On the number of digons in arrangements of pairwise intersecting circles

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September 15, 2025

Abstract

A long-standing open conjecture of Branko Grünbaum from 1972 states that any simple arrangement of n pairwise intersecting pseudocircles in the plane can have at most $2n-2$ digons. Agarwal et al. proved this conjecture for arrangements of pairwise intersecting pseudocircles in which there is a common point surrounded by all pseudocircles. Recently, Felsner, Roch and Scheucher showed that Grünbaum's conjecture is true for arrangements of pairwise intersecting pseudocircles in which there are three pseudocircles every pair of which create a digon. In this paper we prove this over 50-year-old conjecture of Grünbaum for any simple arrangement of pairwise intersecting circles in the plane.

1 Introduction

A family of *pseudocircles* is a set of closed Jordan curves such that every two of them are either disjoint, intersect at exactly one point in which they touch or intersect at exactly two points in which they properly cross each other. The bounded regions whose boundaries are the pseudocircles are called *pseudodiscs*. An *arrangement* $\mathcal{A}(\mathcal{F})$ of a family $\mathcal{F}$ of pseudocircles is the cell complex into which the plane is decomposed by the pseudocircles and consists of *vertices*, *edges* and *faces*. If there are two points that lie on every pseudocircle, then the arrangement is *trivial*. If there is no point that lies on three pseudocircles, then the arrangement is *simple*.

A *digon* is a face in $\mathcal{A}(\mathcal{F})$ whose boundary consists of two edges. The two circles containing the two edges of a digon are said to *support* the digon. We also say that these two circles *create* the digon.

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‡HUN-REN Alfréd Rényi Institute of Mathematics and ELTE Eötvös Loránd University, Budapest, Hungary. Research supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, by the National Research, Development and Innovation Office – NKFIH under the grant K 132696 and FK 132060, by the ÚNKP-23-5 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund and by the ERC Advanced Grant "ERMiD". This research has been implemented with the support provided by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, financed under the ELTE TKP 2021-NKTA-62 funding scheme. keszegh@renyi.hu

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We distinguish two different types of digons. A *lens* is a digon that is equal to the intersection of the two discs supporting it. A *lune* is a digon that is equal to a difference of the two discs supporting it (see Figure 1).

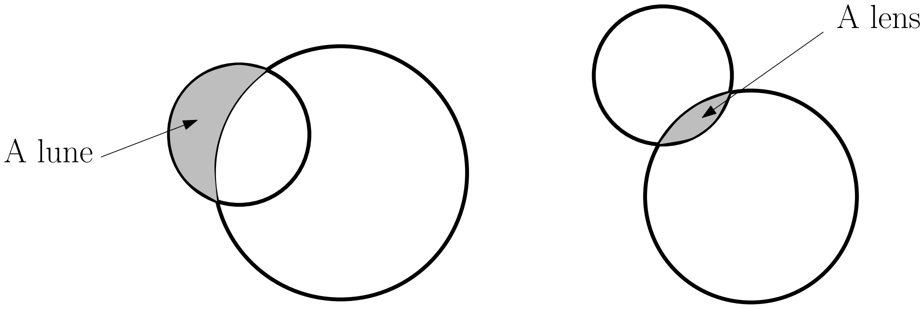

Figure 1: A lens and a lune.

It is easy to see that there are $2n$ digons in a trivial arrangement of n pseudocircles, for $n > 1$. More than 50 years ago Grünbaum conjectured that non-trivial arrangements of *pairwise intersecting* pseudocircles have fewer digons.

Conjecture 1 (Grünbaum's digon conjecture [8, Conjecture 3.6])**.** *Every simple arrangement of $n > 2$ pairwise intersecting pseudocircles has at most $2n - 2$ digons.*

It is possible to show, by small perturbation the pseudocircles near intersection points of three or more curves, that one can assume the family of pseudocircles is simple without decreasing the number of digons in the arrangement, as long as we do not start with a trivial arrangement. By assuming that the arrangement of pseudocircles is simple, we conclude that it is nontrivial for $n > 2$.

Some special cases of Grünbaum's conjecture were settled. Agarwal et al. [1] proved the conjecture for *cylindrical* arrangements, that is, for arrangements in which there is a point that is surrounded by every pseudocircle. Recently, Felsner, Roch and Scheucher [5] showed that the conjecture also holds for simple arrangements in which there are three pseudocircles such that every two of them create a digon.

In this paper we prove Grübnaum's conjecture for any simple arrangement of pairwise intersecting circles in the plane.

Theorem 1. *Every non-trivial simple arrangement of n pairwise intersecting circles has at most $2n - 2$ digons.*

The simple construction in Figure 2 (taken from [8]) shows that the bound in Theorem 1 is best possible for $n \geqslant 4$. There are 5 circles in this construction and 8 lenses. One can generalize the construction for any number of circles by suitably adding more circles to the three smaller circles in the figure.

Before we continue we would like to note that with more care one could prove Theorem 1 only under the assumption that the family of circles is nontrivial, rather than simple. Here, unlike in the pseudocircles case, one can no longer perturb the circles near multiple intersection points and hope to remain with a family of circles. Therefore, it will be easier and more natural to address the case of non-simple arrangements as part of the more general conjecture of Grünbaum for non-simple arrangements of pairwise intersecting pseudocircles. We decided not to address in this paper the rather technical issues that may arise if one does not assume simplicity of the arrangement of circles. In this paper we would like to emphasize the special and beautiful properties of geometric circles related to digons. The assumption that the family of circles is simple allows one to assume, and we will indeed assume this, that

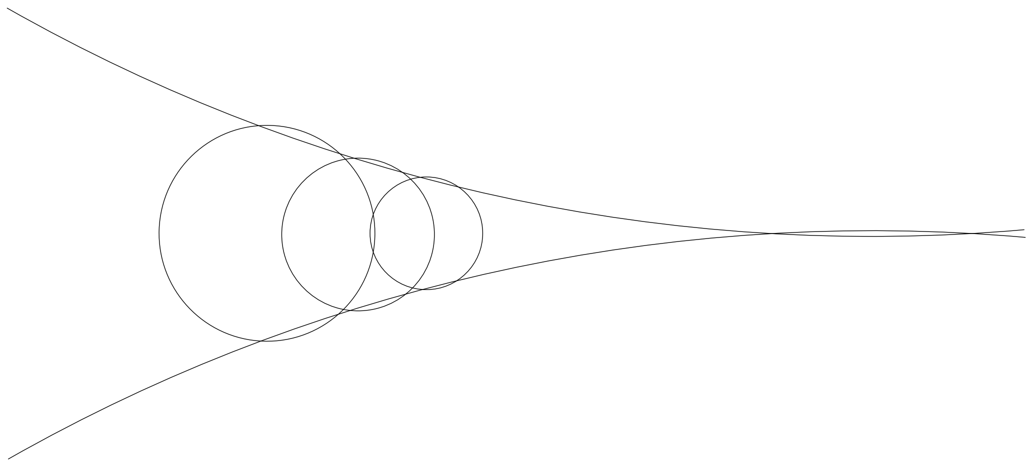

Figure 2: A family of 5 pairwise intersecting circles with 8 lenses.

the set of centers of the circles in question is in general position in the sense that no three of the centers are collinear. This can be done by applying a generic inversion map to the plane. We will not make use these assumptions explicitly in the proofs, but we will use them implicitly.

There has been a lot of research about digons in arrangements of circles (and pseudo-circles) that are not necessarily pairwise intersecting. We will not survey here the vast literature about digons in arrangements of circles and pseudo-circles and on related situations where we allow curves to intersect more than twice and only refer the reader to [7] and the many references therein. The case where circles need not be pairwise intersecting is of completely different nature. We remark that in such a case the best constructions show that it is possible that n circles will determine $\Omega(n^{4/3})$ many lenses. The best known upper bound is $O(n^{3/2}\log n)$ given in [10], that is following the footsteps of [15]. A slightly better upper bound of $O(n^{3/2})$ for the number of touching points among n circles follows from a result of Ellenberg, Solymosi and Zahl [3]. The case of unit circles is of particular interest because of its relation to the celebrated unit distance problem posed by Paul Erdős ([4]). For this problem the best known lower and upper bounds are $\Omega(n^{1+c/\log\log n})$ [4] and $O(n^{4/3})$ [11, 16, 17], respectively.

Going back to families of pairwise intersecting circles, the number of lunes in these arrangements was studied in [2].

Theorem 2. *Any arrangement of n pairwise intersecting circles in the plane has at most $2n-4$ lunes.*

Theorem 2 is used in [2] to derive a linear upper bound, that is not tight, for the number of digons (lunes and lenses) in any arrangement of pairwise intersecting circles in the plane. Specifically, it is shown in [2] that arrangements of n pairwise intersecting circles in the plane contain at most $2n-2$ lunes and at most $18n$ lenses.

The tight bound on the maximum number of lenses in a family of pairwise intersecting circles in the plane is established in [13].

Theorem 3 ([13])**.** *Any arrangement of n pairwise intersecting circles in the plane determines at most $2n-2$ lenses.*

Theorem 2 and Theorem 3 imply immediately an upper bound of $4n-6$ for the number of digons in arrangements of n pairwise intersecting circles in the plane. Hence, Theorem 1 improves this upper bound.

For arrangements of pairwise intersecting *unit circles*, Pinchasi [14] proved that they can have at most n lenses and at most 3 lunes, hence at most $n+3$ digons.

2 Three crucial geometric lemmata

In this section we will explore three crucial geometric lemmata concerning pairwise intersecting circles in the plane. Two of these lammata have been shown in previous works while the third one is new and we bring its proof here.

All the three lemmata are concerned with the geometric graph G on the set of centers of circles in a family $\mathcal{F}$ of pairwise intersecting circles in the plane. The edges in G correspond to the pairs of circles creating digons (either lunes, or lenses, or both) in $\mathcal{A}(\mathcal{F})$.

The first lemma is from [2], where it is used to derive the tight upper bound for the number of lunes in Theorem 2.

Lemma 1 ([2])**.** *Let $\mathcal{F}$ be a finite family of pairwise intersecting circles in the plane. Let G be the geometric graph whose vertices are the centers of the circles in $\mathcal{F}$ such that two vertices are connected by an edge if and only if the corresponding circles in $\mathcal{F}$ create a lune in the arrangement $\mathcal{A}(\mathcal{F})$. Then no pair of edges in G cross, thus G is a planar embedding.*

The second lemma is from [13], where it is used to prove Theorem 3. We recall that two edges in a geometric graph are called *avoiding* if they are opposite edges of a convex quadrilateral. That is, two edges e and f in a geometric graph are avoiding if the lines containing e and f intersect at a point that is neither on e nor nor on f (see Figure 3).

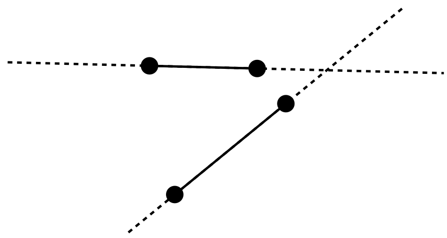

Figure 3: A pair of avoiding edges.

Lemma 2 ([13])**.** *Let $\mathcal{F}$ be a finite family of pairwise intersecting circles in the plane. Let G be the geometric graph whose vertices are the centers of the circles in $\mathcal{F}$ such that two vertices are connected by an edge if and only if the corresponding circles in $\mathcal{F}$ create a lens in the arrangement $\mathcal{A}(\mathcal{F})$. Then G does not contain a pair of avoiding edges.*

Lemma 2 implies immediately Theorem 3 because of a result of Katchalski and Last [9] and Valtr [18] by which a geometric graph on n vertices with no pair of avoiding edges can have at most $2n-2$ edges. We will not make use of this bound, but rather use the observation in Lemma 2 directly.

For the sake of completeness and because the result in [13] is rather recent, we bring here an independent proof, and different from the one in [13], of Lemma 2 in the case where the arrangement of pairwise intersecting circles is simple.

Proof of Lemma 2. Under the contrary assumption, there are four circles c_1, c_2, c_3, and c_4 in $\mathcal{F}$ with centers O_1, O_2, O_3, and O_4, respectively, such that c_1 and c_2 create a lens in $\mathcal{A}(\mathcal{F})$ and also c_3 and c_4 create a lens in $\mathcal{A}(\mathcal{F})$. Moreover, the line segments $[O_1O_2]$ and $[O_3O_4]$ are opposite edges of a convex quadrilateral.

Without loss of generality we assume that the line O_1O_2 is horizontal and O_1 lies to the left of O_2. We may also assume that both O_3 and O_4 lie above the line O_1O_2 such that $O_1O_2O_3O_4$ is a convex quadrilateral (see Figure 6).

We need the following two simple observations. The first is extremely elementary and its proof is left to the reader. The second observation is a common knowledge that is very well known.

Observation 1. *Let c and c' be two intersecting circles with centers O and O' respectively. Then the intersection of c and the ray $\overrightarrow{OO'}$ is the center of the arc on c that is the part of c inside the disc bounded by c' (see Figure 4).*

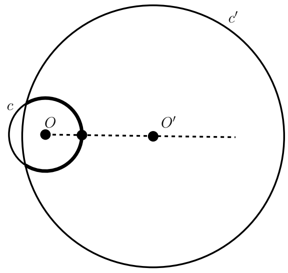

Figure 4: An illustration for Observation 1.

Observation 2. *Let A, B, A', and B' be four points in the plane. If there is a circular disc D that contains A and B but not A' and B', and there is another circular disc D' that contains A' and B' but not A and B, then the two segments $[AB]$ and $[A'B']$ are disjoint.*

Proof. This is clear if the circular discs D and D' are disjoint. If D and D' intersect we observe that the line through the intersection points of their boundaries (notice none of D and D' can be contained in the other) separates the regions $D \setminus D'$ and $D' \setminus D$ (see Figure 5). ∎

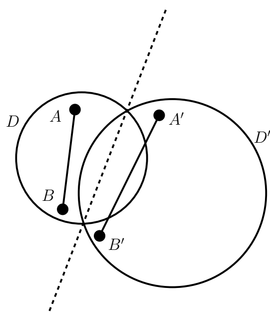

Figure 5: An illustration for Observation 2.

Going back to the proof of Lemma 2, let D_3 and D_4 denote the circular discs bounded by c_3 and c_4, respectively. let S_3 and S_4 denote the centers of the arcs $c_1 \cap D_3$ and $c_1 \cap D_4$, respectively, on the circle c_1. We observe that the two arcs must be disjoint, or else the lens $D_3 \cap D_4$ would have contained a point of c_1, which is impossible.

Similarly, let T_3 and T_4 denote the centers of the arcs $c_2 \cap D_3$ and $c_2 \cap D_4$, respectively, on the circle c_2 and observe that the two arcs must be disjoint.

By observation 1, S_3 is the point of intersection of c_1 with $\overrightarrow{O_2O_3}$. Similarly, S_4 is the point of intersection of c_1 with $\overrightarrow{O_2O_4}$. Because $O_1O_2O_3O_4$ is a convex quadrilateral, it must be that S_3 lies to the right of S_4 on c_1 above the line O_1O_2 (see Figure 6).

We claim that D_4 cannot contain the lens created by c_1 and c_2. Indeed, $c_1 \cap D_4$ is an arc whose center S_4 lies above the line O_1O_2. Therefore, if D_4 contains the lens created by c_1 and c_2, then $c_1 \cap D_4$ must contain all the part of c_1 that is above the line O_1O_2 and to the right of S_4. In particular it must contain the center S_3 of the arc $c_1 \cap D_3$. Consequently, the point S_3 lies both in the interior of D_4 and the interior of D_3. This is a contradiction because the lens $D_3 \cap D_4$ cannot contain in its interior a point of c_1.

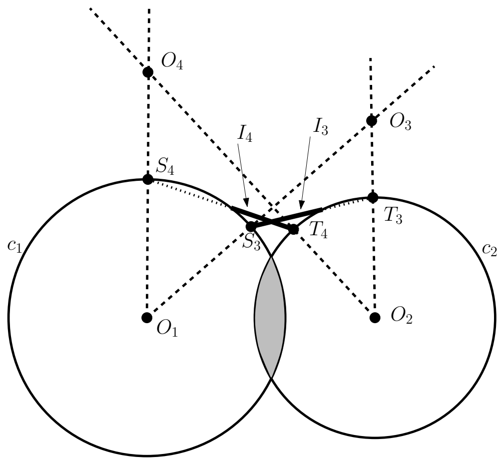

Figure 6: Lemma 2.

We argue similarly about T_3 and T_4. By Observation 1, T_3 is the point of intersection of c_2 with $\overrightarrow{O_2O_3}$. Similarly, T_4 is the point of intersection of c_2 with $\overrightarrow{O_2O_4}$. Because $O_1O_2O_3O_4$ is a convex quadrilateral T_3 lies to the right of T_4 on c_2 above the line O_1O_2. We observe now that D_3 cannot contain the lens created by c_1 and c_2. This is because otherwise D_3 contains the arc $c_2 \cap D_4$ whose center is the point T_4 on c_2. Hence T_4, that is a point on c_2, lies in the interiors of both D_4 and D_3. Consequently c_2 intersects the interior of the lens $D_3 \cap D_4$, which is a contradiction.

Because D_3 and D_4 do not contain the lens $D_1 \cap D_2$, they must be disjoint from it, as the lens $D_1 \cap D_2$ cannot be intersected by any of the circles c_1 and c_2. Hence all four arcs $c_1 \cap D_3, c_1 \cap D_4, c_2 \cap D_3$, and $c_2 \cap D_4$ lie on the boundary of $D_1 \cup D_2$. This boundary is a simple closed curve that we denote by Γ. The curve Γ is the union of the two arcs $c_1 \setminus D_2$ and $c_2 \setminus D_1$.

Because S_3 lies to the right of S_4 on c_1 and T_3 lies to the right of T_4 on c_2, above the line O_1O_2, then the four pairwise disjoint arcs $c_2 \cap D_3, c_2 \cap D_4, c_1 \cap D_3$, and $c_1 \cap D_4$ lie in this counterclockwise cyclic order on Γ. In particular, the arcs $c_1 \cap D_3$ and $c_2 \cap D_3$ separate the arcs $c_1 \cap D_4$ and $c_2 \cap D_4$ on the simple closed curve Γ.

We now claim that the line segments $[S_3T_3]$ and $[S_4T_4]$ must cross. This will lead to a contradiction because these two segments must be disjoint by Observation 2. This is because both points S_3 and T_3 belong to D_3 and not to D_4 while both points S_4 and T_4 belong to D_4 and not to D_3.

To see that the line segments $[S_3T_3]$ and $[S_4T_4]$ must cross, we observe that the segment $[S_3T_3]$ intersects the interior of the region $\mathbb{R}^2 \setminus D_1 \cup D_2$ at a chord I_3 connecting a point on the arc $c_1 \cap D_3$ (this point could be S_3, but not necessarily) with a point on the arc $c_2 \cap D_3$. Similarly, $[S_4T_4]$ intersects the interior of $\mathbb{R}^2 \setminus D_1 \cup D_2$ at a chord I_4 connecting a point on the arc $c_1 \cap D_4$ with a point on the arc $c_2 \cap D_4$.

It now follows that the two endpoints of the chord I_4 separate the two endpoints of the chord I_3 on the simple closed curve Γ, that is also the boundary of $\mathbb{R}^2 \setminus D_1 \cup D_2$ (see Figure 6). Because both I_3 and I_4 are contained in $\mathbb{R}^2 \setminus D_1 \cup D_2$ whose boundary is the simple closed curve Γ, it follows that I_3 and I_4 must cross inside the region $\mathbb{R}^2 \setminus D_1 \cup D_2$. Consequently, $[S_3T_3]$ and $[S_4T_4]$ cross, which is the desired contradiction. ∎

The third and last lemma that we will need for the proof of Theorem 1 is new. Similar to Lemma 1 and Lemma 2, this lemma is concerned too with the geometric graph representing the lunes and lenses in arrangements of pairwise intersecting circles and it is concerned with the mutual relations between lunes and lenses in such arrangements.

Lemma 3. *Let $\mathcal{F}$ be a family of pairwise intersecting circles in the plane. Define a geometric graph G on the set of centers of the circles in $\mathcal{F}$. We connect two vertices (centers) in G with a blue edge if the corresponding circles create a lune in $\mathcal{A}(\mathcal{F})$. We connect two vertices in G with a red edge if the corresponding circles create a lens in $\mathcal{A}(\mathcal{F})$. Then G does not contain a red edge e and a blue edge f such that e and f are disjoint and the line through e intersects f (see Figure 7).*

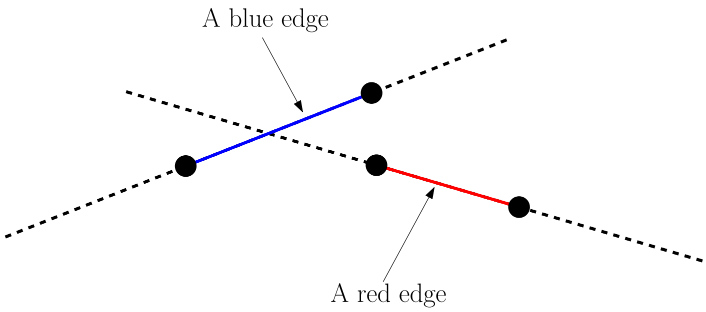

Figure 7: The forbidden position of red and blue edges.

Proof. Under the contrary assumption, there are four circles c_1, c_2, c_3, and c_4 in $\mathcal{F}$ with centers O_1, O_2, O_3, and O_4, respectively, such that c_1 and c_2 create a lune in $\mathcal{A}(\mathcal{F})$ while c_3 and c_4 create a lens in $\mathcal{A}(\mathcal{F})$. Moreover, the line segments $[O_1O_2]$ and $[O_3O_4]$ are disjoint and the line O_3O_4 intersects the line segment $[O_1O_2]$.

Without loss of generality we assume that the lune created by c_1 and c_2 is equal to the disc bounded by c_1 minus the disc bounded by c_2. We can also assume without loss of generality that O_1O_2 is horizontal and O_1 lies to the left of O_2. We may also assume that both O_3 and O_4 lie above the line O_1O_2 and O_3 is closer than O_4 to the line O_1O_2. By our assumptions, the point O_4 must belong to the angle opposite to $\angle O_1O_3O_2$ (see Figure 8).

Let D_3 and D_4 denote the circular discs bounded by c_3 and c_4, respectively. Let S_3 and S_4 denote the centers of the arcs $c_1 \cap D_3$ and $c_1 \cap D_4$, respectively, on the circle c_1. We observe that the two arcs must be disjoint, or else the lens $D_3 \cap D_4$ would have contained a point of c_1. Similarly, let T_3 and T_4 denote the centers of the arcs $c_2 \cap D_3$ and $c_2 \cap D_4$, respectively, on the circle c_2 and observe that the two arcs must be disjoint.

By observation 1, S_3 is the point of intersection of c_1 with $\overrightarrow{O_2O_3}$. Similarly, S_4 is the point of intersection of c_1 with $\overrightarrow{O_2O_4}$. Because O_4 belongs to the angle opposite to $\angle O_1O_3O_2$, it must be that S_3 lies to the right of S_4 on c_1 above the line O_1O_2.

We observe now that D_3 cannot contain the lune created by c_1 and c_2. Indeed, $c_1 \cap D_3$ is an arc whose center S_3 lies above the line O_1O_2. Therefore, if D_3 contains the lune created by c_1 and c_2, then $c_1 \cap D_3$ must contain all the part of c_1 that is above the line O_1O_2 and to the left of S_3. In particular it contains the arc $c_1 \cap D_4$ and its center S_4. Then S_4 is contained in the interiors of both D_3 and D_4 and therefore it is contained in the interior of the lens $D_3 \cap D_4$. This is impossible because S_4 is a point on c_1 that must be disjoint from the lens $D_3 \cap D_4$.

We now argue similarly about T_3 and T_4. By Observation 1, T_3 is the point of intersection of c_2 with $\overrightarrow{O_2O_3}$. Similarly, T_4 is the point of intersection of c_2 with $\overrightarrow{O_2O_4}$. Because O_4 belongs to the angle opposite to $\angle O_1O_3O_2$, it must be that T_4 lies to the right of T_3 on c_2 above the line O_1O_2. We observe now that D_4 cannot contain the lune created by c_1 and c_2. This is because otherwise D_4 contains the point T_3 on c_2. Consequently T_3 is contained in the interiors of both D_4 and D_3. This is impossible because T_3 is a point on c_2 that is disjoint from the lens $D_3 \cap D_4$.

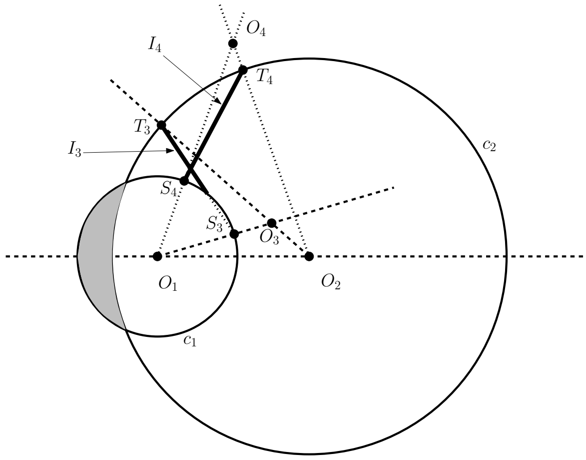

Figure 8: Lemma 3.

Because D_3 and D_4 do not contain the lune $D_1 \setminus D_2$, they must be disjoint from it, as a lune $D_1 \setminus D_2$ cannot be intersected by any of the circles c_1 and c_2. Hence all four arcs $c_1 \cap D_3, c_1 \cap D_4, c_2 \cap D_3$, and $c_2 \cap D_4$ lie on the boundary of $D_2 \setminus D_1$. This boundary is a simple closed curve that we denote by Γ.

It follows now that the pairwise disjoint arcs $c_2 \cap D_4, c_2 \cap D_3, c_1 \cap D_4$, and $c_1 \cap D_3$ lie in this counterclockwise cyclic order on the closed curve Γ. Notice in particular that the arcs $c_2 \cap D_3$ and $c_1 \cap D_3$ separate the arcs $c_2 \cap D_4$ and $c_1 \cap D_4$ on the simple closed curve Γ.

We claim that the line segments $[S_3T_3]$ and $[S_4T_4]$ must cross. This will lead to a contradiction because these two segments must be disjoint by Observation 2. This is because both points S_3 and T_3 belong to D_3 and not to D_4 while both points S_4 and T_4 belong to D_4 and not to D_3.

To see that the line segments $[S_3T_3]$ and $[S_4T_4]$ must cross, we observe that the segment $[S_3T_3]$ intersects the interior of the simply connected region $D_2 \setminus D_1$ at a chord I_3 connecting a point on the arc $c_1 \cap D_3$ (this point could be S_3, but not necessarily) with the point T_3 on the arc $c_2 \cap D_3$.

Similarly, $[S_4T_4]$ intersects the interior of the region $D_2 \setminus D_1$ at a chord I_4 connecting a point on the arc $c_1 \cap D_4$ with the point T_4 on the arc $c_2 \cap D_4$. In particular, the two endpoints of the chord I_4 separate the two endpoints of the chord I_3 on the simple closed curve Γ. Because both I_3 and I_4 are contained in simply connected region $D_2 \setminus D_1$ whose boundary is Γ, then I_3 and I_4 must cross inside the region $D_2 \setminus D_1$ (see Figure 8). Consequently, $[S_3T_3]$ and $[S_4T_4]$ cross, which is the desired contradiction. ∎

The proof of Lemma 3 resembles a lot the proof of Lemma 2 that we brought above. We expect that one can state and prove a unified version of all three lemmata presented in this section for arrangements of pairwise intersecting circles on the sphere.

3 Proof of Theorem 1

We consider the geometric graph G as in the statement of Lemma 3. That is, the vertices of G are the centers of the circles in $\mathcal{F}$. Two centers are connected with a red edge if the corresponding circles create a lens. Two centers are connected with a blue edge if the corresponding circles create a lune.

We may assume without loss of generality that every circle in $\mathcal{F}$ supports some digon, which is either a lens or a lune. For a circle c in $\mathcal{F}$ we say that it is *internal* if it supports a digon that is surrounded by c. We say that c is *external* if it supports a digon (necessarily a lune) that is not surrounded by c.

The following observation is simple and yet important for the proof.

Observation 3. *A circle in $\mathcal{F}$ cannot be both internal and external.*

Proof. Assume to the contrary that $c \in \mathcal{F}$ is both external and internal. Then there is a circle $c_1 \in \mathcal{F}$ such that c_1 and c create a digon (must be a lune) that is not surrounded by c. Similarly, there is a circle $c_2 \in \mathcal{F}$ such that c and c_2 create a digon (could be a lune or a lens) that is surrounded by c.

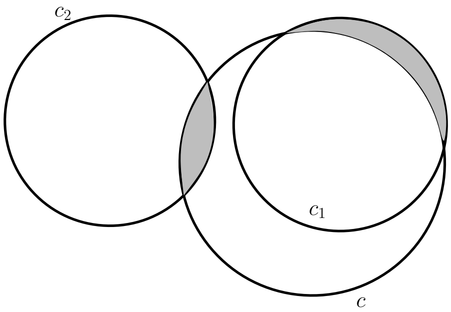

Figure 9: A circle cannot be both internal and external.

We show that c_1 and c_2 cannot cross and thus obtain a contradiction because $\mathcal{F}$ is a family of pairwise intersecting circles (see Figure 9). Indeed, c_1 and c_2 cannot cross inside the disc bounded by c because the part of c_2 there is an edge of a face (digon) in $\mathcal{A}(\mathcal{F})$. In very much the same way c_1 and c_2 cannot cross in the region not bounded by c because the part of c_1 there is an edge of a face (in fact a lune) in $\mathcal{A}(\mathcal{F})$. ∎

As a consequence of Observation 3, we see that the blue edges connect centers of internal circles to centers of external circles, thus forming a bipartite graph. The red edges in G connect centers of pairs of internal circles.

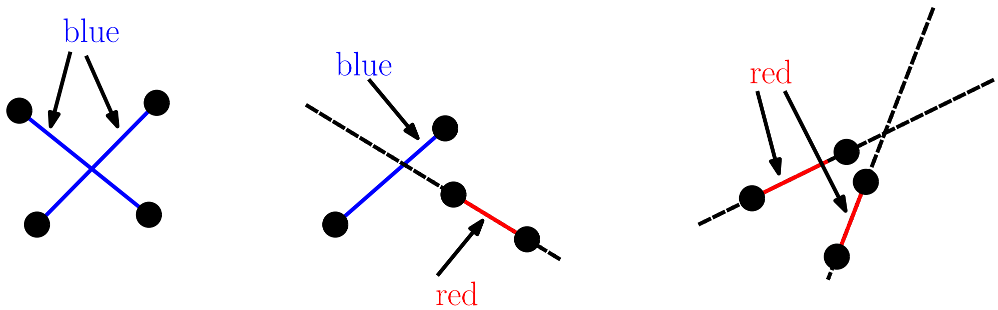

Figure 10: The forbidden pairs of edges in G.

By Lemma 1, no two blue edges cross each other. By Lemma 2, no two red edges are avoiding. By Lemma 3, it is not possible that the complement of a red edge on the line containing it crosses a blue edge (see Figure 10).

With an aid of a nice trick we will move from the graph G to another graph G' that has twice as many edges as G and twice as many vertices. The graph G' will be bipartite and planar. This will show that $2E \leqslant 2(2n) - 4$, where E is the number of edges in G. This implies $E \leqslant 2n - 2$ as desired.

To construct G' we place a sphere S touching the plane from above, thinking of the plane as horizontal and the center of S as the origin O. Then we use a central projection from the center of S and project the plane including the drawing of G to the southern hemishpere of S (see Figure 11).

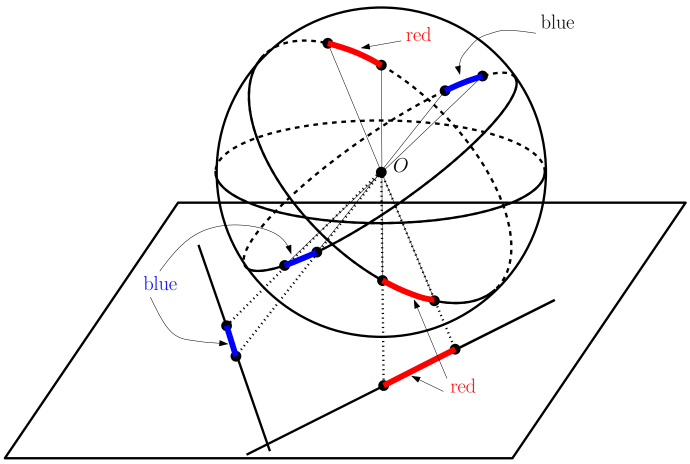

Figure 11: Projecting G on the southern hemisphere and reflecting on the northern hemisphere.

We get a drawing of G on the southern hemisphere of S where the edges are great arcs of S. Next we duplicate the drawing on the northern hemisphere by reflecting the southern hemisphere through the center of S. That is, we take the drawing of G on the southern hemisphere of S and also take the minus of this drawing with respect to the origin O that is also the center of S (see Figure 11).

We thus get two drawings of the graph G on S. One on the southern hemisphere and one on the northern hemisphere (its minus).

Next we perform the following change in the drawing to get the graph G' drawn on S. For every red edge e in the southern hemisphere we replace e and its reflection $-e$ by the complementary arcs on the great circle on S containing them (see Figure 12).

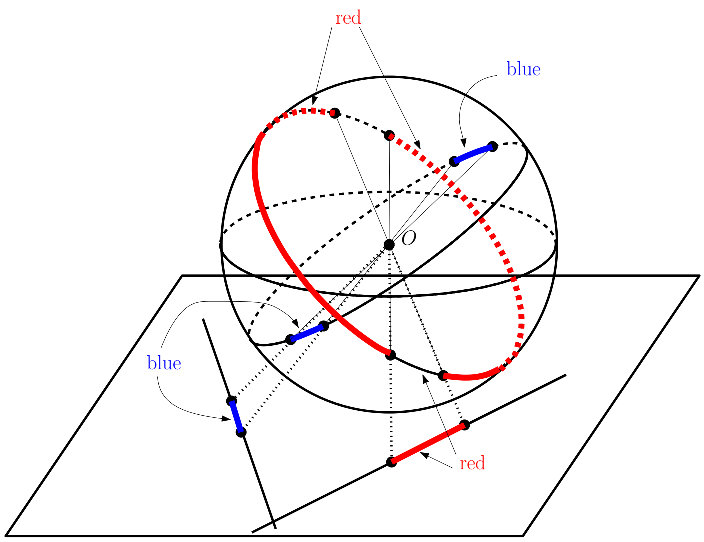

Figure 12: Replacing the red edges by their complements on the great circle.

Therefore, G' is a graph on $2n$ vertices drawn on S with twice as many red and blue edges as in the graph G.

The nice and crucial observation is that no two edges of G' may cross. This is directly follows from the three lemmata Lemma 1, Lemma 2, and Lemma 3. Moreover, the graph G'

is easily seen to be bipartite. Indeed, denote by A the set of external vertices on the southern hemisphere of S and denote by B the set of internal vertices on the southern hemisphere of S. Then the blue edges in G' run between A and B and between $-A$ and $-B$. The red edges in G' run between vertices in B and vertices in $-B$. Therefore, G' is bipartite with the two parts being $A \cup -B$ and $B \cup -A$. This completes the proof of Theorem 1. ∎

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