

Graph minors and metric spaces

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Abstract

We present problems and results that combine graph-minors and coarse geometry. For example, we ask whether every geodesic metric space (or graph) without a fat H minor is quasi-isometric to a graph with no H minor, for an arbitrary finite graph H . We answer this affirmatively for a few small H . We also present a metric analogue of Menger's theorem and König's ray theorem. We conjecture metric analogues of the Erdős–Pósa Theorem and Halin's grid theorem.

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1 Introduction

Gromov's [22] coarse geometry perspective had a groundbreaking impact on geometric group theory, also spreading onto nearby areas. As an example highlight, it has been proved that (any Cayley graph of) every finitely generated group with finite asymptotic dimension embeds coarsely into Hilbert space [38], and therefore it satisfies the Novikov conjecture [22].

Several recent papers study the interplay between Gromov's asymptotic dimension and graph-theoretic notions such as graph minors [5, 7, 17, 18, 25, 34, 35]. With these in the background, we present some results and questions with a strong interplay between geometry and graph theory. We hope that this will evolve into a coherent theory that could be called '*Coarse Graph Theory*' or '*Graph-Theoretic Geometry*'.

We would like to view graphs 'from far away' to see their large-scale geometry and its implications. More generally, the same idea can be applied to Riemannian manifolds, or more generally, arbitrary length spaces. In what follows X

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can stand for a length space, or an infinite graph, or a family of finite graphs, endowed with their graph-distance to turn them into metric spaces; none of our statements are affected by this choice, the reader may choose their preferred setting. Our favourite such problem is

Conjecture 1.1. *Let X be a graph or a length space, and let H be a finite graph. Then X has no K -fat H minor for some $K \in \mathbb{N}$ if and only if X is $f(K)$ -quasi-isometric to a graph with no H minor, where $f : \mathbb{N} \rightarrow \mathbb{N}$ depends on H only.¹*

A quasi-isometry is a generalisation of a bi-Lipschitz map that allows for an additive error; see Section 2.1. This notion lies at the heart of Gromov’s programme [22]. A fat minor is a coarse version of the notion of a graph minor defined as follows. Recall that X has a (usually finite) graph H as a *minor*, if we can find connected *branch sets* and *branch paths* in X , such that after contracting each branch set to a point, and each branch path to an edge, we obtain a copy of H . We say that X has H as a *K -fat minor*, if we can choose the above sets to be pairwise at distance $\geq K$, except that we do not require this for incident branch set–branch edge pairs—which must be at distance 0 by definition; see also Definition 1 below. This notion of fat minor also features in [7], and a slight variation features in [8].

If X has a K -fat H minor for every K , then we say that X has H as an *asymptotic minor*, and write $H \prec^\infty X$. An important feature of asymptotic minors is that they are preserved under quasi-isometries, and in particular they are stable under changing the generating set in a Cayley graph of a group (Observation 2.4).

For example, let L be the square lattice on \mathbb{Z}^2 , and let $L' := L \times K_2$ be its cartesian product with an edge. Then one can easily show that L' has every finite graph as a minor, but it does not have K_5 or $K_{3,3}$ as a 2-fat minor. Thus this notion correctly captures the intuition that L' looks like the plane from a coarse point of view. As another example, the cubic lattice on \mathbb{Z}^3 has every finite graph H as an asymptotic minor; to see this, embed H in \mathbb{R}^3 , fatten it slightly, then zoom \mathbb{Z}^3 out while keeping your H fixed.

We know that Conjecture 1.1 holds for some very small graphs H . If $H = K_2$ (a pair of vertices with an edge), then Conjecture 1.1 becomes the obvious fact that if X has finite diameter then it is quasi-isometric to a point. The case $H = K_3$ is already non trivial and it follows from Manning’s characterization of quasi-trees [31]. We give a self contained exposition of this which entails a shorter proof of Manning’s result (Theorem 3.1). Further alternative characterisations of quasi-trees have been recently obtained by Berger & Seymour [6].

Chepoi, Dragan, Newman, Rabinovich & Vaxès [8] settled the case $H = K_{2,3}$ of Conjecture 1.1 using slightly different terminology. The similar case $H = K_4^-$, i.e. K_4 with an edge removed, has been settled by Fujiwara and the second author [17], along with a stronger result characterising quasi-cacti. In Section 4 we settle the case where H is a star $K_{1,m}$, $m \geq 1$. This is, as far as we know, a new result.

Call two graphs H_1, H_2 *asymptotically equivalent*, if $H_1 \prec^\infty X$ implies $H_2 \prec^\infty X$ and vice-versa for every space X . In Section 5 we show that if two

¹The backward direction is straightforward.

finite graphs H_1, H_2 are homeomorphic as 1-complexes, then they are asymptotically equivalent (the converse is true as well if we allow X to be disconnected). This immediately settles Conjecture 1.1 for paths and cycles H . Moreover, combined with the aforementioned result of [17], it implies that a graph is quasi-isometric to a cactus if and only if it is quasi-isometric to an outerplanar graph (Corollary 5.2), a fact previously proved by Chepoi et al. [8].

In Conjecture 1.1 we are mostly interested in the case where H is finite, but we point out some infinite cases of interest: in Section 6 we prove a coarse analogue of König’s theorem, and in Section 7 we discuss coarse analogues of Halin’s grid theorem.

We observe below (Observation 2.2) that when H is 2-connected in Conjecture 1.1, then instead of a quasi-isometry we can equivalently ask for a bi-Lipschitz map. Bi-Lipschitz embeddings, aka. embeddings of bounded multiplicative distortion, into various targets, most notably tree metrics and normed spaces, are an actively researched topic, see e.g. [8, 15, 32, 36] and references therein. Embeddings of bounded additive distortion are also of interest [8]. This raises:

Question 1.2. *For which finite graphs H is it true that if a graph X has no K -fat H minor for some $K \in \mathbb{N}$, then X admits a map of bounded additive distortion onto a graph with no H minor?*

We know that this is the case when H is a cycle [26] (see also claim (??) in Section 3) or $H = K_{1,3}$.

Ostrovskii & Rosenthal [35] proved that (infinite, locally finite) graphs with a forbidden minor have finite asymptotic dimension. Bonamy et al. [7] improve this by proving that such graphs have asymptotic dimension at most 2 (as was conjectured in [17]). They ask whether forbidding a fat minor implies bounded asymptotic dimension. Note that a positive answer to Conjecture 1.1 would imply this. The following question is similar in spirit.

Conjecture 1.3 (Coarse Hadwiger conjecture). *For every $n \in \mathbb{N}$, if X has no asymptotic K_{n+1} minor, then X has Assouad-Nagata dimension $\text{ANdim}(X)$ at most $n - 1$.*

Since $\text{ANdim}(X)$ can be defined via colouring (Section 2.2)—and $\text{ANdim}(X) = n - 1$ means that n colours suffice—Conjecture 1.3 can be thought of as the coarse version of the Hadwiger conjecture, perhaps the oldest well-known open problem of graph theory, which asserts that every graph with no K_{n+1} minor is n -colourable. If we drop the word ‘asymptotic’ in the statement of Conjecture 1.3, then a better bound $\text{ANdim}(X) \leq 2$ was recently obtained by Distel [12] and by Liu [29]. In fact 2 might be the right upper bound in Conjecture 1.3 as well. A somewhat related result relaxes the Hadwiger conjecture by allowing monochromatic components of bounded size [13].

Apart from its theoretic interest due to the connection with the Hadwiger conjecture, Conjecture 1.3 is also interesting from an algorithmic perspective. Given the theoretical and algorithmic importance of minor-closed graph classes, one hopes that the graphs we encounter in practice come from such a class. This is however rarely the case, because e.g. a little bit of randomness will result in arbitrarily large cliques. Thus it is natural to seek extensions of the

theory of graph minors to a larger setting. This has been a mainstream trend in computer science, see e.g. [21] and references therein. Conjecture 1.3 pursues something similar in a different direction. The kind of practical problems we have in mind are those typically encountered in computational geometry, e.g. the Traveling Salesman Problem (TSP). Computer scientists have developed a notion called *sparse partition schemes*, and have shown how these can be used to solve important practical problems such as the Universal Steiner Tree and the Universal TSP problem. The definition of sparse partition schemes is a slightly more quantitative version of the colourings used to define the Assouad-Nagata dimension (Section 2.2); see [7] and references therein for the precise relationship and further applications). Recall that the presence of large cliques in real-world graphs is an obstruction to using algorithms coming from graph minor theory. Our coarse approach overcomes such obstructions: a clique of any size has diameter 1, and so it looks just like a point when seen from far away.

In the hope to develop more basic tools towards Conjecture 1.1 and Conjecture 1.3, we considered the following coarse version of Menger's theorem:

Conjecture 1.4 (Coarse Menger Theorem). *For every $n \in \mathbb{N}$, there is a constant C_n , such that the following holds for every graph G and every two subsets A, Z of its vertex set. For every $r \in \mathbb{Z}$, either there is a set of n A - Z paths in G at distance at least r from each other, or there is a set S of less than n vertices of G , such that removing the ball of radius $C_n r$ around S separates A from Z in G .*

In Section 8 we will prove the special case $n = 2$ of this conjecture². We warn the reader that Conjecture 1.4 seems very difficult even for $n = 3$. In Section 9 we discuss related problems that might be more accessible.

2 Preliminaries

2.1 Metric spaces

Let (X, d) be a metric space. Given $x \in X$ and $R \in \mathbb{R}_+$, we let $\text{Ball}_x(R)$ denote the ball of radius R around x .

For $Y, Z \subseteq X$ we define $d(Y, Z) := \inf_{y \in Y, z \in Z} d(y, z)$.

A *length space* is a metric space (X, d) such that for every $x, y \in X$ and $\epsilon > 0$ there is an x - y arc of length at most $d(x, y) + \epsilon$. Thus every geodesic metric space, in particular every graph endowed with its standard metric, is a length space.

An (M, A) -*quasi-isometry* between graphs $G = (V, E)$ and $H = (V', E')$ is a map

$f : V \rightarrow V'$ such that the following hold for fixed constants $M \geq 1, A \geq 0$:

- (i) $M^{-1}d(x, y) - A \leq d(f(x), f(y)) \leq Md(x, y) + A$ for every $x, y \in V$, and
- (ii) for every $z \in V'$ there is $x \in V$ such that $d(z, f(x)) \leq A$.

²An alternative proof was independently obtained by Albrechtsen et al. [1], who also state Conjecture 1.4.

Here $d(\cdot, \cdot)$ stands for the graph distance in the corresponding graph G or H . We say that G and H are *quasi-isometric*, if such a map f exists. Quasi-isometries between arbitrary metric spaces are defined analogously. The following is well-known, see e.g. [9, Proposition 3.B.7(6)]:

Observation 2.1. *Every length space is quasi-isometric to a connected graph.*

Proof. **TOPROVE 2** □

When $f : X \rightarrow Y$ between metric spaces satisfies (i) with $A = 0$, then we say that f has bounded *multiplicative distortion*. When $M = 1$, we say that f has bounded *additive distortion*. We observe that in many cases where Y is not fixed, but allowed to vary inside a class of graphs, then a quasi-isometry can be turned into a map of bounded multiplicative distortion:

Observation 2.2. *Let \mathcal{H} be a set of 2-connected graphs, and let $\mathcal{C} = \text{Forb}(\mathcal{H})$ be the class of graphs that do not have any element of \mathcal{H} as a minor. Then a graph X is quasi-isometric to an element of \mathcal{C} , if and only if X admits a map of bounded multiplicative distortion into an element of \mathcal{C} .*

Proof. **TOPROVE 3** □

2.2 Asymptotic dimension

We recall one of the standard definitions of asymptotic dimension $\text{asdim}(X)$ of a metric space (X, d) from e.g. [25]. The reader can think of X as being a graph endowed with its graph distance d . See [4] for a survey of this rich topic.

Let \mathcal{U} be a collection of subsets of (X, d) —usually a cover. For $s > 0$, we say that \mathcal{U} is *s-disjoint*, if $d(U, U') := \inf \{d(x, x') \mid x \in U, x' \in U'\} \geq s$ whenever $U, U' \in \mathcal{U}$ are distinct. More generally, we say that \mathcal{U} is $(n + 1, s)$ -disjoint, if $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ for subcollections \mathcal{U}_i each of which is *s-disjoint*. The indices $1, \dots, n + 1$ are the *colours* of \mathcal{U} . We define the *asymptotic dimension* $\text{asdim}(X)$ as the smallest n such that for every $s \in \mathbb{R}_+$ there is an $(n + 1, s)$ -disjoint cover \mathcal{U} of X with $\sup_{U \in \mathcal{U}} \text{diam}(U) < \infty$. Here, the diameter $\text{diam}(U)$ is measured with respect to the metric d of X . We say that \mathcal{U} is *D-bounded* for any $D > \sup_{U \in \mathcal{U}} \text{diam}(U)$.

The *Assouad-Nagata dimension* $\text{ANdim}(X)$ is defined by putting a stronger restriction on the diameters: we define $\text{ANdim}(X)$ to be the smallest n for which there is $c \in \mathbb{R}$ such that for every $s \in \mathbb{R}_+$ there is an $(n + 1, s)$ -disjoint and cs -bounded cover \mathcal{U} of X .

2.3 Fat minors

Definition 1 (Fat minors). *Let (X, d) be a metric space and Δ a graph. A K -fat Δ -minor of X is a collection of disjoint connected subspaces $B_v, v \in V(\Delta)$ of X (called branch-sets), and disjoint arcs $P_e, e \in E(\Delta)$ (called branch-paths) such that:*

- (i) *Each P_{uw} intersects $\bigcup_{v \in V(\Delta)} B_v$ exactly at its two endpoints, which lie in B_u and B_w .*

- (ii) For any $Y, Z \in \{B_v \mid v \in V(\Delta)\} \cup \{P_e \mid e \in E(\Delta)\}$, we have $d(Y, Z) \geq K$, unless $Y = Z$, or $Y = P_{vw}$ and $Z = B_v$ for some $v, w \in V(\Delta)$, or vice versa.

Definition 2.3 (Asymptotic minors). *Let Δ be a finite graph. We say that X has Δ as an asymptotic minor, and write $\Delta \prec^\infty X$, if X has Δ as a K -fat minor for all $K > 0$.*

Observation 2.4. *The property of having a fixed finite graph H as an asymptotic minor is preserved under quasi-isometries between length spaces.*

In fact, asymptotic minors are preserved under the more general *coarse embeddings* as defined in [22, 36], but we will not use this fact here.

3 Manning’s theorem — quasi-trees

Manning’s theorem [31] says that a graph is quasi-isometric to a tree if and only if it has the following δ -Bottleneck Property for some $\delta \in \mathbb{R}_+$:

Definition 2. *We say that a metric space X has the δ -Bottleneck Property, if for all $x, y \in X$ there is a ‘midpoint’ $m = m(x, y)$ with $d(x, m) = d(y, m) = d(x, y)/2$ such that any path from x to y is at distance less than δ from m .*

Our next result contains Manning’s theorem, and at the same time proves the case $H = K_3$ of Conjecture 1.1:

Theorem 3.1. *The following are equivalent for every graph G*

- (i) G satisfies the δ -Bottleneck Property for some $\delta > 0$;
- (ii) G has no K -fat K_3 -minor for some $K > 0$; and
- (iii) G is quasi-isometric to a tree.

Remark 1: In this formulation our statement only makes sense for graphs G of infinite diameter, but our proof yields explicit relations between the constants involved that make it meaningful for finite G as well.

Remark 2: Krön & Möller [28] provide an alternative characterisation of quasi-trees (for graphs G of infinite diameter). Kerr [26] provides another proof of Manning’s theorem, with multiplicative constant 1 for the quasi-isometry. Similar results had previously been obtained by Chepoi et al. [8]. Our proof of Theorem 3.1 in fact also gives Kerr’s theorem (with a different proof), as we will obtain a $(1, 10K)$ -quasi-isometry to a tree (??). In other words, (i)–(iii) of Theorem 3.1 are equivalent to admitting an embedding into a tree with bounded additive distortion.

We will use the following terminology. Let (X, d) be a metric space. We say that $C \subseteq X$ is M -near-connected for $M \in \mathbb{R}_+$, if for every $x, y \in C$, there is a sequence $x = x_0, x_1, \dots, x_k = y$ of points of C such that $d(x_i, x_{i+1}) \leq M$ for every $i < k$. Such a sequence $P = \{x_i\}$ will be called an M -near path from x to y . An M -near-component is a maximal subset of X that is M -near-connected.

Our method below bears some similarity with [8], but our use of M -near-components instead of components in a layered partition is an essential difference.

Proof. **TOPROVE 5**

□

4 Conjecture 1.1 for stars

In this section we establish the special case of Conjecture 1.1 where H is a *star*, i.e. $H = K_{1,m}$:

Theorem 4.1. *Let X be a length space, and $m \in \mathbb{N}_{>0}$. Then X has no asymptotic $K_{1,m}$ minor if and only if X is quasi-isometric to a graph with no $K_{1,m}$ minor.*

Proof. **TOPROVE 7** □

Remark: In the special case $m = 3$ of Theorem 4.1, the resulting graph G becomes a path or a cycle. In this case, after subdividing each edge of G M times, the quasi-isometry from X to G —obtained by mapping each vertex to its super-box—becomes a map of bounded additive distortion, i.e. a $(1, K')$ -quasi-isometry. This can be proved similarly to (??). We do not see how to extend this to the case $m > 3$.

5 More on Conjecture 1.1

Despite our Conjecture 1.1, we remark that the correspondence between minor-closed families of graphs and quasi-isometry classes is not 1–1. Theorem 5.1 below makes this point clear, but it also provides a tool for studying Conjecture 1.1. Here we think of graphs as topological spaces, by considering the corresponding 1-complex.

Theorem 5.1. *Let Γ, Δ be finite graphs that are homeomorphic as topological spaces, and let X be a length space. Then $\Gamma \prec^\infty X$ if and only if $\Delta \prec^\infty X$.*

Let Θ be the graph obtained from K_4 by removing one edge (or from $K_{2,3}$ by contracting one). Recall that outerplanar graphs can be characterised as $\text{Forb}(K_4, K_{2,3})$. Combining this with Theorem 5.1, and a result of Fujiwara & Papasoglou [17] stating that a graph is quasi-isometric to a cactus if and only if it has no asymptotic Θ minor, we obtain

Corollary 5.2. *The following are equivalent for a length space X :*

- (i) X is quasi-isometric to a cactus;
- (ii) X is quasi-isometric to an outerplanar graph;
- (iii) X has no asymptotic K_4 or $K_{2,3}$ minor, and
- (iv) there is a subdivision Θ' of Θ such that X has no asymptotic Θ' minor.

Here, we say that a graph Δ' is a *subdivision* of a graph Δ , if Δ' is obtained by replacing some of the edges of Δ by independent paths with the same end-vertices.

Proof. **TOPROVE 8** □

Note that two graphs are homeomorphic (as topological spaces) if and only if they are subdivisions of the same graph (which can be obtained by suppressing all vertices of degree 2). Therefore Theorem 5.1 follows by applying the following lemma twice:

Lemma 5.3. *Let Δ' be a subdivision of a finite graph Δ , and let X be a length space. Then $\Delta \prec^\infty X$ if and only if $\Delta' \prec^\infty X$.*

Proof. **TOPROVE 9** □

The converse of Theorem 5.1 is false in general: for example, we can let Γ be K_2 , and Δ be its complement. But if we insist that Γ, Δ are both connected, and allow X to be disconnected, i.e. we allow for infinite distances between pairs of points of X , then the converse becomes true and not hard to prove³. If we insist that X be connected however, then the converse fails: if Γ is a cycle, and Δ the union of two cycles with exactly one common vertex, then Γ, Δ are asymptotically equivalent though non-homeomorphic. Still, we can obtain a converse to Theorem 5.1 (for connected X) by restricting to 2-connected graphs. We leave the details to the interested reader.

6 Coarse König Theorem

König's classical theorem states that every infinite connected graph contains a countably infinite star or a ray as a subgraph [11, Proposition 8.2.1.]. It is a cornerstone of infinite graph theory as it facilitates compactness arguments. In this section we prove a metric analogue.

Let P_n denote the path (in the graph theoretic sense) of length n . Let $\bigvee P_n$ be the graph obtained from the disjoint union of $(P_n)_{n \in \mathbb{N}}$ by identifying the first vertex of each P_n . A *ray* is a 1-way infinite path.

Theorem 6.1. (*Coarse König Theorem*) *Let X be a (connected) length space of infinite diameter. Then X has either the ray or $\bigvee P_n$ as an asymptotic minor.*

We prepare the proof of this with a couple of lemmas of independent interest. A ray P in a graph G is called *divergent*, if $P \cap S$ is finite for every subset S of $V(G)$ with bounded diameter.

Lemma 6.2. *Let G be a graph that has a divergent ray. Then G has the ray as an asymptotic minor.*

Proof. **TOPROVE 10** □

By the Infinite Ramsey theorem, every infinite graph contains either an infinite clique or an infinite independent set. We will use the following corollary of this.

Lemma 6.3. *Let $G_1 \supseteq G_2 \supseteq G_3 \dots$ be graphs with a common infinite vertex set V . Then either some G_i contains an infinite independent set, or there are infinite subsets $S_1 \supseteq S_2 \supseteq S_3 \dots$ of V such that $G_i[S_i]$ is complete for every i .*

Proof. **TOPROVE 11** □

We can now prove the main result of this section.

Proof. **TOPROVE 12** □

³For this, given non-homeomorphic Γ_1, Γ_2 , let $X_i, i = 1, 2$ be the disjoint union of ω copies of Γ_i , with each edge of the n th copy subdivided n times. Then $\Gamma_i \prec^\infty X_i$, but $\Gamma_i \not\prec^\infty X_{3-i}$.

7 Coarse Halin Theorems

For a cardinal k , and a graph H , let $k \cdot H$ denote the disjoint union of k copies of H . Let R denote the ray, i.e. the 1-way infinite path. Halin [23] proved that for every graph G , if $n \cdot R \subset G$ holds for every $n \in \mathbb{N}$, then $\omega \cdot R \subset G$. Does the coarse version hold?

Conjecture 7.1. (*Coarse Halin Ray Theorem*) *Let X be a length space. Suppose $n \cdot R \prec^\infty X$ holds for every $n \in \mathbb{N}$. Then $\omega \cdot R \prec^\infty X$.*

Let \mathcal{H} denote the ‘half’ square grid on $\mathbb{N} \times \mathbb{N}$, and let \mathcal{F} denote the ‘full’ square grid on $\mathbb{Z} \times \mathbb{Z}$. Halin’s grid theorem [23] says that every 1-ended graph satisfying $n \cdot R \subset G$ for every $n \in \mathbb{N}$ contains a subdivision of \mathcal{H} .

Problem 7.2. (*Coarse Halin Grid Theorem*) *Let X be an 1-ended length space. Suppose $\omega \cdot R \prec^\infty X$. Then $\mathcal{H} \prec^\infty X$.*

Problem 7.3. *Let G be a Cayley graph of an 1-ended finitely generated group. Must $\mathcal{H} \prec^\infty G$ hold? Must $\mathcal{F} \prec^\infty G$ hold?*

8 Coarse Menger Theorem

In this section we prove the case $n = 2$ of Conjecture 1.4. In Section 8.3 below we present a corollary for diverging rays in infinite graphs. We restate our special case for convenience:

Theorem 8.1. *Let X be a graph or a geodesic metric space, and A, Z two subsets of X . For every $K > 0$, there is either*

- (i) *a set $S \subset X$ with $\text{diam}(S) \leq K$ such that $X \setminus S$ contains no path joining A to Z , or*
- (ii) *two paths joining A to Z at distance at least $a = K/16 \cdot 17 = K/272$ from each other.*

Moreover, for every $x \in A, z \in Z$, we can choose the paths of (ii) so that both x, z are among their four endpoints.

We prepare the proof with some preliminaries. We will assume that (i) does not hold and we will show that (ii) holds.

We fix throughout the proof a shortest path $\gamma : [0, \ell] \rightarrow X$ joining A to Z .

Definition 8.2 (Bridges). *A bridge B is a union of three successive paths $c_1 \cup b \cup c_2$ such that:*

- *for any $x \in b, y \in \gamma$, we have $d(x, y) \geq K/8$.*
- *If $b \cap A \neq \emptyset$ then c_1 is an endpoint of b and similarly if $b \cap Z \neq \emptyset$ then c_2 is an endpoint of b ; otherwise c_1, c_2 are (geodesic) paths of length $K/8$ joining the endpoints of b to γ .*

We call c_1, c_2 the *legs* of B , and b its *spine*. The two endpoints of B on γ , if they exist, are denoted by B^0, B^1 so that $B^0 < B^1$. The *interval* of B is the interval $[B^0, B^1]$ of γ . If c_1 is a point (of A) then the interval of B is $[\gamma(0), v]$

where v is the endpoint of c_2 , and we define similarly the interval of B when c_2 is a point of Z .

We say that a bridge is *maximal* if its interval is not contained properly in the interval of another bridge.

Proposition 8.3. *Let B be a maximal bridge with spine b . For any point $x \in b$, let x' be a point of γ minimizing $d(x, x')$. Then $x' \in [B^0, B^1]$.*

Proof. **TOPROVE 14** □

We assume that γ is parametrized by arc-length. If $a \in \gamma$ and $r > 0$, we denote by $a - r$ the point at distance r from a on γ which is before a . We define $a + r$ similarly.

Definition 8.4 (Surrounding). *We say that a bridge B d -surrounds a point $x = \gamma(t) \in \gamma$ if the interval $[t_1, t_2]$ of B contains $x - d$ and $x + d$. Here if $t < d$ or $t > \ell - d$ we set $x - d = \gamma(0)$ and $x + d = \gamma(\ell)$ respectively.*

We restrict our attention now to maximal bridges.

Definition 8.5 (Order). *Let B_1, B_2 be maximal bridges. If $B_2^0 > B_1^0$ (in which case $B_2^1 > B_1^1$ holds too by maximality), we write $B_2 > B_1$.*

For $R \in \mathbb{R}$, we say that B_2 R -crosses B_1 if B_2 R -surrounds B_1^1 .

Lemma 8.6. *Assume condition (i) of Theorem 8.1 fails. Then for any $x \in \gamma$ there is some bridge B that $3K/8$ -surrounds x .*

Proof. **TOPROVE 15** □

Lemma 8.7. *Let $B < C$ be maximal bridges with $C^0 > B^1$. Assume that there are $x \in B, y \in C$ such that $d(x, y) \leq K/4$. Then $d(B^1, C^0) < 3K/4$.*

Proof. **TOPROVE 16** □

We choose a smallest set \mathcal{U} of maximal bridges such that any $x \in \gamma$ is $K/4$ -surrounded by some bridge in \mathcal{U} , and

$$\text{every } B \in \mathcal{U} \text{ has an interval of length at least } 3K/4. \quad (1)$$

Such a \mathcal{U} exists by Lemma 8.6. We remark that \mathcal{U} is finite because γ is compact, and any bridge that $3K/4$ -surrounds $x \in \gamma$ also $K/4$ -surrounds any point in an open interval of x on γ .

The minimality of \mathcal{U} implies that the initial points of its bridges are not too dense on γ :

Proposition 8.8. *Let B, C, D be bridges in \mathcal{U} . Suppose that C r -crosses B , and D r -crosses C for some $r \leq K/4$. Then $d(B^0, D^0) \geq r$.*

Proof. **TOPROVE 17** □

A sequence $\mathcal{B} = B_1, \dots, B_n$ of bridges is said to be R -crossing for some $R \in \mathbb{R}$, if B_1 R -surrounds $\gamma(0)$, each $B_i, i \geq 2$ R -crosses B_{i-1} , and $B_n^1 \in Z$. Note that B_n must R -surround $\gamma(1)$ as $\gamma(1) > B_{n-1}^1$. Such a sequence is *perfect*, if in addition its initial points are R -apart, i.e. $d(B_i^0, B_j^0) \geq R$ holds for every $1 \leq i, j \leq n$.

Remark: A perfect R -crossing sequence \mathcal{B} remains a perfect R -crossing sequence if we exchange the roles of A and Z and reverse the direction of γ and the ordering of \mathcal{B} . But we will not use this fact.

It is straightforward to check that if we order the elements of \mathcal{U} by $<$ then the resulting sequence S is a $K/4$ -crossing sequence. Our next lemma says that we can obtain a perfect subsequence by sacrificing half of our crossing constant.

Lemma 8.9. *Let $S = \{B_i\}_{1 \leq i \leq n}$ be a $K/4$ -crossing sequence of bridges. Then there is a perfect $K/8$ -crossing subsequence S' of S .*

Proof. **TOPROVE 18** □

Proposition 8.10. *Suppose that B_1, B_2, B_3, B_4, B_5 are bridges such that each $B_i, i \geq 2$, $K/8$ -crosses B_{i-1} , and B_i^0 lies in the interval of B_1 for every $2 \leq i \leq 5$. Then at least one B_i does not lie in \mathcal{U} .*

Proof. **TOPROVE 19** □

Lemma 8.11. *Suppose that $B_i : i = 1, \dots, n$ are bridges in \mathcal{U} such that B_i crosses B_{i-1} for $i = 2, \dots, n$, and that there are $x \in B_1, y \in B_n$ such that $d(x, y) < K/4$. Then $n \leq 8$.*

Proof. **TOPROVE 20** □

Our plan is to construct the two paths α_1, α_2 joining A to Z required by Theorem 8.1 by combining subpaths of γ with bridges from a sequence S' as in Lemma 8.9. However, we can have bridges B, C in S' arbitrarily close to each other (but not intersecting, as this would contradict maximality), in which case we have to avoid putting B in α_1 and C in α_2 . Therefore, if $d(B, C) < a$, we will use a B - C path α of length less than a to join B, C , and produce a new path called a *meta-bridge*. We may need to iterate the process, joining two nearby meta-bridges into a new meta-bridge. But the following lemma will exploit Lemma 8.11 to upper bound the complexity of meta-bridges.

Formally, we can simply define a *meta-bridge* B to be a path with endpoints in $\gamma \cup A \cup Z$ whose interior is disjoint from $\gamma \cup A \cup Z$. But the meta-bridges we will actually work with will obey further restrictions, being constructed recursively starting from bridges as explained above. Much of the notation we introduced for bridges can be applied verbatim to meta-bridges, and we will do so without further comment.

A *join* of a meta-bridge D is a maximal subpath of D consisting of points that lie on no bridge of S' .

Choose $a = K/(16 \cdot 17)$.

Lemma 8.12. *There is a perfect, $K/8$ -crossing, sequence M of meta-bridges such that*

- (i) *each join J of an element of M has length $\ell(J) < 15a$;*
- (ii) *we have $d(J, \gamma) \geq a$ for each join J of an element of M , and*
- (iii) *if x, y are points of distinct meta-bridges B, C of M , then $d(x, y) \geq a$ unless there are endpoints $B^i, C^{1-i}, i \in \{0, 1\}$ with $d(B^i, C^{1-i}) < K/8$.*

Proof. **TOPROVE 21** □

Remark: The last inequality is the bottleneck for maximizing the distance a between the two paths α_i of Theorem 8.1.

We have collected all the necessary ingredients for the proof of our Theorem 8.1.

Proof. **TOPROVE 23** □

8.1 Alternative construction of the paths — reducing to the classical Menger theorem

This section offers an alternative proof of Theorem 8.1, which is simpler but gives a weaker lower bound a on $d(\alpha_1, \alpha_2)$. The main difference is that instead of constructing α_1, α_2 explicitly, we apply Menger's theorem to an auxiliary graph, consisting of γ -edges and meta-bridges in M .

For this, we first adapt (iii) of Lemma 8.12 by lowering the constant $K/8$ to $K/20$, say, and lowering a proportionately; the proof can be repeated verbatim after this change. This change allows us to deduce the following corollary of Lemma 8.12:

Corollary 8.13. *If B, C are meta-bridges in M or γ -edges, then $d(B, C) \geq a$ unless B, C are incident or as in (iii) of Lemma 8.12 (i.e. they have $K/20$ -close end-points).*

Proof. **TOPROVE 24** □

Proof. **TOPROVE 25** □

8.2 Choosing endpoints

We remark that $\gamma(0), \gamma(1)$ are among the four endpoints of $\alpha_1, y \in \alpha_2$ in the above construction. This can be seen directly from the construction, or by noticing that \mathcal{U} , and hence S' and M , contains at most one bridge with an endpoint in each of A and Z by maximality. Recall that γ was chosen as an arbitrary shortest A - Z path. By modifying X we can strengthen the above conclusion:

Corollary 8.14. *For every $x \in A$ and $z \in Z$, we can choose the paths of Theorem 8.1 so that both x, z are among their four endpoints.*

Proof. **TOPROVE 26** □

8.3 An application — diverging rays

It is not hard to extend Theorem 8.1 to the situation where Z is replaced by ∞ ; that is, we have

Corollary 8.15. *Let X be a locally finite metric graph, and $A \subset X$. For every $K > 0$, there is either*

- (i) *a set $S \subset X$ with $\text{diam}(S) \leq K$ such that $X \setminus S$ contains no path of infinite length starting at A , or*

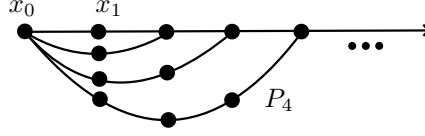


Figure 1: A non-locally-finite graph failing the conclusion of Corollary 8.15.

- (ii) two paths of infinite length starting at A at distance at least $a = K/16 \cdot 17$ from each other.

Proof. **TOPROVE 27** □

Remark: the local-finiteness condition is necessary in Corollary 8.15. A non-locally-finite counterexample can be obtained as follows. Let $R = x_0x_1\ldots$ be a ray, and for each $i > 1$, join x_0 to x_i by a path P_i of length i , so that $P_i \cap P_j = \{x_0\}$ for every $i \neq j$, to obtain a graph X (in which x_0 is the only vertex of infinite degree, see Figure 1). Let A be any infinite set of vertices of X . Easily, no set of finite diameter separates A from infinity in X , i.e. (i) fails. Moreover, if X contains two infinite paths, then so does $X' := X - x_0$. But X' is an 1-ended tree. Thus (ii) fails as well.

We say that two rays R, L in a graph G *diverge*, if for every $n \in \mathbb{N}$ they have tails $R' \subseteq R, L' \subseteq L$ satisfying $d(R', L') > n$.

Theorem 8.16. *Let G be a graph with bounded degrees which has an infinite set of pairwise disjoint rays. Then G has a pair of diverging rays.*

The idea is to apply Corollary 8.15, letting the scale K increase as we progress towards infinity. Instead of increasing K , we will use the equivalent but more convenient method of scaling down the edge-lengths as the distance from a fixed origin tends to infinity. This is the role of the following lemma:

Lemma 8.17. *Let $G = (V, E)$ be a graph with bounded degrees which has an infinite set of pairwise disjoint rays. Then there is a finite vertex-set $A \subset V$ and an assignment of edge-lengths $\ell : E \rightarrow \mathbb{R}_{>0}$ with the following properties:*

- (i) no ball of radius 1 in the corresponding metric d_ℓ separates A from ∞ ,
and
- (ii) $\lim_{e \in E} \ell(e) = 0$.⁴

Let us first see how this implies Theorem 8.16.

Proof. **TOPROVE 28** □

It remains to prove Lemma 8.17.

Proof. **TOPROVE 29** □

⁴This means that for every $\epsilon > 0$ all but finitely many $e \in E$ satisfy $\ell(e) < \epsilon$.

9 Further problems

As we saw in the introduction, Conjecture 1.1 can be very difficult even for a simple graph H . The following special case is of particular interest (Conjecture 1.1 can be formulated with H replaced by a finite set of finite graphs):

Conjecture 9.1 (Coarse Kuratowski–Wagner theorem). *A length space X is quasi-isometric to a planar graph if and only if it has no asymptotic K_5 or $K_{3,3}$ minor.*

Here, and more generally in Conjecture 1.1, we may have to allow the resulting planar graph to be edge-weighted; we do not know whether every planar graph with arbitrary real-valued edge-lengths (used to induce a length space) is quasi-isometric to a planar graph with all edge-lengths equal to 1.

This special case is well-motivated in itself due to widespread interest in planar graph metrics, and due to the interest in groups acting on planar surfaces [20].

Another interesting class of graphs is that of graphs of finite tree width.

Conjecture 9.2 (Coarse grid theorem). *A length space X is quasi-isometric to a graph of finite tree width if and only if it does not have the $n \times n$ grid as an asymptotic minor for some $n \in \mathbb{N}$.*

One can also ask whether under the same hypothesis $asdim(X) \leq 1$ (as this is weaker).

Our Conjecture 1.1 would imply that the fundamental Robertson–Seymour graph minor theory has a coarse geometric generalisation. One can of course formulate weaker versions of our conjecture in the same direction. For example, one could ask whether excluding an infinite family of asymptotic minors is equivalent to excluding a finite such family. In geometry and topology one often asks for the ‘nicest’ metric among all metrics in a homeomorphism class. This is the classical topic of uniformisation of surfaces, and this point of view was crucial in the recent classification of 3-manifolds after Thurston and Perelman. Conjecture 1.1 can be seen as a coarse analog to uniformisation: we ask for a ‘nice’ representative in a quasi-isometry class of graphs.

Our next conjecture is a coarse version of a result of Thomassen [37] saying that an 1-ended vertex transitive graph is either planar or has an infinite clique as a minor.

Conjecture 9.3. *Let G be a locally finite, vertex-transitive graph. Then either G is quasi-isometric to a planar graph, or it contains every finite graph as an asymptotic minor.*

Recall that a graph G is k -planar, if it can be drawn in \mathbb{R}^2 so that each of its edges crosses at most k other edges.

Problem 9.4. *Let G be a locally finite, vertex-transitive graph. Is it true that G is quasi-isometric to a planar graph if and only if it is k -planar for some $k \in \mathbb{N}$?*

Vertex-transitivity is essential here, for otherwise we could let G be a sequence of subdivisions of K_5 .

Our next problem is motivated by the following theorem of Khukhro:

Theorem 9.1 ([27]). *A finitely generated, infinite group Γ is virtually free, if and only if for every finitely generated Cayley graph G of Γ , there is some finite graph that is not a minor of G .*

Conjecture 9.5. *A finitely generated group Γ is virtually planar, if and only if for every finitely generated Cayley graph G of Γ , there is some finite graph that is not an asymptotic minor of G .*

Virtually planar here means that Γ has a finite index subgroup H which has a planar Cayley graph. The forward implication of this is easy: by the Svarc–Milnor lemma, any finitely generated Cayley graphs of H and Γ are quasi-isometric, and so the statement follows from Observation 2.4. (The analogous implication of Theorem 9.1 was proved ‘by hand’ by Ostrovskii & Rosenthal [35].)

As already mentioned, we expect Conjecture 1.4 to be very difficult for $n \geq 3$. The following is a related algorithmic question. Let MM3 denote the algorithmic problem that takes as input a graph G on n vertices and two subsets A, Z of its vertex set, and asks whether there is a triple of A – Z paths that are pairwise at distance at least 3.

Conjecture 9.6. *MM3 is NP-complete.*

A classical result of Erdős & Pósa [14] says that every finite graph has either a $k \cdot K_3$ minor or a set of at most $f(k)$ vertices the removal of which results into a forest, where $f(k) \sim k \log k$ is universal. The following proposes a coarse analogue, motivated by both Conjecture 1.1 and Conjecture 1.4:

Conjecture 9.7 (Coarse Erdős–Pósa Theorem). *There is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ and a universal constant C , such that the following holds for every length space X . If $n \cdot K_3$ is not a K -fat minor of G , then there is a set S of at most $f(n)$ points of X such that the ball of radius CK around S meets all K -fat K_3 minors of X . (Alternatively, X is a graph, and removing $B_{CK}(S)$ results into a quasi-forest.)*

10 Latest progress

Many of the questions of this paper have been answered between the original submission and the revision. The aim of this section is to provide an update.

Nguyen, Scott & Seymour [33] found a counterexample to Conjecture 1.4. This was used by Davies, Hickingbotham, Illingworth & McCarty [10] to disprove Conjecture 1.1. Conjecture 9.6 has been proved by Baligács & MacManus [3].

The case $H = K_4$ of Conjecture 1.1 has been proved by Albrechtsen, Jacobs, Knappe, & Wollan [2]. Special cases of Conjecture 1.4, for $r = 2$, and sparse graphs, have been proved by Gartland, Korhonen & Lokshtanov [19] and by Hendrey, Norin, Steiner & Turcotte [24]. The special case of Conjecture 9.3 for finitely presented Cayley graphs has been proved by MacManus [30]. The “only if” direction of Proposition 9.4 has been proved by Esperet & Giocanti [16].

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