Limitations of Affine Integer Relaxations for Solving Constraint Satisfaction Problems

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Abstract

We show that various recent algorithms for finite-domain constraint satisfaction problems (CSP), which are based on solving their affine integer relaxations, do not solve all tractable and not even all Maltsev CSPs. This rules them out as candidates for a universal polynomial-time CSP algorithm. The algorithms are $\mathbb{Z}$-affine k-consistency, BLP+AIP, BA^k, and CLAP. We thereby answer a question by Brakensiek, Guruswami, Wrochna, and Živný [10] whether BLP+AIP solves all tractable CSPs in the negative. We also refute a conjecture by Dalmau and Opršal [19] (LICS 2024) that every CSP is either solved by $\mathbb{Z}$-affine k-consistency or admits a Datalog reduction from 3-colorability. For the cohomological k-consistency algorithm, that is also based on affine relaxations, we show that it correctly solves our counterexample but fails on an NP-complete template.

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1 Introduction

Constraint satisfaction problems (CSPs) provide a general framework that encompasses a huge variety of different problems, from solving systems of linear equations over Boolean satisfiability to variants of the graph isomorphism problem. We view CSPs as homomorphism problems. A CSP is defined by a relational structure $\mathbf{A}$ called the *template* of the CSP. An *instance* is a structure $\mathbf{B}$ of matching vocabulary and the question is whether there is a homomorphism from $\mathbf{B}$ to $\mathbf{A}$. We only consider *finite-domain* CSPs, i.e., the template $\mathbf{A}$ is always finite. It had long been conjectured by Feder and Vardi [22] that every finite-domain CSP is NP-complete or in P. In 2017, the conjecture was confirmed independently by Bulatov [11] and Zhuk [34]. The complexity of a CSP is determined by the polymorphisms ("higher-dimensional symmetries") of its template. If the template has no a so-called *weak near-unanimity* polymorphism, then the corresponding CSP is NP-complete. For the other case, Bulatov and Zhuk presented sophisticated polynomial-time algorithms. A less involved algorithm had been known earlier for templates with a *Maltsev* polymorphism [12]. None of these algorithms is *universal* in the sense that on input $(\mathbf{B}, \mathbf{A})$ they decide whether $\mathbf{B}$ maps homomorphically into $\mathbf{A}$ in time polynomial in both $|\mathbf{B}|$ and $|\mathbf{A}|$. Instead, these are *families*

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of algorithms, one for each template. The question whether there is a universal, and ideally "simple", algorithm for all tractable CSPs, or even just for all Maltsev CSPs, is still open.

One natural approach towards universal algorithms is via affine relaxations of systems of linear equations over $\{0, 1\}$: Given a template $\mathbf{A}$, an instance $\mathbf{B}$, and possibly a width parameter k, the existence of a homomorphism $\mathbf{B} \to \mathbf{A}$ is encoded into a system of linear equations. If the domain of the variables is relaxed from $\{0, 1\}$ to $\mathbb{Z}$, the system can be solved in polynomial time [27, 30], and the transformation of the CSP into the equation system is also computationally easy. Thus, if this integer relaxation were exact for all tractable CSPs, or at least an interesting subclass thereof, such as all Maltsev CSPs, then computing and solving it would constitute a universal polynomial-time algorithm for that class. Several algorithms based on this idea have been developed in recent years, motivated specifically by the study of *Promise CSPs* [10, 9, 16, 19]. This is a relatively new variant of CSPs which generalize for example the approximate graph coloring problem and are still not very well understood. The algorithms can be applied just the same to classical CSPs, and not even for these, much is known about their power. In the present article, we prove strong limitations for all these algorithms and show that even for Maltsev CSPs, none of them is universal: We construct a template $\mathbf{A}$ whose CSP is *not solved* by these algorithms by providing instances $\mathbf{B}$ that admit no homomorphism to $\mathbf{A}$ but which are accepted by the algorithms. This also refutes a conjecture by Dalmau and Opršal [19], that we expand upon below. Our result is in stark contrast to the situation for *valued CSPs*, an optimization version of CSP. For these, a surprisingly simple linear-algebraic algorithm solves all tractable cases optimally [32].

Let us briefly introduce the algorithms that are addressed by our construction. All of them make use of (slightly) different systems of equations, which can all be reduced to the *width-k affine relaxation*. Given a template structure $\mathbf{A}$, an instance $\mathbf{B}$, and a width $k \in \mathbb{N}$, the variables of the equation system are indexed with partial homomorphisms from induced size-k substructures of $\mathbf{B}$ to $\mathbf{A}$. A solution to the width-k affine relaxation is thus an assignment of numerical values to partial homomorphisms. The equations enforce a consistency condition, i.e., express that partial homomorphisms with overlapping domains receive values that fit together. This is related to, but stronger than, the *k-consistency* method: The k-consistency algorithm is a well-studied simple combinatorial procedure that checks for inconsistencies between local solutions and propagates these iteratively. This solves the *bounded width* CSPs (see e.g. [22, 4, 2]) but is not powerful enough to deal with *all* tractable CSPs [1]. The consistency conditions of the width-k affine relaxation are stronger in the sense that they enforce a *global* notion of consistency rather than a local one. The algorithms that fail to solve our counterexample are the following:

The **$\mathbb{Z}$-affine k-consistency algorithm** [19] (Section 5.1) runs the k-consistency procedure. All non-k-consistent partial homomorphisms are removed from the width-k affine relaxation. The algorithm accepts the instance $\mathbf{B}$ if and only if this modified version of the width-k affine relaxation has an integral solution. Dalmau and Opršal [19] conjectured that for all finite structures $\mathbf{A}$, CSP($\mathbf{A}$) is either Datalog$^{\cup}$-reducible to CSP($\mathbb{Z}$) and thus solved by $\mathbb{Z}$-affine k-consistency for a fixed k, or 3-colorability is Datalog$^{\cup}$-reducible to CSP($\mathbf{A}$) (see Conjecture 17). Assuming P $\neq$ NP, the conjecture implies that every tractable finite-domain CSP is solved by $\mathbb{Z}$-affine k-consistency.

The **BLP+AIP algorithm** by Brakensiek, Guruswami, Wrochna, and Živný [10] (Section 5.2) first solves the width-k affine relaxation over the non-negative rationals, where k is the arity of the template. Next, the integral width-k affine relaxation is checked for a solution, but every variable is set to 0 that is set to 0 by every rational solution. The **BA^k-algorithm** proposed by Ciardo and Živný [15] (Section 5.2) generalizes BLP+AIP:

The width k is not fixed to be the arity of the template but is a parameter of the algorithm, like in $\mathbb{Z}$-affine k-consistency. In [15], it is shown that there is an *NP-complete* (promise) CSP on which the algorithm fails, but no tractable counterexample had been known until now.

The **CLAP algorithm**, due to Ciardo and Živný [16] (Section 5.3), tests in the first step, for each partial homomorphism f, whether f can receive weight exactly 1 in a non-negative rational solution of the width-k affine relaxation, where k is the arity of the template. If not, it is discarded. This is repeated until the process stabilizes. Then the width-k affine relaxation is solved over the integers, where all discarded partial homomorphisms are forced to 0.

▶ **Theorem 1.** *There is a Maltsev template with* 7 *elements that is neither solved by*

(1) $\mathbb{Z}$-affine k-consistency, for every constant $k \in \mathbb{N}$,

(2) BLP+AIP,

(3) BA^k, for every constant $k \in \mathbb{N}$, nor

(4) the CLAP algorithm.

Hence, none of the algorithms solves all tractable CSPs. For the $\mathbb{Z}$-affine k-consistency relaxation and BA^k, the result is even true if k is a function in the instance size that grows at most sublinearly.

In particular, this answers a question of Brakensiek, Guruswami, Wrochna, and Živný [10] whether BLP+AIP solves all tractable CSPs in the negative. It also refutes the aforementioned Conjecture 17 regarding the power of the $\mathbb{Z}$-affine k-consistency relaxation [19], under the assumption that P $\neq$ NP. But we actually show a stronger statement: Namely, 3-colorability is not Datalog$^{\cup}$-reducible to the CSP that we use in the proof of the above theorem (Lemma 19). This is shown via a known inexpressibility result for *rank logic* [25] and disproves the conjecture unconditionally.

To prove Theorem 1, we construct and analyze concrete instances. Our template is a combination of systems of linear equations over the Abelian groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, though the template itself is not a group. Since the affine algorithms reduce CSPs to a problem over the infinite Abelian group $(\mathbb{Z}, +)$, we investigate for which finite groups this is possible: we study what we call *group coset-CSPs* (to distinguish them from equation systems over groups). The template of a coset-CSP consists of a finite group Γ, and its relations are cosets of powers of subgroups of Γ. They always have a Maltsev polymorphism [6]. Coset-CSPs have been studied as "group-CSPs" by Berkholz and Grohe [6, 8] or as "subgroup-CSPs" by Feder and Vardi [22].

▶ **Theorem 2.** *For each of the algorithms $\mathbb{Z}$-affine k-consistency, BLP+AIP, BA^k, and CLAP, the following is true:*

(1) Every coset-CSP over a finite Abelian group is solved (for $\mathbb{Z}$-affine k-consistency, k must be at least the arity of the template structure).

(2) There exists a non-Abelian coset-CSP that is not solved, namely over S_{18}, the symmetric group on 18 *elements (for any constant or even sublinearly growing k).*

(3) There are non-Abelian coset-CSPs that are solved, namely over any 2-nilpotent group of odd order. For example, there are non-Abelian 2*-nilpotent semidirect products $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ of order p^3 for each odd prime p.*

While Assertion 1 is easily derived from the literature [18, 3], it turns out somewhat surprisingly that Abelian groups are not the border of tractability for the affine algorithms: They also work over certain 2-nilpotent groups; these are in a sense the non-Abelian groups that are closest possible to being Abelian. Assertion 2 is shown with a construction that is “semantically equivalent” to the one that we use for Theorem 1, but whose template is a coset-CSP. However, the analysis of the instances is technically much more involved. The construction in Theorem 1 is simpler and yields a smaller template. We show that our first counterexample can be expressed as instances of the *graph isomorphism* problem with *bounded color class size*, that is, the isomorphism problem of vertex-colored graphs, in which each color is only used for a constant number of vertices. This problem is expressible as a coset-CSP over the symmetric group [8]. This also shows that the affine CSP algorithms cannot be adapted to solve the graph isomorphism problem. They fail already on the bounded color class version, which is known to be in P [23].

There exists another highly interesting affine CSP algorithm that we have not addressed so far. This is the *cohomological k-consistency* algorithm due to Ó Conghaile [18] (see Section 5.4). As it turns out, this algorithm is actually able to solve our counterexample correctly. Hence, for all we know, it may be possible that there is a $k \in \mathbb{N}$ such that cohomological k-consistency is a universal polynomial-time algorithm for Maltsev or even all tractable CSPs. However, we can show *without* complexity-theoretic assumptions that it fails on NP-complete CSPs.

▶ **Theorem 3.** *The CSP on which the algorithms in Theorem 1 fail is solved by cohomological k-consistency, for every $k \geq 4$. There exists an NP-complete CSP such that for every constant k, cohomological k-consistency fails to solve it.*

Our Techniques. Our proof of Theorem 1 combines results due to Berkholz and Grohe [8] with a new *homomorphism or-construction* that encodes the disjunction of two CSPs. For a system of linear equations to have an integral solution, it suffices to have a rational *p-solution* and a rational *q-solution* (for p and q coprime), in which all non-zero values are of the form p^z, with $z \in \mathbb{Z}$, or q^z, respectively. Thus, it suffices to design the instances in such a way that these two co-prime rational solutions exist. For the algorithms that involve a width-parameter k, the additional challenge is to make the construction robust so that it works against any choice of k (in our case it works even if k grows with the instance size). The *Tseitin contradictions* [33] over *expander graphs* (see Section 4) achieve this robustness. It is known that these cannot be solved by “local” algorithms, e.g., the k-consistency method, for any constant k [1]. Berkholz and Grohe showed that the width-k relaxation for unsatisfiable Tseitin contradictions over $\mathbb{Z}_p$, for a prime p, still has a p-solution. We combine two unsatisfiable Tseitin systems over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in the aforementioned homomorphism or-construction (Section 3). This yields an unsatisfiable CSP instance whose width-k relaxation has a 2- and a 3-solution and thereby also an integral solution. The reason why this approach fails for the cohomological algorithm (Theorem 3) is that it solves the width-k relaxation when a partial homomorphism is fixed. This fixing of local solutions reduces the homomorphism or-construction to just solving equations over $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively, which the affine relaxation can do. To prove the second part of Theorem 3, we modify the homomorphism or-construction so that cohomology no longer solves it, but this also makes the template NP-complete.

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2 Preliminaries

We write $[k]$ for $\{1, \dots, k\}$. For $k \in \mathbb{N}$ and a set N, let $\binom{N}{\leq k}$ be the set of all subsets of N of size at most k. A **relational vocabulary** τ is a set of relation symbols $\{R_1, \dots, R_k\}$ with associated arities $\mathrm{ar}(R_i)$. A **relational τ-structure** is a tuple $\mathbf{A} = (A, R_1^{\mathbf{A}}, \dots, R_k^{\mathbf{A}})$ of a **universe** A and interpretations of the relation symbols such that $R_i^{\mathbf{A}} \subseteq A^{\mathrm{ar}(R_i)}$ for all $i \in [k]$. We use letters $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ for finite relational structures. Their universes are denoted A, B, and C, respectively. If $\mathbf{A}$ is a structure and $X \subseteq A$, then $\mathbf{A}[X]$ denotes the induced substructure with universe X. For two τ-structures $\mathbf{A}$ and $\mathbf{B}$, we write $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ for the set of **homomorphisms** $\mathbf{A} \to \mathbf{B}$. A **graph** $G = (V, E)$ is a binary $\{E\}$-structure, where we denote its **vertex set** by $V(G)$ and its **edge set** by $E(G)$. Group operations are written as multiplication, for Abelian groups we use additive notation.

CSPs and Polymorphisms. For a finite τ-structure $\mathbf{A}$, denote by $\mathrm{CSP}(\mathbf{A})$ the **CSP with template** $\mathbf{A}$, i.e., the class of finite τ-structures $\mathbf{B}$ such that there is a homomorphism $\mathbf{B} \to \mathbf{A}$. We call a structure $\mathbf{B}$ a $\mathrm{CSP}(\mathbf{A})$-instance if $\mathbf{B}$ has the same vocabulary as $\mathbf{A}$. The complexity of $\mathrm{CSP}(\mathbf{A})$, and also the applicability of certain algorithms, is determined by the **polymorphisms** of the τ-structure $\mathbf{A}$. An ℓ-ary polymorphism is a map $p \colon A^\ell \to A$ such that for every $R \in \sigma$ of arity $r = \mathrm{ar}(R)$ and all $\bar{a}_1, \dots, \bar{a}_\ell \in R^{\mathbf{A}}$, the tuple $(p(a_{11}, a_{21}, \dots, a_{\ell 1}), \dots, p(a_{1r}, a_{2r}, \dots, a_{\ell r}))$ is also in $R^{\mathbf{A}}$ (where a_{ij} denotes the j-th entry of the tuple $\bar{a}_i$). The polymorphisms of a structure are closed under composition. A ternary operation p is **Maltsev** if it satisfies the identity $p(x, x, y) = p(y, x, x) = y$ for all inputs. For a group Γ the map $f(x, y, z) = xy^{-1}z$ is a typical example of a Maltsev operation. The templates with Maltsev polymorphisms form a subclass of all tractable CSPs [12]. For more background on the algebraic approach to CSPs, see for example [5].

Logics, Interpretations, and Reductions. A logic L defines $\mathrm{CSP}(\mathbf{A})$ if there exists an L-formula F such that each instance $\mathbf{B}$ satisfies F if and only if $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$. **Inflationary fixed-point logic** (IFP) is the extension of first-order logic by an operator defining inflationary fixed-points. IFP defines connected components of graphs, which is not possible in pure first-order logic. More details are not needed and we refer to [21, Chapter 8.1]. A **logical interpretation** is a (partial) map from σ-structures to τ-structures defined by logical formulas in the following way. For a σ-structure, a formula using vocabulary σ with k free variables defines the set of all k-tuples of the structure satisfying the formula. An interpretation consists of a formula defining the new universe, and for each relation symbol $R \in \tau$ of a formula defining the relation R in the new structure. Formal details are not needed in this paper, and for more details we refer to [21, Section 11.2] (or the Appendix). Such interpretations can be used as reductions between decision problems. Of particular interest in the context of CSPs are Datalog-interpretations, which can be expressed as IFP-interpretations (again see [21, Theorem 9.1.4]). Dalmau and Opršal [19] consider a variant of these reductions called **Datalog$^\cup$-reductions**. We omit a definition and only note that Datalog$^\cup$-reductions can be expressed as IFP-interpretations, too.

The k-Consistency Algorithm. A well-known heuristic for CSPs is the k-consistency algorithm. For a template $\mathbf{A}$ and an instance $\mathbf{B}$, the k-consistency algorithm computes a map $\kappa_k^{\mathbf{A}}[\mathbf{B}]$ assigning to each $X \in \binom{B}{\leq k}$ a set of partial homomorphisms $\mathbf{B}[X] \to \mathbf{A}$: it is the unique greatest fixed-point that satisfies the following properties for all $Y \subset X \in \binom{B}{\leq k}$.

Forth-Condition: Every $f \in \kappa_k^{\mathbf{A}}[\mathbf{B}](Y)$ extends to some $g \in \kappa_k^{\mathbf{A}}[\mathbf{B}](X)$, that is, $g|_Y = f$.

Down-Closure: For every $g \in \kappa_k^{\mathbf{A}}[\mathbf{B}](X)$, we have $g|_Y \in \kappa_k^{\mathbf{A}}[\mathbf{B}](Y)$.

If $\kappa_k^{\mathbf{A}}[\mathbf{B}](X) = \emptyset$ for some $X \in \binom{B}{\leq k}$, then the algorithm rejects $\mathbf{B}$, otherwise it accepts.

CSP-Relaxation via Affine Systems of Linear Equations. We introduce a system of linear equations due to Berkholz and Grohe [6], which will be used to (approximately) solve CSPs. We transfer hardness results for this system to other systems used in the different algorithms. Let $\mathbf{A}$ be a template structure and $\mathbf{B}$ be an instance. The **width-k affine relaxation** $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ aims to encode (approximately) whether $\mathbf{B}$ is in $\mathrm{CSP}(\mathbf{A})$.

$\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$: variables $x_{X,f}$ for all $X \in \binom{B}{\leq k}$ and all $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$
$\displaystyle\sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}),\\ f\vert_{X\setminus\{b\}}=g}} x_{X,f} = x_{X\setminus\{b\},g}$ for all $X \in \binom{B}{\leq k}, b \in X, g \in \mathrm{Hom}(\mathbf{B}[X\setminus\{b\}], \mathbf{A})$ $x_{\emptyset,\emptyset} = 1$

Here $\emptyset$ is the unique homomorphism $\mathbf{B}[\emptyset] \to \mathbf{A}$. If k is at least the arity of $\mathbf{A}$, then $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$ if and only if $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ has a nonnegative integral solution (and actually a $\{0,1\}$-solution) [6]. We will be mainly interested in integral solutions of $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$, so without the non-negativity restriction. To show the existence of these solutions, we will also consider special rational solutions:

▶ **Definition 4.** *For $p \in \mathbb{Z}$, a p-**solution** of a system of linear equations L with variables* $\mathrm{Var}(\mathsf{L})$ *is a solution* $\Phi\colon \mathrm{Var}(\mathsf{L}) \to \mathbb{Q}$ *of* L *such that, for all $x \in \mathrm{Var}(\mathsf{L})$, $\Phi(x) = 0$ or $\Phi(x) = p^i$ for some $i \in \mathbb{Z}$.*

▶ **Lemma 5** ([6])**.** *If p and q are coprime integers and a system L of linear equations over $\mathbb{Q}$ has a p-solution and a q-solution, then L has an integral solution, which is only non-zero for variables on which the p-solution or the q-solution is non-zero.*

Group Coset-CSPs. Let Γ be a finite group. In a Γ**-coset-CSP** [6, 22] variables range over Γ and the constraints are of the following form. For an r-tuple of variables $\bar{x} = (x_1, \dots, x_r)$, an r-ary Γ**-coset-constraint** is the constraint $\bar{x} \in \Delta\delta$, where $\Delta \leq \Gamma^r$ is a subgroup of Γ^r and $\delta \in \Gamma^r$, that is, $\Delta\delta$ is a right coset of Γ^r. With the term **coset-CSP** we refer to a Γ-coset-CSP in this sense. For every finite group Γ and every arity r, there is a structure $\Gamma^{[r]}$ such that every r-ary Γ-coset-CSP is a $\Gamma^{[r]}$-instance and $\mathrm{CSP}(\Gamma^{[r]})$ contains all solvable r-ary Γ-coset-CSPs. $\mathrm{CSP}(\Gamma^{[r]})$ is always tractable [22]. In fact, being a coset-CSP in this sense is equivalent to having the Maltsev polymorphism $f(x, y, z) = xy^{-1}z$ (see Appendix B for a proof).

▶ **Lemma 6.** *For every finite template $\mathbf{A} = (A, R_1^{\mathbf{A}}, ..., R_m^{\mathbf{A}})$ and every binary operation $\cdot\colon A \times A \to A$ such that $\Gamma = (A, \cdot)$ is a group,*

- *the map $f : \Gamma^3 \to \Gamma$ defined by $f(x, y, z) = xy^{-1}z$ is a polymorphism of $\mathbf{A}$ if and only if*
- *each relation $R_i^{\mathbf{A}}$ is a coset of a subgroup of Γ^r for some $r \in \mathbb{N}$.*

Thus, coset-CSPs are a natural class to study. In particular, being Maltsev, they are tractable even if Γ is non-Abelian. By contrast, solving *systems of linear equations* is NP-complete if (and only if) Γ is non-Abelian [24]. Systems of linear equations over an *Abelian* group Γ can however always be viewed as a Γ-coset-CSP. Hence, when we consider equation systems over Abelian groups in Section 4, we can treat them uniformly as coset-CSPs.

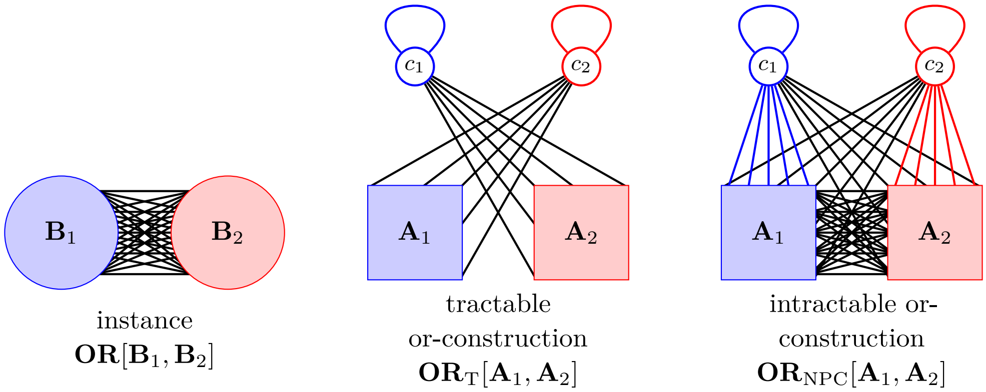

Figure 1 The different homomorphism or-constructions: The picture assumes that the two vocabularies τ_1 and τ_2 are binary and contain a single relation each (blue and red). It shows the instance $\mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$. and the tractable and intractable template construction. The new S-relation is drawn in black, where the edges are all oriented from left to right. Pairs added to the relation of τ_1 or τ_2 are drawn in blue or red, respectively.

3 Homomorphism OR-Construction

For $i \in [2]$, let $\mathbf{A}_i$ and $\mathbf{B}_i$ be nonempty τ_i-structures and τ_1 and τ_2 be disjoint. The $\mathbf{A}_i$ are templates and the $\mathbf{B}_i$ the corresponding instances, and we assume that their universes are disjoint. We aim to define τ-structures $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$ if and only if $\mathbf{B}_i \in \mathrm{CSP}(\mathbf{A}_i)$ for some $i \in [2]$. Let S be a fresh binary relation symbol and $\tau := \tau_1 \cup \tau_2 \cup \{S\}$. The arities are inherited from τ_1 and τ_2. The instance $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$ with universe $B := B_1 \cup B_2$ has relations $S^{\mathbf{B}} := B_1 \times B_2$ and $R^{\mathbf{B}} := R^{\mathbf{B}_i}$ for all $i \in [2]$ and $R \in \tau_i$. We provide two variants of $\mathbf{A}$ with different properties.

The Tractable Case. The first variant of the template will preserve tractability of the $\mathrm{CSP}(\mathbf{A}_i)$ and is called the **tractable homomorphism or-construction**. The template $\mathbf{A} = \mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2]$ is defined as follows. Let c_1 and c_2 be two fresh vertices.

$$\begin{aligned} A &:= A_1 \cup A_2 \cup \{c_1, c_2\} \\ R^{\mathbf{A}} &:= R^{\mathbf{A}_i} \cup \{c_i\}^{\mathrm{ar}(R)} && \text{for all } i \in [2], R \in \tau_i \\ S^{\mathbf{A}} &:= \big(A_1 \times \{c_2\}\big) \cup \big(\{c_1\} \times A_2\big) \end{aligned}$$

The construction is depicted in Figure 1. Intuitively, a homomorphism $\mathbf{B}_i \to \mathbf{A}_i$ induces a homomorphism $\mathbf{B} \to \mathbf{A}$ by mapping all vertices of $\mathbf{B}_{3-i}$ to c_{3-i}. The relation S ensures that every homomorphism of $\mathbf{B} \to \mathbf{A}$ is of this form, which proves the following lemma:

▶ **Lemma 7.** $\mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2] \in \mathrm{CSP}(\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2])$ *if and only if* $\mathbf{B}_i \in \mathrm{CSP}(\mathbf{A}_i)$ *for some* $i \in [2]$.

The next lemmas summarize the properties of the construction, which are required later (full proofs are provided in Appendix C.1): the tractable homomorpism or-construction preserves k-consistency of the $\mathbf{B}_i$ and inherits solutions of the width-k affine relations from the $\mathbf{B}_i$.

▶ **Lemma 8.** *Let* $\mathbf{A} = \mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$, $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$, $k \in \mathbb{N}$, $i \in [2]$, $X \in \binom{B}{\leq k}$, *and* $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$. *If* $f(X \cap B_{3-i}) = \{c_{3-i}\}$ *and* $f|_{X \cap B_i} \in \kappa_k^{\mathbf{A}_i}[\mathbf{B}_i](X \cap B_i)$, *then* $f \in \kappa_k^{\mathbf{A}}[\mathbf{B}](X)$.

▶ **Lemma 9.** *Let* $\mathbf{A} = \mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$, $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$, $i \in [2]$, *and* Φ *be a solution to* $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$. *Then there is a solution* Ψ *to* $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ *defined, for every* $X \in \binom{B}{\leq k}$ *and* $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$, *by* $\Psi(x_{X,f}) = \Phi(x_{X \cap B_i}, f|_{X \cap B_i})$ *if* $f(X \cap B_{3-i}) = \{c_{3-i}\}$ *and* $\Psi(x_{X,f}) = 0$ *otherwise. In particular,* Ψ *is a p-solution or integral, if* Φ *is a p-solution or integral, respectively.*

▶ **Lemma 10.** *If* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *have a Maltsev polymorphism, then* $\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$ *has one.*

Also if the $\mathbf{A}_i$ are not Maltsev, tractability is preserved.

▶ **Lemma 11.** *If* $\mathrm{CSP}(\mathbf{A}_1)$ *and* $\mathrm{CSP}(\mathbf{A}_2)$ *are tractable, then* $\mathrm{CSP}(\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2])$ *is tractable.*

For the case that both $\mathbf{A}_i$ are tractable, we provide a polynomial-time algorithm for $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2])$ in Appendix C. Roughly speaking, the algorithm proceeds as follows: If the instance $\mathbf{B}$ has connected components, they can be treated individually. So we can assume that $\mathbf{B}$ is connected. The algorithm divides the universe B into two sets B_1 and B_2 such that elements in B_i are contained in tuples of a τ_i-relation. If one is contained in both a τ_1- and a τ_2-relation, then $\mathbf{B}$ is a no-instance. Because $\mathbf{B}$ is connected, B_1 and B_2 are connected by the relation S (not necessarily by a biclique). By the construction of the tractable or-construction, for any potential homomorphism there is an $i \in [2]$ such that B_i is mapped to c_i and B_{3-i} to $\mathbf{A}_{3-i}$. This can be tested by the algorithms for $\mathrm{CSP}(\mathbf{A}_1)$ and $\mathrm{CSP}(\mathbf{A}_2)$. Note that this algorithm essentially only computes connected components and calls the decision algorithms of the $\mathrm{CSP}(\mathbf{A}_i)$. Hence, it can be expressed in logics that are at least expressive as IFP.

▶ **Corollary 12.** *Let* L *be a logic that is at least expressive as inflationary fixed-point logic. If* $\mathrm{CSP}(\mathbf{A}_1)$ *and* $\mathrm{CSP}(\mathbf{A}_2)$ *are* L*-definable, then* $\mathrm{CSP}(\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2])$ *is* L*-definable.*

The tractable homomorphism or-construction has a drawback: forbidding a single partial homomorphism mapping vertices of $\mathbf{B}_i$ to c_i resolves the or-construction for the width-k affine relaxation: every such solution to the width-k affine relaxation of $\mathbf{OR}_{\mathrm{T}}[\mathbf{B}_1, \mathbf{B}_2]$ induces a solution for $\mathbf{B}_i$.

▶ **Lemma 13.** *Let* $k \geq 2$, $i \in [2]$, *and* Φ *be a solution to* $\mathsf{L}^{\mathbf{A},k+1}_{\mathsf{CSP}}(\mathbf{B})$. *If there is a set* $X \in \binom{B_i}{\leq k}$ *such that for* $f \colon X \to \{c_i\}$ *it holds that* $\Phi(x_{X,f}) = 0$, *then* $\Phi|_{B_i}$ *is a solution to* $\mathsf{L}^{\mathbf{A}_i,k}_{\mathsf{CSP}}(\mathbf{B}_i)$.

In some situations this will cause problems later. To remedy this problem, we make the construction more flexible at the cost of giving up tractability.

The Intractable Case. The second variant, which is called the **intractable homomorphism or-construction**, $\mathbf{A} = \mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]$ has the same universe as before, but interprets relations differently:

$$\begin{aligned} R^{\mathbf{A}} &:= R^{\mathbf{A}_i} \cup \Big((A_i \cup \{c_i\})^{\mathrm{ar}(R)} \setminus A_i^{\mathrm{ar}(R)} \Big) && \text{for all } i \in [2], R \in \tau_i \\ S^{\mathbf{A}} &:= \big(A_1 \times (A_2 \cup \{c_2\})\big) \cup \big((A_1 \cup \{c_1\} \times A_2\big) \end{aligned}$$

The construction is shown in Figure 1. The definition of S still enforces that a homomorphism $\mathbf{B} \to \mathbf{A}$ has to induce a homomorphism $\mathbf{B}_i \to \mathbf{A}_i$ for one i. Only for one i, the extra vertex c_i can be in the image of a homomorphism. Thus, this variant provides again a homomorphism-or construction, but now two *partial* homomorphisms $\mathbf{B}_1 \to \mathbf{A}_1$ and $\mathbf{B}_2 \to \mathbf{A}_2$ can be combined:

▶ **Lemma 14.** *Let* $\mathbf{A} = \mathbf{OR}_{NPC}[\mathbf{A}_1, \mathbf{A}_2]$*,* $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$*,* $X_i \subseteq B_i$*, and* $f_i \in \mathrm{Hom}(\mathbf{B}_i, \mathbf{A}_i)$ *for both* $i \in [2]$*. The map* $f\colon X_1 \cup X_2 \to A$ *induced by* f_1 *and* f_2 *satisfies* $f \in \mathrm{Hom}(\mathbf{B}[X_1 \cup X_2], \mathbf{A})$*.*

Also the intractable construction inherits solutions of the width-k affine relaxation and preserves k-consistency similar to Lemmas 8 and 9, but now for the combined partial homomorphisms. Instead of requiring that $f(X \cap B_{3-i}) = \{c_{3-i}\}$, we only need $f|_{X \cap B_{3-i}} \in \mathrm{Hom}(\mathbf{B}_{3-i}[X], \mathbf{A}[A_{3-i} \cup \{c_{3-i}\}])$ (proofs in Appendix C.2). Already for easy templates the CSP of the intractable homomorphism-or construction is NP-complete. The reason is that for general instances the S-relation does not have to be a biclique and hence a yes-instance does not have to induce a homomorphism for one $\mathbf{B}_i$ at all. Even for choosing $\mathbf{A}_1$ and $\mathbf{A}_2$ to be the structure with a single element and one nonempty ternary relation, the intractable OR-construction yields an NP-complete CSP (e.g. shown by a reduction from 3-SAT). Many other CSPs can be reduced to this case, for example group-coset-CSPs (full proofs in Appendix C.2).

4 Tseitin Formulas over Abelian Groups and Expanders

A family of 2-connected 3-regular graphs $(G_n)_{n\in\mathbb{N}}$ is a family of **expander graphs** if there is a constant $c > 0$ (the **expansion constant**) such that for all G_n and $X \subseteq E(G_n)$, there is a set $\hat{X} \supseteq X$ of size $|\hat{X}| \leq c|X|$ such that $E(G) \setminus \hat{X}$ is empty or the edge set of a 2-connected subgraph of G. The existence of such families is folklore, see e.g. [7]. Let G be a 2-connected 3-regular expander graph. Fix an orientation H of G, i.e., a directed graph with one direction of each edge of G. Let $V := V(G), E := E(G)$ in this section. For a set $W \subseteq V$, denote by $\delta_-(W) \subseteq E$ the set of all $uv \in E$ such that $(u,v) \in E(H) \cap (V \setminus W) \times W$. Analogously, $\delta_+(W) \subseteq E$ is the set of all edges leaving W, and $\delta(W) := \delta_+(W) \cup \delta_-(W)$. Fix a finite Abelian group Γ. Let $\lambda\colon V \to \Gamma$. Define the Γ-coset-CSP $\mathcal{C}^{H,\Gamma,\lambda}$, or $\mathcal{C}^\lambda$ for short, with variable set $\{y_e \mid e \in E\}$ and linear equations

$$\sum_{e\in\delta_+(v)} y_e - \sum_{e\in\delta_-(v)} y_e = \lambda(v) \qquad \text{for all } v \in V.$$

In the case $\Gamma = \mathbb{Z}_2$, we obtain the classic Tseitin contradictions [33]. The CSP $\mathcal{C}^\lambda$ is solvable if and only if $\sum_{v\in V} \lambda(v) = 0$ [8]. For all sets $W \subseteq V$, the CSP $\mathcal{C}^\lambda$ implies the constraint $C(W)$ defined via

$$\sum_{e\in\delta_+(W)} y_e - \sum_{e\in\delta_-(W)} y_e = \sum_{v\in W} \lambda(v).$$

▶ **Definition 15** (Robustly Consistent Assignments [8]). *For* $\lambda\colon V \to \Gamma$ *and a set* $X \subseteq E$*, a partial assignment* $f\colon X \to \Gamma$ *for* $\mathcal{C}^\lambda$ *is* ℓ***-consistent****, if for every* $W \in \binom{V}{\leq \ell}$ *such that* $\delta(W) \subseteq X$*, the assignment* f *satisfies the constraint* $C(W)$*. Note that* f *is a partial solution if it is* 1*-consistent. We call* f ***robustly consistent*** *if it is* $n/3$*-consistent.*

We review facts about robustly consistent assignments for $\mathcal{C}^\lambda$. The detailed statements and proofs can be found in Appendix D and [8]. The key idea is that the expansion property of G ensures that $\mathcal{C}^\lambda$ is always locally satisfiable, on subinstances of size up to $k = o(|E|)$. This is because the inconsistency can be "shifted around" the graph to any equation outside of the local scope. Thus, for every set X of at most k variables, there is at least one robustly consistent assignment with domain X. Robustly consistent assignments are also not discarded by the k-consistency procedure. In particular, k-consistency always accepts $\mathcal{C}^\lambda$ even if it has no solution.

▶ **Lemma 16** ([8]). *If $k \in o(|E|)$ and Γ is a p-group (i.e., $|\Gamma|$ is a power of p), and $\lambda\colon V \to \Gamma$, then there is a p-solution of $\mathsf{L}_{\mathsf{CSP}}^{k,\Gamma^{[3]}}(\mathcal{C}^{\lambda})$ such that non-robustly consistent partial assignments are set to 0, and each robustly consistent partial solution is mapped to $1/p^{\ell}$ for some $\ell \in \mathbb{N}$.*

The above lemma can also be refined so that the resulting p-solution assigns the value 1 to $x_{X,f}$ for a single robustly consistent partial homomorphism $f\colon X \to \Gamma$ of our choice.

5 Limitations of the Affine Algorithms

All of the affine algorithms are *sound*: they accept all yes-instances. This section shows that many of them are note *complete* on tractable CSPs: they do not reject all no-instances, and thus do not solve the CSP. We consider the tractable homomorphism or-construction $\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$ of the ternary $\mathbb{Z}_2$-coset-CSP and the ternary $\mathbb{Z}_3$-coset-CSP.

5.1 $\mathbb{Z}$-Affine k-Consistency Relaxation

The **$\mathbb{Z}$-affine k-consistency relaxation** [19] solves the following system of affine linear equations over the integers. Let $\mathbf{A}$ be a template, $\mathbf{B}$ be an instance, and κ be a map assigning to every set $X \in \binom{B}{\leq k}$ a set of partial homomorphisms $\mathbf{B}[X] \to \mathbf{A}$. Define the system $\mathsf{L}_{\mathbb{Z}\text{-aff}}^{k,\mathbf{A}}(\mathbf{B}, \kappa)$:

$\mathsf{L}_{\mathbb{Z}\text{-aff}}^{k,\mathbf{A}}(\mathbf{B}, \kappa)$: variables $z_{X,f}$ for all $X \in \binom{B}{\leq k}$ and $f \in \kappa(X)$

$$
\begin{aligned}
z_{X,f} &\in \mathbb{Z} && \text{for all } X \in \tbinom{B}{\leq k} \text{ and } f \in \kappa(X) \\
\sum_{f \in \kappa(X)} z_{X,f} &= 1 && \text{for all } X \in \tbinom{B}{\leq k} \\
\sum_{f \in \kappa(X), f|_Y = g} z_{X,f} &= z_{Y,g} && \text{for all } Y \subset X \in \tbinom{B}{\leq k} \text{ and } g \in \kappa(Y)
\end{aligned}
$$

For a fixed positive integer k, a template structure $\mathbf{A}$, and an instance $\mathbf{B}$, the $\mathbb{Z}$-affine k-consistency relaxation runs the k-consistency algorithm to compute $\kappa_k^{\mathbf{A}}[\mathbf{B}]$, the function that maps each set $X \in \binom{B}{\leq k}$ to the set of k-consistent partial homomorphisms $\mathbf{B}[X] \to \mathbf{A}$. The instance $\mathbf{B}$ is accepted by the algorithm if $\mathsf{L}_{\mathbb{Z}\text{-aff}}^{k,\mathbf{A}}(\mathbf{B}, \kappa_k^{\mathbf{A}}[\mathbf{B}])$ has an integral solution and rejects $\mathbf{B}$ otherwise. Dalmau and Opršal [19] conjectured the following on the power of the $\mathbb{Z}$-affine k-consistency relaxation:

▶ **Conjecture 17** ([19]). *For every finite structure $\mathbf{A}$, either $\mathrm{CSP}(K_3)$ is Datalog$^{\cup}$-reducible to $\mathrm{CSP}(\mathbf{A})$ or $\mathrm{CSP}(\mathbf{A})$ is Datalog$^{\cup}$-reducible to $\mathrm{CSP}(\mathbb{Z})$, where K_3 denotes the triangle.*

Being Datalog$^{\cup}$-reducible to $\mathrm{CSP}(\mathbb{Z})$ implies that $\mathrm{CSP}(\mathbf{A})$ is solved by $\mathbb{Z}$-affine k-consistency for some constant k [19], which we show is not the case for our counterexample.

▶ **Theorem 18.** *For every $k \geq 1$, the $\mathbb{Z}$-affine k-consistency relaxation does not solve $\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$. This is even true if k is not a constant, but an at most sublinear function in the instance size.*

Proof. Let G be a 3-regular 2-connected expander whose number of vertices is sufficiently larger than k, and let H be an orientation of G. Let $p_1 := 2$ and $p_2 := 3$. For $i \in [2]$, let $\Gamma_i := \mathbb{Z}_{p_i}$, and $\lambda_i : V(G) \to \Gamma_i$ be 0 everywhere except at one vertex $v^* \in V(G)$, where we set $\lambda_i(v^*) := 1$. For each $i \in [2]$, we consider the 3-ary Γ_i-coset-CSP $\mathbf{B}_i := \mathcal{C}^{H,\Gamma_i,\lambda_i}$. Let $\mathbf{B} := \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$ and $\mathbf{A} := \mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$ be the corresponding tractable homomorphism or-template. From $\sum_{v \in V(G)} \lambda_i(v) \neq 0$ it follows $\mathbf{B}_i \notin \mathrm{CSP}(\Gamma_i^{[3]})$ for both $i \in [2]$ and thus $\mathbf{B} \notin \mathrm{CSP}(\mathbf{A})$ by Lemma 7. For a set $X \subseteq B$, a partial homomorphism $f\colon \mathbf{B}[X] \to \mathbf{A}$ is robustly consistent if $f|_{B_i}$ is a robustly consistent partial homomorphism $\mathbf{B}_i[X \cap B_i] \to \mathbf{A}_i$ for some $i \in [2]$ and $f(X \cap B_{3-i}) = \{c_{3-i}\}$, where c_{3-i} is the fresh vertex of the or-construction. Robustly consistent partial solutions of the Tseitin systems are not discarded by k-consistency, and by Lemma 8 this is also true for the robustly consistent partial homomorphisms of the or-instance. Then $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$ has a p_i-solution for both $i \in [2]$ by Lemma 16, which is only non-zero for variables indexed by robustly consistent partial homomorphisms. By Lemma 9, $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ has a p_i-solution in which only robustly consistent partial homomorphisms are non-zero, too. By Lemma 5, there is an integral solution to $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$, which is only non-zero for robustly consistent partial solutions of $\mathbf{B}$. Such solutions to $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ imply solutions to $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \kappa^{\mathbf{A}}_k[\mathbf{B}])$. Hence, the $\mathbb{Z}$-affine k-consistency relaxation wrongly accepts $\mathbf{B}$. ◀

▶ **Lemma 19.** $\mathrm{CSP}(K_3)$ *is not Datalog*$^{\cup}$*-reducible to* $\mathrm{CSP}(\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$.

Theorem 18 and Lemma 19 disprove Conjecture 17. To show the lemma, we show that $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ is not Datalog$^{\cup}$-reducible to $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$ for a prime $p \notin \{2, 3\}$. This suffices because $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ is Datalog$^{\cup}$-reducible to $\mathrm{CSP}(K_3)$ [19]. We use results from finite model theory: $\{2, 3\}$-rank logic [20] extends IFP by a rank operator for definable matrices over $\mathbb{Z}_2$ and $\mathbb{Z}_3$. The logic defines $\mathrm{CSP}(\mathbb{Z}_2^{[3]})$ and $\mathrm{CSP}(\mathbb{Z}_3^{[3]})$ and so also $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$ by Corollary 12. If $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ is Datalog$^{\cup}$-reducible to $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$, then $\{2, 3\}$-rank logic solves $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ because IFP expresses Datalog$^{\cup}$-reductions. But $\mathbb{Z}_p$-equation systems are only solvable in $\{2, 3\}$-rank logic if $p \in \{2, 3\}$ [25] (full proof in Appendix E.1).

5.2 BLP+AIP and BA^k

We introduce another well-studied system of equations for CSPs [3, 10] parameterized by the size of partial solutions [14]. Let k be a positive integer, $\mathbf{A}$ a template τ-structure and $\mathbf{B}$ a $\mathrm{CSP}(\mathbf{A})$-instance. We define the system $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{IP}}(\mathbf{B})$ with variable set $\mathcal{V}^{k,\mathbf{A}}(\mathbf{B})$.

$\mathsf{L}^{k,\mathbf{A}}_{\mathsf{IP}}(\mathbf{B})$: variables $\lambda_{X,f}$ for all $X \in \binom{B}{\leq k}$ and $f\colon X \to A$, and variables $\mu_{R,\bar{b},\bar{a}}$ for all $R \in \tau$, $\bar{b} \in R^{\mathbf{B}}$, and $\bar{a} \in R^{\mathbf{A}}$

$$\sum_{f\colon X \to A} \lambda_{X,f} = 1 \qquad \text{for all } X \in \tbinom{B}{\leq k},$$

$$\sum_{\substack{f\colon X \to A,\\ f|_Y = g}} \lambda_{X,f} = \lambda_{Y,g} \qquad \text{for all } Y \subset X \in \tbinom{B}{\leq k}, g\colon Y \to A,$$

$$\sum_{\bar{a} \in R^{\mathbf{A}}, a_{\bar{i}} = \bar{a}'} \mu_{R,\bar{b},\bar{a}} = \lambda_{X(\bar{b}_{\bar{i}}), \bar{b}_{\bar{i}} \mapsto \bar{a}'} \qquad \text{for all } R \in \tau, \bar{a}' \in A^k, \bar{b} \in R^{\mathbf{B}}, \bar{i} \in [\mathrm{ar}(R)]^k,$$

where $a_{\bar{i}}$ and $b_{\bar{i}}$ are the k-tuples $(a_{\bar{i}_1}, ..., a_{\bar{i}_k})$ and $(b_{\bar{i}_1}, ..., b_{\bar{i}_k})$, respectively, $X(\bar{b}_{\bar{i}})$ is the set of entries of $\bar{b}_{\bar{i}}$, and $\bar{b}_{\bar{i}} \mapsto \bar{a}'$ is the function $X(\bar{b}_{\bar{i}}) \to A$ mapping $\bar{b}_{\bar{i}}$ to $\bar{a}'$.

Different domains of the variables (see [10]) are of interest: If we restrict the variables to $\{0, 1\}$, then $\mathsf{L}^{1,\mathbf{A}}_{\mathsf{IP}}(\mathbf{B})$ is solvable if and only if $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$. The relaxation of $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{IP}}(\mathbf{B})$ to nonnegative rationals is the k**-basic linear programming (BLP)** relaxation $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$[3].

[3] The literature [10, 15] only calls $\mathsf{L}^{1,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ the basic linear programming relaxation. For convenience and uniformity, we extend the notion in this paper to arbitrary k.

The affine relaxation of $\mathsf{L}_{\mathsf{IP}}^{k,\mathbf{A}}(\mathbf{B})$ to all integers is the k**-affine integer programming (AIP)** relaxation $\mathsf{L}_{\mathsf{AIP}}^{k,\mathbf{A}}(\mathbf{B})$. By increasing the parameter k, the BLP and AIP relaxations result in the Sherali-Adams LP hierarchy [31] and the affine integer programming hierarchy [14] of the $\{0,1\}$-system, respectively. In contrast to the $\mathbb{Z}$-affine k-consistency relaxation, the BLP+AIP algorithm is not parameterized by the size of partial solutions k. Ciardo and Živný [17, 15] proposed this parameterized version called $\mathbf{BA}^k$, where BA^1 is just the BLP+AIP algorithm.

$\mathsf{BA}^k(\mathbf{A})$-algorithm: input a CSP($\mathbf{A}$)-instance $\mathbf{B}$
1. Compute a relative interior point $\Phi\colon \mathcal{V}^{k,\mathbf{A}}(\mathbf{B}) \to \mathbb{Q}$ in the polytope defined by $\mathsf{L}_{\mathsf{BLP}}^{k,\mathbf{A}}(\mathbf{B})$. The solution Φ has in particular the property that for each variable $x \in \mathcal{V}^{k,\mathbf{A}}(\mathbf{B})$ there is a solution Ψ to $\mathsf{L}_{\mathsf{BLP}}^{\mathbf{A}}(\mathbf{B})$ such that $\Psi(x) \neq 0$ if and only if $\Phi(x) \neq 0$. If such a point does not exist, reject.
2. Refine $\mathsf{L}_{\mathsf{AIP}}^{k,\mathbf{A}}(\mathbf{B})$ by adding the constraints $x = 0$ whenever $\Phi(x) = 0$ for all $x \in \mathcal{V}^{k,\mathbf{A}}(\mathbf{B})$.
3. If the refined system is feasible (over $\mathbb{Z}$), then accept, otherwise reject.

The original presentation of BA^k [17] uses a slightly different system of equations but one can easily verify that our presentation is equisatisfiable. The system in [17] does not have variables $\lambda_{X,f}$ but uses variables $\lambda_{R_k,\bar{b},\bar{a}}$ instead, where R_k is the full k-ary relation. We deviate from the presentation in [17] to keep it consistent with the systems for the other algorithms. We show that BA^k fails on the counterexample provided for $\mathbb{Z}$-affine k-consistency.

▶ **Theorem 20.** *For every integer k, the algorithm $\mathsf{BA}^k(\mathbf{A})$ does not solve $\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$. This is even true if k is not a constant but an at most sublinear function in the instance size.*

The theorem is proved using the same construction as for Theorem 18. One shows, for k at least the arity of $\mathbf{A}$, that a non-negative or integral solution of $\mathsf{L}_{\mathsf{CSP}}^{k,\mathbf{A}}(\mathbf{B})$ implies a solution for $\mathsf{L}_{\mathsf{BLP}}^{k,\mathbf{A}}(\mathbf{B})$ or $\mathsf{L}_{\mathsf{AIP}}^{k,\mathbf{A}}(\mathbf{B})$, respectively. The solution to $\mathsf{L}_{\mathsf{BLP}}^{k,\mathbf{A}_i}(\mathbf{B}_i)$, where $\mathbf{B}_i$ is the Tseitin-system over $p_i = i + 1$, is non-zero and non-negative exactly for the robustly consistent partial homomorphisms. This carries over to the notion of robustly consistent partial homomorphisms for the or-instance $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$. This implies that the interior point computed in Step 1 is non-zero for robustly consistent partial homomorphisms. So they are not set to zero in Step 2 and there is an integral solution (full proof in Appendix E.2).

5.3 The CLAP Algorithm

The CLAP algorithm [16] combines the BLP and the AIP relaxation. It first iteratively reduces the solution space using BLP by fixing partial solutions to 1 and discarding those for which this refined BLP is not solvable. Then BLP+AIP is run on the restricted solution space, where again a partial solution is fixed:

$\mathsf{CLAP}(\mathbf{A})$-algorithm: input a CSP($\mathbf{A}$)-instance $\mathbf{B}$

1. Maintain, for each relation symbol $R \in \tau$ and each tuple $\bar{b} \in R^{\mathbf{B}}$, a set $S_{\bar{b},R} \subseteq R^{\mathbf{A}}$ of possible images of $\bar{b}$ under a homomorphism. Initialize $S_{\bar{b},R} := R^{\mathbf{A}}$.
2. Repeat until no set $S_{\bar{b},R}$ changes anymore: For each $R \in \tau$, $\bar{b} \in R^{\mathbf{B}}$, and $\bar{a} \in S_{\bar{b},R}$, solve $\mathsf{L}^{1,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ together with the following additional constraints:
$$\begin{aligned} \mu_{R,\bar{b},\bar{a}} &= 1, \\ \mu_{R',\bar{b}',\bar{a}'} &= 0 \qquad \text{for all } R' \in \tau, \bar{b}' \in R'^{\mathbf{B}}, \bar{a}' \notin S_{\bar{b}',R'}. \end{aligned}$$
If this system is not feasible, remove $\bar{a}$ from $S_{\bar{b},R}$.
3. If there are $R \in \tau$ and $\bar{b} \in R^{\mathbf{B}}$ such that $S_{\bar{b},R} = \emptyset$, then reject.
4. For each $R \in \tau$, $\bar{b} \in R^{\mathbf{B}}$, and $\bar{a} \in S_{\bar{b},R}$, execute $\mathsf{BA}^1(\mathbf{A})$ (which is BLP+AIP) on $\mathbf{B}$, where we additionally fix
$$\begin{aligned} \mu_{R,\bar{b},\bar{a}} &= 1, \\ \mu_{R',\bar{b}',\bar{a}'} &= 0 \qquad \text{for all } R' \in \tau, \bar{b}' \in R'^{\mathbf{B}}, \bar{a}' \notin S_{\bar{b}',R'} \end{aligned}$$
in Step 1 of $\mathsf{BA}^1(\mathbf{A})$. If $\mathsf{BA}^1(\mathbf{A})$ accepts, then accept.
5. If $\mathsf{BA}^1(\mathbf{A})$ rejects all inputs in the step before, then reject.

Note that the algorithm accepts in Step 4 if for *one* $R \in \tau$, one $\bar{b} \in R^{\mathbf{B}}$, and one $\bar{a} \in S_{\bar{b},R}$ BLP+AIP accepts when fixing the partial solution $\bar{b} \mapsto \bar{a}$ (in contrast to the stronger requirement that for each $R \in \tau$ and $\bar{b} \in R^{\mathbf{B}}$, there is one $\bar{a} \in S_{\bar{b},R}$ for which BLP+AIP accepts). We show that this weaker requirement can actually by omitted and does not change whether CLAP solves CSP(A). This is crucial to show that CLAP fails on the same counterexample.

▶ **Theorem 21.** $\mathsf{CLAP}(\mathbf{A})$ *does not solve* $\mathrm{CSP}(\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$.

We prove this theorem again with the construction from the proof of Theorem 18, where we pick k to be the arity of $\mathbf{A}$. Again we have to argue that the pruning steps never remove robustly consistent partial homomorphisms. Here we also need the property of the Tseitin systems that their relaxation admits a p_i-solution even if we require a given partial homomorphism to receive value 1. A full proof is provided in Appendix E.3. Theorems 18, 20, and 21 **prove Theorem 1**. Our arguments do not exploit that CLAP uses $\mathsf{L}^{1,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ and $\mathsf{BA}^1(\mathbf{A})$ and is not parametrized by a width k. Hence, a parameterized version of CLAP will fail on the same counterexample.

5.4 The Cohomological k-Consistency Algorithm

The cohomological k-consistency algorithm due to Ó Conghaile [18] was originally presented in categorical and cohomological notions but computing cohomology groups is nothing else than solving a system of linear equations over the integers. Therefore, we can state the algorithm in a way similar to the other algorithms seen so far.

Cohomological k-consistency algorithm: input a CSP($\mathbf{A}$)-instance $\mathbf{B}$
1. Maintain, for each $X \in \binom{B}{\leq k}$, a set $\mathcal{H}(X) \subseteq \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$. Initialize $\mathcal{H}(X) := \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$. **2.** Repeat until none of the sets $\mathcal{H}(X)$ changes anymore: **a.** Run the k-consistency algorithm on $\mathcal{H}$ and remove from each $\mathcal{H}(X)$ the partial homomorphisms that fail the forth-condition or down-closure property. **b.** For each $X \in \binom{B}{\leq k}$ and $f \in \mathcal{H}(X)$, check whether $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \mathcal{H})$ has a solution that satisfies $x_{X,f} = 1$ and $x_{X,f'} = 0$ for every $f' \in \mathcal{H}(X) \setminus \{f\}$. If it does not, then remove f from $\mathcal{H}(X)$ for the next iteration of the loop. **3.** If $\mathcal{H}(X) = \emptyset$ for some $X \in \binom{B}{\leq k}$, then reject; otherwise accept.

Step 2(b) of the algorithm approximates whether there is a global homomorphism whose restriction to X is equal to f via solving the $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \mathcal{H})$ in which we set $x_{X,f} = 1$ and $x_{X,f'} = 0$ for all other f'. At least for the template CSP($\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$), the cohomological k-consistency algorithm is strictly more powerful than the previous ones because it correctly rejects the instances:

▶ **Theorem 22.** *If $\mathbf{A}_1, \mathbf{A}_2$ are templates of Abelian coset-CSPs and $k \geq \mathrm{ar}(A_i) + 1$ for both $i \in [2]$, then the k-cohomological algorithm solves* CSP($\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$).

The proof exploits Lemma 13: When a partial homomorphism of $\mathbf{B}_i$ is fixed as in Step 2(b) of the algorithm, then the tractable homomorphism or-construction is resolved: $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \mathcal{H})$ (with the additional constraints) has an integral solution only if $\mathsf{L}^{k,\mathbf{A}_i}_{\mathbb{Z}\text{-aff}}(\mathbf{B}_i, \mathcal{H})$ has an integral solution. But for Abelian coset-CSPs, $\mathsf{L}^{k,\mathbf{A}_i}_{\mathbb{Z}\text{-aff}}(\mathbf{B}_i, \mathcal{H})$ has an integral solution if and only if $\mathbf{B}_i \in$ CSP($\mathbf{A}_i$) (see Section 6). This means that there is no integral solution and the algorithm rejects $\mathbf{B}$. Actually, showing that the cohomological algorithm correctly solves all instances also follows the algorithm for Lemma 11 (proofs in Appendix E.4). While the algorithm correctly solves the tractable homomorphism or-construction, it fails on the *intractable* one. This proves *without* complexity-theoretic assumptions like P $\neq$ NP that this polynomial-time algorithm does not solve all finite-domain CSPs.

▶ **Theorem 23.** *There is an NP-complete template structure $\mathbf{A}$ such that for every k, the cohomological k-consistency algorithm does not solve* CSP($\mathbf{A}$).

The theorem is proved using the same setup as in the counterexample for the other algorithms, but we use the intractable homomorphism or-construction. This makes the crucial difference that, by Lemma 14, two partial homomorphisms of both $\mathbf{B}_1 \to \mathbf{A}_1$ and $\mathbf{B}_2 \to \mathbf{A}_2$ can be combined into a partial homomorphism $\mathbf{B} \to \mathbf{A}$. If both partial homomorphisms are robustly consistent, then also their combination is not ruled out by k-consistency. When a partial solution is fixed in Step 2, there is still a p_i-solution for both $\mathbf{B}_i$. Hence there is an integral solution by Lemma 5. Thus, all robustly consistent assignments survive Step 2. As indicated in Section 3, CSP($\mathbf{A}$) is NP-complete. Theorems 22 and 23 **prove Theorem 3**.

6 Affine Algorithms and Coset-CSPs

The counterexample we have used so far is not a coset-CSP itself, but a combination of two Abelian coset-CSPs in the homomorphism or-construction. We now set out to explore the power of the affine algorithms on coset-CSPs. It follows from [3] that solving $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{AIP}}(\mathbf{B})$ for $k = 1$ (often just called AIP) suffices to solve all Abelian coset-CSPs. Since all algorithms considered are at least as powerful as AIP, this **proves Theorem 2(1)**. But there are non-Abelian groups for which AIP still works: These are 2-nilpotent groups Γ of odd order, for example certain non-Abelian semidirect products $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ for each odd prime p. For these groups, the Maltsev operation $f(x, y, z) = x - y + z$ for a certain Abelian group Δ is a polymorphism of the non-Abelian template. It follows again from [3] that AIP solves the Γ-coset-CSP. We thank Michael Kompatscher for this proof idea. This **proves Theorem 2(3)**.

It remains to **show Theorem 2(2)**, i.e., that the affine algorithms studied in Section 5 also fail on coset-CSPs. The key idea is to translate our counterexample from Section 5 into a coset-CSP such that hardness for the algorithms is preserved. This is achieved via a series of reduction steps: We start again with Tseitin systems $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. These systems, like every coset-CSP, can be reduced, by a variant of the well-known CFI construction [13], to the *graph isomorphism problem for graphs of bounded color class size* [8]. These are vertex-colored graphs in which only a constant number of vertices have the same color. The reduction gives for each $i \in [2]$ two colored graphs $\mathbf{G}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}}$ and $\widetilde{\mathbf{G}}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}}$ that are isomorphic if and only if $\mathbf{B}_i \in \mathrm{CSP}(\mathbb{Z}^{[3]}_{p_i})$. Next, we use an *isomorphism or-construction*, similar to the one in [8]. This yields two colored graphs which are isomorphic if and only if for at least one $i \in [2]$ we have $\mathbf{G}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}} \cong \widetilde{\mathbf{G}}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}}$. Finally, this isomorphism problem is expressed as a binary S_d-coset-CSP [8], where d is the size of the color classes, which is 18 in our case. This S_{18}-coset CSP has a solution if and only if at least one of the initial two Tseitin instances has a solution – which by construction is not the case.

The technical difficulty is showing that the hardness of the instances is preserved via the reduction steps. One can indeed show that the translations from group-coset-CSPs to bounded color class size graph isomorphism and back as well as the isomorphism or-construction preserve solutions of the width-k affine relaxation and k-consistency in a similar way as we did this for the homomorphism or-construction in Section 3. In the end, we are essentially in the same setting as in the proofs in Section 5. We find a suitable notion of robustly consistent partial homomorphisms which are not ruled out by k-consistency. We also get p_i-solutions to the width-k affine relaxation of the S_{18}-coset-CSP, which are only non-zero for robustly consistent partial homomorphisms. Thus, using Lemma 5, we obtain an integral solution. By following the same reasoning as in Section 5, we can show that neither the $\mathbb{Z}$-affine k-consistency relaxation, BA^k, nor CLAP solve $\mathrm{CSP}(S^{[2]}_{18})$. We provide a full proof including a description of the reduction steps in Appendix G.

7 Conclusion

Regarding the question for a universal polynomial-time CSP algorithm, we conclude that most of the affine algorithms from recent years are not powerful enough. Only *cohomological k-consistency* remains as a candidate because it sets local solutions to 1 when solving the affine relaxation. We are aware of another algorithm with this feature, that is only sketched in the literature: This is C(BLP+AIP), a variation of CLAP mentioned in [16]. This algorithm also involves solving the integer relaxation where additionally a local solution is set to 1. We expect that it solves our counterexample, too, but the precise power of C(BLP+AIP), and also of cohomological k-consistency, remains an intriguing open problem. Another question that we have not addressed is the relationship between the different algorithms. It is obvious from the definitions that cohomological k-consistency subsumes $\mathbb{Z}$-affine k-consistency, and that CLAP subsumes BLP+AIP. How the k-consistency based methods compare to the BLP-based ones remains unanswered; it may be that they are incomparable. In particular, we would like to know if cohomological k-consistency strictly subsumes all the other algorithms. In light of our results, this seems likely, but since the cohomological algorithm does not use the BLP, it is not obvious how it compares to, say, BA^k.

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Appendix

A Extended Preliminaries for the Appendix

We write $[k]$ for the set $\{1, \dots, k\}$. For $k \in \mathbb{N}$ and a set N, we write $\binom{N}{\leq k}$ for the set of all subsets of N of size at most k.

A **relational vocabulary** τ is a set of relation symbols $\{R_1, \dots, R_k\}$ with associated arities $\mathrm{ar}(R_i)$. A **relational τ-structure** is a tuple $\mathbf{A} = (A, R_1^{\mathbf{A}}, \dots, R_k^{\mathbf{A}})$ of a **universe** A and interpretations of the relation symbols such that $R_i^{\mathbf{A}} \subseteq A^{\mathrm{ar}(R_i)}$ for all $i \in [k]$. We use letters $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ for finite relational structures. Their universes will be denoted A, B, and C, respectively. If $\mathbf{A}$ is a structure and $X \subseteq A$ a subset of its universe, then $\mathbf{A}[X]$ denotes the induced substructure with universe X.

For two τ-structures $\mathbf{A}$ and $\mathbf{B}$, we write $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ for the set of **homomorphisms** $\mathbf{A} \to \mathbf{B}$ and $\mathrm{Iso}(\mathbf{A}, \mathbf{B})$ for the set of **isomorphisms** $\mathbf{A} \to \mathbf{B}$.

A **graph** $G = (V, E)$ is a binary $\{E\}$-structure, where we denote its **vertex set** by $V(G)$ and its **edge set** by $E(G)$. The graph G is undirected if $E(G)$ is a symmetric relation and we write uv for an edge incident to vertices u and v. Unless specified otherwise, we consider undirected graphs.

We use the letters Γ and Δ for **finite groups** and usually use letters α, β, γ, and δ for group elements. For arbitrary groups, we write the group operation as multiplication. If we specifically consider Abelian groups, we write the group operation as addition. For the **symmetric group** on d elements, we write S_d.

For an **equation system** L over K (where K can be a finite group, $\mathbb{Q}$, $\mathbb{Z}$, or the-like and is specified in the context), we denote the set of its **variables** by $\mathrm{Var}(\mathsf{L})$. We use the letters Φ and Ψ for **assignments** $\mathrm{Var}(\mathsf{L}) \to K$. By a system of linear equations we refer to, unless stated otherwise, a system over the rationals or integers.

CSPs and Polymorphisms. For a finite τ-structure $\mathbf{A}$, denote by $\mathrm{CSP}(\mathbf{A})$ the **CSP with template $\mathbf{A}$**, i.e., the class of finite τ-structures $\mathbf{B}$ such that there is a homomorphisms $\mathbf{B} \to \mathbf{A}$. We call a structure $\mathbf{B}$ a $\mathrm{CSP}(\mathbf{A})$-instance if $\mathbf{B}$ has the same vocabulary as $\mathbf{A}$.

The complexity of $\mathrm{CSP}(\mathbf{A})$, and also the applicability of certain algorithms, is determined by the **polymorphisms** of the σ-structure $\mathbf{A}$. An ℓ-ary polymorphism p is a homomorphism from the ℓ-th power of $\mathbf{A}$ to $\mathbf{A}$. Concretely, this means that $p\colon A^\ell \to A$ satisfies the following for every $R \in \sigma$ or arity $r = \mathrm{ar}(R)$: for all $\bar{a}_1, \dots, \bar{a}_\ell \in R^{\mathbf{A}}$, the tuple $(p(a_{11}, a_{21}, \dots, a_{\ell 1}), \dots, p(a_{1r}, a_{2r}, \dots, a_{\ell r}))$ is also in $R^{\mathbf{A}}$ (where $\bar{a}_{ij}$ denotes the j-th entry of the tuple $\bar{a}_i$). When we speak of applying a polymorphism to a collection of tuples in a relation, it is meant in this sense. The polymorphisms of a structure are closed under composition, so any term that can be built from variables and applications of polymorphisms again defines a polymorphism of the structure. A ternary operation p is **Maltsev** if it satisfies the identity $p(x, x, y) = p(y, x, x) = y$ for all inputs. If $(A, \cdot)$ is a group, then $f(x, y, z) = x \cdot y^{-1} \cdot z$ is a typical example of a Maltsev operation. The templates with Maltsev polymorphisms form a subclass of all tractable CSPs [12]. For more background on the algebraic approach to CSPs, see for example [5].

Logics, Interpretations, and Reductions. **Inflationary fixed-point logic** (IFP) is the extension of first-order logic by an operator that defines inflationary fixed-points. Roughly speaking, this operator defines a k-ary relation R from a formula $F(x_1, \dots, x_k)$ with free variables $x_1, \dots, x_k$, which itself uses R. The fixed-point is iteratively computed starting from the empty relation and adding in each iteration the tuples $(v_1, \dots, v_k)$ of the input structure to R, for which the assignment $x_i \mapsto v_i$ satisfies F. This process is repeated until R stabilizes. This will always occur because R only becomes larger. For a rigorous introduction of the logic we refer to [21], formal details are not needed in this paper. In particular, IFP can define connected components of graphs, which is not possible in pure first-order logic.

Let σ and τ be two relational vocabularies and L a logic. An $L[\sigma, \tau]$**-interpretation** I defines a (partial) map from τ-structures to σ-structures. Given a σ-structure $\mathbf{A}$, the interpretation I defines a structure $I(\mathbf{A})$ in the following way. Starting with the set A^d, the interpretation I provides a formula that defines a subset B of A^d (the set of all d-tuples satisfying this formula). For every relation R_i, the interpretation provides another formula F_i, that defines the relation R_i for $I(\mathbf{A})$ (the set of $\mathrm{ar}(R_i)$-tuples over B satisfying F_i). Finally, the interpretation I can also define an equivalence relation $\cong$ on B, which has to be compatible with the defined relations, to take the quotient of the structure defined so far by $\cong$. This means that each $\cong$-equivalence class gets contracted to a single vertex. If I does not define such an equivalence, it is called **congruence-free**. For more formal details we also refer to [21], but they are not needed.

This notion of logical reduction can also be used as reduction between decision problems. Given two τ_i-structures $\mathbf{A}_i$ (for $i \in [2]$), $\mathrm{CSP}(\mathbf{A}_1)$ is L**-reducible** to $\mathrm{CSP}(\mathbf{A}_2)$ if there is an $L[\tau_1, \tau_2]$-interpretation I such that all $\mathrm{CSP}(\mathbf{A}_1)$ instances $\mathbf{B}$ satisfy that $\mathbf{B} \in \mathrm{CSP}(\mathbf{A}_1)$ if and only if $I(\mathbf{B}) \in \mathrm{CSP}(\mathbf{A}_2)$ (of course this notion applies also to other means of reductions).

Of particular interest in the context of CSP are Datalog-interpretations. Datalog is another logic, weaker than IFP, that we do not introduce in this paper. We only note that every Datalog interpretation can be expressed by an IFP-interpretation (again see [21, Theorem 9.1.4] for details). Dalmau and Opršal [19] also consider a variant of these reductions called **Datalog$^\cup$ reductions**. These are a composition of congruence-free Datalog reductions (without inequality) and a so-called union gadget. Formally, Dalmau and Opršal work with structures with disjoint sorts, and the union gadget allows to take unions of relations and of sorts. When working in IFP, these sorts can for example be encoded with unary relations. An IFP-interpretation can then define the unification of sorts by defining the new unary relation as the union of the relevant unary relations in the input structure, and unions of other relations are also easily IFP-definable. Thus, every Datalog$^\cup$-reduction can be expressed as an IFP-interpretation.

The k-Consistency Algorithm. A well-known heuristic to solve CSPs is the k-consistency algorithm. For a template structure $\mathbf{A}$ and an instance $\mathbf{B}$, the k-consistency algorithm computes a map $\kappa_k^{\mathbf{A}}[\mathbf{B}]$, which assigns to every $X \in \binom{B}{\leq k}$ a set of partial homomorphisms $\mathbf{B}[X] \to \mathbf{A}$, as follows:

k-consistency algorithm for template $\mathbf{A}$: input a $\mathrm{CSP}(\mathbf{A})$-instance $\mathbf{B}$

1. For all $X \in \binom{B}{\leq k}$, initialize $\kappa_k^{\mathbf{A}}[\mathbf{B}](X)$ to be $\mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$.
2. For all $Y \subset X \in \binom{B}{\leq k}$, ensure that $\kappa_k^{\mathbf{A}}[\mathbf{B}](Y)$ and $\kappa_k^{\mathbf{A}}[\mathbf{B}](X)$ are consistent:

   Forth-Condition: Every $f \in \kappa_k^{\mathbf{A}}[\mathbf{B}](Y)$ extends to some $g \in \kappa_k^{\mathbf{A}}[\mathbf{B}](X)$, that is, $g|_Y = f$.

   Down-Closure: For every $g \in \kappa_k^{\mathbf{A}}[\mathbf{B}](X)$, we have $g|_Y \in \kappa_k^{\mathbf{A}}[\mathbf{B}](Y)$.

   Remove all partial homomorphisms violating at least one of the two conditions.
3. Repeat the prior step until nothing changes anymore.
4. If, for some $X \in \binom{B}{\leq k}$, we have $\kappa_k^{\mathbf{A}}[\mathbf{B}](X) = \emptyset$, then reject, and otherwise accept.

The algorithm computes a greatest fixed-point of such partial homomorphisms that satisfy the forth-condition and the down-closure. We remark that there are different versions of the k-consistency algorithm in the literature, in particular there are ones in which the k-consistency algorithm considers partial homomorphisms whose domain has size $k+1$ [1]. We follow the one given in [19].

A.1 CSP-Relaxation using Affine Systems of Linear Equations

We now introduce a system of linear equations, which will be used to (approximately) solve CSPs. The system presented here is due to Berkholz and Grohe [6]. We will transfer hardness results for this system to other systems used in the different algorithms. Let $\mathbf{A}$ be a template structure and $\mathbf{B}$ be a CSP($\mathbf{A}$)-instance. We define the **width-k affine relaxation** $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ with the aim to encode (approximately) whether $\mathbf{B}$ is in CSP($\mathbf{A}$).

$\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$: variables $x_{X,f}$ for all $X \in \binom{B}{\leq k}$ and all $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$

$$\sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}),\\ f|_{X\setminus\{b\}}=g}} x_{X,f} = x_{X\setminus\{b\},g} \quad \text{for all } X \in \binom{B}{\leq k}, b \in X, g \in \mathrm{Hom}(\mathbf{B}[X\setminus\{b\}],\mathbf{A}) \tag{L1}$$

$$x_{\emptyset,\emptyset} = 1 \tag{L2}$$

In Equation L2, $\emptyset$ denotes the unique homomorphism $\mathbf{B}[\emptyset] \to \mathbf{A}$. If k is at least the arity of $\mathbf{A}$, then $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$ if and only if $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ has a nonnegative integral solution (and actually a $\{0,1\}$-solution) [6]. We will be mainly interested in integral solutions of $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$, so without the non-negativity restriction. Such solutions can be computed in polynomial time. To show the existence of these solutions, we will also consider special rational solutions:

▶ **Definition 24** (p-Solution). *For an integer p and a system of linear equations L over $\mathbb{Q}$, a **p-solution** is a solution $\Phi\colon \mathrm{Var}(\mathsf{L}) \to \mathbb{Q}$ of L satisfying for each variable $x \in \mathrm{Var}(\mathsf{L})$ that $\Phi(x) = 0$ or $\Phi(x) = p^i$ for some $i \in \mathbb{Z}$.*

▶ **Lemma 5** ([6]). *If p and q are coprime integers and a system L of linear equations over $\mathbb{Q}$ has a p-solution and a q-solution, then L has an integral solution, which is only non-zero for variables on which the p-solution or the q-solution is non-zero.*

▶ **Lemma 25.** *All solutions Φ of $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ satisfy for all $X \in \binom{B}{\leq k}$, $Y \subseteq X$, and $g \in \mathrm{Hom}(\mathbf{B}[Y], \mathbf{A})$ that*

$$\sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}),\\ f|_{Y}=g}} \Phi(x_{X,f}) = \Phi(x_{Y,g}).$$

In particular, for all $X \in \binom{B}{\leq k}$, we have

$$\sum_{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A})} \Phi(x_{X,f}) = 1.$$

Proof. Let $X = \{v_1, \dots, v_\ell\}$ and $Y = \{v_1, \dots, v_j\}$ for some $j \leq k$. The first claim is proven inductively on $\ell - j$ using equation of Type L1. The second claim follows as special case for $Y = \emptyset$ and Equation L2. ◀

B Details on Group Coset-CSPs

Let Γ be a finite group. We define **Γ-coset-CSPs** [6, 22], a class of CSPs, in which variables range over Γ and the constraints are of the following form. For an r-tuple of variables $\bar{x} = (x_1, \dots, x_r)$, an r-ary **Γ-coset-constraint** is the constraint $\bar{x} \in \Delta\delta$, where $\Delta \leq \Gamma^r$ is a subgroup of Γ^r and $\delta \in \Gamma^r$. Hence, $\Delta\delta$ is a right coset of Γ^r. When we use the term **coset-CSP**, we refer to a Γ-coset-CSP in this sense.

It is known that, for each fixed Γ and each fixed arity r, every r-ary Γ-coset-CSP is polynomial-time solvable [22]. For every finite group Γ and every arity r, there is a structure $\Gamma^{[r]}$ such that every r-ary Γ-coset-CSP can be seen as a $\Gamma^{[r]}$-instance and CSP($\Gamma^{[r]}$) contains all solvable r-ary Γ-coset-CSPs. The tractability of CSP($\Gamma^{[r]}$) can also be seen from the fact that $\Gamma^{[r]}$ admits a Maltsev polymorphism [12] (whose existence was already noted, but not made explicit, in [6]). In fact, the universe of a CSP can be extended to a group such that the CSP is a coset-CSP in this sense if and only if its template has the Maltsev polymorphism $f(x, y, z) = xy^{-1}z$.

▶ **Lemma 6.** *For every finite template $\mathbf{A} = (A, R_1^{\mathbf{A}}, ..., R_m^{\mathbf{A}})$ and every binary operation $\cdot \colon A \times A \to A$ such that $\Gamma = (A, \cdot)$ is a group,*

- *the map $f : \Gamma^3 \to \Gamma$ defined by $f(x, y, z) = xy^{-1}z$ is a polymorphism of $\mathbf{A}$ if and only if*
- *each relation $R_i^{\mathbf{A}}$ is a coset of a subgroup of Γ^r for some $r \in \mathbb{N}$.*

Proof. For the backwards direction, let $\Delta\delta$ be a right coset of Γ^r and consider r-tuples

$$(\alpha_1, \dots, \alpha_r), (\beta_1, \dots, \beta_r), (\gamma_1, \dots, \gamma_r) \in \Delta\delta.$$

Then $(f(\alpha_1, \beta_1, \gamma_1), \dots, f(\alpha_r, \beta_r, \gamma_r)) = (\alpha_1\beta_1^{-1}\gamma_1, \dots, \alpha_r\beta_r^{-1}\gamma_r) \in \Delta\delta\delta^{-1}\Delta\Delta\delta = \Delta\delta$ because Δ is a subgroup of Γ^r. Hence, f is a polymorphism. For the other direction, suppose that f is a polymorphism of the r-ary relation $R_i^{\mathbf{A}}$. We can write $R_i^{\mathbf{A}} = K\gamma$, for some $K \subseteq \Gamma^r, \gamma \in \Gamma^r$ such that K contains the neutral element $\bar{0} \in \Gamma^r$. It remains to show that K is a subgroup of Γ^r. Let $\alpha, \beta \in K$. Then $\alpha\gamma, \gamma, \beta\gamma \in R_i^{\mathbf{A}}$. Apply f to these three tuples in this order. For each $j \in [r]$, we have $f(\alpha_j\gamma_j, \gamma_j, \beta_j\gamma_j) = \alpha_j\beta_j\gamma_j$. Because f is a polymorphism of $R_i^{\mathbf{A}}$, it follows that $\alpha\beta\gamma \in R_i^{\mathbf{A}} = K\gamma$. So $\alpha\beta \in K$, and K is a subgroup. ◀

Thus, coset-CSPs are a natural class to study. In particular, being Maltsev, they are always tractable even if Γ is non-Abelian. By contrast, for *systems of linear equations*, we have NP-completeness if (and only if) Γ is non-Abelian [24]. Systems of linear equations over an *Abelian* group Γ can however be viewed as a Γ-coset-CSP: A linear equation $x_1 + \dots + x_k = \alpha$ for $\alpha \in \Gamma$ is equivalent to the Γ-coset-constraint $(x_1, ..., x_k) \in \Delta\delta_\alpha$, where $\Delta = \{(b_1, ..., b_k) \mid b_1 + \dots + b_k = 0\}$, and $\delta_\alpha = (\alpha, 0, ..., 0)$. Hence, when we consider equation systems over Abelian groups in Section 4, we can treat them uniformly as coset-CSPs. For coset-CSPs over the cyclic group $\mathbb{Z}_p$, we will also need the (first-order definable) reverse translation from coset-CSP to linear equations:

▶ **Lemma 26.** *Let p be a prime and $\mathbf{B}$ an instance of* CSP($\mathbb{Z}_p^{[r]}$). *Then there is a system of linear equations over $\mathbb{Z}_p$ that has a solution if and only if $\mathbf{B} \in$* CSP($\mathbb{Z}_p^{[r]}$). *Moreover, the equation system is definable from $\mathbf{B}$ in first-order logic.*

Proof. Let $\bar{b} \in \Delta\gamma$ be an r-ary constraint in $\mathbf{B}$. Let $\{\delta_1, ..., \delta_m\} \subseteq \Delta$ be a set of generators of the subgroup Δ. Then $\alpha \in \mathbb{Z}_p^r$ is in $\Delta\gamma$ if and only if it satisfies: $\alpha = \left(\sum_{i\in[m]} z_i \cdot \delta_i\right) + \gamma$, for some $z_i \in \mathbb{Z}_p$. In this equation, γ and the δ_i are r-tuples, so we can break this up into r

many equations, one for each $j \in [r]$: $\alpha_j = \left(\sum_{i\in[m]} z_i \cdot \delta_{ij}\right) + \gamma_j$. Each constraint $\bar{b} \in \Delta\gamma$ in $\mathbf{B}$ is translated into this set of r equations, with the z_i being the variables. Formally, we use different variables for each constraint, so the z_i are also indexed with the constraint $\bar{b} \in \Delta\gamma$ in $\mathbf{B}$ that they belong to. For each $\Delta \leq \mathbb{Z}_p^r$, we can use a fixed generating set, so with respect to this, the translation from coset constraints into the equations is first-order definable in $\mathbf{B}$. ◀

It is also known that if Γ is Abelian, then the tractable CSPs over Γ are precisely the Γ-coset-CSPs. For some non-Abelian groups Γ, there exist examples of tractable templates that contain non-coset relations. But, even then, we can only have tractability if the constraints are so-called "nearsubgroup" constraints (see [22]). So the Γ-coset-CSPs that we study here exactly cover the tractable regime for Abelian Γ, and nearly cover it for general Γ.

C Details on the Homomorphism OR-Construction

For $i \in [2]$, let $\mathbf{A}_i$ and $\mathbf{B}_i$ be nonempty τ_i-structures, for which we assume that τ_1 and τ_2 are disjoint. We see the $\mathbf{A}_i$ as template structures and the $\mathbf{B}_i$ as the corresponding instances. We aim to define two structures $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$ if and only if there is an $i \in [2]$ such that $\mathbf{B}_i \in \mathrm{CSP}(\mathbf{A}_i)$.

Let S be a fresh binary relation symbol. Set $\tau := \tau_1 \cup \tau_2 \cup \{S\}$, where the arities of the relations are inherited from τ_1 and τ_2. Our construction is parameterized by subsets $W_i \subseteq A_i$ for each $i \in [2]$. We also let c_1 and c_2 be two fresh vertices. For $i \in [2]$, we let

$$W_i^{\ell,c_i} := \left\{ (u_1, \dots, u_k) \in (A_i \cup \{c_i\})^\ell \;\middle|\; u_j \in W_i, u_k = c_i \text{ for some } j,k \in [k] \right\}$$

be the set of all ℓ-tuples that containing c_i and some element of W_i. We define the τ-structures $\mathbf{A} = \mathbf{OR}[\mathbf{A}_1, \mathbf{A}_2, W_1, W_2]$ and $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$. In the following, we assume that the universes of $\mathbf{A}_1$ and $\mathbf{A}_2$ and the ones of $\mathbf{B}_1$ and $\mathbf{B}_2$ are disjoint (and non-empty), so that the following unions are disjoint unions.

$$\begin{aligned}
A &:= A_1 \cup A_2 \cup \{c_1, c_2\} \\
R^{\mathbf{A}} &:= R^{\mathbf{A}_i} \cup \{c_i\}^{\mathrm{ar}(R)} \cup W_i^{\mathrm{ar}(R),c_i} && \text{for all } i \in [2], R \in \tau_i \\
S^{\mathbf{A}} &:= \Big(A_1 \times (W_2 \cup \{c_2\})\Big) \cup \Big((W_1 \cup \{c_1\}) \times A_2\Big)
\end{aligned}$$

$$\begin{aligned}
B &:= B_1 \cup B_2 \\
R^{\mathbf{B}} &:= R^{\mathbf{B}_i} && \text{for all } i \in [2], R \in \tau_i \\
S^{\mathbf{B}} &:= B_1 \times B_2
\end{aligned}$$

Figure 2 illustrates the construction. The following lemma shows that this definition yields a homomorphism or-construction independently of the sets W_i. As we will see, the choice of W_i controls embeddings of partial homomorphisms and the complexity of the resulting template. We will later work with two concrete instantiations of the sets W_i, one of which yields a tractable and the other an intractable or-construction. But first we prove all properties that hold for any choice of W_i.

▶ **Lemma 27.** $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$ *if and only if there is an* $i \in [2]$ *such that* $\mathbf{B}_i \in \mathrm{CSP}(\mathbf{A}_i)$.

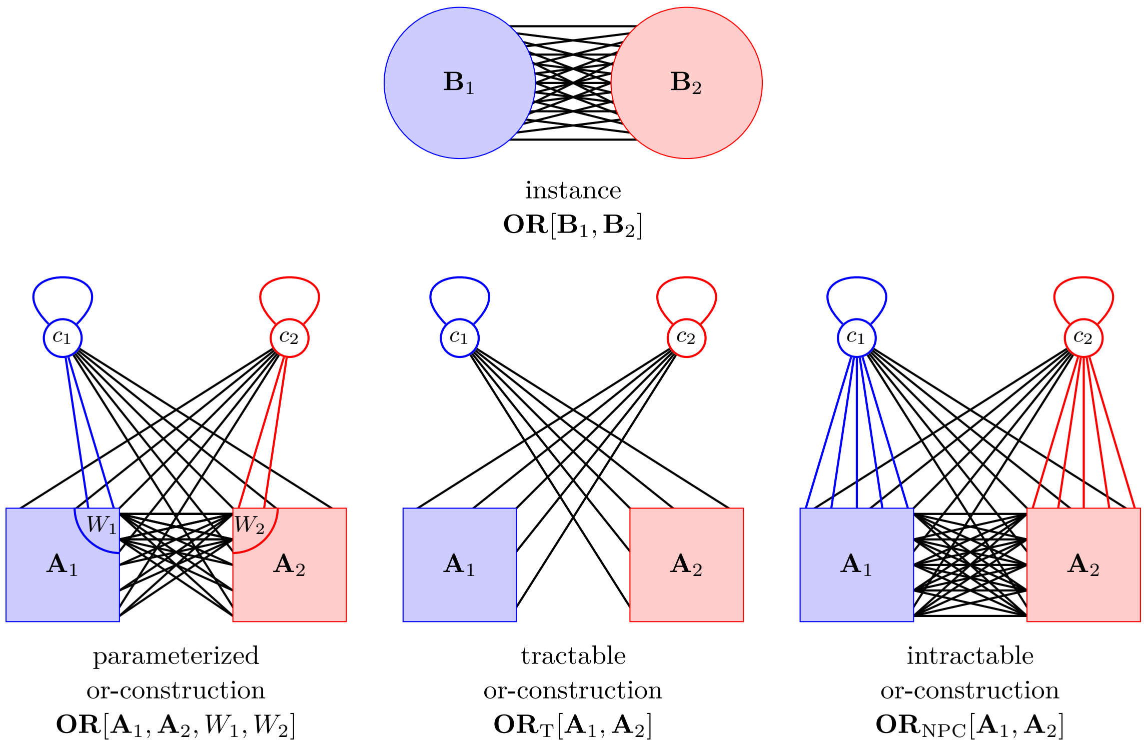

Figure 2 The different homomorphism or-constructions: The picture assumes that the two vocabularies τ_1 and τ_2 are binary and contain a single relation each (blue and red). At the top the instance $\mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$. At the bottom three different version on the templates: the general parameterized construction, and the special cases of the tractable and intractable construction, which will be discussed in Sections C.1 and C.2, respectively. The new S-relation is drawn in black, where the edges are all oriented from left to right. Pairs added to the relation of τ_1 or τ_2 are drawn in blue or red, respectively.

Proof. First, assume that there is a homomorphism $f\colon \mathbf{B}_i \to \mathbf{A}_i$. We define a homomorphism $g\colon \mathbf{B} \to \mathbf{A}$ via

$$g(b) := \begin{cases} f(b) & \text{if } b \in B_i, \\ c_{3-i} & \text{otherwise.} \end{cases}$$

We show that g is a homomorphism. Let $R \in \tau_i$. Then $g(R^{\mathbf{B}}) = g(R^{\mathbf{B}_i}) = f(R_i^{\mathbf{B}}) \subseteq R^{\mathbf{A}_i} \subseteq R^{\mathbf{A}}$. Let $R \in \tau_{3-i}$. Then $g(R^{\mathbf{B}}) = g(R^{\mathbf{B}_{3-i}}) = \{c_{3-i}\}^{\mathrm{ar}(R)} \subseteq R^{\mathbf{A}}$. Finally, we consider the relation S. We have $g(S^{\mathbf{B}}) = g(B_1 \times B_2) \subseteq A_1 \times \{c_2\} \cup \{c_1\} \times A_2 \subseteq S^{\mathbf{A}}$ by the definitions of $\mathbf{A}$ and $\mathbf{B}$.

Second, assume that there is a homomorphism $f\colon \mathbf{B} \to \mathbf{A}$. Because f preserves the relation S, we have $f(B_i) \subseteq A_i \cup \{c_i\}$ for both $i \in [2]$. We claim that for some $i \in [2]$, we actually have $f(B_i) \subseteq A_i$. Assume that for $i \in [2]$ this is not the case, that is, there is some $b \in B_i$ such that $f(b) = c_i$. By definition of $\mathbf{B}$, we have $\{b\} \times B_{3-i} \subseteq S^{\mathbf{B}}$ if $i = 1$ and $B_{3-i} \times \{b\} \subseteq S^{\mathbf{B}}$ if $i = 2$. Because f is a homomorphism and by the definition of $\mathbf{A}$, we have $f(\{b\} \times B_{3-i}) \subseteq \{c_i\} \times A_{3-i}$ if $i = 1$ and similar for $i = 2$. But this in particular implies that $f(B_{3-i}) \subseteq A_{3-i}$ as claimed.

So there is an $i \in [2]$ such that $f(B_i) \subseteq A_i$. We define $g\colon B_i \to A_i$ via $g(b) = f(b)$ for all $b \in B_i$. Because $R^{\mathbf{B}} = R^{\mathbf{B}_i}$ and $g(R^{\mathbf{B}_i}) \subseteq A_i^{\mathrm{ar}(R)}$, and f is a homomorphism, g maps tuples in $R^{\mathbf{B}_i}$ to tuples in $R^{\mathbf{A}_i}$, for all $R \in \tau_i$. Hence the function g is a homomorphism $\mathbf{B}_i \to \mathbf{A}_i$. ◀

We now analyze which partial homomorphisms $\mathbf{B}_i \to \mathbf{A}_i$ can be extended to partial or global homomorphisms $\mathbf{B} \to \mathbf{A}$. For $i \in [2]$, denote by $\mathbf{A}|_i$ the structure $\mathbf{A}[A_i \cup \{c_i\}]$.

▶ **Lemma 28.** *Let $i \in [2]$, $X \subseteq B_i$, and $f \in \mathrm{Hom}(\mathbf{B}_i[X], \mathbf{A}_i)$. Then $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A}|_i)$.*

Proof. The relation S is clearly preserved since $S^{\mathbf{B}[X]} = \emptyset$. Because $R^{\mathbf{B}} \subseteq B_i^{\mathrm{ar}(R)}$, the map f also preserves R. ◀

▶ **Lemma 29.** *Let $i \in [2]$. For every $X \subseteq B_i$ and $f \in \mathrm{Hom}(\mathbf{B}_i[X], \mathbf{A}|_i[(W_i \cup \{c_i\})])$, the map $g\colon B_i \to W_i \cup \{c_i\}$ defined by*

$$g(b) := \begin{cases} f(b) & \text{if } b \in X, \\ c_i & \text{otherwise.} \end{cases}$$

is a homomorphism in $\mathrm{Hom}(\mathbf{B}[B_i], \mathbf{A}|_i[W_i \cup \{c_i\}])$. It satisfies in particular $g|_X = f$.

Proof. Clearly, g preserves S since $S^{\mathbf{B}} \cap B_i^2 = \emptyset$. Let $R \in \tau_i$ and $\bar{b} \in R^{\mathbf{B}} = R^{\mathbf{B}_i}$.

- If all elements of $\bar{b}$ are contained in X, then $g(\bar{b}) = f(\bar{b}) \in R^{\mathbf{A}_i} \subseteq R^{\mathbf{A}}$.
- If no elements of $\bar{b}$ are contained in X, then $g(\bar{b}) \in \{c_i\}^{\mathrm{ar}(R)} \subseteq R^{\mathbf{A}}$.
- Otherwise, some element of $\bar{b}$ is contained in X but not all of them. This means that $g(\bar{b})$ contains c_i at least once and at least one element of W_i. Hence $g(\bar{b}) \in W_i^{\mathrm{ar}(R), c_i} \subseteq R^{\mathbf{A}}$. ◀

▶ **Lemma 30.** *Let $X_i \subseteq B_i$ and $f_i \in \mathrm{Hom}(\mathbf{B}[X_i], \mathbf{A}|_i)$ for both $i \in [2]$. Let $f\colon X_1 \cup X_2 \to A$ be the map defined by each f_i on X_i for $i \in [2]$. If there is an $i \in [2]$ such that $c_i \notin f_i(X_i)$ and $f_{3-i}(B_{3-i}) \subseteq W_{3-i} \cup \{c_{3-i}\}$, then $f \in \mathrm{Hom}(\mathbf{B}[X_1 \cup X_2], \mathbf{A})$.*

Proof. Let $j \in [2]$ and $R \in \tau_j$. Because $R^{\mathbf{B}} \subseteq B_j^{\mathrm{ar}(R)}$, the map f also preserves R. It remains to show that f preserves S. Assume that $c_1 \notin f_1(X_1)$ and $f_2(X_2) \subseteq W_2 \cup \{c_2\}$ (the case for X_1 and X_2 swapped is similar). Then, $f(S^{\mathbf{B}} \cap (X_1 \times X_2)) \subseteq A_1 \times (W_2 \cup \{c_2\}) \subseteq S^{\mathbf{A}}$ and thus the relation S is preserved. ◀

▶ **Corollary 31.** *Let $i \in [2]$ and $f_i \in \mathrm{Hom}(\mathbf{B}_i, \mathbf{A}_i)$. Assume $X \subseteq B_{3-i}$ and let $f_{3-i} \in \mathrm{Hom}(\mathbf{B}_{3-i}[X], \mathbf{A}_{3-i}[W_{3-i}])$. Then there is a $g \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$ that agrees with f_i on B_i and with f_{3-i} on X.*

Proof. By Lemma 28, we have $f_i \in \mathrm{Hom}(\mathbf{B}[B_i], \mathbf{A}|_i)$. By Lemma 29, f_{3-i} extends to some $f' \in \mathrm{Hom}(\mathbf{B}[B_{3-i}], \mathbf{A}|_{3-i}[W_{3-i} \cup \{c_{3-i}\}])$. Together, f_i and f' yield the desired homomorphism g by Lemma 30. ◀

Next, we show that the homomorphism or-construction is compatible with k-consistency and solving $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ in the sense that if k-consistency accepts $\mathbf{B}_i$, or $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$ is satisfiable, then k-consistency accepts $\mathbf{B}$, or $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ is satisfiable, respectively.

▶ **Lemma 32.** *Let $k \in \mathbb{N}$, $X_i \subseteq B_i$ and $f_i \in \mathrm{Hom}(\mathbf{B}[X_i], \mathbf{A}|_i)$ for both $i \in [2]$ such that $|X_1 \cup X_2| \leq k$. Let $f \colon X_1 \cup X_2 \to A$ be the map induced by f_1 and f_2. If there is an $i \in [2]$ such that*

- *$f_i \in \kappa^{\mathbf{A}_i}_k[\mathbf{B}_i](X_i)$ (so in particular $c_i \notin f_i(X_i)$),*
- *$f_{3-i}(X_{3-i}) \subseteq W_{3-i} \cup \{c_{3-i}\}$, and*
- *for $X'_{3-i} := f^{-1}_{3-i}(W_{3-i})$ we have $f_{3-i}|_{X'_{3-i}} \in \kappa^{\mathbf{A}_{3-i}}_k[\mathbf{B}_{3-i}](X'_{3-i})$,*

then $f \in \kappa^{\mathbf{A}}_k[\mathbf{B}](X_1 \cup X_2)$.

Proof. For a set $Z \subseteq B$, let $Z_j := Z \cap B_j$ for both $j \in [2]$. For all $X \subseteq \binom{B}{\leq k}$, let $H(X) \subseteq \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ be the set of all $f \colon X \to A$ for which there is an $i \in [2]$ such that $f|_{X_i} \in \kappa^{\mathbf{A}_i}_k[\mathbf{B}_i](X_i)$, $f_{3-i}(B_{3-i}) \subseteq (W_{3-i} \cup \{c_{3-i}\})$ and for $X'_{3-i} := f^{-1}_{3-i}(W_{3-i})$ we have $f|_{X'_{3-i}} \in \kappa^{\mathbf{A}_{3-i}}_k[\mathbf{B}_{3-i}](X'_{3-i})$. By Lemma 29, the function f_{3-i} satisfies $f_{3-i} \in \mathrm{Hom}(\mathbf{B}_{3-i}[X_{3-i}], \mathbf{A}|_{3-i})$. Hence, the function f is indeed a homomorphism in $\mathrm{Hom}(\mathbf{B}[X_1 \cup X_2], \mathbf{A})$ by Lemma 30. We show that the family of the $H(X)$ satisfy the Forth-Condition and the Down-Closure. Hence, the partial homomorphisms in the $H(X)$ are not discarded by the k-consistency algorithm.

Let $Y \subset X \subseteq \binom{B}{\leq k}$. We first show the Forth-Condition. Let $f \in H(Y)$ and $f_j = f|_{Y_j}$ for both $j \in [2]$. By construction, we have $f|_{X_i} \in \kappa^{\mathbf{A}_i}_k[\mathbf{B}_i](X_i)$, $f_{3-i}(B_{3-i}) \subseteq (W_{3-i} \cup \{c_{3-i}\})$ and for $X'_{3-i} := f^{-1}_{3-i}(W_{3-i})$ that $f|_{X'_{3-i}} \in \kappa^{\mathbf{A}_{3-i}}_k[\mathbf{B}_{3-i}](X'_{3-i})$. By the Forth-Condition for $\mathbf{B}_i$, there is a $g_i \in \kappa^{\mathbf{A}_i}_k[\mathbf{B}_i](X_i)$ such that $g|_{X_i} = f_i$. Let g_{3-i} be the extension of f_{3-i} to X_{3-i} by $f_{3-i}(b) = c_{3-i}$ for all $b \in X_{3-i} \setminus Y_{3-i}$. Then the map $g \colon X_1 \cup X_2 \to A$ induced by g_1 and g_2 is in $H(X)$.

We secondly show the Down-Closure. Let $g \in H(X)$, and $g_j := g|_{Y_j}$ for both $j \in [2]$. Let $g' \colon Y \to A$ be the function induced by g_1 and g_2. By construction, there is an $i \in [2]$ such that we have $g_i \in \kappa^{\mathbf{A}_i}_k[\mathbf{B}_i](X_i)$, $g_{3-i}(B_j) \subseteq (W_{3-j} \cup \{c_{3-j}\}$, and for $X'_{3-i} := g^{-1}_{3-i}(W_{3-i})$ that $g_{3-i}|_{X'_{3-i}} \in \kappa^{\mathbf{A}_{3-i}}_k[\mathbf{B}_{3-i}](X'_{3-i})$. By the Down-Closure for the $\mathbf{B}_j$, we have that $g_i|_{Y_i} \in \kappa^{\mathbf{A}_i}_k[\mathbf{B}_i](Y_i)$ and $g_{3-i}|_{Y_{3-i} \cap X'_{3-i}} \in \kappa^{\mathbf{A}_{3-i}}_k[\mathbf{B}_{3-i}](Y_{3-i} \cap X'_{3-i})$. Clearly, we also have $g_{3-i}|_{Y_{3-i} \cap X'_{3-i}}(B_j) \subseteq (W_{3-j} \cup \{c_{3-j}\})$. So indeed, $g' \in H(Y)$. ◀

▶ **Lemma 33.** *Let $i \in [2]$ and assume that Φ is a solution to $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$. Furthermore, let $h \in \mathrm{Hom}(\mathbf{B}_{3-i}, \mathbf{A}|_{3-i}[W_{3-i} \cup \{c_{3-i}\}])$. Then the following map Ψ is a solution to $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$. Let $X \in \binom{\mathbf{B}}{\leq k}$, $X_j := X \cap B_j$ for $j \in [2]$, and $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$. We set*

$$\Psi(x_{X,f}) := \begin{cases} \Phi(x_{X_i, f|_{X_i}}) & \text{if } f|_{X_i} \in \mathrm{Hom}(\mathbf{B}_i[X_i], \mathbf{A}_i) \text{ and } f|_{X_{3-i}} = h|_{X_{3-i}} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if Φ is an integral solution or a p-solution, then Ψ is an integral solution or p-solution, respectively.

Proof. We show that the equations of Type L1 are satisfied by Ψ. Let $X \in \binom{B}{\leq k}$, $b \in X$, $X_j := X \cap B_j$ for $j \in [2]$, and $g \in \mathrm{Hom}(\mathbf{B}[X \setminus \{b\}], \mathbf{A})$. By definition of Ψ, we have

$$\sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}),\\ f|_{X\setminus\{b\}}=g}} \Psi(x_{X,f}) = \sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}),\\ f|_{X\setminus\{b\}}=g,\\ f|_{X_i} \in \mathrm{Hom}(\mathbf{B}_i[X_i],\mathbf{A}_i),\\ f|_{X_{3-i}}=h|_{X_{3-i}}}} \Phi(x_{X_i,f|_{X_i}}). \tag{$\star$}$$

We make a case distinction.

(1) Assume that $g|_{X_{3-i}\setminus\{b\}} \neq h|_{X_{3-i}\setminus\{b\}}$. Then $f|_{X_{3-i}} \neq h|_{X_{3-i}}$ for every $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ such that $f|_{X\setminus\{b\}} = g$ because $f|_{X_{3-i}\setminus\{b\}} = g|_{X_{3-i}\setminus\{b\}} \neq h|_{X_{3-i}\setminus\{b\}}$. This implies that the summation in $(\star)$ sums over zero many numbers, and thus

$$(\star) = 0 = \Psi(x_{X\setminus\{b\},g}).$$

(2) Assume that $g|_{X_i\setminus\{b\}} \notin \mathrm{Hom}(\mathbf{B}_i[X_i \setminus \{b\}], \mathbf{A}_i)$. Then $f|_{X_i} \notin \mathrm{Hom}(\mathbf{B}_i[X_i], \mathbf{A}_i)$ for every $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A},)$ such that $f|_{X\setminus\{b\}} = g$ because $f|_{X_i\setminus\{b\}} = g|_{X_i\setminus\{b\}} \notin \mathrm{Hom}(\mathbf{B}_i[X_i \setminus \{b\}], \mathbf{A}_i)$. We again have $(\star) = 0 = \Psi(x_{X\setminus\{b\},g})$ as in the case before.

(3) Assume that $b \in B_{3-i}$. By the cases before, we can assume that $g|_{X_{3-i}\setminus\{b\}} = h|_{X_{3-i}\setminus\{b\}}$ and $g|_{X_i} \in \mathrm{Hom}(\mathbf{B}_i[X_i], \mathbf{A}_i)$. Then there is at most one $f' \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ such that $f'|_{X\setminus\{b\}} = g$ and $f'|_{X_{3-i}} = h|_{X_{3-i}}$, namely the extension of g by mapping b to $h(b)$. By Lemmas 28 and 30, we indeed have $f' \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$. Thus, f' is unique. We also have $f'|_{X_i} \in \mathrm{Hom}(\mathbf{B}_i[X_i], \mathbf{A}_i)$ because $f'|_{X_i} = g|_{X_i} \in \mathrm{Hom}(\mathbf{B}_i[X_i], \mathbf{A}_i)$. It follows

$$(\star) = \Phi(x_{X_i,f'|_{X_i}}) = \Phi(x_{X_i,g|_{X_i}}) = \Psi(x_{X,g}).$$

(4) Lastly, we assume that $b \in B_i$. By the cases before, we may assume that $g|_{X_{3-i}} = h|_{X_{3-i}}$ and $g|_{X_i\setminus\{b\}} \in \mathrm{Hom}(\mathbf{B}_i[X_i \setminus \{b\}], \mathbf{A}_i)$. Since, as in the case before, there is a unique extension of a partial homomorphism in $\mathrm{Hom}(\mathbf{B}_i[X_i], \mathbf{A}_i)$ to a partial homomorphism in $\mathrm{Hom}(\mathbf{B}[X], \mathbf{A}_i)$ that agrees with h on X_{3-i}, we have

$$(\star) = \sum_{\substack{f|_{X_i} \in \mathrm{Hom}(\mathbf{B}_i[X_i],\mathbf{A}_i),\\ f|_{X_i\setminus\{b\}}=g|_{X_i\setminus\{b\}},\\ f|_{X_{3-i}}=h|_{X_{3-i}}}} \Phi(x_{X_i,f|_{X_i}}) = \Phi(x_{X_i\setminus b, g|_{X_i\setminus b}}) = \Psi(x_{X,g})$$

because Φ is a solution to $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$.

It remains to verify that Equation L2 is satisfied: $\Psi(x_{\emptyset,\emptyset}) = \Phi(x_{\emptyset,\emptyset}) = 1$ by the definition of Ψ and because Φ is a solution to $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$. ◀

C.1 The Tractable Case

We now consider the case that $W_1 = W_2 = \emptyset$. In this setting, the homomorphism or-construction yields a tractable CSP if $\mathrm{CSP}(\mathbf{A}_i)$ is tractable for both $i \in [2]$. We refer to this construction as the **tractable homomorphism or-construction** and write $\mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2] := \mathbf{OR}[\mathbf{A}_1, \mathbf{A}_2, \emptyset, \emptyset]$. We start with corollaries from the lemmas of the previous subsection:

▶ **Lemma 8.** *Let* $\mathbf{A} = \mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$, $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$, $k \in \mathbb{N}$, $i \in [2]$, $X \in \binom{B}{\leq k}$, *and* $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$. *If* $f(X \cap B_{3-i}) = \{c_{3-i}\}$ *and* $f|_{X\cap B_i} \in \kappa_k^{\mathbf{A}_i}[\mathbf{B}_i](X \cap B_i)$, *then* $f \in \kappa_k^{\mathbf{A}}[\mathbf{B}](X)$.

Proof. Immediately follows from Lemma 32. ◀

▶ **Lemma 9.** *Let* $\mathbf{A} = \mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$, $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$, $i \in [2]$, *and* Φ *be a solution to* $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$. *Then there is a solution* Ψ *to* $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ *defined, for every* $X \in \binom{B}{\leq k}$ *and* $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$, *by* $\Psi(x_{X,f}) = \Phi(x_{X\cap B_i}, f|_{X\cap B_i})$ *if* $f(X \cap B_{3-i}) = \{c_{3-i}\}$ *and* $\Psi(x_{X,f}) = 0$ *otherwise. In particular,* Ψ *is a p-solution or integral, if* Φ *is a p-solution or integral, respectively.*

Proof. Follows from Lemma 33 and the fact that the map $B_{3-i} \to \{c_{3-i}\}$ is a partial homomorphism. ◀

We now prove that the tractable or-construction indeed deserves its name because it generally preserves tractability of $\mathbf{A}_1$ and $\mathbf{A}_2$. In the special case of Maltsev templates, also this stronger condition is preserved:

▶ **Lemma 10.** *If* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *have a Maltsev polymorphism, then* $\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$ *has one.*

Proof. Let $\mathbf{A} = \mathbf{OR}_\mathrm{T}[\mathbf{A}_1, \mathbf{A}_2]$. Let f_1, f_2 be the Maltsev polymorphisms of $\mathbf{A}_1, \mathbf{A}_2$, respectively. Define $f : A^3 \to A$ as follows:

$$f(x,y,z) := \begin{cases} f_i(x,y,z) & \text{if } x,y,z \in A_i, \\ c_i & \text{if } x,y,z \in A_i \cup \{c_i\} \text{ and exactly one or all are equal to } c_i, \\ a & \text{otherwise, where } a \text{ is the left-most input} \\ & \text{not occurring exactly twice} \end{cases}$$

It can be checked that f is (idempotent) Maltsev: If all three inputs are in A_i, then this is inherited from f_i. If all inputs are in $A_i \cup \{c_i\}$, and one or three of them are equal to c_i, then the second case applies and so, $f(y,x,x) = f(x,x,y) = y$ and $f(x,x,x) = x$ are ensured. If all inputs are in $A_i \cup \{c_i\}$ and exactly two of them are equal to c_i, then the third case applies and produces the correct outcome. If the inputs are mixed between $A_1 \cup \{c_1\}$ and $A_2 \cup \{c_2\}$, then also the third case ensures $f(y,x,x) = f(x,x,y) = y$.
The relation $R^{\mathbf{A}}$ is preserved by f: If f is applied to three tuples in $R^{\mathbf{A}_i}$, then it behaves like f_i, which is a polymorphism of $\mathbf{A}_i$, so it will produce a tuple in $R^{\mathbf{A}_i} \subseteq R^{\mathbf{A}}$. If f is applied to two tuples in $R^{\mathbf{A}_i}$ and the third tuple (c_i, c_i, c_i), then the output will be $(c_i, c_i, c_i) \in R^{\mathbf{A}}$. The same applies if all three input tuples are (c_i, c_i, c_i). If two of the input tuples are (c_i, c_i, c_i) and the third one is $\bar{a} \in R^{\mathbf{A}_i}$, then f will output $\bar{a}$, which is correct. Also $S^{\mathbf{A}}$ is preserved by f: Let $\bar{a}_1, \bar{a}_2, \bar{a}_3 \in S^{\mathbf{A}}$. Each of these pairs is either of the type (a, c_2) or (c_1, a), with $a \in A_1$ or $a \in A_2$, respectively. If all three pairs are of the same type, then f will also produce a pair of that type, which is in $S^{\mathbf{A}}$. Otherwise, two of the three pairs are of type, say, (a, c_2), and the other one is of type (c_1, a). Then f produces a pair of the type (c_1, a), with $a \in A_2$, so this is in $S^{\mathbf{A}}$. ◀

In particular, the above lemma shows tractability of the or-construction in the case that $\mathrm{CSP}(\mathbf{A}_1)$ and $\mathrm{CSP}(\mathbf{A}_2)$ have a Maltsev polymorphism. The next lemma shows this for the general case that $\mathrm{CSP}(\mathbf{A}_1)$ and $\mathrm{CSP}(\mathbf{A}_2)$ are tractable by providing a polynomial-time algorithm for $\mathrm{CSP}(\mathbf{OR}_\mathrm{T}[\mathbf{A}_1, \mathbf{A}_2])$. More strongly, we will show, using that algorithm, that if both $\mathrm{CSP}(\mathbf{A}_i)$ are definable in a logic subsuming inflationary fixed-point logic, then so is $\mathrm{CSP}(\mathbf{OR}_\mathrm{T}[\mathbf{A}_1, \mathbf{A}_2])$.

▶ **Lemma 11.** *If* $\mathrm{CSP}(\mathbf{A}_1)$ *and* $\mathrm{CSP}(\mathbf{A}_2)$ *are tractable, then* $\mathrm{CSP}(\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2])$ *is tractable.*

Proof. Let $\mathbf{A} = \mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2]$. Assume $\mathbf{C}$ is a CSP($\mathbf{A}$)-instance. A vertex u of $\mathbf{C}$ is a τ_i-vertex if

- u is in a tuple in a τ_i-relation, or
- there is a pair $(v, w) \in S^{\mathbf{C}}$ such that $u = v$ and $i = 1$ or $u = w$ and $i = 2$.

If there is a vertex that is both a τ_1-vertex and a τ_2-vertex, then $\mathbf{C} \notin \mathrm{CSP}(\mathbf{A})$. So assume this is not the case. If a vertex is neither a τ_1-vertex nor a τ_2-vertex, then this vertex is an isolated vertex. Hence, we can assume that every vertex is either a τ_1-vertex or a τ_2-vertex. A τ_i-component of $\mathbf{C}$ is a connected component of $\mathbf{C}|_{\tau_i}$ consisting only of τ_i-vertices. Note that every τ_i-vertex is in a τ_i-component of $\mathbf{C}$.

Consider the following graph G_S. The vertices of G_S are the τ_1-components and τ_2-components of $\mathbf{C}$. There is an edge between a τ_1-component D and a τ_2-component D', if $D \times D' \cap S^{\mathbf{C}} \neq \emptyset$. Hence, we can also obtain G_S by contracting each τ_i-component into a single vertex and only keeping the relation S. An S-component is a connected component of G_S when viewing S as an undirected edge relation. For an S-component D, denote by $D|_i$ the set of τ_i-vertices contained in D.

We claim that $\mathbf{C} \in \mathrm{CSP}(\mathbf{A})$ if and only if for every S-component D there is an $i \in [2]$, such that $\mathbf{C}[D|_i] \in \mathrm{CSP}(\mathbf{A}_i)$. First assume that $\mathbf{C} \in \mathrm{CSP}(\mathbf{A})$, witnessed by a homomorphism $f \colon \mathbf{C} \to \mathbf{A}$. Let D be an S-component. Then $\mathbf{C}[D] \in \mathrm{CSP}(\mathbf{A})$ because CSPs are closed under induced substructures. If D contains no τ_i-component, then $\mathbf{C}[D|_i]$ is trivially in CSP($\mathbf{A}_i$). So assume D contains both τ_1-components and τ_2-components. Let $i \in [2]$ and $D^i \in D$ be some τ_i-component and assume there is a vertex $u \in D^i$ such that $f(u) \in A_i$. Because D^i is a connected component in $\mathbf{C}|_{\tau_i}$, all vertices in D^i have to be mapped onto A_i by f (the vertex c_i in A is not connected to the A_i vertices). All neighbors of D^i in G_S are τ_{3-i}-components (otherwise, a vertex was both, a τ_1-vertex and a τ_2-vertex). Let D^{3-i} be such a neighbor of D^i. Then there are vertices $u \in D^i$ and $v \in D^{3-i}$ such that $(u, v) \in S^{\mathbf{C}}$ or $(v, u) \in S^{\mathbf{C}}$ (depending on whether $i = 1$ or $i = 2$). By symmetry, assume $(u, v) \in S^{\mathbf{C}}$. Then $f(v) = c_{3-i}$ because vertices of $\mathbf{A}_i$ are only connected to c_{3-i} in $S^{\mathbf{A}}$. Now because D^{3-i} is a connected component of $\mathbf{C}[D|_{3-i}]$, all vertices of D^{3-i} are mapped to c_{3-i} by f. One similarly shows, that if D^i contains a vertex that is mapped to c_i by f, then all vertices in D^i are mapped to c_i and all neighbors of D^i in G_S are mapped onto A_{3-i}.

Hence, we can inductively show, that if for some $i \in [2]$ one τ_i-component in D is mapped onto A_i, then all τ_i-components in D are mapped onto A_i. This implies that $\mathbf{C}[D|_i] \in \mathrm{CSP}(\mathbf{A}_i)$. There has to be one $i \in [2]$ such that a τ_i-component is mapped onto A_i because we have seen that such a component is either mapped onto A_i or onto c_i, that neighbored components are mapped on exactly the other option, and there are both τ_1-components and τ_2-components in C.

Second, assume that for every S-component D there is an $i \in [2]$ such that $\mathbf{C}[D|_i] \in \mathrm{CSP}(\mathbf{A}_i)$. We construct a homomorphism $f \colon \mathbf{C} \to \mathbf{A}$. For an S-component D of $\mathbf{C}$, note that $\mathbf{C}[D]$ is a connected component of $\mathbf{C}$ because all vertices related by a τ_i-relation are in the same τ_i-component and two τ_1 and τ_2 components, which are related by S, are connected in the same S-component. Hence, it suffices to show that $\mathbf{C}[D] \in \mathrm{CSP}(\mathbf{A})$ for every S-component D. Let D be an S-component of $\mathbf{C}$ and $i \in [2]$ such that $\mathbf{C}[D|_i] \in \mathrm{CSP}(\mathbf{A}_i)$. Essentially by the arguments in the proof of Lemma 27, one shows that the extension of a homomorphism $\mathbf{C}[D|_i] \to \mathbf{A}_i$ to all τ_{3-i} vertices by mapping them to c_{3-i} is a homomorphism $\mathbf{C}[D] \to \mathbf{A}$.

So the algorithm deciding CSP($\mathbf{A}$) works as follows. Because CSP($\mathbf{A}_i$) are tractable, we pick polynomial-time algorithms for these CSP. First compute the τ_i-components of $\mathbf{C}$, then the graph G_S, and then check for every S-component D of G_S wether for some $i \in [2]$ the algorithm for $\mathrm{CSP}(\mathbf{A}_i)$ accepts $\mathbf{C}[D|_i]$. If this is the case, then accepts $\mathbf{C}$, and otherwise rejects it. Clearly this algorithm runs in polynomial time and by the reasoning before it correctly decides $\mathrm{CSP}(\mathbf{A})$. ◀

▶ **Corollary 12.** *Let L be a logic that is at least expressive as inflationary fixed-point logic. If $\mathrm{CSP}(\mathbf{A}_1)$ and $\mathrm{CSP}(\mathbf{A}_2)$ are L-definable, then $\mathrm{CSP}(\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2])$ is L-definable.*

Proof. It can be easily seen that the algorithm in the proof of Lemma 11 is L-definable given L-formulas defining $\mathrm{CSP}(\mathbf{A}_1)$ and $\mathrm{CSP}(\mathbf{A}_2)$. The algorithm essentially computes different connected components, which is definable in inflationary fixed-point logic. ◀

We will use the tractable homomorphism or-construction to build tractable CSPs that are not solved correctly by the algorithms in Theorem 1. We consider templates $\mathbf{A}_1$ and $\mathbf{A}_2$ for which the tractable or-construction $\mathbf{A} = \mathbf{OR}_\mathrm{T}[\mathbf{A}_1, \mathbf{A}_2]$ will guarantee the existence of integral solutions to $\mathsf{L}_{\mathsf{CSP}}^{\mathbf{A},k}(\mathbf{B})$ for certain instances $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2] \notin \mathrm{CSP}(\mathbf{A})$. This will in particular be the case even though no such integral solution exists for $\mathsf{L}_{\mathsf{CSP}}^{\mathbf{A}_1,k}(\mathbf{B}_1)$ and $\mathsf{L}_{\mathsf{CSP}}^{\mathbf{A}_2,k}(\mathbf{B}_2)$. However, the *cohomological k-consistency* algorithm will be able to tell that $\mathsf{L}_{\mathsf{CSP}}^{\mathbf{A}_1,k}(\mathbf{B}_1)$ and $\mathsf{L}_{\mathsf{CSP}}^{\mathbf{A}_2,k}(\mathbf{B}_2)$ do not have an integral solution, and this will be sufficient for it to correctly output that $\mathbf{B} \notin \mathrm{CSP}(\mathbf{A})$. The next two lemmas are the technical foundation for this and will be used in the proof of the first part of Theorem 3. The crucial point is that the cohomological algorithm considers solutions to $\mathsf{L}_{\mathsf{CSP}}^{\mathbf{A},k}(\mathbf{B})$ in which for certain sets X, every $f : X \to A$ that has c_i in its image receives value 0.

▶ **Lemma 34.** *Let $k \geq 2$, $i \in [2]$, and Φ be a solution to $\mathsf{L}_{\mathsf{CSP}}^{\mathbf{A},k+1}(\mathbf{B})$. If there is a set $Z \in \binom{B_i}{\leq k}$ such that for every $f \in \mathrm{Hom}(\mathbf{B}[Z], \mathbf{A})$ with $c_i \in f(Z)$ it holds that $\Phi(x_{Z,f}) = 0$, then we have $\Phi(x_{X,g}) = 0$ for every $X \in \binom{B_i}{\leq k}$ and every $g \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ with $c_i \in f(X)$.*

Proof. We first show that every Φ is only non-zero for $x_{X,f}$, for non-empty $X \subseteq B_i$, if either $f(X) \subseteq A_i$ or $f(X) = \{c_i\}$.

▷ Claim 35. Let $X \in \binom{B_i}{\leq k}$, $g \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ such that there is $b \in X$ with $g(b) \in A_i$, and there is $b' \in X$ with $g(b') \notin A_i$. Then $\Phi(x_{X,g}) = 0$.

Proof. We extend the domain and let $Y = X \cup \{y\}$ for an arbitrary $y \in B_{3-i}$. Since Φ is a solution, it satisfies Equation L1:

$$\Phi(x_{X,g}) = \sum_{f \in \mathrm{Hom}(\mathbf{B}[Y],\mathbf{A}), f|_X = g} \Phi(x_{X,f}) = 0.$$

The last equality is due to the fact that the sum is over the empty set. Indeed, every partial homomorphism that maps $b \in B_i$ to an element of A_i has to map $y \in B_{3-i}$ to c_{3-i} (because of the relation S). But then it also has to map $b' \in B_i$ to an element of A_i, again to preserve the relation S between b' and y. So g is not extendable to a partial homomorphism with domain Y. ◁

Similarly, it is not possible to have non-zero values for partial homomorphisms which do not map any element in $X \subseteq B_i$ to A_i.

▷ Claim 36. Let $X \in \binom{B_i}{\leq k}$ be non-empty, $g \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ such that $g(X) \cap (A_{3-i} \cup \{c_{3-i}\}) \neq \emptyset$. Then $\Phi(x_{X,g}) = 0$.

Proof. We argue in the same way as in the proof of the previous claim and extend X by some $y \in B_{3-i}$. Since g maps at least one element $x \in X$ to $A_{3-i} \cup \{c_{3-i}\}$, it is impossible to preserve the directed $S^{\mathbf{B}}$-edge between x and y, so g cannot be extended to a partial homomorphism with domain Y. ◁

From these two claims it follows that for $X \in \binom{B_i}{\leq k}$, the number $\Phi(x_{X,f})$ can only be non-zero if $f(X) \subseteq A_i$ or $f(X) = \{c_i\}$. It thus remains to show that $\Phi(x_{X,f}) = 0$ if $f(X) = \{c_i\}$. For $X = Z$, we have this by assumption of the lemma. For other sets X, we use the next claim to propagate this inductively. In the following, we write $X \mapsto c_i$ for the partial homomorphism with domain X that sends every element to c_i.

▷ Claim 37. Let $X \in \binom{B_i}{\leq k}$ be non-empty such that $\Phi(x_{X,X\mapsto c_i}) = 0$.

(1) Let $b \in B_i \setminus X$ and $Y = X \uplus \{b\}$. Then $\Phi(x_{Y,Y\mapsto c_i}) = 0$.

(2) Let $Y \subseteq X$ be non-empty. Then $\Phi(x_{Y,Y\mapsto c_i}) = 0$.

Proof. We start with the first part of the claim. Again we use Equation L1:

$$
\begin{aligned}
0 = \Phi(x_{X,X\mapsto c_i}) &= \sum_{f \in \mathrm{Hom}(\mathbf{B}[Y],\mathbf{A}), f|_X = g} \Phi(x_{Y,f}) = \\
&= \Phi(x_{Y,Y\mapsto c_i}) + \sum_{f \in \mathrm{Hom}(\mathbf{B}[Y],\mathbf{A}), f|_X = g, f(b) \neq c_i} \Phi(x_{Y,f}) \\
&= \Phi(x_{Y,Y\mapsto c_i}) + 0.
\end{aligned}
$$

The last equality uses Claim 1 and 2, which tell us that Φ is zero whenever $f(Y) \neq \{c_i\}$ or $f(Y) \not\subseteq A_i$. The second part of the claim is proved similarly:

$$
\begin{aligned}
\Phi(x_{Y,Y\mapsto c_i}) &= \sum_{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}), f(Y) = \{c_i\}} \Phi(x_{X,f}) = \\
&= \Phi(x_{X,X\mapsto c_i}) + 0 = 0.
\end{aligned}
$$

Again we use that by Claim 1 and 2, all extensions of $Y \mapsto c_i$ which do not map all of X to c_i receive the value 0 in Φ. ◁

The lemma follows by inductively applying Claim 3 and noting that by Claim 1 and 2, $\Phi(x_{X,f}) = 0$ whenever $f(X) \not\subseteq A_i$ or $f(X) \neq \{c_i\}$. ◀

▶ **Lemma 13.** *Let $k \geq 2$, $i \in [2]$, and Φ be a solution to $\mathsf{L}^{\mathbf{A},k+1}_{\mathsf{CSP}}(\mathbf{B})$. If there is a set $X \in \binom{B_i}{\leq k}$ such that for $f \colon X \to \{c_i\}$ it holds that $\Phi(x_{X,f}) = 0$, then $\Phi|_{B_i}$ is a solution to $\mathsf{L}^{\mathbf{A}_i,k}_{\mathsf{CSP}}(\mathbf{B}_i)$.*

Proof. We have to argue that $\Psi|_{B_i}$ satisfies all equations of Type L1 in $\mathsf{L}^{\mathbf{A}_i,k}_{\mathsf{CSP}}(\mathbf{B}_i)$. So let $X \in \binom{B_i}{\leq k}, b \in X, g \in \mathrm{Hom}(\mathbf{B}_i[X \setminus \{b\}], \mathbf{A}_i)$. The associated equation is

$$
\sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}_i[X],\mathbf{A}_i), \\ f|_{X\setminus\{b\}} = g}} x_{X,f} = x_{X\setminus\{b\},g} \tag{$\star$}
$$

We know that Φ satisfies the corresponding equation in $\mathsf{L}^{\mathbf{A},k+1}_{\mathsf{CSP}}(\mathbf{B})$, so

$$
\sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}), \\ f|_{X\setminus\{b\}} = g}} \Phi(x_{X,f}) = \Phi(x_{X\setminus\{b\},g})
$$

Lemma 34 implies that only those extensions f of g with $f(b) \in A_i$ can have a non-zero value in Φ (this is also true if g has empty domain). These are precisely the f that appear in the sum in ($\star$). Thus, Φ also satisfies ($\star$). Finally, $\Psi|_{B_i}$ trivially satisfies Equation L2. ◀

From this lemma it follows that the tractable homomorphism or-construction cannot be used to make CSPs harder for algorithms that solve $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ over the integers and fix local solutions. To deal with this and be able to prove the second part of Theorem 3, we now sacrifice tractability of the homomorphism or-construction, which will also make it harder for the cohomological algorithm.

C.2 The Intractable Case

This section deals with the case that $W_1 = A_1$ and $W_2 = A_2$. In this case, the homomorphism or-construction has the drawback to yield an NP-complete CSP even if $\mathrm{CSP}(\mathbf{A}_1)$ and $\mathrm{CSP}(\mathbf{A}_2)$ are tractable. But it has the benefit that more partial homomorphisms can be extended to global ones, in particular if $\mathbf{B}_1 \in \mathrm{CSP}(\mathbf{A}_1)$ and $\mathbf{B}_2 \notin \mathrm{CSP}(\mathbf{A}_2)$, we can still extend partial homomorphisms $\mathbf{B}_2 \to \mathbf{A}_2$ to global homomorphisms. We set $\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2] := \mathbf{OR}[\mathbf{A}_1, \mathbf{A}_2, A_1, A_2]$. We refer to this as the **intractable homomorphism or-construction**. We again start with corollaries from the general or-construction.

▶ **Lemma 14.** *Let $\mathbf{A} = \mathbf{OR}_{NPC}[\mathbf{A}_1, \mathbf{A}_2]$, $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$, $X_i \subseteq B_i$, and $f_i \in \mathrm{Hom}(\mathbf{B}_i, \mathbf{A}_i)$ for both $i \in [2]$. The map $f\colon X_1 \cup X_2 \to A$ induced by f_1 and f_2 satisfies $f \in \mathrm{Hom}(\mathbf{B}[X_1 \cup X_2], \mathbf{A})$.*

Proof. The lemma is a consequence of Lemma 30. ◀

▶ **Lemma 38.** *Assume $\mathbf{A} = \mathbf{OR}_{NPC}[\mathbf{A}_1, \mathbf{A}_2]$ and $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$. Let $k \in \mathbb{N}$, $X_i \subseteq B_i$ and $f_i \in \mathrm{Hom}(\mathbf{B}_i, \mathbf{A}_i)$ for both $i \in [2]$ such that $|X_1 \cup X_2| \leq k$. If for some $i \in [2]$ we have $f_i \in \kappa^{\mathbf{A}_i}_k[\mathbf{B}_i](X_i)$, then the map $f\colon X_i \cup X_2 \to A$ induced by f_1 and f_2 satisfies $f \in \kappa^{\mathbf{A}}_k[\mathbf{B}](X_1 \cup X_2)$.*

Proof. The lemma follows from Lemma 32. ◀

▶ **Lemma 39.** *Assume $\mathbf{A} = \mathbf{OR}_{NPC}[\mathbf{A}_1, \mathbf{A}_2]$ and $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$. Let $i \in [2]$, let Φ be a solution to $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$, and let $Y \subseteq B_{3-i}$ and $h \in \mathrm{Hom}(\mathbf{B}_{3-i}[Y], \mathbf{A}_{3-i})$. Then there is a solution Ψ to $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ such that for every $X \in \binom{\mathbf{B}}{\leq k}$ and $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$, $\Psi(x_{X,f}) \neq 0$ implies $\Psi(x_{X,f}) = \Phi(x_{X \cap B_i, f|_{X \cap B_i}})$ and $f|_{X \cap Y} = h|_{X \cap Y}$. In particular, if $\Psi(x_{Y,h}) = 1$ and Φ is an integral solution or a p-solution, then Ψ is an integral solution or p-solution, respectively.*

Proof. Let $\hat{h}$ be the extension of h to a homomorphism $\mathbf{B}_{3-i} \to \mathbf{A}[A_{3-i} \cup \{c_{3-i}\}]$ by Lemma 29. Then the claimed solution exists by Lemma 33. It satisfies $\Psi(x_{Y,h}) = 1$ because $\Phi(x_{\emptyset,\emptyset}) = 1$ since Φ is a solution to $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}_i)$. ◀

The following lemmas show that the intractable homomorphism or-construction yields an NP-complete CSP if for both $\mathbf{A}_i$ there are non-trivial no-instances for $\mathrm{CSP}(\mathbf{A}_i)$. We start to consider the very simple case that both $\mathbf{A}_i$ are of size 1 and contain a ternary relation that is empty. For a ternary relational symbol R, denote by $\mathbf{1}_R$ such a $\{R\}$-structure.

▶ **Lemma 40.** *Let R_i be ternary relation symbols and $\mathbf{A}_i := \mathbf{1}_{R_i}$. Then $\mathrm{CSP}(\mathbf{OR}_{NPC}[\mathbf{A}_1, \mathbf{A}_2])$ is NP-complete.*

Proof. We show that monotone 3-SAT, that variant of 3-SAT in which in every clause the variables all have to be either positive or all negated, polynomial-time-reduces (and in particular also Karp-reduces) to $\mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$. Let F be a 3-CNF formula with variables $V = \{x_1, \ldots, x_n\}$ such that in each clause the variables are either all positive or all negative. We define an instance $\mathbf{B}$ of $\mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$. Let R_1 and R_2 be the ternary relation symbols in τ_1 and τ_2. For every variable x_i, we add two vertices x_i and $\bar{x}_i$ and add the pair $(x_i, \bar{x}_i)$ to the $S^{\mathbf{B}}$-relation. For every clause $\{x_1, x_2, x_3\}$ in which the variables are positive, we add the triple (x_1, x_2, x_3) to $R_1^{\mathbf{B}}$. For every clause $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, in which the variables are negated, we add the triple $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ to $R_2^{\mathbf{B}}$.

We show that $\mathbf{B} \in \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$ if and only if F is satisfiable. Assume that the universe of $\mathbf{A}_i$ is $A_i = \{a_i\}$ for both $i \in [2]$. We first assume that F is satisfiable and let $\Phi \colon V \to \{0, 1\}$ be a satisfying assignment. We show that the following map g is a homomorphism $\mathbf{B} \to \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$: for a variable $x \in V$ define

$$g(x) := \begin{cases} a_1 & \text{if } \Phi(x) = 0, \\ c_1 & \text{if } \Phi(x) = 1, \end{cases}$$

$$g(\bar{x}) := \begin{cases} a_2 & \text{if } \Phi(x) = 1, \\ c_2 & \text{if } \Phi(x) = 0. \end{cases}$$

Now consider a triple $(x_1, x_2, x_2) \in R_1^{\mathbf{B}}$. Then for at least one $j \in [3]$, we have that $\Phi(x_j) = 1$ because Φ satisfies all clauses. Hence, the tuple $g(x_1, x_2, x_3)$ contains c_1 at least once. And so, $g(x_1, x_2, x_3) \in R^{\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]}$. Similarly, consider a triple $(\bar{x}_1, \bar{x}_2, \bar{x}_2) \in R_2^{\mathbf{B}}$. Then for at least one $j \in [3]$, we have that $\Phi(x_j) = 0$ because Φ satisfies all clauses. Hence, the tuple $g(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ contains c_2 at least once. And so, $g(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in R_2^{\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]}$.

Second, assume that $\mathbf{B} \in \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$ and let $g \in \mathrm{Hom}(\mathbf{B}, \mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$. Consider the assignment $\Phi \colon V \to \{0, 1\}$ that is defined as follows:

$$\Phi(x) := \begin{cases} 1 & \text{if } g(x) = c_1, \\ 0 & \text{otherwise.} \end{cases}$$

We show that Φ satisfies F. We first see, that if $g(x) = c_1$, then $g(\bar{x}) = a_2$ and similarly, if $g(x) = c_2$, then $g(\bar{x}) = a_1$ because $(x, \bar{x}) \in S^{\mathbf{B}}$. Let $\{x_1, x_2, x_3\}$ be a clause of F in which all variables occur positively. Because g is a homomorphism, g maps one x_i to c_1 and hence $\Phi(x_i) = 1$ and Φ satisfies the clause $\{x_1, x_2, x_3\}$. Let $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ be a clause of F in which all variables occur negatively. Because g is a homomorphism, g maps one $\bar{x}_i$ to c_2. Hence, g cannot map x_i to c_1 and thus $g(x_i) = 0$. Thus, Φ satisfies the clause $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$. Because we considered arbitrary clauses, Φ satisfies F. ◀

The next lemma shows that many CSPs reduce to the one in the following lemma. We call a no-instance of a CSP **inclusion-wise minimal** if every proper induced subinstance of it is a yes-instance. The following lemma requires that each $\mathbf{A}_i$ has a inclusion-wise minimal no-instance of order at least 3. This covers many CSPs, for example many kinds of equation systems, group coset CSPs, but also $\mathrm{CSP}(P_1)$ with K_3 as inclusion-wise minimal no instance (where P_1 is the path of length one and K_3 the triangle).

▶ **Lemma 41.** *Let R_i be ternary relation symbols and $\mathbf{A}_i$ be τ_i-structures for $i \in [2]$. If for both $i \in [2]$, there is a inclusion-wise minimal τ_i-structure $\mathbf{C}_i \notin \mathrm{CSP}(\mathbf{A}_i)$ of order at least 3, then $\mathbf{OR}_{NPC}[\mathbf{1}_{R_1}, \mathbf{1}_{R_2}]$ is Karp-reducible to $\mathbf{OR}_{NPC}[\mathbf{A}_1, \mathbf{A}_2]$.*

Proof. For $i \in [2]$, let $\mathbf{C}_i$ be a inclusion-wise minimal τ_i-structure $\mathbf{C}_i \notin \mathrm{CSP}(\mathbf{A}_i)$ of order at least 3. Pick a partition of C_i into C_i^1, C_i^2, and C_i^3. Such a partition exists because $\mathbf{C}_i$ has order at least 3.

Let $\mathbf{B}$ be an $\mathbf{OR}_{\mathrm{NPC}}[1_{R_1}, 1_{R_2}]$-instance. We define a $\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]$-instance $\mathbf{B}'$. For every tuple $\bar{u} = (u_1, u_2, u_3) \in R_i^{\mathbf{B}}$, introduce a fresh copy $\mathbf{C}_i^{\bar{u}}$ of $\mathbf{C}_i$. We denote the corresponding partition of $C_i^{\bar u}$ by $C_i^{\bar u,1}$, $C_i^{\bar u,2}$, and $C_i^{\bar u,3}$. For every pair $(u_1, u_2) \in S^{\mathbf{B}}$, add the pairs (v_1, v_2) to $S^{\mathbf{B}'}$ such that for each $i \in [2]$ there exists a tuple $\bar w = (w_1, w_2, w_3) \in R_i^{\mathbf{B}}$ for which $w_j = u_i$ and $v_i \in C_i^{\bar w, j}$ for some $j \in [3]$.

We show that $\mathbf{B} \in \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{1}_{R_1}, \mathbf{1}_{R_2}])$ if and only if $\mathbf{B}' \in \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$. First assume that $\mathbf{B} \in \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{1}_{R_1}, \mathbf{1}_{R_2}])$. Let $g \in \mathrm{Hom}(\mathbf{B}, \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{1}_{R_1}, \mathbf{1}_{R_2}]))$. For every tuple $\bar u = (u_1, u_2, u_3) \in R_i^{\mathbf{B}}$, define $W_{\bar u} := \{u_i \mid g(u_i) = c_i\}$, where c_i is the fresh vertex added to $\mathbf{1}_{R_i}$ in the homomorphism or-construction $\mathbf{OR}_{\mathrm{NPC}}[\mathbf{1}_{R_1}, \mathbf{1}_{R_2}]$. Let $i \in [2]$ and $\bar u \in R_i^{\mathbf{B}}$. Because g is a homomorphism, we have $W_{\bar u} \neq \emptyset$. Pick a homomorphism $f_{\bar u} \in \mathrm{Hom}(\mathbf{C}_i^{\bar u}[C_i^{\bar u} \setminus W_{\bar u}], \mathbf{A}_i)$. Such a homomorphism exists because $\mathbf{C}_i$ is inclusion-wise minimal. Now define a map h from B' to $\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]$ via

$$h(x) = \begin{cases} f_{\bar u}(x) & \text{if } x \in C_i^{\bar u, j}, \bar u = (u_1, u_2, u_3), \text{ and } g(u_j) \in A_i, \\ c_i' & \text{if otherwise } x \in C_i^{\bar u, j} \text{ and } g(u_j) \notin A_i. \end{cases}$$

Here, c_i' denotes the the fresh vertex added to $\mathbf{A}_i$ in the homomorphism or-construction $\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]$. We show that h is indeed a homomorphism. Let $\bar u \in R_i^{\mathbf{B}}$. Then $h|_{C_i^{\bar u} \setminus W_{\bar u}} = f_{\bar u}$ and $h(W_{\bar u}) = \{c_i'\}$. Hence $h|_{C_i^{\bar u}} \in \mathrm{Hom}(B'[C^{\bar u}], \mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$ because extending a partial homomorphism by mapping to c_i' always yields a partial homomorphism in the intractable construction. It remains to show that h also preserves the S relation. Let $(v_1, v_2) \in S^{\mathbf{B}'}$. Then there exists a pair $(u_1, u_2) \in S^{\mathbf{B}}$ such that for each i there exists a tuple $\bar w = (w_1, w_2, w_3) \in R_i^{\mathbf{B}}$ for which $w_j = u_i$ and $v_i \in C_i^{\bar w, j}$ for some $j \in [3]$. Because g is a homomorphism, g cannot map both u_i to c_i. Hence, for one $i \in [2]$, we have $g(u_i) \in A_i$ and thus $h(v_i) \in A_i$. We conclude that $h(u_1, u_2) \in S^{\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]}$.

Second, assume that $\mathbf{B}' \in \mathrm{CSP}(\mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$ and let $g \in \mathrm{Hom}(\mathbf{B}, \mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2])$. We define a map h from B to $\mathbf{OR}_{\mathrm{NPC}}[\mathbf{1}_{R_1}, \mathbf{1}_{R_2}]$ via

$$h(u) := \begin{cases} c_i & \text{if } g(v) = c_i' \text{ for some } \bar w = (w_1, w_2, w_3) \in R_i^{\mathbf{B}}, u = w_j, v \in C_i^{\bar w, j}, \\ 1_i & \text{otherwise.} \end{cases}$$

Here 1_i denotes the unique universe member of $\mathbf{1}_{R_i}$. We show that h is a homomorphism.

Let $\bar u = (u_1, u_2, u_3) \in R_i^{\mathbf{B}}$. Because g is a homomorphism, there is an $j \in [3]$ and a $v \in C_i^{\bar u, j}$ such that $g(v) = c_i'$ because $\mathbf{C}_i \notin \mathrm{CSP}(\mathbf{A}_i)$ but removing any vertex from $\mathbf{C}_i$ turns it into a yes-instance. Hence, $h(u_j) = c_i$. We conclude that $h(\bar u) \in R_i^{\mathbf{OR}_{\mathrm{NPC}}[1_{R_1}, 1_{R_2}]}$. It remains to show that g preserves the S relation. Let $(u_1, u_2) \in S^{\mathbf{B}}$. We see that for all tuples $\bar w^i = (w_1^i, w_2^i, w_3^i) \in R_i^{\mathbf{B}}$ such that $w_{j_i}^i = u_i$ and all vertices $v_i \in C_i^{\bar u, j_i}$ (for both $i \in [2]$), if $g(v_i) = c_i'$, then $g(v_{3-i}) \neq c_{3-i}$ by the construction of $S^{\mathbf{B}'}$. This means that there is at most $i \in [2]$, such that h maps u_i to c_i. This implies $h(u_1, u_2) \in S^{\mathbf{OR}_{\mathrm{NPC}}[1_{R_1}, 1_{R_2}]}$. ◀

D Details on the Tseitin-Systems

In the following, we assume the set-up from Section 4. The value $k \in \mathbb{N}$ is thought of as a constant, but unless explicitly stated otherwise, the following lemmas also hold in case that k is a function in $|E|$ that grows at most sublinearly.

▶ **Lemma 42.** *For all $\lambda \colon V \to \Gamma$ and $X \in \binom{E}{\leq k}$, there is a robustly consistent partial assignment $f \colon X \to \Gamma$ for $\mathcal{C}^\lambda$.*

Proof. For $|X| = 1$, it is clear that there are robustly consistent partial solutions. It then follows from Lemmas 4.4 and 4.5 in [8] that such a robustly consistent partial solution can always be extended while maintaining robust consistency, as long as the domain size is sublinear in $|E|$. ◀

When we view $\mathcal{C}^\lambda$ as a homomorphism problem, then for every $X \subseteq E$, a partial solution $f\colon X \to \Gamma$ of $\mathcal{C}^\lambda$ (so in particular a robustly consistent partial assignment) is a homomorphism in $\mathrm{Hom}(\mathcal{C}^\lambda, \Gamma^{[3]})$ and vice versa (recall that $\Gamma^{[3]}$ is the template structure for 3-ary Γ-coset-CSPs).

▶ **Lemma 43.** *For every $\lambda\colon V \to \Gamma$, the k-consistency algorithm does not rule out any robustly consistent partial assignments. This means that, for every $X \in \binom{E}{\leq k}$, every robustly consistent assignment contained in $\mathrm{Hom}(\mathcal{C}^\lambda[X], \Gamma^{[3]})$ is contained in $\kappa_k^{\Gamma^{[3]}}[\mathcal{C}^\lambda](X)$.*

Proof. For a set $X \in \binom{E}{\leq k}$, denote by $\mathrm{Hom}(\mathcal{C}^\lambda[X], \Gamma^{[3]})_{n/3} \subseteq \mathrm{Hom}(\mathcal{C}^\lambda[X], \Gamma^{[3]})$ the set of robustly consistent homomorphisms in $\mathrm{Hom}(\mathcal{C}^\lambda[X], \Gamma^{[3]})$. In order to prove the lemma, we show that the collection

$$\left\{ \mathrm{Hom}(\mathcal{C}^\lambda[X], \Gamma^{[3]})_{n/3} \;\middle|\; X \in \binom{E}{\leq k} \right\}$$

of robustly consistent partial homomorphisms satisfies the down-closure and the forth-condition of k-consistency. For the down-closure, let $f \in \mathrm{Hom}(\mathcal{C}^\lambda[X], \Gamma^{[3]})_{n/3}$. By robust consistency, the partial solution f satisfies $C(W)$ for every $W \in \binom{V}{\leq n/3}$ such that $\delta(W) \subseteq X$. Then the restriction of f to any subset of X still satisfies $C(W)$ for every $W \in \binom{V}{\leq n/3}$ such that $\delta(W)$ is in its domain. So this restriction is also robustly consistent. For the forth-condition, let $f \in \mathrm{Hom}(\mathcal{C}^\lambda[X], \Gamma^{[3]})_{n/3}$, for some $|X| < k$. Let $y \in E \setminus X$. We need to show that there exists an $f' \in \mathrm{Hom}(\mathcal{C}^\lambda[X \cup \{y\}], \Gamma^{[3]})_{n/3}$ that extends f. This is again proven in Lemmas 4.4 and 4.5 in [8], as long as k is at most sublinear in $|E|$. ◀

▶ **Corollary 44.** *For all $\lambda\colon V \to \Gamma$ and $X \in \binom{E}{\leq k}$, we have $\kappa_k^{\Gamma^{[3]}}[\mathcal{C}^\lambda](X) \neq \emptyset$.*

Proof. Combine Lemmas 42 and 43. ◀

For a prime p, a p-**group** is a group in which the order of every element is a power of p. For instance, $\mathbb{Z}_2$ is a 2-group and $\mathbb{Z}_3$ a 3-group.
We now need to consider the case when one robustly consistent partial solution $f : Z \to \Gamma$, with $Z \in \binom{E(G)}{\leq k}$ is fixed. We show that in this case the system $\mathsf{L}_{\mathsf{CSP}}^{k,\Gamma^{[3]}}(\mathcal{C}^\lambda)$ has a solution in which $x_{Z,f}$ is set to 1. Fix a set $\hat{Z} \supseteq Z$ such that $E(G) \setminus \hat{Z}$ is the edge set of a 2-connected subgraph of G. By the expansion property, we can choose it such that $|\hat{Z}| \leq c \cdot |Z|$, where c is the expansion constant.

▶ **Lemma 45.** *There is an assignment $h : \hat{Z} \to \Gamma$ such that $h|_Z = f$, and $h \in \mathrm{Hom}(\mathcal{C}^\lambda[\hat{Z}], \Gamma^{[3]})$ is robustly consistent.*

Proof. Since $|\hat{Z}| \leq c \cdot |X| \leq ck$, we can use Lemmas 4.4 and 4.5 in [8] to extend f to a robustly consistent h with domain $\hat{Z}$. This is in particular a partial homomorphism. ◀

Fix this partial solution $h\colon \hat{Z} \to \Gamma$ for $\mathcal{C}^\lambda$ given by Lemma 45 in the following. Let $G' = (V', E')$ be the graph obtained from G by deleting all edges in $\hat{Z}$ and all vertices that are not in the 2-connected component of $G - \hat{Z}$. Similarly, obtain the directed graph H' from H by deleting the same (directed) edges and vertices. Let $\lambda' : V' \to \Gamma$ be defined as follows. For every $v \in V'$, set

$$\lambda'(v) := \lambda(v) - \sum_{e \in \delta_+(v) \cap \hat{Z}} h(y_e) + \sum_{e \in \delta_-(v) \cap \hat{Z}} h(y_e).$$

With this definition, $\mathcal{C}^{H',\Gamma,\lambda'}$ is the CSP that we obtain from $\mathcal{C}^\lambda$ by fixing values for the variables in $\hat{Z}$ according to h from Lemma 45. In what follows, let $\mathbf{C} = \mathcal{C}^{H,\Gamma,\lambda}$ and $\mathbf{C}' = \mathcal{C}^{H',\Gamma,\lambda'}$. The graph G' is still an expander:

▶ **Lemma 46.** *Let $k \in \mathbb{N}$ and $(G_n)_{n\in\mathbb{N}}$ a family of expander graphs with expansion constant c. Fix a set $X_n \in \binom{E(G_n)}{\leq k}$ for every n. Let G'_n be the 2-connected subgraph of G_n that remains after removing the edges $\hat{X}_n \supseteq X_n$ (and potentially isolated vertices) from G_n. Then $(G'_n)_{n\in\mathbb{N}}$ is also a family of expander graphs (with a different expansion constant that depends on k and c).*

Proof. Let $Y \subseteq E(G'_n)$. Then $Y \cup \hat{X}_n \subseteq E(G_n)$ and $|Y \cup \hat{X}| \leq ck + |Y|$. Because G is an expander, there exists $\hat{Y} \supseteq Y \cup \hat{X}$ such that $E(G) - \hat{Y} = E(G') - (\hat{Y} \setminus \hat{X})$ is the edge set of a 2-connected subgraph. Moreover, $|\hat{Y}| \leq c \cdot |Y \cup \hat{X}| = c^2k + c|Y|$. Thus $\hat{Y} \setminus \hat{X}$ is the superset of Y that witnesses the expansion property for G'. ◀

▶ **Lemma 47.** *If $\mathsf{L}^{k,\Gamma^{[3]}}_{\mathsf{CSP}}(\mathbf{C}')$ has a p-solution Φ, then $\mathsf{L}^{k,\Gamma^{[3]}}_{\mathsf{CSP}}(\mathbf{C})$ has a p-solution Ψ such that*

(1) if $\Phi(x_{X',f'}) = 0$, then $\Psi(x_{X,f}) = 0$, for every X with $X \cap E' = X'$ and $f|_{E'} = f'$.

(2) for all sets of variables $X \in \binom{E\setminus E'}{\leq k}$ of the system $\mathbf{C}$ and for all partial homomorphisms $f \in \mathrm{Hom}(\mathbf{C}[X], \Gamma^{[3]})$, we have $\Psi(x_{X,f}) = 1$ if f agrees with h, and $\Psi(x_{X,f}) = 0$, otherwise.

Proof. Define Ψ as follows. We identify the variable set of $\mathbf{C}' = \mathcal{C}^{H',\Gamma,\lambda'}$ with E'. For all $X \in \binom{E}{\leq k}$ and $f \in \mathrm{Hom}(\mathbf{C}[X], \Gamma^{[3]})$, we set

$$\Psi(x_{X,f}) := \begin{cases} \Phi(x_{X\cap E', f|_{E'}}) & \text{if } f|_{X\setminus E'} = h|_{X\setminus E'} \text{ or if } X \subseteq E', \\ 0 & \text{otherwise.} \end{cases}$$

It remains to show that Ψ is a solution for $\mathsf{L}^{k,\Gamma^{[3]}}_{\mathsf{CSP}}(\mathbf{C})$. For Equation L2, this is clear. Now consider an equation of Type L1: Let $X \in \binom{E}{\leq k}$, $b \in X$, and $g \in \mathrm{Hom}(\mathbf{C}, \Gamma^{[3]})$. We need to show

$$\sum_{\substack{f \in \mathrm{Hom}(\mathbf{C}[X], \Gamma^{[3]}),\\ f|_{X\setminus\{b\}} = g}} \Psi(x_{X,f}) = \Psi(x_{X\setminus\{b\},g}).$$

If $g|_{(X\setminus b)\setminus C}$ does not agree with h, then both sides of the equation are mapped to zero by Ψ. Hence it remains the case that $g|_{(X\setminus b)\setminus E'}$ does agree with h. For every $f \in \mathrm{Hom}(\mathbf{C}[X], \Gamma^{[3]})$, it holds: If $f|_{X\setminus E'} = h|_{X\setminus E'}$, then $f|_{E'} \in \mathrm{Hom}(\mathbf{C}'[X \cap E'], \Gamma^{[3]})$. This is due to the definition of λ'. Thus we have

$$\begin{aligned} \sum_{\substack{f \in \mathrm{Hom}(\mathbf{C}[X], \Gamma^{[3]}),\\ f|_{X\setminus\{b\}} = g}} \Psi(x_{X,f}) &= \sum_{\substack{f \in \mathrm{Hom}(\mathbf{C}'[X\cap E'], \Gamma^{[3]}),\\ f|_{X\cap E'\setminus\{b\}} = g}} \Phi(x_{X\cap E', f|_{E'}}) \\ &= \Phi(x_{(X\cap E')\setminus\{b\}, g|_{E'}}) = \Psi(x_{X\setminus\{b\},g}). \end{aligned}$$

Therefore, Ψ is a solution of $\mathsf{L}^{k,\Gamma^{[3]}}_{\mathsf{CSP}}(\mathbf{C})$. For every $X \in E \setminus E'$, we have $\Psi(x_{X,f}) = 0$ if f disagrees with h, and $\Psi(x_{X,f}) = \Phi(x_{X\cap C, f|_C}) = \Phi(x_{\emptyset,\emptyset}) = 1$, otherwise. ◀

▶ **Corollary 48.** *Let $k \in \mathbb{N}$ be a constant and explicitly not a function of $|E|$. Let $Z \in \binom{E(G)}{\leq k}$ and assume that Γ is a p-group. If $f \in \mathrm{Hom}(\mathbf{C}[Z], \Gamma)$ is robustly consistent, then $\mathsf{L}^{k,\Gamma}_{\mathsf{CSP}}(\mathbf{C})$ has a p-solution Ψ such that*

- *Ψ is 0 for partial assignments that are not robustly consistent.*
- *$\Psi(x_{Z,f}) = 1$.*

Proof. With Lemma 45, we extend f to $h \in \mathrm{Hom}(\mathbf{C}[\hat{Z}], \Gamma)$. The graph G' is still an expander graph (follows from Lemma 46). Hence Lemma 16 can be applied and gives us a p-solution for $\mathsf{L}^{k,\Gamma^{[3]}}_{\mathsf{CSP}}(\mathbf{C}')$, to which we can apply Lemma 47 to get a p-solution for $\mathsf{L}^{k,\Gamma^{[3]}}_{\mathsf{CSP}}(\mathbf{C})$. This has the property that it is zero for assignments which are not robustly consistent and it is 1 for $x_{Z,f}$. ◀

E Details on the Limitations of the Affine Algorithms

This section introduces the algorithms from Theorem 1 and shows that each of them fails to solve the same CSP: the tractable homomorphism or-construction of ternary $\mathbb{Z}_2$-coset-CSP and ternary $\mathbb{Z}_3$-coset-CSP $\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$.

E.1 $\mathbb{Z}$-Affine k-Consistency Relaxation

We first consider the $\mathbb{Z}$**-affine** k**-consistency relaxation**, an algorithm proposed by Dalmau and Opršal [19]. The algorithm considers the following system of affine linear equations over the integers. Let $\mathbf{A}$ be a template structure, $\mathbf{B}$ be an instance, and κ be a map that assigns to every set $X \in \binom{B}{\leq k}$ a set of partial homomorphisms $\mathbf{B}[X] \to \mathbf{A}$. The system $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \kappa)$ is defined as follows:

$\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \kappa)$: variables $z_{X,f}$ for all $X \in \binom{B}{\leq k}$ and $f \in \kappa(X)$

$$z_{X,f} \in \mathbb{Z} \qquad \text{for all } X \in \tbinom{B}{\leq k} \text{ and } f \in \kappa(X) \tag{Z1}$$

$$\sum_{f \in \kappa(X)} z_{X,f} = 1 \qquad \text{for all } X \in \tbinom{B}{\leq k} \tag{Z2}$$

$$\sum_{f \in \kappa(X), f|_Y = g} z_{X,f} = z_{Y,g} \qquad \text{for all } Y \subset X \in \tbinom{B}{\leq k} \text{ and } g \in \kappa(Y) \tag{Z3}$$

Recall that $\kappa^{\mathbf{A}}_k[\mathbf{B}]$ denotes the output of the k-consistency algorithm, which is a function that assigns partial homomorphisms to each set $X \in \binom{B}{\leq k}$. The $\mathbb{Z}$-affine k-consistency relaxation runs, for a fixed positive integer k and a template structure $\mathbf{A}$, as follows:

$\mathbb{Z}$-affine k-consistency relaxation for template $\mathbf{A}$: input a CSP($\mathbf{A}$)-instance $\mathbf{B}$

1. Compute $\kappa^{\mathbf{A}}_k[\mathbf{B}]$ using the k-consistency algorithm.
2. Accept if the system $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \kappa^{\mathbf{A}}_k[\mathbf{B}])$ is solvable and reject otherwise.

Dalmau and Opršal [19] made the following conjecture on the power of the $\mathbb{Z}$-affine k-consistency relaxation:

▶ **Conjecture 17** ([19]). *For every finite structure $\mathbf{A}$, either $\mathrm{CSP}(K_3)$ is Datalog$^{\cup}$-reducible to $\mathrm{CSP}(\mathbf{A})$ or $\mathrm{CSP}(\mathbf{A})$ is Datalog$^{\cup}$-reducible to $\mathrm{CSP}(\mathbb{Z})$, where K_3 denotes the triangle.*

The latter case of the conjecture considers a reduction to systems of linear equation systems over the integers. This condition in particular implies that $\mathrm{CSP}(\mathbf{A})$ is solved by the $\mathbb{Z}$-affine k-consistency algorithm for some constant k [19].

▶ **Theorem 18.** *For every $k \geq 1$, the $\mathbb{Z}$-affine k-consistency relaxation does not solve $\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$. This is even true if k is not a constant, but an at most sublinear function in the instance size.*

Proof. Let G be a 3-regular 2-connected expander graph whose order is sufficiently larger than k, and let H be an arbitrary orientation of G. Let $p_1 := 2$ and $p_2 := 3$. For $i \in [2]$, let $\Gamma_i := \mathbb{Z}_{p_i}$, and let $\lambda_i : V(G) \to \Gamma_i$ be defined to be 0 everywhere except at one arbitrarily chosen vertex $v^* \in V(G)$, where we set $\lambda_i(v^*) := 1$. For each $i \in [2]$, we consider the 3-ary Γ_i-coset-CSPs $\mathbf{B}_i := \mathcal{C}^{H,\Gamma_i,\lambda_i}$. Let $\mathbf{B} := \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$ be the homomorphism or-instance and $\mathbf{A} := \mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$ be the corresponding tractable homomorphism or-template. From $\sum_{v\in V(G)} \lambda_i(v) \neq 0$ it follows that $\mathbf{B}_i \notin \mathrm{CSP}(\Gamma_i^{[3]})$ for both $i \in [2]$. Hence, $\mathbf{B} \notin \mathrm{CSP}(\mathbf{A})$ by Lemma 27. By Lemma 43, the k-consistency algorithm does not rule out any robustly consistent partial solutions for $\mathbf{B}_1$ and $\mathbf{B}_2$. For a set $X \subseteq B$, we call a partial homomorphism $f\colon \mathbf{B}[X] \to \mathbf{A}$ robustly consistent, if for some $i \in [2]$, the restriction $f|_{B_i}$ is a robustly consistent partial homomorphism $\mathbf{B}_i[X \cap B_i] \to \mathbf{A}_i$ and $f(X \cap B_{3-i}) = \{c_{3-i}\}$, where c_{3-i} is the additional vertex added for $\mathbf{B}_{3-i}$ in the homomorphism or-construction. Then it follows from Lemma 8 that also for every size-k subinstance of $\mathbf{B}$, the robustly consistent partial solutions are not ruled out by k-consistency. Since by Corollary 44 every size-k subinstance of $\mathbf{B}_1$ and of $\mathbf{B}_2$ admits a partial robustly consistent solution, k-consistency fails to reject $\mathbf{B}$. Lemma 16 shows that a p_i-solution for the affine relaxation of $\mathbf{B}_i$ exists, for each i. This solution is only non-zero for robustly consistent partial solutions of $\mathbf{B}_i$. Applying Lemma 9 to the respective p_i-solution results in a p_i-solution for $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$, where the only non-zero entries are extensions of robustly consistent partial solutions of $\mathbf{B}$. By Lemma 5, the p_1-solution and the p_2-solution imply an integral solution Φ for $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$, which again is only non-zero for robustly consistent partial solutions of $\mathbf{B}$. Lemma 25 implies that such a solution to $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ also satisfies the equations of Type Z2, and Equation L1 is equivalent to Equation Z3. Thus, Φ defines an integral solution to $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-}\mathsf{aff}}(\mathbf{B}, \kappa^{\mathbf{A}}_k[\mathbf{B}])$, which is only non-zero for robustly consistent partial solutions. Hence, the $\mathbb{Z}$-affine k-consistency relaxation wrongly accepts $\mathbf{B}$. All the lemmas we need in the proof are also true if k is not constant but a sublinear function of the instance size. ◀

▶ **Lemma 19.** $\mathrm{CSP}(K_3)$ *is not Datalog*$^\cup$*-reducible to* $\mathrm{CSP}(\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$.

Proof. Let $p \notin \{2,3\}$ be a prime and $r \geq 3$ be an arity. Then $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ is Datalog$^\cup$-reducible to $\mathrm{CSP}(K_3)$ because every finite-domain CSP is Datalog$^\cup$-reducible to $\mathrm{CSP}(K_3)$ (see [19]). We claim that $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ is not Datalog$^\cup$-reducible to $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$, which by transitivity of Datalog$^\cup$-reducibility [19] implies the lemma. Suppose for the sake of a contradiction that $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ is Datalog$^\cup$-reducible to $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$. *Rank logic* [20] extends inflationary fixed-point logic by operators to define the rank of definable matrices over finite prime fields. For a set of primes P, the characteristic-P fragment only provides these operators for finite prime fields whose characteristic is contained in P. For all $i \in \{2,3\}$, characteristic-$\{2,3\}$ rank logic defines $\mathrm{CSP}(\mathbb{Z}_i^{[3]})$ by Lemma 26, that is, there is a formula of the logic that is satisfied by a $\mathrm{CSP}(\mathbb{Z}_i^{[3]})$-instance if and only if it is contained in $\mathrm{CSP}(\mathbb{Z}_i^{[3]})$. By Corollary 12, $\mathrm{CSP}(\mathbf{B})$ is definable in characteristic-$\{2,3\}$ rank logic. Because Datalog$^\cup$-reductions are definable in inflationary fixed-point logic, $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$ is definable in characteristic-$\{2,3\}$ rank logic. This contradicts the result by Grädel and Pakusa [25] that non-isomorphic Cai-Fürer-Immerman graphs over the group $\mathbb{Z}_p$ are indistinguishable in characteristic-P rank logic whenever $p \notin P$. Additionally, distinguishing these graphs is first-order-reducible to solving a system of (ternary) linear equations in $\mathbb{Z}_p$ [25, Lemma 18], so to a $\mathrm{CSP}(\mathbb{Z}_p^{[r]})$-instance. Therefore, the latter is not definable in characteristic-$\{2,3\}$ rank logic. ◀

Theorem 18 and Lemma 19 disprove Conjecture 17.

E.2 BLP+AIP and BA^k

Before considering the next algorithm, we first introduce a well-studied system of equations for CSPs [3, 10], or, more precisely, a variant of it parameterized by the size of partial solutions [14]. Let k be a positive integer, $\mathbf{A}$ a template τ-structure and $\mathbf{B}$ a CSP($\mathbf{A}$)-instance. We define the system $\mathsf{L}_{\mathsf{IP}}^{k,\mathbf{A}}(\mathbf{B})$ with variable set $\mathcal{V}^{k,\mathbf{A}}(\mathbf{B})$.

$\mathsf{L}_{\mathsf{IP}}^{k,\mathbf{A}}(\mathbf{B})$: variables $\lambda_{X,f}$ for all $X \in \binom{B}{\leq k}$ and $f\colon X \to A$, and variables $\mu_{R,\bar{b},\bar{a}}$ for all $R \in \tau$, $\bar{b} \in R^{\mathbf{B}}$, and $\bar{a} \in R^{\mathbf{A}}$

$$\sum_{f\colon X \to A} \lambda_{X,f} = 1 \qquad \text{for all } X \in \tbinom{B}{\leq k}, \tag{B1}$$

$$\sum_{\substack{f\colon X \to A,\\ f|_Y = g}} \lambda_{X,f} = \lambda_{Y,g} \qquad \text{for all } Y \subset X \in \tbinom{B}{\leq k}, g\colon Y \to A, \tag{B2}$$

$$\sum_{\bar{a} \in R^{\mathbf{A}}, a_{\bar{i}} = \bar{a}'} \mu_{R,\bar{b},\bar{a}} = \lambda_{X(\bar{b}_{\bar{i}}), \bar{b}_{\bar{i}} \mapsto \bar{a}'} \qquad \text{for all } R \in \tau, \bar{a}' \in A^k, \bar{b} \in R^{\mathbf{B}}, \bar{i} \in [\mathrm{ar}(R)]^k, \tag{B3}$$

where $a_{\bar{i}}$ and $b_{\bar{i}}$ denote the k-tuples $(a_{\bar{i}_1}, ..., a_{\bar{i}_k})$ and $(b_{\bar{i}_1}, ..., b_{\bar{i}_k})$, respectively, $X(\bar{b}_{\bar{i}})$ denotes the set of entries of $\bar{b}_{\bar{i}}$, and $\bar{b}_{\bar{i}} \mapsto \bar{a}'$ denotes the partial homomorphism sending $\bar{b}_{\bar{i}}$ to $\bar{a}'$.

We consider different domains of the variables (see [10]):

- If we restrict the variables to $\{0, 1\}$, then $\mathsf{L}_{\mathsf{IP}}^{1,\mathbf{A}}(\mathbf{B})$ is solvable if and only if $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$.
- The relaxation of $\mathsf{L}_{\mathsf{IP}}^{k,\mathbf{A}}(\mathbf{B})$ to nonnegative rationals is the **basic linear programming (BLP)** relaxation $\mathsf{L}_{\mathsf{BLP}}^{k,\mathbf{A}}(\mathbf{B})$.
- The affine relaxation of $\mathsf{L}_{\mathsf{IP}}^{k,\mathbf{A}}(\mathbf{B})$ to all integers is the **affine integer programming (AIP)** relaxation $\mathsf{L}_{\mathsf{AIP}}^{k,\mathbf{A}}(\mathbf{B})$.

By increasing the parameter k, the BLP and AIP relaxations result in the Sherali-Adams LP hierarchy and the affine integer programming hierarchy of the $\{0, 1\}$-system, respectively.

Brakensiek, Guruswami, Wrochna, and Živný [10] use a certain combination of $\mathsf{L}_{\mathsf{BLP}}^{1,\mathbf{A}}(\mathbf{B})$ and $\mathsf{L}_{\mathsf{AIP}}^{1,\mathbf{A}}(\mathbf{B})$ to formulate the **BLP+AIP algorithm**. Similar to the $\mathbb{Z}$-affine k-consistency relaxation, the BLP+AIP algorithm tries to solve CSP($\mathbf{A}$) in the sense that it is sound. However, it may wrongly answer $\mathbf{B} \in \mathrm{CSP}(\mathbf{A})$. The question is whether the BLP+AIP algorithm is also complete for tractable CSPs. In contrast to the $\mathbb{Z}$-affine k-consistency relaxation, the BLP+AIP algorithm is not parameterized by the size of partial solutions k. This parameterized version was proposed by Ciardo and Živný [17, 15] and is called $\mathbf{BA}^k$, where BA^1 is just the BLP+AIP algorithm.

We now formally introduce this parameterized algorithm. Let k be a positive integer.

$\mathsf{BA}^k(\mathbf{A})$-algorithm: input a $\mathrm{CSP}(\mathbf{A})$-instance $\mathbf{B}$

1. Compute a relative interior point $\Phi\colon \mathcal{V}^{k,\mathbf{A}}(\mathbf{B}) \to \mathbb{Q}$ in the polytope defined by $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$. The solution Φ has in particular the property that for each variable $x \in \mathcal{V}^{k,\mathbf{A}}(\mathbf{B})$ there is a solution Ψ to $\mathsf{L}^{\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ such that $\Psi(x) \neq 0$ if and only if $\Phi(x) \neq 0$. If such a point does not exist, reject.
2. Refine $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{AIP}}(\mathbf{B})$ by adding the constraints
$$x = 0 \quad \text{whenever} \quad \Phi(x) = 0 \qquad \text{for all } x \in \mathcal{V}^{k,\mathbf{A}}(\mathbf{B}).$$
3. If the refined system is feasible (over $\mathbb{Z}$), then accept, otherwise reject.

The original presentation of BA^k in [17] uses a slightly different system of equations but one can verify that our presentation is indeed equivalent. The system in [17] does not have variables $\lambda_{X,f}$ but uses variables $\lambda_{R_k,\bar{b},\bar{a}}$ instead, where R_k is the full k-ary relation. These have equivalent semantics. Our equation $(B1)$ corresponds to equation (1) in [17], and our equations $(B2), (B3)$ are expressed by equation (2) in [17]. We deviate from the original presentation to keep it consistent with the systems for the other algorithms.

We show that BA^k fails on the counterexample provided for $\mathbb{Z}$-affine k-consistency. To do so, we relate solutions of $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ to solutions of $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ or $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{AIP}}(\mathbf{B})$.

▶ **Lemma 49.** *Let $\mathbf{A}$ and $\mathbf{B}$ be τ-structures and $k \geq \mathrm{ar}(\tau)$. If $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ has a solution Φ over the non-negative rationals or the integers, then the following map Ψ is a solution to $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ or $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{AIP}}(\mathbf{B})$, respectively:*

$$\Psi(\lambda_{X,f}) := \begin{cases} \Phi(x_{X,f}) & \text{if } f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A}), \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } X \in \binom{\mathbf{B}}{\leq k}, f\colon X \to A,$$
$$\Psi(\mu_{R,\bar{b},\bar{a}}) := \Phi(x_{X(\bar{b}),\bar{b}\mapsto\bar{a}}) \qquad \text{for all } R \in \tau, \bar{a} \in R^{\mathbf{A}}, \bar{b} \in R^{\mathbf{B}},$$

where $X(\bar{b})$ denotes the set of elements appearing in the tuple $\bar{b}$ and $\bar{b} \mapsto \bar{a}$ denotes the partial homomorphism sending $\bar{b}$ to $\bar{a}$.

Proof. Lemma 25 implies that equations of Type B1 and B2 are satisfied. For $R \in \tau$, $\bar{a}' \in A^k$, $\bar{b} \in R^{\mathbf{B}}$, and $\bar{i} \in [\mathrm{ar}(R)]^k$, we let Y be the set of entries of $\bar{b}_{\bar{i}}$, and consider the homomorphism $g := \bar{b}_{\bar{i}} \mapsto \bar{a}'$. Then Lemma 25 also implies that equations of Type B3 are satisfied. It is clear that non-negativity or integrality, respectively, of the solution is not changed. ◀

▶ **Theorem 20.** *For every integer k, the algorithm $\mathsf{BA}^k(\mathbf{A})$ does not solve $\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$. This is even true if k is not a constant but an at most sublinear function in the instance size.*

Proof. Let $p_1 = 2$, $p_2 = 3$, and $\mathbf{A} := \mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$. As in the proof of Theorem 18, we consider the ternary Tseitin CSP no-instances over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ constructed for sufficiently large expander graphs. Again, let $\mathbf{B}_i$ be the $\mathbb{Z}_{p_i}$ instance for both $i \in [2]$. We again have that $\mathbf{B}_i \notin \mathrm{CSP}(\mathbb{Z}_{p_i}^{[3]})$ for both $i \in [2]$ and hence $\mathbf{B} := \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2] \notin \mathrm{CSP}(\mathbf{A})$.

By Lemma 16, for each $i \in [2]$, there is a p_i-solution Φ_i for $\mathsf{L}^{k,\mathbb{Z}_{p_i}^{[3]}}_{\mathsf{CSP}}(\mathbf{B}_i)$ which sets exactly the robustly consistent partial solution to a non-zero value. With Lemma 9, each Φ_i gives rise to a solution Ψ_i of $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ that is non-zero exactly for all partial homomorphisms $f\colon X \to A$ that are robustly consistent for $X \cap B_i$ and map $X \cap B_{3-i}$ to c_{3-i}. We call these partial homomorphisms $\mathbf{B} \to \mathbf{A}$ also robustly consistent. Let F be the set of all robustly consistent partial homomorphisms $\mathbf{B}[X] \to \mathbf{A}$ for all $X \in \binom{B}{\leq k}$. The relative interior point computed in Step 1 of the BA^k-algorithm exists (because the system is solvable) and is in particular non-zero for every $f \in F$. By Lemma 5, the p_1-solution Ψ_1 and the p_2-solution Ψ_2 for $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ can be combined to an integral solution that is only non-zero for partial homomorphisms in F. Therefore by Lemma 49, the system $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{AIP}}(\mathbf{B})$ also has such an integral solution. This solution satisfies the refined constraints from Step 2 of the BA^k-algorithm. Hence, the algorithm incorrectly accepts the unsatisfiable instance $\mathbf{B}$. All lemmas used in this proof are also true if k is a sublinear function in the instance size. ◀

E.3 The CLAP Algorithm

This subsection considers the CLAP algorithm, introduced by Ciardo and Živný [16]. Let $\mathbf{A}$ be a template τ-structure.

$\mathsf{CLAP}(\mathbf{A})$-algorithm: input a CSP($\mathbf{A}$)-instance $\mathbf{B}$

1. Maintain, for each pair of a relation symbol $R \in \tau$ and a tuple $\bar{b} \in R^{\mathbf{B}}$, a set $S_{\bar{b},R} \subseteq R^{\mathbf{A}}$ of possible images of $\bar{b}$ under a homomorphism. Initialize $S_{\bar{b},R} := R^{\mathbf{A}}$ for all $R \in \tau$ and $\bar{b} \in R^{\mathbf{B}}$.
2. Repeat until no set $S_{\bar{b},R}$ changes anymore: For each $R \in \tau$, $\bar{b} \in R^{\mathbf{B}}$, and $\bar{a} \in S_{\bar{b},R}$, solve $\mathsf{L}^{1,\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ together with the following additional constraints:
$$\begin{aligned} \mu_{R,\bar{b},\bar{a}} &= 1, \\ \mu_{R,\bar{b}',\bar{a}'} &= 0 \qquad\qquad \text{for all } R' \in \tau, \bar{b}' \in R'^{\mathbf{B}}, \bar{a}' \notin S_{\bar{b}',R'}. \end{aligned}$$
If this system is not feasible, remove $\bar{a}$ from $S_{\bar{b},R}$.
3. If there are $R \in \tau$ and $\bar{b} \in R^{\mathbf{B}}$ such that $S_{\bar{b},R} = \emptyset$, then reject.
4. For each $R \in \tau$, $\bar{b} \in R^{\mathbf{B}}$, and $\bar{a} \in S_{\bar{b},R}$, execute $\mathsf{BA}^1(\mathbf{A})$ (which is BLP+AIP) on $\mathbf{B}$, where we additionally fix
$$\begin{aligned} \mu_{R,\bar{b},\bar{a}} &= 1, \\ \mu_{R,\bar{b}',\bar{a}'} &= 0 \qquad\qquad \text{for all } R' \in \tau, \bar{b}' \in R'^{\mathbf{B}}, \bar{a}' \notin S_{\bar{b}',R'} \end{aligned}$$
in Step 1 of $\mathsf{BA}^1(\mathbf{A})$ (and thus also implicitly in $\mathsf{L}^{1,\mathbf{A}}_{\mathsf{AIP}}(\mathbf{B})$ in Step 2 of $\mathsf{BA}^1(\mathbf{A})$). If $\mathsf{BA}^1(\mathbf{A})$ accepts, then accept.
5. If $\mathsf{BA}^1(\mathbf{A})$ rejects all inputs in the step before, then reject.

To simplify the analysis, we consider a variant of the CLAP algorithm, which we call CLAP′.

$\mathsf{CLAP}'(\mathbf{A})$-algorithm: input a CSP($\mathbf{A}$)-instance $\mathbf{B}$

Execute Steps 1 to 3 of $\mathsf{CLAP}(\mathbf{A})$. Then execute

4′. Execute $\mathsf{BA}^1(\mathbf{A})$ on $\mathbf{B}$ where we additionally fix
$$\mu_{R',\bar{b}',\bar{a}'} = 0 \qquad\qquad \text{for all } R' \in \tau, \bar{b}' \in R'^{\mathbf{B}}, \bar{a}' \notin S_{\bar{b}',R'}.$$
Accept if $\mathsf{BA}^1(\mathbf{A})$ accepts this input and reject otherwise.

It is immediate that $\mathsf{CLAP}'(\mathbf{A})$ does not solve more CSPs than $\mathsf{CLAP}(\mathbf{A})$. We show that it actually solves the same:

▶ **Lemma 50.** *For every structure $\mathbf{A}$, $\mathsf{CLAP}(\mathbf{A})$ solves $\mathrm{CSP}(\mathbf{A})$ if and only if $\mathsf{CLAP}'(\mathbf{A})$ solves $\mathrm{CSP}(\mathbf{A})$.*

Proof. Let $\mathbf{A}$ be a template τ-structure. It is clear that if $\mathsf{CLAP}'(\mathbf{A})$ solves $\mathrm{CSP}(\mathbf{A})$, then also $\mathsf{CLAP}(\mathbf{A})$ solves $\mathrm{CSP}(\mathbf{A})$. We show that if $\mathsf{CLAP}'(\mathbf{A})$ does not solve $\mathrm{CSP}(\mathbf{A})$, then $\mathsf{CLAP}(\mathbf{A})$ does not solve $\mathrm{CSP}(\mathbf{A})$, either. Let $\mathbf{B}$ be a τ-structure such that $\mathbf{B} \notin \mathrm{CSP}(\mathbf{A})$, but $\mathsf{CLAP}'(\mathbf{A})$ accepts $\mathrm{CSP}(\mathbf{A})$. We create a modified variant of $\mathbf{B}$ as follows. Let $T \in \tau$ be some relation symbol of arity r that is non-empty in $\mathbf{A}$ (if $\mathbf{A}$ contains only empty relations, then CLAP and CLAP' can trivially solve $\mathrm{CSP}(\mathbf{A})$). Let $\mathbf{B}'$ be the disjoint union of $\mathbf{B}$ and the r-element τ-structure, for which one r-tuple of distinct elements $\bar{x}$ is contained in T. Obviously, we have $\mathbf{B}' \notin \mathrm{CSP}(\mathbf{A})$. We show that $\mathsf{CLAP}(\mathbf{A})$ accepts $\mathbf{B}'$. Since $\mathbf{B}'$ is a disjoint union, after Steps 1 to 3, the sets $S_{\bar{b},R}$ on input $\mathbf{B}'$ will contain at least the elements as on input $\mathbf{B}$. The set $S_{\bar{x},T}$ will be equal to $T^{\mathbf{A}}$ because fixing the assignment of $\bar{x}$ does not restrict any other partial homomorphisms, and since CLAP' accepts $\mathbf{B}$, the system $\mathsf{L}^{\mathbf{A}}_{\mathsf{BLP}}(\mathbf{B})$ is solvable when an image of $\bar{x}$ is fixed. In particular, no set $S_{\bar{b},R}$ will be empty after Step 2. Hence, Step 3 is passed successfully. Now for Step 4, we consider the relation T and the tuple $\bar{x}$. We consider the execution of BA^1, where an arbitrary image of $\bar{x}$ contained in T is fixed. Because $\mathbf{B}'$ is a disjoint union and the mapping of $\bar{x}$ is a valid homomorphism from the attached structure to $\mathbf{A}$ and because $\mathsf{BA}^1(\mathbf{A})$ accepts in Step 4′, $\mathsf{BA}^1(\mathbf{A})$ will accept in Step 4 for the tuple $\bar{x}$. Hence, $\mathsf{CLAP}(\mathbf{A})$ wrongly accepts $\mathbf{B}'$, which means that it does not solve $\mathrm{CSP}(\mathbf{A})$. ◀

▶ **Theorem 21.** $\mathsf{CLAP}(\mathbf{A})$ *does not solve* $\mathrm{CSP}(\mathbf{OR}_T[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$.

Proof. We prove the result for $\mathsf{CLAP}'(\mathbf{A})$, which is sufficient by Lemma 50. Let $k = 3$. As in the proofs of Theorems 20 and 18, we consider ternary Tseitin systems over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ for a sufficiently large 3-regular 2-connected expander graph. Let again $\mathbf{B}_1$ and $\mathbf{B}_2$ be these instances, which for $p_i = i + 1$ are no-instances for $\mathrm{CSP}(\mathbb{Z}_{p_i}^{[3]})$ for both $i \in [2]$. Again, let $\mathbf{B} := \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$ and $\mathbf{A} := \mathrm{CSP}(\mathbf{OR}_\mathrm{T}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}])$. So, $\mathbf{B} \notin \mathrm{CSP}(\mathbf{A})$.

By Corollary 48, for each $i \in [2]$, and every $f \in \mathrm{Hom}(\mathbf{B}_i[X], \mathbf{A}_i)$ that is robustly consistent, there exists a p_i-solution $\Phi_{i,f}$ to $\mathsf{L}^{k,\mathbb{Z}_{p_i}^{[3]}}_{\mathsf{CSP}}(\mathbf{B}_i)$ which sets $x_{X,f}$ to 1 and every partial homomorphism that is not robustly consistent to 0. By Lemma 9, each $\Phi_{i,f}$ translates into a p_i-solution $\Psi_{i,f}$ for $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ that sets every partial homomorphism to 1 which agrees with f on B_i and sends the B_{3-i}-part of its domain to c_{3-i}. We call such partial homomorphisms (for both $i \in [2]$) again also robustly consistent. Let F denote the set of al robustly consistent partial solutions $\mathbf{B}[X] \to \mathbf{A}$ for every $X \in \binom{B}{\leq k}$. Now consider Step 2 of CLAP. The algorithm adds in particular the equation $\mu_{R,\bar{b},\bar{a}} = 1$ to the systems considered in $\mathsf{BA}^1(\mathbf{A})$. If there is some $f \in F$ that contains the assignment $\bar{b} \mapsto \bar{a}$, then $\Psi_{i,f}$ gives us a solution for $\mathsf{BA}^1(\mathbf{A})$ (via Lemma 49) that also satisfies $\mu_{R,\bar{b},\bar{a}} = 1$. By Lemmas 42 and 9, for every $R \in \tau(\mathbf{A})$ and $\bar{b} \in R^{\mathbf{A}}$, there is at least one $\bar{a} \in R^{\mathbf{A}}$ such that we can find an $f \in F$ containing $\bar{b} \mapsto \bar{a}$. All tuples $\bar{a}$ that are removed from $S_{\bar{b},R}$ in Step 2 do not satisfy that $\bar{b} \mapsto \bar{a}$ is contained in an $f \in F$. This means that $\bar{b} \mapsto \bar{a}$ is not part of a robustly consistent partial solution of $\mathbf{B}_1$ or $\mathbf{B}_2$. Thus, it will be set to zero by all the $\Phi_{i,f}$ and $\Psi_{i,f}$, and hence, these solutions also satisfy the extra equations $\mu_{R,\bar{b},\bar{a}} = 0$ in Step 2. In total, Step 3 is of CLAP is passed successfully, and the only tuples $\bar{a}$ that are removed from $S_{\bar{b},R}$ are such that $\bar{b} \mapsto \bar{a}$ is not part of a robustly consistent partial solution. In Step 4′, $\mathsf{CLAP}'(\mathbf{A})$ will then accept: The proof of Theorem 20 shows that $\mathsf{BA}^1(\mathbf{A})$ accepts $\mathbf{B}$, and it can be seen that this proof also holds if we set $\mu_{R,\bar{b}',\bar{a}'} = 0$ for all partial solutions $\bar{b}' \mapsto \bar{a}'$ that are not robustly consistent. ◀

In contrast to the $\mathbb{Z}$-affine k-consistency relaxation and the BA^k algorithms, CLAP is not parameterized by a width k. However, we did not exploit this fact and our techniques could also be applied to a version of CLAP parameterized by a width.

We can prove Lemma 50 because CLAP immediately accepts if Step 4 is passed successfully for at least one tuple. One could modify CLAP so that Step 4 has to find one possible image for all $R \in \tau$ and all $\bar{b} \in R^{\mathbf{B}}$. This would still be a sound algorithm. Ciardo and Živný [16] already noted this possibility when introducing CLAP, and moreover suggested a possibly even stronger version: replace BLP with BLP+AIP in Step 2, which in turn would make Steps 4 and 5 unnecessary. The authors refer to this algorithm as C(BLP+AIP) but considered CLAP because it allows to characterize the CSPs solved by CLAP in terms of the polymorphisms of the template structure $\mathbf{A}$. Whether a similar characterization for C(BLP+AIP) is possible is an open question. We do not study C(BLP+AIP) in this article but suspect that it has similar properties as the cohomological algorithm, which we turn to next. In particular, we believe that Theorem 21 is not true for C(BLP+AIP).

E.4 The Cohomological k-Consistency Algorithm

We review the cohomological k-consistency algorithm due to Ó Conghaile [18]. It combines techniques of the algorithms we have seen so far – the iterative approach of k-consistency with solving the AIP in *every* iteration. The name references *cohomology* because solving the AIP (also in the other algorithms) can be interpreted as checking for the existence of a cohomological obstruction in the presheaf of partial homomorphisms. More details on this interpretation can be found in [18]. The algorithm itself is straightforward and can be stated without the categorical terminology:

Cohomological k-consistency algorithm: input a CSP($\mathbf{A}$)-instance $\mathbf{B}$

1. Maintain, for each $X \in \binom{B}{\leq k}$, a set $\mathcal{H}(X) \subseteq \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$. Initialize $\mathcal{H}(X) := \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$.
2. Repeat until none of the sets $\mathcal{H}(X)$ changes anymore:
   a. Run the k-consistency algorithm on $\mathcal{H}$ to remove from each $\mathcal{H}(X)$ the partial homomorphisms that fail the forth-condition or down-closure property.
   b. For each $X \in \binom{B}{\leq k}$ and $f \in \mathcal{H}(X)$, check whether $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \mathcal{H})$ has a solution that satisfies $x_{X,f} = 1$ and $x_{X,f'} = 0$ for every $f' \in \mathcal{H}(X) \setminus \{f\}$. If it does not, then remove f from $\mathcal{H}(X)$ for the next iteration of the loop.
3. If $\mathcal{H}(X) = \emptyset$ for some $X \in \binom{B}{\leq k}$, then reject; otherwise accept.

Step 2(b) of the algorithm tries to check whether there is a global homomorphism whose restriction to X is equal to f – and this check is approximated by solving the AIP in which we set $x_{X,f} = 1$ and $x_{X,f'} = 0$ for all other f'.

At least for the template CSP($\mathbf{OR}_{\mathrm{T}}[\mathbb{Z}_2^{[3]}, \mathbb{Z}_3^{[3]}]$) that we have used as a counterexample for the other algorithms, we can say that cohomological k-consistency is strictly more powerful than all of them because it actually solves that template correctly (Theorem 22).

Nonetheless, we can also show a limitation of this algorithm: It fails to solve the *intractable* homomorphism or-construction on $\mathbb{Z}_2$ and $\mathbb{Z}_3$. This proves *without* any complexity theoretical assumptions like P $\neq$ NP that this polynomial time algorithm does not solve all finite-domain CSPs. Thus, cohomological k-consistency is a potential candidate for a universal polynomial time algorithm for all Maltsev or even all tractable CSPs. We leave this as an intriguing question that can hopefully be settled in the near future.

▶ **Theorem 22.** *If $\mathbf{A}_1, \mathbf{A}_2$ are templates of Abelian coset-CSPs and $k \geq \mathrm{ar}(A_i) + 1$ for both $i \in [2]$, then the k-cohomological algorithm solves* CSP($\mathbf{OR}_T[\mathbf{A}_1, \mathbf{A}_2]$).

Proof. We first argue that the cohomological k-consistency algorithm correctly rejects or-instances $\mathbf{B} = \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2] \notin$ CSP($\mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2]$). Let $\mathbf{A} = \mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2]$.

We argue that when the algorithm terminates, $\mathcal{H}(X) = \emptyset$ for at least one $X \in \binom{B}{\leq k}$. Consider some $X \subseteq B_1$ that is exactly the set of entries of some tuple $\bar{b} \in R^{\mathbf{B}_1}$, for some $R \in \tau(\mathbf{B}_1)$. Let $f : X \to A_1$ be an arbitrary partial homomorphism. Then by Lemma 13, $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$ does not have an integral solution Φ that satisfies $\Phi(x_{X,f}) = 1$ and $\Phi(x_{X,f'}) = 0$ for every $f' \in \mathcal{H}(X) \setminus \{f\}$: By the lemma, such a Φ would in particular be a solution to $\mathsf{L}^{\mathbf{A}_1,k-1}_{\mathsf{CSP}}(\mathbf{B}_1)$. But it is known that for Abelian coset-CSPs, the existence of an integral solution to $\mathsf{L}^{\mathbf{A}_1,k-1}_{\mathsf{CSP}}(\mathbf{B}_1)$, where $k-1 \geq \mathrm{ar}(\mathbf{A}_1)$, is equivalent to $\mathbf{B}_1 \in \mathrm{CSP}(\mathbf{A}_1)$ (see also Theorem 51). Thus, since $\mathbf{B}_1 \notin \mathrm{CSP}(\mathbf{A}_1)$, such a Φ cannot exist. Then in particular, $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \mathcal{H})$ does not have such an integral solution, even if $\mathcal{H}(X) = \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$, as initially. Hence, all f with $f(X) \subseteq A_1$ are removed from $\mathcal{H}(X)$ in this iteration. Since X is the entry set of $\bar{b} \in R^{\mathbf{B}_1}$, and $R^{\mathbf{A}} \subseteq A_1^3 \cup \{(c_1, c_1, c_1)\}$, the only other partial homomorphism in $\mathcal{H}(X)$ is the one with $f(X) = \{c_1\}$. This is the only homomorphism that may still be in $\mathcal{H}(X)$ after the first iteration. We can also consider another $X' \subseteq B_2$ that is the set of entries of some $\bar{b}' \in R'^{\mathbf{B}_2}$, and the same argument shows that after the first iteration, there is at most the partial homomorphism f with $f(X') = \{c_2\}$ in $\mathcal{H}(X')$. Then consider the set $\{x, x'\}$ for some $x \in X, x' \in X'$. After k-consistency is run in the second iteration, $\mathcal{H}(\{x, x'\})$ will be empty. This is because $\mathcal{H}(X)$ and $\mathcal{H}(X')$ enforce that x is mapped to c_1 and x' is mapped to c_2, but every partial homomorphism in $\mathrm{Hom}(\mathbf{B}[x, x'], \mathbf{A})$ maps either x or x' to an element of A_1 or A_2, respectively.

We now not only want to show that the cohomological k-consistency algorithm correctly rejects or-instances but every $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2])$-instance. Assume that $\mathbf{B}$ is an unsatisfiable $\mathrm{CSP}(\mathbf{OR}_{\mathrm{T}}[\mathbf{A}_1, \mathbf{A}_2])$-instance. To show that the algorithm rejects $\mathbf{B}$, we follow the algorithm to solve this CSP provided in the proof of Lemma 11. For $i \in [2]$, let $B_i \subseteq B$ be the set of all τ_i-vertices. We can assume that B_1 and B_2 form a partition of B: a vertex with is neither in B_1 nor in B_2 is isolated, and a vertex in $B_1 \cap B_2$ is part in relations of both τ_1 and τ_2, which does not exist in $\mathbf{A}$ and thus k-consistency would immediately reject $\mathbf{B}$. The proof of Lemma 11 shows that if $\mathbf{B}$ is unsatisfiable, then there is some S-component D in the graph G_S that contains both an unsatisfiable τ_1- and unsatisfiable τ_2-component (we refer to that proof for the terminology). Therefore we know that such an S-component D exists and we will now argue that the cohomological algorithm detects it.

We first note that k-consistency detects τ_i-components in the following sense: for every $i \in [2]$ and $X \subseteq B_i$ that is also a subset of a single τ_i-component, k-consistency discards all partial homomorphisms $f \colon \mathbf{B}[X] \to \mathbf{A}$ such that there are b and b' with $f(b) \in A_i$ and $f(b') = c_i$. This is the case because b and b' are connected via relations in τ_i, but $f(b)$ and $f(b')$ are not connected via τ_i-relations in the tractable homomorphism-or construction, which can easily be detected by k-consistency because $k \geq \mathrm{ar}(\mathbf{A}_i) + 1$.

For the now following Step 2 in the cohomological algorithm, we show that unsatisfiable τ_i-components (for both $i \in [2]$) are detected: Let $i \in [2]$ and D be a τ_i-component such that $\mathbf{B}[D] \notin \mathrm{CSP}(\mathbf{A}_i)$. Let $X \subseteq D$ be of size at most k. If we now fix a partial homomorphism $f \colon \mathbf{B}[X] \to \mathbf{A}_i$ by setting its variable to 1, we in particular set the partial homomorphism $g \colon \mathbf{B}[X] \to \mathbf{A}$ with $g(X) = \{c_i\}$ to 0. Now consider Lemma 34 but only for the component D. One can easily show that k-consistency has already discarded the partial homomorphisms for which we showed in Claim 35 and 36 in the proof of Lemma 34 that their variable is set to 0. Then similarly to Claim 37 in that proof, a solution to $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \mathcal{H})$ where we set f to 1 and the other partial homomorphisms to 0, has to induce a solution to $\mathsf{L}^{k,\mathbf{A}_i}_{\mathsf{CSP}}(\mathbf{B}[D])$. But for $\mathbf{A}_i$, we know that AIP solves $\mathrm{CSP}(\mathbf{A}_i)$. This implies that there is no solution to $\mathsf{L}^{k,\mathbf{A}}_{\mathbb{Z}\text{-aff}}(\mathbf{B}, \mathcal{H})$ that fixes f in this sense. So similarly to the case of proper or-instances above, we can show that f gets discarded by the cohomological algorithm.

Now consider the next iteration of the algorithm, in which k-consistency is executed again. As with τ_i-components, the k-consistency algorithm detects S-components. So let D be an S-component in which neither all τ_1-components are solvable nor all τ_2-components are solvable. Let D^i be an unsolvable τ_i-component in D and D^{3-i} a neighbored τ_{3-i}-component of D^i in G_S, which means that there are vertices $u \in D^i$ and $v \in D^{3-i}$ connected via an S-edge. All partial homomorphisms mapping vertices in D^i to A_i have already been discarded, which means that only maps to c_i remain. That means that all homomorphisms $f\colon \{u, v\} \to B$ which map v to c_{3-i} are also discarded because the map $uv \mapsto c_i c_{3-i}$ is not a partial homomorphism. In particular, the partial homomorphism $v \to c_{3-i}$ is discarded. By the reasoning before, all partial homomorphisms from D^{3-i} to c_{3-i} get discarded by k-consistency. This is then propagated to the τ_i-components which are neighbors of D^{3-i} in the sense that in those, the partial homomorphisms to A_i are discarded and only those to c_i remain. This propagation continues through the whole S-component D. Because there is an unsatisfiable τ_1-component and an unsatisfiable τ_2-component in D, at some point all partial homomorphisms of some vertex in D get discarded. But that means that k-consistency rejects the input and so the cohomological k-consistency algorithm correctly rejects $\mathbf{B}$. ◀

After this positive result about the power of cohomological k-consistency, we now turn to the NP-complete counterexample.

▶ **Theorem 23.** *There is an NP-complete template structure* $\mathbf{A}$ *such that for every* k*, the cohomological* k*-consistency algorithm does not solve* $\mathrm{CSP}(\mathbf{A})$*.*

Proof. Let $p_1 = 2$, $p_2 = 3$ and let $\mathbf{A}_i = \mathbb{Z}_{p_i}^{[3]}$ be the template structure for ternary $\mathbb{Z}_{p_i}$-coset CSPs (and this in particular for ternary linear equations over $\mathbb{Z}_{p_i}$). Set $\mathbf{A} := \mathbf{OR}_{\mathrm{NPC}}[\mathbf{A}_1, \mathbf{A}_2]$. Let $k \in \mathbb{N}$ be arbitrary. We show that the cohomological k-consistency algorithm does not solve $\mathrm{CSP}(\mathbf{A})$. Let G be a sufficiently large 3-regular 2-connected expander graph and H be an orientation of G. Let $\lambda_i\colon V(G) \to \mathbb{Z}_2$ be zero apart from one arbitrary vertex that is mapped to 1. Let $\mathbf{B}_i := \mathcal{C}^{H,\mathbb{Z}_{p_i},\lambda_i}$ and $\mathbf{B} := \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$. Consider the following family of sets of partial homomorphisms: for each $X \in \binom{B}{\leq k}$ define $\mathcal{H}(X) \subseteq \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ as follows. For $i \in [2]$, let $X_i = X \cap B_i$. Let $\mathcal{H}_i(X_i)$ be the set of all robustly consistent homomorphisms $\mathbf{B}_i[X_i] \to \mathbf{A}_i$. All $f_1 \in \mathcal{H}_1(X_1)$ and $f_2 \in \mathcal{H}_2(X_2)$ induce a partial homomorphism $f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ by Lemma 14. Let $\mathcal{H}(X)$ be the sets of all these homomorphisms. We show that this family of partial homomorphisms is stable under the cohomological k-consistency algorithm. By Lemma 43, the robustly consistent partial homomorphisms are not ruled out by k-consistency for each $\mathbf{B}_i$ and by Lemma 38, the induced ones for $\mathbf{B}$ are also not ruled out by k-consistency on the intractable homomorphism-or construction. Let $X \in \binom{B}{\leq k}$ and $f \in \mathcal{H}(X)$ be induced by $f_i \in \mathcal{H}_i(X_i)$. By Corollary 48, $\mathsf{L}_{\mathsf{CSP}}^{k,\mathbf{A}_1}(\mathbf{B}_1)$ has a solution Φ such that $\Phi(x_{X_1,f_1}) = 1$. By Lemma 39, $\mathsf{L}_{\mathsf{CSP}}^{k,\mathbf{A}}(\mathbf{B})$ has a solution Ψ such that $\Psi(x_{X,f}) = \Phi(x_{X_1,f_1}) = 1$. Hence indeed, this family of partial homomorphisms is stable under the cohomological k-consistency algorithm. Since by Lemma 45 none of the sets $\mathcal{H}(X)$ is empty, $\mathbf{B}$ is accepted by the cohomological k-consistency algorithm. However, $\mathbf{B} \notin \mathrm{CSP}(\mathbf{A})$ by Lemma 27 and because $\mathbf{B}_i \notin \mathrm{CSP}(\mathbf{A}_i)$ for both $i \in [2]$.

To show that $\mathrm{CSP}(\mathbf{A})$ is NP-complete, it suffices by Lemmas 40 and 41 that there is an inclusion-wise minimal no-instance $\mathbf{C}_i \notin \mathrm{CSP}(\mathbf{A}_i)$ with 3 elements for every $i \in [2]$. This is, e.g., achieved by the equations $x_1 + x_2 + x_3 = 1$ and $x_1 + x_2 + x_3 = 0$, which over both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ constitute an inclusion-wise minimal no-instance: deleting one variable removes all equations. ◀

F Details on Affine Algorithms on Coset-CSPs

The counterexample we have used so far is not a coset-CSP itself, but a combination of two Abelian coset-CSPs. We now set out to explore the power of the affine algorithms on coset-CSPs, in particular because the algorithms themselves solve a CSP over the infinite Abelian group $(\mathbb{Z}, +)$.

One of the simplest possible affine algorithms just checks for solvability of the basic affine integer relaxation (AIP) of a CSP. This relaxation is the system $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{AIP}}(\mathbf{B})$ for $k = 1$ introduced in Section 5.2, and every algorithm we study here clearly solves at least those CSPs that AIP can solve. It can be derived from the literature that already AIP suffices to solve all Abelian coset-CSPs.

▶ **Theorem 51.** *If Γ is Abelian, then AIP solves* $\mathrm{CSP}(\Gamma^{[r]})$ *for every $r \in \mathbb{N}$.*

Proof. Theorem 7.19 in [3] characterizes the power of AIP in terms of the polymorphisms of the template $\Gamma^{[r]}$: The CSP is solved correctly by AIP if and only if $\Gamma^{[r]}$ has alternating functions of all odd arities as polymorphisms. The exact definition of an alternating function is not needed here but it suffices to know that in any Abelian group, the $(2n+1)$-ary function $a(x_1, ..., x_{2n+1}) = x_1 - x_2 + x_3 - \cdots + x_{2n+1}$ is alternating [3, Example 7.17]. We know that $\Gamma^{[r]}$ has the polymorphism $f(x, y, z) = x - y + z$, so we can use f to generate a $2n + 1$-ary function a as above for every $n \in \mathbb{N}$, e.g., $a(x_1, x_2, x_3, x_4, x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$, and likewise for higher n. ◀

But Abelian problems are not the demarcation line for the power of affine algorithms. The following result shows that there exist non-Abelian groups for which AIP still works; these are certain 2-nilpotent groups, so intuitively, they are as close to being Abelian as possible. Formally, a group Γ is *2-nilpotent* if its commutator subgroup is contained in its center, i.e. the *commutator* $\alpha^{-1}\beta^{-1}\alpha\beta$ of any two $\alpha, \beta \in \Gamma$ commutes with all elements of Γ.

▶ **Theorem 52.** *If Γ is 2-nilpotent and of odd order, then AIP solves* $\mathrm{CSP}(\Gamma^{[r]})$ *for every $r \in \mathbb{N}$.*

Proof. We begin with some background. Let $\Gamma = (G, \cdot)$. For $\alpha, \beta \in \Gamma$, their commutator is defined as $[\alpha, \beta] = \alpha^{-1} \cdot \beta^{-1} \cdot \alpha \cdot \beta$. The commutator subgroup of Γ is denoted $[\Gamma, \Gamma]$ and is defined as the group generated by all $[\alpha, \beta]$, for $\alpha, \beta \in \Gamma$. Let m be the exponent of $[\Gamma, \Gamma]$, the least common multiple of the order of all element in $[\Gamma, \Gamma]$. This is odd because $|\Gamma|$ and hence $|[\Gamma, \Gamma]|$ is odd. For 2-nilpotent groups of odd order, we can apply the so-called "Baer trick" [29, 26] to obtain an Abelian group reduct. Define $(G, +)$ as the group on the same universe as Γ but with the operation defined as $x + y := xy[x, y]^{(m-1)/2}$. As shown in the proof of [29, Corollary 5.2], $(G, +)$ is Abelian. The goal is now to show that the operation $f(x, y, z) = x - y + z$ in $(G, +)$ is a polymorphism of $\Gamma^{[r]}$. Once we have that, we can obtain alternating operations of all odd arities exactly as in the proof of Theorem 51. To start with, it is easy to check that the inverse $-x$ in $G(+)$ is x^{-1}. Thus we have

$$\begin{aligned}
x - y + z &= xy^{-1}[x, y^{-1}]^{(m-1)/2} \cdot z[xy^{-1}[x, y^{-1}]^{(m-1)/2}, z]^{(m-1)/2} \\
&= xy^{-1}z[x, y^{-1}]^{(m-1)/2} \cdot [xy^{-1}, z]^{(m-1)/2} \cdot [[x, y^{-1}]^{(m-1)/2}, z]^{(m-1)/2} \\
&= xy^{-1}z[x, y^{-1}]^{(m-1)/2} \cdot [xy^{-1}, z]^{(m-1)/2}.
\end{aligned}$$

Here we used that commutators in 2-nilpotent groups are central, the commutator identity $[xy, z] = [x, z] \cdot [y, z]$ that holds in this form in 2-nilpotent groups, and the fact that a commutator that has another commutator as one of its arguments is the neutral element in any 2-nilpotent group. Let $d(x, y, z) := xy^{-1}z$, $s := d(x, y, z)$, and $t := d(z, y, x)$. We show by induction on c that the following identity holds:

$$d(...d(d(t,s,t),s,t)...,s,t) = xy^{-1}z[x,y^{-1}]^c \cdot [xy^{-1},z]^c,$$

where d appears c times in the equation. Then for $c = (m-1)/2$, this identity gives us a term for $x - y + z$ that just uses the Maltsev polymorphism of $\Gamma^{[r]}$. For $c = 1$, we get

$$\begin{aligned} xy^{-1}z[x,y^{-1}] \cdot [xy^{-1},z] &= xy^{-1}z \cdot x^{-1}yxy^{-1} \cdot yx^{-1}z^{-1}xy^{-1}z \\ &= xy^{-1}z \cdot x^{-1}yz^{-1} \cdot xy^{-1}z = d(t,s,t). \end{aligned}$$

For the inductive step, we have

$$\begin{aligned} xy^{-1}z[x,y^{-1}]^{c+1} \cdot [xy^{-1},z]^{c+1} &= \underbrace{d(...d(d(t,s,t),s,t)...,s,t)}_{c \text{ ds}} \cdot [x,y^{-1}] \cdot [xy^{-1},z] \\ &= \underbrace{d(...d(d(t,s,t),s,t)...,s,t)}_{c \text{ ds}} \cdot x^{-1}yz^{-1} \cdot xy^{-1}z \\ &= \underbrace{d(...d(d(t,s,t),s,t)...,s,t)}_{c+1 \text{ ds}} \end{aligned}$$

This finishes the proof of the theorem. A known example of a 2-nilpotent group of odd order is $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, which is of order 27. ◀

G A Group Coset-CSP Counterexample via Graph Isomorphism

This section shows that the affine algorithms studied in Section 5 also fail on group-coset-CSPs. The key idea is to exploit that group coset-CSPs are inter-reducible with the *bounded color class size graph isomorphism problem* [8]. For every constant d, this is the task to decide whether two vertex-colored graphs, in which at most d many vertices have the same color, are isomorphic. Instead of the homomorphism or-construction, we use an *isomorphism or-construction*. We first reduce our aforementioned Tseitin instances of $\mathrm{CSP}(\mathbb{Z}_2^{[3]})$ and $\mathrm{CSP}(\mathbb{Z}_3^{[3]})$ to bounded color class size graph isomorphism. Using the isomorphism or-construction, we combine these two isomorphism problems in the same fashion as we did with homomorphisms. Finally, the resulting bounded color class size graph isomorphism problem is translated back into a group coset-CSP over the symmetric group, which, on d elements, is denoted by S_d.

▶ **Theorem 53.** *For every $d \geq 18$ and every constant or at most sublinearly growing k, neither the $\mathbb{Z}$-affine k-consistency relaxation, BA^k, nor CLAP solve* $\mathrm{CSP}(S_d^{[2]})$.

The proof of this theorem spans the rest of this section.

G.1 Bounded Color Class Structure Isomorphism and Group Coset-CSPs

A **colored relational structure** is a pair $(\mathbf{A}, \chi_{\mathbf{A}})$ of a relational structure $\mathbf{A}$ and a function $\chi_{\mathbf{A}} \colon A \to \mathfrak{C}$, for some finite set of colors $\mathfrak{C}$. A **color class** of $\mathbf{A}$ is a maximal set $V \subseteq A$ of elements of the same color. The **color class size** of $\mathbf{A}$ is the maximal size of the color classes of $\mathbf{A}$. For a color $c \in \mathfrak{C}$, denote by $\mathbf{A}[c]$ the substructure induced on the vertices in the c-color class. For a set of colors $C \subseteq \mathfrak{C}$, write $\mathbf{A}[C]$ for $\mathbf{A}[\bigcup C]$. An isomorphism between colored structures has to preserve colors, that is, it maps vertices of one color to vertices of the same color. For two possibly colored relational structures $\mathbf{A}$ and $\mathbf{B}$ we write $\mathrm{Iso}(\mathbf{A},\mathbf{B})$ for the set of **isomorphisms** $\mathbf{A} \to \mathbf{B}$. Let $d \in \mathbb{N}$ be a constant. Instances of the **d-bounded color class size structure isomorphism problem** are pairs $(\mathbf{A},\mathbf{B})$ of relational structures of color class size at most d. The problem asks whether there is a color-preserving isomorphism from $\mathbf{A}$ to $\mathbf{B}$. Polynomial time reductions in both directions between this problem and group coset-CSPs [8] are presented in the following.

Reducing Coset-CSPs to Bounded Color Class Size Isomorphism. Let Γ be a finite group and $\mathbf{B}$ be an r-ary Γ-coset-CSP instance. We encode $\mathbf{B}$ into a colored graph $\mathbf{G}_\Gamma^\mathbf{B}$ as follows: For every variable x of $\mathbf{B}$, we add a vertex (x,γ) for every $\gamma \in \Gamma$. We call x the origin of (x,γ) and color all vertices with origin x with a fresh color c_x. For every constraint $C\colon (x_1,\dots,x_r) \in \Delta\delta$, add a vertex $(C,\gamma_1,\dots,\gamma_r)$ for all $(\gamma_1,\dots,\gamma_r) \in \Delta\delta$. We call C the origin of these vertices and color all vertices with origin C with a fresh color c_C. We then add edges $\{(x_i,\gamma_i),(C,\gamma_1,\dots,\gamma_r)\}$, which we color with fresh colors c_i', for all $i \in [r]$ (which formally is encoded in a fresh binary relation symbol). Note that, since $\mathbf{G}_\Gamma^\mathbf{B}$ is a graph, its arity is always 2 independent of the arity of $\mathbf{B}$.

We now derive the homogeneous Γ-coset-CSP $\tilde{\mathbf{B}}$ from $\mathbf{B}$ as follows: we replace every constraint $C\colon (x_1,\dots,x_r) \in \Delta\delta$ of $\mathbf{B}$ with the constraint $\tilde{C}\colon (x_1,\dots,x_r) \in \Delta$ in $\tilde{\mathbf{B}}$. For $\tilde{\mathbf{B}}$, we obtain the graph $\widetilde{\mathbf{G}}_\Gamma^\mathbf{B}$ by the construction before, where we identify the colors c_C and $c_{\tilde{C}}$ for every constraint C. The graphs $\mathbf{G}_\Gamma^\mathbf{B}$ and $\widetilde{\mathbf{G}}_\Gamma^\mathbf{B}$ are the **CFI graphs** over Γ for $\mathbf{B}$. If $\mathbf{B}$ is the Tseitin equation system over $\mathbb{Z}_2$, the obtained CFI graphs correspond to the known CFI graphs introduced by Cai, Fürer, and Immerman [13], which have found many applications in finite model theory and other areas since then.

▶ **Lemma 54** ([6]). *Let Γ be a finite group and $\mathbf{B}$ an r-ary Γ-coset-CSP instance.*

(1) $\mathbf{G}_\Gamma^\mathbf{B}$ and $\widetilde{\mathbf{G}}_\Gamma^\mathbf{B}$ have color class size at most the maximum $|\Delta|$ over all subgroups Δ occurring in constraints of $\mathbf{B}$, which is in particularly bounded by $|\Gamma|^r$.

(2) $\mathbf{G}_\Gamma^\mathbf{B} \cong \widetilde{\mathbf{G}}_\Gamma^\mathbf{B}$ if and only if $\mathbf{B} \in \mathrm{CSP}(\Gamma^{[r]})$.

Reducing Bounded Color Class Size Isomorphism to Coset-CSPs. Let $(\mathbf{A},\mathbf{B})$ be an instance of the d-bounded color class structure isomorphism problem. We encode isomorphisms between $\mathbf{A}$ and $\mathbf{B}$ as solutions of the following S_d-coset-CSP. Denote the set of colors $\mathbf{A}$ and $\mathbf{B}$ by $\mathfrak{C}$. We also assume that $\ell_c := |\mathbf{A}[c]| = |\mathbf{B}[c]|$ for each color $c \in \mathfrak{C}$. Otherwise, $\mathbf{A}$ and $\mathbf{B}$ are trivially not isomorphic. For every $c \in \mathfrak{C}$, we add a variable y_c. First, we add constraints that ensure that y_c is actually a variable over S_{ℓ_c}:

$$y_c \in \{\, \gamma \in S_d \mid \gamma(j) = j \text{ for all } \ell_c \le j \le d \,\}.$$

It is clear that this set is a subgroup of S_d and hence we indeed added S_d-constraints. Next, for every $c \in \mathfrak{C}$, we assume that the vertices of $\mathbf{A}[c]$ are $u_{c,1},\dots,u_{c,\ell_c}$ and the ones of $\mathbf{B}[c]$ are $v_{c,1},\dots,v_{c,\ell_c}$. We pick, for every set $C = \{c_1,...,c_{r'}\}$ of $r' \le r$ color classes, an isomorphism $\varphi_C\colon \mathbf{A}[C] \to \mathbf{B}[C]$ if it exists. We identify φ_C with a permutation in $\bigtimes_{i\in[r']} S_{\ell_{c_i}}$: The i-th component of this tuple of permutations maps j to k if $\varphi_C(u_{c_i,j}) = v_{c_i,k}$. Similarly, we can identify an automorphism $\psi \in \mathrm{Aut}(\mathbf{A}[C])$ with a permutation in $\bigtimes_{i\in[r']} S_{\ell_{c_i}}$. If for some C such an isomorphism φ_C does not exist, then $\mathbf{A} \not\cong \mathbf{B}$ and we just add some unsatisfiable constraints and are done (e.g., use two cosets $\{1\}\gamma$, $\{1\}\delta$ for $\gamma \neq \delta$). Via these identifications, we add the r'-ary S_d-constraint

$$(y_{c_1},\dots,y_{c_r'}) \in \mathrm{Aut}(\mathbf{A}[C])\varphi_C.$$

We denote the resulting S_d-coset-CSP by $\mathbf{BI}(\mathbf{A};\mathbf{B})$. For a set C of colors of $\mathbf{A}$ and $\mathbf{B}$, we denote by $\mathbf{BI}(\mathbf{A};\mathbf{B})[C]$ the subsystem induced by all variables y_c for which $c \in C$.

▶ **Lemma 55** ([6, 28]). *For all r-ary colored structures $\mathbf{A}$ and $\mathbf{B}$ of color class size d, the structure $\mathbf{BI}(\mathbf{A};\mathbf{B})$ is an r-ary S_d-coset-CSP such that $\mathbf{BI}(\mathbf{A};\mathbf{B}) \in \mathrm{CSP}(S_d^{[r]})$ if and only if $\mathbf{A} \cong \mathbf{B}$.*

G.2 Isomorphism OR-Construction on Structures

We present an isomorphism or-construction. For a sequence of colored structures $\mathbf{B}_1, \dots, \mathbf{B}_j$, we write $\langle \mathbf{B}_1, \dots, \mathbf{B}_j \rangle$ for the structure that encodes this sequence and which is defined as follows: Assume $\mathbf{B}_i$ uses $\mathfrak{C}_i = [\ell_i]$ as set of colors. First, we increment each color of the vertices of $\mathbf{B}_i$ by $m_i := \sum_{k=1}^{i-1} \ell_i$. Next, we extend each $\mathbf{B}_i$ by a new binary relation that is interpreted as B_i^2. We now start with the disjoint union of all $\mathbf{B}_i$, where we call vertices of $\mathbf{B}_i$ **entry-i vertices**. We add a new binary relation symbol such that for all $i < j$ we add an edge between all entry-i and entry-j vertices to this relation.

Now let $\mathbf{B}_1^0, \dots, \mathbf{B}_j^0$ and $\mathbf{B}_1^1, \dots, \mathbf{B}_j^1$ be two sequences of colored structures. We define a pair of structures $(\mathbf{C}^0, \mathbf{C}^1) = \mathbf{OR}^{\mathrm{ISO}}_{i \in [j]}[\mathbf{B}_i^0, \mathbf{B}_i^1]$ as follows. For each $k \in \{0, 1\}$, define

$$\mathbf{C}^k := \biguplus \left\{ \langle \mathbf{B}_1^{a_1}, \dots, \mathbf{B}_j^{a_j} \rangle \;\middle|\; a_1 + \dots + a_j \equiv k \mod 2 \right\},$$

where we call the $\langle \mathbf{B}_1^{a_1}, \dots, \mathbf{B}_j^{a_j} \rangle$ **components**.

▶ **Lemma 56.** *Let $\mathbf{B}_1^0, \dots, \mathbf{B}_j^0$ and $\mathbf{B}_1^1, \dots, \mathbf{B}_j^1$ be two sequences of colored structures of color class size at most d. Then for $\mathbf{OR}^{ISO}_{i \in [j]}[\mathbf{B}_i^0, \mathbf{B}_i^1]$ we have*

(1) $\mathbf{C}^0 \cong \mathbf{C}^1$ if and only if there exists an $i \in [j]$ such that $\mathbf{B}_i^0 \cong \mathbf{B}_i^1$, and

(2) $\mathbf{C}^0$ and $\mathbf{C}^1$ have color class size at most $2^{j-1} d$.

Proof. The first claim was (for a slightly different encoding of sequences of graphs) shown in [8]. For the second claim, we note that the encoding of a sequence does not increase the color class size and that there are 2^{j-1} such sequences in the disjoint union. ◀

G.3 Instances of the Counterexample

To obtain instances of $\mathrm{CSP}(S_d^{[2]})$ that are hard for the affine algorithms, we start with Tseitin systems over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ and then chain together the former constructions. From now on, fix a positive integer k. As in the proofs in Section A.1, let G be a 3-regular 2-connected expander graph whose order is sufficiently larger than the width parameter k. Let H be an arbitrary orientation of G. Let $p_1 := 2$ and $p_2 := 3$. For $i \in [2]$, let $\lambda_i \colon V(G) \to \mathbb{Z}_{p_i}$ be defined to be 0 everywhere except at one arbitrarily chosen vertex $v^* \in V(G)$, where we set $\lambda_i(v^*) := 1$. For each $i \in [2]$, we consider the 3-ary $\mathbb{Z}_{p_i}$-coset-CSPs $\mathbf{B}_i := \mathcal{C}^{H, \mathbb{Z}_{p_i}, \lambda_i}$. We apply the reduction to graph isomorphism to obtain for each $i \in [2]$ a pair of colored graphs $(\mathbf{G}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}}, \widetilde{\mathbf{G}}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}})$ such that $\mathbf{G}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}} \cong \widetilde{\mathbf{G}}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}}$ if and only if $\mathbf{B}_i \in \mathrm{CSP}(\mathbb{Z}_{p_i}^{[3]})$. By construction, $\mathbf{B}_i \notin \mathrm{CSP}(\mathbb{Z}_{p_i}^{[3]})$, so the corresponding graphs are non-isomorphic. Now apply the graph isomorphism or-construction $(\mathbf{C}^0, \mathbf{C}^1) = \mathbf{OR}^{\mathrm{ISO}}_{i \in [2]}[\mathbf{G}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}}, \widetilde{\mathbf{G}}^{\mathbf{B}_i}_{\mathbb{Z}_{p_i}}]$ so that $\mathbf{C}^0 \cong \mathbf{C}^1$ if and only if $\mathbf{G}^{\mathbf{B}_1}_{\mathbb{Z}_{p_1}} \cong \widetilde{\mathbf{G}}^{\mathbf{B}_1}_{\mathbb{Z}_{p_1}}$ or $\mathbf{G}^{\mathbf{B}_2}_{\mathbb{Z}_{p_2}} \cong \widetilde{\mathbf{G}}^{\mathbf{B}_2}_{\mathbb{Z}_{p_2}}$. Since neither of these are isomorphic, we have $\mathbf{C}^0 \not\cong \mathbf{C}^1$. The two graphs $\mathbf{C}^0$ and $\mathbf{C}^1$ have bounded color class size and it can in fact be checked that this size is 18: The color class size of $\widetilde{\mathbf{G}}^{\mathbf{B}_2}_{\mathbb{Z}_{p_2}}$ is upper bounded by 9 because there exist 9 triples in $\mathbb{Z}_3$ whose sum in $\mathbb{Z}_3$ is 0. The color class size of $\widetilde{\mathbf{G}}^{\mathbf{B}_1}_{\mathbb{Z}_{p_1}}$ is smaller. The isomorphism or-construction applied to two graphs doubles the color class size. So with the reduction of bounded color class size isomorphism to a coset-CSP as described above, the problem "$\mathbf{C}^0 \cong \mathbf{C}^1$?" is turned into the instance $\mathbf{BI}(\mathbf{C}^0;\mathbf{C}^1)$ of CSP($S_d^{[2]}$) for every $d \geq 18$. This instance does not admit a solution because $\mathbf{C}^0 \not\cong \mathbf{C}^1$. However, we can show that $\mathsf{L}_{\mathsf{CSP}}^{k,S_d^{[2]}}(\mathbf{BI}(\mathbf{C}^0;\mathbf{C}^1))$ has an integral solution.

To do so, we pull the notion of a robustly consistent partial homomorphism of the Tseitin-systems from Section 4 through all the constructions, so through the translation of group coset-CSP into bounded color class isomorphism, through the isomorphism or-construction, and through the reverse translation of bounded color class isomorphism to group coset-CSPs over symmetric groups.

- Partial homomorphisms of the Tseitin system induce partial isomorphisms of the graph encoding.
- Partial isomorphisms of the graph encoding induce partial isomorphisms in the isomorphism or-construction.
- Finally, partial isomorphisms of the isomorphism or-construction induce partial homomorphisms of the encoding as a group coset-CSP over S_d.

The reverse direction is not always true. But for the partial isomorphisms or homomorphisms, for which this is true, we can transfer the notion of robust consistency: A partial homomorphism $\mathbf{BI}(\mathbf{C}^0;\mathbf{C}^1) \to S_d^{[2]}$ is robustly consistent if it is induced by a robustly consistent partial homomorphism $\mathcal{C}^{H,\mathbb{Z}_{p_i},\lambda_i} \to \mathbb{Z}_{p_i}^{[r]}$ (we will make this notion precise in the following). We show that the properties of robustly consistent homomorphisms from Section 4 transfer to the group coset-CSP setting in the end:

- Robustly consistent partial solutions of the final $S_d^{[2]}$-coset-CSP are also not ruled out by k-consistency.
- A p_i-solution to the width-k affine relaxation of the Tseitin system over $\mathbb{Z}_{p_i}$ translates to a p_i-solution to the width-k affine relaxation for the final S_d-coset-CSP. In particular, only variables for robustly consistent partial homomorphisms are non-zero in the solution.
- Thus, the width-k affine relaxation of the S_d-coset-CSP also has an integral solution.

So essentially, the proofs in Section 5 translate to the S_d-coset-CSP. These arguments are the technically tedious part of the proof of Theorem 53. We prove this in detail in the following subsection but the key source of hardness is the same as in Section 4.

G.4 Proof of Theorem 53

First of all, we show that p-solutions to the width-k affine relaxation for any Γ-coset-CSP translate to p-solutions of the width-k affine relaxation for the CSP(S_d)-formulation of the corresponding graph isomorphism instance.

▶ **Lemma 57.** *Let $k \in \mathbb{N}$, Γ be a finite group, $\mathbf{B}$ an r-ary Γ-coset-CSP, and d be the maximum color class size of $\mathbf{G}_\Gamma^\mathbf{B}$. If $\mathsf{L}_{\mathsf{CSP}}^{kr,\Gamma^{[r]}}(\mathbf{B})$ has a p-solution, then $\mathsf{L}_{\mathsf{CSP}}^{k,S_d^{[2]}}(\mathbf{BI}(\mathbf{G}_\Gamma^\mathbf{B};\widetilde{\mathbf{G}}_\Gamma^\mathbf{B}))$ has a p-solution.*

Proof. Let $\mathbf{L} = \mathbf{BI}(\mathbf{G}_\Gamma^\mathbf{B};\widetilde{\mathbf{G}}_\Gamma^\mathbf{B})$ and let Φ be a p-solution of $\mathsf{L}_{\mathsf{CSP}}^{kd,\Gamma^{[r]}}(\mathbf{B})$. We define a p-solution Ψ for $\mathsf{L}_{\mathsf{CSP}}^{k,S_d^{[2]}}(\mathbf{L})$ as follows. Let $\mathfrak{C}$ be the colors of $\mathbf{G}_\Gamma^\mathbf{B}$. Assign with a set of colors $Y \subseteq \mathfrak{C}$ the set $\hat{Y}$ of the corresponding elements of $\mathbf{B}$: If Y contains a color of a variable vertex (x,γ), add x to $\hat{Y}$. If Y contains a color of a constraint vertex $(C,(\gamma_1,\dots,\gamma_{r'}))$ for $C\colon (x_1,\dots,x_{r'}) \in \Delta\delta$, add $x_1,\dots,x_{r'}$ to $\hat{Y}$. Note that $|\hat{Y}| \leq r|Y|$ because variables vertices for different variables, or constraint vertices for different constraints, have different colors, respectively.

Let $Y \in \binom{\mathfrak{C}}{\leq k}$ and $f \in \mathrm{Hom}(\mathbf{B}[\hat{Y}], \Gamma^{[r]})$. We define a bijection $f' \colon V(\mathbf{G}_\Gamma^{\mathbf{B}}[Y]) \to V(\widetilde{\mathbf{G}}_\Gamma^{\mathbf{B}}[Y])$ via

$$
\begin{aligned}
f'((x,\gamma)) &:= \big(x, \gamma f(x)^{-1}\big) && \text{for all } (x,\gamma) \in V(\mathbf{G}_\Gamma^{\mathbf{B}}[Y]),\\
f'((C,\gamma_1,\dots,\gamma_r)) &:= \big(C, \gamma_1 f(x_1)^{-1}, \dots, \gamma_r f(x_r)^{-1}\big) && \text{for all } (C,\gamma_1,\dots,\gamma_r) \in V(\mathbf{G}_\Gamma^{\mathbf{B}}[Y]).
\end{aligned}
$$

Since f is a partial homomorphism, the map f' indeed maps to vertices of $\widetilde{\mathbf{G}}_\Gamma^{\mathbf{B}}[Y]$ and moreover is a partial isomorphism: If f satisfies a constraint C of $\mathbf{B}$, then $(f(x_1), \dots, f(x_r)) \in \Delta\delta$. So for all $(\gamma_1, \dots, \gamma_r) \in \Delta\delta$, we have $(\gamma_1 f(x_1)^{-1}, \dots, \gamma_r f(x_r)^{-1}) \in \Delta$. This is exactly the homogeneous version of the constraint, which occurs in $\widetilde{\mathbf{G}}_\Gamma^{\mathbf{B}}$. Thus, $f' \in \mathrm{Iso}(\mathbf{G}_\Gamma^{\mathbf{B}}[Y], \widetilde{\mathbf{G}}_\Gamma^{\mathbf{B}}[Y])$. Hence, f' induces a partial homomorphism $\hat{f} \in \mathrm{Hom}(\mathbf{L}[Y], S_d^{[2]})$. For all $g \in \mathrm{Hom}(\mathbf{L}[Y], S_d^{[2]})$, define

$$
\Psi(x_{Y,g}) := \begin{cases} \Phi(x_{\hat{Y},f}) & \text{if } g = \hat{f} \text{ for some } f \in \mathrm{Hom}(\mathbf{B}[\hat{Y}], \Gamma^{[r]}),\\ 0 & \text{otherwise.} \end{cases}
$$

We say that the **partial homomorphism** g **corresponds to** f in the equation above. Likewise, that **variable** $x_{Y,g}$ **of** $\mathsf{L}_{\mathsf{CSP}}^{k,S_d^{[2]}}(\mathbf{BI}(\mathbf{G}_\Gamma^{\mathbf{B}}; \widetilde{\mathbf{G}}_\Gamma^{\mathbf{B}}))$ **corresponds** to the variable $x_{\hat{Y},f}$ of $\mathsf{L}_{\mathsf{CSP}}^{kr,\Gamma^{[r]}}(\mathbf{B})$.

We show that Ψ is a solution to $\mathsf{L}_{\mathsf{CSP}}^{k,S_d^{[2]}}(\mathbf{L})$, which then is obviously a p-solution. We first consider the equations of Type L1. Let $Y \in \binom{\mathfrak{C}}{\leq k}$, $c \in Y$, and $g' \in \mathrm{Hom}(\mathbf{L}[Y \setminus \{c\}], S_d^{[2]})$. First assume that there is a $g \in \mathrm{Hom}(\mathbf{B}[\widehat{Y \setminus \{c\}}], \Gamma^{[r]})$ such that $g' = \hat{g}$. Then, possibly exploiting Lemma 25,

$$
\sum_{\substack{f' \in \mathrm{Hom}(\mathbf{L}[Y], S_d^{[2]}),\\ f'|_{Y \setminus \{c\}} = \hat{g}}} \Psi(x_{Y,f'}) = \sum_{\substack{f \in \mathrm{Hom}(\mathbf{B}[\hat{Y}], \Gamma^{[r]}),\\ f|_{\widehat{Y \setminus \{c\}}} = g}} \Phi(x_{\hat{Y},f}) = \Phi(x_{\hat{Y},g}) = \Psi(y_{Y,\hat{g}}).
$$

Second assume that there is no $g \in \mathrm{Hom}(\mathbf{B}[\widehat{Y \setminus \{c\}}], \Gamma^{[r]})$ such that $g' = \hat{g}$. Then for every partial homomorphism $f' \in \mathrm{Hom}(\mathbf{L}[Y], S_d^{[2]})$ (via the identification of permutation on each color class with the S_d-variables) such that $f'|_{Y \setminus \{c\}} = g'$, there is also no $f \in \mathrm{Hom}(\mathbf{B}[\hat{Y}], \Gamma^{[r]})$ such that $f' = \hat{f}$. Hence both sides of Equation L1 are 0.

Finally, consider Equation L2: we have $\Psi(x_{\emptyset,\emptyset}) = \Psi(_{\emptyset,\hat{\emptyset}}) = \Phi(x_{\emptyset,\emptyset}) = 1$ because the empty homomorphism $\emptyset \colon \mathbf{B}[\emptyset] \to \Gamma^{[r]}$ induces the empty homomorphism $\hat{\emptyset} \colon \mathbf{L}[\emptyset] \to S_d^{[2]}$. ◀

In the setting of the former proof, we show that if a partial homomorphism of $\mathbf{B}$ is not discarded by the k-consistency algorithm, then the corresponding one of $\mathbf{L}$ is not discarded, either.

▶ **Lemma 58.** *Let $k \in \mathbb{N}$, let Γ be a finite group and $\mathbf{B}$ an r-ary Γ-coset-CSP, and let $\mathbf{L} = \mathbf{BI}(\mathbf{G}_\Gamma^{\mathbf{B}}; \widetilde{\mathbf{G}}_\Gamma^{\mathbf{B}})$. Let $\mathfrak{C}$ be the set of colors of $\mathbf{G}_\Gamma^{\mathbf{B}}$. For all $X \in \binom{B}{rk}$, $Y \in \binom{\mathfrak{C}}{k}$, $f \in \mathrm{Hom}(\mathbf{B}[X], \Gamma^{[r]})$, and $g \in \mathrm{Hom}(\mathbf{L}[Y], \Gamma^{[r]})$ such that f corresponds to g (in the sense of the proof of Lemma 57), if $f \in \kappa_{rk}^{\Gamma^{[r]}}[\mathbf{B}](X)$ then $g \in \kappa_k^{S_d^{[2]}}[\mathbf{L}](Y)$.*

Proof. Recall that if f corresponds to g, we have $\hat{Y} = X$ and $g = \hat{f}$. We consider the sets of partial homomorphisms $\{\hat{f} \mid f \in \kappa_{rk}^{\Gamma^{[t]}}[\mathbf{B}](\hat{X})\}$ for every $X \in \binom{\mathfrak{C}}{\leq k}$ and show that they satisfy the down-closure and forth-condition property.

Let $Y \subseteq X \in \binom{\mathfrak{C}}{\leq k}$. To show the down-closure, let $f \in \kappa_{rk}^{\Gamma[t]}[\mathbf{B}](\hat{X})$. Because $Y \subseteq X$, we have $\hat{Y} \subseteq \hat{X}$. From the down-closure of $\kappa_{rk}^{\Gamma[r]}[\mathbf{B}]$ follows that $f|_{\hat{Y}} \in \kappa_{rk}^{\Gamma[t]}[\mathbf{B}](\hat{Y})$. By the construction of the corresponding homomorphisms, we have $\hat{f}|_Y = \widehat{f|_{\hat{Y}}}$.

The forth-condition is similarly inherited from $\kappa_{rk}^{\Gamma[r]}[\mathbf{B}]$: Let $g \in \kappa_{rk}^{\Gamma[t]}[\mathbf{B}](\hat{Y})$. Then g extends to some $f \in \kappa_{rk}^{\Gamma[t]}[\mathbf{B}](\hat{H})$ by the forth-condition of $\kappa_{rk}^{\Gamma[r]}[\mathbf{B}]$. Again be the construction of corresponding homomorphisms, we have that $\hat{h}$ extends $\hat{g}$. ◀

The previous lemmas establish the link between coset-CSPs and their isomorphism formulation. The next step is to deal with the isomorphism or-construction. We extend the notion of entry-ℓ vertices from the encoding of sequences to the isomorphism or-construction: For $\ell \in [j]$, we call a vertex of $\mathbf{C}^0$ or $\mathbf{C}^1$ an entry-ℓ vertex if it is a an entry-ℓ vertex of some component $\langle \mathbf{B}_1^{a_1}, \dots, \mathbf{B}_j^{a_j} \rangle$.

For the following, fix an $i \in [j]$. We now describe how partial isomorphisms between $\mathbf{B}_i^0$ and $\mathbf{B}_i^1$ can be extended to partial isomorphisms of $\mathbf{C}_0$ and $\mathbf{C}_1$. We fix a bijection b between the components of $\mathbf{C}^0$ and $\mathbf{C}^1$, that is, between the structures $\langle \mathbf{B}_1^{a_1}, \dots, \mathbf{B}_j^{a_j} \rangle$ with even and odd sum of the a_ℓ, such that identified components only differ in entry i:

$$b(\langle \mathbf{B}_1^{a_1}, \dots, \mathbf{B}_i^{a_i}, \dots, \mathbf{B}_j^{a_j} \rangle) = \langle \mathbf{B}_1^{a_1}, \dots, \mathbf{B}_i^{1-a_i}, \dots, \mathbf{B}_j^{a_j} \rangle.$$

Using the identity map on $\mathbf{B}_\ell^0$ and $\mathbf{B}_\ell^1$ for all $\ell \neq i$, the bijection b induces a bijection $\hat{b}$ between the vertices of these components apart from the entry-i vertices.

Let X be a set of colors of $\mathbf{C}^0$ and $\mathbf{C}^1$ and denote by $X|_i$ the set of colors of $\mathbf{B}_i^0$ and $\mathbf{B}_i^1$ that occur (after the possible renaming to encode sequences) in X. We now define the function $\iota_i^X\colon \operatorname{Iso}(\mathbf{B}_i^0[X|_i], \mathbf{B}_i^1[X|_i]) \to \operatorname{Iso}(\mathbf{C}^0[X], \mathbf{C}^1[X])$ for every set of colors X as follows: For a partial isomorphism $f \in \operatorname{Iso}(\mathbf{B}_i^0[X|_i], \mathbf{B}_i^1[X|_i])$, the function $\iota_i^X(f)$ is defined as follows:

- Let v be an entry-i vertex of a component $D = \langle \mathbf{B}_1^{a_1}, \dots, \mathbf{B}_i^{a_j} \rangle$. If $a_i = 0$, then $\iota_i^X(f)$ maps v to an entry-i vertex of $b(D)$ according to f (when seeing v as a vertex of $\mathbf{B}_i^0$). If $a_i = 1$, then we proceed as in the previous case using f^{-1} instead of f.
- Otherwise, $\iota_i^X(f)$ maps v to $\hat{b}(v)$.

Intuitively, $\iota_i^X(f)$ maps all components in $\mathbf{C}^0[X]$ to the corresponding ones in $\mathbf{C}^1[X]$ according to b and uses f or f^{-1}, respectively, for the i-th entry.

▶ **Lemma 59.** *Fix $k \in \mathbb{N}$, let $\mathbf{B}_1^0, \dots, \mathbf{B}_j^0$ and $\mathbf{B}_1^1, \dots, \mathbf{B}_j^1$ be two sequences of colored structures of arity at most r and color class size at most d, and let $(\mathbf{C}^0, \mathbf{C}^1) = \mathbf{OR}_{i\in[j]}^{ISO}[\mathbf{B}_i^0, \mathbf{B}_i^1]$. Assume $\mathfrak{C}$ is the set of colors of $\mathbf{C}^0$ and $\mathbf{C}^1$, $\mathbf{L} = \mathbf{BI}(\mathbf{C}^0; \mathbf{C}^1)$, and $\mathbf{L}_i = \mathbf{BI}(\mathbf{B}_i^0; \mathbf{B}_i^1)$ for all $i \in [j]$. If, for some $i \in [j]$, the equation system $\mathsf{L}_{\mathsf{CSP}}^{k, S_d^{[r]}}(\mathbf{L}_i)$ has a p-solution Φ, then the equation system $\mathsf{L}_{\mathsf{CSP}}^{k, S_d^{[r]}}(\mathbf{L})$ has the p-solution Ψ defined, for all $X \in \binom{\mathfrak{C}}{\leq k}$ and $g \in \operatorname{Hom}(\mathbf{L}[X], S_d^{[r]})$, via*

$$\Psi(x_{X,g}) := \begin{cases} \Phi(x_{X|_i, f}) & \text{if } \iota_i^X(f(X|_i)) = g(X) \text{ for some } f \in \operatorname{Hom}(\mathbf{L}_i[X|_i], S_d^{[r]}), \\ 0 & \text{otherwise.} \end{cases}$$

*We say that the **partial homomorphism** g **corresponds** to f in the equation above and that the **variable** $x_{X,g}$ **of** $\mathsf{L}_{\mathsf{CSP}}^{k, S_d^{[r]}}(\mathbf{L})$ **corresponds** to the variable $x_{X|_i,f}$ of $\mathsf{L}_{\mathsf{CSP}}^{k, S_d^{[r]}}(\mathbf{L}_i)$.*

Proof. First consider the equations of Type L1: Recall that $\mathbf{L}$ has a variable for every color class. Let $X \in \binom{\mathfrak{C}}{\leq k}$ be a set of at most k colors, let $c \in X$ be a color, and $g \in$

$\mathrm{Hom}(\mathbf{L}[X \setminus \{c\}], S_d^{[r]})$. First assume that there is an $f \in \mathrm{Hom}(\mathbf{L}_i[X|_i], S_d^{[r]})$ such that $\iota_i^X(f) = g$. Then

$$\begin{aligned}\sum_{\substack{h \in \mathrm{Hom}(\mathbf{L}[X], S_d^{[r]}),\\ h|_{X\setminus\{c\}}=g}} \Psi(x_{X,h}) &= \sum_{\substack{h \in \mathrm{Hom}(\mathbf{L}_i[X|_i], S_d^{[r]}),\\ \iota_i^X(h)|_{X\setminus\{c\}}=g}} \Phi(x_{X|_i,h}) \\ &= \sum_{\substack{h \in \mathrm{Hom}(\mathbf{L}_i[X|_i], S_d^{[r]}),\\ h|_{X\setminus\{c\}}=f}} \Phi(x_{X|_i,h}) \\ &= \Phi(x_{X|_i,f}) = \Psi(x_{X,g}).\end{aligned}$$

Assume otherwise that there is no such f. Then $\Psi(x_{X,g}) = 0$. But in this case, every partial homomorphism $h \in \mathrm{Hom}(\mathbf{L}[X], S_d^{[r]})$ is not in the image of ι_i, which means that both sides of Equation L1 are zero. It remains to check Equation L2. Since the empty homomorphism is the image of the empty homomorphism under ι_k^i, Equation L2 for $\mathsf{L}_{\mathsf{CSP}}^{k,S_d^{[r]}}(\mathbf{L})$ follows from Equation L2 for $\mathsf{L}_{\mathsf{CSP}}^{k,S_d^{[r]}}(\mathbf{L}_i)$. It is clear that the solution is a p-solution. ◀

The former lemma showes that p-solutions for one entry translate to a p-solution of the or-construction. We now show a similar statement for the k-consistency algorithm.

▶ **Lemma 60.** *Fix $k \in \mathbb{N}$, let $\mathbf{B}_1^0, \dots, \mathbf{B}_j^0$ and $\mathbf{B}_1^1, \dots, \mathbf{B}_j^1$ be two sequences of colored structures of arity at most r and color class size at most d, and let $(\mathbf{C}^0, \mathbf{C}^1) = \mathbf{OR}_{i\in[j]}^{ISO}[\mathbf{B}_i^0, \mathbf{B}_i^1]$. Let $\mathfrak{C}$ be the set of colors of $\mathbf{C}^0$ and $\mathbf{C}^1$, and let $\mathbf{L} = \mathbf{BI}(\mathbf{C}^0; \mathbf{C}^1)$. For every $i \in [j]$, let $\mathfrak{C}_i$ be the set of colors of $\mathbf{C}^0$ and $\mathbf{C}^1$, and let $\mathbf{L}_i = \mathbf{BI}(\mathbf{B}_i^0; \mathbf{B}_i^1)$. Let $i \in [j]$, $X \in \binom{\mathfrak{C}_i}{\le k}$, and $f \in \mathrm{Hom}(\mathbf{L}_i[X], S_d^{[r]})$. If $f \in \kappa_k^{S_d^{[r]}}[\mathbf{L}_i]$, then for every $g \in \mathrm{Hom}(\mathbf{L}[Y], S_d^{[r]})$ (for some $Y \in \binom{\mathfrak{C}}{\le k}$) such that f corresponds to g, we have that $g \in \kappa_k^{S_d^{[r]}}[\mathbf{L}]$.*

Proof. We show that the sets of partial homomorphisms $\iota_i^X(\kappa_k^{S_d^{[r]}}[\mathbf{L}](X|_i))$ for all $X \in \binom{\mathfrak{C}}{\le k}$ satisfy the down-closure and forth-condition property. Then they have to in included in the greatest fixed-point computed by the k-consistency algorithm.

The down-closure is inherited from $\kappa_k^{S_d^{[r]}}[\mathbf{L}_i]$: Let $Y \subset X \in \binom{\mathfrak{C}}{\le k}$ and $f \in \iota_i^X(\kappa_k^{S_d^{[r]}}[\mathbf{L}](X|_i)$. Then there is a $g \in \kappa_k^{S_d^{[r]}}[\mathbf{L}](X|_i)$ such that $\iota_i^X(g) = f$. By the forth-condition for $\kappa_k^{S_d^{[r]}}[\mathbf{L}_i]$, we have that $g|_{Y|_i} \in \kappa_k^{S_d^{[r]}}[\mathbf{L}_i](Y|_i)$. Hence $h = \iota_i^Y(g|_{Y|_i}) \in \iota_i^Y(\kappa_k^{S_d^{[r]}}[\mathbf{L}](Y|_i)$. Because $\hat{b}$ is a bijection and by the definition of ι_i^X, it follows $f|_Y = h$.

To show the forth-condition, let $Y \subset X \in \binom{\mathfrak{C}}{\le k}$ and $g \in \iota_i^Y(\kappa_k^{S_d^{[r]}}[\mathbf{L}](Y|_i))$. Now there is an $f \in \kappa_k^{S_d^{[r]}}[\mathbf{L}](Y|_i))$ such that $\iota_i^Y(f) = g$. Because $Y|_i \subset X|_i$ (and both sets are of size at most k), there is an $h \in \kappa_k^{S_d^{[r]}}[\mathbf{L}](X|_i))$ that extends g. But then $\iota_i^X(h) \in \kappa_k^{S_d^{[r]}}[\mathbf{L}](X|_i))$ and $\iota_i^X(h)$ extends g by the definition of ι_i^X. ◀

Finally, we are ready to prove Theorem 53.

Proof of Theorem 53. Fix a $k \in \mathbb{N}$ and $d \ge 18$. We construct $S_d^{[2]}$-instances as described in Section G.3. Set $p_1 = 2$ and $p_1 = 3$. Robustly consistent partial homomorphism of the Tseitin system $\mathbf{B}_i := \mathcal{C}^{H,\mathbb{Z}_{p_i},\lambda_i}$ are not ruled out by k-consistency by Lemma 43. Lemma 58 shows that the corresponding partial homomorphisms of $\mathbf{L}_i := \mathbf{BI}(\mathbf{G}_{\mathbb{Z}_{p_i}}^{\mathbf{B}_i}; \widetilde{\mathbf{G}}_{\mathbb{Z}_{p_i}}^{\mathbf{B}_i})$ for both $i \in [2]$ also survive k-consistency. Let $(\mathbf{C}^0, \mathbf{C}^1) = \mathbf{OR}_{i\in[2]}^{\mathrm{ISO}}[\mathbf{G}_{\mathbb{Z}_{p_i}}^{\mathbf{B}_i}, \widetilde{\mathbf{G}}_{\mathbb{Z}_{p_i}}^{\mathbf{B}_i}]$. Lemma 60 shows that also the corresponding partial homomorphisms of $\mathbf{BI}(\mathbf{C}^0; \mathbf{C}^1)$ are not ruled out by the k-consistency algorithm.

Now consider solutions of the width-k affine relaxation. By Lemma 16, there is a p_i-solution for $\mathsf{L}_{\mathsf{CSP}}^{k,\mathbb{Z}_{p_i}^{[3]}}(\mathbf{B}_i)$ for both $i \in [2]$ in which only variables for robustly consistent partial homomorphisms are non-zero. Lemma 57 shows that such p_i-solutions also exist for $\mathsf{L}_{\mathsf{CSP}}^{k,S_{q_i}^{[2]}}(\mathbf{L}_i)$, where $q_i = p_i^2$, for both $i \in [2]$. These solutions are non-zero only for variables of partial homomorphisms that correspond to robustly consistent ones of the Tseitin systems. The domain size p_i^2 comes from Lemma 54 and the fact that the coset-constraints of the ternary Tseitin systems use subgroups of order p_i^2. Lemma 59 provides p_i-solutions for $\mathsf{L}_{\mathsf{CSP}}^{k,S_{18}^{[2]}}(\mathbf{BI}(\mathbf{C}^0; \mathbf{C}^1))$ for both $i \in [2]$, for which again only variables are set to a non-zero value for partial homomorphisms corresponding to robustly consistent ones. Here the domain size is $2\max\{p_1^2, p_2^2\} = 18$ and comes from Lemma 56.

Now for the $\mathbb{Z}$-affine k-consistency relaxation, the proof proceeds exactly as the one of Theorem 18. For BA^k, we proceed as in the proof of Theorem 20, where we again note that the p_i-solutions exactly set the variables for partial homomorphisms corresponding to robustly consistent ones of the Tseitin system to a non-zero value. And finally for CLAP, we proceed as in the proof of 21. Here we again use Lemma 48 to show that we can set the variable of a single robustly consistent solution in $\mathsf{L}_{\mathsf{CSP}}^{k,\mathbb{Z}_{p_i}^{[3]}}(\mathbf{B}_i)$ to 1, which then travels through Lemmas 57 and 59 to the corresponding variable of $\mathsf{L}_{\mathsf{CSP}}^{k,S_{18}^{[2]}}(\mathbf{BI}(\mathbf{C}^0; \mathbf{C}^1))$. ◀