The Ultimate Signs of Second-Order Holonomic Sequences

Fugen Hagihara ✉
Graduate School of Science, Kyoto University, Japan

Akitoshi Kawamura ✉
Research Institute for Mathematical Sciences, Kyoto University, Japan

Abstract

A real-valued sequence $f = \{f(n)\}_{n\in\mathbb{N}}$ is said to be second-order holonomic if it satisfies a linear recurrence $f(n+2) = P(n)f(n+1) + Q(n)f(n)$ for all sufficiently large n, where $P, Q \in \mathbb{R}(x)$ are rational functions. We study the ultimate sign of such a sequence, i.e., the repeated pattern that the signs of $f(n)$ follow for sufficiently large n. For each P, Q we determine all the ultimate signs that f can have, and show how they partition the space of initial values of f. This completes the prior work by Neumann, Ouaknine and Worrell, who have settled some restricted cases. As a corollary, it follows that when P, Q have rational coefficients, f either has an ultimate sign of length 1, 2, 3, 4, 6, 8 or 12, or never falls into a repeated sign pattern. We also give a partial algorithm that finds the ultimate sign of f (or tells that there is none) in almost all cases.

2012 ACM Subject Classification Mathematics of computing → Discrete mathematics

Keywords and phrases Holonomic sequences, ultimate signs, Skolem Problem, Positivity Problem

Funding This work was supported by JSPS KAKENHI Grant Numbers JP18H03203, JP23K28036, JP25KJ1559, ISHIZUE 2025 of Kyoto University, and JST SPRING Grant Number JPMJSP2110.

Acknowledgements We thank Kohki Baku at Faculty of Science, Kyoto University, for helping us find the explicit formula (7) in Example 5. We also thank the anonymous referees for knowledgeable comments, which helped us clarify the explanation on previous work and our results.

This is a full version of the same-name paper accepted to ICALP 2025.

1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of all natural numbers. A sequence $f = \{f(n)\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ of real numbers is called a *holonomic sequence* (of order $r \in \mathbb{N}$) if there are real-coefficient rational functions $P_0, \dots, P_{r-1} \in \mathbb{R}(x)$ such that f satisfies the linear recurrence

$$f(n+r) = P_{r-1}(n)f(n+r-1) + \cdots + P_0(n)f(n) \tag{1}$$

for all sufficiently large $n \in \mathbb{N}$. Holonomic sequences arise in various areas of mathematics. For instance, solutions of linear differential equations with polynomial coefficients are generating functions of holonomic sequences [25] (see also [4, Appendix B.4]), and for a "proper hypergeometric term" $F(n,k)$, which involves binomial coefficients $\binom{n}{k}$, the sum $f(n) = \sum_{k\in\mathbb{Z}} F(n,k)$ is holonomic if it converges for all $n \in \mathbb{N}$ [21].

An important computational problem concerning holonomic sequences is the *Ultimate Sign Problem* [16]: Given (rational-coefficient) rational functions $P_0, \dots, P_{r-1} \in \mathbb{Q}(x)$ without poles in $\mathbb{N}$ and (rational-valued) initial values $f(0), \dots, f(r-1) \in \mathbb{Q}$, find an ultimate sign, defined as follows, of the unique sequence f having these initial values and satisfying (1) for all $n \in \mathbb{N}$, and an index $N \in \mathbb{N}$ at which this ultimate sign is reached. Although we assume that f satisfies the recurrence (1) not only for $n \geq I$ for some $I \in \mathbb{N}$ but also for all n, it is not different in computability from the problem of finding the ultimate sign and the index N from the coefficients $P_0, \dots, P_{r-1}$, initial values $f(I), \dots, f(I+r-1)$ and I.

▶ **Definition 1.** *A sequence* $f \in \mathbb{R}^{\mathbb{N}}$ *is said to have an* ultimate sign $(s_0, \dots, s_{\tau-1}) \in \{+, -, 0\}^*$ *at* $N \in \mathbb{N}$ *if* $\operatorname{sgn} f(n) = s_{n \bmod \tau}$ *for all* $n \geq N$*, where* $\operatorname{sgn}\colon \mathbb{R} \to \{+, -, 0\}$ *is the function that maps each real number to its sign.*

For instance, the sequence $\{(-1)^n(n-2)\}_{n\in\mathbb{N}} = -2, 1, 0, -1, 2, -3, \dots$ has the ultimate sign $(+, -)$ at 3. Note that if f has the ultimate sign s at N, then it also has any repetition of s as an ultimate sign, and it does so at any index $\geq N$; but we could of course ask for the *shortest* ultimate sign s and the *least* index N without changing the computability of the problem.

The Ultimate Sign Problem is a generalization of several important problems about signs of holonomic sequences. One of the most famous problems is the *Skolem Problem*, which asks whether $f(n) = 0$ for some n (see [19, § 4] for an argument that it reduces to the Ultimate Sign Problem). Its decidability has been studied for almost 90 years [7]. The *Positivity Problem* asking whether $f(n) > 0$ for all n and the *Ultimate Positivity Problem* asking whether f has the ultimate sign $(+)$ are also well studied, with applications to automated inequality proving [6]; see also subsequent works [9, 22, 23] and a SageMath implementation [18].

When the coefficients $P_0, \dots, P_{r-1}$ are constant, f is called a C-finite sequence (or a linear recurrence sequence). The Skolem Problem for C-finite sequences of order $r \leq 4$ [27, 28] and the (Ultimate) Positivity Problem for C-finite sequences of order $r \leq 5$ [20] are known to be decidable, whereas the decidability for higher order C-finite sequences is open.

For holonomic sequences, when $r = 1$ (i.e., when f is a hypergeometric sequence), the Ultimate Sign Problem is easy since for given $P_0 \in \mathbb{Q}(x)$, we can effectively compute an index $N \in \mathbb{N}$ such that $P_0(n)$ has a constant sign for $n \geq N$. When $r = 2$, i.e., when f satisfies a recurrence of the form

$$f(n+2) = P(n)f(n+1) + Q(n)f(n), \tag{2}$$

the decidability of Skolem and (Ultimate) Positivity Problem for some subclasses is known in the context of the Membership Problem [17] and the Threshold Problem [10], respectively. [16, Theorem 7] shows that the Ultimate Sign Problem for another subclass is computable. However, the computability for general second-order holonomic sequences remains unknown. To make progress on this open problem, we study the ultimate signs of all second-order holonomic sequences.

Our first main contribution is to classify all pairs $(P, Q) \in \mathbb{R}(x)^2$ by the ultimate signs f can have, and show how the ultimate signs partition the space of initial values of f (Theorem 4). This result resolves all remaining cases in [16, Theorem 1], which handles the restricted case where P, Q are polynomials, P is non-constant and $\deg Q \leq \deg P$. In addition, this result implies that when P, Q have rational coefficients, the shortest ultimate sign of f, if it has one, is either of length 1, 2, 3, 4, 6, 8 or 12 (Corollary 6).

Our second contribution is to give a partial algorithm that solves the Ultimate Sign Problem for second-order holonomic sequences and halts on almost all inputs (Theorem 10). This extends a similar result [16, Theorem 3] for the restricted case mentioned above. This result can be also stated as a reduction theorem: for second-order holonomic sequences, the Ultimate Sign Problem Turing-reduces to the Minimality Problem, which asks the minimality of a given f, i.e., whether $f(n)/g(n) \to 0$ for all linearly independent solutions g of the same recurrence. In this sense our result extends [11, Theorem 3.1], which shows that the Positivity Problem Turing-reduces to the Minimality Problem. Note that, unfortunately, the decidability of Minimality Problem is unknown whereas many researchers numerically calculate minimal holonomic sequences and apply them to numerical analysis of some special functions (for example [5, 3]).

As a byproduct of our arguments, we amend some gaps in the proof of [16], slightly modifying its Theorem 7. This will be discussed in Section 2.3.

Related work

A lot of previous works describe their results in terms of continued fractions, which have a strong connection to second-order holonomic sequences. We illustrate the connection between those works and one of our main theorems in Sections 2.1.1 and 2.1.2.

Not only the ultimate signs, but also other periodicities of signs of holonomic (or C-finite) sequences are investigated. Closely related to the Skolem Problem, the periodicity of the zeros of C-finite (and for some holonomic) sequences is well-known as the Skolem-Mahler-Lech theorem [2]. Almagor et al. [1] give some sufficient conditions for C-finite sequences to have an "almost periodic sign", a loose property of sign periodicity.

Kooman [13] studies the asymptotic behaviour of complex solutions of the recurrence (2), where P and Q are not necessarily rational functions. His results helped us see the big picture of our main theorems.

2 Results

The Ultimate Sign Problem asks about the ultimate signs of f that satisfies (2) for all n. Such f is identified by the coefficient pair (P, Q) and the initial value $(f(0), f(1))$.

▶ **Definition 2.** *Let P, $Q \in \mathbb{R}(x)$ be rational functions without poles in $\mathbb{N}$. A sequence $f \in \mathbb{R}^{\mathbb{N}}$ is* (P, Q)-holonomic *if it satisfies* (2)*. The pair $(f(0), f(1)) \in \mathbb{R}^2$ is called the* initial value *of f.*

The Ultimate Sign Problem for $(0, Q)$- or $(P, 0)$-holonomic sequences is easy, so we assume $P \neq 0$ and $Q \neq 0$. By shifting the index by finitely many terms, we may assume that P, Q have no zeros in $\mathbb{N}$. This shifting changes the ultimate sign and the initial value of f in such a simple way that it does not affect the computability of the Ultimate Sign Problem. We adopt this assumption in all the following theorems.

2.1 Ultimate signs

Our first main theorem lists the ultimate signs that (P, Q)-holonomic sequences f can have, and shows how the ultimate signs partition the space of initial values of f for each of the following types (Definition 3) of (P, Q). For $R \in \mathbb{R}(x) \setminus \{0\}$, let $\deg R$ denote $d \in \mathbb{Z}$ satisfying $|R(x)| = \Theta(x^d)$ and call the ultimate sign of $\{R(n)\}_{n \in \mathbb{N}}$ that of R.

▶ **Definition 3.** *We classify $(P, Q) \in (\mathbb{R}(x) \setminus \{0\})^2$ into the following types. Let $d := \deg \frac{Q(x)}{P(x)P(x-1)}$ and (s) $(s \in \{+, -\})$ be the ultimate sign of $\frac{Q(x)}{P(x)P(x-1)}$.*

- *If $s = +$ and $d > 2$, then we say that (P, Q) is of ∞-O* loxodromic type.
- *If $s = +$ and $d \leq 2$, then we say that (P, Q) is of ∞-Ω* loxodromic type.
- *If $s = -$ and $d \leq 0$, then let $\alpha_0, \alpha_1, \alpha_2$ be real numbers satisfying*

$$\frac{Q(x)}{P(x)P(x-1)} = \alpha_0 + \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + O(x^{-3}). \tag{3}$$

- *If* $(\alpha_0, \alpha_1, \alpha_2) \geq (-\frac{1}{4}, 0, -\frac{1}{16})$ *in lexicographic order, then we say that* (P, Q) *is of* hyperbolic type.
- *Otherwise,* $\alpha_0 \leq -\frac{1}{4}$*, so there is a real number* $\theta \in [0, \frac{1}{2})$ *such that* $\alpha_0 = -\frac{1}{4\cos^2\theta\pi}$.

(1) *If* θ *is a positive rational number and* $\alpha_1 = 0$*, then we say that* (P, Q) *is of* θ*-O* elliptic type.

(2) *Otherwise, we treat* (P, Q) *together with the next case.*

- *If* $s = -$ *and* $d = 1, 2$*, or it is the case of* (2) *above, then we say that* (P, Q) *is of* $\mathbb{Q}$-Ω elliptic type.
- *If* $s = -$ *and* $d > 2$*, then we say that* (P, Q) *is of* $\frac{1}{2}$*-O* elliptic type.

This classification consists of the distinctions between *loxodromic type* (∞-O loxodromic type and ∞-Ω loxodromic type), hyperbolic type and *elliptic type* (θ-O elliptic type and $\mathbb{Q}$-Ω elliptic type), and between *O type* (∞-O loxodromic type and θ-O elliptic type) and Ω *type* (∞-Ω loxodromic type and $\mathbb{Q}$-Ω elliptic type). The highly non-trivial border between hyperbolic type and elliptic type is well-studied in the context of the convergence of continued fractions (Theorem 9).

The terminologies of "O" and "Ω" come from big O and Ω notations. They represent whether $\frac{Q(x)}{P(x)P(x-1)}$ is near or apart from a certain value (∞ for loxodromic type, $-\frac{1}{4\cos^2\theta\pi}$ for θ-O elliptic type and $-\frac{1}{4\cos^2 q\pi}$ for all $q \in (0, \frac{1}{2}] \cap \mathbb{Q}$ for $\mathbb{Q}$-Ω elliptic type).

The terminologies of loxodromic, hyperbolic and elliptic come from the classification of linear fractional transformations. If P and Q are constant, the linear fractional transformation $z \mapsto \frac{1}{P+Qz}$ maps the ratio $f(n)/f(n+1)$ between the two neighbouring terms of the (P, Q)-holonomic sequence to the next ratio $f(n+1)/f(n+2)$, and is said to be elliptic, parabolic, hyperbolic and loxodromic when $\frac{Q}{P^2}$ is in $(-\infty, -\frac{1}{4})$, $\{-\frac{1}{4}\}$, $(-\frac{1}{4}, 0)$ and $(0, \infty)$, respectively (with slight variations among authors – some (cf. [14, §4.1.3]) treat hyperbolic as a subclass of loxodromic, while some (cf. [24, § 4.7]) treat loxodromic as a subclass of hyperbolic).

This classification is a little complicated, but considering the case of constant P, Q, they are reasonable that the boundary between hyperbolic type and elliptic type is approximately at $-\frac{1}{4}$ and that θ-O elliptic type and $\mathbb{Q}$-Ω elliptic type are distinguished in such a way. If P and Q are constant, we can explicitly solve the recurrence (2) for f:

$$f(n) = \begin{cases} \dfrac{\alpha^n}{\alpha - \beta}\big(f(1) - \beta f(0)\big) + \dfrac{\beta^n}{\alpha - \beta}\big(-f(1) + \alpha f(0)\big) & \text{if } \alpha \neq \beta, \\ n\alpha^n\big(\alpha^{-1} f(1) - f(0)\big) + \alpha^n f(0) & \text{if } \alpha = \beta, \end{cases} \tag{4}$$

where α and β are the roots of the quadratic polynomial $x^2 - Px - Q$. When $\frac{Q}{P^2} \geq -\frac{1}{4}$, we have $\alpha, \beta \in \mathbb{R}$ and f has an ultimate sign of length 1 or 2. On the other hand, when $\frac{Q}{P^2} < -\frac{1}{4}$, the roots α and β are conjugate imaginary numbers. Then we can rewrite the formula (4) into $f(n) = Ar^n \sin(n\theta\pi + B)$, where $A, B, r \in \mathbb{R}$ are constants independent of n and $\theta \in (0, \frac{1}{2})$ is a constant satisfying $\frac{Q}{P^2} = -\frac{1}{4\cos^2\theta\pi}$. f has an ultimate sign of length τ for τ (≥ 4) such that $\tau\theta \in 2\mathbb{Z}$ if $\theta \in \mathbb{Q}$, whereas f has no ultimate signs if $\theta \notin \mathbb{Q}$. Our first main result (Theorem 4) is an extension of this fact, although we do not have explicit formulas like (4) for non-constant P, Q.

Since the set $I_{P,Q}(s)$ of initial values $(f(0), f(1))$ leading f to the ultimate sign s is closed under linear combinations with positive coefficients, it is a convex linear cone and thus specified by an (open, closed or half-open) interval $p(I_{P,Q}(s))$ on the unit circle S^1, where

$$p\colon \mathbb{R}^2 \setminus \{(0,0)\} \to S^1;\ (x, y) \mapsto (x, y)/\sqrt{x^2 + y^2} \tag{5}$$

is the projection. Thus, we will state the theorem by describing how S^1 is partitioned into intervals $p(I_{P,Q}(s))$. It is also obvious that flipping the sign of the initial value flips each element of the ultimate sign, so that $I_{P,Q}(-s)$ is just $I_{P,Q}(s)$ flipped around the origin.

We omit the parentheses and write $I_{P,Q}(+,-)$, say, for $I_{P,Q}((+,-))$.

Rather than considering all $P, Q \in \mathbb{R}(x)$, we state the theorem assuming the ultimate sign of P is $(+)$ because otherwise the ultimate sign of f can be obtained easily from that of the $(-P, Q)$-holonomic sequence $\{(-1)^n f(n)\}_{n\in\mathbb{N}}$ with initial value $(f(0), -f(1))$.

▶ **Theorem 4.** *Let P, $Q \in \mathbb{R}(x)$ be rational functions without zeros or poles in $\mathbb{N}$, and suppose that the ultimate sign of P is $(+)$. For each $s \in \{+,-,0\}^*$, we write $p(I_{P,Q}(s))$ for the set of $f_0 \in S^1$ such that the (P,Q)-holonomic sequence with initial value f_0 has the ultimate sign s.*

(I) *If (P,Q) is of ∞-O loxodromic type, S^1 is partitioned into closed intervals $p(I_{P,Q}(+,-))$, $p(I_{P,Q}(-,+))$ which have non-empty interiors and non-empty open intervals $p(I_{P,Q}(+))$, $p(I_{P,Q}(-))$.*

(II) *If (P,Q) is of ∞-Ω loxodromic type, S^1 is partitioned into singletons $p(I_{P,Q}(+,-))$, $p(I_{P,Q}(-,+))$ and non-empty open intervals $p(I_{P,Q}(+))$, $p(I_{P,Q}(-))$.*

(III) *If (P,Q) is of hyperbolic type, S^1 is partitioned into half-open intervals $p(I_{P,Q}(+))$, $p(I_{P,Q}(-))$.*

(IV) *If (P,Q) is of $\frac{k}{r}$-O elliptic type, where r and k are coprime positive integers, let*

$$s_j = \left(\operatorname{sgn}\sin\frac{j-ik+0.5}{r}\pi\right)_{i=0,\dots,2r-1}, \qquad t_j = \left(\operatorname{sgn}\sin\frac{j-ik}{r}\pi\right)_{i=0,\dots,2r-1}$$

for each $j = 0, \dots, 2r-1$.

- *If $\frac{Q(x)}{P(x)P(x-1)}$ is constant, S^1 is partitioned into $p(I_{P,Q}(t_0))$, $p(I_{P,Q}(s_0)), \dots$, $p(I_{P,Q}(t_{2r-1}))$, $p(I_{P,Q}(s_{2r-1}))$, arranged in this order (clockwise or anticlockwise), of which $p(I_{P,Q}(t_j))$ are singletons and $p(I_{P,Q}(s_j))$ are non-empty open intervals.*
- *Otherwise, S^1 is partitioned into non-empty half-open intervals $p(I_{P,Q}(s_0)), \dots$, $p(I_{P,Q}(s_{2r-1}))$, arranged in this order, where for each $j = 0, \dots, 2r-1$, the intersection of the closures of $p(I_{P,Q}(s_j))$ and $p(I_{P,Q}(s_{j+1}))$ (where $s_{2r} = s_0$) belongs to $p(I_{P,Q}(s_{j+1}))$ if $\frac{Q(x)}{P(x)P(x-1)}$ is eventually increasing (i.e., increasing for sufficiently large x), and to $p(I_{P,Q}(s_j))$ if it is eventually decreasing.*

(V) *If (P,Q) is of $\mathbb{Q}$-Ω elliptic type, then no non-zero (P,Q)-holonomic sequence has an ultimate sign.*

In Part (IV), the value 0.5 can be replaced by any value between 0 and 1. If (P,Q) is of $\frac{1}{2}$-O elliptic type, then $\frac{Q(x)}{P(x)P(x-1)}$ necessarily decreases eventually.

In Parts (I), (II), (III) and (IV), the union of the boundaries of the sets $I(s)$ is a finite union of lines. Following [16], which handles restricted cases of (II) and (III) with $\deg \frac{Q(x)}{P(x)P(x-1)} \leq -1$, we call these lines the *critical lines*.

▶ **Example 5.** Let $P(x) = \frac{x+2}{x+1}$ and $Q(x) = -\frac{x+3}{x+1}$, so that $\frac{Q(x)}{P(x)P(x-1)} = -1 + \frac{2}{x^2+3x+2}$ is decreasing and (P,Q) is $\frac{1}{3}$-O elliptic. Part (IV) of Theorem 4 states that non-zero (P,Q)-holonomic sequences f in this case have ultimate signs

$$\begin{aligned} &s_0 = (+,-,-,-,+,+), && s_1 = (+,+,-,-,-,+), && s_2 = (+,+,+,-,-,-), \\ &s_3 = (-,+,+,+,-,-), && s_4 = (-,-,+,+,+,-), && \text{or } s_5 = (-,-,-,+,+,+), \end{aligned} \tag{6}$$

and that the set $I_{P,Q}(s_j)$ of initial values that result in each ultimate sign s_j is the area between two halves of critical lines and includes the boundary facing $I_{P,Q}(s_{j+1})$ (where we

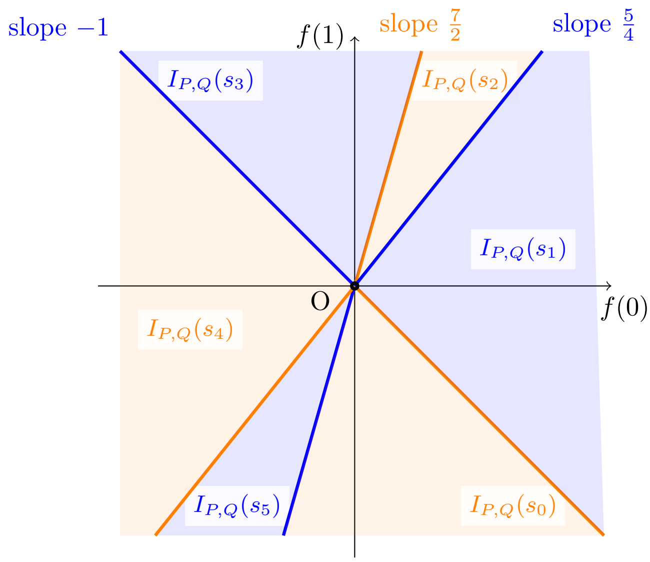

Figure 1 The set of initial values $(f(0), f(1))$ of $\left(\frac{x+2}{x+1}, -\frac{x+3}{x+1}\right)$-holonomic sequences f having each of the ultimate signs in (6).

write $s_6 = s_0$). For this particular example, we can verify this by finding $I_{P,Q}(s_j)$ explicitly, as we see by induction n that the solution of (2) is

$$f(n) = \begin{cases} (-1)^m \left(\left(\frac{7}{2}m+1\right) f(0) - mf(1)\right) & \text{if } n = 3m, \\ (-1)^m \left(mf(0) + (m+1)f(1)\right) & \text{if } n = 3m+1, \\ (-1)^{m+1} \left(\left(\frac{5}{2}m+3\right) f(0) - 2(m+1)f(1)\right) & \text{if } n = 3m+2, \end{cases} \tag{7}$$

so that $I_{P,Q}(s_0)$, ..., $I_{P,Q}(s_5)$ are as depicted in Figure 1.

Note that the solution (7) is a normal form of a hypergeometric sequence in the sense of [26] and can be found algorithmically.

Restricting Theorem 4 to rational-coefficient (P, Q), we obtain the following:

▶ **Corollary 6.** *Suppose that P, $Q \in \mathbb{Q}(x)$ have no zeros or poles in $\mathbb{N}$. Then every (P, Q)-holonomic sequence has an ultimate sign of length 1, 2, 3, 4, 6, 8 or 12, if it has an ultimate sign at all.*

Proof. We may assume that P has the ultimate sign $(+)$, because, as mentioned immediately before Theorem 4, a (P, Q)-holonomic sequence f' for P having $(-)$ can be written as $f' = \{(-1)^n f(n)\}_{n\in\mathbb{N}}$ for a $(-P, Q)$-holonomic sequence f, and hence, if f has an ultimate sign of length τ, then f' has an ultimate sign of length τ (if τ is even) or 2τ (if τ is odd).

Of the five cases in Theorem 4, the only one that does not immediately imply our claim is (IV), namely when (P, Q) is of $\frac{k}{r}$-O elliptic type for some coprime positive integers r and k. If $(r, k) = (2, 1)$, we are done. Otherwise, $-\frac{1}{4\cos^2 \frac{k}{r}\pi} = \lim_{x\to\infty} \frac{Q(x)}{P(x)P(x-1)} \in \mathbb{Q}$. Since $\cos^2 \frac{k}{r}\pi = \frac{1}{2}\left(\cos \frac{2k}{r}\pi + 1\right)$, we have $\cos \frac{2k}{r}\pi \in \mathbb{Q}$, and thus $\cos \frac{2}{r}\pi \in \mathbb{Q}$ since r and k are coprime. The corollary follows from the fact that the only possibilities for such r are 2, 3, 4, 6, since f will then have an ultimate sign of length $2r \in \{4, 6, 8, 12\}$ by (IV). This fact is known as (a version of) Niven's theorem, but we present its proof for the sake of completeness.

If r were a multiple of 8, then $\cos\left(\frac{2}{r}\pi \cdot \frac{r}{8}\right) = \frac{1}{\sqrt{2}}$ would be rational, which is a contradiction. Thus there is $j \in \{0, 1, 2\}$ such that $2^{-j}r$ is odd. Since $\cos \frac{2}{r}\pi$ is rational, so is $\cos \frac{2^{j+1}}{r}\pi$. The *Chebyshev polynomial* $T \in \mathbb{Z}[x]$ of order $2^{-j}r$ is the polynomial such that $T(\cos\theta) = \cos 2^{-j}r\theta$ for any $\theta \in \mathbb{R}$, whose leading coefficient and constant term are known to be a non-negative power of 2 and 0 respectively. It follows from $T\left(\cos \frac{2^{j+1}}{r}\pi\right) - 1 = 0$ that $\left|\cos \frac{2^{j+1}}{r}\pi\right|$ is a non-positive power of 2. One can get $r = 2, 3, 4, 6$ by some calculation using $\frac{1}{2} < \cos \frac{2^{j+1}}{r}\pi$ when r is large. ◀

We can derive from Theorem 4 another corollary (Corollary 20 in Section 3.2). Appropriate subsequences of second-order holonomic sequences are again second-order holonomic sequences. That corollary describes the types of the coefficients of the recurrence which the subsequences satisfy.

2.1.1 Connection to continued fractions

In this section, we discuss the connection between Theorem 4 and convergence theorems of continued fractions

$$\operatorname*{K}_{k=0}^{n} \frac{Q(k)}{P(k)} = \cfrac{Q(0)}{P(0) + \cfrac{Q(1)}{P(1) + \ddots + \cfrac{Q(n)}{P(n)}}}.$$

Note that continued fractions take values in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with $x/\infty = 0$ for $x \in \mathbb{R}$ and $x/0 = \infty$ for $x \in \mathbb{R} \setminus \{0\}$. See [14] about their deep theory and application. Continued fractions are closely related to second-order holonomic sequences through the next proposition, which can be verified by induction on n (simultaneously for all P and Q):

▶ **Proposition 7.** *Let $P, Q \in \mathbb{R}(x)$ have no poles in $\mathbb{N}$ and A and B be the (P, Q)-holonomic sequences with initial values $(1, 0)$ and $(0, 1)$ respectively. Then*

$$\operatorname*{K}_{k=0}^{n} \frac{Q(k)}{P(k)} = \frac{A(n+2)}{B(n+2)} \tag{8}$$

in $\hat{\mathbb{R}}$ for all $n \in \mathbb{N}$.

For this reason, $A(n)$ and $B(n)$ are called the nth canonical numerator and denominator, respectively. We can interpret Theorem 4 to a convergence theorem of subsequences $\{p(A(n), B(n))\}_{n \equiv i \pmod{\tau}}$, $i = 0, \dots, \tau - 1$, of $p(A(n), B(n))$ where p is the projection (5) and $\tau \geq 1$ is a suitable integer below.

Let τ be 2, 1, 1, $2r$ in Theorem 4 (I), (II), (III), (IV), respectively. Then the set $I_i(+)$ of initial values of (P, Q)-holonomic sequence f such that $\{f(n)\}_{n \equiv i \pmod{\tau}}$ has the ultimate sign $(+)$ is a half-plane on $\mathbb{R}^2$. Since f satisfies

$$f(n) = A(n)f(0) + B(n)f(1) = \sqrt{A(n)^2 + B(n)^2}\; p(A(n), B(n)) \cdot (f(0), f(1)), \tag{9}$$

$\{p(A(n), B(n))\}_{n \equiv i \pmod{\tau}}$ converges to the midpoint of the interval $p(I_i(+))$. Similarly, it can be derived that, for any $\tau \geq 1$, one of $\{p(A(n), B(n))\}_{n \equiv i \pmod{\tau}}$ must diverge in the case of Theorem 4 (V). In this sense, Theorem 4 is a convergence theorem of the subsequences of $p(A(n), B(n))$.

By the discussion above, the slopes of the critical lines in Theorem 4 (I), (II), (III), (IV) can be represented by the limits of $\left\{-\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\equiv i \pmod{\tau}} = \left\{-\frac{A(n)}{B(n)}\right\}_{n\equiv i \pmod{\tau}}$, $i = 0, \dots, \tau-1$, and thus the convergence of subsequences of $\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}$ follows.

▶ **Theorem 8.** *Let P, $Q \in \mathbb{R}(x)$ be rational functions without zeros or poles in $\mathbb{N}$. First, in* (I), (II), (III) *and* (IV) *of Theorem 4, the slopes of the critical lines are exactly the accumulation points of the continued fraction $\left\{-\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\in\mathbb{N}}$. Second, the accumulation of the continued fraction is as follows:*

(1) *If (P, Q) is of ∞-O loxodromic type, the subsequences $\left\{\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\equiv i \pmod 2}$, $i = 0, 1$, converge in $\hat{\mathbb{R}}$ to distinct values.*

(2) *If (P, Q) is of ∞-Ω loxodromic or hyperbolic type, the sequence $\left\{\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\in\mathbb{N}}$ converges in $\hat{\mathbb{R}}$.*

(3) *If (P, Q) is of $\frac{k}{r}$-O elliptic type, where r and k are coprime positive integers, the sequences $\left\{\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\equiv i \pmod r}$, $i = 0, \dots, r-1$, converge in $\hat{\mathbb{R}}$ to distinct values.*

(4) *If (P, Q) is of $\mathbb{Q}$-Ω elliptic type, then for no positive integer τ and no $i \in \{0, \dots, \tau-1\}$ does the sequence $\left\{\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\equiv i \pmod \tau}$ converge in $\hat{\mathbb{R}}$.*

We consider the "gap-r subsequences" $\left\{\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\equiv i \pmod r}$ instead of the gap-$2r$ subsequences in (3) because the limit of $\{p(A(n), B(n))\}_{n\equiv i \pmod{2r}}$ is equal to the limit of $\{p(A(n), B(n))\}_{n\equiv i+\tau \pmod{2r}}$ except for multiplication by ± 1.

Part (1) of this theorem is included in [14, Theorems 3.12 and 3.13]. Part (3) is similar to [14, Lemma 4.28]. Part (2) can be derived from the following well-known convergence theorem. Although Parts (1), (2) and (3) follow from Theorem 4, Part (4) does not follow from Theorem 4 alone since it states divergence instead of convergence. We prove (4) in Section 4.1 using the convergence theorem below and Corollary 20 (2) (in Section 3.2), which is derived from Theorem 4.

▶ **Theorem 9** ([12, Theorem 7.1]). *Let P, $Q \in \mathbb{R}(x)$ be rational functions without zeros or poles in $\mathbb{N}$. The continued fraction $\left\{\mathrm{K}_{k=0}^{n}\frac{Q(k)}{P(k)}\right\}_{n\in\mathbb{N}}$ converges in $\hat{\mathbb{R}}$ if and only if (P, Q) is of ∞-Ω loxodromic or hyperbolic type.*

2.1.2 Connection to monotonic convergence of continued fractions

If we identify the ultimate sign of B, we can extend the convergence of subsequences of $\frac{A(n)}{B(n)}$ to that of $p(A(n), B(n))$. But this is not enough to prove each part of Theorem 4; we need monotonic convergence theorems. This is because Theorem 4 even describes the ultimate signs of holonomic sequences with initial values on the critical lines, and therefore figures out not only the convergence of subsequences of $p(A(n), B(n))$, but also the direction in which the subsequences of $p(A(n), B(n))$ converge to their limits.

[14, Theorems 3.12 and 3.13] and [11, Lemma 3.4] are monotonic convergence theorems for (P, Q) of ∞-O, -Ω loxodromic type and of hyperbolic type, respectively, and both literature identify the ultimate sign of B in their cases. Hence Theorem 4 (I) and (II) can be derived from the former literature, and (III) can be derived from the latter.

2.2 Computing the ultimate sign

The partial algorithm in the following theorem tells us, for given $(P, Q) \in \mathbb{Q}(x)^2$ and $f_0 \in \mathbb{Q}^2$, the index $N \in \mathbb{N}$ at which the (P, Q)-holonomic sequence with initial value f_0, whenever it terminates. Note that once we get N, we can obtain the ultimate sign itself by looking at the signs of a finite number of terms $f(N)$, $f(N+1)$, ... according to Theorem 4.

▶ **Theorem 10.** *There exists a partial algorithm that,*

- *given $P, Q \in \mathbb{Q}(x)$ without zeros or poles in $\mathbb{N}$, together with a pair $f_0 \in \mathbb{Q}^2$,*
- *terminates if and only if the (P,Q)-holonomic sequence f with initial value f_0 has an ultimate sign and it is stable in the sense that there is a neighbourhood $\mathcal{N} \subseteq \mathbb{Q}^2$ of f_0 such that all (P,Q)-holonomic sequences with initial value in $\mathcal{N}$ have the same ultimate sign, and*
- *whenever it terminates, outputs an index at which f has its ultimate sign.*

Note that the type of (P,Q) can be computed from P and Q, and hence, although the partial algorithm does not terminate when $f_0 = (0,0)$ or when (P,Q) is $\mathbb{Q}$-Ω elliptic (because of Theorem 4 (V)), we could make it terminate also on these inputs and declare the non-existence of an ultimate sign in the latter case.

This partial algorithm terminates on "most" inputs since, for (P,Q) of ∞-O, -Ω loxodromic, hyperbolic and θ-O type, the (P,Q)-holonomic sequence f with initial value f_0 has an unstable ultimate sign if and only if f_0 is on the finitely many critical lines delimiting the areas $I_{P,Q}(s)$ in Theorem 4. For a small but substantial class of (P,Q), it is known that all $f_0 \in \mathbb{Q}^2 \setminus \{(0,0)\}$ lead f to a stable ultimate sign, or in other words, the slopes of the critical lines are irrational, which is the main topic of Section 2.3. However there is no known general method to determine the stability, and it is a wide-open problem whether we can make the algorithm terminate on all inputs [11, 8, 16].

Theorem 10 is stated for rational-coefficient P, Q and rational-valued f_0, so that the problem is computationally meaningful. By studying the proofs in some detail we could, however, modify the statement appropriately so that the partial algorithm accepts inputs involving real numbers represented as infinite sequences of approximations, in a way analogous to the discussion in [15] about signs of C-finite sequences.

▶ **Example 11.** Let us compare the values of the sums

$$\sum_{0 \le k \le \frac{n+1}{2},\ k \in 2\mathbb{Z}} k\binom{n+1-k}{k} \quad \text{and} \quad \sum_{0 \le k \le \frac{n+1}{2},\ k \in 2\mathbb{Z}+1} k\binom{n+1-k}{k}$$

using the partial algorithm in Theorem 10. It suffices to identify the ultimate sign of the difference $f(n) := \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^k k \binom{n+1-k}{k}$ of the two sums and at which index f has it. By creative telescoping [21, Chapter 6], we find that f is the (P,Q)-holonomic sequence with initial value $(0,-1)$, where (P,Q) is as in Example 5. Now we can input this into our partial algorithm and it tells that f has the ultimate sign $(+,-,-,-,+,+)$ at 1, i.e., when $n \ge 1$, the former sum is greater than and less than the latter sum if $n \equiv 0,4,5 \pmod 6$ and if $n \equiv 1,2,3 \pmod 6$, respectively.

Unlike the above case, for the same (P,Q), the partial algorithm never terminates with initial values on the critical lines in Figure 1, such as $(1,-1)$, $(4,5)$ and $(2,7)$.

Let us consider another example: compare

$$\sum_{0 \le k \le n,\ k \in 2\mathbb{Z}} k\binom{n}{k}^3 \quad \text{and} \quad \sum_{0 \le k \le n,\ k \in 2\mathbb{Z}+1} k\binom{n}{k}^3.$$

Taking a similar process, we find that the difference $g(n) := \sum_{k=0}^{n} (-1)^k k \binom{n}{k}^3$ is the (R,S)-holonomic sequence with initial value $(0,-1)$, where

$$(R(x), S(x)) = \left(\frac{18x^2+36x+12}{(x+1)(x+2)(6x^2+4x+1)}, -\frac{3(3x+2)(3x+1)(6x^2+16x+11)}{(x+1)(x+2)(6x^2+4x+1)} \right).$$

Note that (R, S) is of $\frac{1}{2}$-O elliptic type. Our partial algorithm proves that g has the ultimate sign $(+,-,-,+)$ at 1, i.e., when $n \geq 1$, the former sum is greater than and less than the latter sum if $n \equiv 0, 3 \pmod 4$ and if $n \equiv 1, 2 \pmod 4$, respectively.

However, for this (R, S), we do not know how to identify the critical lines. Algorithm Hyper [21, Chapter 8] declared that there is no explicit formula like (7) (more precisely, "closed form"), so the above discussion for (P, Q) does not work for (R, S). By numerical analysis using our partial algorithm, we find that the slope of the critical line between $I_{R,S}(+,-,-,+)$ and $I_{R,S}(+,+,-,-)$ is in the interval $(-2.452, -2.434)$, and the one between $I_{R,S}(+,+,-,-)$ and $I_{R,S}(-,+,+,-)$ is in $(4.8094, 4.816)$.

Theorem 10 can be described in a reduction form that is an extension of [11, Theorem 3.1]:

▶ **Theorem 12.** *For second-order holonomic sequences, the Ultimate Sign Problem Turing-reduces to the Minimality Problem.*

2.3 Input set admitting a total algorithm

The main predecessor to our work [16, Theorem 1, 3 and 7] relies on [16, Lemma 14] whose proof contained an error in the calculation of an inverse image. Their classification and the partial algorithm [16, Theorem 1 and 3] analogous to our Theorems 4 and 10 are correct after all, as our theorems imply. In this section, we state Theorem 13, an amendment of [16, Theorem 7]. Note that our theorem is slightly weaker than the original one due to one more gap in the proof. We mention this in detail after proving Theorem 13 in Section 4.2.

Theorem 13 gives a sufficient condition on $P, Q \in \mathbb{Q}(x)$ for all non-zero (P, Q)-holonomic sequences $f \in \mathbb{Q}^{\mathbb{N}} \setminus \{0\}$ to have stable ultimate signs. This gives a nontrivial input set on which the Ultimate Sign Problem is solvable by the partial algorithm in Theorem 10.

The restriction of P and Q to $\mathbb{Z}[x]$ instead of $\mathbb{Q}(x)$ is no essential loss of generality: For P, $Q \in \mathbb{Q}(x)$, let $P_1, P_2, Q_1, Q_2 \in \mathbb{Z}[x]$ satisfy $P = \frac{P_1}{P_2}$ and $Q = \frac{Q_1}{Q_2}$. Then we can apply the theorem on P' and Q', where $P'(x) = P_1(x+1)Q_2(x+1)$ and $Q'(x) = Q_1(x+1)Q_2(x)P_2(x+1)P_2(x)$. The ultimate sign of a (P, Q)-holonomic sequence f is stable if and only if that of the (P', Q')-holonomic sequence $\{f(n+1)\prod_{k=0}^{n-1} P_2(k)Q_2(k)\}_{n\in\mathbb{N}}$ is stable.

▶ **Theorem 13.** *Let $P(x) = p_0x^d + p_1x^{d-1} + \cdots + p_d \in \mathbb{Z}[x]$ and $Q(x) = q_0x^d + q_1x^{d-1} + \cdots + q_d \in \mathbb{Z}[x]$ be polynomials without zeros in $\mathbb{N}$. Suppose that $p_0 > 0$ and $d \geq 1$ (where q_0 might be zero). Then, if P and Q satisfy either of the following conditions, any (P, Q)-holonomic sequence $f \in \mathbb{Q}^{\mathbb{N}} \setminus \{0\}$ has a stable ultimate sign.*

(1) $|q_0| < p_0$

(2) $|q_0| = p_0$ *and the two conditions below hold for* $s := \operatorname{sgn} q_0 \in \{1, -1\}$*:*

- $Q(x) - sP(x) \neq 1$ *in* $\mathbb{Z}[x]$,
- $\begin{cases} sq_1 - p_1 - s < p_0 & \text{if } d = 1, \\ sq_1 - p_1 < p_0 & \text{if } d \geq 2. \end{cases}$

3 Proof of the Main Results

In this section, we prove Theorems 4, 10, and 12. All the proofs of the lemmas in the following Sections 3.1 and 3.2 are postponed to Section 3.3.

3.1 Proof of Theorem 4

Let us first focus on identifying the lengths of the ultimate signs that (P, Q)-holonomic sequences can have and get an overview of the proof of Theorem 4. Lemmas 14 and 15 below, by types of (P, Q), characterize (P, Q) admitting (P, Q)-holonomic sequences with ultimate signs of lengths 1 and 2, respectively. Then only lengths $\tau \geq 3$ are left. For each $\tau \geq 3$, we will introduce a special recurrence such that we can decide if $F \in \mathbb{R}^{\mathbb{N}}$ satisfying the recurrence has a (shortest) ultimate sign of length τ (Lemma 16). Next, by types of (P, Q), we characterize (P, Q) and τ that allow all (P, Q)-holonomic sequences f to be transformed to F satisfying the special recurrence and having the same ultimate sign as f (Lemma 18). Finally we show that, for the other (P, Q) and $\tau \geq 3$, no non-zero (P, Q)-holonomic sequences have the shortest ultimate sign of length τ in the proof of Theorem 4 (V). Note that some lemmas below are superfluous for identifying the lengths of ultimate signs, but required to identify the ultimate signs themselves and how they partition the space of the initial values.

▶ **Lemma 14.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and P have the ultimate sign* $(+)$.

(1) $I_{P,Q}(+) \neq \varnothing \iff (P, Q)$ *is of loxodromic type or hyperbolic type.*

(2) *If (P, Q) is of hyperbolic type, then $I_{P,Q}(+) \cup I_{P,Q}(-) = \mathbb{R}^2 \setminus \{(0, 0)\}$.*

Similar results to the above lemma appear in, e.g., [11].

The following lemma is relatively easy and similar propositions appear in context of continued fractions (e.g., [14, Theorem 3.12]).

▶ **Lemma 15.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and P have the ultimate sign* $(+)$.

(1) $I_{P,Q}(+, -) \neq \varnothing \iff (P, Q)$ *is of loxodromic type.*

(2) *$p(I_{P,Q}(+, -))$ is a closed interval.*

(3) *If (P, Q) is of loxodromic type, then $I_{P,Q}(+) \cup I_{P,Q}(-) \cup I_{P,Q}(+, -) \cup I_{P,Q}(-, +) = \mathbb{R}^2 \setminus \{(0, 0)\}$.*

Now we introduce the special recurrence mentioned in the first paragraph of this section. For a (not necessarily holonomic) sequence $F \in \mathbb{R}^{\mathbb{N}}$, consider a *single-term-feedback recurrence*

$$F(n + \tau) - F(n) = R(n)F(n + 1), \tag{10}$$

where τ is an integer ≥ 2 and $R \in \mathbb{R}^{\mathbb{N}}$. This recurrence expresses the difference between two neighbouring terms in the gap-τ subsequences $\{F(n)\}_{n\equiv i \pmod \tau}$, $i = 0, \dots, \tau - 1$, as a single term in the next subsequence $\{F(n)\}_{n\equiv i+1 \pmod \tau}$ multiplied by the coefficient R. In the following lemma, we treat the case where $|R(n)|$ rapidly decreases in (1) and the case where $|R(n)|$ does not rapidly decrease in (2).

▶ **Lemma 16.** *Let $F \in \mathbb{R}^{\mathbb{N}}$ satisfy the single-term-feedback recurrence* (10) *for a coefficient $R \in \mathbb{R}^{\mathbb{N}}$ and an integer $\tau \geq 2$.*

(1) *(restricted case of [12, Theorem 6]) Suppose $R(n) = O(n^{-1-\varepsilon})$ for some $\varepsilon > 0$.*

- **(1a)** *Each of the gap-τ subsequences $\{F(n)\}_{n\equiv i \pmod \tau}$, $i = 0, \dots, \tau - 1$, converges.*
- **(1b)** *If $F \neq 0$, then there is $i \in \{0, \dots, \tau - 1\}$ for which $\{F(n)\}_{n\equiv i \pmod \tau}$ does not converge to* 0.

(2) *Suppose that $|R(n)| = \Omega(n^{-1})$ and R has an ultimate sign* $(+)$ *or* $(-)$. *If F has an ultimate sign of length τ, then F also has an ultimate sign of length ≤ 2.*

(3) *Suppose that R has an ultimate sign (q), $q \in \{+,-,0\}$. Let $i \in \{0,\ldots,\tau-1\}$. If a subsequence $\{F(n)\}_{n\equiv i+1 \pmod{\tau}}$ of F has the ultimate sign (s), $s \in \{+,-,0\}$ and $\{F(n)\}_{n\equiv i \pmod{\tau}}$ converges to 0, then $\{F(n)\}_{n\equiv i \pmod{\tau}}$ has the ultimate sign $(-qs)$.*

In the situation of (1), F has an ultimate sign of length τ as follows. If $F = 0$, it is obvious. If $F \neq 0$, then by (1a) and (1b), there is i such that $\{F(n)\}_{n\equiv i \pmod{\tau}}$ has the ultimate sign $(+)$ or $(-)$. Then $\{F(n)\}_{n\equiv i-1 \pmod{\tau}}$ also has $(+)$ or $(-)$ if it converges to a non-zero real number. It has $(+)$, $(-)$ or (0) even if it converges to zero by (3). Thus, by induction, every gap-τ subsequence of F has ultimate sign of length 1, meaning that F has an ultimate sign of length τ. On the other hand, in the situation of (2), F does not have the shortest ultimate sign of length $\tau \geq 3$.

Part (1) of Lemma 16 is known for a larger class of recurrences [12, Theorem 6]. Our restriction to the single-term-feedback recurrence allows (2) and (3) to hold.

Now we want to find sequences $T, R \in \mathbb{R}^{\mathbb{N}}$ such that for each (P,Q)-holonomic sequence f, the transformed sequence $F(n) := T(n)f(n)$ has the same ultimate sign as f and satisfies the recurrence (10). F and f have the same ultimate sign if and only if T has the ultimate sign $(+)$. To find the condition on T and R for F to satisfy the recurrence (10), we use $A^{(\tau)}, B^{(\tau)} \in \mathbb{R}(x)$ below.

► **Definition 17.** *For $P, Q \in \mathbb{R}(x)$ without zeros or poles in $\mathbb{N}$, there uniquely exist $A^{(\tau)}, B^{(\tau)} \in \mathbb{R}(x)$ such that any (P,Q)-holonomic sequence f satisfies the recurrence*

$$f(n+\tau) = B^{(\tau)}(n)f(n+1) + A^{(\tau)}(n)f(n) \tag{11}$$

for all $n \in \mathbb{N}$. Let us call $A^{(\tau)}$ and $B^{(\tau)}$ the generalized τth canonical numerator *and* denominator *(of (P,Q)) respectively.*

These are generalizations of the notions of τth canonical numerator A and denominator B in Proposition 7 since $(A^{(\tau)}(0), B^{(\tau)}(0)) = (A(\tau), B(\tau))$. We can generalize Equation (8) to $\mathrm{K}_{k=n}^{n+\tau} \frac{Q(k)}{P(k)} = \frac{A^{(\tau+2)}(n)}{B^{(\tau+2)}(n)}$. Equation (11) is a generalization of the equation $f(\tau) = B(\tau)f(1) + A(\tau)f(0)$ that A and B satisfy for any (P,Q)-holonomic sequence f.

Let $\tau \geq 2$ and $T, R \in \mathbb{R}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, by Equation (11), $F(n) = T(n)f(n)$ satisfy Equation (10) for all (P,Q)-holonomic sequences f if and only if

$$T(n+\tau)A^{(\tau)}(n) = T(n), \quad R(n)T(n+1) = B^{(\tau)}(n)T(n+\tau). \tag{12}$$

To allow T to have the ultimate sign $(+)$, we want $A^{(\tau)}$ to have $(+)$. In addition, to apply Lemma 16 (1) for $F(n) = T(n)f(n)$, the absolute value of the coefficient $|R(n)|$ has to decrease rapidly. The next lemma shows that there exists τ satisfying these conditions if and only if (P,Q) is of O type.

► **Lemma 18.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$, and P have the ultimate sign $(+)$. Let $\tau \geq 2$ be an integer and $A^{(\tau)}$ and $B^{(\tau)}$ be the τth generalized canonical numerator and denominator, respectively.*

(1) *Assume that $T, R \in \mathbb{R}^{\mathbb{N}}$ satisfy (12) and $T(n) \neq 0$ for all sufficiently large n. Then $\left|\frac{T(n+1)}{T(n)}\right| = \Theta(|A^{(\tau)}(n)|^{-1/\tau})$. Especially, $|R(n)| = \Theta\left(|B^{(\tau)}(n)||A^{(\tau)}(n)|^{-1+1/\tau}\right)$.*

(2) *The following are equivalent.*

- **(2a)** *$A^{(\tau)}$ has the ultimate sign $(+)$ and $|B^{(\tau)}(n)|A^{(\tau)}(n)^{-1+1/\tau} = O(n^{-1-\varepsilon})$ for some $\varepsilon > 0$.*
- **(2b)** *(P,Q) is of θ-O elliptic type and $\tau\theta \in 2\mathbb{Z}$, or (P,Q) is of ∞-O loxodromic type and $\tau \in 2\mathbb{Z}$.*

Now we are ready to show Theorem 4.

Proof of Theorem 4 (I) **and** (II)**.** By Lemma 15 (3), it remains to prove that $p(I_{P,Q}(+,-))$ has width if (P,Q) is of ∞-O loxodromic type and does not if (P,Q) is of ∞-Ω loxodromic type. In other words, we should prove the existence of a (P,Q)-holonomic sequence with the stable ultimate sign $(+,-)$ in the former case and the non-existence in the latter case.

Define $T, R \in \mathbb{R}^{\mathbb{N}}$ as they satisfy $T(n), R(n) > 0$ and the relation (12) for $\tau = 2$ for all sufficiently large n. (Note that $A^{(2)} = Q$ and $B^{(2)} = P$.) Then, for all (P,Q)-holonomic sequences f and all sufficiently large n, the transformed sequences $F(n) := T(n)f(n)$ satisfy the single-term-feedback recurrence (10) for $\tau = 2$, i.e.,

$$F(n+2) - F(n) = R(n)F(n+1). \tag{13}$$

Since $A^{(2)}(n) = Q(n) > 0$ for all sufficiently large n, Lemma 18 implies $R(n) = O(n^{-1-\varepsilon})$ for some $\varepsilon > 0$ if (P,Q) is of ∞-O loxodromic type and $R(n) = \Omega(n^{-1})$ if (P,Q) is of ∞-Ω loxodromic type.

If (P,Q) is of ∞-O loxodromic type and so $R(n) = O(n^{-1-\varepsilon})$, we can define a linear map L that maps a (P,Q)-holonomic sequence f to

$$L(f) := \left(\lim_{\substack{n\equiv 0 \pmod 2,\\ n\to\infty}} T(n)f(n), \quad \lim_{\substack{n\equiv 1 \pmod 2,\\ n\to\infty}} T(n)f(n) \right) \in \mathbb{R}^2$$

by Lemma 16 (1a). By Lemma 16 (1b), L is injective. Since the domain and range of L are both two-dimensional, L is bijective. Hence, for example, $L^{-1}(1,-1)$ is a (P,Q)-holonomic sequence that has the stable ultimate sign $(+,-)$.

If (P,Q) is of ∞-Ω loxodromic type, take a (P,Q)-holonomic sequence f with the ultimate sign $(+,-)$. Let us show that this is unstable. It suffices to show that $T(n)f(n) = O(1)$ and $\lim_{n\to\infty} T(n)g(n) = \infty$ where g is a (P,Q)-holonomic sequence with the ultimate sign $(+)$ (because it follows that, for any $\delta > 0$, the perturbations $f + \delta g$ of f have the ultimate sign $(+)$). For all sufficiently large n, since $R(n) > 0$ and $F(n) = T(n)f(n)$ satisfies Equation (13), $F(2n)$ (> 0) is monotonically decreasing and $F(2n+1)$ (< 0) is monotonically increasing. So $F(n) = O(1)$. On the other hand, $F'(n) := T(n)g(n)$ (> 0), a sequence satisfying the same recurrence, eventually increasing. Especially $F'(n) = \Omega(1)$. Since $R(n) = \Omega(n^{-1})$, we have $F'(n+2) - F'(n) = \Omega(n^{-1})$. Thus $\lim_{n\to\infty} F'(n) = \infty$. ◀

Proof of Theorem 4 (III)**.** $I_{P,Q}(+)$, $I_{P,Q}(-)$ are both connected and $I_{P,Q}(+) = -I_{P,Q}(-)$. The statement follows from this and Lemma 14 (2). ◀

Proof of Theorem 4 (V)**.** Suppose, for a contradiction, that a non-zero (P,Q)-holonomic sequence f has an ultimate sign $(s_0, \ldots, s_{\tau-1})$.

Let $\tau \geq 3$ first. Let $A^{(\tau)}$ and $B^{(\tau)}$ be the generalized τth canonical numerator and denominator. It follows from Lemma 18 (2) that $A^{(\tau)}$ has the ultimate sign $(-)$ or (0), or that $A^{(\tau)}$ has $(+)$ and $|B^{(\tau)}(n)|A^{(\tau)}(n)^{-1+1/\tau} = \Omega(n^{-1})$. Let us first consider the former case. Let (b) $(b \in \{+,-,0\})$ be the ultimate sign of $B^{(\tau)}$. Comparing the signs of the three terms in Equation (11), we have $s_i = bs_{i+1}$ for all $i = 0, \ldots, \tau-1$, where $s_\tau := s_0$, and so f has an ultimate sign of length ≤ 2. Next, let us consider the latter case. We can choose $T, R \in \mathbb{R}^{\mathbb{N}}$ satisfying $T(n) > 0$ and the relation (12) for all sufficiently large n. Then we have $|R(n)| = \Omega(n^{-1})$. The transformed sequence $F(n) := T(n)f(n)$ satisfies the recurrence (10) for all sufficiently large n. It follows from Lemma 16 (2) that F has an ultimate sign of length ≤ 2, and so does f.

Now it remains to consider the case $\tau = 1, 2$. By Lemma 14 (1) and Lemma 15 (1), f does not have ultimate signs of length 1 or 2. ◀

It remains to show (IV). Let (P, Q) be of θ-O elliptic type. As already mentioned, for τ such that $\tau\theta \in 2\mathbb{Z}$, all (P, Q)-holonomic sequences f have ultimate signs of length τ. Now we need to determine which ultimate signs (of length τ) f can have. This will be derived from the following lemma.

▶ **Lemma 19.** *Take (P, Q) as in Lemma 18 and assume that it is of $\frac{k}{r}$-O elliptic type.*

(1) *The generalized $2r$th canonical denominator $B^{(2r)}$ has the ultimate sign $(+)$, $(-)$ and (0) if $\frac{Q(x)}{P(x)P(x-1)}$ is eventually increasing, if it is eventually decreasing and if it is constant, respectively.*

(2) *By Lemma 18 (2), we can choose $T \in \mathbb{R}^{\mathbb{N}}$ such that $T(n) > 0$ and the relation (12) for $\tau = 2r$ hold for all sufficiently large n. Then, for each $j = 0, \ldots, \tau - 1$, there exists a (P, Q)-holonomic $f^{(j)}$ such that for all $i \in \{0, \ldots, \tau - 1\}$, $\{T(n)f^{(j)}(n)\}_{n\equiv i \pmod{2r}}$ converges to a real number of sign $\operatorname{sgn}\sin\frac{j-ik}{r}\pi$.*

Proof of Theorem 4 (IV). Take T and $f^{(0)}, \ldots, f^{(2r-1)}$ as in Lemma 19 (2). Let $f^{(2r)} := f^{(0)}$. (P, Q)-holonomic sequences of the form $f = af^{(j)} + bf^{(j+1)}$ $(a, b > 0)$ have the ultimate sign s_j since each $\{T(n)f(n)\}_{n\equiv i \pmod{2r}}$, $i = 0, \ldots, 2r - 1$, converges to a real number of sign $\operatorname{sgn}\sin\frac{j-ik+0.5}{r}\pi$. Then we have $\{\text{initial values of } af^{(j)} + bf^{(j+1)} \mid a, b > 0\} \subseteq I_{P,Q}(s_j)$. It remains to prove that $f^{(j)}$ has the ultimate sign s_j, s_{j-1} and t_j if $\frac{Q(x)}{P(x)P(x-1)}$ is eventually increasing, if it is eventually decreasing and if it is constant, respectively.

For $i, j \in \{0, \ldots, 2r - 1\}$ and $q \in \{0, \pm 1\}$, let $u_{i,j,q} := \operatorname{sgn}\sin\frac{j-ik+q/2}{r}\pi$. Then what we want to prove is that $f^{(j)}$ has the ultimate sign $(u_{i,j,q})_{i=0,\ldots,2r-1}$, where $(\operatorname{sgn} q)$ is the ultimate sign of $B^{(2r)}$ in Lemma 19 (1). We will show that the subsequence $\{T(n)f^{(j)}(n)\}_{n\equiv i \pmod{2r}}$ has the ultimate sign $u_{i,j,q}$ for each i.

If $j - ik \not\equiv 0 \pmod r$, then this subsequence converges to a real number of sign $u_{i,j,0}$ $(\neq 0)$. Therefore it has the ultimate sign $(u_{i,j,0}) = (u_{i,j,q})$. If $j - ik \equiv 0 \pmod r$, then this subsequence converges to 0. Define $R \in \mathbb{R}^{\mathbb{N}}$ by the relation (12). R has the ultimate sign $(\operatorname{sgn} q)$ and $F(n) = T(n)f^{(j)}(n)$ satisfies (10). It follows from Lemma 16 (3) that this subsequence has the ultimate sign $(-\operatorname{sgn} q u_{i+1,j,0}) = (\operatorname{sgn} q(-1)^{\frac{j-ik}{r}}) = (u_{i,j,q})$. ◀

3.2 Proof of Theorems 10 and 12

Theorems 10 and 12 are algorithmic claims stating that the ultimate signs can be partially computed in each sense. We could prove them by analyzing the proof of Theorem 4 quantitatively. But instead of carrying out such analysis for each case of Theorem 4 separately, we choose to do so just for the hyperbolic type (Lemma 21 below), and argue that all other types (having ultimate signs) reduce to it in the sense of the following Corollary 20.

From the original recurrence (2), we can obtain, for each positive integer τ, a "gap-τ recurrence"

$$f(n + 2\tau) = P_\tau(n)f(n + \tau) + Q_\tau(n)f(n), \tag{14}$$

where P_τ and Q_τ are rational functions. Specifically, they can be written as

$$P_\tau = \frac{B^{(2\tau)}}{B^{(\tau)}}, \qquad Q_\tau = A^{(2\tau)} - \frac{B^{(2\tau)}}{B^{(\tau)}}A^{(\tau)} \tag{15}$$

using the generalized canonical numerators $A^{(0)}$, $A^{(1)}$, ...and denominators $B^{(0)}$, $B^{(1)}$, ...of (P, Q) (see Definition 17), assuming that $B^{(\tau)}$ is non-zero. (Note that if $B^{(\tau)} =$

0, we have $f(n+\tau) = A^{(\tau)}(n)f(n)$, in which case the ultimate sign of f can be found easily.) Thus, the subsequence $\{f(\tau n+N)\}_{n\in\mathbb{N}}$ of f, for any number $N \in \mathbb{N}$ greater than all zeros of $B^{(\tau)}$, is the $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$-holonomic sequence with initial value $(f(N), f(N+\tau))$. The following corollary to Theorem 4 says that with a right choice of τ, this $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of hyperbolic type, unless (P,Q) is of $\mathbb{Q}$-Ω elliptic type.

▶ **Corollary 20.** *Suppose that P, $Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$. Let $A^{(0)}$, $A^{(1)}$, ... and $B^{(0)}$, $B^{(1)}$, ... be the generalized canonical numerators and denominators, respectively.*

(1) *Suppose that (P,Q) is either of loxodromic type or of $\frac{k}{r}$-O elliptic type for some coprime positive integers r and k. Let $\tau = 2$ in the former case, and $\tau = 2r$ in the latter case. Suppose that $B^{(\tau)}$ and $B^{(2\tau)}$ are non-zero. Then P_τ and Q_τ defined by* (15) *are non-zero, and $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of hyperbolic type for all $N \in \mathbb{N}$.*

(2) *Suppose that (P,Q) is of $\mathbb{Q}$-Ω elliptic type. Then $B^{(\tau)}$ is non-zero, P_τ and Q_τ defined by* (15) *are also non-zero, and $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of $\mathbb{Q}$-Ω elliptic type for all $N \in \mathbb{N}$ and $\tau \geq 1$.*

Proof. (1) $P_\tau, Q_\tau \neq 0$ follows from $B^{(\tau)}, B^{(2\tau)} \neq 0$. Since the type of $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ does not depend on N, it suffices to prove this corollary only for N which is larger than any zero and pole of P_τ, Q_τ, $B^{(\tau)}$. Since $\{(f(N), f(N+\tau)) \mid f \text{ is a } (P,Q)\text{-holonomic sequence}\} = \mathbb{R}^2$ by $B^{(\tau)}(N) \neq 0$, when f runs on the set of all (P,Q)-holonomic sequences, $\{f(\tau n+N)\}_{n\in\mathbb{N}}$ runs on the set of all $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$-holonomic sequences. By Theorem 4, any $\{f(\tau n+N)\}_{n\in\mathbb{N}}$ has an ultimate sign $(+)$, $(-)$, or (0). Hence, again by Theorem 4, $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of hyperbolic type.

(2) By Theorem 4 (V), no non-zero (P,Q)-holonomic sequence has an ultimate sign. Therefore $B^{(\tau)} \neq 0$ for any τ. Since the type of $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ does not depend on N, it suffices to prove this corollary for one N. Non-zero (P,Q)-holonomic sequences do not have ultimate signs, so there exists at least one $N \in \mathbb{N}$ such that the subsequence $\{f(\tau n+N)\}_{n\in\mathbb{N}}$ does not have any ultimate signs. Therefore $P_\tau, Q_\tau \neq 0$, and it follows from Theorem 4 that $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of $\mathbb{Q}$-Ω elliptic type. ◀

▶ **Lemma 21** (A quantitative version of Lemma 14)**.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$.*

(1) *The following are equivalent.*

(1a) *(P,Q) is of loxodromic or hyperbolic type.*

(1b) *There exists $q \in \mathbb{R}^{\mathbb{N}}$ with ultimate sign $(+)$ that satisfies*

$$q(n)(1-q(n+1)) \geq -\frac{Q(n)}{P(n)P(n-1)} \tag{16}$$

for all sufficiently large $n \in \mathbb{N}$.

(2) *If* (1b) *holds, then it holds for the sequence q defined by $q(0) = q(1) = 1$ and $q(n) = \frac{1}{2} + \frac{1}{4n} + \frac{1}{4n\log n}$, $n \geq 2$.*

(3) *Let (P,Q) be of hyperbolic type and P have the ultimate sign $(+)$. Take any q in* (1b)*. Take $N \in \mathbb{N}$ such that P, q, Q have their ultimate signs at N and the condition* (16) *is satisfied for any $n \geq N$. Let f be a (P,Q)-holonomic sequence. Then if*

$$f(n) \neq 0 \text{ and } \frac{f(n+1)}{f(n)} > q(n)P(n-1) \tag{17}$$

holds for some $n \geq N$, this condition also holds for $n+1, n+2, \ldots$. In particular, f has an ultimate sign $(+)$ or $(-)$ at n.

The sequence q in Lemma 21 (2) is what appears in the proof of [11, Lemma 3.4].

Proof of Theorem 10. The desired partial algorithm simply diverges when the input (P,Q) is of $\mathbb{Q}$-Ω elliptic type. For the input (P,Q) of hyperbolic type together with $f_0 \in \mathbb{Q}^2$, define $q \in \mathbb{R}^{\mathbb{N}}$ as in Lemma 21 (2) and execute the following procedure:

1. If P has the ultimate sign $(-)$, then write $f_0 = (a,b)$, and let $P := -P$ and $f_0 := (a,-b)$.
2. Calculate any N as in Lemma 21 (3).
3. Let f be the (P,Q)-holonomic sequence with initial value f_0. For $n = N, N+1, \dots,$ check the condition (17), and if it is satisfied then output n.

Let us show that if this procedure halts, then the output is correct and the (P,Q)-holonomic sequence f with initial value f_0 has a stable ultimate sign. Without loss of generality, we can assume that P has the ultimate sign $(+)$. It follows from Lemma 21 (3) that f has an ultimate sign at the output n when the procedure halts. Moreover, since $\operatorname{sgn} f(n)$ and the condition (17) are robust under small perturbations of the initial value of f, the ultimate sign of f is stable.

Conversely, let us assume that the (P,Q)-holonomic sequence f with initial value f_0 has a stable ultimate sign. By Lemma 14 (2), f has $(+)$ or $(-)$. Without loss of generality, we can assume it is $(+)$. Let N be the number obtained in step 2 of the procedure with input P, Q, f_0. It follows from the stability of the ultimate sign of f that there exists a (P,Q)-holonomic sequence g such that

- $g(N) > 0$,
- g satisfies the condition (17) for $n = N$, where f is replaced by g,
- A small perturbation $f - g$ of (the initial value of) f has the same ultimate sign $(+)$ as f.

We want to show that $F(n) := g(n)/\prod_{k=N}^{n-1} q(k)P(k-1) \to \infty (n \to \infty)$ since then we have $\lim_{n\to\infty} f(n)/\prod_{k=N}^{n-1} q(k)P(k-1) = \infty$ and the condition (17) holds for some n. By the assumption of g and Lemma 21 (3), g (and so F) has the ultimate sign $(+)$ at N. Recurrence $g(n+2) = P(n)g(n+1) + Q(n)g(n)$ and the condition (16) yield that $F(n+2) - F(n+1) \geq (q(n+1)^{-1} - 1)(F(n+1) - F(n))$ for all $n \geq N$. Note that $F(N+1) - F(N) > 0$. Then we have $F(n+2) - F(n+1) = \Omega\left(\prod_{k=0}^{n}(q(k+1)^{-1} - 1)\right)$. Since $q(k+1)^{-1} - 1 = 1 - \frac{1}{k} - \frac{1}{k \log k} + O(k^{-2})$, it follows that $\prod_{k=0}^{n}(q(k+1)^{-1} - 1) = \Theta\left(\frac{1}{n \log n}\right)$. (Herein we used $\prod_{k=2}^{n}(1 + \frac{\alpha}{k} + \frac{\beta}{k \log k}) = \Theta(n^{\alpha}(\log n)^{\beta})$ for arbitrary $\alpha, \beta \in \mathbb{R}$.) Thus $F(n+2) - F(n+1) = \Omega\left(\frac{1}{n \log n}\right)$, and so $F(n) = \Omega(\log\log n)$, which proves $F(n) \to \infty$.

Finally, when the input (P,Q) is of loxodromic type or $\frac{k}{r}$-O elliptic type, define τ as in Corollary 20. If the τth generalized canonical denominator $B^{(\tau)}$ or the 2τth one $B^{(2\tau)}$ is 0, it is easy to make our partial algorithm behave as in Theorem 10. Assume that $B^{(\tau)}, B^{(2\tau)} \neq 0$, and define P_τ, Q_τ by (15). Let $N_0 \in \mathbb{N}$ be larger than any pole of P_τ and Q_τ. Since all $(P_\tau(\tau x + N), Q_\tau(\tau x + N))$ for $N = N_0, \dots, N_0 + \tau - 1$ are of hyperbolic type, we can execute the aforementioned procedure with inputs $(P_\tau(\tau x + N), Q_\tau(\tau x + N))$ and $(f(N), f(N+\tau))$ for each N, which each halts if and only if $\{f(\tau n + N)\}_{n\in\mathbb{N}}$ has a stable ultimate sign of length 1. All of these τ executions thus halt if and only if f has a stable ultimate sign of length τ. ◀

Proof of Theorem 12. Take inputs $f_0 \in \mathbb{Q}^2$ and $(P,Q) \in \mathbb{Q}(x)^2$ for the Ultimate Sign Problem for second-order holonomic sequences. Let f be the (P,Q)-holonomic sequence with initial value f_0. Without loss of generality, we can assume that P, Q are non-zero, (P,Q) is not of $\mathbb{Q}$-Ω type and $f_0 \neq (0,0)$ (otherwise the problem is easy). We can also assume that P,

Q have no zeros in $\mathbb{N}$. As in the proof of Theorem 10, we only have to consider the case of (P, Q) of hyperbolic type, by taking a suitable subsequence.

Assume that one has an oracle for the Minimality Problem for second-order holonomic sequences. This oracle tells us whether f has an unstable ultimate sign, since it is equivalent to the minimality of f for (P, Q) of hyperbolic type.

If f has a stable ultimate sign, execute the partial algorithm in Theorem 10. Otherwise, take q as in Lemma 21 (2), and calculate and output N of (3) in the same lemma. Let us show that this output is correct. If $f(n) = 0$ for some $n \geq N$, then $f(n+1) \neq 0$ and $\frac{f(n+2)}{f(n+1)} = P(n) > q(n+1)P(n)$, which is the condition (17) for $n+1$. This implies that f has a stable ultimate sign at $n+1$, which is a contradiction. If f has no zeros $\geq N$ and satisfies $\frac{f(n+1)}{f(n)} < 0$ for some $n \geq N$, then

$$\frac{f(n+2)}{f(n+1)} = P(n) + Q(n)\frac{f(n)}{f(n+1)} \geq P(n) > q(n+1)P(n), \tag{18}$$

resulting in the same as above. Thus, $\frac{f(n+1)}{f(n)} > 0$ for all $n \geq N$. ◀

3.3 Proof of the lemmas

Proof of Lemma 21. (1a) $\Longrightarrow$ (1b) and (2) These follow from the inequality

$$q(n)(1 - q(n+1)) \geq \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16n^2(\log n)^2} - \frac{1}{n^3}$$

for all $n \geq 3$, where q is defined as in (2). (You can show this inequality using $n^{-1} - \frac{1}{2}n^{-2} \leq \log(1 + n^{-1}) \leq n^{-1}$.)

(1b) $\Longrightarrow$ (1a) Suppose, for a contradiction, that (1b) holds and (P, Q) is of elliptic type. Take q in (1b). Then there exists $C > \frac{1}{16}$ such that for all sufficiently large n,

$$q(n)(1 - q(n+1)) > \frac{1}{4} + \frac{C}{n^2}. \tag{19}$$

Especially, we have $0 < q(n) < 1$ for all sufficiently large n. If $q(n) < q(n+1)$, then $q(n)(1-q(n+1)) < q(n+1)(1-q(n+1)) \leq \frac{1}{4}$, which contradicts the equation above. Hence q is eventually decreasing, and $\alpha := \lim_{n\to\infty} q(n) \geq 0$ exists. Letting $n \to \infty$ in Equation (19) gives $\alpha(1-\alpha) \geq \frac{1}{4}$, so $\alpha = \frac{1}{2}$. Define $p(n)$ so that $q(n) = \frac{1}{2} + p(n)/n$. Then p has the ultimate sign $(+)$, and by the inequality (19) we have

$$\frac{n}{2}(p(n) - p(n+1)) + \frac{1}{2}p(n) - p(n)p(n+1) > \frac{C}{n^2}(n+1)n > C. \tag{20}$$

If $p(n) < p(n+1)$, we have the left-hand side of the above inequality $\leq \frac{1}{2}p(n) - p(n)^2 \leq \frac{1}{16}$, which contradicts $C > \frac{1}{16}$. Therefore $p(n)$ is eventually decreasing, and $\beta := \lim_{n\to\infty} p(n) \in \mathbb{R}$ exists. If $\liminf_{n\to\infty} n(p(n) - p(n+1)) > 0$, then we have $p(n) - p(n+1) = \Omega(n^{-1})$ and so $\lim_{n\to\infty} p(n) = -\infty$, which contradicts the existence of β. Thus $\liminf_{n\to\infty} n(p(n) - p(n+1)) \leq 0$. Taking $\liminf_{n\to\infty}$ of inequality (20) yields $\frac{1}{2}\beta - \beta^2 \geq C > \frac{1}{16}$. This is a contradiction.

(3) Use the condition (17) and the inequality (16) to obtain

$$\frac{f(n+2)}{f(n+1)} = P(n) + Q(n)\frac{f(n)}{f(n+1)} > P(n) + \frac{Q(n)}{q(n)P(n-1)} \geq q(n+1)P(n).$$

Then the assertion follows by induction. ◀

Proof of Lemma 14. (1) It suffices to prove $I_{P,Q}(+) \neq \varnothing \iff$ Lemma 21 (1b). $I_{P,Q}(+) \neq \varnothing$, i.e., there exists a (P,Q)-holonomic sequence f with the ultimate sign $(+)$, if and only if there exists $q \in \mathbb{R}^{\mathbb{N}}$ with the ultimate sign $(+)$ such that, for all sufficiently large n,

$$q(n+1) = 1 - \frac{-Q(n)}{P(n)P(n-1)} \cdot \frac{1}{q(n)},$$

which is an equation obtained from the recurrence (2) by rewriting with $q(n) = f(n+1)/(f(n)P(n-1))$. Since the right-hand side monotonically increases as $q(n)(>0)$ increases, the existence of such q is equivalent to the existence of q that satisfies

$$0 < q(n+1) \leq 1 - \frac{-Q(n)}{P(n)P(n-1)} \cdot \frac{1}{q(n)},$$

an inequality version of the above equation, for all sufficiently large n. This is (1b).

(2) Take q and N as in Lemma 21 (3). We want to prove that $\operatorname{sgn} f(n)$, $n \geq \max\{N, 1\}$, changes at most once for any non-zero (P,Q)-holonomic sequence f. Assume that $\operatorname{sgn} f(n)$ changes into $\operatorname{sgn} f(n+1) \neq 0$ at $n \geq \max\{N,1\}$. Then, by a similar discussion in the proof of Theorem 12, the inequality (18) holds. By Lemma 21 (3), the sign of f does not change after $n+1$. ◀

Proof of Lemma 15. Assume first that (P,Q) is not of loxodromic type. If a (P,Q)-holonomic sequence f has the ultimate sign $(+,-)$, then we find a contradiction by comparing the signs of the three terms of the equation $f(n+2) = P(n)f(n+1) + Q(n)f(n)$. Hence $I_{P,Q}(+,-) = \varnothing$, which proves (1) and (2). There is nothing to prove for (3).

Next, assume that (P,Q) is of loxodromic type. Take $N \in \mathbb{N}$ at which P, Q have the ultimate sign $(+)$. For any non-zero (P,Q)-holonomic sequence f, if successive terms $f(n)$, $f(n+1)$ $(n \geq N)$ have the same sign, then all the following terms have the same sign. It follows from this that the non-empty closed subset (of $\mathbb{R}^2 \setminus \{(0,0)\}$) $I_n = \{f_0 \in \mathbb{R}^2 \setminus \{(0,0)\} \mid$ The (P,Q)-holonomic sequence f with initial value f_0 satisfies $f(2n) \geq 0$ and $f(2n+1) \leq 0.\}$ is decreasing for $n \geq N$. Since $p(I_n)$ are decreasing non-empty closed intervals, $\bigcap_{n \geq N} p(I_n)$ is also non-empty closed intervals. This proves (1) and (2). Finally let us show (3). Take a non-zero (P,Q)-holonomic sequence f with the initial value $f_0 \notin I_{P,Q}(+,-) \cup I_{P,Q}(-,+) = \left(\bigcap_{n \geq N} I_n\right) \cup \left(\bigcap_{n \geq N}(-I_n)\right)$. Then there exists $n \geq N$ such that $f_0 \notin I_n \cup (-I_n)$, i.e., $\operatorname{sgn} f(2n) = \operatorname{sgn} f(2n+1)$. Thus f has the ultimate sign $(+)$ or $(-)$. ◀

Proof of Lemma 16

Proof of Lemma 16 (1). Set $S_{N,n} := \sum_{k=N}^{n-1} |R(n)|$. There exists $S_{N,\infty} := \lim_{n\to\infty} S_{N,n} \in \mathbb{R}$. By taking large N, we can assume $S_{N,\infty} < \frac{1}{2}$. For any $n \in \mathbb{N}$ and $N' \in \{N, N+1, \ldots, N+\tau-1\}$ such that $n \geq N'$ and $n \equiv N' \pmod{\tau}$, let us prove the following upper bound of the variation of F by course-of-values induction on n.

$$|F(n) - F(N')| \leq 2S_{N,n} \max_{N \leq I < N+\tau} |F(I)| \tag{21}$$

If $n = N'$, it is obvious. Assume $n > N'$. Set $C := \max_{N \leq I < N+\tau} |F(I)|$. By the induction hypothesis and the recurrence (10), we have

$$\begin{aligned}|F(n) - F(N')| &\leq |F(n) - F(n-\tau)| + |F(n-\tau) - F(N')| \\ &\leq |R(n-\tau)|\,|F(n-\tau+1)| + 2S_{N,n-\tau}C.\end{aligned}$$

Let us find an upper bound of $|F(n-\tau+1)|$ in this inequality. Consider $N'' \in \{N, N+1, \dots, N+\tau-1\}$ such that $N'' \equiv n-\tau+1 \pmod{\tau}$. Then the induction hypothesis and $S_{N,\infty} < \frac{1}{2}$ give a bound:

$$|F(n-\tau+1)| \le |F(n-\tau+1) - F(N'')| + |F(N'')| \le 2S_{N'',n-\tau+1}C + C \le 2C.$$

Since $|R(n-\tau)| \le S_{n-\tau,n}$, we obtain the bound (21).

For any N such that $S_{N,\infty} < \frac{1}{2}$ and any $n \in \mathbb{N}$, $N' \in \{N, N+1, \dots, N+\tau-1\}$ such that $n \ge N'$ and $n \equiv N' \pmod{\tau}$, by the bound (21), we especially have

$$|F(n) - F(N')| \le 2S_{N,\infty} \max_{N \le I < N+\tau} |F(I)|. \tag{22}$$

(1a) By (22), F is bounded. Therefore $\left\{ \max_{N \le I < N+\tau} |F(I)| \right\}_{N\in\mathbb{N}}$ is also bounded. Since $S_{N,\infty} \to 0$ as $N \to \infty$, it follows from the bound (22) that each $\{F(n)\}_{n\equiv i \pmod{\tau}}$, $i = 0, \dots, \tau-1$ is a Cauchy sequence.

(1b) Take I that realizes the "max" on the right-hand side of (22) and set $N' = I$. Then $|F(n) - F(I)| \le 2S_{N,\infty}|F(I)|$. Letting $N \to \infty$ with keeping the condition $n \equiv I \pmod{\tau}$ in this inequality, we find that the limit of $\{F(n)\}_{n\equiv I \pmod{\tau}}$ is not 0, since $2S_{N,\infty} < 1$. ◀

Proof of Lemma 16 (3). The right-hand side of the recurrence (10) has the sign qs for sufficiently large n with $n \equiv i \pmod{\tau}$. Therefore $\{F(n)\}_{n\equiv i \pmod{\tau}}$ is eventually increasing if qs is positive, eventually decreasing if negative, and constant if 0. Thus, it has the ultimate sign $(-qs)$. ◀

Proof of Lemma 16 (2). Since the right-hand side of Equation (10) has a constant sign for sufficiently large n with $n \equiv i \pmod{\tau}$, the subsequence $\{F(n)\}_{n\equiv i \pmod{\tau}}$ is eventually monotonic. Then there exists $L_i := \lim_{\substack{n\equiv i \pmod{\tau} \\ n\to\infty}} F(n) \in \mathbb{R} \cup \{\pm\infty\}$ for each $i = 0, \dots, \tau-1$.

Let us show that $L_0 = \cdots = L_{\tau-1} = 0$ or $L_0, \dots, L_{\tau-1} \in \{\pm\infty\}$. We assume $L_i \ne 0$ for some i and show $L_0, \dots, L_{\tau-1} \in \{\pm\infty\}$. By $R(n) = \Omega(n^{-1})$ and $L_i \ne 0$, the recurrence (10) yields $F(n+\tau) - F(n) = \Omega(n^{-1})$ where $n \equiv i-1 \pmod{\tau}$. Therefore $L_{i-1} \in \{\pm\infty\}$. The same discussion proves $L_{i-2}, L_{i-3}, \dots \in \{\pm\infty\}$ by induction on i.

By what we showed above and a similar discussion in the proof of Lemma 16 (3), for each $i = 0, \dots, \tau-1$, the ultimate sign of the subsequence $\{F(n)\}_{n\equiv i \pmod{\tau}}$ and whether the previous subsequence $\{F(n)\}_{n\equiv i-1 \pmod{\tau}}$ eventually increases or decreases are determined by whether $\{F(n)\}_{n\equiv i \pmod{\tau}}$ eventually increases or decreases. Therefore, there exists $s \in \{0, \pm1\}$ such that s is independent of i and the ultimate sign of $\{F(n)\}_{n\equiv i-1 \pmod{\tau}}$ is that of $\{F(n)\}_{n\equiv i \pmod{\tau}}$ multiplied by s. Thus, F has an ultimate sign of length ≤ 2. ◀

Proof of Lemmas 18 and 19

Proof of Lemma 18 (1). There exists $N \in \mathbb{N}$ such that $A^{(\tau)}(n) \ne 0$ for all $n \ge N$ since $T(n) \ne 0$. By $|T(n)| = \Theta\left(\prod_{\substack{k\equiv n (\mathrm{mod}\ \tau), \\ N \le k \le n-\tau}} \frac{1}{|A^{(\tau)}(k)|} \right)$, we have

$$\left| \frac{T(n+1)}{T(n)} \right| = \Theta\left(|A^{(\tau)}(n)|^{-1/\tau} \prod_{\substack{k\equiv n \pmod{\tau}, \\ N \le k \le n-\tau}} \left| \frac{A^{(\tau)}(k+\tau)^{\frac{1}{\tau}} A^{(\tau)}(k)^{1-\frac{1}{\tau}}}{A^{(\tau)}(k+1)} \right| \right).$$

Each factor $\frac{A^{(\tau)}(k+\tau)^{\frac{1}{\tau}}A^{(\tau)}(k)^{1-\frac{1}{\tau}}}{A^{(\tau)}(k+1)}$ of the product is $1+O(k^{-2})$, so the product converges as $n\to\infty$. Therefore $\left|\frac{T(n+1)}{T(n)}\right| = \Theta\left(|A^{(\tau)}(n)|^{-1/\tau}\right)$. Especially,

$$|R(n)| = \left|B^{(\tau)}(n)\frac{T(n+\tau)}{T(n+\tau-1)}\cdots\frac{T(n+2)}{T(n+1)}\right| = \Theta(|B^{(\tau)}(n)||A^{(\tau)}(n)|^{1-1/\tau}). \quad ◀$$

To prove Lemma 18 (2) and Lemma 19, let us study the properties of the generalized τth canonical numerator and denominator.

▶ **Lemma 22.** *Let $P, Q \in \mathbb{R}(x)^2$ have no zeros or poles in $\mathbb{N}$. The generalized ith canonical denominators $B^{(i)} \in \mathbb{R}(x)$ of (P, Q) satisfy the recurrence*

$$B^{(i+2)}(x) = P(x)B^{(i+1)}(x+1) + Q(x+1)B^{(i)}(x+2), \quad (B^{(0)}, B^{(1)}) = (0, 1). \tag{23}$$

The generalized ith canonical numerator ($i \geq 1$) is $A^{(i)}(x) = Q(x)B^{(i-1)}(x+1)$.

Proof. For any (P, Q)-holonomic sequence f, the term $f(n+i+2)$ is expressed by $f(n)$ and $f(n+1)$ as follows:

$$\begin{aligned} f((n+1)+(i+1)) &= B^{(i+1)}(n+1)f(n+2) + A^{(i+1)}(n+1)f(n+1) \\ &= \Big(P(n)B^{(i+1)}(n+1) + A^{(i+1)}(n+1)\Big)f(n+1) + Q(n)B^{(i+1)}(n+1)f(n) \end{aligned}$$

Comparing this to Equation (11) for $\tau = i+2$, we obtain the lemma by induction on i. ◀

Let us calculate the ultimate sign of $B^{(i)}$ and $\deg B^{(i)}$. Let $\deg 0 := -\infty$.

▶ **Lemma 23.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and P have the ultimate sign $(+)$. Let $i \geq 1$ be an integer and $B^{(i)}$ be the generalized ith canonical denominator. Let $L := \lim_{x\to\infty} \frac{Q(x)}{P(x)P(x-1)} \in [-\infty, \infty]$. Then $\deg B^{(i)}$ is:*

$$\begin{cases} (i-1)\deg P + \deg\left(\frac{Q(x)}{P(x)P(x-1)} + \frac{1}{4\cos^2\theta\pi}\right) & \text{if } L = -\frac{1}{4\cos^2\theta\pi} \in (-\infty, -\frac{1}{4}) \text{ and } i\theta \in \mathbb{Z}, \\ (i-1)\deg P + \lfloor\frac{i-1}{2}\rfloor \max\left\{0, \deg\frac{Q(x)}{P(x)P(x-1)}\right\} & \text{Otherwise.} \end{cases}$$

Let $q \in \{+, -, 0\}$ be $+$ if $\frac{Q(x)}{P(x)P(x-1)}$ is eventually increasing, $-$ if eventually decreasing, and 0 if constant. Then the ultimate sign of $B^{(i)}$ is:

$$\begin{cases} (+) & \text{if } L \geq -\frac{1}{4}, \\ (\operatorname{sgn}\sin i\theta) & \text{if } L = -\frac{1}{4\cos^2\theta\pi} \in [-\infty, -\frac{1}{4}) \text{ and } i\theta \notin \mathbb{Z}, \\ (\operatorname{sgn}(-1)^{i\theta}q) & \text{if } L = -\frac{1}{4\cos^2\theta\pi} \in [-\infty, -\frac{1}{4}) \text{ and } i\theta \in \mathbb{Z}. \end{cases}$$

Proof. Since $\frac{B^{(i)}(x)}{P(x+i-2)\cdots P(x)}$ is the generalized ith canonical denominator of $\left(1, \frac{Q(x)}{P(x)P(x-1)}\right)$, we can assume $P = 1$ without loss of generality. If $L = \pm\infty$, it follows by induction on i from the recurrence (23) for $P = 1$ that $\lim_{x\to\infty} B^{(i)}(x)/Q(x)^{\lfloor\frac{i-1}{2}\rfloor} = \begin{cases} 1 & (i \notin 2\mathbb{Z}) \\ \frac{i}{2} & (i \in 2\mathbb{Z}) \end{cases}$. This proves the lemma in this case.

If $L \in (-\infty, \infty)$, let $b_i := \lim_{x\to\infty} B^{(i)}(x)$. Letting $x \to \infty$ in the recurrence (23) for $P = 1$, we have

$$b_{i+2} = b_{i+1} + Lb_i. \tag{24}$$

If $L \in [-\frac{1}{4}, \infty)$, then $b_i > 0$ for all $i \geq 1$, so the lemma follows. Assume $L \in (-\infty, -\frac{1}{4})$ and let $L = -\frac{1}{4\cos^2\theta\pi}$. Then $b_i = \frac{\sin i\theta\pi}{\sin\theta\pi(2\cos\theta\pi)^{i-1}}$. This proves the claim in the case where $i\theta \notin \mathbb{Z}$. Moreover, by induction on i, it follows from the recurrence (23) for $P = 1$ that

$$B^{(i)}(x) = b_i - \frac{2\varepsilon(x)}{\tan^2\theta\pi}\left(\frac{i\cos(i-1)\theta\pi}{(2\cos\theta\pi)^{i-1}} - b_i\right) + O\left(x^{-1}\varepsilon(x)\right),$$
$$\varepsilon(x) := Q(x) + \frac{1}{4\cos^2\theta\pi}.$$

If $i\theta \in \mathbb{Z}$, then the above expression is $B^{(i)}(x) = \frac{2\varepsilon(x)}{\tan^2\theta\pi}(-1)^{i\theta+1}\frac{i\cos\theta\pi}{(2\cos\theta\pi)^{i-1}} + O\left(x^{-1}\varepsilon(x)\right)$. The lemma follows from this and $\operatorname{sgn}\varepsilon(x) = -\operatorname{sgn} q$ for large x. ◀

Proof of Lemma 18 (2)**.** One can verify this lemma using $A^{(\tau)}(x) = Q(x)B^{(\tau-1)}(x+1)$ (by Lemma 22) and Lemma 23. ◀

Proof of Lemma 19. (1) This immediately follows from Lemma 23.

(2) By Lemma 16 (1), there exist linear maps L_i, $i = 0, \ldots, 2r-1$ that map (P, Q)-holonomic sequences f to $L_i(f) := \lim_{\substack{n \equiv j \pmod{2r},\\ n \to \infty}} T(n)f(n)$. Let $j \in \{0, \ldots, 2r-1\}$ and take $j' \in \{0, \ldots, r-1\}$ such that $j'k \equiv j \pmod{r}$. Since the range of $L_{j'}$ has a lower dimension than its domain, $L_{j'}$ is not injective. Therefore there exists a non-zero (P, Q)-holonomic sequence $f^{(j)}$ such that $L_{j'}(f^{(j)}) = 0$. Without loss of generality, we can assume that $\operatorname{sgn} L_{j'+1}(f^{(j)}) \in \{0, \operatorname{sgn}(-1)^{\frac{j'k-j}{r}+1}\}$. (Otherwise we consider $-f^{(j)}$ instead of $f^{(j)}$.)

Let us first show that $\operatorname{sgn} L_{j'+i}(f^{(j)}) = \operatorname{sgn} L_{j'+1}(f^{(j)}) \sin\frac{ik}{r}\pi$. Let $A^{(0)}, A^{(1)}, \ldots$ and $B^{(0)}, B^{(1)}, \ldots$ be the generalized canonical numerators and denominators. Then we have

$$T(n+i)f^{(j)}(n+i) = \frac{T(n+i)}{T(n+1)}B^{(i)}(n)T(n+1)f^{(j)}(n+1) + \frac{T(n+i)}{T(n)}A^{(i)}(n)T(n)f^{(j)}(n). \quad (25)$$

It suffices to show that the right-hand side converges to a real number whose sign is $\operatorname{sgn} L_{j'+1}(f^{(j)}) \sin\frac{ik}{r}\pi$ as $n \to \infty$ keeping the condition $n \equiv j' \pmod{2r}$. It follows from Lemma 18 (1) and Lemma 22 that

$$\frac{T(n+i)}{T(n+1)}|B^{(i)}(n)| = \Theta\left(\left(Q(n)B^{(2r-1)}(n+1)\right)^{-(i-1)/2r} B^{(i)}(n)\right),$$
$$\frac{T(n+i)}{T(n)}|A^{(i)}(n)| = \Theta\left(\left(Q(n)B^{(2r-1)}(n+1)\right)^{-i/2r} Q(n)B^{(i-1)}(n+1)\right).$$

Then, using Lemma 23, one can verify the followings:

- $\frac{T(n+i)}{T(n)}A^{(i)}(n) = O(1)$,
- $\frac{T(n+i)}{T(n+1)}|B^{(i)}(n)| = \Theta(1)$ if $i \neq 0, r$, and
- $\lim_{n\to\infty} \frac{T(n+i)}{T(n+1)}B^{(i)}(n) = 0$ if $i = 0, r$.

Hence, the right-hand side of the equation (25) converges to a real number of the desired sign. (Note that we used $L_{j'}(f^{(j)}) = 0$ and the ultimate sign of $B^{(i)}$ shown in Lemma 23.)

By Lemma 16 (1b), there exists i such that $L_{j'+i}(f^{(j)}) \neq 0$. Since $\operatorname{sgn} L_{j'+i}(f^{(j)}) = \operatorname{sgn} L_{j'+1}(f^{(j)}) \sin\frac{ik-j}{r}\pi$, it follows that $L_{j'+1}(f^{(j)}) \neq 0$, and so $\operatorname{sgn} L_{j'+1}(f^{(j)}) = \operatorname{sgn}(-1)^{\frac{j'k-j}{r}+1}$. Therefore, we have $\operatorname{sgn} L_{j'+i}(f^{(j)}) = \operatorname{sgn}(-1)^{\frac{j'k-j}{r}+1}\sin\frac{-ik}{r}\pi$. Replacing i by $i - j'$, we obtain $\operatorname{sgn} L_i(f^{(j)}) = \operatorname{sgn}\sin\frac{j-ik}{r}\pi$. ◀

4 Proof of the Other Results

In this section, we prove Theorems 8 and 13.

4.1 Proof of Theorem 8

As we pointed out in Section 2.1.1, the first half and Parts (1), (2) and (3) of the second half of Theorem 8 follow from Theorem 4. We will prove Part (4) here.

Proof of Theorem 8 (4). By Proposition 7, it suffices to show that $\{A(n)/B(n)\}_{n\equiv i \pmod{\tau}}$ diverges in $\hat{\mathbb{R}}$ for any τ and i, where A and B are (P,Q)-holonomic sequences with initial values $(1,0)$, $(0,1)$, respectively. Define P_τ and Q_τ as in (15). Then, the subsequences $A(\tau n+i)$ and $B(\tau n+i)$ are $(P_\tau(\tau x+i), Q_\tau(\tau x+i))$-holonomic sequences. From Corollary 20 (2), $(P_\tau(\tau x+i), Q_\tau(\tau x+i))$ is of $\mathbb{Q}$-Ω elliptic type. The divergence of $\{A(n)/B(n)\}_{n\equiv i \pmod{\tau}}$ follows from Theorem 9 and Proposition 7. ◀

4.2 Proof of Theorem 13

By the assumption of the theorem, we have $\deg \frac{Q(x)}{P(x)P(x-1)} \leq -1$, so (P,Q) is of ∞-Ω loxodromic type or hyperbolic type. Then, by Theorem 4, (P,Q)-holonomic sequences $g \in \mathbb{R}^{\mathbb{N}}$ with unstable ultimate signs form a one-dimensional linear subspace in the linear space of all (P,Q)-holonomic sequences. Therefore, $g(n+1)$ and $g(n)$ must satisfy a linear relation as shown below. To keep the statement simple, let $R(x) := \frac{Q(x)}{P(x)P(x-1)}$ and consider the $(1,R)$-holonomic sequence $f(n) = \frac{g(n)}{P(n-1)\cdots P(-1)}$ with an unstable ultimate sign instead of g.

▶ **Lemma 24.** *Let $R \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and satisfy $\deg R \leq -1$. Then, for all sufficiently large $n \in \mathbb{N}$, there exists $h(n) \in [1-R(n+1)-3R(n+1)^2, 1-R(n+1)+3R(n+1)^2]$ such that any $(1,R)$-holonomic sequence f whose ultimate sign is unstable satisfies the relation*

$$f(n+1) = -R(n)h(n)f(n). \tag{26}$$

The relation (26) corresponds to the equation (6) in [16]. Instead of using [16, Lemma 14], whose proof contains a gap, we use Theorem 4 and Lemma 22 to prove this lemma.

Proof. Let $A^{(0)}$, $A^{(1)}$, ... and $B^{(0)}, B^{(1)}, \ldots$ be the generalized canonical numerators and denominators of $(1,R)$. Let f be a $(1,R)$-holonomic sequence whose ultimate sign is unstable. Dividing Equation (11) (with its Q replaced by R) by $B^{(i)}(n)$ and using $A^{(i)}(x) = R(x)B^{(i-1)}(x+1)$ in Lemma 22, we have $\frac{f(n+i)}{B^{(i)}(n)} = f(n+1) + R(n)\frac{B^{(i-1)}(n+1)}{B^{(i)}(n)}f(n)$. Hence showing the existence and estimate of

$$h(n) := \lim_{\tau\to\infty} \frac{B^{(\tau-1)}(n+1)}{B^{(\tau)}(n)} \tag{27}$$

and $\lim_{\tau\to\infty} \frac{f(n+\tau)}{B^{(\tau)}(n)} = 0$ completes this proof. Take $N \in \mathbb{N}$ such that $|R(n)|$ is monotonically decreasing and less than $\frac{1}{9}$ for all $n \geq N$.

First, we show that $\frac{B^{(i)}(n+1)}{B^{(i+1)}(n)}$ is contained in the closed interval $[1-R(n+1)-3R(n+1)^2, 1-R(n+1)+3R(n+1)^2]$ with center $1-R(n+1)$ and radius $3R(n+1)^2$ for all $i \geq 2$ and $n \geq N$, by induction on i. We use the inequality

$$1-r-3r^2 \leq \left(1+r+\frac{4}{3}r^2\right)^{-1} \leq (1+r)^{-1} \leq \left(1+r-\frac{4}{3}r^2\right)^{-1} \leq 1-r+3r^2 \tag{28}$$

for any $r \in [-\frac{1}{9}, \frac{1}{9}]$. If $i = 2$, then $\frac{B^{(2)}(n+1)}{B^{(3)}(n)} = (1+R(n+1))^{-1}$. Comparing the very middle of (28) to its very left- and right-hand sides (with $r = R(n+1)$), we get the claim. Let us prove the claim for $i+1$, assuming that the claim holds for i. Replace (P,Q) of (23) by $(1,R)$, and divide it by $B^{(i+1)}(n+1)$, then we have

$$\frac{B^{(i+2)}(n)}{B^{(i+1)}(n+1)} = 1 + R(n+1)\frac{B^{(i)}(n+2)}{B^{(i+1)}(n+1)}. \tag{29}$$

$\frac{B^{(i)}(n+2)}{B^{(i+1)}(n+1)}$ in the right-hand side is contained in the closed interval with center 1 and radius $\frac{4}{3}|R(n+1)|$ since $|R(n+2)| \leq |R(n+1)| < \frac{1}{9}$. So the both sides of (29) are in the closed interval with center $1+R(n+1)$, radius $\frac{4}{3}R(n+1)^2$. By the very left "$\leq$" and the very right "$\leq$" of (28) where $r = R(n+1)$, it follows that $\frac{B^{(i+1)}(n+1)}{B^{(i+2)}(n)}$ is in the closed interval with center $1-R(n+1)$ and radius $3R(n+1)^2$.

As shown above, $h(n)$ is in the closed interval with center $1-R(n+1)$ and radius $3R(n+1)^2$, if $h(n)$ exists. Next, we prove the existence of $h(n)$. Since $\frac{B^{(i)}(n+1)}{B^{(i+1)}(n)} \in [1-R(n+1)-3R(n+1)^2, 1-R(n+1)+3R(n+1)^2] \subseteq [\frac{1}{2}, 2]$ where $n \geq N$ and $i \geq 2$, the existence of $h(n)$ is equivalent to the convergence of the inverse $\frac{B^{(i+1)}(n)}{B^{(i)}(n+1)}$. By (29), we have

$$\begin{aligned}
&\left|\frac{B^{(i+2)}(n)}{B^{(i+1)}(n+1)} - \frac{B^{(i+1)}(n)}{B^{(i)}(n+1)}\right| \\
&= |R(n+1)|\left|\frac{B^{(i)}(n+2)}{B^{(i+1)}(n+1)} - \frac{B^{(i-1)}(n+2)}{B^{(i)}(n+1)}\right| \\
&= |R(n+1)|\frac{B^{(i)}(n+2)}{B^{(i+1)}(n+1)}\frac{B^{(i-1)}(n+2)}{B^{(i)}(n+1)}\left|\frac{B^{(i+1)}(n+1)}{B^{(i)}(n+2)} - \frac{B^{(i)}(n+1)}{B^{(i-1)}(n+2)}\right| \\
&\leq \frac{4}{9}\left|\frac{B^{(i+1)}(n+1)}{B^{(i)}(n+2)} - \frac{B^{(i)}(n+1)}{B^{(i-1)}(n+2)}\right| \\
&\leq \cdots \leq \left(\frac{4}{9}\right)^{i-2}\left|\frac{B^{(4)}(n+i-1)}{B^{(3)}(n+i)} - \frac{B^{(3)}(n+i-1)}{B^{(2)}(n+i)}\right| = O\left(\left(\frac{4}{9}\right)^{i}\right).
\end{aligned}$$

This shows that $\left\{\frac{B^{(i+1)}(n)}{B^{(i)}(n+1)}\right\}_{i\in\mathbb{N}}$ is a Cauchy sequence and converges.

Finally we prove $\lim\limits_{i\to\infty}\frac{f(n+i)}{B^{(i)}(n)} = 0$. Recall $\frac{B^{(i)}(n+1)}{B^{(i+1)}(n)} \in [\frac{1}{2}, 2]$ for $n \geq N$ and $i \geq 2$. Then $\frac{1}{B^{(i)}(n)} = \frac{B^{(i-1)}(n+1)}{B^{(i)}(n)}\frac{B^{(i-2)}(n+2)}{B^{(i-1)}(n+1)}\cdots\frac{B^{(2)}(n+i-2)}{B^{(3)}(n+i-3)} = O(2^i)$ $(i \to \infty)$. Now it remains to show $f(n+i) = O\left(\left(\frac{2}{5}\right)^i\right)$, i.e., $f(n) = O\left(\left(\frac{2}{5}\right)^n\right)$ $(n \to \infty)$. Let $f \neq 0$, since it is obvious if $f = 0$. $(1,R)$ is of ∞-Ω loxodromic type or hyperbolic type by the assumption $\deg R \leq -1$.

Let us first assume that $(1,R)$ is of ∞-Ω loxodromic type. It follows from Theorem 4 (II) that f has the ultimate sign $(+,-)$ or $(-,+)$. For all $n \geq N$ at which f has the ultimate sign, $R(n)f(n)$ and $f(n+2)$ have the same sign and $f(n+1)$ has the different sign, so it follows from $f(n+2) = f(n+1) + R(n)f(n)$ that $|f(n+2)| < |R(n)f(n)| \leq \frac{1}{9}|f(n)|$. Hence $f(n) = O\left(\left(\frac{1}{3}\right)^n\right) = O\left(\left(\frac{2}{5}\right)^n\right)$.

Let us second assume that $(1,R)$ is of hyperbolic type. Once $\frac{f(N'+1)}{f(N')} > \frac{2}{5}$ holds for some $N' \geq N$, then $\frac{f(N'+2)}{f(N'+1)} = 1 + R(N')\frac{f(N')}{f(N'+1)} > \frac{13}{18} > \frac{2}{5}$, so $\frac{f(n+1)}{f(n)} > \frac{2}{5}$ holds for all $n \geq N'$. Such f has a stable ultimate sign $(\operatorname{sgn} f(N'))$, which contradicts the assumption of this lemma. Hence $\frac{f(n+1)}{f(n)} \leq \frac{2}{5}$ for all $n \geq N$. In addition, if f has an ultimate sign at n, then $\frac{f(n+1)}{f(n)} > 0$ because the ultimate sign is $(+)$ or $(-)$ according to Theorem 4 (III). These two inequalities imply $f(n) = O\left(\left(\frac{2}{5}\right)^n\right)$. ◀

We are now ready to prove Theorem 13.

Proof of Theorem 13. Without loss of generality, we can assume $P(-1) \neq 0$. Let us take a (P, Q)-holonomic sequence $g \in \mathbb{Q}^{\mathbb{N}}$ with an unstable ultimate sign, and show $g = 0$. By multiplying a positive integer by the initial value of g, we assume $g \in \mathbb{Z}^{\mathbb{N}}$. Applying Lemma 24 to $R(x) := \frac{Q(x)}{P(x)P(x-1)}$ and $f(n) := \frac{g(n)}{P(n-2)\cdots P(-1)}$, we obtain

$$g(n+1) = -\frac{Q(n)h(n)}{P(n)} g(n), \tag{30}$$

where $h(n) = 1 - \frac{Q(n+1)}{P(n+1)P(n)} + O(n^{-2})$.

(1) $\left|\frac{Q(n)h(n)}{P(n)}\right| < 1$ holds for all sufficiently large n since $\lim_{n\to\infty} h(n) = 1$. Therefore, $|g(n+1)| < |g(n)|$ or $g(n) = 0$, which implies $g(n) = 0$ for sufficiently large n. Since Q has no zeros in $\mathbb{N}$, we get $g = 0$.

(2) Let us first show $g(n)/n \to 0$. The absolute value of the coefficient in (30) is estimated as

$$\frac{|Q(n)|h(n)}{P(n)} = 1 + \frac{|Q(n)| - P(n) - \frac{|Q(n)|Q(n+1)}{P(n+1)P(n)}}{P(n)} + O(n^{-2}).$$

If $d = 1$, then this estimate turns out to be $1 + \frac{sq_1 - p_1 - s}{p_0} n^{-1} + O(n^{-2})$. If $d \geq 2$, then $1 + \frac{sq_1 - p_1}{p_0} n^{-1} + O(n^{-2})$. Since $\prod_{k=1}^{n} \left(1 + \alpha k^{-1} + O(k^{-2})\right) = O(n^{\alpha})$ for all $\alpha \in \mathbb{R}$, it follows from (30) that

$$g(n) = \begin{cases} O\left(n^{\frac{sq_1 - p_1 - s}{p_0}}\right) & (d = 1) \\ O\left(n^{\frac{sq_1 - p_1}{p_0}}\right) & (d \geq 2) \end{cases}.$$

By the assumption on p_0, p_1, q_1, we have $g(n)/n \to 0$.

Since $g(n)/n \to 0$ and $d \geq 1$, it follows that $0 = \lim_{n\to\infty} g(n+2)/n^d = \lim_{n\to\infty} (P(n)g(n+1) + Q(n)g(n))/n^d = \lim_{n\to\infty} (p_0 g(n+1) + q_0 g(n))$. Since $p_0 g(n+1) + q_0 g(n) \in \mathbb{Z}$, we have $p_0 g(n+1) + q_0 g(n) = 0$ for all sufficiently large n. Then $g(n+1) = -sg(n)$ follows from this and $sq_0 = p_0$. Substituting this into the recurrence (2), we get $g = 0$, by the assumption of $Q(x) - sP(x) \neq 1$. ◀

We changed the assumption of the theorem from the original $sq_1 - p_1 - s < 3p_0$ (if $d = 1$) and $sq_1 - p_1 < (d+2)p_0$ (if $d \geq 2$) to our stronger one to fill in the gap at the top of page 13 in [16] as shown in the last paragraph of the proof above. We did not make any other changes to the original proof in [16, § 3.3].

References

1 Shaull Almagor, Toghrul Karimov, Edon Kelmendi, Joël Ouaknine, and James Worrell. Deciding ω-regular properties on linear recurrence sequences. *Proceedings of the ACM on Programming Languages*, 5(POPL):1–24, January 2021. `doi:10.1145/3434329`.

2 Jason P. Bell, Stanley N. Burris, and Karen Yeats. On the set of zero coefficients of a function satisfying a linear differential equation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 153(2):235–247, September 2012. `doi:10.1017/S0305004112000114`.

3 Alfredo Deaño, Javier Segura, and Nico M. Temme. Computational properties of three-term recurrence relations for Kummer functions. *Journal of Computational and Applied Mathematics*, 233(6):1505–1510, January 2010. `doi:10.1016/j.cam.2008.03.051`.

4 Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge, 4th edition, 2013.

5 Walter Gautschi. Computational aspects of three-term recurrence relations. *SIAM Review*, 9(1):24–82, 1967. doi:10.1137/1009002.

6 Stefan Gerhold and Manuel Kauers. A procedure for proving special function inequalities involving a discrete parameter. In *Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation - ISSAC '05*, pages 156–162, Beijing, China, 2005. ACM Press. doi:10.1145/1073884.1073907.

7 Vesa Halava, Tero Harju, Mika Hirvensalo, and Juhani Karhumäki. Skolem's problem - on the border between decidability and undecidability. Technical Report 683, Turku Centre for Computer Science, 2005.

8 Alaa Ibrahim and Bruno Salvy. Positivity certificates for linear recurrences. In David P. Woodruff, editor, *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 982–994. Society for Industrial and Applied Mathematics, 2024.

9 Manuel Kauers and Veronika Pillwein. When can we detect that a P-finite sequence is positive? In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation - ISSAC '10*, page 195, Munich, Germany, 2010. ACM Press. doi:10.1145/1837934.1837974.

10 George Kenison. The threshold problem for hypergeometric sequences with quadratic parameters. In Karl Bringmann, Martin Grohe, Gabriele Puppis, and Ola Svensson, editors, *51st International Colloquium on Automata, Languages, and Programming (ICALP 2024)*, volume 297 of *Leibniz International Proceedings in Informatics (Lipics)*, pages 145:1–145:20, Dagstuhl, Germany, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2024.145.

11 George Kenison, Oleksiy Klurman, Engel Lefaucheux, Florian Luca, Pieter Moree, Joël Ouaknine, Markus A. Whiteland, and James Worrell. On positivity and minimality for second-order holonomic sequences. In Filippo Bonchi and Simon J. Puglisi, editors, *46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021)*, volume 202 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 67:1–67:15, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.MFCS.2021.67.

12 Robert J. Kooman. *Convergence Properties of Recurrence Sequences*. Number 83 in CWI Tract. Centrum voor Wiskunde en Informatica, Amsterdam, 1991.

13 Robert-Jan Kooman. An asymptotic formula for solutions of linear second-order difference equations with regularly behaving coefficients. *Journal of Difference Equations and Applications*, 13(11):1037–1049, November 2007. doi:10.1080/10236190701414462.

14 Lisa Lorentzen and Haakon Waadeland. *Continued Fractions*. Atlantis Studies in Mathematics for Engineering and Science. North-Holland ; World Scientific, Amsterdam : [Singapore ; Hackensack, NJ], 2nd edition, 2008.

15 Eike Neumann. Decision problems for linear recurrences involving arbitrary real numbers. *Logical Methods in Computer Science*, Volume 17, Issue 3:6880, August 2021. doi:10.46298/lmcs-17(3:16)2021.

16 Eike Neumann, Joël Ouaknine, and James Worrell. Decision problems for second-order holonomic recurrences. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming (ICALP 2021)*, volume 198 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 99:1–99:20, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2021.99.

17 Klara Nosan, Amaury Pouly, Mahsa Shirmohammadi, and James Worrell. The Membership Problem for Hypergeometric Sequences with Rational Parameters. In *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*, pages 381–389, Villeneuve-d'Ascq France, July 2022. ACM. doi:10.1145/3476446.3535504.

18 Philipp Nuspl. C-finite and C^2-finite Sequences in SageMath. Technical report, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, June 2022. doi:10.35011/RISC.22-06.

19 Joël Ouaknine and James Worrell. Decision problems for linear recurrence sequences. In Alain Finkel, Jérôme Leroux, and Igor Potapov, editors, *Reachability Problems*, pages 21–28, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg.

20 Joël Ouaknine and James Worrell. Positivity Problems for Low-Order Linear Recurrence Sequences. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 366–379. Society for Industrial and Applied Mathematics, January 2014. `doi:10.1137/1.9781611973402.27`.

21 Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger. *A=B*. A K Peters, Wellesley, Mass, 1996.

22 Veronika Pillwein. Termination conditions for positivity proving procedures. In *Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation*, pages 315–322, Boston Maine USA, June 2013. ACM. `doi:10.1145/2465506.2465945`.

23 Veronika Pillwein and Miriam Schussler. An efficient procedure deciding positivity for a class of holonomic functions. *ACM Communications in Computer Algebra*, 49(3):90–93, November 2015. `doi:10.1145/2850449.2850458`.

24 John G. Ratcliffe. *Foundations of Hyperbolic Manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer International Publishing, Cham, 2019. `doi:10.1007/978-3-030-31597-9`.

25 R. P. Stanley. Differentiably Finite Power Series. *European Journal of Combinatorics*, 1(2):175–188, 1980. `doi:10.1016/S0195-6698(80)80051-5`.

26 Bertrand Teguia Tabuguia. Hypergeometric-type sequences. *Journal of Symbolic Computation*, 125:102328, November 2024. `doi:10.1016/j.jsc.2024.102328`.

27 Mignotte Tijdeman, R. The distance between terms of an algebraic recurrence sequence. *Journal für die reine und angewandte Mathematik*, 349:63–76, 1984.

28 N. K. Vereshchagin. Occurrence of zero in a linear recursive sequence. *Mathematical Notes of the Academy of Sciences of the USSR*, 38(2):609–615, August 1985. `doi:10.1007/BF01156238`.