Holant* Dichotomy on Domain Size 3: A Geometric Perspective

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Abstract

Holant problems are a general framework to study the computational complexity of counting problems. It is a more expressive framework than counting constraint satisfaction problems (CSP) which are in turn more expressive than counting graph homomorphisms (GH). In this paper, we prove the first complexity dichotomy of $\text{Holant}_3^*(\mathcal{F})$ where $\mathcal{F}$ is an arbitrary set of symmetric, real valued constraint functions on domain size 3. We give an explicit tractability criterion and prove that, if $\mathcal{F}$ satisfies this criterion then $\text{Holant}_3^*(\mathcal{F})$ is polynomial time computable, and otherwise it is #P-hard, with no intermediate cases. We show that the geometry of the tensor decomposition of the constraint functions plays a central role in the formulation as well as the structural internal logic of the dichotomy.

1 Introduction

Holant problems were introduced in [13] as a broad framework to study the computational complexity of counting problems. Counting CSP is a special case of Holant problems [17, 4, 3, 18, 12, 20, 8, 5]. In turn, counting CSP includes counting graph homomorphisms (GH), introduced by Lovász [25, 23], which is a special case with a single binary constraint function. Typical Holant problems include counting all matchings, counting perfect matchings #PM (including all weighted versions), counting cycle covers, counting edge colorings, and many other natural problems. It is strictly more expressive than GH; for example, it is known that #PM cannot be expressed in the framework of GH [21, 9].

The complexity classification program of counting problems is to classify as broad a class of problems as possible according to their inherent computational complexity within these frameworks. Let $\mathcal{F}$ be a set of (real or complex valued) constraint functions defined on some domain set D. It defines a Holant problem $\text{Holant}(\mathcal{F})$ as follows. An input consists of a graph $G = (V, E)$, where each $v \in V$ has an associated $\mathbf{F} \in \mathcal{F}$, with incident edges to v labeled as input variables of $\mathbf{F}$. The output is the sum of products of evaluations of the constraint functions over all assignments over D for the variables. The goal of the complexity classification of Holant problems is to classify the complexity of $\text{Holant}(\mathcal{F})$. A complexity dichotomy theorem for counting problems classifies every problem in a broad class of problems $\mathcal{F}$ to be either polynomial time solvable or #P-hard.

There has been tremendous progress in the classification of counting GH and counting CSP [17, 18, 12, 3, 20, 8, 5, 19, 1, 22, 7]. Much progress was also made in the classification of Holant problems, particularly on the Boolean domain ($|D| = 2$), i.e., when variables take 0-1 values (but constraint functions take arbitrary values, such as partition functions from statistical physics). This includes the dichotomy for all complex-valued symmetric constraint functions [10] and for all real-valued not necessarily symmetric constraint functions [26]. On the other hand, obtaining higher domain Holant dichotomy has been far more challenging. There is a huge increase in difficulty in proving

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dichotomy theorems for domain size > 2, as already seen in decision CSP of domain size 3, a major achievement by Bulatov [2]. Toward proving these dichotomies one often first considers restricted classes of Holant problems assuming some particular set of constraint functions are present. Two sets stand out: (1) the set of equality functions $\mathcal{EQ}$ of all arities (this is the class of all counting CSP problems) and (2) the set of all unary functions $\mathcal{U}$, i.e., functions of arity one. Indeed, $\#\mathrm{CSP}(\mathcal{F}) = \mathrm{Holant}(\mathcal{F} \cup \mathcal{EQ})$; i.e., counting CSP are the special case of Holant problems with $\mathcal{EQ}$ assumed to be present. In this paper we study (2): $\mathrm{Holant}_3^*(\mathcal{F}) := \mathrm{Holant}_3(\mathcal{F} \cup \mathcal{U})$, for an arbitrary set $\mathcal{F}$ of symmetric real-valued constraint functions on domain size 3.

Previously there were only two significant Holant dichotomies on higher domains. One is for a single ternary constraint function that has a strong symmetry property called domain permutation invariance [11]. That work also solves a decades-old open problem of the complexity of counting edge colorings. The other is a dichotomy for $\mathrm{Holant}_3^*(f)$ where f is a single symmetric complex-valued ternary constraint function on domain size 3 [14]. Extending this dichotomy to an arbitrary constraint function, or more ambitiously, to a set of constraint functions has been a goal for more than 10 years without much progress.

In this paper, we extend the result in [14] to an arbitrary set of real-valued symmetric constraint functions. In [24] an interesting observation was made that an exceptional form of complex-valued tractable constraint functions does not occur when the function is real-valued. By restricting ourselves to a set $\mathcal{F}$ of real-valued constraint functions, we can bypass a lot of difficulty associated with this exceptional form. Another major source of intricacy is related to the interaction of binary constraint functions with other constraint functions in $\mathcal{F}$. We introduce a new geometric perspective that provides a unifying principle in the formulation as well as a structural internal logic of what leads to tractability and what leads to #P-hardness. After discovering some new tractable classes of functions aided by the geometric perspective, we are able to prove a $\mathrm{Holant}_3^*(\mathcal{F})$ dichotomy. This dichotomy is dictated by the geometry of the tensor decomposition of constraint functions.

Suppose $\mathbf{G}$ is a binary constraint function and $\mathbf{F}$ is a tenary constraint function, with $\mathbf{F} = \mathbf{u}^{\otimes 3} + \mathbf{v}^{\otimes 3}$ its tensor decomposition. One of the simplest constructions possible with $\mathbf{G}$ and $\mathbf{F}$ is to connect $\mathbf{G}$ at the three edges of $\mathbf{F}$; the resulting constraint function is $\mathbf{G}^{\otimes 3}\mathbf{F}$ which has tensor decomposition $(\mathbf{Gu})^{\otimes 3} + (\mathbf{Gv})^{\otimes 3}$. We see that this gadget construction plays nicely with the tensor decomposition. Generalizing this idea, suppose $\mathcal{B}$ is a set of binary constraint functions and $\mathcal{T}$ is a set of ternary constraint functions. Let $\langle \mathcal{B} \rangle$ be the monoid generated by $\mathcal{B}$. We may consider the orbit $\mathcal{O}$ of $\mathcal{T}$ under the monoid action of $\langle \mathcal{B} \rangle$, such that $\mathbf{G} \in \langle \mathcal{B} \rangle$ acts on $\mathbf{F} \in \mathcal{T}$ by $\mathbf{G} : \mathbf{F} \mapsto \mathbf{G}^{\otimes 3}\mathbf{F}$. Although the constraint functions in $\mathcal{O}$ are the results of a very simple gadget construction, we show that $\mathcal{O}$ contains sufficient information about the interaction of binary constraint functions and other constraint functions, and the simplicity allows us to analyze it by considering the geometry of the vectors of the tensor decomposition of the constraint functions in $\mathcal{O}$.

Compared to the Boolean domain dichotomy theorem, stated in explicit recurrences on the values of the signatures (see Theorem 2.12 in [6]) the dichotomy theorem (Theorem 3.1) we wish to prove has a more non-explicit form, which is also more conceptual. This is informed by the geometric perspective, but it also causes some difficulty in its proof, when we try to extend to a set of constraint functions of arbitrary arities. We introduce a new technique to overcome this difficulty. First (and this is quite a surprise), it turns out that a dichotomy of two constraint functions of arity 3 is easier to state and prove than the dichotomy of one binary and one ternary constraint functions. Also they can be proven independently of each other. This is a departure from all previous proofs of dichotomy theorems in this area. Second, using the unary constraint functions available in Holant^*, any symmetric constraint function $\mathbf{F}$ of arity 4 defines a linear transformation from $\mathbb{R}^3$ to the space of symmetric constraint functions of arity 3, which corresponds to the ternary constraint functions constructible by connecting a unary function to $\mathbf{F}$. In particular, the image

$$\begin{array}{ccccccc} & & & f_{\mathsf{BBB}} & & & \\ & & f_{\mathsf{BBG}} & & f_{\mathsf{BBR}} & & \\ & f_{\mathsf{BGG}} & & f_{\mathsf{BGR}} & & f_{\mathsf{BRR}} & \\ f_{\mathsf{GGG}} & & f_{\mathsf{GGR}} & & f_{\mathsf{GRR}} & & f_{\mathsf{RRR}} \end{array}$$

Figure 1: Notation for expressing a symmetric ternary domain 3 constraint functions. This notation can be extended for higher arity signatures by using a larger triangle.

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & -1 & & \\ & 2 & & -1 & & 5 & \\ 2 & & -1 & & 5 & & -7 \end{array}$$

(a) $\mathbf{F}_1$

$$\begin{array}{ccccccc} & & & -7 & & & \\ & & 5 & & -1 & & \\ & 5 & & 5 & & 2 & \\ -7 & & -1 & & 2 & & 2 \end{array}$$

(b) $\mathbf{G}_1$

$$\begin{array}{ccccccc} & & & -2 & & & \\ & & 1 & & 1 & & \\ & 1 & & -2 & & 1 & \\ -2 & & 1 & & 1 & & -2 \end{array}$$

(c) $\mathbf{H}_1$

$$\begin{array}{ccccc} & & 3+2\sqrt{2} & & \\ & -3+2\sqrt{2} & & -\sqrt{2} & \\ 3+2\sqrt{2} & & -\sqrt{2} & & -4\sqrt{2} \end{array}$$

(d) $\mathbf{B}_1$

Figure 2: Ternary constraint functions $\mathbf{F}_1$, $\mathbf{G}_1$, $\mathbf{H}_1$, and a binary constraint function $\mathbf{B}_1$.

of this map, $\mathscr{F}$, is a linear subspace. In particular, the image $\mathscr{F}$ of this map is a linear subspace. Considering the space $\mathscr{F}$ instead of specific sub-functions allows us to bypass the difficulty from the non-explicit form of the dichotomy statement, which is in terms of tensor decompositions up to an orthogonal transformation. We show that a dichotomy of two ternary constraint functions and the fact that $\mathscr{F}$ is closed under linear combinations imply that $\mathscr{F}$ must be of a very special form for $\text{Holant}_3^*(\mathscr{F})$ to be tractable, which in turn implies that $\mathbf{F}$ must possess a certain regularity.

While the tractability criterion in Theorem 3.1 is stated in a conceptual and succinct way, the tractable cases are actually quite rich and varied. We present here specific examples of new tractable cases. Denote the domain by $D = \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$. We use the notation in Fig. 1 to denote a symmetric ternary constraint function on domain D. Consider the four constraint functions $\mathbf{F}_1, \mathbf{G}_1, \mathbf{H}_1, \mathbf{B}_1$ in Fig. 2. It is not obvious that $\text{Holant}_3^*(\mathbf{F}_1, \mathbf{G}_1, \mathbf{H}_1, \mathbf{B}_1)$ is polynomial-time computable.

We apply the orthogonal transform $T = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -2 \\ \sqrt{3} & -\sqrt{3} & 0 \end{bmatrix}$ which transforms $\mathbf{F}_1, \mathbf{G}_1$ and $\mathbf{H}_1$ to be supported in $\{\mathsf{B}, \mathsf{G}\}^*, \{\mathsf{B}, \mathsf{R}\}^*, \{\mathsf{G}, \mathsf{R}\}^*$ respectively, Their tensor decompositions have a revealing structure. Ignoring the scalar constants, we have[1]

$$\begin{aligned} \mathbf{F}'_1 &= T^{\otimes 3}\mathbf{F}_1 = 3\sqrt{3}(1,0,0)^{\otimes 3} + 6\sqrt{6}(0,1,0)^{\otimes 3} = 3\sqrt{3}\,\mathbf{e}_1^{\otimes 3} + 6\sqrt{6}\,\mathbf{e}_2^{\otimes 3} \\ \mathbf{G}'_1 &= T^{\otimes 3}\mathbf{G}_1 = (1,0,i)^{\otimes 3} + (1,0,-i)^{\otimes 3} = (\mathbf{e}_1 + i\mathbf{e}_3)^{\otimes 3} + (\mathbf{e}_1 - i\mathbf{e}_3)^{\otimes 3} \\ \mathbf{H}'_1 &= T^{\otimes 3}\mathbf{H}_1 = (0,1,i)^{\otimes 3} + (0,1,-i)^{\otimes 3} = (\mathbf{e}_2 + i\mathbf{e}_3)^{\otimes 3} + (\mathbf{e}_2 - i\mathbf{e}_3)^{\otimes 3} \end{aligned}$$

The vectors in tensor decompositions show that geometrically, $\mathbf{F}'_1$, $\mathbf{G}'_1$, and $\mathbf{H}'_1$ are associated with three coordinate planes. The function $\mathbf{B}'_1 = T^{\otimes 2}\mathbf{B}_1$ written in matrix form is $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the (i,j) entry is the function value $\mathbf{B}'_1(i,j)$, for i,j in the new domain set. Applying Theorem 3.1 we can conclude that $\{\mathbf{F}_1, \mathbf{G}_1, \mathbf{H}_1, \mathbf{B}_1\}$ is in tractable class $\mathscr{E}$.

For the second example we apply the orthogonal transform $T = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 2 \\ -\sqrt{3} & \sqrt{3} & 0 \end{bmatrix}$ to the constraint functions in Fig. 3.

$$T^{\otimes 3}\mathbf{F}_2 = 3\sqrt{3}((1,i,0)^{\otimes 3} + (1,-i,0)^{\otimes 3}) + 4\sqrt{2}\mathbf{e}_3^{\otimes 3}, \quad T^{\otimes 3}\mathbf{G}_2 = (\sqrt{3}, \sqrt{6}, 0)^{\otimes 3} + 6\sqrt{2}\mathbf{e}_3^{\otimes 3}$$

and $T^{\otimes 3}\mathbf{H}_2$ and $T^{\otimes 3}\mathbf{B}_2$ can be written in matrix form $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ respectively, up to scalar constants. Applying Theorem 3.1 we can conclude that $\{\mathbf{F}_2, \mathbf{G}_2, \mathbf{H}_2, \mathbf{B}_2\}$ is in tractable class $\mathscr{D}$.

[1] Complex numbers do appear, even though the signatures are all real valued. This is similar to eigenvalues.

$$\begin{array}{ccccccc} & & & -3 & & & \\ & & 1 & & -5 & & \\ & -3 & & -5 & & 2 & \\ 1 & & -5 & & 2 & & 10 \end{array}$$

(a) $\mathbf{F}_2$

$$\begin{array}{ccccccc} & & & 5 & & & \\ & & 11 & & 4 & & \\ & 5 & & 4 & & 2 & \\ 11 & & 4 & & 2 & & 1 \end{array}$$

(b) $\mathbf{G}_2$

$$\begin{array}{ccccc} & & 4+2\sqrt{2} & & \\ & -2+2\sqrt{2} & & -4+\sqrt{2} & \\ 4+2\sqrt{2} & & -4+\sqrt{2} & & -2-4\sqrt{2} \end{array}$$

(c) $\mathbf{H}_2$

$$\begin{array}{ccccc} & & 2-2\sqrt{2} & & \\ & 0 & & 2+\sqrt{2} & \\ -2+2\sqrt{2} & & -2-\sqrt{2} & & 0 \end{array}$$

(d) $\mathbf{B}_2$

Figure 3: Ternary constraint functions $\mathbf{F}_2$, $\mathbf{G}_2$ and bianry constraint functions $\mathbf{H}_2, \mathbf{B}_2$.

Our new algorithm also solves some natural problems. Consider the following problem. For $n \in \mathbb{N}$, $i \neq j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$, and any $a, b \in \mathbb{R}$, let $\mathsf{PARITY}_{a,b}^{n,i,j} : \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}^n \to \mathbb{R}$ be the function

$$\mathsf{PARITY}_{a,b}^{n,i,j}(\mathbf{x}) = \begin{cases} a & \text{if } \mathbf{x} \in \{i,j\}^n \text{ and } \mathbf{x} \text{ contains even number of } i \\ b & \text{if } \mathbf{x} \in \{i,j\}^n \text{ and } \mathbf{x} \text{ contains odd number of } i \\ 0 & \text{otherwise} \end{cases}$$

Let $(\neq)_{pq;r} : \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}^2 \to \{0, 1\}$ for distinct $p, q, r \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$ be the function

$$(\neq)_{pq;r}(x, y) = \begin{cases} 1 & \text{if } x, y \in \{p, q\} \text{ and } x \neq y \\ 1 & \text{if } x = y = r \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{F} = \{\mathsf{PARITY}_{a,b}^{n,i,j} : i \neq j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}, a, b \in \mathbb{R}\} \cup \{(\neq)_{pq;r} : p, q, r \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}\}$. There is a related constraint satisfaction decision problem, where

$$\mathcal{F}^b = \{\mathsf{PARITY}_{a,b}^{n,i,j} : i \neq j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}, a, b \in \{0, 1\}\} \cup \{(\neq)_{pq;r} : p, q, r \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}\},$$

and we ask if an $\mathcal{F}^b$ signature grid has a nonzero assignment. It is not even immediately obvious whether this decision problem is solvable in polynomial time. Theorem 3.1 tells us that $\mathcal{F}$ is in class $\mathscr{E}$ and thus $\text{Holant}_3^*(\mathcal{F})$ is computable in polynomial time, which implies that the decision problem is also solvable in polynomial time.

2 Preliminaries

2.1 Definitions

Definitions of Holant problem and gadget are introduced in this subsection.

Let D be a finite domain set, and $\mathcal{F}$ be a set of constraint functions, called signatures. Each $\mathbf{F} \in \mathcal{F}$ is a mapping from $D^k \to \mathbb{C}$ for some arity k. If the image of $\mathbf{F}$ is contained in $\mathbb{R}$, we say $\mathbf{F}$ is real-valued.

A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$ where each vertex is labeled by a function $\mathbf{F}_v \in \mathcal{F}$ and π is the labeling. The arity of $\mathbf{F}_v$ must match the degree of v. The Holant problem on instance Ω is to evaluate

$$\text{Holant}_\Omega = \sum_\sigma \prod_{v \in V} \mathbf{F}_v(\sigma|_{E(v)}), \tag{1}$$

where the sum is over all edge assignments $\sigma : E \to D$ and $E(v)$ is the edges adjacent to v, and $\mathbf{F}_v(\sigma|_{E(v)})$ is the evaluation of $\mathbf{F}_v$ on the ordered input tuple $\sigma|_{E(v)}$.

A signature $\mathbf{F}_v$ is listed by its values lexicographically as a table, or it can be expressed as a tensor in $(\mathbb{C}^{|D|})^{\otimes \deg(v)}$. We can identify a unary function $\mathbf{F}(x) : D \to \mathbb{C}$ with a vector $\mathbf{u} \in$

$\mathbb{C}^{|D|}$. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ of dimension $|D|$, the tensor product $\mathbf{u} \otimes \mathbf{v}$ is a vector in $\mathbb{C}^{|D|^2}$, with entries $u_i v_j$ for $1 \leq i, j \leq |D|$. For matrices $A = (a_{ij})$ and $B = (b_{kl})$ the tensor product (or Kronecker product) $A \otimes B$ is defined similarly; it has entries $a_{ij} b_{kl}$ indexed by $((i,k),(j,l))$ lexicographically. We write $\mathbf{u}^{\otimes k}$ for $\mathbf{u} \otimes \cdots \otimes \mathbf{u}$ with k copies of $\mathbf{u}$. $A^{\otimes k}$ is similarly defined. We have $(A \otimes B)(A' \otimes B') = (AA' \otimes BB')$ whenever the matrix products are defined. In particular, $A^{\otimes k}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = A\mathbf{u}_1 \otimes \cdots \otimes A\mathbf{u}_k$ when the matrix-vector products $A\mathbf{u}_i$ are defined.

A signature $\mathbf{F}$ of arity k is *degenerate* if $\mathbf{F} = \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k$ for some vectors $\mathbf{u}_i$. Such a signature is very weak; there is no interaction between the variables. If every signature in $\mathcal{F}$ is degenerate, then Holant_Ω for any $\Omega = (G, \mathcal{F}, \pi)$ is computable in polynomial time in a trivial way: Simply split every vertex v into $\deg(v)$ vertices each assigned a unary $\mathbf{F}_i$ and connected to the incident edge. Then Holant_Ω becomes a product over each component of a single edge. Thus degenerate signatures are weak and should be properly understood as made up by unary signatures. To concentrate on the essential features that differentiates tractability from intractability, Holant^* was introduced in [12, 13]. These are the problems where all unary signatures are assumed to be present, i.e. $\text{Holant}^*(\mathcal{F}) = \text{Holant}(\mathcal{F} \cup \mathcal{U})$ where $\mathcal{U}$ is the set of all unary signatures. We note that for real valued $\mathcal{F}$ the complexity of $\text{Holant}^*(\mathcal{F})$ is unchanged whether we use real valued or complex valued $\mathcal{U}$ [24](Lemma 9), and hence in this paper we use real valued $\mathcal{U}$. In the proof of #P-hardness, we freely use complex valued unary functions and apply the known Holant^* dichotomy theorems that may use complex valued unary functions.

Our proof uses the notion of a *gadget*. Consider a type of graph $G = (V, I, E)$ where I and E are two kinds of edges. Edges in I are ordinary internal edges with two endpoints in V. Edges in E are external edges (also called dangling edges) which have only one end point in V. Such a graph can be made into a part of a larger graph as follows: Given a graph G' and a vertex v of G', we may replace v with a graph G by merging the external edges of E to the incident edges of v.

A $\mathcal{F}$ gadget consists of a graph $G = (V, I, E)$ and a labeling π where each vertex $v \in V$ is labeled by $\mathbf{F}_v \in \mathcal{F}$. We may view G as a function $\mathbf{F}_G$, such that if we replace a vertex v of a graph G' by G, the Holant value of the resulting instance is as if we assign F_G to v. For this to hold, $\mathbf{F}_G$ must be such that for an assignment $\tau : E \to D$,

$$\mathbf{F}_G(\tau) = \sum_{\sigma} \prod_{u \in V} \mathbf{F}_u(\tau\sigma|_{E(u)}) \,,$$

where the sum is over all edge assignments $\sigma : I \to D$ and $\tau\sigma$ is the combined assignment on $E \cup I$.

2.2 Holographic Transformation

To describe the idea of holographic transformations, it is convenient to consider bipartite graphs. For a general graph, we can always transform it into a bipartite graph while preserving the Holant value, as follows: for each edge in the graph, we replace it by a path of length 2, and assign to the new vertex the binary Equality function $(=_2)$.

We use the notation $\text{Holant}(\mathcal{R}|\mathcal{G})$ to denote the Holant problem on bipartite graphs $H = (U, V, E)$, where each signature for a vertex in U or V is from $\mathcal{R}$ or $\mathcal{G}$, respectively. An input instance for the bipartite Holant problem is a bipartite signature grid and is denoted as $\Omega = (H; \mathcal{R}|\mathcal{G}; \pi)$. Signatures in $\mathcal{R}$ are considered as row vectors (or covariant tensors); signatures in $\mathcal{G}$ are considered as column vectors (or contravariant tensors).

For a $|D| \times |D|$ matrix T and a signature set $\mathcal{F}$, define

$$T\mathcal{F} = \{\mathbf{G} : \exists \mathbf{F} \in \mathcal{F} \text{ of arity n, such that } \mathbf{G} = T^{\otimes n}\mathbf{F}\} \,,$$

and similarly for $\mathcal{F}T$. Whenever we write $T^{\otimes n}\mathbf{F}$ or $T\mathcal{F}$, we view the signatures as column vectors; similarly $\mathbf{F}T^{\otimes n}$ or $\mathcal{F}T$ as row vectors. A holographic transformation by T is the following operation: given a signature grid $\Omega = (H; \mathcal{R}|\mathcal{G}; \pi)$, for the same graph H, we get a new grid $\Omega' = (H; \mathcal{R}T|T^{-1}\mathcal{G}; \pi)$ by replacing each signature in $\mathcal{R}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{R}T$ or $T^{-1}\mathcal{G}$.

Theorem 2.1 (Valiant's Holant Theorem [27])**.** *If there is a holographic transformation mapping signature grid Ω to Ω', then* $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

Therefore, an invertible holographic transformation does not change the complexity of the Holant problem in the bipartite setting. Furthermore, if T is orthogonal, then $(=_2)T^{\otimes 2} = T^{\intercal}IT = I$, so it preserves binary equality. This means that an orthogonal holographic transformation can be used freely in the standard setting.

Corollary 2.2. *Suppose T is an orthogonal matrix, $T^{\intercal}T = I$, and let $\Omega = (G, \mathcal{F}, \pi)$ be a signature grid. Under a holographic transformation by T, we get a new signature grid $\Omega' = (G, T\mathcal{F}, \pi)$ and* $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

2.3 Notation

For two nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, we write $\mathbf{x} \sim \mathbf{y}$ to denote projective equality, i.e. $\mathbf{x} = \lambda\mathbf{y}$ for some nonzero $\lambda \in \mathbb{C}$. For two tuples of vectors $X = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$ and $Y = (\mathbf{y}_1, \ldots, \mathbf{y}_m)$, we write $X \sim Y$ if $\mathbf{x}_i \sim \mathbf{y}_i$ for all $1 \leq i \leq m$ after some reordering. Throughout this paper, the symbol $\langle \mathbf{u}, \mathbf{v} \rangle$ for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ denotes the dot product, i.e. $\langle \mathbf{u}, \mathbf{v} \rangle = \sum u_i v_i$. We say $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

For the ease of notation, we do not distinguish between column vectors and row vectors in this paper when the intention is clear from the context.

A signature $\mathbf{F}$ of arity k is *symmetric* if $\mathbf{F}(x_1, \ldots, x_k) = \mathbf{F}(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ for all $\sigma \in S_k$, the symmetric group. It can be shown that a symmetric signature is degenerate if and only if $\mathbf{F} = \mathbf{u}^{\otimes k}$ for some unary $\mathbf{u}$. In this paper, if not further specified, a signature $\mathbf{F}$ is assumed to be real-valued, symmetric, and on domain 3.

We consider a signature $\mathbf{F}$ and its nonzero multiple $c\mathbf{F}$ as the same signature, since replacing $\mathbf{F}$ by $c\mathbf{F}$ only introduces a easily computable global factor in the Holant value.

A symmetric signature $\mathbf{F}$ on k Boolean variables $\{0, 1\}$ can be expressed as $[f_0, f_1, \ldots, f_k]$ where f_i is the value of $\mathbf{F}$ on inputs of Hamming weight i. In this paper, we focus on signatures on domain size 3, and we use the symbols $\{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$ to denote the domain elements. A binary signature $\mathbf{F}$ (not necessarily symmetric) can be expressed as a $|D| \times |D|$ matrix $M_{\mathbf{F}}$, where the entry $(i, j) \in D \times D$ is the value of $\mathbf{F}(i, j)$. For the ease of notation, we use the term matrix and binary signature interchangeably, and use $\mathbf{F}$ to refer to both a signature and its matrix $M_{\mathbf{F}}$. To fix an ordering, binary signature on domain 3 is expressed as the following:

$$\mathbf{F} = \begin{bmatrix} f_{\mathsf{BB}} & f_{\mathsf{BG}} & f_{\mathsf{BR}} \\ f_{\mathsf{GB}} & f_{\mathsf{GG}} & f_{\mathsf{GR}} \\ f_{\mathsf{RB}} & f_{\mathsf{RG}} & f_{\mathsf{RR}} \end{bmatrix}.$$

If $\mathbf{F}$ is a symmetric signature, then $\mathbf{F}$ is a symmetric matrix.

Let $\mathbf{G}$ be a binary signature and $\mathbf{F}$ be a symmetric signature of arity $k \geq 2$. We use $\mathbf{G}^{\otimes k}\mathbf{F}$ to denote the gadget constructed by attaching a $\mathbf{G}$ at the edges of $\mathbf{F}$. An example for $k = 3$ is shown in Fig. 4. If $\mathbf{F}$ is written in a tensor form, i.e. $\mathbf{F} = \mathbf{v}_1^{\otimes k} + \cdots + \mathbf{v}_s^{\otimes k}$ for $\mathbf{v}_i \in \mathbb{C}^{|D|}$ we can easily

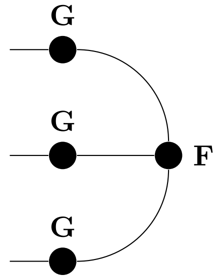


Figure 4: Gadget $\mathbf{G}^{\otimes 3}\mathbf{F}$

check that $\mathbf{G}^{\otimes k}\mathbf{F} = (\mathbf{G}\mathbf{v}_1)^{\otimes k} + \cdots + (\mathbf{G}\mathbf{v}_s)^{\otimes k}$. This gadget construction will be used throughout the paper.

Another gadget construction common in this paper is connecting a unary signature. For the ease of notation, we identify a vector $\mathbf{u} \in \mathbb{C}^{|D|}$ with a unary signature on domain D. Let $\mathbf{u} \in \mathbb{C}^{|D|}$ and $\mathbf{F}$ be a symmetric signature on domain D of arity k. Then, $\langle \mathbf{F}, \mathbf{u} \rangle$ is the arity $k-1$ gadget obtained by connecting $\mathbf{u}$ to any edge of $\mathbf{F}$. Since $\mathbf{F}$ is symmetric, the choice of the edge does not matter.

We use Holant_2 to denote the Holant problem on Boolean domain $\{0, 1\}$, and Holant_3 to denote the Holant problem on domain $\{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$. We say two sets of signatures $\mathcal{F}$ and $\mathcal{G}$ are *compatible* if $\text{Holant}(\mathcal{F} \cup \mathcal{G})$ is tractable.

For a domain 3 signature, we use the symbol $\mathbf{F}^{*\to\{i,j\}}$ to denote the domain 2 signature obtained by restricting the inputs of $\mathbf{F}$ to be from $\{i, j\} \subset \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$. We extend this notation to set of signatures $\mathcal{F}$: $\mathcal{F}^{*\to\{i,j\}} := \{\mathbf{F}^{*\to\{i,j\}} : \mathbf{F} \in \mathcal{F}\}$. When we take a domain restriction of a domain 3 signature to Boolean domain, we will identify the domains in the following way:

- $\mathbf{F}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is viewed as identifying B as 0 and G as 1.
- $\mathbf{F}^{*\to\{\mathsf{B},\mathsf{R}\}}$ is viewed as identifying B as 0 and R as 1.
- $\mathbf{F}^{*\to\{\mathsf{G},\mathsf{R}\}}$ is viewed as identifying G as 0 and R as 1.

Let $\mathbf{F}$ be a domain 3 signature. We define $\operatorname{supp}\mathbf{F}$ to be the set of inputs for which $\mathbf{F}$ is nonzero. We say $\mathbf{F}$ is EBD (a signature defined essentially on a Boolean domain) if $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$, $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B}, \mathsf{R}\}^*$ or $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{G}, \mathsf{R}\}^*$. Note that if $\operatorname{supp}\mathbf{F} = \{\mathsf{B}\}^*$ for a symmetric $\mathbf{F}$, then $\mathbf{F} = a\mathbf{e_1}^{\otimes n}$, and thus degenerate.

We say $\mathbf{F}$ is *domain separated* to $\{\mathsf{B}, \mathsf{G}\}$ and $\{\mathsf{R}\}$, written $\mathbf{F}$ is $\mathsf{BG|R}$, if $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B}, \mathsf{G}\}^* \cup \{\mathsf{R}\}^*$. In other words, $\mathbf{F}$ is zero on inputs that take values from both $\{\mathsf{B}, \mathsf{G}\}$ and $\{\mathsf{R}\}$. It is possible that $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B}\}^* \cup \{\mathsf{G}\}^* \cup \{\mathsf{R}\}^*$, in which case $\mathbf{F} = a\mathbf{e_1}^{\otimes n} + b\mathbf{e_2}^{\otimes n} + c\mathbf{e_3}^{\otimes n}$ for some $a, b, c \in \mathbb{R}$. We call such $\mathbf{F}$ a GenEQ signature. We similarly define domain separation to $\{i, j\}$ and $\{k\}$ and write $ij|k$ for any distinct $i, j, k \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$. We also refer to a matrix $\mathsf{BG|R}$ if it can be viewed as a $\mathsf{BG|R}$ binary signature. For example, $M = \left[\begin{smallmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{smallmatrix}\right]$ is a $\mathsf{BG|R}$ matrix.

Let M be a $\mathsf{BG|R}$ matrix. We can easily check the following two facts. If $\mathbf{F}$ is a $\mathsf{BG|R}$ signature of arity k, then $M^{\otimes k}\mathbf{F}$ is $\mathsf{BG|R}$ as well. If $\mathbf{G}$ is a signature of arity k such that $\operatorname{supp}\mathbf{G} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$, then $\operatorname{supp} M^{\otimes k}\mathbf{G} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$ as well.

Denote by $\mathcal{E}$ the set of all functions $\mathbf{F}$ such that if $\mathbf{F}$ has arity n, then $\operatorname{supp}\mathbf{F} \subseteq \{a, b, c\}$ for $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), c = (c_1, \ldots, c_n) \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}^n$ such that for all $1 \le i \le n$, a_i, b_i, c_i are all distinct. We think of $\mathcal{E}$ as a generalized form of GenEQ function to not necessarily symmetric functions.

We use $\mathcal{D}$ to denote the set of 3×3 matrices such that the first two columns are linearly dependent and also the first two rows are linearly dependent. In other words,

$$\mathcal{D} = \left\{ \begin{bmatrix} - & x\mathbf{v} & - \\ - & y\mathbf{v} & - \\ - & \mathbf{u} & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ x'\mathbf{v}' & y'\mathbf{v}' & \mathbf{u}' \\ | & | & | \end{bmatrix} : x, y, x', y' \in \mathbb{R}, \mathbf{v}, \mathbf{u}, \mathbf{v}', \mathbf{u}' \in \mathbb{R}^3 \right\}.$$

We can easily check that $\mathcal{D}$ is closed under multiplication. If M is a $\mathsf{BG|R}$ matrix, we can see that $M\mathcal{D}, \mathcal{D}M \subseteq \mathcal{D}$. Also, the symmetric matrices in $\mathcal{D}$ have the following form: $\begin{bmatrix} ax^2 & axy & bx \\ axy & ay^2 & by \\ bx & by & c \end{bmatrix}$ for some $a, b, c, x, y \in \mathbb{R}$.

We use $\mathsf{GenPerm}$ to denote the 3×3 generalized permutation matrices, which are matrices such that each row and column contains at most one nonzero real value. We use O_h to denote the symmetry group of an octahedron. As a subgroup of the real 3×3 orthogonal group $O(3)$, O_h consists of the matrices

$$\begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_2 \\ 0 & \epsilon_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \\ \epsilon_3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \epsilon_1 \\ \epsilon_2 & 0 & 0 \\ 0 & \epsilon_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \epsilon_1 \\ 0 & \epsilon_2 & 0 \\ \epsilon_3 & 0 & 0 \end{bmatrix}$$

for $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$. Thus, O_h has order 48. The symmetric matrices in O_h are of the form

$$\begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_2 \\ 0 & \epsilon_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_2 & 0 \\ \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \epsilon_2 \\ 0 & \epsilon_1 & 0 \\ \epsilon_2 & 0 & 0 \end{bmatrix}$$

for $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

We call a signature/matrix of the form $\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & 0 \end{bmatrix}$ for some $a, b, c, d \in \mathbb{R}$ as a $\mathsf{Swap}_{\mathsf{BG;R}}$ signature/matrix. The intuition is that a signature of this form swaps the domains $\{\mathsf{B}, \mathsf{G}\}$ and $\{\mathsf{R}\}$ in a degenerate way. Similarly, we call $\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$ as $\mathsf{Swap}_{\mathsf{BR;G}}$ and $\begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ d & 0 & 0 \end{bmatrix}$ as $\mathsf{Swap}_{\mathsf{GR;B}}$.

2.4 Symmetric Tensor Rank

We follow [16]. $\mathsf{S}^k(\mathbb{C}^n)$ is the set of order-k symmetric tensor over $\mathbb{C}^n$. In our setting, we may think of it as the set of symmetric complex-valued k-arity signatures over a domain of size n.

Definition 2.3 (Definition 4.1 of [16]). *The symmetric rank of $A \in \mathsf{S}^k(\mathbb{C}^n)$ is defined as*

$$\operatorname{rank}(A) := \min\{s : A = \sum_{i=1}^{s} \mathbf{y}_i^{\otimes k}\}$$

By Lemma 4.2 in [16], symmetric rank always exists.

Corollary 2.4 (Corollary 4.4 of [16]). *Let $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{C}^n$ be r pairwise linearly independent vectors. Then, for any $k \geq r - 1$, the rank-1 symmetric tensors*

$$\mathbf{v}_1^{\otimes k}, \ldots, \mathbf{v}_r^{\otimes k} \in \mathsf{S}^k(\mathbb{C}^n)$$

are linearly independent.

Lemma 2.5 (Lemma 5.1 of [16]). *Let $\mathbf{y}_1, \ldots, \mathbf{y}_s \in \mathbb{C}^n$ be linearly independent and $k \geq 2$ an integer. Then, the symmetric tensor defined by $A := \sum_{i=1}^{s} \mathbf{y}_i^{\otimes k}$ has $\operatorname{rank}(A) = s$.*

We derive a simple result on the uniqueness by adapting the proof of Lemma 2.5.

Proposition 2.6. *Let $k \geq 3$. Let $\mathbf{y}_1, \ldots, \mathbf{y}_s, \mathbf{z}_1, \ldots, \mathbf{z}_s \in \mathbb{C}^n$ such that $\{\mathbf{y}_i\}_{1 \leq i \leq s}$ and $\{\mathbf{z}_i\}_{1 \leq i \leq s}$ are two sets of linearly independent vectors. Suppose*

$$\sum_{i=1}^{s} \mathbf{y}_i^{\otimes k} = \sum_{i=1}^{s} \mathbf{z}_i^{\otimes k}. \tag{2}$$

Then, $(\mathbf{y}_1, \ldots, \mathbf{y}_s) \sim (\mathbf{z}_1, \ldots, \mathbf{z}_s)$.

Proof. TOPROVE 0 $\square$

2.5 Known Dichotomy Theorems

A $\mathrm{Holant}_2^*(\mathcal{F})$ dichotomy on a set of symmetric, complex-valued signatures $\mathcal{F}$ is known.

Definition 2.7 (Definition 2.9 in [6]). *A signature $[x_0, x_1, \ldots, x_n]$, where $n \geq 2$, has type $\mathrm{I}(a, b)$, if there exist a and b (not both 0), such that $ax_k + bx_{k+1} = ax_{k+2}$ for $0 \leq k \leq n-2$. We say it it is of type II, if $x_k = -x_{k+2}$ for $0 \leq k \leq n-2$.*

Theorem 2.8 (Theorem 2.12 in [6]). *Let $\mathcal{F}$ be a set of nondegenerate symmetric signatures over $\mathbb{C}$ in Boolean variables. Then $\mathrm{Holant}_2^*(\mathcal{F})$ is computable in polynomial time for the following three classes of $\mathcal{F}$. In all other cases, $\mathrm{Holant}_2^*(\mathcal{F})$ is #P-hard.*

1. *Every signature in $\mathcal{F}$ is of arity ≤ 2.*
2. *There exists a and b (not both 0, depending only on $\mathcal{F}$), such that every signature in $\mathcal{F}$ either (1) has type $\mathrm{I}(a, b)$ or (2) has arity 2 and is of the form $[2a\lambda, b\lambda, -2a\lambda]$.*
3. *Every signature in $\mathcal{F}$ either (1) has type II or (2) has arity 2 and is of the form $[\lambda, 0, \lambda]$.*

A $\mathrm{Holant}_3^*(\mathbf{F})$ dichotomy for a single symmetric, complex-valued signature of arity 3 is known, but it has an exceptional tractable case, the case 3.

Theorem 2.9 (Theorem 3.1, 3.2 in [14]). *Let $\mathbf{F}$ be a symmetric, complex valued, ternary signature over domain $\{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$. Then $\mathrm{Holant}_3^*(\mathbf{F})$ #P-hard unless $\mathbf{F}$ is one of the following forms, in which case the problem is solvable in polynomial time.*

1. *There exists three vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{C}^3$ such that $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = 0$, $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle = 0$, and $\langle \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle = 0$, and*
$$\mathbf{F} = \boldsymbol{\alpha}^{\otimes 3} + \boldsymbol{\beta}^{\otimes 3} + \boldsymbol{\gamma}^{\otimes 3}$$
2. *There exists three vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathbb{C}^3$ such that $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}_1 \rangle = 0$, $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}_2 \rangle = 0$, $\langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \rangle = 0$, $\langle \boldsymbol{\beta}_2, \boldsymbol{\beta}_2 \rangle = 0$ and*
$$\mathbf{F} = \boldsymbol{\alpha}^{\otimes 3} + \boldsymbol{\beta}_1^{\otimes 3} + \boldsymbol{\beta}_2^{\otimes 3}$$
3. *There exists two vectors $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{C}^3$ and a signature $\mathbf{F}_{\boldsymbol{\beta}}$ of arity 3, such that $\boldsymbol{\beta} \neq 0$, $\langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle = 0$, $\langle \mathbf{F}_{\boldsymbol{\beta}}, \boldsymbol{\beta} \rangle = 0$ and*
$$\mathbf{F} = \mathbf{F}_{\boldsymbol{\beta}} + \boldsymbol{\beta}^{\otimes 2} \otimes \boldsymbol{\gamma} + \boldsymbol{\beta} \otimes \boldsymbol{\gamma} \otimes \boldsymbol{\beta} + \boldsymbol{\gamma} \otimes \boldsymbol{\beta}^{\otimes 2}$$

These cases are equivalent to an existence of an orthogonal transformation T, such that

1. *For some $a, b, c \in \mathbb{C}$*
$$T^{\otimes 3}\mathbf{F} = a\mathbf{e}_1^{\otimes 3} + b\mathbf{e}_2^{\otimes 3} + c\mathbf{e}_3^{\otimes 3}$$

2. *For some* $c \neq 0$ *and* $\lambda \in \mathbb{C}$,

$$cT^{\otimes 3}\mathbf{F} = \boldsymbol{\beta}_0^{\otimes 3} + \overline{\boldsymbol{\beta}_0}^{\otimes 3} + \lambda \mathbf{e}_3^{\otimes 3}$$

where $\boldsymbol{\beta}_0 = \frac{1}{\sqrt{2}}(1, i, 0)^{\intercal}$ *and* $\overline{\boldsymbol{\beta}}$ *is its conjugate* $\frac{1}{\sqrt{2}}(1, -i, 0)^{\intercal}$.

3. *For* $\epsilon \in \{0, 1\}$,

$$T^{\otimes 3}\mathbf{F} = \mathbf{F}_0 + \epsilon\left(\boldsymbol{\beta}^{\otimes 2} \otimes \overline{\boldsymbol{\beta}} + \boldsymbol{\beta} \otimes \overline{\boldsymbol{\beta}} \otimes \boldsymbol{\beta} + \overline{\boldsymbol{\beta}} \otimes \boldsymbol{\beta}^{\otimes 2}\right)$$

where $\mathbf{F}_0$ *satisfies the annihilation condition* $\langle \mathbf{F}_0, \boldsymbol{\beta}_0 \rangle = 0$.

It is shown in [24] that when $\mathbf{F}$ is assumed to be a real-valued signature, case 3 does not occur.

Theorem 2.10 (Theorem 2 in [24])**.** *Let* $\mathbf{F}$ *be a real-valued symmetric ternary function over domain* $\{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$. *Then,* $\operatorname{Holant}_3^*(\mathbf{F})$ *is* #P*-hard unless the function* $\mathbf{F}$ *is expressible as one of the following two forms, in which case the problem is in* P.

$\mathfrak{A}$. $\mathbf{F} = \boldsymbol{\alpha}^{\otimes 3} + \boldsymbol{\beta}^{\otimes 3} + \boldsymbol{\gamma}^{\otimes 3}$ *where* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3$ *and* $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle = \langle \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle = \langle \boldsymbol{\gamma}, \boldsymbol{\alpha} \rangle = 0$.

$\mathfrak{B}$. $\mathbf{F} = \boldsymbol{\alpha}^{\otimes 3} + \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}$ *where* $\boldsymbol{\alpha} \in \mathbb{R}^3$, $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle = 0$.

This is equivalent to the existence of a real orthogonal transformation T*, such that*

$\mathfrak{A}$. $T^{\otimes 3}\mathbf{F} = a\mathbf{e_1}^{\otimes 3} + b\mathbf{e_2}^{\otimes 3} + c\mathbf{e_3}^{\otimes 3}$ *for some* $a, b, c \in \mathbb{R}$.

$\mathfrak{B}$. $cT^{\otimes 3}\mathbf{F} = \epsilon(\boldsymbol{\beta_0}^{\otimes 3} + \overline{\boldsymbol{\beta_0}}^{\otimes 3}) + \lambda \mathbf{e_3}^{\otimes 3}$ *where* $\boldsymbol{\beta_0} = (1, i, 0)^{\intercal}$, $\epsilon \in \{0, 1\}$ *and for some* $c, \lambda \in \mathbb{R}$ *and* $c \neq 0$.

2.6 Miscellaneous

Proposition 2.11. *Suppose* $\mathbf{v}^{\otimes k}$ *for a nonzero* $\mathbf{v} \in \mathbb{C}^n$ *is real-valued. Then, there exists a nonzero* $\mathbf{u} \in \mathbb{R}^n$ *and nonzero* $\lambda \in \mathbb{R}$ *such that* $\mathbf{v}^{\otimes k} = \lambda \mathbf{u}^{\otimes k}$.

Proof. TOPROVE 1 □

Therefore, we may always assume that a real-valued, symmetric, degenerate signature is a tensor power of a real vector by Proposition 2.11 and rescaling.

A characterization of a signature that will be often used in this paper is in terms of the symmetric tensor decomposition. Proposition 2.10, 2.11, and the tractability proof of Theorem 2.12 of [6] yields the following:

Proposition 2.12. *A nondegenerate Boolean domain signature* $\mathbf{F}$ *is of type* $\mathrm{I}(a, b)$ *if and only if* $\mathbf{F} = c\begin{bmatrix}\alpha \\ \beta\end{bmatrix}^{\otimes n} + d\begin{bmatrix}\gamma \\ \delta\end{bmatrix}^{\otimes n}$ *where* $\alpha\gamma + \beta\delta = 0$ *and* $a = \alpha\gamma = -\beta\delta$ *and* $b = \alpha\delta + \beta\gamma$. *Also, the set of orthogonal vectors* $\left\{\begin{bmatrix}\alpha \\ \beta\end{bmatrix}, \begin{bmatrix}\gamma \\ \delta\end{bmatrix}\right\}$ *is uniquely determined by* a, b *up to a scalar multiple.*

Proposition 2.13. *A nondegenerate Boolean domain signature* $\mathbf{F}$ *is of type* II *if and only if* $\mathbf{F} = c\begin{bmatrix}1 \\ i\end{bmatrix}^{\otimes n} + d\begin{bmatrix}1 \\ -i\end{bmatrix}^{\otimes n}$. *A real valued* $\mathbf{F}$ *is of type* II *if and only if* $\mathbf{F} = (\mathbf{u} + i\mathbf{v})^{\otimes n} + (\mathbf{u} - i\mathbf{v})^{\otimes n}$ *for orthogonal* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ *with* $\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle$.

Proof. TOPROVE 2 □

The dichotomy theorems with symmetric tensor decomposition and Proposition 2.6 imply the following criteria for determining the hardness of a signature from the tensor decomposition.

Corollary 2.14. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^2$ be linearly independent and $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \neq 0$. Then, the following are #P-hard.*

1. $\operatorname{Holant}_2^*(\boldsymbol{\alpha}^{\otimes 3} + \boldsymbol{\beta}^{\otimes 3})$
2. $\operatorname{Holant}_2^*((\boldsymbol{\alpha} + i\boldsymbol{\beta})^{\otimes 3} + (\boldsymbol{\alpha} - i\boldsymbol{\beta})^{\otimes 3})$

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3$ be linearly independent and $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \neq 0$. Then, the following are #P-hard.

1. $\operatorname{Holant}_3^*(\boldsymbol{\alpha}^{\otimes 3} + \boldsymbol{\beta}^{\otimes 3})$
2. $\operatorname{Holant}_3^*(\boldsymbol{\alpha}^{\otimes 3} + \boldsymbol{\beta}^{\otimes 3} + \boldsymbol{\gamma}^{\otimes 3})$
3. $\operatorname{Holant}_3^*((\boldsymbol{\alpha} + i\boldsymbol{\beta})^{\otimes 3} + (\boldsymbol{\alpha} - i\boldsymbol{\beta})^{\otimes 3})$
4. $\operatorname{Holant}_3^*((\boldsymbol{\alpha} + i\boldsymbol{\beta})^{\otimes 3} + (\boldsymbol{\alpha} - i\boldsymbol{\beta})^{\otimes 3} + \boldsymbol{\gamma}^{\otimes 3})$

Proof. TOPROVE 3 □

3 Statement of the Dichotomy Theorem

Let $\mathcal{F}$ be a set of nondegenerate, real-valued, symmetric signatures over domain $\{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$.

Theorem 3.1. $\operatorname{Holant}_3^*(\mathcal{F})$ *is computable in polynomial time if there exists a real orthogonal T, such that one of the following conditions holds. In all other cases,* $\operatorname{Holant}_3^*(\mathcal{F})$ *is #P-hard.*

$\mathscr{A}$. *Every signature in $\mathcal{F}$ has arity ≤ 2.*

$\mathscr{B}$. $T\mathcal{F} \subseteq \mathcal{E}$.

$\mathscr{C}$. (a) *For all $\mathbf{F} \in T\mathcal{F}$ of arity ≥ 3, $\operatorname{supp} \mathbf{F} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$, and*
   (b) *For all binary $\mathbf{G} \in T\mathcal{F}$, either $\mathbf{G} \in \mathcal{D}$ or $\mathbf{G}$ is $\mathsf{BG}|\mathsf{R}$, and*
   (c) $\operatorname{Holant}_2^*((T\mathcal{F})^{*\to\{\mathsf{B},\mathsf{G}\}})$ *is tractable.*

$\mathscr{D}$. (a) *For all $\mathbf{F} \in T\mathcal{F}$ of arity ≥ 3, $\mathbf{F}$ is $\mathsf{BG}|\mathsf{R}$, and*
   (b) *For all binary $\mathbf{G} \in T\mathcal{F}$, either $\mathbf{G}$ is $\mathsf{BG}|\mathsf{R}$ or $\mathbf{G}$ is $\mathsf{Swap}_{\mathsf{BG};\mathsf{R}}$, and*
   (c) $\operatorname{Holant}_2^*((T\mathcal{F})^{*\to\{\mathsf{B},\mathsf{G}\}})$ *is tractable.*

$\mathscr{E}$. *Let $\mathcal{F}_{ij} = \{\mathbf{F} \in T\mathcal{F} : \operatorname{supp} \mathbf{F} \subseteq \{i, j\}^*\}$. Let $\mathcal{R} = T\mathcal{F} - (\mathcal{F}_{\mathsf{BG}} \cup \mathcal{F}_{\mathsf{BR}} \cup \mathcal{F}_{\mathsf{GR}})$.*
   (a) *$\mathcal{R} \subseteq \mathbb{R}O_h$, and $\langle \mathcal{R}' \rangle \subseteq O_h$, where $\langle \mathcal{R}' \rangle$ is the monoid generated by $\mathcal{R}' = \mathbb{R}\mathcal{R} \cap O_h$, and*
   (b) *For all $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$, $\operatorname{Holant}_2^*(\langle \mathcal{R}' \rangle^{*\to\{i,j\}} \cup \mathcal{F}_{ij})$ is tractable, and*
   (c) *For all $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$, $\operatorname{Holant}_2^*((\bigcup_{\mathbf{G} \in \langle \mathcal{R}' \rangle} \mathbf{G}(T\mathcal{F}))^{*\to\{i,j\}})$ is tractable.*

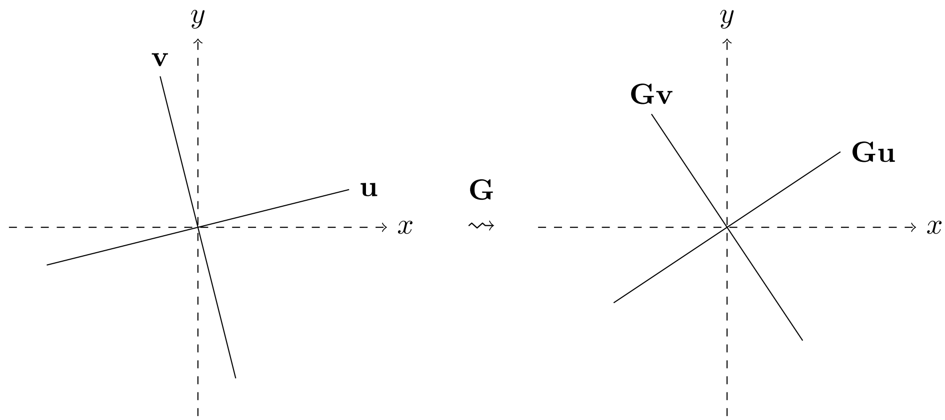


Figure 5: Visualization of type $\mathrm{I}(a, b)$.

In classes $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$, we refer to the $\operatorname{Holant}_2^*$ tractability. By the Boolean domain dichotomy, it is necessary that those $\operatorname{Holant}_2^*$ problems are tractable. However, it is not immediately clear why they are sufficient conditions.

In case (b) of class $\mathscr{E}$, it seems like we may need to use the asymmetric $\operatorname{Holant}_2^*$ dichotomy (which is known [15]), because while the signatures in $\mathcal{R}$ are symmetric, $\langle\mathcal{R}\rangle$ may contain asymmetric signatures. We claim that is not necessary. Let $\mathbf{G} \in O_h$. If $\mathbf{G}^{*\to\{i,j\}}$ is nondegenerate and asymmetric, then $\mathbf{G}^{*\to\{i,j\}} = \pm\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. We can deduce that $\mathbf{G}^{*\to\{i,j\}}$ is *universally compatible*, i.e., when appended to any $\operatorname{Holant}_2^*$ tractable class results in a tractable set, without referring to any asymmetric $\operatorname{Holant}_2^*$ dichotomy. For type $\mathrm{I}(a, b)$ (see Definition 2.7 and Theorem 2.8), we see that $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] = \frac{1}{4a^2+b^2}\left[\begin{smallmatrix} b & -2a \\ -2a & -b \end{smallmatrix}\right]\left[\begin{smallmatrix} 2a & b \\ b & -2a \end{smallmatrix}\right]$. The first matrix in the symmetric signature notation is $[b, -2a, -b]$, which satisfies the recurrence $a \cdot b + b(-2a) = a \cdot (-b)$. The second matrix in the symmetric signature notation is $[2a, b, -2a]$, which is the other specified form. For type II, we see that $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and $[1, 0, -1]$ and $[0, 1, 0]$ are type II.

This is not a coincidence. We may view the $\operatorname{Holant}_2^*$ dichotomy in a geometric way. Consider the tractable type $\mathrm{I}(a, b)$. There are two norm 1 orthogonal vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ such that any signature of arity ≥ 3 of type $\mathrm{I}(a, b)$ is of the form $c\mathbf{u}^{\otimes n} + d\mathbf{v}^{\otimes n}$. Essentially, the type $\mathrm{I}(a, b)$ signatures of arity ≥ 3 can be represented by two orthogonal lines in the $\mathbb{R}^2$ plane. Consider the gadget construction of connecting a binary signature $\mathbf{G}$ to all of the edges of a signature. In the tensor decomposition form, we have $\mathbf{G}^{\otimes n}(c\mathbf{u}^{\otimes n} + d\mathbf{v}^{\otimes n}) = c(\mathbf{G}\mathbf{u})^{\otimes n} + d(\mathbf{G}\mathbf{v})^{\otimes n}$. For the binary signatures, we can check that the tractable signatures correspond to the linear transformations that fix the union of the two orthogonal lines as a set. This is easily verified for the case of type $\mathrm{I}(0, 1)$, when $\mathbf{u} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$, since $[*, 0, *]$ signatures correspond to scaling $\mathbf{e}_1$ and $\mathbf{e}_2$, while $[0, *, 0]$ signatures correspond to reflection along $x = y$ line. Since type $\mathrm{I}(a, b)$ for different a, b are all related by a holographic transformation by a rotation, it is sufficient to check for type $\mathrm{I}(0, 1)$. The asymmetric signature above is a $\pi/2$ rotation, which fixes any pair of orthogonal lines, so it is tractable with all $\mathrm{I}(a, b)$ types. Another geometric view is that a $\pi/2$ rotation can be written as composition of a reflection along the line at the $\pi/4$ angle between $\mathbf{u}$ and $\mathbf{v}$ and the reflection along $\mathbf{u}$. The two signatures that we used to construct the $\pi/2$ rotation above are precisely these geometric transformations.

The tractable type II may be viewed as a circle in the $\mathbb{R}^2$ plane. The justification is that by Proposition 6.2, Corollary 6.3, and Proposition 7.3, any type II signature can be written as $(\mathbf{u} + i\mathbf{v})^{\otimes n} + (\mathbf{u} - i\mathbf{v})^{\otimes n}$ for two orthogonal $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ of the same norm, and all such signatures can be constructed from a single type II signature. So, in a sense, after normalization, type II signatures span the whole unit circle. And the compatible binary signature are the linear transformations that

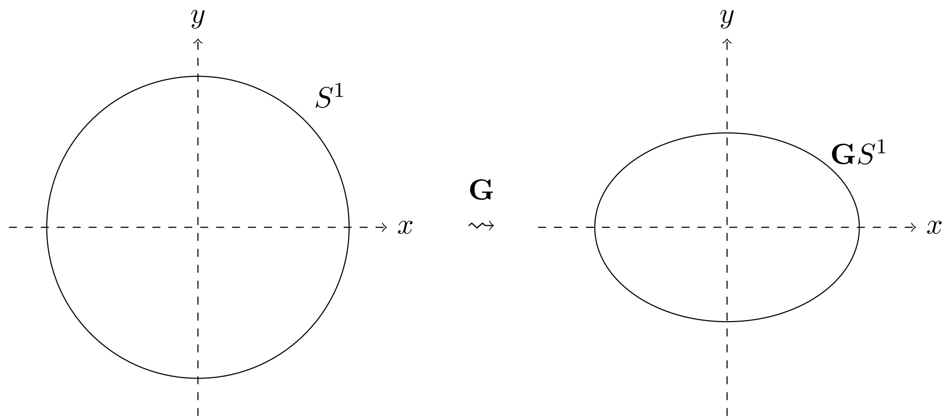


Figure 6: Visualization of type II.

transform a circle to a circle. These signatures are precisely the scalar multiples of the elements in the orthogonal group $O(2)$. We see that the signature $[x, y, -x]$ corresponds to a scalar multiple of a reflection matrix. Also, since any rotation can be written as a product of two reflections, we have that the rotation matrices viewed as signatures, possibly asymmetric, are also compatible with type II signatures.

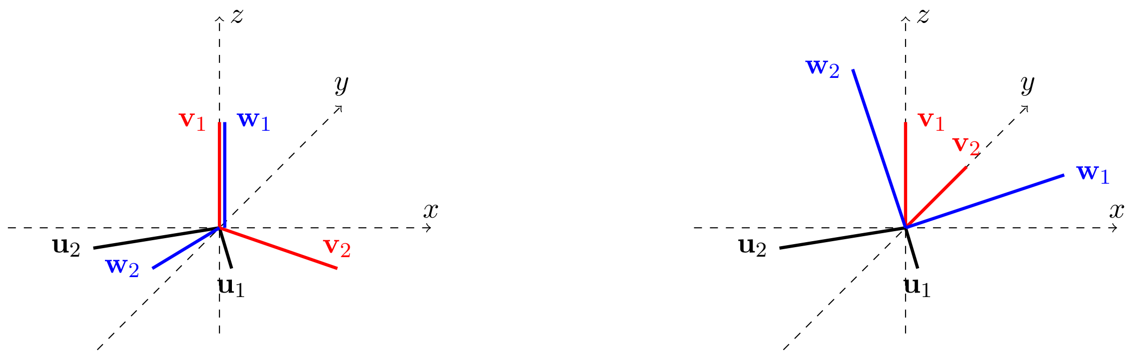
A similar analogy can be made for domain 3 signatures as well, since the tensor decomposition forms in Theorem 2.10 are also about orthogonality of the vectors. Similar to the Boolean domain tractable signatures, a tractable domain 3 signature also can be represented as a set of vectors in the three dimensional space. For instance, let $\mathbf{u}_1^{\otimes 3} + \mathbf{u}_2^{\otimes 3}$, $\mathbf{v}_1^{\otimes 3} + \mathbf{v}_2^{\otimes 3}$ and $\mathbf{w}_1^{\otimes 3} + \mathbf{w}_2^{\otimes 3}$, be three signatures. The idea is depicted in Fig. 7, and this intuition is formalized in Section 9. Class $\mathscr{C}$ says that the signatures of arity 3 or higher must live in some plane, and we can assume it is the xy-plane after an orthogonal transformation. The compatible binary signatures are those fixing the xy-plane ($\mathsf{BG|R}$) or behaving like a degenerate transformation on the xy plane. The class $\mathscr{D}$ says that the signatures of arity 3 or higher are formed by xy-plane and a z-axis line. The compatible binary signatures, after the same orthogonal transformation, are those fixing the xy-plane and z-axis line ($\mathsf{BG|R}$) or mapping between the xy-plane and the z-axis ($\mathsf{Swap}_{\mathsf{BG;R}}$). The class $\mathscr{E}$ says that the signatures of arity 3 or higher must live in one of xy-plane, yz-plane, or xz-plane. The compatible binary signatures are those permuting the three planes without stretching. Such transformations are precisely the group O_h.

This idea of viewing a binary signature as a transformation is also captured in the final proof in Section 9, where we define a set $\mathcal{O}$ which is the set of all gadgets constructible by connecting a chain of binary signatures on a ternary signature. In terms of the vectors in the tensor decomposition, $\mathcal{O}$ can be viewed as the orbit under the monoid action of the binary signatures.

4 Tractability

In this section, we prove tractability.

Any 3×3 orthogonal matrix T keeps the binary equality $(=_2)$ over $\{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$ unchanged, namely $T^{\intercal} I_3 T = I_3$ in matrix notation. Hence $\operatorname{Holant}_3^*(\mathcal{F})$ is tractable if and only if $\operatorname{Holant}_3^*(T\mathcal{F})$ is tractable. Also, we may always assume that the given signature grid is connected, as the Holant value of any signature grid is the product over connected components.



(a) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_2, \mathbf{w}_2$ are on the xy plane and $\mathbf{v}_1, \mathbf{w}_1$ are on the z axis.

(b) $\mathbf{u}_1, \mathbf{u}_2$ are on xy plane, $\mathbf{v}_1, \mathbf{v}_2$ are on the yz plane and $\mathbf{w}_1, \mathbf{w}_2$ are on the xz plane.

Figure 7: Geometric idea behind the dichotomy of $\text{Holant}_3^*(\mathcal{F})$

4.1 Class $\mathscr{A}$

Class $\mathscr{A}$ is when every signature in $\mathcal{F}$ has arity ≤ 2. Then, the graph of the signature grid Ω is a disjoint union of paths and cycles. By matrix multiplication, we can compute the Holant value for a path. The Holant value for a cycle is obtained by taking the trace of a path.

4.2 Class $\mathscr{B}$

Class $\mathscr{B}$ is when there exists an orthogonal T such that $T\mathcal{F} \subseteq \mathcal{E}$. We prove the tractability of $\text{Holant}_3^*(\mathcal{E})$. Note that $\mathcal{U} \subseteq \mathcal{E}$, so we argue the tractability of $\text{Holant}_3(\mathcal{E})$. Let Ω be a signature grid and e any edge. For any $\mathsf{B}, \mathsf{G}, \mathsf{R}$ assignment to e, the assignment must propagate uniquely to all edges, which implies that there are at most three assignments to the whole grid that can result in a nonzero Holant value. Therefore, $\text{Holant}_3(\mathcal{E})$ is polynomial time computable.

4.3 Class $\mathscr{C}$

Class $\mathscr{C}$ is when there exists an orthogonal T such that $T\mathcal{F}$ has the following properties:

1. For all $\mathbf{F} \in T\mathcal{F}$ of arity 3 or higher, $\operatorname{supp} \mathbf{F} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$.
2. For all binary $\mathbf{G} \in T\mathcal{F}$, either $\mathbf{G} \in \mathcal{D}$ or $\mathbf{G}$ is $\mathsf{BG}|\mathsf{R}$.
3. $\text{Holant}_2^*((T\mathcal{F})^{*\to\{\mathsf{B},\mathsf{G}\}})$ is tractable.

Let Ω be a connected signature grid over $T\mathcal{F}$. We may assume Ω is not in Class $\mathscr{A}$. If a unary signature is connected to another unary signature then this is the entire Ω and we are done. If a unary signature is connected to a binary signature then they become another unary constraint which we can compute its signature. So, by induction, we can assume any unary remaining is connected to some signature $\mathbf{F} \in T\mathcal{F}$ of arity ≥ 3, which has $\operatorname{supp} \mathbf{F} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$, and thus we can replace the unary restricted to $\{\mathsf{B}, \mathsf{G}\}^*$. If there is a chain of binary signatures $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_k$, we replace it by a single binary signature $\mathbf{G}$ by taking the matrix product. If all $\mathbf{G}_i$ are $\mathsf{BG}|\mathsf{R}$, then $\mathbf{G}$ is also $\mathsf{BG}|\mathsf{R}$. In addition, the matrix of $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is equal to the product of the matrices of $\mathbf{G}_i^{*\to\{\mathsf{B},\mathsf{G}\}}$. If there is some i such that $\mathbf{G}_i \in \mathcal{D}$, then $\mathbf{G} \in \mathcal{D}$ as well, since $\mathcal{D}$ is closed under multiplication and also closed under left or right multiplication by a $\mathsf{BG}|\mathsf{R}$ matrix. Note that if $\mathbf{G} \in \mathcal{D}$, then $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is degenerate. Now since any binary $\mathbf{G}$ remaining can only be connected to signatures $\mathbf{F}$ of arity ≥ 3 with $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B},\mathsf{G}\}^*$, we can replace $\mathbf{G}$ by $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$. Thus we obtain an equivalent signature grid Ω' as a Holant_2^* instance on domain $\{\mathsf{B},\mathsf{G}\}$. Then condition 3. implies that the Holant value of Ω' can be computed in polynomial time.

4.4 Class $\mathscr{D}$

Class $\mathscr{D}$ is when there exists an orthogonal T such that $T\mathcal{F}$ has the following properties:

1. For all $\mathbf{F} \in T\mathcal{F}$ of arity 3 or higher, $\mathbf{F}$ is $\mathsf{BG|R}$.
2. For all binary $\mathbf{G} \in T\mathcal{F}$, either $\mathbf{G}$ is $\mathsf{BG|R}$ or $\mathbf{G}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$.
3. $\text{Holant}_2^*((T\mathcal{F})^{*\to\{\mathsf{B},\mathsf{G}\}})$ is tractable.

Suppose we are given a $T\mathcal{F}$ signature grid Ω. If Ω does not use any binary signature of $\mathsf{Swap}_{\mathsf{BG;R}}$, then all signatures are $\mathsf{BG|R}$. Then, if any edge gets assigned B or G, all other edges must also be assigned B or G for the assignment to result in a nonzero value. Similarly, any assignment of R to an edge must propagate as R to all other edges for the assignment to result in a nonzero value. Therefore, the Holant value is sum of all assignments taking values in $\{\mathsf{B},\mathsf{G}\}$ and an assignment that only assigns R. The first sum can be computed in polynomial since $\text{Holant}_2^*((T\mathcal{F})^{*\to\{\mathsf{B},\mathsf{G}\}})$ is tractable, and the second value is computable in polynomial time.

Now, suppose Ω contains $\mathsf{Swap}_{\mathsf{BG;R}}$ signatures. First, note that a product of two $\mathsf{Swap}_{\mathsf{BG;R}}$ signatures is $\mathsf{BG|R}$ signature that is degenerate on $\{\mathsf{B},\mathsf{G}\}$.

$$\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ g & h & 0 \end{bmatrix} = \begin{bmatrix} ag & ah & 0 \\ bg & bh & 0 \\ 0 & 0 & ce+df \end{bmatrix}.$$

Hence, we may reduce any chain $\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_k$ of $\mathsf{Swap}_{\mathsf{BG;R}}$ signatures by matrix multiplication to a length 2 chain $\mathbf{G}, \mathbf{G}_k$ if k is odd and a single $\mathbf{G}$ if k is even, where $\mathbf{G}$ is a $\mathsf{BG|R}$ signature that is degenerate on $\{\mathsf{B},\mathsf{G}\}$. In particular, $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is compatible with $\mathcal{F}^{*\to\{\mathsf{B},\mathsf{G}\}}$. Therefore, we may assume that we have a signature grid Ω' such that any $\mathsf{Swap}_{\mathsf{BG;R}}$ signature is connected to two $\mathsf{BG|R}$ signatures. Now, suppose we gather each connected component of $\mathsf{BG|R}$ signatures as a cluster, so that the connections between clusters are by $\mathsf{Swap}_{\mathsf{BG;R}}$ signatures. If we imagine a graph with vertices being the clusters and edges being the $\mathsf{Swap}_{\mathsf{BG;R}}$ signatures, then this graph must be bipartite. Otherwise, suppose there is an odd cycle of clusters $C_1, C_2, \cdots, C_k, C_1$. In any nonzero assignment, each cluster can take only values from $\{\mathsf{B},\mathsf{G}\}$ or R. If C_1 gets $\{\mathsf{B},\mathsf{G}\}$, then C_2 must have R because of the $\mathsf{Swap}_{\mathsf{BG;R}}$ connection. Therefore, if there is an odd cycle, C_1 will have an incoming R, resulting in a zero evaluation. Similar argument shows that an assignment of R to C_1 evaluates to zero. Also, by the same argument, there cannot be a self loop by $\mathsf{Swap}_{\mathsf{BG;R}}$ signature within a single cluster.

Let the bipartition be $L \sqcup R$. There are only two types of nonzero assignments: (1) all clusters in L get $\{\mathsf{B},\mathsf{G}\}$ and all clusters in R get R; (2) all clusters in L get R and all clusters in R get $\{\mathsf{B},\mathsf{G}\}$. For both types of assignments, the $\mathsf{Swap}_{\mathsf{BG;R}}$ signatures connecting a cluster $C_L \in L$ and $C_R \in R$ factors into two degenerate signatures. In particular, suppose $\mathbf{G} = \left[\begin{smallmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & 0 \end{smallmatrix}\right]$ is a $\mathsf{Swap}_{\mathsf{BG;R}}$ signature connecting C_L and C_R. Let Ω' be obtained from Ω by removing $\mathbf{G}$ and connecting the unary $(a, b, 0)$ to C_L and $(0, 0, 1)$ to C_R, as described in Fig. 8. Then, since C_L only takes values in $\{\mathsf{B},\mathsf{G}\}$ and C_R only takes values in R, the sum of type (1) assignments is the same as the Holant value of Ω'. This equivalence can be shown formally by a calculation. Let e_L denote the edge between C_L and $\mathbf{G}$ and e_R denote the edge between $\mathbf{G}$ and C_R. Let c_{ij} be the contribution from

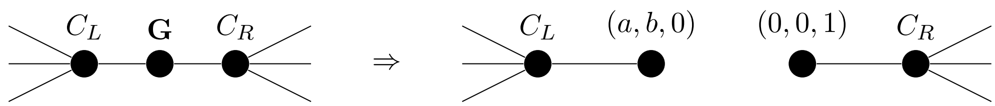


Figure 8: Factorization of $\mathsf{Swap}_{\mathsf{BG};\mathsf{R}}$ in type (1) assignment

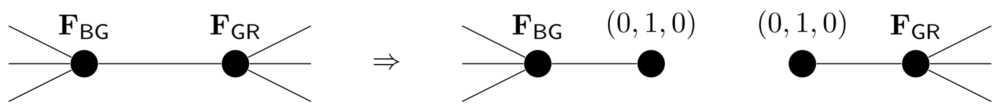


Figure 9: Factorization of an edge between EBD signatures of different support

rest of the grid when $e_L = i$ and $e_R = j$. In a type (1) assignment, e_L can only take value $\{\mathsf{B}, \mathsf{G}\}$ and e_R can only take value R. The sum of all type (1) assignments is then:

$$\mathbf{G}(\mathsf{B}, \mathsf{R}) \cdot c_{\mathsf{BG}} + \mathbf{G}(\mathsf{G}, \mathsf{R}) \cdot c_{\mathsf{GR}} = a \cdot c_{\mathsf{BG}} + b \cdot c_{\mathsf{GR}}\,.$$

The Holant value of Ω' is:

$$a \cdot 1 \cdot c_{\mathsf{BG}} + b \cdot 1 \cdot c_{\mathsf{GR}}\,.$$

Note that we may use the same c_{ij} since the two new unary signatures force the edge assignment in L to be from $\{\mathsf{B}, \mathsf{G}\}$ and the edge assignment in R to be R. Therefore, to evaluate the sum of type (1) assignments, we may factor all the $\mathsf{Swap}_{\mathsf{BG};\mathsf{R}}$ signatures in this way, evaluate in $\{\mathsf{B}, \mathsf{G}\}$ domain Holant$_2^*$ on L and assign R to all of R, and multiply the two resulting values. Both can be done in polynomial time since $\mathcal{F}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is assumed to be tractable. Similar argument shows that the sum of type (2) assignments can be also computed in polynomial time. Then, the Holant value is just the sum of those two values.

4.5 Class $\mathscr{E}$

Class $\mathscr{E}$ is when there exists an orthogonal T such that $T\mathcal{F}$ has the following properties. Let $\mathcal{F}_{ij} = \{\mathbf{F} \in T\mathcal{F} : \operatorname{supp} \mathbf{F} \subseteq \{i, j\}^*\}$. Let $\mathcal{R} = T\mathcal{F} - (\mathcal{F}_{\mathsf{BG}} \cup \mathcal{F}_{\mathsf{BR}} \cup \mathcal{F}_{\mathsf{GR}})$. Let $\mathcal{R}' = \{\lambda \mathbf{G} : \lambda \in \mathbb{R}, \mathbf{G} \in \mathcal{R}$ such that $\lambda \mathbf{G} \in O_h\}$.

1. For any $\mathbf{G} \in \mathcal{R}$, there is some $\lambda \in \mathbb{R}$ such that $\lambda \mathbf{G} \in O_h$.
2. For all $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$, Holant$_2^*(\langle \mathcal{R}' \rangle^{*\to\{i,j\}} \cup \mathcal{F}_{ij})$ is tractable.
3. For all $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$, Holant$_2^*((\bigcup_{\mathbf{G} \in \langle \mathcal{R}' \rangle} \mathbf{G}(T\mathcal{F}))^{*\to\{i,j\}})$ is tractable.

Suppose we are given a $T\mathcal{F}$ signature grid Ω. If Ω does not use any signature from $\mathcal{R}$, then all signatures in Ω are EBD. Then, whenever there is an edge between two signatures with different support, the edge factors as pinning. For example, if there is an edge e between a signature $\mathbf{F}_{\mathsf{BG}} \in \mathcal{F}_{\mathsf{BG}}$ and $\mathbf{F}_{\mathsf{GR}} \in \mathcal{F}_{\mathsf{GR}}$, we may remove the edge and connect the unary $(0, 1, 0)$ to both $\mathbf{F}_{\mathsf{BG}}$ and $\mathbf{F}_{\mathsf{GR}}$, as shown Fig. 9. This is because e can only take value G in any nonzero assignment. Also, if there is a connection between any unary signature $\mathbf{u}$ and a EBD signature supported on $\{i, j\}$, then we may replace $\mathbf{u}$ with $\mathbf{u}^{*\to\{ij\}}$. We obtain a new signature grid Ω' after factoring all such edges, and each of the connected components of Ω' is an instance of Holant$_2^*(\mathcal{F}_{ij})$ for some $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$. By assumption, the Holant value of each of the connected components can be computed in polynomial time.

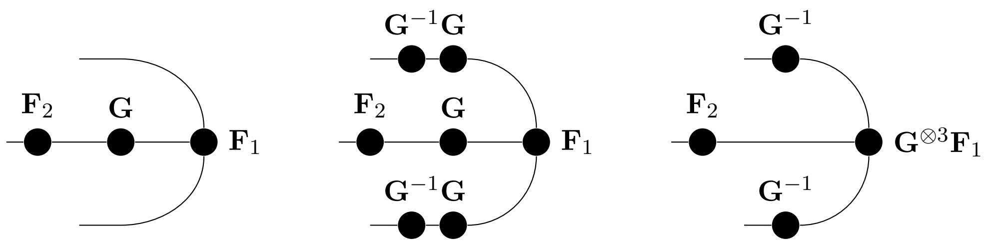


Figure 10: Local holographic transformation

Now, suppose Ω contains signatures from $\mathcal{R}$. We may replace those signatures by the corresponding scalar multiple in $\mathcal{R}'$. Recall that the signatures in $\mathcal{R}'$ are not EBD, but they are binary signatures of signed swap matrices by the assumption that $\mathcal{R}' \subseteq O_h$. By the above, we may assume that in Ω, there is no edge between EBD signatures of different supports. First, we reduce any chain of $\mathcal{R}'$ signatures into a single binary signature in $\langle \mathcal{R}' \rangle \subseteq O_h$. Reducing any connection of $\mathcal{R}'$ signature with a unary signature to a unary, we obtain a new grid Ω' in which any $\langle \mathcal{R}' \rangle$ signature is between two signatures from $\mathcal{F}_{ij}$ and $\mathcal{F}_{i'j'}$. We imagine Ω' is composed of clusters C where each cluster is a connected component of $\mathcal{F}_{ij}$ signatures for some $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$, and each cluster has outgoing edges of $\langle \mathcal{R}' \rangle$ signatures.

Suppose $\mathbf{G} \in \langle \mathcal{R}' \rangle$ is a self loop on a cluster C. Since its two end points are from C and all signatures in C are EBD on the same support $\{i, j\}$, we may replace $\mathbf{G}$ with $\mathbf{G}^{* \to \{i,j\}}$. By assumption, $\mathbf{G}^{* \to \{i,j\}}$ is compatible with signatures in C, so we may absorb it into C.

Suppose $\mathbf{G} \in \langle \mathcal{R}' \rangle$ connects two clusters C_1 and C_2 with supports $\{i_1, j_1\}$ and $\{i_2, j_2\}$ respectively. Suppose $\mathbf{F}_1$ and $\mathbf{F}_2$ are the signatures in C_1 and C_2 connected by $\mathbf{G}$ respectively. Let the arity of $\mathbf{F}_1$ be n. Then, we replace all the edges of $\mathbf{F}_1$, except the one connecting to $\mathbf{G}$, with $\mathbf{G}^{-1}\mathbf{G}$. Essentially, we are performing a local holographic transformation, so this does not change the Holant value. The process is described in Fig. 10 for arity 3 case. We now have $\mathbf{G}^{\otimes n}\mathbf{F}_1$ connected directly to $\mathbf{F}_2$. If $|\operatorname{supp} \mathbf{G}^{\otimes n}\mathbf{F}_1 \cap \operatorname{supp} \mathbf{F}_2| \leq 1$, the edge factors in to pinning. Otherwise, by the assumption, $\mathbf{G}^{\otimes n}\mathbf{F}_1$ is compatible with $\mathbf{F}_2$, so we may absorb it into the C_2 cluster.

We choose a cluster to begin with, and repeatedly absorb its neighboring signatures or factor the edge into pinning by the above process. Each step of the above process can be done in polynomial time. Then, in the end, we will be left with multiple connected components in which all the signatures are EBD, and their Holant_2^* values can be computed in polynomial time by the assumption.

5 Outline of Hardness

For the rest of the paper, we prove the hardness results. After a dichotomy of a single ternary signature [14], a natural next step is proving a dichotomy of a pair of ternary and binary signatures (as binary signatures are the 'simplest' signatures after unary signatures), and use it to prove further theorems. However, for domain size 3 in the Holant setting, binary signatures actually allow nontrivial, and somewhat unanticipated, interactions with other signatures. Also, it turns out that a dichotomy for a pair of binary and ternary signatures, while certainly needed on its own, is not easily applicable for showing further dichotomies. We circumvent this difficulty by proving a dichotomy of a pair of ternary signatures directly.

In Section 6, we show the dichotomy of $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$, in which $\mathbf{F}$ is a ternary signature and $\mathbf{G}$ is a binary signature. In Section 7, we show the dichotomy of $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$, in which $\mathbf{F}$ and

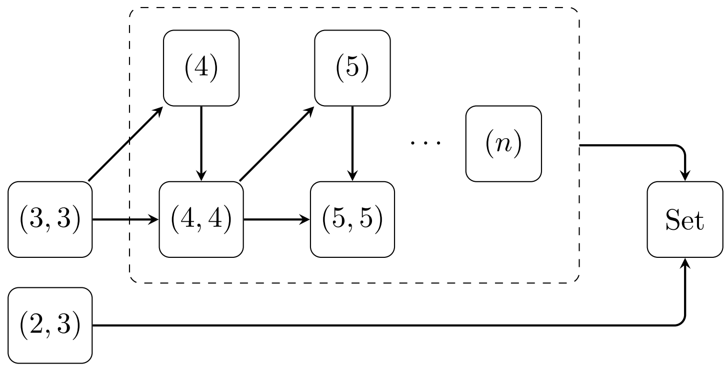


Figure 11: Logical dependency diagram: (n) refers to a dichotomy of $\operatorname{Holant}_3^*(\mathbf{F})$ for an arity-n signature $\mathbf{F}$. (n, m) refers to a dichotomy of $\operatorname{Holant}_3^*(\mathbf{F}, \mathbf{G})$ of an arity-n signature $\mathbf{F}$ and an arity-m signature $\mathbf{G}$. 'Set' refers to the dichotomy of an arbitrary set of signatures.

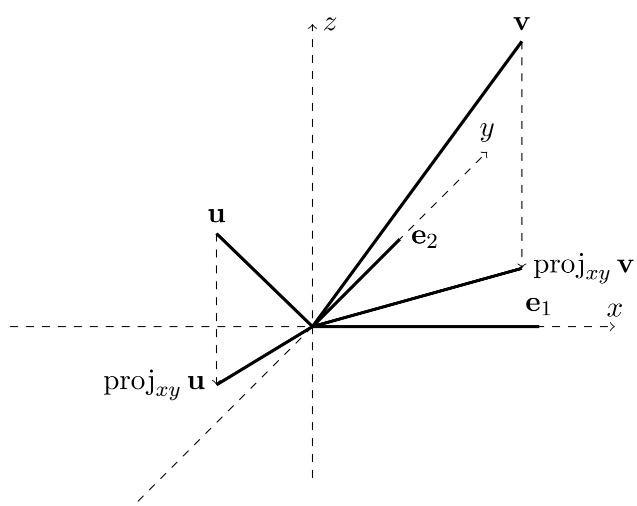


Figure 12: Geometric intuition of domain restriction

$\mathbf{G}$ are both ternary signatures. In Section 8, we show the dichotomy of $\operatorname{Holant}_3^*(\mathbf{F})$ for a signature $\mathbf{F}$ of arbitrary arity ≥ 3. In Section 9, we show the dichotomy of $\operatorname{Holant}_3^*(\mathcal{F})$ for an arbitrary set of signatures $\mathcal{F}$. A logical dependency is described by the diagram in Fig. 11. In each case, we only consider signatures of rank ≥ 2. This is because rank 1 signatures are degenerate and thus can be factored into unary signatures. Since Holant^* assumes the existence of unary signatures, degenerate signatures do not impact the complexity.

In the proof of the dichotomy theorems, by applying the known dichotomy theorems, we argue that to escape #P-hardness the signatures must be of the tractable forms of Theorem 3.1. Therefore, the signatures are either tractable or fail to escape hardness, meaning they are #P-hard. Hence, we obtain a dichotomy.

The main intuition behind the proof of hardness is the geometry of the vectors in a tensor decomposition of a signature. We always start with a canonical form of a signature (after a suitable orthogonal transformation), for example, $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}$. By Proposition 6.5 and Proposition 6.4, $\mathbf{F}$ can realize $(=_{\mathsf{BG}})$ and take domain restriction to $\{\mathsf{B}, \mathsf{G}\}$. If the other signature is $\mathbf{G} = \mathbf{u}^{\otimes 3} + \mathbf{v}^{\otimes 3}$ for orthogonal $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the domain restriction is $\mathbf{G}^{* \to \{\mathsf{B}, \mathsf{G}\}} = (\operatorname{proj}_{xy} \mathbf{u})^{\otimes 3} + (\operatorname{proj}_{xy} \mathbf{v})^{\otimes 3}$. Essentially, most of the arguments boil down to showing that the angle between $\operatorname{proj}_{xy} \mathbf{u}$ and $\operatorname{proj}_{xy} \mathbf{v}$ must be

$\pi/2$ (with one exception), otherwise, $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is #P-hard by the Boolean domain dichotomy. The other tractable possibility is when $\operatorname{proj}_{xy}\mathbf{u} \sim \operatorname{proj}_{xy}\mathbf{v}$, which can only happen if the plane that contains $\mathbf{u}$ and $\mathbf{v}$ contains the z-axis, which corresponds to the case when $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is degenerate.

6 A Pair of Ternary and Binary Signatures

We show the dichotomy of $\operatorname{Holant}_3^*(\mathbf{F},\mathbf{G})$ for a ternary signature $\mathbf{F}$ and a binary signature $\mathbf{G}$. We may assume that $\mathbf{F}$ is of the tractable form in Theorem 2.10, and we separate into four cases, in which $\mathbf{F}$ is: type $\mathfrak{A}$ and rank 2; type $\mathfrak{B}$ and rank 2; type $\mathfrak{A}$ and rank 3; type $\mathfrak{B}$ and rank 3. Note that we do not need to consider a rank 1 signature since such signature is degenerate. In addition, we may assume that $\mathbf{F}$ is in the canonical form of $a\mathbf{e}_1^{\otimes 3}+b\mathbf{e}_2^{\otimes 3}+c\mathbf{e}_3^{\otimes 3}$ or $\boldsymbol{\beta}_0^{\otimes 3}+\overline{\boldsymbol{\beta}_0}^{\otimes 3}+\lambda\mathbf{e}_3^{\otimes 3}$ for some $a,b,c,\lambda\in\mathbb{R}$.

We prove some simple propositions about normalizing the individual scalar constants of the tensor decomposition and realizing similar signatures. This will allow us to assume that a type $\mathfrak{A}$ signature has the form $a\mathbf{e}_1^{\otimes 3}+b\mathbf{e}_2^{\otimes 3}+c\mathbf{e}_3^{\otimes 3}$ for $a,b,c\in\{0,1\}$ and a type $\mathfrak{B}$ signature has the form $\boldsymbol{\beta}_0^{\otimes 3}+\overline{\boldsymbol{\beta}_0}^{\otimes 3}+\lambda\mathbf{e}_3^{\otimes 3}$ for $\lambda\in\{0,1\}$.

Proposition 6.1. *Let $\mathbf{F}=\mathbf{v}_1^{\otimes 3}+\mathbf{v}_2^{\otimes 3}+\mathbf{v}_3^{\otimes 3}$ such that $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\in\mathbb{R}^3$ are pairwise orthogonal. Let $\mathcal{F}$ be any set of signatures. Let $a,b,c\in\mathbb{R}$ be arbitrary. If $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\neq 0$, then*

$$\operatorname{Holant}_3^*(\mathcal{F}\cup\{a\mathbf{v}_1^{\otimes 3}+b\mathbf{v}_2^{\otimes 3}+c\mathbf{v}_3^{\otimes 3}\})\le_T \operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{F}\}).$$

If $\mathbf{v}_1,\mathbf{v}_2\neq 0$ and $\mathbf{v}_3=0$, then for any $a,b\in\mathbb{R}$,

$$\operatorname{Holant}_3^*(\mathcal{F}\cup\{a\mathbf{v}_1^{\otimes 3}+b\mathbf{v}_2^{\otimes 3}\})\le_T \operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{F}\}).$$

Proof. TOPROVE 4 ☐

Proposition 6.2. *Let $\mathbf{F}=\boldsymbol{\beta}^{\otimes 3}+\overline{\boldsymbol{\beta}}^{\otimes 3}+\lambda\mathbf{e}_3^{\otimes 3}$ for $\lambda\in\mathbb{R}$ where $\boldsymbol{\beta}=\frac{1}{\sqrt{2}}(1,i,0)$. Let $\mathcal{F}$ be any set of signatures. If $\lambda\neq 0$, then*

$$\operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{G}\})\le_T \operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{F}\})$$

for any real valued symmetric ternary $\mathsf{BG|R}$ *signature $\mathbf{G}$ such that $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is of type* II, *i.e., $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}=[x,y,-x,-y]$ for some $x,y\in\mathbb{R}$.*

If $\lambda=0$, then

$$\operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{G}\})\le_T \operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{F}\})$$

for any real valued symmetric ternary signature $\mathbf{G}$ such that $\operatorname{supp}\mathbf{G}\subseteq\{\mathsf{B},\mathsf{G}\}^*$ *and $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is of type* II, *i.e., $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}=[x,y,-x,-y]$ for some $x,y\in\mathbb{R}$.*

Proof. TOPROVE 5 ☐

Corollary 6.3. *Let $\mathbf{F}=\boldsymbol{\beta}^{\otimes 3}+\overline{\boldsymbol{\beta}}^{\otimes 3}+\lambda\mathbf{e}_3^{\otimes 3}$ for $\lambda\in\mathbb{R}$ where $\boldsymbol{\beta}=\frac{1}{\sqrt{2}}(1,i,0)$. Let $\mathbf{G}=T^{\otimes 3}\mathbf{F}$ for some real 3×3 orthogonal* $\mathsf{BG|R}$ *matrix. In particular, $\mathbf{G}=(\mathbf{u}+i\mathbf{v})^{\otimes 3}+(\mathbf{u}-i\mathbf{v})^{\otimes 3}+\lambda\mathbf{e}_3^{\otimes 3}$ such that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{e}_3$ form an orthonormal basis of $\mathbb{R}^3$. Let $\mathcal{F}$ be any set of signatures. Then,*

$$\operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{F}\}) =_T \operatorname{Holant}_3^*(\mathcal{F}\cup\{\mathbf{G}\}).$$

The main strategy of proving hardness is by showing that domain restriction can be realized within $\mathcal{F}$ and using the Boolean domain dichotomy.

Proposition 6.4. *Let $\mathcal{F}$ be a set of signatures such that $(=_{\mathsf{BG}}) \in \mathcal{F}$. Then,*

$$\operatorname{Holant}_2^*(\mathcal{F}^{*\to\{\mathsf{B},\mathsf{G}\}}) \leq_T \operatorname{Holant}_3^*(\mathcal{F}).$$

Proof. TOPROVE 6 □

Proposition 6.5. *Let $\mathbf{F} = (a,b,0)^{\otimes 3} + (c,d,0)^{\otimes 3}$ for $a,b,c,d \in \mathbb{R}$ such that $(a,b,0)$ and $(c,d,0)$ are nonzero orthogonal vectors. Let $\mathcal{F}$ be any set of signatures containing $\mathbf{F}$. Then,*

$$\operatorname{Holant}_3^*(\mathcal{F} \cup \{(=_{\mathsf{BG}})\}) \leq_T \operatorname{Holant}_3^*(\mathcal{F}).$$

Proof. TOPROVE 7 □

Proposition 6.6. *Let $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}$ where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1,i,0)$. Let $\mathcal{F}$ be any set of signatures containing $\mathbf{F}$. Then,*

$$\operatorname{Holant}_3^*(\mathcal{F} \cup \{(=_{\mathsf{BG}})\}) \leq_T \operatorname{Holant}_3^*(\mathcal{F}).$$

Proof. TOPROVE 8 □

6.1 Rank 2 Type $\mathfrak{A}$

Let $\mathbf{F}$ be a type $\mathfrak{A}$ signature of rank 2. After reordering the domain, we may assume that $\mathbf{F} = a\mathbf{e}_1^{\otimes 3} + b\mathbf{e}_2^{\otimes 3}$, for nonzero $a,b \in \mathbb{R}$. By Proposition 6.1, we may assume $a,b = 1$.

Lemma 6.7. *Let $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}$. Let $\mathbf{G}$ be a nondegenerate, real-valued, symmetric, binary domain 3 signature. Then, $\operatorname{Holant}_3^*(\mathbf{F},\mathbf{G})$ is computable in polynomial time if one of the following conditions holds. Otherwise, it is #P-hard.*

1. *$\mathbf{G}$ is $\mathsf{BG|R}$ and $\operatorname{Holant}_2^*(\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}, [1,0,0,1])$ is tractable.*
2. *$\mathbf{G}$ is $\mathsf{BR|G}$ or $\mathsf{GR|B}$.*
3. *$\mathbf{G}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$, $\mathsf{Swap}_{\mathsf{GR;B}}$, or $\mathsf{Swap}_{\mathsf{BR;G}}$.*
4. *$\mathbf{G}$ is in $\mathcal{D}$.*
5. *$\mathbf{G} = c\begin{bmatrix} 1 & x & -x\alpha \\ x & x^2 & \alpha \\ -x\alpha & \alpha & 0 \end{bmatrix}$ for $\alpha = \pm\sqrt{1+x^2}$ and nonzero $c, x \in \mathbb{R}$.*

Proof. TOPROVE 9 □

6.2 Rank 3 Type $\mathfrak{A}$

We first prove a lemma about $\mathcal{D}$ signatures when a rank 2 GenEQ with a different support is present.

Lemma 6.8. *Let $\mathbf{F} = \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ and $\mathbf{G} \in \mathcal{D}$ be symmetric. Then, $\operatorname{Holant}_3^*(\mathbf{F},\mathbf{G})$ is computable in polynomial time if $\operatorname{Holant}_2^*(\mathbf{G}^{*\to\{\mathsf{G},\mathsf{R}\}}, [1,0,0,1])$ is tractable and $\mathbf{G}$ is degenerate, $\mathsf{BG|R}$, $\mathsf{Swap}_{\mathsf{BG;R}}$, or EBD. Otherwise, it is #P-hard.*

Proof. TOPROVE 10 □

Let $\mathbf{F}$ be a type $\mathfrak{A}$ signature of rank 3, so $\mathbf{F} = a\mathbf{e}_1^{\otimes 3} + b\mathbf{e}_2^{\otimes 3} + c\mathbf{e}_3^{\otimes 3}$, for nonzero $a, b, c \in \mathbb{R}$. By Proposition 6.1, we may assume $a, b, c = 1$.

Lemma 6.9. *Let $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. Let $\mathbf{G}$ be a nondegenerate, real-valued, symmetric, binary domain 3 signature. Then, $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is computable in polynomial time if one of the following conditions holds. Otherwise, it is #P-hard.*

1. *For some $i, j, k \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$, $\mathbf{G}$ is $ij|k$ and $\text{Holant}_2^*(\mathbf{G}^{*\to\{i,j\}}, [1, 0, 0, 1])$ is tractable.*

2. *$\mathbf{G}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$, $\mathsf{Swap}_{\mathsf{GR;B}}$, or $\mathsf{Swap}_{\mathsf{BR;G}}$.*

Proof. TOPROVE 11 □

6.3 Rank 2 Type $\mathfrak{B}$

Lemma 6.10. *Let $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}$ where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)^{\intercal}$. Let $\mathbf{G}$ be a nondegenerate, real-valued, symmetric, binary domain 3 signature. Then, $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is computable in polynomial time if one of the following conditions holds. Otherwise, it is #P-hard.*

1. *$\mathbf{G}$ is $\mathsf{BG}|\mathsf{R}$ and $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is tractable with type II signatures.*

2. *$\mathbf{G}$ is $c\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{bmatrix}$ or $c\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 0 \end{bmatrix}$ for $\alpha = \pm 1$ and nonzero $c \in \mathbb{R}$.*

3. *$\mathbf{G}$ is in $\mathcal{D}$.*

4. *$\mathbf{G} = c\begin{bmatrix} 1 & x & -x\alpha \\ x & x^2 & \alpha \\ -x\alpha & \alpha & 0 \end{bmatrix}$ for $\alpha = \pm\sqrt{1 + x^2}$ and nonzero $c, x \in \mathbb{R}$.*

Proof. TOPROVE 12 □

6.4 Rank 3 Type $\mathfrak{B}$

Let $\mathbf{F}$ be a type $\mathfrak{B}$ signature of rank 3, so $\mathbf{F} = (1, i, 0)^{\otimes 3} + (1, -i, 0)^{\otimes 3} + \lambda\mathbf{e}_3^{\otimes 3}$ for some $\lambda \in \mathbb{R}$. By Proposition 6.2, we may assume that $\lambda = 1$.

Lemma 6.11. *Let $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)^{\intercal}$. Let $\mathbf{G}$ be a nondegenerate, real-valued, symmetric, binary domain 3 signature. Then, $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is computable in polynomial time if one of the following conditions holds. Otherwise, it is #P-hard.*

1. *$\mathbf{G}$ is $\mathsf{BG}|\mathsf{R}$ and $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is tractable with type II signatures.*

2. *$\mathbf{G}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$.*

Proof. TOPROVE 13 □

7 Two Ternary Signatures

We show the dichotomy of $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ for two ternary signatures $\mathbf{F}$ and $\mathbf{G}$. After an orthogonal transformation, we may assume that $\mathbf{F}$ is in the canonical form in Theorem 2.10. We separate into the cases when at least one of the signatures is rank 3 and when both are rank 2. Note that we do not need to consider a rank 1 signature since such signature is degenerate. Further, we separate into the cases when $\mathbf{F}$ is of type $\mathfrak{A}$ and type $\mathfrak{B}$. Similar to Section 6, we normalize the individual constants occurring in tensor decomposition using Proposition 6.1 and Proposition 6.2. Note that in the proof in this section, we do not refer to the dichotomy theorems proved in Section 6.

7.1 Rank 2 Type $\mathfrak{A}$

Lemma 7.1. *Let $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}$. Let $\mathbf{G}$ be a nondegenerate, real-valued, symmetric, ternary signature of rank 2. Then, $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is computable in polynomial time if one of the following conditions holds. Otherwise, it is #P-hard.*

1. *$\mathbf{G}$ is GenEQ.*
2. *For some nonzero $c \in \mathbb{R}$, $c\mathbf{G} = \mathbf{e}_i^{\otimes 3} + \mathbf{v}^{\otimes 3}$ for $\mathbf{v} \in \mathbb{R}^3$ such that $\langle \mathbf{e}_i, \mathbf{v} \rangle = 0$.*
3. *$\mathbf{G} = \mathbf{u}^{\otimes 3} + \mathbf{v}^{\otimes 3}$ for $\mathbf{v} = (v_1, v_2, v_3), \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ such that (u_1, u_2) and (v_1, v_2) are linearly dependent and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.*
4. *$\mathbf{G} = (\mathbf{u} + i\mathbf{v})^{\otimes 3} + (\mathbf{u} - i\mathbf{v})^{\otimes 3}$ for $\mathbf{v} = (v_1, v_2, v_3), \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ such that (u_1, u_2) and (v_1, v_2) are linearly dependent, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle$.*

Proof. TOPROVE 14 □

7.2 Rank 3 Type $\mathfrak{A}$

Lemma 7.2. *Let $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. Let $\mathbf{G}$ be a nondegenerate, real-valued, symmetric, ternary signature. Then, $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is computable in polynomial time if one of the following conditions holds. Otherwise, it is #P-hard.*

1. *$\mathbf{G}$ is GenEQ.*
2. *For some nonzero $c \in \mathbb{R}$, $c\mathbf{G} = \mathbf{e}_i^{\otimes 3} + \mathbf{v}^{\otimes 3}$ for $\mathbf{v} \in \mathbb{R}^3$ such that $\langle \mathbf{e}_i, \mathbf{v} \rangle = 0$.*

Proof. TOPROVE 15 □

7.3 Rank 2 Type $\mathfrak{B}$

Before the main lemma, we prove a simple proposition about a type II Boolean domain signature.

Proposition 7.3. *Let $\mathbf{F}$ be a real-valued type II Boolean domain signature. If $\mathbf{F} = (\mathbf{u} + i\mathbf{v})^{\otimes 3} + (\mathbf{u} - i\mathbf{v})^{\otimes 3}$ for some linearly independent $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, then it must be the case that $\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.*

Proof. TOPROVE 16 □

Lemma 7.4. *Let $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}$ where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)$. Let $\mathbf{G}$ be a nondegenerate, real-valued, symmetric, ternary signature of rank 2. Then, $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is computable in polynomial time if one of the following conditions holds. Otherwise, it is #P-hard.*

1. $\mathbf{G}$ *is* $\mathsf{BG|R}$ *and* $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ *is compatible with type* II *signatures.*

2. $\mathbf{G} = \mathbf{u}^{\otimes 3} + \mathbf{v}^{\otimes 3}$ *for* $\mathbf{v} = (v_1, v_2, v_3), \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ *such that* (u_1, u_2) *and* (v_1, v_2) *are linearly dependent and* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

3. $\mathbf{G} = (\mathbf{u} + i\mathbf{v})^{\otimes 3} + (\mathbf{u} - i\mathbf{v})^{\otimes 3}$ *for* $\mathbf{v} = (v_1, v_2, v_3), \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ *such that* (u_1, u_2) *and* (v_1, v_2) *are linearly dependent,* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ *and* $\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle$.

Proof. TOPROVE 17 □

7.4 Rank 3 Type $\mathfrak{B}$

Lemma 7.5. *Let* $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ *where* $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)$. *Let* $\mathbf{G}$ *be a nondegenerate, real-valued, symmetric, ternary signature. Then,* $\mathrm{Holant}_3^*(\mathbf{F}, \mathbf{G})$ *is computalbe in polynomial time if* $\mathbf{G}$ *is* $\mathsf{BG|R}$ *and* $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ *is compatible with type* II *signatures. Otherwise, it is* #P*-hard.*

Proof. TOPROVE 18 □

8 A Single Signature Dichotomy

In this section, we prove a dichotomy of $\mathrm{Holant}_3^*(\mathbf{F})$ for an $\mathbf{F}$ of arbitrary arity n. The case $n \leq 3$ was proved in [14]. Let $n \geq 4$. Similar to the Boolean domain case, it is natural to expect that higher arity tractable signatures also have the same tensor decomposition form, i.e. $a\mathbf{e}_1^{\otimes n} + b\mathbf{e}_2^{\otimes n} + c\mathbf{e}_3^{\otimes n}$ or $\boldsymbol{\beta}^{\otimes n} + \overline{\boldsymbol{\beta}}^{\otimes n} + \lambda\mathbf{e}_3^{\otimes n}$ after some orthogonal transformation. We show that indeed this is the case.

The proof is an induction on the arity. First, we show that a signature of arity 4 must be of the same form, using the dichotomy of two ternary signatures. To do so, we use the fact that the set $\{\langle \mathbf{F}, \mathbf{u} \rangle : \mathbf{u} \in \mathbb{R}^3\}$ is a vector space. That allows us to add signatures, and we argue that unless the signatures are perfectly aligned, the sum of signatures cannot be tractable using the dichotomy of two ternary signatures of Section 7. The reason for this roundabout strategy, compared to the proof of the Boolean domain case, is that it is hard to characterize how a non-tractable signature may look like, since the dichotomy theorems we have are not explicitly defined on the entries of the signatures, but talk about the tensor decomposition form.

Once we show that arity 4 signatures must have tensor decomposition of at most 3 linearly independent vectors, then we argue that the dichotomy of two arity 4 signatures must be essentially same as the dichotomy of the two ternary signatures, since unary signatures allow us to decrease the arity, while preserving the vectors in the tensor decomposition. Finally, we notice that in the proof of the above two statements, the fact that the arities of the signatures were 3 and 4 does not matter, and the proof can be made inductive. The logical dependency is visualized in the dashed box of Fig. 11.

8.1 Subspace of Signatures

Let $\mathbf{F}$ be a real-valued symmetric signature of arity 4. Consider the set $\mathscr{F} = \{\langle \mathbf{F}, \mathbf{u} \rangle : \mathbf{u} \in \mathbb{R}^3\}$. Note that $\mathbf{u} \mapsto \langle \mathbf{F}, \mathbf{u} \rangle$ is a linear map from $\mathbb{R}^3$ to $\mathsf{S}^3(\mathbb{C}^3)$, the space of complex-valued symmetric signatures of arity 3. In particular, $\mathscr{F}$ is a vector space, so it is closed under linear combinations. Also, $\mathscr{F}$ only consists of real-valued signatures if $\mathbf{F}$ is a real-valued signature. Let $\mathsf{S}^3_{\mathbb{R}}$ denote the set of all real-valued symmetric ternary signatures.

We show a dichotomy for an arbitrary subspace $\mathscr{F} \subseteq \mathsf{S}^3_{\mathbb{R}}$. For a nonempty $\mathscr{F}$, each ternary signature must be a tractable signature, otherwise the problem is already #P-hard. We prove this dichotomy for a subspace $\mathscr{F}$ by considering each canonical form such a tractable signature can take under a holographic transformation T. Note that $T\mathscr{F}$ is also a subspace.

Proposition 8.1. *Let $\mathbf{u} \in \mathbb{R}^2$ be a nonzero vector. Let $\mathbf{F} = \mathbf{v}_1^{\otimes 3} + \mathbf{v}_2^{\otimes 3}$ be a Boolean domain signature for nonzero $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ such that $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. Let $\mathbf{G} = \mathbf{F} + \mathbf{u}^{\otimes 3}$. Then, $\text{Holant}_2^*(\mathbf{F}, \mathbf{G})$ is #P-hard unless $\mathbf{u} \sim \mathbf{v}_1$ or $\mathbf{u} \sim \mathbf{v}_2$.*

Proof. TOPROVE 19 □

Proposition 8.2. *Let $(a, b) \in \mathbb{R}^2$ be a nonzero vector. Let $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}$ be a Boolean domain signature where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i)^{\mathsf{T}}$. Let $\mathbf{G} = \mathbf{F} + (a, b)^{\otimes 3}$. Then, $\text{Holant}_2^*(\mathbf{F}, \mathbf{G})$ is #P-hard.*

Proof. TOPROVE 20 □

Proposition 8.3. *Let $\mathbf{F}$ and $\mathbf{G}$ be nondegenerate, real valued, symmetric, ternary signatures such that $\operatorname{supp} \mathbf{F} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$ and $\operatorname{supp} \mathbf{G} \subseteq \{\mathsf{G}, \mathsf{R}\}^*$. Let $\mathbf{H} = \mathbf{F} + \mathbf{G}$. Then, $\text{Holant}_3^*(\mathbf{F}, \mathbf{G}, \mathbf{H})$ is #P-hard unless $\mathbf{F}, \mathbf{G}$ are both* GenEQ.

Proof. TOPROVE 21 □

8.1.1 Rank 3 Type $\mathfrak{A}$

Note that in the following lemmas in Section 8.1, we do not normalize the individual constants using Proposition 6.1 and Proposition 6.2. This is because the although the normalized signature is *realizable* using $\mathscr{F}$, it may not *belong* to $\mathscr{F}$.

Lemma 8.4. *Let $\mathscr{F}$ be a subspace of $\mathsf{S}^3_{\mathbb{R}}$. Suppose $\mathbf{F} = c_1\mathbf{e}_1^{\otimes 3} + c_2\mathbf{e}_2^{\otimes 3} + c_3\mathbf{e}_3^{\otimes 3} \in \mathscr{F}$ for nonzero $c_1, c_2, c_3 \in \mathbb{R}$. Then, $\text{Holant}_3^*(\mathscr{F})$ is #P-hard unless every $\mathbf{G} \in \mathscr{F}$ is a* GenEQ.

Proof. TOPROVE 22 □

8.1.2 Rank 3 Type $\mathfrak{B}$

Lemma 8.5. *Let $\mathscr{F}$ be a subspace of $\mathsf{S}^3_{\mathbb{R}}$. Suppose $\mathbf{F} = c(\boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}) + \lambda\mathbf{e}_3^{\otimes 3} \in \mathscr{F}$ where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)^{\mathsf{T}}$ and nonzero $c, \lambda \in \mathbb{R}$. Then, $\text{Holant}_3^*(\mathscr{F})$ is #P-hard unless every $\mathbf{G} \in \mathscr{F}$ is such that $\mathbf{G}$ is* $\mathsf{BG}|\mathsf{R}$ *and $\mathbf{G}^{* \to \{\mathsf{B},\mathsf{G}\}} = [x, y, -x, -y]$ for some $x, y \in \mathbb{R}$.*

Proof. TOPROVE 23 □

8.1.3 Rank 2 Type $\mathfrak{A}$

Lemma 8.6. *Let $\mathscr{F}$ be a subspace of $\mathsf{S}^3_{\mathbb{R}}$ such that all signatures in $\mathscr{F}$ are of rank at most 2. Suppose $\mathbf{F} = c_1\mathbf{e}_1^{\otimes 3} + c_2\mathbf{e}_2^{\otimes 3} \in \mathscr{F}$ for nonzero $c_1, c_2 \in \mathbb{R}$. Then, $\text{Holant}_3^*(\mathscr{F})$ is #P-hard unless every $\mathbf{G} \in \mathscr{F}$ is $d_1\mathbf{e}_1^{\otimes 3} + d_2\mathbf{e}_2^{\otimes 3}$ for some $d_1, d_2 \in \mathbb{R}$.*

Proof. TOPROVE 24 □

8.1.4 Rank 2 Type $\mathfrak{B}$

Lemma 8.7. *Let $\mathscr{F}$ be a subspace of $\mathsf{S}^3_{\mathbb{R}}$ such that all signatures in $\mathscr{F}$ are of rank at most 2. Suppose $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3} \in \mathscr{F}$ where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)^{\mathsf{T}}$. Then, $\text{Holant}_3^*(\mathscr{F})$ is #P-hard unless every $\mathbf{G} \in \mathscr{F}$ is such that $\operatorname{supp} \mathbf{G} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$ and $\mathbf{G}^{* \to \{\mathsf{B},\mathsf{G}\}} = [x, y, -x, -y]$ for some $x, y \in \mathbb{R}$.*

Proof. TOPROVE 25 □

8.1.5 Rank 1

Lemma 8.8. *Let $\mathscr{F}$ be a subspace of $\mathsf{S}^3_{\mathbb{R}}$ such that all signatures in $\mathscr{F}$ are of rank at most* 1*. Then, there exists some $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{F} = \{\lambda \mathbf{v}^{\otimes 3} : \lambda \in \mathbb{R}\}$.*

Proof. TOPROVE 26 ☐

8.2 A Single Signature of Arity 4

Lemma 8.9. *Let $\mathbf{F}$ be a nondegenerate, real-valued, symmetric signature of arity* 4*. Then,* $\text{Holant}^*_3(\mathbf{F})$ *is computable in polynomial time if there exists some real orthogonal matrix T such that one of the following conditions holds. Otherwise, it is* #P*-hard.*

1. $T^{\otimes 4}\mathbf{F} = a\mathbf{e}_1^{\otimes 4} + b\mathbf{e}_2^{\otimes 4} + c\mathbf{e}_3^{\otimes 4}$ *for some $a, b, c \in \mathbb{R}$.*
2. $T^{\otimes 4}\mathbf{F} = \boldsymbol{\beta}^{\otimes 4} + \overline{\boldsymbol{\beta}}^{\otimes 4} + \lambda\mathbf{e}_3^{\otimes 4}$ *where $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)^\intercal$ for some $\lambda \in \mathbb{R}$.*

Proof. TOPROVE 27 ☐

8.3 Arbitrary Arity

Proposition 8.10. *Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ be pairwise orthogonal vectors. Let $\mathbf{F}_k = \mathbf{v}_1^{\otimes k} + \mathbf{v}_2^{\otimes k} + \mathbf{v}_3^{\otimes k}$. Then, for any $k \geq 3$ and any set of signatures $\mathscr{F}$,*

$$\text{Holant}^*_3(\mathscr{F} \cup \{\mathbf{F}_k\}) =_T \text{Holant}^*_3(\mathscr{F} \cup \{\mathbf{F}_i : i \geq 3\})\,.$$

Proof. TOPROVE 28 ☐

Proposition 8.11. *Let $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)^\intercal$ and $\lambda \in \mathbb{R}$. Let $\mathbf{F}_k = \boldsymbol{\beta}^{\otimes k} + \overline{\boldsymbol{\beta}}^{\otimes k} + \lambda\mathbf{e}_3^{\otimes k}$. Then, for any $k \geq 3$ and any set of signatures $\mathscr{F}$,*

$$\text{Holant}^*_3(\mathscr{F} \cup \{\mathbf{F}_k\}) =_T \text{Holant}^*_3(\mathscr{F} \cup \{\mathbf{F}_i : i \geq 3\})\,.$$

Proof. TOPROVE 29 ☐

We state a meta claim about the dichotomy theorems of two signatures of same arity.

Claim 1. *Let $\mathbf{F}, \mathbf{G}$ be a pair of real-valued symmetric arity* 4 *signatures. Then, the dichotomy statement of* $\text{Holant}^*_3(\mathbf{F}, \mathbf{G})$ *is same as that of two ternary signatures in Lemmas 7.1, 7.2, 7.4 and 7.5, after changing the tensor powers from* 3 *to* 4*.*

Proof. TOPROVE 30 ☐

One can inductively prove the following using the arguments in Section 8.1 and Lemma 8.9:

Lemma 8.12. *Let $\mathbf{F}$ be a nondegenerate real-valued symmetric signature of arity $n \geq 3$. Then,* $\text{Holant}^*_3(\mathbf{F})$ *is tractable if there exists some real orthogonal matrix T such that one of the following conditions holds.*

1. $T^{\otimes n}\mathbf{F} = a\mathbf{e}_1^{\otimes n} + b\mathbf{e}_2^{\otimes n} + c\mathbf{e}_3^{\otimes n}$ *for some $a, b, c \in \mathbb{R}$.*
2. $T^{\otimes n}\mathbf{F} = \boldsymbol{\beta}^{\otimes n} + \overline{\boldsymbol{\beta}}^{\otimes n} + \lambda\mathbf{e}_3^{\otimes n}$ *where $\beta = \frac{1}{\sqrt{2}}(1, i, 0)^\intercal$ for some $\lambda \in \mathbb{R}$.*

Otherwise $\text{Holant}^*_3(\mathbf{F})$ *is* #P*-hard.*

From now on, for notational convenience, we may only consider signatures of arity 2 and 3 when proving a dichotomy of a set of signatures $\mathcal{F}$.

Corollary 8.13. *Let $\mathcal{F}$ be an arbitrary set of nondegenerate, real-valued, symmetric signatures. Suppose that for all $\mathbf{F} \in \mathcal{F}$, the arity of $\mathbf{F}$ is ≥ 3 and $\operatorname{Holant}_3^*(\mathbf{F})$ is tractable. Then, there exists a set $\mathcal{F}'$ such that $\mathcal{F}'$ consists of nondegenerate, real-valued, symmetric signatures of arity 3 and $\operatorname{Holant}_3^*(\mathcal{F}') =_T \operatorname{Holant}_3^*(\mathcal{F})$.*

Proof. TOPROVE 31 □

9 Set of Signatures

We are now almost ready to prove the main theorem of the paper, a complexity dichotomy for an arbitrary set of real-valued symmetric signatures, Theorem 3.1.

Proposition 9.1. *Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ be nonzero vectors such that (u_1, u_2) and (v_1, v_2) are linearly dependent and $\langle u, v\rangle = 0$. Then, $\mathbf{u} \sim (px, py, -q)$ and $\mathbf{v} \sim (qx, qy, p)$ for some p, q, x, y such that $x^2 + y^2 = 1$ and $p^2 + q^2 = 1$.*

Proof. TOPROVE 32 □

Proposition 9.2. *Let $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}$ and $\mathbf{G} = \mathbf{e}_1^{\otimes 3} + (0, a, b)^{\otimes 3}$ for nonzero $a, b \in \mathbb{R}$. Let $\mathbf{H}$ be a nondegenerate, real-valued, symmetric, ternary signature of rank 2. Then, $\operatorname{Holant}_3^*(\mathbf{F}, \mathbf{G}, \mathbf{H})$ is tractable if $\mathbf{H}$ is $\mathsf{GR|B}$. Otherwise, it is #P-hard.*

Proof. TOPROVE 33 □

Proposition 9.3. *Let $\mathbf{F} = (a, b, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ for nonzero $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Let $\mathbf{G}$ be a nondegenerate symmetric signature in $\mathcal{D}$. Then, $\operatorname{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is #P-hard unless one of the following conditions holds.*

1. *$\mathbf{G}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$.*
2. *$\mathbf{G}$ is $\mathsf{BG|R}$.*
3. *for $T = \begin{bmatrix} -b & a & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $T^{\otimes 2}\mathbf{G}$ is EBD.*

Proof. TOPROVE 34 □

We are now ready to formalize the idea behind Fig. 7. We define a notion of a plane of a rank 2 signature. A rank 2 signature of type $\mathfrak{A}$ or type $\mathfrak{B}$ of Theorem 2.10 has a symmetric tensor decomposition such that the two vectors occurring inside it are orthogonal, i.e. $\mathbf{v}_1^{\otimes 3} + \mathbf{v}_2^{\otimes 3}$ or $(\mathbf{v}_1 + i\mathbf{v}_2)^{\otimes 3} + (\mathbf{v}_1 - i\mathbf{v}_2)^{\otimes 3}$. The two vectors are unique up to a scalar, so the plane of a signature $\mathbf{F}$, denoted $P_{\mathbf{F}} = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$ is well defined.

Definition 9.4. *Let $\mathbf{F}$ be a rank 2 signature of type $\mathfrak{A}$ or type $\mathfrak{B}$. If $\mathbf{F}$ is type $\mathfrak{A}$, we may write $\mathbf{F} = \mathbf{v}_1^{\otimes 3} + \mathbf{v}_2^{\otimes 3}$. If $\mathbf{F}$ is type $\mathfrak{B}$, we may write $\mathbf{F} = (\mathbf{v}_1 + i\mathbf{v}_2)^{\otimes 3} + (\mathbf{v}_1 - i\mathbf{v}_2)^{\otimes 3}$. We denote the plane of $\mathbf{F}$ as $P_{\mathbf{F}} = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$.*

Note that $\mathbf{F}$ is EBD if and only if $P_{\mathbf{F}}$ is one of the coordinate planes, i.e. xy-plane, yz-plane or xz-plane. We will say that two planes are orthogonal if their normal vectors are orthogonal. Note that this notion is well defined because a normal vector to a plane in 3 dimensional space is unique up to scalar.

Proposition 9.5. *Let $\mathcal{F}$ be a set of rank 2 signatures of type $\mathfrak{A}$ or type $\mathfrak{B}$. The following two statements are equivalent:*

1. *For any orthogonal T, there is some signature $\mathbf{F} \in T\mathcal{F}$ that is not EBD.*
2. *There exist $\mathbf{F}, \mathbf{G} \in \mathcal{F}$ such that $P_{\mathbf{F}}$ and $P_{\mathbf{G}}$ are not equal or orthogonal.*

Proof. TOPROVE 35 □

The next lemma encapsulates the essence of the dichotomy theorems of two rank 2 ternary signatures in Section 7.

Lemma 9.6. *Let $\mathcal{F}$ be a set of rank 2 signatures of type $\mathfrak{A}$ or type $\mathfrak{B}$. Suppose that for every orthogonal matrix T, neither of the following holds:*

1. *for all $\mathbf{F} \in T\mathcal{F}$, $\mathbf{F}$ is $\mathsf{GR|B}$.*
2. *all signatures in $T\mathcal{F}$ are EBD.*

Then, $\operatorname{Holant}_3^*(\mathcal{F})$ *is* #P*-hard.*

Proof. TOPROVE 36 □

9.1 Proof of the Main Theorem

We are now ready to prove the main theorem. We restate it here for an easy reference.

Theorem 3.1. $\operatorname{Holant}_3^*(\mathcal{F})$ *is computable in polynomial time if there exists a real orthogonal T, such that one of the following conditions holds. In all other cases,* $\operatorname{Holant}_3^*(\mathcal{F})$ *is* #P*-hard.*

$\mathscr{A}$. *Every signature in $\mathcal{F}$ has arity ≤ 2.*

$\mathscr{B}$. *$T\mathcal{F} \subseteq \mathcal{E}$.*

$\mathscr{C}$. (a) *For all $\mathbf{F} \in T\mathcal{F}$ of arity ≥ 3, $\operatorname{supp} \mathbf{F} \subseteq \{\mathsf{B}, \mathsf{G}\}^*$, and*
(b) *For all binary $\mathbf{G} \in T\mathcal{F}$, either $\mathbf{G} \in \mathcal{D}$ or $\mathbf{G}$ is $\mathsf{BG|R}$, and*
(c) $\operatorname{Holant}_2^*((T\mathcal{F})^{*\to\{\mathsf{B},\mathsf{G}\}})$ *is tractable.*

$\mathscr{D}$. (a) *For all $\mathbf{F} \in T\mathcal{F}$ of arity ≥ 3, $\mathbf{F}$ is $\mathsf{BG|R}$, and*
(b) *For all binary $\mathbf{G} \in T\mathcal{F}$, either $\mathbf{G}$ is $\mathsf{BG|R}$ or $\mathbf{G}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$, and*
(c) $\operatorname{Holant}_2^*((T\mathcal{F})^{*\to\{\mathsf{B},\mathsf{G}\}})$ *is tractable.*

$\mathscr{E}$. *Let $\mathcal{F}_{ij} = \{\mathbf{F} \in T\mathcal{F} : \operatorname{supp} \mathbf{F} \subseteq \{i,j\}^*\}$. Let $\mathcal{R} = T\mathcal{F} - (\mathcal{F}_{\mathsf{BG}} \cup \mathcal{F}_{\mathsf{BR}} \cup \mathcal{F}_{\mathsf{GR}})$.*
(a) *$\mathcal{R} \subseteq \mathbb{R}O_h$, and $\langle \mathcal{R}' \rangle \subseteq O_h$, where $\langle \mathcal{R}' \rangle$ is the monoid generated by $\mathcal{R}' = \mathbb{R}\mathcal{R} \cap O_h$, and*
(b) *For all $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$,* $\operatorname{Holant}_2^*(\langle \mathcal{R}' \rangle^{*\to\{i,j\}} \cup \mathcal{F}_{ij})$ *is tractable, and*
(c) *For all $i, j \in \{\mathsf{B}, \mathsf{G}, \mathsf{R}\}$,* $\operatorname{Holant}_2^*((\bigcup_{\mathbf{G} \in \langle \mathcal{R}' \rangle} \mathbf{G}(T\mathcal{F}))^{*\to\{i,j\}})$ *is tractable.*

We will assume that $\text{Holant}_3^*(\mathcal{F})$ is not #P-hard, and show that then, $\mathcal{F}$ must fall into one of the tractable classes in the theorem statement. We may assume that for all $\mathbf{F} \in \mathcal{F}$, $\text{Holant}_3^*(\mathbf{F})$ is tractable. By Proposition 6.1, Proposition 6.2 and Corollary 8.13, we may assume that all signatures are either ternary or binary, and all ternary signatures are written in the tensor decomposed form with unit vectors.

We first explain the organization of the proof. $\mathcal{O}$ is defined in Eq. (3) as the orbit of the ternary signatures under the monoid action by the binary signatures. We list the assumptions to be used subsequently.

1. $\mathcal{F}$ contains a rank 3 type $\mathfrak{B}$ signature.
2. $\mathcal{F}$ contains a rank 3 type $\mathfrak{A}$ signature.
3. $\mathcal{O}$ can be transformed such that all signatures are supported on $\{\mathsf{B}, \mathsf{G}\}$.
4. $\mathcal{O}$ can be transformed such that all signatures are in $\mathcal{E}$.
5. $\mathcal{O}$ can be transformed such that all signatures are $\mathsf{BG|R}$.
6. $\mathcal{O}$ can be transformed such that all signatures are EBD.

With these assumptions, we can show that the following sequence covers all possible cases of $\text{Holant}_3^*(\mathcal{F})$ being tractable using the dichotomy statements proven so far.

- Section 9.1.1: Assume 1 holds.
- Section 9.1.2: Assume 1 does not hold. Assume 2 holds.
- Section 9.1.4: Assume 1, 2 do not hold. Assume 3 holds.
- Section 9.1.5: Assume 1, 2, 3 do not hold. Assume 4 holds.
- Section 9.1.6: Assume 1, 2, 3, 4 do not hold. Assume 5 holds.
- Section 9.1.7: Assume 1, 2, 3, 4, 5 do not hold. Assume 6 holds.

Let $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(1, i, 0)^{\mathsf{T}}$ for the rest of this paper.

9.1.1 Rank 3 Type $\mathfrak{B}$

Suppose $\mathcal{F}$ contains a rank 3 type $\mathfrak{B}$ signature. Then, there exists some real orthogonal matrix T such that $T\mathcal{F}$ contains $\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. We may assume that all other ternary signatures in $T\mathcal{F}$ are of the tractable form in Lemma 7.5. Also, we may assume that all binary signatures in $T\mathcal{F}$ are of the tractable form in Lemma 6.11. Therefore, for all $\mathbf{G} \in T\mathcal{F}$, $\mathbf{G}$ is either $\mathsf{BG|R}$ and $\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}$ is tractable with II signature, or $\mathbf{G}$ is binary and is $\mathsf{Swap}_{\mathsf{BG;R}}$. This means $\mathcal{F}$ is in the tractable class $\mathscr{D}$.

9.1.2 Rank 3 Type $\mathfrak{A}$

Suppose $\mathcal{F}$ contains a rank 3 type $\mathfrak{A}$ signature. Then, there exists some real orthogonal matrix T such that $T\mathcal{F}$ contains $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. We may assume that all other ternary signatures in $T\mathcal{F}$ are of the tractable form in Lemma 7.2. Also, we may assume that all binary signatures in $T\mathcal{F}$ are of the tractable form in Lemma 6.9. If $T\mathcal{F} \subseteq \mathcal{E}$, then we are done since $\mathcal{F}$ is in the tractable class $\mathscr{B}$. So, assume that $T\mathcal{F} \not\subseteq \mathcal{E}$.

Suppose there exists a ternary signature $\mathbf{G} \in T\mathcal{F} \setminus \mathcal{E}$. Since $\text{Holant}_3^*(\mathbf{F}, \mathbf{G})$ is assumed to be not #P-hard, by Lemma 7.2, $\mathbf{G}$ must be of the form $\mathbf{e}_i^{\otimes 3} + \mathbf{v}^{\otimes 3}$ for some $\mathbf{v} \in \mathbb{R}^3$ such that $\langle \mathbf{e}_i, \mathbf{v} \rangle = 0$ and $\mathbf{v} \not\sim \mathbf{e}_j$ for all j. Without loss of generality, assume $i = 3$, so $\mathbf{G} = (a, b, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ for some nonzero $a, b \in \mathbb{R}$. By Proposition 9.2, all other non-GenEQ ternary signatures in $T\mathcal{F}$ must also be of the form $(c, d, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ for some $c, d \in \mathbb{R}$. Suppose there is a binary signature $\mathbf{H} \in T\mathcal{F}$. Then, it must be domain separated or one of $\mathsf{Swap}_{\mathsf{BG;R}}$, $\mathsf{Swap}_{\mathsf{BR;G}}$, or $\mathsf{Swap}_{\mathsf{GR;B}}$. We claim that $\mathbf{H}$ must be $\mathsf{BG|R}$ or $\mathsf{Swap}_{\mathsf{BG;R}}$. By Proposition 9.3, if $\mathbf{H}$ is $\mathsf{Swap}_{\mathsf{BR;G}}$ or $\mathsf{Swap}_{\mathsf{GR;B}}$, then it must be $\mathsf{Swap}_{\mathsf{BG;R}}$ or $\mathsf{BG|R}$ to be tractable. Suppose $\mathbf{H}$ is $\mathsf{BR|G}$ and not $\mathsf{BG|R}$. Then, $\mathbf{H} = \left[\begin{smallmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & w \end{smallmatrix}\right]$ for some $x, y, z, w \in \mathbb{R}$ with $y \neq 0$. If $z \neq 0$, then it must be the case that $x = 0$ since we may realize $\mathbf{H}^{\otimes 3}(\mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}) = (x, 0, y)^{\otimes 3} + (0, z, 0)^{\otimes 3}$, which is not $\mathsf{BG|R}$. Then by Proposition 9.2, $\text{Holant}_3^*(\mathbf{G}, \mathbf{H}^{\otimes 3}(\mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3})$ is #P-hard, contrary to the assumption. Similarly, if $z \neq 0$, then it must be the case that $w = 0$ to be not #P-hard. Then, $\mathbf{H}^{\otimes 3}\mathbf{G} = (0, bz, ay)^{\otimes 3} + (y, 0, 0)^{\otimes 3}$, which again is not of $\mathsf{BG|R}$ form. So, it must be the case that $z = 0$ to be not #P-hard.

If $z = 0$, then, $\mathbf{H}^{\otimes 3}\mathbf{G} = (ax, 0, ay)^{\otimes 3} + (y, 0, w)^{\otimes 3}$. If $x, w = 0$, then $\mathbf{H}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$. Otherwise, $(\mathbf{H}^{\otimes 3}\mathbf{G})^{*\to\{\mathsf{B},\mathsf{R}\}} = (ax, ay)^{\otimes 3} + (y, w)^{\otimes 3}$, and it is not degenerate if $\mathbf{H}$ is not degenerate. Since $(x, w) \neq (0, 0)$, $\text{Holant}_2^*(\mathbf{F}^{*\to\{\mathsf{B},\mathsf{R}\}}, (\mathbf{H}^{\otimes 3}\mathbf{G})^{*\to\{\mathsf{B},\mathsf{R}\}})$ is #P-hard because $(\mathbf{H}^{\otimes 3}\mathbf{G})^{*\to\{\mathsf{B},\mathsf{R}\}})$ is not a GenEQ.

Therefore, any binary $\mathsf{BR|G}$ signature that is not $\mathsf{BG|R}$ must be $\mathsf{Swap}_{\mathsf{BG;R}}$. Similar argument shows that any binary $\mathsf{GR|B}$ signature that is not $\mathsf{BG|R}$ must be $\mathsf{Swap}_{\mathsf{BG;R}}$. Hence $\mathcal{F}$ is in the tractable case $\mathscr{D}$.

Suppose there is no ternary signature in $T\mathcal{F} \setminus \mathcal{E}$. Suppose there is a binary signature $\mathbf{G} \in T\mathcal{F} \setminus \mathcal{E}$. Suppose $\mathbf{G}$ is $\mathsf{BG|R}$, so $\mathbf{G} = \left[\begin{smallmatrix} x & y & 0 \\ y & z & 0 \\ 0 & 0 & w \end{smallmatrix}\right]$ for some $x, y, z, w \in \mathbb{R}$ with $y \neq 0$. Also, we must have $x, z \neq 0$, since if $x, z = 0$, $\mathbf{G} \in \mathcal{E}$, and if only one of them is 0, $\text{Holant}_2^*(\mathbf{G}^{*\to\{\mathsf{B},\mathsf{G}\}}, [1, 0, 0, 1])$ is #P-hard. That means $[x, y, z]$ must be degenerate, and for $\mathbf{G}$ to be nondegenerate, we need $w \neq 0$. Then, $\mathbf{G}^{\otimes 3}(\mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}) = (y, z, 0)^{\otimes 3} + (0, 0, w)^{\otimes 3}$, so we go back to the previous case of having some non-GenEQ ternary signature.

Suppose all binary signatures in $T\mathcal{F} \setminus \mathcal{E}$ are $\mathsf{Swap}_{\mathsf{BG;R}}$, $\mathsf{Swap}_{\mathsf{BR;G}}$, $\mathsf{Swap}_{\mathsf{GR;B}}$. If $\mathbf{G}$ is a such $\mathsf{Swap}_{\mathsf{BG;R}}$ signature, then $\mathbf{G} = \left[\begin{smallmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & 0 \end{smallmatrix}\right]$. By assumption, we must have $x, y \neq 0$, and $\mathbf{G}^{\otimes 3}(\mathbf{e_2}^{\otimes 3} + \mathbf{e}_3^{\otimes 3}) = (x, y, 0)^{\otimes 3} + (0, 0, y)^{\otimes 3}$, so we go back to the previous case.

Therefore, if $T\mathcal{F} \not\subseteq \mathcal{E}$ and is tractable, it must be in class $\mathscr{D}$.

9.1.3 Rank 2

Suppose $\mathcal{F}$ does not contain a rank 3 ternary signature. If there is no rank 2 ternary signature as well, then $\mathcal{F}$ only consists of binary signatures, so it is in the tractable class $\mathscr{A}$. So, assume that there exists some rank 2 ternary signature.

Let $\mathcal{T}$ be the set of ternary signatures in $\mathcal{F}$. Let $\mathcal{B}$ be the set of binary signatures in $\mathcal{F}$. Let $\langle \mathcal{B} \rangle$ be the monoid generated by $\mathcal{B}$ under multiplication. We define $\mathcal{O}$ to be

$$\mathcal{O} := \left\{ \mathbf{G}^{\otimes 3}\mathbf{F} : \mathbf{F} \in \mathcal{T}, \mathbf{G} \in \langle \mathcal{B} \rangle, \mathbf{G}^{\otimes 3}\mathbf{F} \text{ is non-degenerate} \right\}. \tag{3}$$

Combinatorially, $\mathcal{O}$ is the set of all gadgets constructible from connecting the same chain of binary signatures to the three edges of a ternary signatures, $(\mathbf{G}_1\mathbf{G}_2 \cdots \mathbf{G}_k)^{\otimes 3}\mathbf{F}$ for some $\mathbf{G}_i \in \mathcal{B}$ and $\mathbf{F} \in \mathcal{T}$. It can also be viewed as the orbit (ignoring degenerate signatures) of $\mathcal{T}$ under the monoid action of $\langle \mathcal{B} \rangle$, where the action is defined by $\mathbf{G} : \mathbf{F} \mapsto \mathbf{G}^{\otimes 3}\mathbf{F}$ for $\mathbf{G} \in \langle \mathcal{B} \rangle$ and $\mathbf{F} \in \mathcal{T}$. Note that $\mathcal{O}$ is a set of symmetric ternary signatures, and if $\mathbf{G} \in \mathcal{B}$ and $\mathbf{F} \in \mathcal{O}$, then $\mathbf{G}^{\otimes 3}\mathbf{F} \in \mathcal{O}$ as well.

In addition, for any orthogonal matrix T, $T\mathcal{O}$ is the result of the same construction with $T\mathcal{F}$ and $\langle T\mathcal{B}\rangle$. Also, we have $\operatorname{Holant}_3^*(\mathcal{O}\cup\mathcal{B}) =_T \operatorname{Holant}_3^*(\mathcal{F})$.

9.1.4 Rank 2 Class $\mathscr{C}$

Suppose there exists an orthogonal T such that $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B},\mathsf{G}\}^*$ for all $\mathbf{F}\in T\mathcal{O}$. We claim that $\mathcal{F}$ must be in the tractable class $\mathscr{C}$.

1. Suppose there is $\mathbf{F}\in T\mathcal{F}$ such that it is of type $\mathfrak{A}$. Since $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B},\mathsf{G}\}^*$, we may apply a $\mathsf{BG|R}$ orthogonal matrix S such that $\mathbf{F}' = S^{\otimes 3}\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}$. Note that all signatures in $\mathcal{O}' = ST\mathcal{O}$ are supported on $\{\mathsf{B},\mathsf{G}\}$ since S is $\mathsf{BG|R}$. By Proposition 6.5, Proposition 6.4 and Theorem 2.8, it must be the case that all signatures in $\mathcal{O}'$ are also GenEQ. In particular, all signatures in $\mathcal{F}' = ST\mathcal{F}$ are also GenEQ.

   We need to show that $\mathcal{B}' = ST\mathcal{B}$ only consists of binary signatures that are $\mathsf{BG|R}$ or $\mathcal{D}$. Suppose $\mathbf{G}\in\mathcal{B}'$. If $\mathbf{G}^{\otimes 3}\mathbf{F}'$ is non-degenerate, then $\mathbf{G}^{\otimes 3}\mathbf{F}'\in\mathcal{O}'$, so by assumption we need $\operatorname{supp}\mathbf{G}^{\otimes 3}\mathbf{F}' \subseteq \{\mathsf{B},\mathsf{G}\}^*$. This condition is equivalent to the statement that if the first two columns of $\mathbf{G}$ are linearly independent, then the third row of the first two columns must be 0. In other words, if $\mathbf{G}\notin\mathcal{D}$, then it must be $\mathsf{BG|R}$.

2. Suppose there is $\mathbf{F}\in T\mathcal{F}$ such that it is of type $\mathfrak{B}$. Since $\operatorname{supp}\mathbf{F} \subseteq \{\mathsf{B},\mathsf{G}\}^*$, we may apply a $\mathsf{BG|R}$ orthogonal matrix S such that $\mathbf{F}' = S^{\otimes 3}\mathbf{F} = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}$. By Proposition 6.6, Proposition 6.4, and Theorem 2.8, it must be the case that all signatures in $\mathcal{O}'$ are also type II. In particular, all signatures in $\mathcal{F}' = ST\mathcal{F}$ are also type II.

   Same argument as case (1) shows that $ST\mathcal{B}$ only consists of binary signatures that are $\mathsf{BG|R}$ or $\mathcal{D}$.

9.1.5 Rank 2 Class $\mathscr{B}$ or $\mathscr{D}$

From now on, we assume that under any orthogonal transformation T, there exists some $\mathbf{F}\in T\mathcal{O}$ such that $\operatorname{supp}\mathbf{F} \not\subseteq \{\mathsf{B},\mathsf{G}\}^*$. Suppose there exists some orthogonal T such that $T\mathcal{O}\subseteq\mathcal{E}$. We claim that $\mathcal{F}$ must be in the tractable class $\mathscr{B}$ or $\mathscr{D}$. Let $\mathcal{O}' = T\mathcal{O}$. We may assume that there exists $\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}\in\mathcal{O}'$ after permuting the domains. Also, by the assumption that not all signatures can be supported on $\{\mathsf{B},\mathsf{G}\}$, we may assume that there is some $\mathbf{G} = \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}\in\mathcal{O}'$. We need to show that one of the following holds: (1) every signature in $\mathcal{B}' = T\mathcal{B}$ is in $\mathcal{E}$; or (2) every signature in $\mathcal{B}'$ is $\mathsf{BR|G}$ or $\mathsf{Swap}_{\mathsf{BR;G}}$.

Suppose $\mathbf{H}\in\mathcal{B}'$, and let $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ be the columns of $\mathbf{H}$. Then $\mathbf{H}^{\otimes 3}\mathbf{F} = \mathbf{v}_1^{\otimes 3} + \mathbf{v}_2^{\otimes 3}$. If $\mathbf{H}^{\otimes 3}\mathbf{F} = \mathbf{v}_1^{\otimes 3} + \mathbf{v}_2^{\otimes 3}$ is not degenerate, then it must be the case that it is also a GenEQ since $\mathbf{H}^{\otimes 3}\mathbf{F}\in\mathcal{O}'\subseteq\mathcal{E}$. This implies that $\mathbf{v}_1\sim\mathbf{e}_i$ and $\mathbf{v}_2\sim\mathbf{e}_j$ for $i\neq j$. Since $\mathbf{H}$ is symmetric, this implies $\mathbf{H}\in\mathcal{E}$ immediately, except in one case of $\mathbf{v}_1\sim\mathbf{e}_1$ and $\mathbf{v}_2\sim\mathbf{e}_3$, where $\mathbf{H} = \left[\begin{smallmatrix} a & 0 & 0\\ 0 & 0 & b\\ 0 & b & c\end{smallmatrix}\right]$. Since $\mathbf{H}^{\otimes 3}\mathbf{G} = (0,0,b)^{\otimes 3} + (0,b,c)^{\otimes 3}$ is in $\mathcal{E}$ by assumption, it must be the case that $b=0$ or $c=0$, which implies $\mathbf{H}\in\mathcal{E}$. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. Then we may look at $\mathbf{H}^{\otimes 3}\mathbf{G} = \mathbf{v}_2^{\otimes 3} + \mathbf{v}_3^{\otimes 3}$. If $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, then by the same argument, $\mathbf{H}\in\mathcal{E}$. So the only uncovered case is if $\mathbf{v}_2$ and $\mathbf{v}_3$ are also linearly dependent. If $\mathbf{v}_2\neq 0$, then $\mathbf{H}$ is degenerate. Otherwise, if $\mathbf{v}_2 = 0$, then $\mathbf{H}$ must have the form $\left[\begin{smallmatrix} a & 0 & b\\ 0 & 0 & 0\\ b & 0 & c\end{smallmatrix}\right]$.

There are two possible cases. If $\mathbf{e}_1^{\otimes 3} + \mathbf{e}_3^{\otimes 3}\in\mathcal{O}'$, then by the same analysis, the above $\mathbf{H}$ must be degenerate or in $\mathcal{E}$. Therefore, $\mathcal{B}'\subseteq\mathcal{E}$, and thus we get the tractable class $\mathscr{B}$. Otherwise, if $\mathbf{e}_1^{\otimes 3} + \mathbf{e}_3^{\otimes 3}\notin\mathcal{O}'$, $\mathcal{B}'$ may contain binary signatures that are supported on $\{\mathsf{B},\mathsf{R}\}$. Then, we claim that every signature in $\mathcal{B}'$ must be $\mathsf{BR|G}$ or $\mathsf{Swap}_{\mathsf{BR;G}}$. We have already shown above that the binary signatures in $\mathcal{B}'$ are either $\mathsf{GenPerm}$ or possibly supported on $\{\mathsf{B},\mathsf{R}\}$. Therefore, only cases we need to further rule out are the symmetric $\mathsf{GenPerm}$ matrices not supported on $\{\mathsf{B},\mathsf{R}\}$, which are of the form $\mathbf{H}_1 = \left[\begin{smallmatrix} 0 & x & 0 \\ x & 0 & 0 \\ 0 & 0 & y \end{smallmatrix}\right]$ and $\mathbf{H}_2 = \left[\begin{smallmatrix} y & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{smallmatrix}\right]$ for nonzero x, y, since if $y = 0$, they are both $\mathsf{Swap}_{\mathsf{BR;G}}$, and if $x = 0$, they are both degenerate. We see that then $\mathbf{H}_1^{\otimes 3}\mathbf{G} = x^3\mathbf{e}_1^{\otimes 3} + y^3\mathbf{e}_3^{\otimes 3}$ and $\mathbf{H}_2^{\otimes 3}\mathbf{F} = y^3\mathbf{e}_1^{\otimes 3} + x^3\mathbf{e}_3^{\otimes 3}$, which can realize $\mathbf{e}_1^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. Then, we may assume it is in $\mathcal{O}'$ to show that $\mathcal{B}' \subseteq \mathcal{E}$. Otherwise such symmetric $\mathsf{GenPerm}$ matrices cannot exist in $\mathcal{B}'$, so $\mathcal{F}$ must be in the tractable class $\mathscr{D}$.

9.1.6 Rank 2 Class $\mathscr{D}$ or $\mathscr{E}$

From now on, we further assume that under any orthogonal transformation T, there exists some $\mathbf{F} \in T\mathcal{O}$ such that $\mathbf{F}$ is not a GenEQ. By Lemma 9.6, we may assume that there is some orthogonal T such that all signatures in $T\mathcal{O}$ are $\mathsf{BG|R}$.

1. Suppose there is no signature in $T\mathcal{O}$ that is supported on $\{\mathsf{B},\mathsf{G}\}$. This means that all signatures in $T\mathcal{O}$ must have the form $(a, b, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ for some $a, b \in \mathbb{R}$. Note that no type $\mathfrak{B}$ signature can exist since a rank 2 type $\mathfrak{B}$ signature cannot be $\mathsf{BG|R}$ without being supported on $\{\mathsf{B},\mathsf{G}\}$. Fix one such $\mathbf{F} = (a, b, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3} \in T\mathcal{O}$. We may assume that $a^2 + b^2 = 1$. We may apply a $\mathsf{BG|R}$ orthogonal transformation $S = \left[\begin{smallmatrix} -b & a & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ so that $\mathbf{F}' = S^{\otimes 3}\mathbf{F} = \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. Note that every signature in $\mathcal{O}' = ST\mathcal{O}$ is $\mathsf{BG|R}$. By the assumption that $\mathcal{O}$ cannot be transformed into $\mathcal{E}$, we may assume that there exists some $\mathbf{G}' \in \mathcal{O}$ such that $\mathbf{G}' = (c, d, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ for nonzero $c, d \in \mathbb{R}$. We claim that $\mathcal{F}$ must be in the tractable class $\mathscr{D}$. We need to show that all binary signatures in $\mathcal{B}' = ST\mathcal{B}$ are $\mathsf{BG|R}$ or $\mathsf{Swap}_{\mathsf{BG;R}}$.

   Suppose $\mathbf{H} \in \mathcal{B}'$ and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the columns of $\mathbf{H}$. If $\mathbf{v}_2, \mathbf{v}_3$ are linearly independent, then we need $\mathbf{H}^{\otimes 3}\mathbf{F}' = \mathbf{v}_2^{\otimes 3} + \mathbf{v}_3^{\otimes 3}$ to also have the form $(e, f, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. This means that either we have $\mathbf{v}_2 \sim (e, f, 0)$ and $\mathbf{v}_3 \sim \mathbf{e}_3$ or $\mathbf{v}_2 \sim \mathbf{e}_3$ and $\mathbf{v}_3 \sim (e, f, 0)$. The first case implies $\mathbf{H}$ is $\mathsf{BG|R}$ by symmetry. The second case implies $\mathbf{H}$ is of the form $\left[\begin{smallmatrix} x & 0 & e \\ 0 & 0 & f \\ e & f & 0 \end{smallmatrix}\right]$ for some x. If $x = 0$, then $\mathbf{H}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$, so assume otherwise. Then, $\mathbf{H}^{\otimes 3}\mathbf{G}' = (cx, 0, ce + df)^{\otimes 3} + (e, f, 0)^{\otimes 3}$. If this is degenerate, then we must have $f = 0$ and $e = 0$, which implies $\mathbf{H}$ is degenerate. Otherwise, e must be 0 for the two vectors to be orthogonal, but if $f \neq 0$, then this signature is not $\mathsf{BG|R}$, contrary to assumption.

   Now, suppose $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent. Then, $\mathbf{H}^{\otimes 3}\mathbf{G}' = (c\mathbf{v}_1 + d\mathbf{v}_2)^{\otimes 3} + \mathbf{v}_3^{\otimes 3}$. If $c\mathbf{v}_1 + d\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, we need either $c\mathbf{v}_1 + d\mathbf{v}_2 \sim (e, f, 0)$ and $\mathbf{v}_3 \sim \mathbf{e}_3$ or $c\mathbf{v}_1 + d\mathbf{v}_2 \sim \mathbf{e}_3$ and $\mathbf{v}_3 \sim (e, f, 0)$. The first case implies that $\mathbf{H}$ is $\mathsf{BG|R}$ by symmetry of $\mathbf{H}$. The second case implies $\mathbf{H}$ is of the form $\left[\begin{smallmatrix} * & * & e \\ * & * & f \\ e & f & 0 \end{smallmatrix}\right]$ and linear dependence of $\mathbf{v}_2, \mathbf{v}_3$ further implies $f = 0$. So, $\mathbf{H}$ is $\left[\begin{smallmatrix} x & y & e \\ y & 0 & 0 \\ e & 0 & 0 \end{smallmatrix}\right]$ for some $x, y \in \mathbb{R}$. For $c\mathbf{v}_1 + d\mathbf{v}_2 \sim \mathbf{e}_3$ to be true we must have $y = 0$, which then implies $x = 0$, so $\mathbf{H}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$. If $c\mathbf{v}_1 + d\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent, then $\mathbf{v}_1, \mathbf{v}_3$ are linearly dependent as well, so $\mathbf{H}$ is degenerate.

2. Now, we may assume there is some signature in $T\mathcal{O}$ that is supported on $\{\mathsf{B},\mathsf{G}\}$. Assume $\mathbf{F}$ is supported on $\{\mathsf{B},\mathsf{G}\}$ and is of type $\mathfrak{A}$. By applying a $\mathsf{BG|R}$ orthogonal transformation S, we have $\mathbf{F}' = S^{\otimes 3}\mathbf{F} = \mathbf{e}_1^{\otimes 3} + \mathbf{e}_2^{\otimes 3}$. Let $\mathcal{O}' = ST\mathcal{O}$. By the assumptions, we may assume that there is some $\mathbf{G}' \in \mathcal{O}'$ such that $\mathbf{G}' = (a, b, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ for nonzero $a, b \in \mathbb{R}$. We claim that $\mathcal{F}$ must be in the tractable class $\mathscr{D}$ or $\mathscr{E}$. We will narrow down the possible forms of

binary signatures in $\mathcal{B}' = ST\mathcal{B}$. We will show that either: $\mathcal{B}'$ consists of $\mathsf{BG|R}$ and $\mathsf{Swap}_{\mathsf{BG;R}}$ signatures; or $\mathcal{O}'$ and $\mathcal{B}'$ can be transformed so that every signature becomes EBD or in O_h.

Let $\mathbf{H} \in \mathcal{B}'$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ the columns. Assume $\mathbf{H}$ is not $\mathsf{BG|R}$. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Then $\mathbf{H}^{\otimes 3}\mathbf{F}' = \mathbf{v}_1^{\otimes 3} + \mathbf{v}_2^{\otimes 3}$ is not supported on $\{\mathsf{B}, \mathsf{G}\}$ since $\mathbf{H}$ is not $\mathsf{BG|R}$. Therefore, it must be the case that $\mathbf{v}_1 \sim (e, f, 0)$ and $\mathbf{v}_2 \sim \mathbf{e}_3$ or $\mathbf{v}_1 \sim \mathbf{e}_3$ and $\mathbf{v}_2 \sim (e, f, 0)$ for some $e, f \in \mathbb{R}$ to be $\mathsf{BG|R}$. The first case implies $\mathbf{H} = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & x \\ 0 & x & y \end{bmatrix}$ for some x, y with $x \neq 0$. Then, $\mathbf{H}^{\otimes 3}\mathbf{G}' = (ae, 0, bx)^{\otimes 3} + (0, x, y)^{\otimes 3}$. Since $x \neq 0$, $\mathbf{H}^{\otimes 3}\mathbf{G}'$ cannot be degenerate. So, $\mathbf{H}^{\otimes 3}\mathbf{G}'$ must be $\mathsf{BG|R}$, which implies that either $e, y = 0$ or $x = 0$. In the first case, $\mathbf{H}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$, and in the second case, $\mathbf{H}$ is $\mathsf{BG|R}$.

If $\mathbf{v}_1 \sim \mathbf{e}_3$ and $\mathbf{v}_2 \sim (e, f, 0)$, then $\mathbf{H}$ must be of the form $\begin{bmatrix} 0 & 0 & x \\ 0 & f & 0 \\ x & 0 & y \end{bmatrix}$ for some x, y with $x \neq 0$. Then, $\mathbf{H}^{\otimes 3}\mathbf{G}' = (0, bf, ax)^{\otimes 3} + (x, 0, y)$. This cannot be degenerate since $a, x \neq 0$. Since $\mathbf{H}^{\otimes 3}\mathbf{G}' \in \mathcal{O}'$, we must have $f = 0$ and $y = 0$, implying that $\mathbf{H}$ is a $\mathsf{Swap}_{\mathsf{BG;R}}$.

So far, we have shown that if $\mathbf{H} \notin \mathcal{D}$, then it is $\mathsf{Swap}_{\mathsf{BG;R}}$ or $\mathsf{BG|R}$. If $\mathbf{H} \in \mathcal{D}$, then by Proposition 9.3, for $\operatorname{Holant}_3^*(\mathbf{G}', \mathbf{H})$ to be not #P-hard, $\mathbf{H}$ must be $\mathsf{BG|R}$, $\mathsf{Swap}_{\mathsf{BG;R}}$, or for $U = \begin{bmatrix} -b & a & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $U^{\otimes 2}\mathbf{H}$ is EBD. Note that being $\mathsf{BG|R}$ and $\mathsf{Swap}_{\mathsf{BG;R}}$ are invariant under $\mathsf{BG|R}$ transform, so every signature in $U\mathcal{B}'$ are $\mathsf{BG|R}$, $\mathsf{Swap}_{\mathsf{BG;R}}$, or EBD. We will now analyze the EBD signatures in $U\mathcal{B}'$. Also note that $U^{\otimes 3}\mathbf{G}' = \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$.

Suppose $\mathbf{H} \in U\mathcal{B}'$ and is supported on $\{\mathsf{G}, \mathsf{R}\}$. Since $U^{\otimes 3}\mathbf{G}'$ is a GenEQ on $\{\mathsf{G}, \mathsf{R}\}$, it must be the case that $\mathbf{H}^{*\to\{\mathsf{G},\mathsf{R}\}}$ is degenerate, $[0, *, 0]$ or $[*, 0, *]$. $\mathbf{H}^{*\to\{\mathsf{G},\mathsf{R}\}}$ being degenerate implies that $\mathbf{H}$ is degenerate. Others imply that $\mathbf{H}$ is $\mathsf{Swap}_{\mathsf{BG;R}}$ and $\mathsf{BG|R}$ respectively. If there is no $\mathbf{H} \in U\mathcal{B}'$ such that $\operatorname{supp}\mathbf{H} \subseteq \{\mathsf{B}, \mathsf{R}\}^*$, then we are in the tractable case $\mathscr{D}$.

Now, assume that $\mathbf{H} = \begin{bmatrix} x & 0 & y \\ 0 & 0 & 0 \\ y & 0 & z \end{bmatrix} \in U\mathcal{B}'$ such that it is not $\mathsf{BG|R}$ or $\mathsf{Swap}_{\mathsf{BG;R}}$. This means x, z cannot be both 0 and $y \neq 0$. If such $\mathbf{H}$ exists, then we show $\mathcal{F}$ must be in the tractable class $\mathscr{E}$. Note that since U is $\mathsf{BG|R}$, every signature in $U\mathcal{O}'$ is still $\mathsf{BG|R}$ by assumption.

We need to show two things: (a) in $U\mathcal{O}'$, there is no signature of the form $\mathbf{I} = (c, d, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$ for nonzero c, d and hence all signatures are EBD; (b) the binary signatures in $U\mathcal{B}'$ are EBD or in O_h. For (a), suppose such signature exists, and we have $\mathbf{H}^{\otimes 3}\mathbf{I} = (cx, 0, cy)^{\otimes 3} + (y, 0, z)^{\otimes 3}$. By assumption, this needs to be $\mathsf{BG|R}$ since $\mathbf{H}$ is nondegenerate, but that would require $y = 0$ or $x, z = 0$ contrary to assumption. For (b), we already know that the binary signatures that are not EBD are of the form $\mathsf{BG|R}$ or $\mathsf{Swap}_{\mathsf{BG;R}}$. If they are not EBD, then by composing with $U^{\otimes 3}\mathbf{G}' = \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$, we get a ternary signature of the above form in (a) which we already have shown to be not in $U\mathcal{O}'$.

3. We assume now that $\mathbf{F} \in T\mathcal{O}$ is supported on $\{\mathsf{B}, \mathsf{G}\}$ and is of type $\mathfrak{B}$. By the assumptions, we may assume that there is some $\mathbf{G} \in T\mathcal{O}$ that is not supported on $\{\mathsf{B}, \mathsf{G}\}$. It must have the form $(a, b, 0)^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$. Let $S = \begin{bmatrix} -b & a & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and let $\mathcal{O}' = ST\mathcal{O}$ and $\mathcal{B}' = ST\mathcal{B}$. Then, $\mathbf{G}' = S^{\otimes 3}\mathbf{G} = \mathbf{e}_2^{\otimes 3} + \mathbf{e}_3^{\otimes 3}$, and by Corollary 6.3, we may assume that there is $\mathbf{F}' \in \mathcal{O}'$ such that $\mathbf{F}' = \boldsymbol{\beta}^{\otimes 3} + \overline{\boldsymbol{\beta}}^{\otimes 3}$.

   By Lemma 6.10, the only other possible forms of binary signatures are: (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 0 \end{bmatrix}$ for $\alpha = \pm 1$; (b) $\mathcal{D}$; (c) $\begin{bmatrix} 1 & x & -x\gamma \\ x & x^2 & \gamma \\ -x\gamma & \gamma & 0 \end{bmatrix}$ for $\gamma = \pm\sqrt{1 + x^2}$ and a nonzero $x \in \mathbb{R}$. Let $\mathbf{H} \in \mathcal{B}'$. For (a) and (c), we see that $\mathbf{H}^{\otimes 3}\mathbf{F}$ is not $\mathsf{BG|R}$. For (b), if $\mathbf{H} \in \mathcal{D}$, then by Lemma 6.8 with

$\mathbf{G}'$, $\mathbf{H}$ must be $\mathsf{BG|R}$, $\mathsf{Swap}_{\mathsf{BG;R}}$, or EBD. Only case we need to consider is the signatures supported on $\{\mathsf{G},\mathsf{R}\}$. We may write $\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & y & z \end{bmatrix}$, and since it must be tractable with $\mathbf{G}'$, $[x,y,z]$ must be $[0,*,0]$, $[*,0,*]$ or degenerate. However, it cannot be degenerate since then $\mathbf{H}$ is degenerate. If $y=0$, then $\mathbf{H}$ is $\mathsf{BG|R}$. If $x,z=0$, then it is $\mathsf{Swap}_{\mathsf{BG;R}}$.

We have shown so far that all signatures that are not supported on $\{\mathsf{B},\mathsf{R}\}$ are either $\mathsf{BG|R}$ or $\mathsf{Swap}_{\mathsf{BG;R}}$. The existence of a binary signature supported on $\{\mathsf{B},\mathsf{R}\}$ determines the tractable class between $\mathscr{D}$ and $\mathscr{E}$, and the argument is similar to case (2).

9.1.7 Rank 2 Class $\mathscr{E}$

From now on, we further assume that under any orthogonal transformation T, there always is a non $\mathsf{BG|R}$ signature in $T\mathcal{O}$. By Lemma 9.6, this implies that there must exist some T such that all the signatures in $\mathcal{O}' = T\mathcal{O}$ are EBD to escape #P-hardness. Let $\mathcal{O}_{ij}$ be the signatures in $\mathcal{O}'$ such that are supported on $\{i,j\}$. By assumption, it must be the case that at least two $\mathcal{O}_{ij}$ are nonempty, and not GenEQ. Without loss of generality, we may assume that $\mathcal{O}_{\mathsf{BG}}$ and $\mathcal{O}_{\mathsf{GR}}$ are not empty and contains non-GenEQ, say $\mathbf{F} \in \mathcal{O}_{\mathsf{BG}}$ and $\mathbf{G} \in \mathcal{O}_{\mathsf{GR}}$. We claim that all binary signatures in $\mathcal{B}' = T\mathcal{B}$ must be either EBD or $\mathsf{GenPerm}$. Then, using the fact that $\mathbf{F},\mathbf{G}$ are not GenEQ, we will show that only special forms of $\mathsf{GenPerm}$ signatures can be in $\mathcal{B}'$ to escape #P-hardness.

We will argue in the following way: by definition, if $\mathbf{A} \in \mathcal{O}'$ and $\mathbf{H} \in \mathcal{B}'$, then $\mathbf{H}^{\otimes 3}\mathbf{A} \in \mathcal{O}'$; therefore, $\mathbf{H}^{\otimes 3}\mathbf{F}$ and $\mathbf{H}^{\otimes 3}\mathbf{G}$ must be EBD or degenerate. In particular, $\mathbf{H}$ must be a symmetric matrix that maps $P_{\mathbf{F}}$ (xy-plane) to a coordinate plane or to a single vector. Since $\mathbf{H}$ is a linear transformation, it must then send $\mathbf{e}_1$ and $\mathbf{e}_2$ to some axes or collapse them to a single vector. This allows us to analyze the first two columns of $\mathbf{H}$, and similarly we can analyze the second and third columns using $\mathbf{G}$.

1. Suppose $\mathbf{H}$ sends the xy-plane to a single vector. This means that the first two columns of $\mathbf{H}$ are linearly dependent and hence $\mathbf{H} \in \mathcal{D}$. We may write $\mathbf{H} = \begin{bmatrix} ax^2 & axy & bx \\ axy & ay^2 & by \\ bx & by & c \end{bmatrix}$.
   (a) Suppose the yz-plane also goes to a single vector. Then the second and third columns must also be linearly dependent. If the second column is nonzero, then $\mathbf{H}$ is degenerate. If the second column is zero, then $\mathbf{H}$ is EBD.
   (b) Suppose the yz-plane stays on yz-plane. Then, it must be the case that $axy, bx = 0$. If $x=0$, then $\operatorname{supp}\mathbf{H} \subseteq \{\mathsf{G},\mathsf{R}\}^*$. If $x \neq 0$, then b must be 0, and $ay=0$. If $a=0$, then $\mathbf{H}$ is degenerate. If $y=0$, then $\operatorname{supp}\mathbf{H} \subseteq \{\mathsf{B},\mathsf{R}\}^*$.
   (c) Suppose the yz-plane goes to the xy-plane. Then, it must be the case that $by, c=0$. If $y=0$, then, $\operatorname{supp}\mathbf{H} \subseteq \{\mathsf{B},\mathsf{R}\}^*$. If $b=0$, then, $\operatorname{supp}\mathbf{H} \subseteq \{\mathsf{B},\mathsf{G}\}^*$.
   (d) Suppose the yz-plane goes to the xz-plane. Then, it must be the case that $ay^2, by = 0$. If $y=0$, then $\operatorname{supp}\mathbf{H} \subseteq \{\mathsf{B},\mathsf{R}\}^*$. If $a,b=0$, then $\mathbf{H}$ is degenerate.

2. Suppose $\mathbf{H}$ sends xy-plane to xy-plane. Then, $\mathbf{H} = \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & d \end{bmatrix}$. We may assume $d \neq 0$ since otherwise $\operatorname{supp}\mathbf{H} \subseteq \{\mathsf{B},\mathsf{G}\}^*$.
   (a) Suppose yz-plane goes to a single vector. Then, either $b,c=0$ or $d=0$ and $\mathbf{H}$ is EBD.
   (b) Suppose yz-plane stays on yz-plane. Then, $b=0$, so $\mathbf{H}$ is a $\mathsf{GenPerm}$.
   (c) Suppose yz-plane goes to xy-plane. Then $d=0$ so $\mathbf{H}$ is EBD.

(d) Suppose yz-plane goes to xz-plane. Then, $c = 0$. If $b = 0$, then $\mathbf{H}$ is a $\mathsf{GenPerm}$, so suppose $b \neq 0$. That means $\mathcal{F}_{\mathsf{BR}}$ is not empty since $\mathbf{H}^{\otimes 3}\mathbf{G} = (b, 0, 0)^{\otimes 3} + (0, 0, d)^{\otimes 3} \in \mathcal{O}_{\mathsf{BR}}$. So, we may also analyze where xz-plane goes to, which is determined by the first and third columns. Since $b, d \neq 0$, the two columns are linearly independent, and thus xz-plane does not go to a line. Therefore, it must go to a coordinate plane, but since $b \neq 0$, we must have $a = 0$. Then $\mathbf{H}$ is a $\mathsf{GenPerm}$.

3. Suppose $\mathbf{H}$ sends xy-plane to yz-plane. Then, $\mathbf{H} = \left[\begin{smallmatrix} 0 & 0 & b \\ 0 & a & c \\ b & c & d \end{smallmatrix}\right]$.

   (a) Suppose yz-plane goes to a single vector. Then, it must be the case that $b = 0$ and $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{G}, \mathsf{R}\}^*$.

   (b) Suppose yz-plane stays on yz-plane. Then, $b = 0$, so $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{G}, \mathsf{R}\}^*$.

   (c) Suppose yz-plane goes to xy-plane. Then, $c, d = 0$, so $\mathbf{H}$ is a $\mathsf{GenPerm}$.

   (d) Suppose yz-plane goes to xz-plane. Then, $a, c = 0$ so $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{B}, \mathsf{R}\}^*$.

4. Suppose $\mathbf{H}$ sends xy-plane to xz-plane. Then, $\mathbf{H} = \left[\begin{smallmatrix} a & 0 & b \\ 0 & 0 & c \\ b & c & d \end{smallmatrix}\right]$. We may assume that the first two columns are linearly independent, so $\mathbf{H}^{\otimes 3}\mathbf{F} \in \mathcal{O}_{\mathsf{BR}}$.

   (a) Suppose yz-plane goes to a single vector. Then, it must be the case that $b, c = 0$ and $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{B}, \mathsf{R}\}^*$.

   (b) Suppose yz-plane stays on yz-plane. Then, $b = 0$. Since $\mathcal{O}_{\mathsf{BR}} \neq \emptyset$, we may look at where the xz-plane goes to. If it goes to a single line, then we must also have $a, c = 0$, so $\mathbf{H}$ is degenerate. If xz-plane goes to another plane, then we must have either $d = 0$, $a = 0$, or $c = 0$. If $d = 0$, $\mathbf{H}$ is $\mathsf{GenPerm}$. If $a = 0$, $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{G}, \mathsf{R}\}^*$. If $c = 0$, $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{B}, \mathsf{R}\}^*$.

   (c) Suppose yz-plane goes to xy-plane. Then, $c, d = 0$, so $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{B}, \mathsf{R}\}^*$.

   (d) Suppose yz-plane goes to xz-plane. Then, $c = 0$, so $\operatorname{supp} \mathbf{H} \subseteq \{\mathsf{B}, \mathsf{R}\}^*$.

Now, we will argue that if a symmetric $\mathsf{GenPerm}$ matrices is not EBD, then the absolute values of the nonzero coefficients must be the same for $\operatorname{Holant}_3^*(\mathcal{F})$ to be not #P-hard. After normalization, any symmetric $\mathsf{GenPerm}$ matrix that is not EBD has the form

$$\begin{bmatrix} 0 & x & 0 \\ x & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & x \\ 0 & 1 & 0 \\ x & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{bmatrix}.$$

for some nonzero x. The $2k$-th power of each of them are

$$\begin{bmatrix} x^{2k} & 0 & 0 \\ 0 & x^{2k} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x^{2k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{2k} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^{2k} & 0 \\ 0 & 0 & x^{2k} \end{bmatrix}.$$

By assumption, the signatures $\mathbf{F}$ and $\mathbf{G}$ are not of the GenEQ types. Then, by Theorem 2.8, $\operatorname{Holant}_2^*([x^{2k}, 0, 1], \mathbf{F}^{*\to\{\mathsf{B},\mathsf{G}\}})$ and $\operatorname{Holant}_2^*([x^{2k}, 0, 1], \mathbf{G}^{*\to\{\mathsf{G},\mathsf{R}\}})$ are not tractable for big enough k if $x \neq \pm 1$. We see that all the above three matrices are ruled out for $x \neq \pm 1$, and hence the only non EBD symmetric binary signatures that $\mathcal{B}'$ can possibly contain are the scalar multiples of elements of O_h. Therefore, we have proven that $\mathcal{F}$ is in class $\mathscr{E}$.

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