dimensional $\mathcal{L}$-interpretation I_T from $\Sigma_1^{(k)}$ to Σ_2 for some $k \in \mathbb{N}$. It maps a Σ_1-structure G to the set of Σ_2-structures $T(G) := \{I_T(G^+) : G^+ \text{ is a } k\text{-coloring of } G\}$. For classes $\mathcal{C}$ and $\mathcal{D}$ we define $T(\mathcal{C}) := \bigcup_{G \in \mathcal{C}} T(G)$ and say $\mathcal{C}$ *$\mathcal{L}$-transduces* $\mathcal{D}$ if there exists an $\mathcal{L}$-transduction T such that $\mathcal{D} \subseteq T(\mathcal{C})$. Transductions play the role of interpretations in the monadic setting, and we have the following analog of Lemma 2.3.

Lemma 2.4. *Fix $\mathcal{L} \in \{\mathrm{MSO}, \mathrm{CMSO}\}$. Every graph class that $\mathcal{L}$-transduces the class of all paths is monadically $\mathcal{L}$-unstable.*

2.6 Flips

Fix a graph G and a partition $\mathcal{P}$ of its vertices. For every vertex $v \in V(G)$ we denote by $\mathcal{P}(v)$ the unique part $Q \in \mathcal{P}$ satisfying $v \in Q$. Let $F \subseteq \mathcal{P}^2$ be a symmetric relation. We define $H := G \oplus (\mathcal{P}, F)$ to be the graph with vertex set $V(G)$, and edges defined by the following condition, for distinct $u, v \in V(G)$:

$$uv \in E(H) \Leftrightarrow \begin{cases} uv \notin E(G) & \text{if } (\mathcal{P}(u), \mathcal{P}(v)) \in F, \\ uv \in E(G) & \text{otherwise.} \end{cases}$$

We call H a *$\mathcal{P}$-flip* of G. If $\mathcal{P}$ has at most k parts, we say that H a *k-flip* of G. Flips are reversible: $\big(G \oplus (\mathcal{P}, F)\big) \oplus (\mathcal{P}, F) = G$; and hereditary: for every k-flip H of G and $A \subseteq V(G)$, $H[A]$ is also a k-flip of $G[A]$.

It will be convenient to work with flip-partitions that are minimal in the following sense.

Definition 2.5. Let H and G be two graphs on the same vertex set. We call a partition $\mathcal{P}$ of $V(G)$ and a relation $F \subseteq \mathcal{P}^2$ *irreducible (H, G)-flip-witnesses* if

- $H = G \oplus (\mathcal{P}, F)$,
- H is not a $(|\mathcal{P}| - 1)$-flip of G,
- F does not include (Q, Q) for any part $Q \in \mathcal{P}$ with $|Q| = 1$.

Clearly for every two graphs H and G on the same vertex set, there exist irreducible (H, G)-flip-witnesses. In particular, as we work with loopless graphs, every flip relation F can be modified to satisfy the last condition stating that no singleton part is flipped with itself. For the rest of this subsection, we argue that the irreducible flip-witnesses are uniquely determined.

Lemma 2.6. *Let H and G be two graphs on the same vertex set and $\mathcal{P}$ and F be irreducible (H, G)-witnesses. For every two distinct parts $Q_1, Q_2 \in \mathcal{P}$ there exists a part $Q_\Delta \in \mathcal{P}$ such that*

$$(Q_1, Q_\Delta) \in F \Leftrightarrow (Q_2, Q_\Delta) \notin F.$$

We say Q_Δ discerns *Q_1 and Q_2.*

Proof. TOPROVE 1 □

Lemma 2.7. *Let H and G be two graphs on the same vertex set, let $\mathcal{P}$ and F be irreducible (H, G)-flip-witnesses, and let $\mathcal{P}'$ and F' any partition and relation such that $H = G \oplus (\mathcal{P}', F')$. Then $\mathcal{P}$ is a coarsening of $\mathcal{P}'$.*

Proof. TOPROVE 2 □

Lemma 2.8. *For every two graphs H and G on the same vertex set, the irreducible (H, G)-flip-witnesses are uniquely determined.*

Proof. TOPROVE 3 □

For every two graph H and G on the same vertex set, we call $\mathcal{P}$ the *irreducible (H, G)-flip-partition* and F the *irreducible (H, G)-flip-relation*, if $\mathcal{P}$ and F are the unique irreducible (H, G)-flip-witnesses, as justified by Lemma 2.8.

2.7 Shrub-Depth and SC-Depth

Shrub-depth [20, 21] is a parameter for graph classes. Unlike for example tree-width, one cannot meaningfully measure the shrub-depth of a single graph. In this paper we will not define shrub-depth, but instead work with the functionally equivalent *SC-depth*, which was introduced in [20] and is definable for single graphs. We first need to define the eponymous notion of a *set-complementation*. A graph H is a set complementation of a graph G, if H can be obtained from G by complementing the edges on a subset of vertices. In the language of flips: $H = G \oplus (\{A, V(G) \setminus A\}, \{(A, A)\})$ for some set $A \subseteq V(G)$.

The single vertex graph K_1 has SC-depth 0. A graph has SC-depth at most d if it is a set-complementation of a disjoint union of (arbitrarily many) graphs of SC-depth at most $d - 1$. The SC-depth of a graph G is the smallest value d such that G has SC-depth at most d. A graph class $\mathcal{C}$ has *bounded SC-depth* if there is $d \in \mathbb{N}$ such that every graph in $\mathcal{C}$ has SC-depth at most d.

Theorem 2.9 ([20])**.** *A graph class has bounded SC-depth if and only if it has bounded shrub-depth.*

2.8 Distances and Flatness

The *length* of a path is the number of edges it contains, i.e., P_t has length $t - 1$. For a graph G and two vertices $u, v \in V(G)$ we define $\mathrm{dist}_G(u, v)$ to be the length of a shortest path between u and v in G. If no such path exists, then u and v are in different connected components of G, and we define $\mathrm{dist}_G(u, v) := \infty$. For $r \in \mathbb{N}$, a graph G, and a vertex $v \in V(G)$ let $N_r^G[v] := \{u \in V(G) : \mathrm{dist}_G(u, v) \leqslant r\}$ be the *(closed) radius-r neighborhood of v*. We drop the superscript G if it is clear from the context. For $r \in \mathbb{N}$, we call a set $A \subseteq V(G)$ *distance-r independent* in G if $\mathrm{dist}_G(u, v) > r$ for every two distinct vertices $u, v \in A$. Similarly, A is *distance-∞ independent* in G if no two vertices of A are in the same connected component of G.

Definition 2.10. For $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$, a set A of vertices in a graph is *(r, k)-flip-flat*, if there exists a k-flip H of G in which A is distance-r independent. A graph class $\mathcal{C}$ is *r-flip-flat*, if there are *margins* $k_r \in \mathbb{N}$ and $M_r : \mathbb{N} \to \mathbb{N}$, such that for every $G \in \mathcal{C}$ and $m \in \mathbb{N}$, every set $W \subseteq V(G)$ of size $|W| \geqslant M_r(m)$ contains an (r, k_r)-flip-flat subset of size m.

We refer to [28] for an introduction to flatness properties. Crucially, flip-flatness characterizes both monadic FO-stability [13] and bounded shrub-depth [14].

Theorem 2.11 ([13] and [14]). *For every graph class $\mathcal{C}$:*

- *$\mathcal{C}$ is FO-stable if and only if $\mathcal{C}$ is r-flip-flat for every $r \in \mathbb{N}$.*
- *$\mathcal{C}$ has bounded shrub-depth if and only if $\mathcal{C}$ is ∞-flip-flat.*

3 Forbidden Induced Subgraphs

Flipped half-graphs H_t and flipped tP_ts were defined in the introduction (Definitions 1.4 and 1.5). We say a graph class $\mathcal{C}$ is *pattern-free*, if there is a $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes as induced subgraphs every flipped H_t and every flipped tP_t. The goal of this section is to show the following.

Proposition 3.1. *Every pattern-free class is ∞-flip-flat.*

Before we dive into the proof of Proposition 3.1, let us quickly sketch an argument showing the reverse direction: every ∞-flip-flat class is pattern-free. We will be brief here, as this direction will later also be implied by our other proofs (see Figure 4). By Theorems 2.9 and 2.11, it suffices to show that every class that is not pattern-free has unbounded SC-depth. It is well known that (flipped) half-graphs have unbounded SC-depth. To show the same for flipped tP_ts, it suffices to prove the following two lemmas. Together they imply that any SC-depth decomposition of a flipped tP_t has a branch whose depth is unbounded in t.

Lemma 3.2. *Every set-complementation of a flipped sP_t contains a flipped tP_t for $s := t \cdot 2^t$.*

Proof. TOPROVE 4 □

Lemma 3.3. *Every flipped tP_s has a connected component containing a flipped tP_t for $s := t^2+t-1$.*

Proof. TOPROVE 5 □

3.1 Establishing Monadic FO-Stability

As a first step, we will show that pattern-free classes are monadically FO-stable. This will follow from a known characterization of monadic FO-stability by forbidden induced subgraphs. We first introduce the required definitions.

For $r \geqslant 1$, the *star r-crossing* of order t is the r-subdivision of $K_{t,t}$ (the biclique of order t). More precisely, it consists of *roots* $a_1, \ldots, a_t$ and $b_1, \ldots, b_t$ together with t^2 many pairwise vertex-disjoint r-vertex paths $\{\pi_{i,j} : i, j \in [t]\}$, whose endpoints we denote as $\mathrm{start}(\pi_{i,j})$ and $\mathrm{end}(\pi_{i,j})$. Each root a_i is adjacent to $\{\mathrm{start}(\pi_{i,j}) : j \in [t]\}$, and each root b_j is adjacent to $\{\mathrm{end}(\pi_{i,j}) : i \in [t]\}$. See Figure 5. The *clique r-crossing* of order t is the graph obtained from the star r-crossing of order t by turning the neighborhood of each root into a clique. In order to define *flipped* versions of star/clique r-crossings, we partition their vertices into *layers* $\mathcal{L} = \{L_0, \ldots, L_{r+1}\}$: The 0th layer consists of the vertices $\{a_1, \ldots, a_t\}$. The lth layer, for $l \in [r]$, consists of the lth vertices of the paths $\{\pi_{i,j} : i, j \in [t]\}$. Finally, the $(r+1)$th layer consists of the vertices $\{b_1, \ldots, b_n\}$. A *flipped* star/clique r-crossing is a graph obtained from a star/clique r-crossing by performing a flip where the parts of the flip-partition are the layers of the r-crossing. The following characterization was originally proven in [11], but we refer to [28] for this formulation.

Theorem 3.4 ([11]). *A graph class is monadically FO-stable if and only if for every $r \geqslant 1$ there exists $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes as induced subgraphs*

- *all flipped star r-crossings of order t, and*

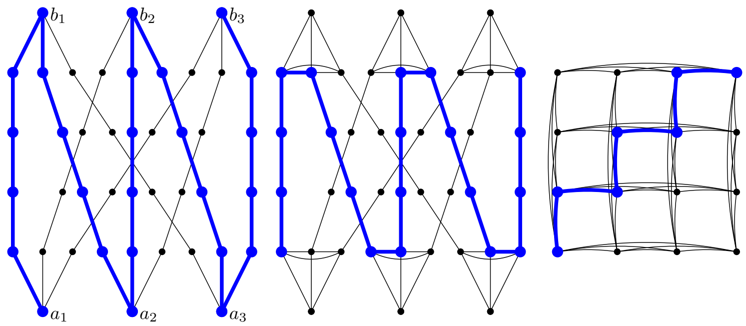

Figure 5: From left to right: the star 4-crossing of order 3, the clique 4-crossing of order 3, and the rook graph of order 4. Highlighted in blue we show how each of the three patterns of order k contains an induced path on at least k vertices.

- *all flipped clique r-crossings of order t, and*
- *all flipped H_t.*

Lemma 3.5. *For every $r \geqslant 1$ and $t \in \mathbb{N}$, the star r-crossing and the clique r-crossing of order t both contain an induced P_t.*

Proof. TOPROVE 6 □

Lemma 3.6. *For every monadically FO-unstable graph class $\mathcal{C}$ there exists $k \in \mathbb{N}$ such that $\mathcal{C}$ contains as induced subgraphs either*

- *a flipped H_t for every $t \in \mathbb{N}$, or*
- *a k-flip of P_t for every $t \in \mathbb{N}$.*

Proof. TOPROVE 7 □

Lemma 3.7. *Every k-flip of $(m \cdot k^t)P_t$ contains an induced flipped mP_t, for every $k, m, t \in \mathbb{N}$.*

Proof. TOPROVE 8 □

By cutting a long path into multiple shorter ones, we observe that $P_{t'}$ contains an induced mP_t for $t' := m \cdot (t+1)$. Combining this observation with Lemma 3.7 yields the following corollary.

Corollary 3.8. *Every k-flip of $P_{t'}$ contains an induced flipped tP_t for every $k, t \in \mathbb{N}$ and $t' := t \cdot k^t \cdot (t+1)$.*

Combining this with Lemma 3.6 yields the following.

Proposition 3.9. *Every pattern-free class is monadically FO-stable.*

3.2 Proving ∞-Flip-Flatness

Lemma 3.10. *For every $m, t \in \mathbb{N}$, graph G, and distance-$2t$ independent set A of size $2m$ in G either*

- *a size m subset of A is distance-∞ independent in G, or*
- *G contains an induced mP_t.*

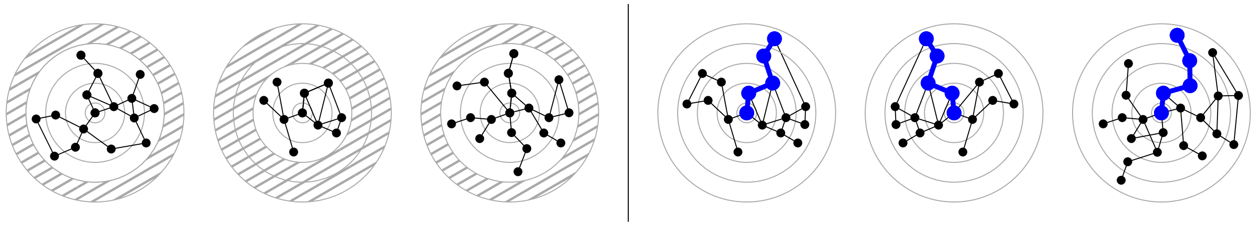

Figure 6: The two cases from Lemma 3.10. The circles are the layering of a BFS tree around the vertices of A. There is a large subset $A' \subseteq A$ whose outermost layers are either all empty (left panel), or all non-empty (right panel). Therefore, the vertices of A' are either in pairwise different connected components, or the vertices of A' are the endpoints of pairwise vertex-disjoint long induced paths.

Proof. TOPROVE 9 □

Using Lemma 3.7, the previous lemma lifts to flipped graphs.

Corollary 3.11. *For every $k, m, t \in \mathbb{N}$, graph G, k-flip H of G, and distance-$2t$ independent set A of size $2k^t \cdot m$ in H either*

- *a size $k^t \cdot m$ subset of A is distance-∞ independent in H, or*
- *G contains an induced flipped mP_t.*

We are now ready to prove Proposition 3.1, which we restate for convenience.

Proposition 3.1. *Every pattern-free class is ∞-flip-flat.*

Proof. TOPROVE 10 □

4 Interpreting Paths

In this section we will prove the following.

Proposition 4.1. *There is a 1-dimensional FO-interpretation I such that for every hereditary graph class $\mathcal{C}$, that for every $t \in \mathbb{N}$ contains either a flipped H_t or a flipped tP_t, the class of all paths is contained in $I(\mathcal{C})$.*

Notably, a single interpretation works for all classes that contain these patterns. We first show that the paths of length at most 3 appear already as induced subgraphs.

Lemma 4.2. *Every flipped $2P_3$ and every flipped H_3 contains P_3 as an induced subgraph.*

Proof. TOPROVE 11 □

4.1 Interpreting Paths in Flipped tP_ts

Two vertices u and v are *twins* in a graph G, if $N_1^G[u] \setminus \{u, v\} = N_1^G[v] \setminus \{u, v\}$. This relation is FO-definable:

$$\text{twins}(x, y) := \forall z : (z \neq x \wedge z \neq y) \to (E(z, x) \leftrightarrow E(z, y)).$$

Observation 4.3. Let H be a $\mathcal{P}$-flip of G. Two vertices $u, v \in V(G)$ that are contained in the same part of $\mathcal{P}$ are twins in G if and only if they are twins in H.

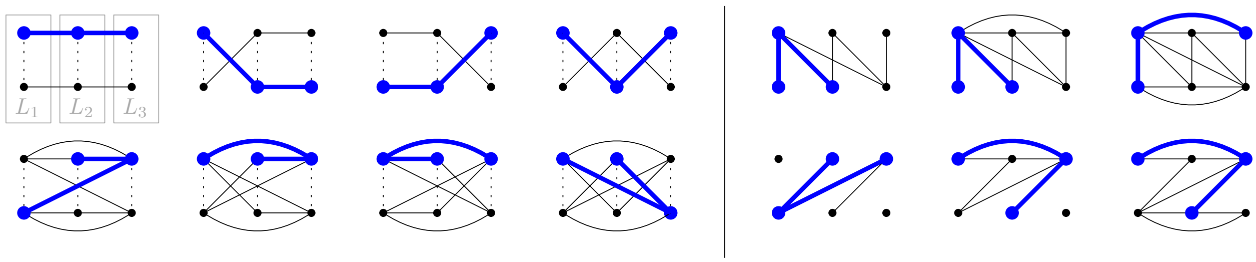

Figure 7: On the left: An enumeration of all flipped $2P_3$. To reduce the number of cases, we do not account for flips that flip a layer L_i with itself. Edges that could possibly be created by such flips are marked as dashed. In each flipped $2P_3$ we have highlighted an induced P_3 that contains no dashed edges. This proves that every flipped $2P_3$ contains an induced P_3. On the right: An enumeration of all flipped H_3, each with a highlighted induced P_3.

Lemma 4.4. *There exists an FO-interpretation I such that for every $t \geqslant 4$, every flipped $5P_t$ contains an induced subgraph H such that $I(H) = P_t$.*

Moreover, H contains at least 8 vertices that have a twin in H.

The "moreover" part of the lemma is later used to construct a single interpretation that can distinguish whether the input graph is an induced subgraph of a flipped tP_t or of a flipped H_t.

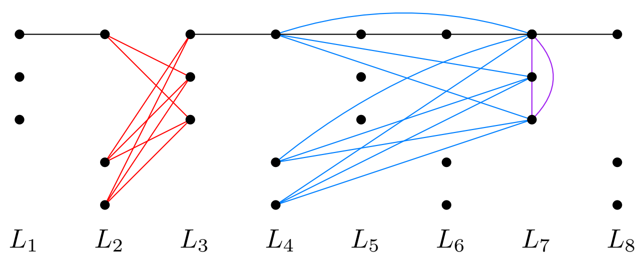

Figure 8: The induced subgraph of a flipped $5P_8$ in which the interpretation I can interpret P_8. In this example the following layers were flipped: L_2 with L_3 (red), L_4 with L_7 (blue), L_7 with L_7 (purple).

Proof. TOPROVE 12 ☐

4.2 Interpreting Paths in Flipped Half-Graphs

Definition 4.5. We call a flipped H_t *clean* if the adjacency between $A := \{a_1, \dots, a_t\}$ and $B := \{b_1, \dots, b_t\}$ was not flipped. More precisely a graph G is a clean flipped H_t if $G = H_t \oplus (\{A, B\}, F)$ for some $F \subseteq \{(A, A), (B, B)\}$.

Lemma 4.6. *Every flipped H_{t+1} contains an induced clean flipped H_t.*

Proof. TOPROVE 13 ☐

Lemma 4.7. *There exists an FO-interpretation I such that for every $t \in \mathbb{N}$, every flipped H_{t+4} contains an induced subgraph G such that $I(G) = P_t$.*

Moreover, G contains exactly 4 vertices that have a twin in G.

Proof. TOPROVE 14 ☐

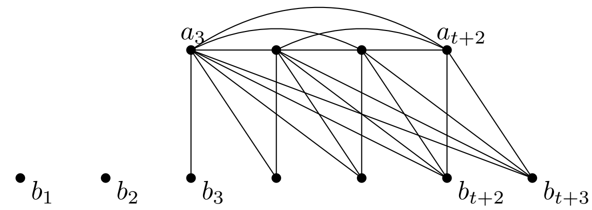

Figure 9: The graph G from the proof of Lemma 4.7 for $t = 4$. It is an induced subgraph of a flipped H_{t+3}. In this example, the set A is flipped with itself, which is the reason why there is a clique on the A vertices.

4.3 Wrapping Up the Interpretation

We are now ready to prove Proposition 4.1, which we restate for convenience.

Proposition 4.1. *There is a 1-dimensional FO-interpretation I such that for every hereditary graph class $\mathcal{C}$, that for every $t \in \mathbb{N}$ contains either a flipped H_t or a flipped tP_t, the class of all paths is contained in $I(\mathcal{C})$.*

Proof. TOPROVE 15 □

4.4 Transducing Paths in Flipped $3P_t$s

In this subsection we will sharpen our induced subgraph characterization and show the following.

Proposition 4.8. *Every graph class $\mathcal{C}$, that for every $t \in \mathbb{N}$ contains as induced subgraphs either a flipped H_t or a flipped $3P_t$, is monadically MSO-unstable.*

Note that we do not need to assume that $\mathcal{C}$ is hereditary, since we can simulate taking induced subgraphs by colors in the monadic setting.

We have already seen how to 1-dimensionally FO-interpret all paths in hereditary classes containing arbitrarily large flipped half-graphs (Lemmas 4.2 and 4.7). By simulating taking induced subgraphs using colors, this gives us an FO-transduction which producing all paths from every (not necessarily hereditary) class containing arbitrarily large flipped half-graphs. Together with Lemma 2.4, this reduces proving Proposition 4.8 to the following.

Lemma 4.9. *There is an FO-transduction T such that for every graph G that contains an induced flipped $3P_t$, we have $P_t \in T(G)$.*

Proof. TOPROVE 16 □

5 Monadic Stability

The goal of this section is to show the following.

Proposition 5.1. *Every class of bounded shrub-depth is monadically CMSO-stable.*

It was already shown in [21] that no class of bounded shrub-depth CMSO-transduces the class of all paths. This reduces proving Proposition 5.1 to the following.

Lemma 5.2. *Every monadically CMSO-unstable class CMSO-transduces the class of all paths.*

For every CMSO-unstable class $\mathcal{C}$, there is a CMSO-formula $\varphi(\bar{x},\bar{y})$ that orders arbitrarily large sequences of tuples in a coloring of $\mathcal{C}$. This is already close to what we want: the successor relation definable through φ forms a "path" on these tuples. However, we cannot directly produce this path by a transduction, as transductions only work with singletons and not with tuples. The trick is to use the color predicates that are available in the monadic setting, to show the existence of a formula $\varphi'(x,y)$ with single free variables that has the order-property on another coloring of $\mathcal{C}$. For FO-formulas, it was shown that this is possible by Baldwin and Shelah [3, Lem. 8.1.3] (see also [2, Thm. 2.2]), and Simon proved a strengthening of the statement that involves parameters instead of color predicates [36]. Both proofs work in a model theoretic setting on infinite structures. However, their core is combinatorial and generalizes from FO to more expressive logics without problems, which yields the following lemma.

Lemma 5.3. *For every logic $\mathcal{L}$ that extends FO and every class of Σ-structures $\mathcal{C}$, $\mathcal{C}$ is monadically $\mathcal{L}$-stable if and only if $\mathcal{C}$ does not $\mathcal{L}$-transduce the class of all half-graphs.*

The proof of Lemma 5.3 is mostly a translation of Simon's result [36] from the infinite to the finite setting and can be found in the appendix.

In Lemma 5.3, we say a logic $\mathcal{L}$ *extends FO* if for every relational symbol $R(\bar{x})$, FO$[\Sigma\cup\{R\}]$-formula $\varphi(\bar{y})$, and $\mathcal{L}[\Sigma]$-formula $\rho(\bar{x})$, the formula $\varphi'(\bar{y})$ obtained by replacing each occurrence of the relation $R(\bar{x})$ in φ with $\rho(\bar{x})$ is a $\mathcal{L}[\Sigma]$-formula. This definition is a bit fuzzy, as we do not define what it means to be a *logic*, but the reader will agree that FO, MSO, and CMSO all extend FO. The same holds for other natural extensions of FO, such as separator logic [34, 4].

Lemma 5.3 implies Lemma 5.2 as follows: every monadically CMSO-unstable graph class $\mathcal{C}$ CMSO-transduces the class of all half-graphs $\mathcal{H}$. Clearly, $\mathcal{H}$ CMSO-transduces the class of all paths $\mathcal{P}$. Using Lemma 2.2, we obtain a CMSO-transduction from $\mathcal{C}$ to $\mathcal{P}$.

An Alternative Proof of Proposition 5.1. Let us briefly mention a second way of proving Proposition 5.1, by showing that ∞-flip-flat classes are monadically CMSO-stable. Through iteration, we can prove a variant of ∞-flip-flatness for tuples. Then we can use Feferman-Vaught style theorems [29] to show that for a fixed $k\in\mathbb{N}$, no CMSO-formula can order arbitrarily large (∞,k)-flip-flat sequence of tuples. This mirrors the use of Gaifman's locality theorem [17] in the setting of monadically FO-stable classes [13].

Monadic Dependence. *Dependence* is another model theoretic dividing line, which generalizes stability. A formula $\varphi(\bar{x},\bar{y})$ has the *ℓ-independence-property* on a structure G, if there exist elements $\bar{a}_i\in V(G)^{|\bar{x}|}$ for each $i\in[\ell]$ and $\bar{b}_S\in V(G)^{|\bar{y}|}$ for each $S\subseteq[\ell]$ such that for $i\in[\ell], S\subseteq[\ell]$

$$G\models\varphi(\bar{a}_i,\bar{b}_S)\Leftrightarrow i\in S.$$

Similarly, φ has the *independence-property* on a class of structures $\mathcal{C}$, if for every $\ell\in\mathbb{N}$, there exists a structure in $\mathcal{C}$ on which φ has the ℓ-independence-property. We say $\mathcal{C}$ is *$\mathcal{L}$-dependent*[5] if no $\mathcal{L}$-formula has the independence-property on $\mathcal{C}$. The definition of a *monadically* $\mathcal{L}$-independent class is as expected. Simon proves his result not only for stability, but also for dependence [36]. The translation to the finite that we do in the appendix yields the following analog of Lemma 5.3 for dependence.

Lemma 5.4. *For every logic $\mathcal{L}$ that extends FO and every class of Σ-structures $\mathcal{C}$, $\mathcal{C}$ is monadically $\mathcal{L}$-dependent if and only if $\mathcal{C}$ does not $\mathcal{L}$-transduce the class of all graphs.*

[5] Dependence is also known as *NIP*, which stands for "Not the Independence Property".

FO-dependence has recently gained traction in structural graph theory [14, 6, 5], and we believe that Lemma 5.4 will be useful for the future study of (C)MSO-dependence, which we comment on in the outlook section of this paper.

6 The Expressive Power of MSO

The goal of this section is to prove the following.

Proposition 6.1. *MSO is more expressive than FO on every hereditary graph class, that for every $t \in \mathbb{N}$ contains either a flipped H_t or a flipped tP_t,*

By the pigeonhole principle and hereditariness, it is sufficient to prove the above lemma in two separate cases: either the graph class contains arbitrarily large flipped H_t or it contains arbitrarily large flipped tP_t. We will do so in the upcoming Lemmas 6.8 and 6.9. For this purpose we first show how interpretations can be used to prove inexpressibility results.

6.1 Interpretations and Inexpressibility

Lemma 6.2. *For every FO-interpretation I from Σ_1 to Σ_2 and $q \in \mathbb{N}$ there exists $q' \in \mathbb{N}$ such that for every two Σ_1-structures G and H*

$$\mathrm{tp}(G, \mathrm{FO}_{q'}) = \mathrm{tp}(H, \mathrm{FO}_{q'}) \Rightarrow \mathrm{tp}(I(G), \mathrm{FO}_q) = \mathrm{tp}(I(H), \mathrm{FO}_q).$$

Proof. TOPROVE 17 □

Denote by $\mathfrak{L}_t$ the *linear order of length* t represented as the structure with universe $[t]$ and a single binary relation $<$ interpreted as expected. FO cannot distinguish long linear orders:

Theorem 6.3 ([27, Thm. 3.6])**.** *For every $q \geqslant 0$ and $s, t \geqslant 2^q$, we have* $\mathrm{tp}(\mathfrak{L}_s, \mathrm{FO}_q) = \mathrm{tp}(\mathfrak{L}_t, \mathrm{FO}_q)$.

Lemma 6.4. *There exists a 1-dimensional FO-interpretation I with $I(\mathfrak{L}_t) = P_t$ for every $t \in \mathbb{N}$.*

Proof. TOPROVE 18 □

Lemma 6.5. *There is an MSO-sentence φ_{even} such that for every $t \in \mathbb{N}$, $P_t \models \varphi_{\mathrm{even}}$ if and only if t is even.*

Proof. TOPROVE 19 □

Corollary 6.6. *There is an MSO-sentence $\varphi_{\mathrm{sameParity}}$ such that for every graph G that is a disjoint union of paths, $G \models \varphi_{\mathrm{sameParity}}$ if and only if*

- *all connected components of G are paths of even length, or*
- *all connected components of G are paths of odd length.*

Proof. TOPROVE 20 □

As an example, we will now show how to use interpretations to separate MSO from FO on the class of all paths.

Lemma 6.7. *MSO is more expressive than FO on the class of all paths.*

Proof. TOPROVE 21 □

6.2 Separating FO and MSO on Flipped Half-Graphs

Lemma 6.8. *Let $\mathcal{C}$ be a hereditary graph class that contains a flipped H_t for every $t \in \mathbb{N}$. Then MSO is more expressive than FO on $\mathcal{C}$.*

Proof. TOPROVE 22 $\square$

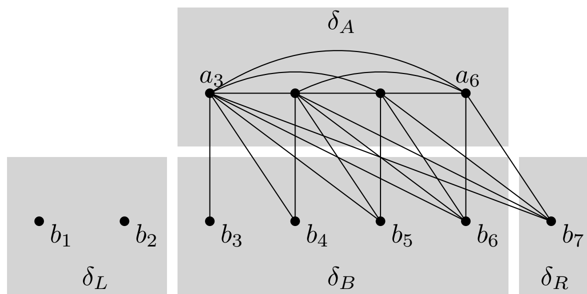

Figure 10: $H_4^\star$ together with a partitioning of its vertex set by the domain formulas.

6.3 Separating FO and MSO on Flipped tP_ts

In this subsection we prove the following.

Lemma 6.9. *Let $\mathcal{C}$ be a hereditary graph class that contains a flipped tP_t for every $t \in \mathbb{N}$. Then MSO is more expressive than FO on $\mathcal{C}$.*

We recall the definition of a (flipped) mP_t as we need to precisely refer to its vertices.

Definition 1.4. P_t is the t-vertex path and mP_t is the disjoint union of m many P_t, with vertices $[m] \times [t]$, where (i, j) is the jth vertex on the ith path. A *flipped* mP_t is an $\mathcal{L}$-flip of mP_t for the partition $\mathcal{L} = \{L_1, \dots, L_t\}$ of the paths into *layers*, where $L_j = \{(1, j), \dots, (m, j)\}$ contains the jth vertices of all the paths for $j \in [t]$.

Let G be a flipped mP_t with $m \geqslant 2$ and $t \geqslant 3$. For $i \in \{1, 2\}$, we denote by $\mathrm{nibble}_i(G)$ the graph obtained by removing from G the vertices $(1, 1)$ and (i, t). Crucially, FO cannot distinguish between the two nibbles of a flipped mP_t for large t, as made precise by the following lemma.

Lemma 6.10. *For every $q \geqslant 1$ and every graph G that is a flipped mP_t with $m \geqslant 2$ and $t \geqslant 3^q$*

$$\mathrm{tp}(\mathrm{nibble}_1(G), \mathrm{FO}_q) = \mathrm{tp}(\mathrm{nibble}_2(G), \mathrm{FO}_q).$$

The proof of Lemma 6.10 requires some additional tooling and is deferred to Section 6.4.

Lemma 6.11. *There exists an MSO sentence φ such that for every $m \geqslant 9$, $t \geqslant 3$, and graph G that is a flipped mP_t*

$$\mathrm{nibble}_1(G) \models \varphi \quad \textit{and} \quad \mathrm{nibble}_2(G) \not\models \varphi.$$

Proof. TOPROVE 23 $\square$

We can now prove Lemma 6.9.

Lemma 6.9. *Let $\mathcal{C}$ be a hereditary graph class that contains a flipped tP_t for every $t \in \mathbb{N}$. Then MSO is more expressive than FO on $\mathcal{C}$.*

Proof. TOPROVE 24 $\square$

6.4 Proof of Lemma 6.10

An *isomorphism type* for a signature Σ is an equivalence class for the "is isomorphic to" relation on Σ-structures, i.e., two Σ-structures have the same isomorphism type if and only if they are isomorphic. The signature Σ will often be clear from the context and omitted.

For a k-colored graph G, $r \in \mathbb{N}$, $v \in V(G)$, the *r-ball marked at v in G* is the substructure induced by the radius-r neighborhood $N_r[v]$ in G, where v is marked as a constant. This means the r-ball marked at v is a structure over the signature $\Gamma^{(k)} \cup \{c\}$ with an additional constant symbol c. For an isomorphism type τ, a k-colored graph G, and $r \in \mathbb{N}$, we write $\#(\tau, r, G)$ to denote the number of vertices v in G such that the r-ball marked at v in G has isomorphism type τ.

The following theorem was proven by Fagin, Stockmeyer, and Vardi [16, Thm. 4.3] in the more general setting of arbitrary finite structures (and not just colored graphs). It is based on the ideas of Hanf [23] for infinite structures. See also [27, Thm. 4.24].

Theorem 6.12 (Finite Hanf locality)**.** *For every $q, d \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, k-colored graphs G_1 and G_2 with maximum degree d if*

$$\#(\tau, 3^{q-1}, G_1) = \#(\tau, 3^{q-1}, G_2) \quad \text{or} \quad \big(\#(\tau, 3^{q-1}, G_1) \geqslant m \text{ and } \#(\tau, 3^{q-1}, G_2) \geqslant m\big)$$

for every isomorphism type τ, then $\mathrm{tp}(G_1, \mathrm{FO}[\Gamma^{(k)}]_q) = \mathrm{tp}(G_2, \mathrm{FO}[\Gamma^{(k)}]_q)$.

We will only use the theorem in cases where we can guarantee that the number of isomorphism types is exactly the same. We can therefore use the following simplified version of the theorem, obtained by setting d to be the maximum of the maximum degrees of G_1 and G_2.

Corollary 6.13. *For every $q, k \in \mathbb{N}$ and every two k-colored graphs G_1 and G_2, if $\#(\tau, 3^{q-1}, G_1) = \#(\tau, 3^{q-1}, G_2)$ for every isomorphism type τ, then $\mathrm{tp}(G_1, \mathrm{FO}[\Gamma^{(k)}]_q) = \mathrm{tp}(G_2, \mathrm{FO}[\Gamma^{(k)}]_q)$.*

We have gathered all the ingredients to prove Lemma 6.10, which we restate for convenience.

Lemma 6.10. *For every $q \geqslant 1$ and every graph G that is a flipped mP_t with $m \geqslant 2$ and $t \geqslant 3^q$*

$$\mathrm{tp}(\mathrm{nibble}_1(G), \mathrm{FO}_q) = \mathrm{tp}(\mathrm{nibble}_2(G), \mathrm{FO}_q).$$

Proof. TOPROVE 25 □

7 Outlook: Stronger Obstructions and MSO-Dependence

In this work, we have characterized graph classes of bounded shrub-depth by forbidden induced subgraphs. This has allowed us to derive further characterizations through logic, for instance by the model theoretic property of MSO-stability, and by comparing the expressive power of FO and MSO. While the obstructions we found were sufficient to achieve our goal of characterizing MSO-stability, we initially had stronger obstructions in mind, which we could neither prove nor refute:

Conjecture 7.1. *For every graph class $\mathcal{C}$ of unbounded shrub-depth there exists $k \in \mathbb{N}$ such that $\mathcal{C}$ contains as induced subgraphs either*

- *a flipped H_t for every $t \in \mathbb{N}$, or*
- *a k-flip of P_t for every $t \in \mathbb{N}$.*

As we have seen in Lemma 3.6, this conjecture holds for every class that is monadically FO-unstable. Even stronger, the k-flips of P_ts appearing there can be assumed to be *periodic*. This means, the flip-partition splits the paths in a repeating pattern. This is due to the fact that we have extracted the k-flips of P_t, by "snaking" along the known obstructions characterizing monadic FO-stability (see Figure 5). The advantage of these periodic flips is that they can be finitely described: for every monadically FO-unstable class $\mathcal{C}$ there exists an algorithm that given $t \in \mathbb{N}$, returns a size-t obstruction (a flipped H_t or a k-flip of P_t) that is contained in $\mathcal{C}$. We currently do not know whether classes of unbounded shrub-depth admit such algorithms, as we have little control over how the tP_ts are flipped. Having such a finite description would help to lift algorithmic hardness results from the class of all paths to every hereditary classes of unbounded shrub-depth. One such result is by Lampis, who shows that there is no fpt MSO model checking algorithm for the class of all paths whose runtime dependence is elementary on the size of the input formula, under the complexity theoretic assumption E $\neq$ NE [26]. Lifting this hardness result to hereditary classes of unbounded shrub-depth would yield another characterization, as it is known that every class of bounded shrub-depth admits elementary fpt MSO model checking [18, 20].

A second interesting question concerns the model theoretic notion of *dependence* (see Section 5 for a definition). Similar to FO-stable classes, also FO-dependent classes have recently been shown to admit nice combinatorial characterizations [14]. This raises the question whether also MSO-dependence can be combinatorially characterized. It is natural to conjecture the following.

Conjecture 7.2. *For every hereditary graph class $\mathcal{C}$, the following are equivalent.*

1. *$\mathcal{C}$ has bounded clique-width (or equivalently bounded rank-width).*
2. *$\mathcal{C}$ is MSO-dependent.*
3. *$\mathcal{C}$ is monadically MSO-dependent.*
4. *$\mathcal{C}$ is CMSO-dependent.*
5. *$\mathcal{C}$ is monadically CMSO-dependent.*

It is already known that a class has bounded clique-width if and only if it does not CMSO-transduce the class of all graphs [10]. By Lemma 5.4, this confirms the equivalence (1) $\Leftrightarrow$ (5) of the conjecture. We remark that, again by Lemma 5.4, both of the two equivalence (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3) would imply that every class of unbounded clique-width MSO-transduces the class of all graphs. In turn, this would imply the longstanding Seese's conjecture stating that the MSO-theory of every graph class of unbounded clique-width is undecidable [35, 9]. This suggests that Conjecture 7.2 is a tough nut to crack.

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A Monadic Stability via Transductions

The goal of this section is to prove the following lemma used in Section 5.

Lemma 5.3. *For every logic $\mathcal{L}$ that extends FO and every class of Σ-structures $\mathcal{C}$, $\mathcal{C}$ is monadically $\mathcal{L}$-stable if and only if $\mathcal{C}$ does not $\mathcal{L}$-transduce the class of all half-graphs.*

Fix a logic $\mathcal{L}$ and a signature Σ. For $\ell \in \mathbb{N} \cup \{\infty\}$, an $\mathcal{L}[\Sigma]$-formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ has the *bipartite ℓ-order-proprety* on a Σ-structure G if there exist tuples $\bar{a}_i \in V(G)^{|\bar{x}|}$, $\bar{b}_i \in V(G)^{|\bar{y}|}$ for every $i \in [\ell]$, and $\bar{c} \in V(G)^{|\bar{z}|}$, such that for all $i, j \in [\ell]$

$$G \models \varphi(\bar{a}_i, \bar{b}_j, \bar{c}) \Leftrightarrow i \leqslant j.$$

The formula φ has the *bipartite order-property* on a class of Σ-structures $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is a structure in $\mathcal{C}$ on which φ has the bipartite ℓ-order-property. Note that whenever a formula $\varphi(\bar{x}, \bar{y})$ has the ℓ-order-property witnessed by tuples $\bar{d}_i$ on a structure G, then φ also has the bipartite ℓ-order-property on G witnessed by the tuples $\bar{a}_i = \bar{b}_i = \bar{d}_i$ and where the parameters $\bar{c}$ are the empty tuple.

A *theory* is a set of FO-sentences. A structure M is a *model* of a theory T, if M satisfies all sentences in T. The compactness theorem states that a theory T has a model if and only if every finite subset of T has a model. A formula $\varphi(\bar{x}, \bar{z}, \bar{y})$ has the *bipartite order-property* on a theory T if there exists a model M of T on which φ has the bipartite ∞-order-property.

The following theorem is by Simon [36].

Theorem A.1 ([36])**.** *For every theory T, if there is an FO-formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ that has the bipartite order-property on T, then there also exists an FO-formula $\psi(x, y, \bar{z}')$ with singleton variables x and y that has the bipartite order-property on T.*

We translate it from the infinite setting of theories to classes of finite structures, and extend it to more expressive logics.

Lemma A.2. *For every logic $\mathcal{L}$ that extends FO, signature Σ, $\mathcal{L}[\Sigma]$-formula $\varphi(\bar{x}, \bar{y}, \bar{z})$, and class of Σ-structures $\mathcal{C}$, if φ has the bipartite order-property on $\mathcal{C}$, then there also exists an $\mathcal{L}[\Sigma]$-formula $\psi(x, y, \bar{z}')$ with singleton variables x and y that has the bipartite order-property on $\mathcal{C}$.*

Proof. TOPROVE 26 ☐

We are now ready to prove Lemma 5.3

Proof. TOPROVE 27 ☐

In [36], Simon also proves an analog of Theorem A.1, showing that also the *independence-property* (cf. Section 5) is always witnessed by formulas $\psi(x, y, \bar{z})$ with two singleton free variables and parameters. Following the proofs above, we obtain the following lemma.

Lemma 5.4. *For every logic $\mathcal{L}$ that extends FO and every class of Σ-structures $\mathcal{C}$, $\mathcal{C}$ is monadically $\mathcal{L}$-dependent if and only if $\mathcal{C}$ does not $\mathcal{L}$-transduce the class of all graphs.*