ON THE INVERSE PROBLEM OF THE k-TH DAVENPORT CONSTANTS FOR GROUPS OF RANK 2

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Abstract. For a finite abelian group G and a positive integer k, let $\mathsf{D}_k(G)$ denote the smallest integer ℓ such that each sequence over G of length at least ℓ has k disjoint nontrivial zero-sum subsequences. It is known that $\mathsf{D}_k(G) = n_1 + kn_2 - 1$ if $G \cong C_{n_1} \oplus C_{n_2}$ is a rank 2 group, where $1 < n_1 \mid n_2$. We investigate the associated inverse problem for rank 2 groups, that is, characterizing the structure of zero-sum sequences of length $\mathsf{D}_k(G)$ that can not be partitioned into $k+1$ nontrivial zero-sum subsequences.

1. Introduction

Let $(G, +, 0)$ be a finite abelian group. By a sequence S over G, we mean a finite sequence of terms from G which is unordered, repetition of terms allowed. We say that S is a zero-sum sequence if the sum of its terms equals zero and denote by $|S|$ the length of the sequence.

Let k be a positive integer. We denote by $\mathsf{D}_k(G)$ the smallest integer ℓ such that every sequence over G of length at least ℓ has k disjoint nontrivial zero-sum subsequences. We call $\mathsf{D}_k(G)$ the k-th Davenport constant of G, while the Davenport constant $\mathsf{D}(G) = \mathsf{D}_1(G)$ is one of the most important zero-sum invariants in Combinatorial Number Theory and, together with Erdős-Ginzburg-Ziv constant, η-constant, etc., has been studied a lot (see [39, 40, 1, 29, 49, 50, 30, 16, 21, 41, 5, 43, 6, 14, 7, 22, 38]). This variant $\mathsf{D}_k(G)$ of the Davenport constant was introduced and investigated by F. Halter-Koch [37], in the context of investigations on the asymptotic behavior of certain counting functions of algebraic integers defined via factorization properties (see the monograph [27, Section 6.1], and the survey article [19, Section 5]). In 2014, K. Cziszter and M. Domokos ([9, 8]) introduced the generalized Noether Number $\beta_k(G)$ for general groups, which equals $\mathsf{D}_k(G)$ when G is abelian (see [11, 12, 10] for more about this direction). Knowledge of those constants is highly relevant when applying the inductive method to determine or estimate the Davenport constant of certain finite abelian groups (see [13, 4, 3, 42]).

In 2010, M. Freeze and W. Schmid ([17]) showed that for each finite abelian group G we have $\mathsf{D}_k(G) = \mathsf{D}_0(G) + k \exp(G)$ for some $\mathsf{D}_0(G) \in \mathbb{N}_0$ and all sufficiently large k. In fact, it is known that for groups of rank at most two, and for some other types of groups, an equality of the form

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$\mathsf{D}_k(G) = \mathsf{D}_0(G) + k\exp(G)$ for some $\mathsf{D}_0(G) \in \mathbb{N}_0$ holds for all k. In particular, for a rank two abelian group $G = C_m \oplus C_n$, where $m \mid n$, we have $\mathsf{D}_k(G) = m + kn - 1$ ([27, Theorem 6.1.5]). Yet, it fails for elementary 2 and 3-groups of rank at least 3 (see [13, 4]). In general, computing (even bounding) $\mathsf{D}_k(G)$ is quite more complicated than for $\mathsf{D}(G)$, in particular for (elementary) p-groups, while $\mathsf{D}(G)$ is know for p-groups.

In zero-sum theory, the associated inverse problems of zero-sum invariants study the structure of extremal sequences that do not have enough zero-sum subsequences with the prescribed properties. The inverse problems of the Davenport constant, the η-constant, and the Erdős-Ginzburg-Ziv constant are central topics (see [51, 52, 45, 46, 23, 24, 15, 34, 35, 47, 48, 31, 36]). The associated inverse problem of $\mathsf{D}_k(G)$ is to characterize the maximal length zero-sum sequences that can not be partitioned into $k+1$ nontrivial zero-sum subsequences. In particular, the inverse problem of $\mathsf{D}(G)$ is to characterize the structure of minimal zero-sum subsequences of length $\mathsf{D}(G)$, which was accomplished for groups of rank 2 in a series of papers [44] [18] [20] [47] [2], where a minimal zero-sum sequence is a zero-sum sequence that can not be partitioned into two nontrivial zero-sum subsequences. Those inverse results can be used to construct minimal generating subsets in Invariant Theory (see [11, Proposition 4.7]).

Let $\mathcal{B}(G)$ be the set of all zero-sum sequences over G. We define

$$\mathcal{M}_k(G) = \{S \in \mathcal{B}(G) \colon S \text{ can not be partitioned into } k+1 \text{ nontrivial zero-sum subsequences}\}.$$

Then it is easy to see that $\mathsf{D}_k(G) = \max\{|S| \colon S \in \mathcal{M}_k(G)\}$. In this paper, we investigate the inverse problem of general Davenport constant $\mathsf{D}_k(G)$ for all rank 2 groups, that is, to study the structure of sequences of $\mathcal{M}_k(G)$ of length $\mathsf{D}_k(G)$. In 2003, Gao and Geroldinger ([18, Theorem 7.1]) studied the inverse problem of $\mathsf{D}_k(G)$ for $G = C_n \oplus C_n$ under some assumptions of G, which had been confirmed later. We reformulate this result in the following and a proof will be given in Section 3.

Theorem 1.1. *Let $G = C_n \oplus C_n$ with $n \ge 2$, let $k \ge 1$, and let $U \in \mathcal{B}(G)$ with $|U| = \mathsf{D}_k(G)$. Then $U \in \mathcal{M}_k(G)$ if and only if there exists a basis (e_1, e_2) of G such that it has one of the following two forms.*

(I)
$$U = e_1^{k_1 n - 1} \prod_{i \in [1, k_2 n]} (x_i e_1 + e_2), \quad \textit{where}$$

(a) $k_1, k_2 \in \mathbb{N}$ *with* $k_1 + k_2 = k + 1$,

(b) $x_1, \ldots, x_{k_2 n} \in [0, n-1]$ *and* $x_1 + \ldots + x_{k_2 n} \equiv 1 \mod n$.

(II)
$$U = e_1^{an} e_2^{bn-1} (x e_1 + e_2)^{cn-1} (x e_1 + 2e_2), \quad \textit{where}$$

(a) $x \in [2, n-2]$ *with* $\gcd(x, n) = 1$,

(b) $a, b, c \ge 1$ *with* $a + b + c = k + 1$.

Note that in this case, we have $k \ge 2$.

For general groups, we have the following main theorem.

Theorem 1.2. *Let $G = C_{n_1} \oplus C_{n_2}$ with $1 < n_1 \mid n_2$ and $n_1 < n_2$, let $k \ge 1$, and let $U \in \mathcal{B}(G)$ with $|U| = \mathsf{D}_k(G)$. Then $U \in \mathcal{M}_k(G)$ if and only if it has one of the following four forms.*

(I)
$$U = e_1^{\operatorname{ord}(e_1)-1} \prod_{i \in [1, k \operatorname{ord}(e_2)]} (x_i e_1 + e_2), \quad \textit{where}$$

(a) (e_1, e_2) *is a basis for G with* $\operatorname{ord}(e_1) = n_1$ *and* $\operatorname{ord}(e_2) = n_2$,
(b) $x_1, \ldots, x_{k \operatorname{ord}(e_2)} \in [0, \operatorname{ord}(e_1) - 1]$ *and* $x_1 + \ldots + x_{k \operatorname{ord}(e_2)} \equiv 1 \mod \operatorname{ord}(e_1)$.

(II)
$$U = e_1^{k \operatorname{ord}(e_1)-1} \prod_{i \in [1, \operatorname{ord}(e_2)]} (x_i e_1 + e_2), \quad \textit{where}$$

(a) (e_1, e_2) *is a basis for G with* $\operatorname{ord}(e_1) = n_2$ *and* $\operatorname{ord}(e_2) = n_1$,
(b) $x_1, \ldots, x_{\operatorname{ord}(e_2)} \in [0, \operatorname{ord}(e_1) - 1]$ *and* $x_1 + \ldots + x_{\operatorname{ord}(e_2)} \equiv 1 \mod \operatorname{ord}(e_1)$.

(III)
$$U = g_1^{n_1-1} \prod_{i \in [1, kn_2]} (-x_i g_1 + g_2), \quad \textit{where}$$

(a) (g_1, g_2) *is a generating set of G with* $\operatorname{ord}(g_1) > n_1$ *and* $\operatorname{ord}(g_2) = n_2$,
(b) $x_1, \ldots, x_{kn_2} \in [0, n_1 - 1]$ *with* $x_1 + \ldots + x_{kn_2} = n_1 - 1$.

(IV)
$$U = e_1^{sn_1-1} \prod_{i \in [1, kn_2-(s-1)n_1]} ((1 - x_i)e_1 + e_2), \quad \textit{where}$$

(a) (e_1, e_2) *is a basis of G with* $\operatorname{ord}(e_1) = n_2$ *and* $\operatorname{ord}(e_2) = n_1$,
(b) $s \in [2, kn_2/n_1 - 1]$,
(c) $x_1, \ldots, x_{kn_2-(s-1)n_1} \in [0, n_1 - 1]$ *with* $x_1 + \ldots + x_{kn_2-(s-1)n_1} = n_1 - 1$.

2. Notation and Preliminaries

Our notations and terminology are consistent with [25] and [32]. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set the discrete interval $[a, b] = \{x \in \mathbb{Z} : a \le x \le b\}$. Throughout this paper, all abelian groups will be written additively, and for $n \in \mathbb{N}$, we denote by C_n a cyclic group with n elements.

Let G be a finite abelian group. It is well-known that $|G| = 1$ or $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$, where $r = \mathsf{r}(G) \in \mathbb{N}$ is the *rank* of G, and $n_r = \exp(G)$ is the *exponent* of G. We denote by $|G|$ the *order* of G, and $\operatorname{ord}(g)$ the *order* of $g \in G$.

Let $\mathcal{F}(G)$ be the free abelian (multiplicatively written) monoid with basis G. Then sequences over G could be viewed as elements of $\mathcal{F}(G)$. A sequence $S \in \mathcal{F}(G)$ could be written as

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \in \mathbb{N}_0$ is the multiplicity of g in S. We call

- $\operatorname{supp}(S) = \{g \in G \colon \mathsf{v}_g(S) > 0\} \subset G$ the *support* of S, and
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S.

Let $t \in \mathbb{N}$. We denote

$$\Sigma_{\le t}(S) = \left\{ \sum_{i \in I} g_i \colon I \subseteq [1, l] \text{ with } 1 \le |I| \le t \right\} .$$

A sequence $T \in \mathcal{F}(G)$ is called a subsequence of S if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G$, and denoted by $T \mid S$. If $T \mid S$, then we denote

$$T^{-1} S = \prod_{g \in G} g^{\mathsf{v}_g(S) - \mathsf{v}_g(T)} \in \mathcal{F}(G) \,.$$

Let $T_1, T_2 \in \mathcal{F}(G)$. We set

$$T_1 T_2 = \prod_{g \in G} g^{\mathsf{v}_g(T_1) + \mathsf{v}_g(T_2)} \in \mathcal{F}(G) \,.$$

If $T_1, \ldots, T_t \in \mathcal{F}(G)$ such that $T_1 \cdot \ldots \cdot T_t \,|\, S$, where $t \ge 2$, then we say $T_1, \ldots, T_t$ are disjoint subsequences of S.

Every map of abelian groups $\phi : G \to H$ extends to a map from the sequences over G to the sequences over H by setting $\phi(S) = \phi(g_1) \cdot \ldots \cdot \phi(g_l)$. If ϕ is a homomorphism, then $\phi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \mathsf{ker}(\phi)$.

We denote by $\mathsf{E}(G)$ the Gao's constant which is the smallest integer ℓ such that every sequence over G of length ℓ has a zero-sum subsequence of length $|G|$ and by $\eta(G)$ the smallest integer ℓ such that every sequence over G of length ℓ has a zero-sum subsequence T of length $1 \le |T| \le \exp(G)$. Let $\mathsf{d}(G)$ be the maximal length of a sequence over G that has no zero-sum subsequence. Then it is easy to see that $\mathsf{d}(G) = \mathsf{D}(G) - 1$. The following result is well-known and we may use it without further mention.

Lemma 2.1. *Let G be a finite abelian group. Then $\mathsf{E}(G) = |G| + \mathsf{d}(G) \le 2|G| - 1$.*

Proof. See [27, Propositions 5.7.9.2 and 5.1.4.4]. □

We also need the following lemmas.

Lemma 2.2. *Let G be a finite abelian group. If $\mathsf{D}(G) = |G|$, then G is cyclic and for every minimal zero-sum sequence S over G of length $|G|$, there exists $g \in G$ with $\operatorname{ord}(g) = |G|$ such that $S = g^{|G|}$.*

Proof. Let $n = \exp(G)$. By [32, Theorem 5.5.5], we have $\mathsf{D}(G) \le n + n \log \frac{|G|}{n}$, whence $\mathsf{D}(G) = |G|$ implies that G is cyclic. The remaining assertion follows from [27, Theorem 5.1.10.1]. □

Lemma 2.3. *Let G be a finite abelian group and let $H \subset G$ be a proper subgroup. Then $\mathsf{D}_k(H) < \mathsf{D}_k(G)$ for all $k \in \mathbb{N}$.*

Proof. The assertion follows from [27, Lemma 6.1.3]. □

Theorem 2.4. *Let $G = C_{n_1} \oplus C_{n_2}$ with $n_1 \mid n_2$, where $n_1, n_2 \in \mathbb{N}$, and let $k \in \mathbb{N}$. Then $\eta(G) = 2n_1 + n_2 - 2$ and $\mathsf{D}_k(G) = n_1 + kn_2 - 1$. In particular, $\mathsf{D}(G) = n_1 + n_2 - 1$.*

Proof. The assertion follows from [27, Theorems 5.8.3 and 6.1.5]. □

Theorem 2.5. *Let $G = C_n \oplus C_{mn}$ with $n \ge 2$ and $m \ge 1$. A sequence S over G of length $\mathsf{D}(G) = n + mn - 1$ is a minimal zero-sum sequence if and only if it has one of the following two forms:*

(I)
$$S = e_1^{\mathrm{ord}(e_1)-1} \prod_{i=1}^{\mathrm{ord}(e_2)} (x_i e_1 + e_2),$$
where
(a) *$\{e_1, e_2\}$ is a basis of G,*
(b) *$x_1, \ldots, x_{\mathrm{ord}(e_2)} \in [0, \mathrm{ord}(e_1) - 1]$ and $x_1 + \ldots + x_{\mathrm{ord}(e_2)} \equiv 1 \mod \mathrm{ord}(e_1)$.*
In this case, we say that S is of type I(a) or I(b) according to whether $\mathrm{ord}(e_2) = n$ or $\mathrm{ord}(e_2) = mn > n$.

(II)
$$S = f_1^{sn-1} f_2^{(m-s)n+\epsilon} \prod_{i=1}^{n-\epsilon} (-x_i f_1 + f_2),$$
where
(a) *$\{f_1, f_2\}$ is a generating set for G with $\mathrm{ord}(f_2) = mn$ and $\mathrm{ord}(f_1) > n$,*
(b) *$\epsilon \in [1, n-1]$ and $s \in [1, m-1]$,*
(c) *$x_1, \ldots, x_{n-\epsilon} \in [1, n-1]$ with $x_1 + \ldots + x_{n-\epsilon} = n - 1$,*
(d) *either $s = 1$ or $nf_1 = nf_2$, with both holding when $m = 2$, and*
(e) *either $\epsilon \ge 2$ or $nf_1 \ne nf_2$.*
In this case, we say that S is of type II.

Proof. The characterization of minimal zero-sum sequences of maximal length over groups of rank two was done in a series of papers by Gao, Geroldinger, Grynkiewicz, Reiher, and Schmid. For the formulation used above we refer to [26, Main Proposition 5.4]. □

Lemma 2.6. *Let G be a finite abelian group, let H be a cyclic subgroup of G, and let $\varphi\colon G \to G/H$ be the canonical epimorphism. If $S \in \mathcal{M}_k(G)$, then $\varphi(S) \in \mathcal{M}_{k|H|}(G/H)$.*

Proof. Suppose $S \in \mathcal{M}_k(G)$. Assume to the contrary that $\varphi(S) \notin \mathcal{M}_{k|H|}(G/H)$. Then we can decompose $S = T_1 \cdot \ldots \cdot T_{k|H|+1}$ such that $\varphi(T_i)$, $i \in [1, k|H|+1]$, are nontrivial zero-sum sequences. Therefore $S_\sigma := \sigma(T_1) \cdot \ldots \cdot \sigma(T_{k|H|+1})$ is a zero-sum sequence over H with length $k|H|+1 > \mathsf{D}_k(H)$. It follows by the definition of $\mathsf{D}_k(H)$ that S_σ and hence S are both a product of $k+1$ nontrivial zero-sum subsequences, a contradiction to $S \in \mathcal{M}_k(G)$. □

3. Proof of main theorems

Proposition 3.1. *Let G be a finite abelian group of rank at most 2, let $k \in \mathbb{N}$, and let S be a zero-sum sequence over G of length $\mathsf{D}_k(G)$. Then $S \in \mathcal{M}_k(G)$ if and only if $0 \notin \Sigma_{\le \exp(G)-1}(S)$.*

Proof. Suppose $0 \notin \Sigma_{\le \exp(G)-1}(S)$. Assume to the contrary that $S = T_1 \cdot \ldots \cdot T_{k+1}$, where T_i is a nontrivial zero-sum subsequence for each $i \in [1, k+1]$. Then $|T_i| \ge \exp(G)$ for each $i \in [1, k+1]$, ensuring $\mathsf{D}_k(G) = |S| \ge (k+1)\exp(G)$. If G is cyclic, then $\mathsf{D}_k(G) = k\exp(G) < (k+1)\exp(G)$ (by Theorem 2.4), a contradiction. If $\mathsf{r}(G) = 2$, then $\mathsf{D}_k(G) = k\exp(G) + |G|/\exp(G) - 1 < (k+1)\exp(G)$ (by Theorem 2.4), a contradiction.

Suppose $S \in \mathcal{M}_k(G)$. Assume to the contrary that $0 \in \Sigma_{\le \exp(G)-1}(S)$. Then S has a zero-sum subsequence T with $1 \le |T| \le \exp(G)-1$. If $k = 1$, then it follows from $|S| = \mathsf{D}(G) > \exp(G)-1$ that $S \notin \mathcal{A}(G)$, a contradiction. Thus we may assume that $k \ge 2$ and hence $T^{-1}S \in \mathcal{M}_{k-1}(G)$. If G is cyclic, then Theorem 2.4 implies

$$(k-1)\exp(G) + 1 = \mathsf{D}_k(G) - (\exp(G) - 1) \le |T^{-1}S| \le \mathsf{D}_{k-1}(G) = (k-1)\exp(G)\,,$$

a contradiction. If $\mathsf{r}(G) = 2$, then Theorem 2.4 implies

$$\begin{aligned}(k-1)\exp(G) + |G|/\exp(G) &= \mathsf{D}_k(G) - (\exp(G) - 1)\\ \le |T^{-1}S| \le \mathsf{D}_{k-1}(G) &= (k-1)\exp(G) + |G|/\exp(G) - 1\,,\end{aligned}$$

a contradiction. $\square$

We first investigate the associated inverse problem for cyclic groups.

Theorem 3.2. *Let G be cyclic, let $k \in \mathbb{N}$, and let S be a zero-sum sequence over G of length $\mathsf{D}_k(G)$. Then $S \in \mathcal{M}_k(G)$ if and only if there exists $g \in G$ with $\mathrm{ord}(g) = |G|$ such that $S = g^{k|G|}$.*

Proof. Note that $\mathsf{D}_k(G) = k|G|$ (by Theorem 2.4). If $S = g^{k|G|}$ for some $g \in G$ with $\mathrm{ord}(g) = |G|$, then the minimal zero-sum subsequence of S can only be of the form $g^{|G|}$, whence S is a product of at most k zero-sum subsequnces. It follows that $S \in \mathcal{M}_k(G)$.

Suppose $S \in \mathcal{M}_k(G)$. Let T be a minimal zero-sum subsequence of S. By Proposition 3.1, we have $\exp(G) \le |T| \le \mathsf{D}(G)$, whence $|T| = |G|$ (since G is cyclic). It follows from Lemma 2.2 that there exists $g \in G$ with $\mathrm{ord}(g) = |G|$ such that $T = g^{|G|}$. Assume to the contrary that there exists $h \,|\, T^{-1}S$ such that $h = sg$ with $s \in [2, n]$, ensuring that $g^{|G|-s}h$ is a zero-sum subsequence of S with length $|G| - s + 1 \le |G| - 1$, a contradiction to Proposition 3.1. Therefore $\mathrm{supp}(T^{-1}S) \subset \{g\}$ and hence $S = g^{k|G|}$. $\square$

Next, we prove Theorem 1.1 which could be handled easily by Proposition 3.1 and [18, Theorem 7.1].

Proof of Theorem 1.1. If U is of type I, then since $\mathrm{supp}(U) \subset \{e_1\} \cup \langle e_1 \rangle + e_2$ and $\mathrm{ord}(e_1) = \mathrm{ord}(e_2) = n$, we obtain that $0 \notin \Sigma_{\le n-1}(U)$. It follows from Proposition 3.1 that $U \in \mathcal{M}_k(G)$.

Suppose U is of type II. Assume to the contrary that there exists a nontrivial zero-sum subsequence T of U with $|T| \le n-1$. If $xe_1+2e_2 \nmid T$, then $\mathrm{supp}(T) \subset \{e_1\} \cup \langle e_1\rangle + e_2$ and hence $|T| \ge n$, a contradiction. Thus $T = (xe_1+2e_2)e_1^{\alpha}e_2^{\beta}(xe_1+e_2)^{\gamma}$ for some $\alpha, \beta, \gamma \in \mathbb{N}_0$, whence $2+\beta+\gamma \equiv 0 \mod n$ and $x(1+\gamma)+\alpha \equiv 0 \mod n$. Since $|T| = 1+\alpha+\beta+\gamma \le n-1$, we obtain that $\alpha = 0$, $\beta+\gamma = n-2$, and $n \mid x(1+\gamma)$. It follows from $\gcd(x,n) = 1$ that $n \mid 1+\gamma$, a contradiction to $\gamma+\beta = n-2$. Thus $0 \notin \Sigma_{\le n-1}(U)$. It follows from Proposition 3.1 that $U \in \mathcal{M}_k(G)$.

Suppose $U \in \mathcal{M}_k(G)$. By [29], the group $G = C_n \oplus C_n$ has Property B (see [25, Chapter 5] for the definition of Property B) and by [19, Theorem 6.7.2], every sequence over G of length $3n-2$ has a zero-sum subsequence of length n or $2n$. Thus all the assumptions of [18, Theorem 7.1] are fulfilled and hence the assertions follows from [18, Theorem 7.1]. □

Lemma 3.3. *Let $G = C_n \oplus C_n$ with $n \ge 2$ and let $k \ge 2$. If $S \in \mathcal{F}(G)$ is a zero-sum sequence with $|S| = (k+1)n-1$ and $0 \notin \Sigma_{\le n-1}(S)$, then there is a basis (e_1, e_2) for G such that either*

1. $\mathrm{supp}(S) \subseteq \{e_1\} \cup \big(\langle e_1\rangle + e_2\big)$ *and* $\mathsf{v}_{e_1}(S) \equiv -1 \mod n$, *or*
2. $S = e_1^{an}e_2^{bn-1}(xe_1+e_2)^{cn-1}(xe_1+2e_2)$ *for some $x \in [2, n-2]$ with $\gcd(x,n) = 1$, and some $a, b, c \ge 1$ with $k+1 = a+b+c$.*

Proof. By Theorem 2.4, we have $\mathsf{D}_k(G) = (k+1)n-1$. Since $0 \notin \Sigma_{\le n-1}(S)$, it follows from Proposition 3.1 that $S \in \mathcal{M}_k(G)$. Now the assertion follows from Theorem 1.1. Moreover, there is a direct proof of this lemma under the assumption of $G = C_n \oplus C_n$ having Property B (see [33, Lemma 3.2]). □

The following lemma shows two special cases of Theorem 1.2.

Lemma 3.4. *Let $G = C_{n_1} \oplus C_{n_2}$ with $1 < n_1 \mid n_2$ and $n_1 < n_2$, let $k \ge 2$, and let $U \in \mathcal{M}_k(G)$ with $|U| = \mathsf{D}_k(G)$.*

1. *If there is some $e_1 \in \mathrm{supp}(U)$ such that $\mathrm{ord}(e_1) = n_1$ and $\mathsf{v}_{e_1}(U) \ge n_1-1$, then there exists $e_2 \in G$ with $\mathrm{ord}(e_2) = n_2$ such that (e_1, e_2) is a basis of G and*
$$U = e_1^{n_1-1}\prod_{i\in[1,kn_2]}(x_ie_1+e_2)\,,$$
where $x_1, \ldots, x_{kn_2} \in [0, n_1-1]$ and $x_1+\ldots+x_{kn_2} \equiv 1 \mod n_1$.
2. *If there is some $e_2 \in \mathrm{supp}(U)$ such that $\mathrm{ord}(e_2) = n_2$ and $\mathsf{v}_{e_2}(U) \ge kn_2-1$, then there exists $e_1 \in G$ with $\mathrm{ord}(e_1) = n_1$ such that (e_1, e_2) is a basis of G and*
$$U = e_2^{kn_2-1}\prod_{i\in[1,n_1]}(e_1+x_ie_2)\,,$$
where $x_1, \ldots, x_{n_1} \in [0, n_2-1]$ and $x_1+\ldots+x_{n_1} \equiv 1 \mod n_2$.

Proof. 1. Suppose there exists $e_1 \in \mathrm{supp}(U)$ with $\mathrm{ord}(e_1) = n_1$ and $\mathsf{v}_{e_1}(U) \ge n_1-1$. Let
$$H = \langle e_1\rangle$$

and let $\phi_H : G \to G/H$ be the canonical epimorphism. Define T by

$$U = e_1^{n_1-1}T, \text{ where } T \in \mathcal{F}(G). \tag{3.1}$$

Then $\phi_H(T)$ is zero-sum over G/H of length $\mathsf{D}_k(G) - (n_1 - 1) = kn_2$ (by Theorem 2.4). Assume to the contrary that $0 \in \Sigma_{\le n_2-n_1}(\phi_H(T))$. Then there exists a nontrivial subsequence T' of T with $|T'| \le n_2 - n_1$ such that $\sigma(T') = se_1$ for some $s \in [1, n_1]$. It follows that $e_1^{n_1-s}T'$ is zero-sum of length $n_1 - s + |T'| \le n_1 - 1 + n_2 - n_1 \le n_2 - 1$, a contradiction to Proposition 3.1. Thus $0 \notin \Sigma_{\le n_2-n_1}(\phi_H(T))$.

By Lemma 2.1, we have $\mathsf{E}(G/H) \le 2|G/H| - 1 = 2n_2 - 1$ and by repeatedly using this result, we can factorize $T = T_1 \cdot \ldots \cdot T_k$ such that $|T_i| = n_2$ and $\phi_H(T_i)$ is zero-sum for every $i \in [1, k]$. If there exists $i \in [1, k]$ such that $\phi_H(T_i)$ is not minimal, then $T_i = T_i^{(1)}T_i^{(2)}$ with $|T_i^{(1)}| \ge |T_i^{(2)}| \ge 1$ such that both $\phi_H(T_i^{(1)})$ and $\phi_H(T_i^{(2)})$ are zero-sum, whence $|T_i^{(2)}| \le \frac{n_2}{2} \le n_2 - n_1$, a contradiction to $0 \notin \Sigma_{\le n_2-n_1}(\phi_H(T))$. Thus for each $i \in [1, k]$, the sequence $\phi_H(T_i)$ is a minimal zero-sum subsequence of length $n_2 = |G/H|$, ensuring by Lemma 2.2 that G/H must be cyclic. It follows from Lemma 2.2 that there exists $e_2 \in G$ such that $\phi_H(e_2)$ is a generator of G/H and $\phi_H(T_1) = \phi_H(e_2)^{n_2}$. Assume that there exists $j \in [2, k]$ such that $\phi_H(T_j) \ne \phi_H(e_2)^{n_2}$, then there exists $s \in [2, n_2 - 1]$ with $\gcd(s, n_2) = 1$ such that $\phi_H(T_j) = (s\phi_H(e_2))^{n_2}$. Note that $s \ge 2$. By letting $t \in \mathbb{N}$ be minimal such that $t(s - 1) \ge n_1$, we have $\phi_H(e_2)^{n_2-ts}(s\phi_H(e_2))^t \mid \phi_H(T_1T_j)$ is zero-sum of length $n_2 - ts + t \le n_2 - n_1$, a contradiction to $0 \notin \Sigma_{\le n_2-n_1}(\phi_H(T))$. Therefore $\phi_H(T) = \phi_H(e_2)^{kn_2}$. Moreover, $\operatorname{ord}(\phi_H(e_2)) = n_2$ ensures that $\operatorname{ord}(e_2)$ is a multiple of $n_2 = \exp(G)$, which is the maximal order of an element from G. This forces $\operatorname{ord}(e_2) = \operatorname{ord}(\phi_H(e_2)) = n_2$, which combined with $G = \langle e_1, e_2\rangle$ and $\operatorname{ord}(e_1) = n_1$ ensures that $G = \langle e_1\rangle \oplus \langle e_2\rangle$ with $\operatorname{ord}(e_2) = n_2$.

Let $\pi_2 : G \to \langle e_2\rangle$ be the projection homomorphism (with kernel $H = \langle e_1\rangle$) given by $\pi_2(xe_1 + ye_2) = ye_2$. Since we now know $H = \langle e_1\rangle$ is a direct summand in G, we can identify π_2 with ϕ_H, whence $\pi_2(T) = e_2^{kn_2}$, ensuring $\operatorname{supp}(T) \subset \langle e_1\rangle + e_2$. Combined with (3.1), the assertion now readily follows from U being zero-sum.

2. Suppose there exists $e_2 \in \operatorname{supp}(U)$ with $\operatorname{ord}(e_2) = n_2$ and $\mathsf{v}_{e_2}(U) \ge kn_2 - 1$. Then

$$U = e_2^{kn_2-1}T, \text{ where } T \in \mathcal{F}(G) \text{ with } |T| = n_1 .$$

Since $e_2^{(k-1)n_2}$ is a product of $k - 1$ zero-sum subsequences of length n_2, it follows from $U \in \mathcal{M}_k(G)$ that $e_2^{n_2-1}T$ must be a minimal zero-sum sequence. Let

$$H = \langle e_2\rangle$$

and since $\exp(G) = n_2$, we have H is a direct summand in G and hence there exists $e_1 \in G$ with $\operatorname{ord}(e_1) = n_1$ such that $G = H \oplus \langle e_1\rangle$. Let $\pi_1 : G \to \langle e_1\rangle$ be the projection homomorphism (with kernel $H = \langle e_2\rangle$) given by $\pi_2(xe_1 + ye_2) = xe_1$. Then $\pi_1(T)$ is zero-sum over G/H of length n_1. Assume that $\pi_1(T)$ is not minimal. Then $T = T^{(1)}T^{(2)}$ with both $\pi_1(T^{(1)})$ and $\pi_1(T^{(2)})$ nontrivial zero-sum. Say $\sigma(T^{(1)}) = se_2$ for some $s \in [1, n_2]$. Then $e_2^{n_2-s}T^{(1)}$ is a proper nontrivial zero-sum subsequence of $e_2^{n_2-1}T$, a contradiction. Thus $\pi_1(T)$ is a minimal zero-sum sequence over $\langle e_1\rangle$ of length n_1, and hence there exists $s\in[1,n_1-1]$ with $\gcd(s,n_1)=1$ such that $\pi_1(T)=(se_1)^{n-1}$. By replacing the basis (e_1,e_2) with (se_1,e_2), the assertion now readily follows from U being zero-sum. □

Now we are ready to prove our main Theorem 1.2.

Proof of Theorem 1.2. Let

$$n=n_1 \quad\text{and}\quad n_2=mn, \qquad \text{with } m\ge 2.$$

Then

$$G=C_n\oplus C_{mn}.$$

Suppose $k=1$. By Theorem 2.5, it suffices to show that type II sequences in Theorem 2.5 is equivalent to type III and type IV sequences in Theorem 1.2.

Let $S=f_1^{sn-1}f_2^{(m-s)n+\epsilon}\prod_{i=1}^{n-\epsilon}(-x_if_1+f_2)$ be a type II sequence in Theorem 2.5. If $s=1$, then it is easy to see that S is of type III in Theorem 1.2. If $s\ge 2$, then II.(d) in Theorem 2.5 implies that $nf_1=nf_2$. Since (f_1,f_2) is a generating set with $\mathrm{ord}(f_2)=mn$, we obtain (f_2-f_1,f_1) is basis of G with $\mathrm{ord}(f_2-f_1)=n$ and $\mathrm{ord}(f_1)=mn$. Set $g_1=f_1$ and $g_2=f_2-f_1$. Then

$$S=g_1^{sn-1}(g_1+g_2)^{(m-s)n+\epsilon}\prod_{i=1}^{n-\epsilon}(-x_ig_1+g_1+g_2)=g_1^{sn-1}\prod_{i=1}^{(m-s+1)n}((1-x_i)g_1+g_2)\,,$$

where $x_i=0$ for all $i\in[n-\epsilon+1,(m-s+1)n]$, whence S is of type IV in Theorem 1.2.

For the inverse, let S be a type III or type IV sequence in Theorem 1.2. By letting $n-\epsilon$ be the number of x_i's that is not zero, it is to easy to see that S is a type II sequence in Theorem 2.5.

Now we assume that $k\ge 2$. Since $\mathsf{D}_k(G)=kn_2+n_1-1$ by Theorem 2.4, we see that all sequences given in (I), (II), (III), or (IV) have length $\mathsf{D}_k(G)$. It is straightforward to check that any sequence U satisfying the conditions given in (I) or (II) has $0\notin\Sigma_{\le mn-1}(U)$, whence $U\in\mathcal{M}_k(G)$ follows from Proposition 3.1. Let us next verify that type III and type IV sequences U have $0\notin\Sigma_{\le mn-1}(U)$, and then $U\in\mathcal{M}_k(G)$ follows from Proposition 3.1.

Let U be a type III sequence. Consider a nontrivial minimal zero-sum subsequence $T\mid U$. It is sufficient to show $|T|\ge n_2$. After renumbering if necessary, we may assume that $T=g_1^u\prod_{i=1}^{v}(-x_ig_1+g_2)$, where $u\in[0,n_1-1]$ and $v\in[0,kn_2]$. Thus $0=\sigma(T)=(u-\sum_{i=1}^{v}x_i)g_1+vg_2$. Since (g_1,g_2) is a generating set with $\mathrm{ord}(g_2)=n_2$, we obtain $u-\sum_{i=1}^{v}x_i$ is a multiple of n_1. It follows from $|u-\sum_{i=1}^{v}x_i|\in[0,n-1]$ that $u-\sum_{i=1}^{v}x_i=0$ and hence v is a multiple of $\mathrm{ord}(g_2)$. Since $v=0$ implies $u=0$ and hence $|T|=u+v=0$, it follows from T is nontrivial that $v\ge\mathrm{ord}(g_2)=n_2$ and hence $|T|\ge v\ge n_2$.

Let U be a type IV sequence. Consider a nontrivial minimal zero-sum subsequence $T \mid U$. It is sufficient to show $|T| \ge n_2$. Suppose

$$T = e_1^a \prod_{i \in I} ((1-x_i)e_1 + e_2)$$

for some $a \in [0, sn-1]$ and nonempty $I \subset [1, (km-s+1)n]$ with $n \mid |I|$. By considering the sum of e_1-coordinates, it follows that $a + |I| - \sum_{i \in I} x_i \equiv 0 \mod mn$, and hence $a \equiv \sum_{i\in I} x_i \mod n$. Set $|I| = s_1 n$ and $a = s_2 n + \sum_{i \in I} x_i$, where $s_1 \in [1, km-s+1]$ and $s_2 \in [0, s-1]$. Then $(s_1+s_2)n = a + |I| - \sum_{i\in I} x_i \equiv 0 \mod mn$, whence $s_1 + s_2 \equiv 0 \mod m$. It follows from

$$s_1 + s_2 \ge 1 \quad \text{and} \quad |T| = a + |I| = (s_1+s_2)n + \sum_{i \in I} x_i \le \mathsf{D}(G) = mn + n - 1 (\text{by Theorem 2.4})$$

that $s_1 + s_2 = m$ and $|T| \ge mn = n_2$.

It remains to show that every sequence in $\mathcal{M}_k(G)$ must have the form either given by (I), (II), (III), or (IV).

Let $U \in \mathcal{M}_k(G)$ of length $|U| = kmn + n - 1$ and suppose

$$U \quad \text{does not have the form of type I or II.} \tag{3.2}$$

We need to show that U has the form of type III or IV.

Let $\varphi : G \to G$ be a homomorphism with

$$\varphi(G) = \mathsf{im}\varphi \cong C_n \oplus C_n \quad \text{and} \quad \ker \varphi \cong C_m.$$

For instance, if (e_1, e_2) were a basis for G with $\mathrm{ord}(e_1) = n$ and $\mathrm{ord}(e_2) = mn$, then the map $xe_1 + ye_2 \mapsto xe_1 + yme_2$ is one such a map.

We define a **block decomposition** of U to be a tuple $W = (W_0, W_1, \ldots, W_{km-1})$, where

$$U = W_0 W_1 \cdot \ldots \cdot W_{km-1}$$

with each $\varphi(W_i)$ a nontrivial zero-sum for $i \in [0, km-1]$.

A1. Let $W = (W_0, \ldots, W_{km-1})$ be a block decomposition of U.

1. For all $i \in [0, km-1]$, we have $\varphi(W_i)$ is minimal, $\sigma(W_i)$ is a generator of $\ker(\varphi)$, and
$$\sigma(W_0) \cdot \ldots \cdot \sigma(W_{km-1}) = \sigma(W_0)^{km}.$$
2. If there exist subsequences $S \mid W_i$ and $T \mid W_j$ with $i \ne j$ such that $\sigma(\varphi(S)) = \sigma(\varphi(T))$, then $\sigma(S) = \sigma(T)$.
3. If there are distinct blocks W_i and W_j having terms $g \in \mathrm{supp}(W_i)$ and $h \in \mathrm{supp}(W_j)$ with $\varphi(g) = \varphi(h)$, then all terms from U equal to $\varphi(g)$ are equal.

*Proof of **A1**.* 1. We have $W_\sigma := \sigma(W_0)\cdot\ldots\cdot\sigma(W_{km-1})$ is a sequence over $\ker(\varphi)$ of length $km = \mathsf{D}_k(\ker(\varphi))$. Since $U \in \mathcal{M}_k(G)$, we have $W_\sigma \in \mathcal{M}_k(\ker(\varphi))$. It follows from Theorem 3.2 that $W_\sigma = \sigma(W_0)^{km}$ with $\sigma(W_0)$ a generator of $\ker(\varphi)$.

Assume to the contrary that there exists some $i \in [0, km-1]$ such that $\varphi(W_i)$ is not minimal. Then we can decompose $W_i = W_i^{(1)} W_i^{(2)}$ such that both $\varphi(W_i^{(1)})$ and $\varphi(W_i^{(2)})$ are nontrivial zero-sum. It follows that $W_\sigma^* := \sigma(W_i)^{-1}\sigma(W_i^{(1)})\sigma(W_i^{(2)})W_\sigma$ is a sequence over $\ker(\varphi)$ of length $km+1 > \mathsf{D}_k(\ker(\varphi))$, whence $W_\sigma^* \notin \mathcal{M}_k(\ker(\varphi))$, a contradiction to $U \in \mathcal{M}_k(G)$.

2. Suppose there exist subsequences $S \mid W_i$ and $T \mid W_j$ with $i \neq j$ such that $\sigma(\varphi(S)) = \sigma(\varphi(T))$. Then we can define $W_i' = S^{-1}W_iT$ and $W_j' = T^{-1}W_jS$. Setting $W_s' = W_s$ for all $s \neq i, j$, we then obtain a new block decomposition $W' = (W_0', W_1', \ldots, W_{km-1}')$ with associated sequence $W_\sigma' = \sigma(W_i)^{-1}\sigma(W_j)^{-1}W_\sigma\sigma(W_i')\sigma(W_j')$. Since $k \geq 2$ and $m \geq 2$, we have $km-1 \geq 3$ and it follows by Item 1 that $W_\sigma' = \sigma(W_s)^{km}$ for some $s \neq i, j$, whence $W_\sigma' = W_\sigma$. Therefore $\sigma(W_i') = \sigma(W_i)$, ensuring $\sigma(S) = \sigma(T)$.

3. Suppose there are distinct blocks W_i and W_j having terms $g \in \operatorname{supp}(W_i)$ and $h \in \operatorname{supp}(W_j)$ with $\varphi(g) = \varphi(h)$. It follows by Item 2 that $g = h$. In such case, the assertion follows by doing this for all g and h contained in distinct blocks with $\varphi(g) = \varphi(h)$. □(**A1**)

Since $U \in \mathcal{M}_k(G)$, we have $\varphi(U) \in \mathcal{M}_{km}(\varphi(G))$ by Lemma 2.6. In view of Proposition 3.1, we have that

$$0 \notin \Sigma_{\leq n-1}(\varphi(G)). \tag{3.3}$$

Hence, since $\varphi(G) \cong C_n \oplus C_n$, we conclude from Lemma 3.3 that there is some basis $(\overline{e}_1, \overline{e}_2)$ for $\varphi(G) \cong C_n \oplus C_n$ such that either

$$\operatorname{supp}(\varphi(U)) \subset \{\overline{e}_1\} \cup \big(\langle \overline{e}_1 \rangle + \overline{e}_2\big), \tag{3.4}$$

or else

$$\varphi(U) = (\overline{e}_1)^{an}(\overline{e}_2)^{bn-1}(u\overline{e}_1 + \overline{e}_2)^{cn-1}(u\overline{e}_1 + 2\overline{e}_2), \tag{3.5}$$

for some $u \in [2, n-2]$ with $\gcd(u, n) = 1$, and some $a, b, c \geq 1$.

We distinguish two cases depending on whether (3.4) or (3.5) holds.

CASE 1: $\varphi(U) = (\overline{e}_1)^{an}(\overline{e}_2)^{bn-1}(u\overline{e}_1 + \overline{e}_2)^{cn-1}(u\overline{e}_1 + 2\overline{e}_2)$, for some $u \in [2, n-2]$ with $\gcd(u, n) = 1$, and some $a, b, c \geq 1$.

Since $u \in [2, n-2]$ with $\gcd(u, n) = 1$, it follows that $n \geq 5$. In view of the hypothesis of CASE 1, we have $(a+b+c)n - 1 = |U| = kmn + n - 1$, implying

$$a + b + c = km + 1. \tag{3.6}$$

Set

$$\overline{e}_3 = u\overline{e}_1 + \overline{e}_2, \quad \text{so} \quad \overline{e}_2 = (n-u)\overline{e}_1 + \overline{e}_3,$$

and note that $\overline{e}_1 = u^*(\overline{e}_2 - \overline{e}_3)$, where $u^* \in [2, n-2]$ is the multiplicative inverse of $-u$ modulo n, so

$$u^* u \equiv -1 \mod n \quad \text{with } u^* \in [2, n-2].$$

In view of the hypothesis of CASE 1, there is a block decomposition $W = (W_0, W_1, \ldots, W_{km-1})$ of U with

$$\varphi(W_0) = (\overline{e}_1)^{n-1}(\overline{e}_2)^{u^*}(\overline{e}_3)^{n-u^*}, \quad \varphi(W_1) = (\overline{e}_3)^{u^*-1}(\overline{e}_2)^{n-u^*-1}\overline{e}_1(\overline{e}_2 + \overline{e}_3), \quad \text{and}$$
$$\varphi(W_i) \in \{(\overline{e}_1)^n, (\overline{e}_2)^n, (\overline{e}_3)^n\} \quad \text{for } i \in [2, km-1].$$

Let $z \in \operatorname{supp}(\varphi(U)) = \{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_2 + \overline{e}_3\}$. If $z = \overline{e}_2 + \overline{e}_3$, then we trivially have $g = h$ for all $g, h \in \operatorname{supp}(U)$ with $\varphi(g) = \varphi(h) = z$, since there is a unique term $g \in \operatorname{supp}(U)$ with $\varphi(g) = z$. Otherwise, since $\mathsf{v}_{\overline{e}_j}(W_i) > 0$ for all $j \in [1,3]$ and $i \in [0,1]$, it follows that there are distinct block W_i and W_j, for some $i, j \in [0, km-1]$, with terms $g \in \operatorname{supp}(W_i)$ and $h \in \operatorname{supp}(W_j)$ such that $\varphi(g) = \varphi(h) = z$. Then **A1**.3 implies

$$g, h \in \operatorname{supp}(U) \;\text{ with } \varphi(g) = \varphi(h) \quad \text{implies } g = h.$$

As a result, we can find representatives e_1 and e_2 for $\overline{e}_1$ and $\overline{e}_2$, and $\alpha, \beta \in \ker \varphi$, such that

$$\operatorname{supp}(U) = \{e_1,\; e_2,\; e_3 + \alpha,\; e_2 + e_3 + \beta\},$$

where $e_3 := ue_1 + e_2$, $\varphi(e_1) = \overline{e}_1$, $\varphi(e_2) = \overline{e}_2$, $\varphi(e_3 + \alpha) = \overline{e}_3 = u\overline{e}_1 + \overline{e}_2$, and $\varphi(e_2 + e_3 + \beta) = \overline{e}_2 + \overline{e}_3 = u\overline{e}_1 + 2\overline{e}_2$.

Since $u, u^* \in [2, n-2]$, it follows that there are subsequences $e_1^u e_2 \mid W_0$ and $e_3 + \alpha = ue_1 + e_2 + \alpha \mid W_1$. By **A1**.2, we have $ue_1 + e_2 = \sigma(e_1^u e_2) = ue_1 + e_2 + \alpha$, whence $\alpha = 0$. Likewise, there are subsequences $e_1^u e_2^2 \mid W_0$ and $e_2 + e_3 + \beta = ue_1 + 2e_2 + \beta \mid W_1$. By **A1**.2, we have $ue_1 + 2e_2 = \sigma(e_1^u e_2^2) = ue_1 + 2e_2 + \beta$, whence $\beta = 0$. As a result, $\operatorname{supp}(U) = \{e_1,\; e_2,\; ue_1 + e_2,\; ue_1 + 2e_2\}$, which together with the hypotheses of CASE 1 gives

$$U = e_1^{an} e_2^{bn-1}(ue_1 + e_2)^{cn-1}(ue_1 + 2e_2),$$
$$W_0 = e_1^{n-1} e_2^{u^*}(ue_1 + e_2)^{n-u^*} \quad \text{and} \quad W_1 = e_1 e_2^{n-u^*-1}(ue_1 + e_2)^{u^*-1}(ue_1 + 2e_2). \tag{3.7}$$

From (3.7), we have $\operatorname{supp}(U) \subset \langle e_1, e_2 \rangle$. If this were a proper subgroup of $G = C_n \oplus C_{mn}$, then $\mathsf{D}_k(G) = |U| \le \mathsf{D}_k(\langle e_1, e_2 \rangle) < \mathsf{D}_k(G)$ (by Lemma 2.3), a contradiction. Therefore we instead conclude that

$$\langle e_1, e_2 \rangle = G = C_n \oplus C_{mn}. \tag{3.8}$$

If $T \mid W_0 W_1$ is any proper, nontrivial subsequence with $\varphi(T)$ zero-sum, then we can set $W_0' = T$, define W_1' by $W_0' W_1' = W_0 W_1$ and set $W_i' = W_i$ for all $i \ge 2$ to thereby obtain a new block decomposition W'. Since $km \ge 4$, **A1**.1 ensures that $\sigma(W_0') = g_0$, where $g_0 := \sigma(W_0)$ is a generator for $\ker \varphi \cong C_m$. This shows that

(3.9)

any proper, nontrivial subsequence $T \mid W_0 W_1$ with $\varphi(T)$ zero-sum has $\sigma(T) = g_0 := \sigma(W_0)$.

In particular, since $e_1^n \mid W_0W_1$ and $e_1^{n-u}e_2^{n-1}(ue_1+e_2) \mid W_0W_1$, we have

$$ne_1 = \sigma(e_1^{n-u}e_2^{n-1}(ue_1+e_2)) = ne_1 + ne_2 = g_0,$$

forcing $ne_2 = 0$. In view of $\mathrm{ord}(\overline{e}_2) = n$, we have $\mathrm{ord}(e_2) = n$. Since $\mathsf{v}_{e_2}(U) = bn-1 \geq n-1$, it follows from Lemma 3.4.1 that U has the form of type I, a contradiction to our assumption of (3.2).

CASE 2: $\mathrm{supp}(\varphi(U)) \subset \{\overline{e}_1\} \cup \big(\langle \overline{e}_1\rangle + \overline{e}_2\big)$.

Let $W = (W_0, \ldots, W_{km-1})$ be a block decomposition of U. We say W is **refined** if $|W_i| \leq n$ for each $i \in [1, km-1]$. Since $|U| = (km-2)n+3n-1 \geq (km-2)n+3n-2 = (km-2)n+\eta(\varphi(G))$ with $\sigma(U) = 0$ by Theorem 2.4, repeated application of the definition $\eta(\varphi(G))$ to the sequence $\varphi(U)$ shows that U has a refined block decomposition.

Let $W = (W_0, \ldots, W_{km-1})$ be a refined block decomposition of U. In view of **A1**.1, we have $\varphi(W_0)$ is a minimal zero-sum sequence of terms from $\varphi(G) \cong C_n \oplus C_n$, thus with $|W_0| \leq \mathsf{D}(C_n \oplus C_n) = 2n-1$. Since each $|W_i| \leq n$ for $i \in [1, km-1]$, we have $2n-1 \geq |W_0| = |U| - \sum\limits_{i=1}^{km-1} |W_i| \geq kmn + n - 1 - (km-1)n = 2n-1$, forcing equality to hold in all estimates. In particular, we now conclude that

$$|W_0| = 2n-1 \quad \text{and} \quad |W_i| = n \quad \text{ for all } i \in [1, km-1], \tag{3.10}$$

for any refined block decomposition W, with $\varphi(W_0)$ always a minimal zero-sum of length $2n-1$.

In view of the hypothesis of CASE 2, any zero-sum subsequence of $\varphi(U)$ must have the number of terms from $\langle \overline{e}_1\rangle + \overline{e}_2$ congruent to 0 modulo n. In particular,

$$\varphi(W_0) = (\overline{e}_1)^{n-1} \prod_{i\in[1,n]} (-x_i\overline{e}_1 + \overline{e}_2), \tag{3.11}$$

for some $x_1, \ldots, x_n \in \mathbb{Z}$ with $x_1 + \ldots + x_n \equiv n-1 \mod n$ and, for every $j \in [1, km-1]$, either

$$\varphi(W_j) = (\overline{e}_1)^n \;\text{ or }\; \varphi(W_j) = \prod_{i\in[1,n]} (-y_i\overline{e}_1 + \overline{e}_2), \tag{3.12}$$

for some $y_1, \ldots, y_n \in \mathbb{Z}$ with $y_1 + \ldots + y_n \equiv 0 \mod n$.

A2. There is some $e_1 \in G$ such that every $g \in \mathrm{supp}(U)$ with $\varphi(g) = \overline{e}_1$ has $g = e_1$.

*Proof of **A2**.* Let W be a refined block decomposition. Then (3.11) implies that $\mathsf{v}_{\overline{e}_1}(\varphi(W_0)) = n-1 \geq 1$. If there exists $i \in [1, km-1]$ such that $\overline{e}_1 \in \mathrm{supp}(\varphi(W_i))$, then the assertion follows by **A1**.3. Thus we may assume that $\mathrm{supp}(W_i) \subset \langle \overline{e}_1\rangle + \overline{e}_2$ for all $i \in [1, km-1]$. If $n = 2$, the assertion is trivial. Suppose $n \geq 3$.

Let $i \in [1, km-1]$ and $h \in \mathrm{supp}(W_i)$ be arbitrary, say with $\varphi(h) = y\overline{e}_1 + \overline{e}_2$. Suppose there is some $g \in \mathrm{supp}(W_0)$ with $\varphi(g) = x\overline{e}_1 + \overline{e}_2$ and $x \notin \{y, y+1\} \mod n$. Then, letting $z \in [1, n-2]$ be the integer such that $z + x \equiv y \mod n$, it follows that there is a subsequence $Tg \mid W_0$ with

$\varphi(Tg) = (\bar{e}_1)^z(x\bar{e}_1 + \bar{e}_2)$ and $\sigma(\varphi(Tg)) = (z + x)\bar{e}_1 + \bar{e}_2 = y\bar{e}_1 + \bar{e}_2 = \varphi(h)$. Note **A1**.2 implies that

$$\sigma(T) + \sigma(g) = \sigma(h).$$

Since $|T| = z \in [1, n-2]$, there are terms $f_1 \in \mathrm{supp}(T)$ and $f_2 \in \mathrm{supp}(T^{-1}W_0)$ with $\varphi(f_1) = \varphi(f_2) = \bar{e}_1$. Repeating this argument using the subsequence $T' = f_1^{-1}Tf_2$ in place of T, we again find that

$$f_2 - f_1 + \sigma(h) = f_2 - f_1 + \sigma(T) + \sigma(g) = \sigma(T') + \sigma(g) = \sigma(h),$$

implying that $f_1 = f_2$. Doing this for all $f_1 \in \mathrm{supp}(T)$ and $f_2 \in \mathrm{supp}(T^{-1}W_0)$ with $\varphi(f_1) = \varphi(f_2) = \bar{e}_1$ would then yield the desired conclusion for **A2**. So we can instead assume every $g \in \mathrm{supp}(W_0)$ with $\varphi(g) \in \langle \bar{e}_1 \rangle + \bar{e}_2$ has either $\varphi(g) = y\bar{e}_1 + \bar{e}_2$ or $\varphi(g) = (y+1)\bar{e}_1 + \bar{e}_2$. In view of (3.11), both possibilities must occur (since $x_1 + \ldots + x_n \equiv 1 \mod n$ in (3.11)), forcing

$$\mathrm{supp}(\varphi(W_0)) = \{\bar{e}_1,\ y\bar{e}_1 + \bar{e}_2,\ (y+1)\bar{e}_1 + \bar{e}_2\}. \tag{3.13}$$

Moreover, the above must be true for any $i \in [1, km-1]$ and $h \in \mathrm{supp}(W_i)$. Since $n \ge 3$, the value of y is uniquely forced by (3.13), which means

$$\varphi(W_i) = (y\bar{e}_1 + \bar{e}_2)^n \quad \text{for all } i \in [1, km-1]. \tag{3.14}$$

Suppose there are two terms $g_1g_2 \mid W_0$ with $\varphi(g_1) = \varphi(g_2) = (y+1)\bar{e}_1 + \bar{e}_2$. Let $h_1h_2 \mid W_1$ be a length two subsequence. Then there is a subsequence $Tg_1g_2 \mid W_0$ with $\varphi(Tg_1g_2) = (\bar{e}_1)^{n-2}((y+1)\bar{e}_1 + \bar{e}_2)^2$. Thus $\sigma(\varphi(Tg_1g_2)) = \sigma(\varphi(h_1h_2))$ and by **A1**.2, we conclude that $\sigma(T) + g_1 + g_2 = h_1 + h_2$. Since $1 \le |T| = n-2 < n-1$, we can find $f_1 \in \mathrm{supp}(T)$ and $f_2 \in \mathrm{supp}(T^{-1}W_0)$ with $\varphi(f_1) = \varphi(f_2) = \bar{e}_1$ and argue as before to conclude that $f_1 = f_2$. Doing this for all f_1 and f_2 then yields the desired conclusion for **A2**. So we can instead assume that $\mathsf{v}_{(y+1)\bar{e}_1+\bar{e}_2}(\varphi(U)) = 1$, implying via (3.13) and (3.11) that $\mathsf{v}_{y\bar{e}_1+\bar{e}_2}(\varphi(W_0)) = n-1$. Combined with (3.14), we find that $\mathsf{v}_{y\bar{e}_1+\bar{e}_2}(\varphi(U)) = kmn - 1$. Moreover, by **A1**.3, all $kmn-1$ of the corresponding terms from U must be equal to the same element (say) $g_0 \in G$, whence $\mathsf{v}_{g_0}(U) \ge kmn-1$, forcing $\mathrm{ord}(g_0) = mn$ else $\mathsf{v}_{g_0}(U) \ge (k+1)\,\mathrm{ord}(g_0)$ implies U contains $k+1$ disjoint zero-sum subsequences of length $\mathrm{ord}(g_0)$, contradicting the hypothesis that $U \in \mathcal{M}_k(G)$. Applying Lemma 3.4 shows U has the form of type II, a contradiction to our assumption (3.2). $\square$(**A2**)

In view of **A2**, we can decompose

$$U = e_1^{\mathsf{v}_{\bar{e}_1}(\varphi(U))}U^*$$

with $U^* \mid U$ the subsequence consisting of all terms g with $\varphi(g) \in \langle \bar{e}_1 \rangle + \bar{e}_2$ (view those as U^*-terms). Note that $\mathsf{v}_{e_1}(U) \ge \mathsf{v}_{e_1}(W_0) = n-1$. If $\mathrm{ord}(e_1) = n$, then Lemma 3.4.1 implies that U has the form of type I, a contradiction to our assumption (3.2). Thus $\mathrm{ord}(e_1) > n$. Since $\varphi(e_1) = \bar{e}_1$ with $\mathrm{ord}(\bar{e}_1) = n$, it follows that $\mathrm{ord}(e_1)$ is a multiple of n. Thus

$$\mathrm{ord}(e_1) \ge 2n. \tag{3.15}$$

If $\varphi(W_j) = (\bar{e}_1)^n$ for all $j \in [1, km-1]$, then $\mathsf{v}_{e_1}(U) = n - 1 + (km-1)n = kmn - 1$ follows from **A2**. Since $U \in \mathcal{M}_k(G)$, we have $\mathsf{v}_{e_1}(U) = kmn - 1 \le k\,\mathrm{ord}(e_1)$, ensuring $\mathrm{ord}(e_1) = mn$. Now Lemma 3.4.2 implies that U has the from of type II, a contradiction to our assumption (3.2). Thus we can assume

(3.16) there is at least one block W_j with $j \ge 1$ containing some U^*-term.

Let $e_2 \,|\, W_j$ be some U^*-term. Then $\varphi(e_2) \in \langle \bar{e}_1 \rangle + \bar{e}_2$. We note that the hypotheses of CASE 2 hold with the basis $(\bar{e}_1, \bar{e}_2)$ replaced by the basis $(\varphi(e_1), \varphi(e_2))$. Thus, by replacing $\bar{e}_2$ by $\varphi(e_2)$, we may assume that $\varphi(e_2) = \bar{e}_2$. Let $I = [0, n-1]$ be the discrete interval of length n. Each U^*-term g can be written uniquely as $g = -\iota(g)e_1 + e_2 + \psi(g)$ for some $\iota(g) \in I \subset \mathbb{Z}$ and $\psi(g) \in \ker \varphi$.

A3. Suppose $W = (W_0, \ldots, W_{km-1})$ is a refined block decomposition for U. If $g \in \mathrm{supp}(W_0)$ and $h \in \mathrm{supp}(W_j)$ are U^*-terms, where $j \ge 1$, then

$$\psi(g) - \psi(h) = \begin{cases} 0 & \text{if } \iota(g) \ge \iota(h), \\ -ne_1 & \text{if } \iota(g) < \iota(h). \end{cases}$$

*Proof of **A3**.* Let $\iota(g) = x$ and $\iota(h) = y$, and let $\psi(g) = \alpha$ and $\psi(h) = \beta$. Then

$$g = -xe_1 + e_2 + \alpha \quad \text{and} \quad h = -ye_1 + e_2 + \beta.$$

If $x \ge y$, let $z = x - y \in [0, n-1]$. If $x < y$, let $z = x - y + n \in [1, n-1]$. In both cases, we have $\sigma(\varphi(e_1^z g)) = (z-x)\bar{e}_1 + \bar{e}_2 = -y\bar{e}_1 + \bar{e}_2 = \varphi(h)$, so **A1**.2 implies that $(z-x)e_1 + e_2 + \alpha = \sigma(e_1^z g) = h = -ye_1 + e_2 + \beta$, whence

$$\alpha - \beta = (x - y - z)e_1.$$

If $x \ge y$, then $z = x - y$, implying $\psi(g) - \psi(h) = \alpha - \beta = 0$. If $x < y$, then $z = x - y + n$, implying $\psi(g) - \psi(h) = \alpha - \beta = -ne_1$. $\square$(**A3**)

Note that e_2 is a U^*-term of some W_j with $\iota(e_2) = 0$ and $\psi(e_2) = 0$. Let $W_0^* \,|\, W_0$ be the subsequence consisting of all U^*-terms. Thus for every term g of W_0^*, we have $\iota(g) \ge 0 = \iota(e_2)$ and hence **A3** implies that $\psi(g) = \psi(e_2) = 0$, that is,

(3.17) for every term g of W_0^*, we have $\psi(g) = 0$.

Let h be a U^*-term of $W_0^{-1}U$. Then there exists $j \in [1, km-1]$ such that $h \,|\, W_j$. By (3.12), we have $\iota(h^{-1}W_0^* W_j) \mod n$ is a sequence of $2n-1$ terms from a cyclic group of order n, thus containing a zero-sum sequence of length n, say $\sigma(\iota(W_j')) \equiv 0 \mod n$ with $W_j' \mid h^{-1}W_0^* W_j$ and $|W_j'| = n$. Define W_0' by $W_0' W_j' = W_0 W_j$ and set $W_i' = W_i$ for all $i \ne 0, j$. Then $W' = (W_0', W_1', \ldots, W_{km-1}')$ is a refined block decomposition of U with $h \in \mathrm{supp}(W_0')$ by construction. Since $h \in \mathrm{supp}(W_0')$, we must have $g \in \mathrm{supp}(W_j')$ for some $g \in \mathrm{supp}(W_0^*)$. If $\iota(g) > \iota(h)$, then, applying **A3** to the block decomposition W and W', it follows that $\psi(g) - \psi(h) = 0$ and $\psi(h) - \psi(g) = -ne_1$, a contradiction to (3.15). If $\iota(g) < \iota(h)$, then, applying **A3** to the block decomposition W and W', it follows that $\psi(g)-\psi(h)=-ne_1$ and $\psi(h)-\psi(g)=0$, a contradiction to (3.15). Therefore $\iota(g)=\iota(h)$ and by applying **A3** to the block decomposition W, we obtain $\psi(h)=\psi(g)=0$ by (3.17). Therefore

$$\begin{aligned}&\text{for every } U^*\text{-term } h \text{ of } W_0^{-1}U, \text{we have } \psi(h)=0 \text{ and}\\ &\text{there exists a term } g \text{ of } W_0^* \text{ such that } \iota(g)=\iota(h).\end{aligned} \tag{3.18}$$

Note e_2 is a U^*-term of $W_0^{-1}U$. Then (3.18) implies that there exists $g\,|\,W_0^*$ such that $\iota(g)=\iota(e_2)=0$. Assume that there exists a U^*-term h of $W_0^{-1}U$ such that $\iota(h)>0=\iota(g)$. In view of (3.18) and (3.17), we have $\psi(h)=\psi(g)=0$. By applying **A3** to the block decomposition W, we obtain $0=\psi(g)-\psi(h)=-ne_1$, a contradiction to (3.15). Thus for every U^*-term h of $W_0^{-1}U$, we have $\iota(h)=0$ and combined with (3.18), we obtain $h=e_2$. Therefore, for every $j\in[1,km-1]$, we either have

$$W_j=e_1^n \quad\text{or}\quad W_j=e_2^n. \tag{3.19}$$

In view of (3.16), we let

$$s\in[0,km-2]$$

be the number of blocks W_j equal to e_1^n. It follows from (3.11) and (3.12) that

$$U=e_1^{(s+1)n-1}e_2^{(km-s-1)n}\prod_{i\in[1,n]}(-x_ie_1+e_2) \tag{3.20}$$

for some $x_1,\dots,x_n\in[0,n-1]$ with $x_1+\ldots+x_n\equiv n-1 \mod n$.

Assume that $x_1+\ldots+x_n\neq n-1$. Then $x_1+\ldots+x_n\geq 2n-1$ and hence there will be some minimal index $t\in[2,n-1]$ such that $x_1+\ldots+x_t\geq n$. By the minimality of t, we have $x_1+\ldots+x_{t-1}\leq n-1$, which combined with $x_t\in[1,n-1]$ ensures that $x_1+\ldots+x_t\in[1,2n-2]$. Hence $x_1+\ldots+x_t=n+r$ for some $r\in[0,n-2]$. Let $j\in[1,km-1]$ such that $W_j=e_2^n$ (which exists by (3.16)). Since $\sigma\big(\varphi\big(e_1^r\prod_{i\in[1,t]}(-x_ie_1+e_2)\big)\big)=t\overline{e}_2=\sigma(\varphi(e_2^t))$, it follows from **A1**.2 that $-ne_1+te_2=\sigma(e_1^r\prod_{i\in[1,t]}(-x_1e_1+e_2))=\sigma(e_2^t)=te_2$, whence $ne_1=0$, contradicting (3.15). So we instead conclude that

$$x_1+\ldots+x_n=n-1.$$

As already noted, there is some block $W_j=e_2^n$, with $j\in[1,km-1]$. By **A1**.1, we obtain $ne_2=\sigma(W_j)=g_0$ is a generator for $\ker(\varphi)\cong C_m$, ensuring that $\operatorname{ord}(ne_2)=\operatorname{ord}(g_0)=m$. We also have $\varphi(e_2)=\overline{e}_2$ with $\operatorname{ord}(\overline{e}_2)=n$, ensuring that $\operatorname{ord}(e_2)=n\operatorname{ord}(ne_2)=nm=\exp(G)$. Since $\operatorname{supp}(U)\subset\langle e_1,e_2\rangle$, we obtain that $|U|=\mathsf D_k(G)\leq\mathsf D_k(\langle e_1,e_2\rangle)\leq\mathsf D_k(G)$. It follows from Lemma 2.3 that $G=\langle e_1,e_2\rangle$ and hence (e_1,e_2) is a generating set of G with $\operatorname{ord}(e_1)>n$ and $\operatorname{ord}(e_2)=mn$.

If $s=0$, then U has the form of type III by writing e_2 as $-0e_1+e_2$. Suppose $s\geq 1$. Then, in view of (3.19), there is some block $W_i=e_1^n$ with $i\in[1,km-1]$, while there is some block $W_j=e_2^n$ with $j\in[1,km-1]$. By **A1**.1, we have $ne_2=\sigma(W_j)=\sigma(W_i)=ne_1$. Since $\operatorname{ord}(e_1)$

is a multiple of n, we have $\mathrm{ord}(e_1) = n\,\mathrm{ord}(ne_1) = n\,\mathrm{ord}(ne_2) = mn$. Note that $(e_1, e_2 - e_1)$ is a generating set of G. It follows from $n(e_2 - e_1) = 0$ that $(e_2 - e_1, e_1)$ is a basis of G. Letting $f_1 = e_1$ and $f_2 = e_2 - e_1$, we have

$$U = f_1^{(s+1)n-1}(f_1 + f_2)^{(km-s-1)n}\prod_{i=1}^{n}((1-x_i)f_1 + f_2) \quad \text{with } s+1 \in [2, km-1]$$

and hence U has the form of type IV by writing $f_1 + f_2$ as $(1-0)f_1 + f_2$. □

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