On r-wise t-intersecting uniform families

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Abstract

We consider families, $\mathcal{F}$ of k-subsets of an n-set. For integers $r \geq 2$, $t \geq 1$, $\mathcal{F}$ is called r-wise t-intersecting if any r of its members have at least t elements in common. The most natural construction of such a family is the full t-star, consisting of all k-sets containing a fixed t-set. In the case $r = 2$ the Exact Erdős-Ko-Rado Theorem shows that the full t-star is largest if $n \geq (t+1)(k-t+1)$. In the present paper, we prove that for $n \geq (2.5t)^{1/(r-1)}(k-t)+k$, the full t-star is largest in case of $r \geq 3$. Examples show that the exponent $\frac{1}{r-1}$ is best possible. This represents a considerable improvement on a recent result of Balogh and Linz.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set. Let $2^{[n]}$ denote the power set of $[n]$ and let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. A subset $\mathcal{F} \subset \binom{[n]}{k}$ is called a *k-uniform family*.

The central notion of this paper is that of r-wise t-intersecting.

Definition 1.1. For positive integers r, t, $r \geq 2$, a family $\mathcal{F} \subset 2^{[n]}$ is called r-wise t-intersecting if $|F_1 \cap F_2 \cap \ldots \cap F_r| \geq t$ for all $F_1, F_2, \ldots, F_r \in \mathcal{F}$.

Let us define

$$m(n, r, t) = \max\left\{|\mathcal{F}| \colon \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-intersecting}\right\},$$
$$m(n, k, r, t) = \max\left\{|\mathcal{F}| \colon \mathcal{F} \subset \binom{[n]}{k} \text{ is } r\text{-wise } t\text{-intersecting}\right\}.$$

Let us define the so-called Frankl families (cf. [7])

$$\mathcal{A}_i(n, r, t) = \{A \subset [n] \colon A \cap [t+ri] \geq t + (r-1)i\},\ 0 \leq i \leq \frac{k-t}{r},$$
$$\mathcal{A}_i(n, k, r, t) = \mathcal{A}_i(n, t) \cap \binom{[n]}{k}.$$

Since $\mathcal{A}_i(n, r, t)$ consists of the sets A satisfying $|[t+ri] \setminus A| \leq i$, that is, sets that leave out at most i elements out of the first $t + ri$, $|A_1 \cap \ldots \cap A_r \cap [t+ri]| \geq t + ri - ri \geq t$ for all $A_1, \ldots, A_r \in \mathcal{A}_i(n, r, t)$.

Conjecture 1.2 ([7]).

$$m(n,r,t) = \max_i |\mathcal{A}_i(n,r,t)|; \tag{1.1}$$

$$m(n,k,r,t) = \max_i |\mathcal{A}_i(n,k,r,t)|. \tag{1.2}$$

Let us note that for $r = 2$ the statement (1.1) is a consequence of the classical Katona Theorem [21].

Theorem 1.3 (The Katona Theorem [21]).

$$m(n,2,t) = |\mathcal{A}_{\lfloor \frac{n-t}{2} \rfloor}(n,2,t)|.$$

The case $r = 2$ of (1.2) was a longstanding conjecture. It was proved in [15] for a wide range and it was completely established by the celebrated Complete Intersection Theorem of Ahlswede and Khachatrain [2].

A family $\mathcal{F} \subset \binom{[n]}{k}$ is called a *t-star* if there exists $T \subset [n]$ with $|T| = t$ such that $T \subset F$ for all $F \in \mathcal{F}$. The family $\{F \in \binom{[n]}{k} : T \subset F\}$ with some $T \in \binom{[n]}{t}$ is called a *full t-star.*

Let us recall a part of it that was proved earlier.

Theorem 1.4 (Exact Erdős-Ko-Rado Theorem [5], [9], [25]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 2-wise t-intersecting family. Then for $n \geq (t+1)(k-t+1)$,*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Moreover, for $n > (t+1)(k-t+1)$ equality holds if and only if $\mathcal{F}$ is the full t-star.

Theorem 1.4 motivates the following question that is the central problem of the present paper: determine or estimate $n_0(k,r,t)$, the minimal integer n_0 such that for all $n \geq n_0$ and all r-wise t-intersecting families $\mathcal{F} \subset \binom{[n]}{k}$, $|\mathcal{F}| \leq |\mathcal{A}_0(n,k,r,t)| = \binom{n-t}{k-t}$. Theorem 1.4 shows $n_0(k,2,t) = (t+1)(k-t+1)$.

Since the value $\binom{n-t}{k-t}$ is independent of r, it should be clear that $n_0(k,r,t)$ is a monotone decreasing function of r. Thus $n_0(k,r,t) \leq n_0(k,2,t) = (t+1)(k-t+1)$. For $t = 1$ the exact value of $m(n,k,r,t)$ and thereby $n_0(k,r,t)$ is known (cf. [6]):

$$m(n,k,r,1) = \begin{cases} \binom{n-1}{k-1}, & \text{if } n \geq \frac{r}{r-1}k \\ \binom{n}{k}, & \text{if } n < \frac{r}{r-1}k. \end{cases} \tag{1.3}$$

Recently, Balogh and Linz [3] showed that

$$n_0(k,r,t) < (t+r-1)(k-t-r+3).$$

The main result of the present paper is

Theorem 1.5. *For $r = 3, 4$,*

$$n_0(k,r,t) \leq (2.5t)^{\frac{1}{r-1}}(k-t) + k. \tag{1.4}$$

For $r \geq 5$,

$$n_0(k,r,t) \leq (2t)^{\frac{1}{r-1}}(k-t) + k. \tag{1.5}$$

Let us show that (1.5) is essentially best possible for $t \geq 2^r - r$ and r sufficiently large. Precisely, for $t \geq 2^r - r$ we have

$$\left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} (k-t) < n_0(k,r,t) \leq (2t)^{\frac{1}{r-1}} (k-t) + k.$$

Let us prove the lower bound by showing that $|\mathcal{A}_1(n,k,r,t)| > \binom{n-t}{k-t}$ for $n = \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} (k-t-r+2) + t + r - 2$. Note that

$$|\mathcal{A}_1(n,k,r,t)| = \binom{n-t-r}{k-t-r} + (t+r)\binom{n-t-r}{k-t-r+1} = \binom{n-t-r}{k-t-r}\left(1 + \frac{(t+r)(n-k)}{k-t-r+1}\right)$$

and

$$\begin{aligned}
\frac{|\mathcal{A}_1(n,k,r,t)|}{\binom{n-t}{k-t}} &= \frac{(k-t)(k-t-1)\dots(k-t-r+1)}{(n-t)(n-t-1)\dots(n-t-r+1)}\left(1 + \frac{(t+r)(n-k)}{k-t-r+1}\right) \\
&= \frac{(k-t)(k-t-1)\dots(k-t-r+2)}{(n-t)(n-t-1)\dots(n-t-r+2)} \frac{(t+r)n - (k+1)(t+r-1)}{n-t-r+1} \\
&> \left(\frac{k-t-r+2}{n-t-r+2}\right)^{r-1} \frac{(t+r)n - (k+1)(t+r-1)}{n-t-r+1}.
\end{aligned}$$

If $t \geq 2^r - r$ then $n = \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} (k-t-r+2) + t + r - 2 \geq 2k - t - r + 2$. Let us assume $k \geq t + r$ (this is no real restriction, cf. Proposition 1.9 below). It follows that

$$\frac{(t+r)n - (k+1)(t+r-1)}{n-t-r+1} \geq (t+r)\frac{n-k-1+\frac{k+1}{t+r}}{n-t-r+1} > \frac{(t+r)(n-k)}{n-t-r+1} > \frac{t+r}{2}.$$

Thus,

$$\frac{|\mathcal{A}_1(n,k,r,t)|}{\binom{n-t}{k-t}} > \left(\frac{k-t-r+2}{n-t-r+2}\right)^{r-1} \frac{t+r}{2} = 1.$$

Therefore for $t \geq 2^r - r$ we obtain that

$$\begin{aligned}
n_0(k,r,t) &> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} (k-t-r+2) + t + r - 2 \\
&> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} (k-t) + \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} \left(2\left(\frac{t+r}{2}\right)^{\frac{r-2}{r-1}} - r\right) \\
&> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} (k-t) + \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} \left(2^{r-1} - r\right) \\
&> \left(\frac{t+r}{2}\right)^{\frac{1}{r-1}} (k-t).
\end{aligned}$$

Our next result determines $m(n,k,3,2)$ for $n > 2k \geq 4$.

Theorem 1.6. *For $n > 2k \geq 4$,*

$$m(n,k,3,2) = \binom{n-2}{k-2}. \tag{1.6}$$

Moreover, in case of equality $\mathcal{F}$ is the full 2-star.

Let us note that Balogh and Linz [3] proved this for $n \geq 4(k-2)$ and in the much older paper [16] the weaker result $m(n,k,3,2) = (1+o(1))\binom{n-2}{k-2}$ was established for $k < 0.501n$.

Let us give two more numerical examples.

Proposition 1.7. *For $n \geq 2k$,*

$$m(n,k,4,3) = \binom{n-3}{k-3} \text{ and } m(n,k,4,4) = \binom{n-4}{k-4}.$$

The next result establishes the analogue of (1.6) for a wide range of the pair (r,t).

Theorem 1.8. *Let $n \geq \max\left\{2k, \frac{t(t-1)}{2\log 2} + 2t - 1\right\}$ and $t \leq 2^{r-2}\log 2 - 2$. Then*

$$m(n,k,r,t) = \binom{n-t}{k-t}. \tag{1.7}$$

Moreover, in case of equality $\mathcal{F}$ is the full t-star.

Let us show that for $k \leq t+r-2$ the only r-wise t-intersecting family is the t-star.

Proposition 1.9. *Suppose that $\mathcal{G}$ is an r-wise t-intersecting k-graph that is not a t-star ($|\cap \mathcal{G}| < t$). Then $k \geq t+r$ or $k = t+r-1$ and $\mathcal{G} \subset \binom{Y}{k}$ for some $(k+1)$-element set Y.*

Proof. TOPROVE 0

Based on Proposition 1.9 in the sequel we always assume that $n \geq k \geq t+r$.

As to the corresponding problem for the non-uniform case, Erdős-Ko-Rado [5] proved $m(n,2,1) = 2^{n-1}$. Then the first author [8] established $m(n,3,2) = 2^{n-2}$. After several partial results the proof of the following result was concluded in [14]:

$$m(n,r,t) = 2^{n-t} \text{ if and only if } t \leq 2^r - r - 1. \tag{1.8}$$

We call a family $\mathcal{F} \subset \binom{[n]}{k}$ *non-trivial* if $\cap\{F \colon F \in \mathcal{F}\} = \emptyset$. Define

$$m^*(n,r,t) = \max\left\{|\mathcal{F}| \colon \mathcal{F} \subset 2^{[n]} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\right\},$$
$$m^*(n,k,r,t) = \max\left\{|\mathcal{F}| \colon \mathcal{F} \subset \binom{[n]}{k} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\right\}.$$

Theorem 1.10 (Brace-Daykin-Frankl Theorem (cf. [4] for $t=1$ and [12] for $t \geq 2$)). *For $t+r \leq n$ and $t < 2^r - r - 1$,*

$$m^*(n,r,t) = |\mathcal{A}_1(n,r,t)| = (t+r+1)2^{n-t-r}. \tag{1.9}$$

Let us recall some notations and useful results. For $i \in [n]$, define

$$\mathcal{F}(i) = \{F \setminus \{i\} \colon i \in F \in \mathcal{F}\},\ \mathcal{F}(\bar{i}) = \{F \colon i \notin F \in \mathcal{F}\}.$$

For $P \subset Q \subset [n]$, define

$$\mathcal{F}(Q) = \{F \setminus Q \colon Q \subset F\},\ \mathcal{F}(P,Q) = \{F \setminus Q \colon F \cap Q = P\}.$$

Let X be a finite set. For any $\mathcal{F} \subset \binom{X}{k}$ and $1 \leq b < k$, define the *bth shadow* $\partial^{(b)}\mathcal{F}$ as

$$\partial^{(b)}\mathcal{F} = \left\{E \in \binom{X}{k-b} \colon \text{there exists } F \in \mathcal{F} \text{ such that } E \subset F\right\}.$$

If $b=1$ then we simply write $\partial\mathcal{F}$ and call it *the shadow* of $\mathcal{F}$. Define the *up shadow* $\partial^+\mathcal{F}$ as

$$\partial^+\mathcal{F} = \left\{G \in \binom{X}{k+1}\colon \text{ there exists } F\in\mathcal{F} \text{ such that } F\subset G\right\}.$$

Sperner [24] proved the following result.

Theorem 1.11 ([24]). *For* $\mathcal{F}\subset\binom{[n]}{k}$,

$$\frac{|\partial^+\mathcal{F}|}{\binom{n}{k+1}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}}. \tag{1.10}$$

For $\mathcal{A},\mathcal{B}\subset\binom{[n]}{k}$, we say that $\mathcal{A},\mathcal{B}$ are *cross-intersecting* if $A\cap B\neq\emptyset$ for all $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

Theorem 1.12 ([18]). *Let* $\mathcal{A},\mathcal{B}\subset\binom{[n]}{k}$ *be cross-intersecting. Then for* $n\geq 2k$,

$$|\mathcal{A}|+|\mathcal{B}| \leq \binom{n}{k}. \tag{1.11}$$

We need the following version of the Kruskal-Katona Theorem.

Theorem 1.13 ([23, 22]). *Let* n,k,m *be positive integers with* $k\leq m\leq n$ *and let* $\mathcal{F}\subset\binom{[n]}{k}$ *and. If* $|\mathcal{F}|>\binom{m}{k}$ *then*

$$|\partial\mathcal{F}| > \binom{m}{k-1}.$$

We also need an inequality concerning the bth shadow of an r-wise t-intersecting family.

Theorem 1.14 ([13]). *Let* $\mathcal{F}\subset\binom{[n]}{k}$ *be an* r*-wise* t*-intersecting family. Then for* $0<b\leq t$ *we have*

$$|\partial^{(b)}\mathcal{F}| \geq |\mathcal{F}| \min_{0\leq i\leq\frac{k-t}{r-1}} \frac{\binom{ri+t}{i+b}}{\binom{ri+t}{i}}. \tag{1.12}$$

2 Shifting and lattice paths

In [5], Erdős, Ko and Rado introduced a very powerful tool in extremal set theory, called shifting. For $\mathcal{F}\subset\binom{[n]}{k}$ and $1\leq i<j\leq n$, define the shifting operator

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F)\colon F\in\mathcal{F}\},$$

where

$$S_{ij}(F) = \begin{cases} F' := (F\setminus\{j\})\cup\{i\}, & \text{if } j\in F, i\notin F \text{ and } F'\notin\mathcal{F};\\ F, & \text{otherwise.}\end{cases}$$

It is well known (cf. [11]) that the shifting operator preserves the size of $\mathcal{F}$ and the r-wise t-intersecting property. Thus one can apply the shifting operator to $\mathcal{F}$ when considering $m(n,k,r,t)$.

A family $\mathcal{F}\subset\binom{[n]}{k}$ is called *shifted* if $S_{ij}(\mathcal{F})=\mathcal{F}$ for all $1\leq i<j\leq n$. It is easy to show (cf. [11]) that every family can be transformed into a shifted family by applying the shifting operator repeatedly. Thus we can always assume that the family $\mathcal{F}$ is shifted when detemining $m(n,k,r,t)$.

Let us define the shifting partial order. Let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$ be two distinct k-sets with $a_1 < a_2 < \dots < a_k$ and $b_1 < b_2 < \dots < b_k$. We say that A *precedes* B in shifting partial order, denoted by $A \prec B$ if $a_i \leq b_i$ for $i = 1, 2, \dots, k$.

Let us recall two properties of shifted families:

Lemma 2.1 (cf. [11]). *If $\mathcal{F} \subset \binom{[n]}{k}$ is a shifted family, then $A \prec B$ and $B \in \mathcal{F}$ always imply $A \in \mathcal{F}$.*

Lemma 2.2 ([11]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted family. Then $\mathcal{F}$ is r-wise t-intersecting if and only if for every $F_1, \dots, F_r \in \mathcal{F}$ there exists s such that*

$$\sum_{1 \leq i \leq r} |F_i \cap [s]| \geq (r-1)s + t. \tag{2.1}$$

Note that $\sum\limits_{1 \leq i \leq r} |F_i \cap [s]| \leq rs$ implies $s \geq t$ if such an s exists. For completeness let us include the proof.

Proof. TOPROVE 1

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. For any $F_1, \dots, F_r \in \mathcal{F}$, define $s(F_1, \dots, F_r)$ to be the minimum s such that

$$\sum_{1 \leq i \leq r} |F_i \cap [s]| \geq (r-1)s + t.$$

Set $s := s(F_1, \dots, F_r)$. Then we must have

$$\sum_{1 \leq i \leq r} |F_i \cap [s]| = (r-1)s + t.$$

Indeed, if $\sum\limits_{1 \leq i \leq r} |F_i \cap [s]| \geq (r-1)s + t + 1$ then

$$\sum_{1 \leq i \leq r} |F_i \cap [s-1]| \geq (r-1)s + t + 1 - r \geq (r-1)(s-1) + t,$$

contradicting the minimality of s. Set $F_1 = F_2 = \dots = F_r = F$ for $F \in \mathcal{F}$, we obtain $r|F \cap [s]| = (r-1)s + t$. It follows that $\frac{s-t}{r} =: i$ is an integer. Then $s = t + ri$ and

$$\frac{(r-1)s + t}{r} = t + \frac{(r-1)(s-t)}{r} = t + (r-1)i.$$

Thus $|F \cap [t + ri]| \geq t + (r-1)i$ holds and we get the following corollary.

Corollary 2.3 ([11]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. Then for every $F \in \mathcal{F}$, there exists $i \geq 0$ so that $|F \cap [t + ri]| \geq t + (r-1)i$.*

In [9] a bijection between subsets and certain lattice paths was established. For $F \in \binom{[n]}{k}$, define $P(F)$ to be the lattice path in the two-dimensional integer grid $\mathbb{Z}^2$ starting at origin as follows. In the ith step for $i = 1, 2, \dots, n$, from the current point (x, y) the path $P(F)$ goes to $(x, y+1)$ if $i \in F$ and goes to $(x+1, y)$ if $i \notin F$. Since $|F| = k$, there are exactly k vertical steps. Thus the end point of $P(F)$ is $(n-k, k)$.

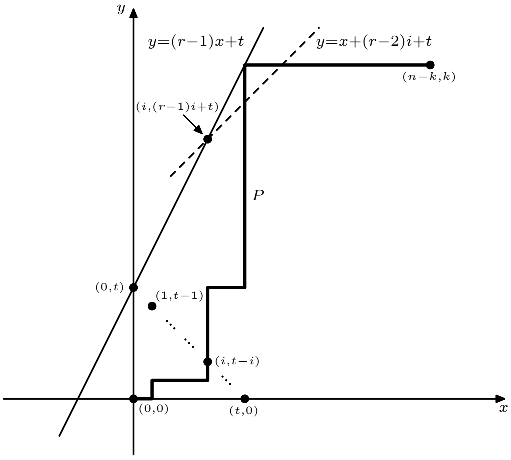

Figure 1: The lattice path P goes through $(i, t-i)$ and hits the line $y = (r-1)x + t$.

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. By Corollary 2.3 we infer that $P(F)$ hits $y = (r-1)x + t$ for every $F \in \mathcal{F}$. For $F \in \mathcal{F}$, define $i(F)$ to be the minimum integer i such that $|F \cap [t + ri]| = t + (r-1)i$. Define

$$\mathcal{F}_i = \{F \in \mathcal{F} \colon i(F) = i\}, i = 0, 1, 2, \ldots, \left\lfloor \frac{k-t}{r-1} \right\rfloor.$$

By Corollary 2.3, $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{\lfloor \frac{k-t}{r-1} \rfloor}$ form a partition of $\mathcal{F}$.

The next lemma gives a universal bound ont the size of an r-wise t-intersecting family for $n \geq 2k - t$.

Lemma 2.4. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family with $r \geq 3$ and $n \geq 2k - t$. Then*

$$|\mathcal{F}| \leq \sum_{0 \leq i \leq t} \binom{t}{i} \binom{n-t}{k-t-(r-1)i}. \tag{2.2}$$

Moreover,

$$\sum_{i \geq 1} |\mathcal{F}_i| \leq \sum_{1 \leq i \leq t} \binom{t}{i} \binom{n-t}{k-t-(r-1)i}. \tag{2.3}$$

Proof. TOPROVE 2

Fact 2.5. *Suppose $\mathcal{F} \subset 2^{[n]}$ is r-wise t-intersecting but $\mathcal{F}$ is not a t-star. Then for $2 \leq s < r$, $\mathcal{F}$ is s-wise $(t + r - s)$-intersecting.*

Proof. TOPROVE 3

Corollary 2.6. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family with $r \geq 3$. If $\mathcal{F}$ is not a t-star, then*

$$|\mathcal{F}| \leq \sum_{0 \leq i \leq t} \binom{t}{i} \binom{n-t}{k-t-(r-1)i} - \binom{n-t-1}{k-t}. \tag{2.4}$$

Proof. TOPROVE 4

3 Proof of Theorem 1.5

Proof. TOPROVE 5

4 The probability of hitting the line, uniform vs non-uniform

We need the following version of the Chernoff bound for the binomial distribution.

Theorem 4.1 ([20]). *Let $X \in Bi(n,p)$ and $\lambda = np$. Then*

$$Pr(X < \lambda - a) \le e^{-\frac{a^2}{2\lambda}}. \tag{4.1}$$

We call $P(n)$ a *p-random walk of length n* if it starts at origin and goes up a unit with probability p and goes right a unit with probability $1-p$ at each step. Let $f(n,r,t,p)$ be the probability that a p-random walk $P(n)$ hits the line $y=(r-1)x+t$. Set $f(r,t,p) = \lim_{n\to\infty} f(n,r,t,p)$. That is, $f(r,t,p)$ is the probability that an infinite p-random walk hits the line $y=(r-1)x+t$.

Lemma 4.2 ([11],[12]). *(i) $f(n,r,t,p) \le f(n+1,r,t,p)$.*

(ii) $f(n+1,r,t,p) = pf(n,r,t-1,p) + (1-p)f(n,r,t+r-1,p)$.

(iii)

$$f(r,t,p) = \gamma^t,$$

where γ is the unique root of $x = p + (1-p)x^r$ in the open interval $(0,1)$.

(iv) Let α_r be the unique root of $x = \frac{1}{2} + \frac{1}{2}x^r$. Then

$$\alpha_3 = \frac{\sqrt{5}-1}{2},\ \frac{1}{2} < \alpha_r < \frac{1}{2} + \frac{1}{2^r} \text{ for } r \ge 4.$$

Moreover,

$$\frac{1}{2^r - r} < \alpha_r^r \le \frac{1}{2^r - r - 1} \text{ for } r \ge 3. \tag{4.2}$$

Let us define another type of random walk. We call $Q(n,i)$ a *uniform random walk* if it is chosen uniformly from all lattice paths from $(0,0)$ to $(n-i,i)$. Let $g(n,i,r,t)$ be the probability that a uniform random walk $Q(n,i)$ hits the line $y=(r-1)x+t$.

Proposition 4.3. *(i) $g(n,i,r,t) \le g(n,i+1,r,t)$.*

(ii) $g(n+1,k,r,t) \le g(n,k,r,t)$.

(iii) For $r \ge 3$ and $t \ge 2$, $g(2k,k,r,t) \le g(2k+2,k+1,r,t)$.

(iv) $\lim_{k\to\infty} g(2k,k,r,t) \le f(r,t,\frac{1}{2})$.

Proof. TOPROVE 6

Proposition 4.4. *For $n \ge 2k$,*

$$m(n,k,r,t) \le \alpha_r^t \binom{n}{k}, \tag{4.3}$$

where α_r is the unique root of $x = \frac{1}{2} + \frac{1}{2}x^r$ in the interval $(0,1)$.

Proof. TOPROVE 7

5 Proof of Theorem 1.6

Let us prove a useful corollary of Theorem 1.14.

Corollary 5.1. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise t-intersecting family. If $t \geq 4$ then $|\partial^{(2)}\mathcal{F}| > 4|\mathcal{F}|$. If $t \geq 7$ then $|\partial^{(4)}\mathcal{F}| > 16|\mathcal{F}|$.*

Proof. TOPROVE 8

Fact 5.2. *For $n \geq \frac{\sqrt{4t+9}-1}{2}k$, $|\mathcal{A}_1(n,k,3,t)| < \binom{n-t}{k-t}$. For $n = \left(\frac{\sqrt{4t+9}-1}{2} - \epsilon\right)k$ with some $0 < \epsilon < \frac{1}{10}$ and $k \geq \frac{t^2+2t}{2\epsilon}$, $|\mathcal{A}_1(n,k,3,t)| > \binom{n-t}{k-t}$.*

Proof. TOPROVE 9

Proof. TOPROVE 10

6 Proof of Proposition 1.7 and Theorem 1.8

Let us prove a useful inequality.

Lemma 6.1. *For $n > \frac{rk-t}{r-1}$,*

$$m(n,k,r,t) \leq m(n-1,k,r,t) + m(n-1,k-1,r,t). \tag{6.1}$$

Proof. TOPROVE 11

Lemma 6.2. *Suppose that $m(n,k,r,t) = \binom{n-t}{k-t}$ then*

$$m(n,k-1,r,t) = \binom{n-t}{k-1-t}.$$

Proof. TOPROVE 12

Let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family. We say that $\mathcal{F}$ is *saturated* if any addition of an extra k-set to $\mathcal{F}$ would destroy the r-wise t-intersecting property. We say $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r \subset \binom{[n]}{k}$ are cross t-intersecting if $|F_1 \cap F_2 \cap \ldots \cap F_r| \geq t$ for all $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$, …, $F_r \in \mathcal{F}_r$.

Lemma 6.3. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted and saturated r-wise t-intersecting family. Let $\mathcal{G}_i = \mathcal{F}([t+1] \setminus \{i\}, [t+1])$, $i = 1,2,3,\ldots,t$. If $\mathcal{F}$ is not a t-star, then $\mathcal{G}_i = \mathcal{G}_j$ for all $1 \leq i < j \leq t$.*

Proof. TOPROVE 13

Lemma 6.4. *For $k \geq 3$,*

$$m(2k,k,4,3) = \binom{n-3}{k-3}.$$

Proof. TOPROVE 14

Lemma 6.5. *For $k \geq 4$,*
$$m(2k, k, 4, 4) = \binom{n-4}{k-4}.$$

Proof. TOPROVE 15

Proof. TOPROVE 16

Lemma 6.6. *If $k \geq \frac{t(t-1)}{4\log 2} + t - 1$ and $t \leq 2^{r-2}\log 2 - 2$, then*

$$m(2k, k, r, t) = \binom{n-t}{k-t}. \tag{6.2}$$

Moreover, in case of equality $\mathcal{F}$ is the full t-star.

Proof. TOPROVE 17

Proof. TOPROVE 18

7 Concluding remarks

The area of research concerning r-wise t-intersecting non-uniform families is quite large and there are several results we could not even mention. The case of uniform families, that is, adding a new parameter k, increases this variety. In the present paper we stayed mostly in the range $k \leq \frac{1}{2}n$. However, it is completely legitimate to consider the range $k \sim cn$ for any fixed $c < 1$ as long as $c \leq \frac{r-1}{r}$.

If one wants to extend the results to such a range it seems to be essential to answer the following question.

Problem 7.1. *Let $c < \frac{r-1}{r}$ and denote by $p(n, k, r, t)$ the probability that a random lattice path from $(0, 0)$ to $(n-k, k)$ hits the line $y = (r-1)x + t$. Let α be the unique root of $c - x + (1-c)x^r = 0$ in $(0, 1)$. Does the inequality*

$$p(n, k, r, t) < \alpha^t \text{ holds always if } k \leq cn? \tag{7.1}$$

It seems to be rather difficult to determine the exact value of $n_0(k, r, t)$. Based on Fact 5.2, let us make the following:

Conjecture 7.2. For $n \geq \frac{\sqrt{4t+9}-1}{2}k$,
$$m(n, k, 3, t) = \binom{n-t}{k-t}.$$

Another important problem would be to determine $m^*(n, k, r, 1)$, the uniform version of the Brace-Daykin Theorem (the case $t = 1$ of Theorem 1.10). In the case $r = 2$ the solution is given by the Hilton-Milner Theorem [19].

Let us recall the Hilton-Milner-Frankl Theorem. Define

$$\mathcal{B}(n, k, r, t) = \left\{ B \in \binom{[n]}{k} : [t+r-2] \subset B,\ B \cap [t+r-1, k+1] \neq \emptyset \right\}$$
$$\cup \left\{ [k+1] \setminus \{j\} : 1 \leq j \leq t+r-2 \right\}.$$

Theorem 7.3 (Hilton-Milner-Frankl Theorem [19, 10, 1]). *For $n \geq (k-t+1)(t+1)$,*

$$m^*(n,k,2,t) = \max\left\{|\mathcal{A}_1(n,k,2,t)|, |\mathcal{B}(n,k,2,t)|\right\}. \tag{7.2}$$

Note that both families $\mathcal{A}_1(n,k,2,t)$ and $\mathcal{B}(n,k,2,t)$ are r-wise $(t+2-r)$-intersecting, in particular, $(t+1)$-wise 1-intersecting. Thus in the range $(k-t+1)(t+1) < n$, i.e., $k < \frac{n}{t+1} + t - 1$,

$$m^*(n,k,r,t+2-r) = m^*(n,k,2,t).$$

However the case $k \sim cn$ with $\frac{1}{t+1} < c < \frac{r-1}{r}$ appears to be much harder. In [17] the following was proved.

Theorem 7.4 ([17]). *Let $0 < \varepsilon < \frac{1}{10}$. For $n \geq \frac{4}{\varepsilon^2} + 7$ and $\left(\frac{1}{2}+\varepsilon\right)n \leq k \leq \frac{3n}{5} - 3$,*

$$m^*(n,k,3,1) = |\mathcal{A}_1(n,k,3,1)|.$$

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