The Ultimate Signs of Second-Order Holonomic Sequences

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Abstract

A real-valued sequence $f = \{f(n)\}_{n\in\mathbb{N}}$ is said to be second-order holonomic if it satisfies a linear recurrence $f(n+2) = P(n)f(n+1) + Q(n)f(n)$ for all sufficiently large n, where $P, Q \in \mathbb{R}(x)$ are rational functions. We study the ultimate sign of such a sequence, i.e., the repeated pattern that the signs of $f(n)$ follow for sufficiently large n. For each P, Q we determine all the ultimate signs that f can have, and show how they partition the space of initial values of f. This completes the prior work by Neumann, Ouaknine and Worrell, who have settled some restricted cases. As a corollary, it follows that when P, Q have rational coefficients, f either has an ultimate sign of length 1, 2, 3, 4, 6, 8 or 12, or never falls into a repeated sign pattern. We also give a partial algorithm that finds the ultimate sign of f (or tells that there is none) in almost all cases.

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1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of all natural numbers. A sequence $f = \{f(n)\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ of real numbers is called a *holonomic sequence* (of order $r \in \mathbb{N}$) if there are real-coefficient rational functions $P_0, \dots, P_{r-1} \in \mathbb{R}(x)$ such that f satisfies the linear recurrence

$$f(n+r) = P_{r-1}(n)f(n+r-1) + \cdots + P_0(n)f(n) \tag{1}$$

for all sufficiently large $n \in \mathbb{N}$. Holonomic sequences arise in various areas of mathematics. For instance, solutions of linear differential equations with polynomial coefficients are generating functions of holonomic sequences [25] (see also [4, Appendix B.4]), and for a "proper hypergeometric term" $F(n, k)$, which involves binomial coefficients $\binom{n}{k}$, the sum $f(n) = \sum_{k\in\mathbb{Z}} F(n, k)$ is holonomic if it converges for all $n \in \mathbb{N}$ [21].

An important computational problem concerning holonomic sequences is the *Ultimate Sign Problem* [16]: Given (rational-coefficient) rational functions $P_0, \dots, P_{r-1} \in \mathbb{Q}(x)$ without poles in $\mathbb{N}$ and (rational-valued) initial values $f(0), \dots, f(r-1) \in \mathbb{Q}$, find an ultimate sign, defined as follows, of the unique sequence f having these initial values and satisfying (1) for all $n \in \mathbb{N}$, and an index $N \in \mathbb{N}$ at which this ultimate sign is reached. Although we assume that f satisfies the recurrence (1) not only for $n \geq I$ for some $I \in \mathbb{N}$ but also for all n, it is not different in computability from the problem of finding the ultimate sign and the index N from the coefficients $P_0, \dots, P_{r-1}$, initial values $f(I), \dots, f(I+r-1)$ and I.

▶ **Definition 1.** *A sequence* $f \in \mathbb{R}^{\mathbb{N}}$ *is said to have an* ultimate sign $(s_0, \dots, s_{\tau-1}) \in \{+, -, 0\}^*$ *at* $N \in \mathbb{N}$ *if* $\operatorname{sgn} f(n) = s_{n \bmod \tau}$ *for all* $n \geq N$*, where* $\operatorname{sgn}\colon \mathbb{R} \to \{+, -, 0\}$ *is the function that maps each real number to its sign.*

For instance, the sequence $\{(-1)^n(n-2)\}_{n\in\mathbb{N}} = -2, 1, 0, -1, 2, -3, \dots$ has the ultimate sign $(+, -)$ at 3. Note that if f has the ultimate sign s at N, then it also has any repetition of s as an ultimate sign, and it does so at any index $\geq N$; but we could of course ask for the *shortest* ultimate sign s and the *least* index N without changing the computability of the problem.

The Ultimate Sign Problem is a generalization of several important problems about signs of holonomic sequences. One of the most famous problems is the *Skolem Problem*, which asks whether $f(n) = 0$ for some n (see [19, § 4] for an argument that it reduces to the Ultimate Sign Problem). Its decidability has been studied for almost 90 years [7]. The *Positivity Problem* asking whether $f(n) > 0$ for all n and the *Ultimate Positivity Problem* asking whether f has the ultimate sign $(+)$ are also well studied, with applications to automated inequality proving [6]; see also subsequent works [9, 22, 23] and a SageMath implementation [18].

When the coefficients $P_0, \dots, P_{r-1}$ are constant, f is called a C-finite sequence (or a linear recurrence sequence). The Skolem Problem for C-finite sequences of order $r \leq 4$ [27, 28] and the (Ultimate) Positivity Problem for C-finite sequences of order $r \leq 5$ [20] are known to be decidable, whereas the decidability for higher order C-finite sequences is open.

For holonomic sequences, when $r = 1$ (i.e., when f is a hypergeometric sequence), the Ultimate Sign Problem is easy since for given $P_0 \in \mathbb{Q}(x)$, we can effectively compute an index $N \in \mathbb{N}$ such that $P_0(n)$ has a constant sign for $n \geq N$. When $r = 2$, i.e., when f satisfies a recurrence of the form

$$f(n+2) = P(n)f(n+1) + Q(n)f(n), \tag{2}$$

the decidability of Skolem and (Ultimate) Positivity Problem for some subclasses is known in the context of the Membership Problem [17] and the Threshold Problem [10], respectively. [16, Theorem 7] shows that the Ultimate Sign Problem for another subclass is computable. However, the computability for general second-order holonomic sequences remains unknown. To make progress on this open problem, we study the ultimate signs of all second-order holonomic sequences.

Our first main contribution is to classify all pairs $(P, Q) \in \mathbb{R}(x)^2$ by the ultimate signs f can have, and show how the ultimate signs partition the space of initial values of f (Theorem 4). This result resolves all remaining cases in [16, Theorem 1], which handles the restricted case where P, Q are polynomials, P is non-constant and $\deg Q \leq \deg P$. In addition, this result implies that when P, Q have rational coefficients, the shortest ultimate sign of f, if it has one, is either of length 1, 2, 3, 4, 6, 8 or 12 (Corollary 6).

Our second contribution is to give a partial algorithm that solves the Ultimate Sign Problem for second-order holonomic sequences and halts on almost all inputs (Theorem 10). This extends a similar result [16, Theorem 3] for the restricted case mentioned above. This result can be also stated as a reduction theorem: for second-order holonomic sequences, the Ultimate Sign Problem Turing-reduces to the Minimality Problem, which asks the minimality of a given f, i.e., whether $f(n)/g(n) \to 0$ for all linearly independent solutions g of the same recurrence. In this sense our result extends [11, Theorem 3.1], which shows that the Positivity Problem Turing-reduces to the Minimality Problem. Note that, unfortunately, the decidability of Minimality Problem is unknown whereas many researchers numerically calculate minimal holonomic sequences and apply them to numerical analysis of some special functions (for example [5, 3]).

As a byproduct of our arguments, we amend some gaps in the proof of [16], slightly modifying its Theorem 7. This will be discussed in Section 2.3.

Related work

A lot of previous works describe their results in terms of continued fractions, which have a strong connection to second-order holonomic sequences. We illustrate the connection between those works and one of our main theorems in Sections 2.1.1 and 2.1.2.

Not only the ultimate signs, but also other periodicities of signs of holonomic (or C-finite) sequences are investigated. Closely related to the Skolem Problem, the periodicity of the zeros of C-finite (and for some holonomic) sequences is well-known as the Skolem-Mahler-Lech theorem [2]. Almagor et al. [1] give some sufficient conditions for C-finite sequences to have an "almost periodic sign", a loose property of sign periodicity.

Kooman [13] studies the asymptotic behaviour of complex solutions of the recurrence (2), where P and Q are not necessarily rational functions. His results helped us see the big picture of our main theorems.

2 Results

The Ultimate Sign Problem asks about the ultimate signs of f that satisfies (2) for all n. Such f is identified by the coefficient pair (P, Q) and the initial value $(f(0), f(1))$.

▶ **Definition 2.** *Let P, $Q \in \mathbb{R}(x)$ be rational functions without poles in $\mathbb{N}$. A sequence $f \in \mathbb{R}^{\mathbb{N}}$ is* (P, Q)-holonomic *if it satisfies* (2)*. The pair $(f(0), f(1)) \in \mathbb{R}^2$ is called the* initial value *of f.*

The Ultimate Sign Problem for $(0, Q)$- or $(P, 0)$-holonomic sequences is easy, so we assume $P \neq 0$ and $Q \neq 0$. By shifting the index by finitely many terms, we may assume that P, Q have no zeros in $\mathbb{N}$. This shifting changes the ultimate sign and the initial value of f in such a simple way that it does not affect the computability of the Ultimate Sign Problem. We adopt this assumption in all the following theorems.

2.1 Ultimate signs

Our first main theorem lists the ultimate signs that (P, Q)-holonomic sequences f can have, and shows how the ultimate signs partition the space of initial values of f for each of the following types (Definition 3) of (P, Q). For $R \in \mathbb{R}(x) \setminus \{0\}$, let $\deg R$ denote $d \in \mathbb{Z}$ satisfying $|R(x)| = \Theta(x^d)$ and call the ultimate sign of $\{R(n)\}_{n \in \mathbb{N}}$ that of R.

▶ **Definition 3.** *We classify $(P, Q) \in (\mathbb{R}(x) \setminus \{0\})^2$ into the following types. Let $d := \deg \frac{Q(x)}{P(x)P(x-1)}$ and (s) $(s \in \{+, -\})$ be the ultimate sign of $\frac{Q(x)}{P(x)P(x-1)}$.*

- *If $s = +$ and $d > 2$, then we say that (P, Q) is of* ∞-O loxodromic type.
- *If $s = +$ and $d \leq 2$, then we say that (P, Q) is of* ∞-Ω loxodromic type.
- *If $s = -$ and $d \leq 0$, then let $\alpha_0, \alpha_1, \alpha_2$ be real numbers satisfying*

$$\frac{Q(x)}{P(x)P(x-1)} = \alpha_0 + \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + O(x^{-3}). \tag{3}$$

- *If* $(\alpha_0, \alpha_1, \alpha_2) \geq (-\frac{1}{4}, 0, -\frac{1}{16})$ *in lexicographic order, then we say that* (P, Q) *is of* hyperbolic type.
- *Otherwise,* $\alpha_0 \leq -\frac{1}{4}$*, so there is a real number* $\theta \in [0, \frac{1}{2})$ *such that* $\alpha_0 = -\frac{1}{4\cos^2\theta\pi}$*.*

(1) *If* θ *is a positive rational number and* $\alpha_1 = 0$*, then we say that* (P, Q) *is of* θ*-O* elliptic type.

(2) *Otherwise, we treat* (P, Q) *together with the next case.*

- *If* $s = -$ *and* $d = 1, 2$*, or it is the case of* (2) *above, then we say that* (P, Q) *is of* $\mathbb{Q}$*-*Ω elliptic type*.*
- *If* $s = -$ *and* $d > 2$*, then we say that* (P, Q) *is of* $\frac{1}{2}$*-O* elliptic type*.*

This classification consists of the distinctions between *loxodromic type* (∞-O loxodromic type and ∞-Ω loxodromic type), hyperbolic type and *elliptic type* (θ-O elliptic type and $\mathbb{Q}$-Ω elliptic type), and between *O type* (∞-O loxodromic type and θ-O elliptic type) and Ω *type* (∞-Ω loxodromic type and $\mathbb{Q}$-Ω elliptic type). The highly non-trivial border between hyperbolic type and elliptic type is well-studied in the context of the convergence of continued fractions (Theorem 9).

The terminologies of "O" and "Ω" come from big O and Ω notations. They represent whether $\frac{Q(x)}{P(x)P(x-1)}$ is near or apart from a certain value (∞ for loxodromic type, $-\frac{1}{4\cos^2\theta\pi}$ for θ-O elliptic type and $-\frac{1}{4\cos^2 q\pi}$ for all $q \in (0, \frac{1}{2}] \cap \mathbb{Q}$ for $\mathbb{Q}$-Ω elliptic type).

The terminologies of loxodromic, hyperbolic and elliptic come from the classification of linear fractional transformations. If P and Q are constant, the linear fractional transformation $z \mapsto \frac{1}{P+Qz}$ maps the ratio $f(n)/f(n+1)$ between the two neighbouring terms of the (P, Q)-holonomic sequence to the next ratio $f(n+1)/f(n+2)$, and is said to be elliptic, parabolic, hyperbolic and loxodromic when $\frac{Q}{P^2}$ is in $(-\infty, -\frac{1}{4})$, $\{-\frac{1}{4}\}$, $(-\frac{1}{4}, 0)$ and $(0, \infty)$, respectively (with slight variations among authors – some (cf. [14, §4.1.3]) treat hyperbolic as a subclass of loxodromic, while some (cf. [24, § 4.7]) treat loxodromic as a subclass of hyperbolic).

This classification is a little complicated, but considering the case of constant P, Q, they are reasonable that the boundary between hyperbolic type and elliptic type is approximately at $-\frac{1}{4}$ and that θ-O elliptic type and $\mathbb{Q}$-Ω elliptic type are distinguished in such a way. If P and Q are constant, we can explicitly solve the recurrence (2) for f:

$$f(n) = \begin{cases} \dfrac{\alpha^n}{\alpha-\beta}\big(f(1) - \beta f(0)\big) + \dfrac{\beta^n}{\alpha-\beta}\big(-f(1) + \alpha f(0)\big) & \text{if } \alpha \neq \beta, \\ n\alpha^n\big(\alpha^{-1}f(1) - f(0)\big) + \alpha^n f(0) & \text{if } \alpha = \beta, \end{cases} \tag{4}$$

where α and β are the roots of the quadratic polynomial $x^2 - Px - Q$. When $\frac{Q}{P^2} \geq -\frac{1}{4}$, we have $\alpha, \beta \in \mathbb{R}$ and f has an ultimate sign of length 1 or 2. On the other hand, when $\frac{Q}{P^2} < -\frac{1}{4}$, the roots α and β are conjugate imaginary numbers. Then we can rewrite the formula (4) into $f(n) = Ar^n \sin(n\theta\pi + B)$, where $A, B, r \in \mathbb{R}$ are constants independent of n and $\theta \in (0, \frac{1}{2})$ is a constant satisfying $\frac{Q}{P^2} = -\frac{1}{4\cos^2\theta\pi}$. f has an ultimate sign of length τ for τ (≥ 4) such that $\tau\theta \in 2\mathbb{Z}$ if $\theta \in \mathbb{Q}$, whereas f has no ultimate signs if $\theta \notin \mathbb{Q}$. Our first main result (Theorem 4) is an extension of this fact, although we do not have explicit formulas like (4) for non-constant P, Q.

Since the set $I_{P,Q}(s)$ of initial values $(f(0), f(1))$ leading f to the ultimate sign s is closed under linear combinations with positive coefficients, it is a convex linear cone and thus specified by an (open, closed or half-open) interval $p(I_{P,Q}(s))$ on the unit circle S^1, where

$$p\colon \mathbb{R}^2 \setminus \{(0,0)\} \to S^1;\ (x, y) \mapsto (x, y)/\sqrt{x^2 + y^2} \tag{5}$$

is the projection. Thus, we will state the theorem by describing how S^1 is partitioned into intervals $p(I_{P,Q}(s))$. It is also obvious that flipping the sign of the initial value flips each element of the ultimate sign, so that $I_{P,Q}(-s)$ is just $I_{P,Q}(s)$ flipped around the origin.

We omit the parentheses and write $I_{P,Q}(+,-)$, say, for $I_{P,Q}((+,-))$.

Rather than considering all $P, Q \in \mathbb{R}(x)$, we state the theorem assuming the ultimate sign of P is $(+)$ because otherwise the ultimate sign of f can be obtained easily from that of the $(-P, Q)$-holonomic sequence $\{(-1)^n f(n)\}_{n\in\mathbb{N}}$ with initial value $(f(0), -f(1))$.

▶ **Theorem 4.** *Let P, $Q \in \mathbb{R}(x)$ be rational functions without zeros or poles in $\mathbb{N}$, and suppose that the ultimate sign of P is $(+)$. For each $s \in \{+,-,0\}^*$, we write $p(I_{P,Q}(s))$ for the set of $f_0 \in S^1$ such that the (P,Q)-holonomic sequence with initial value f_0 has the ultimate sign s.*

(I) *If (P,Q) is of ∞-O loxodromic type, S^1 is partitioned into closed intervals $p(I_{P,Q}(+,-))$, $p(I_{P,Q}(-,+))$ which have non-empty interiors and non-empty open intervals $p(I_{P,Q}(+))$, $p(I_{P,Q}(-))$.*

(II) *If (P,Q) is of ∞-Ω loxodromic type, S^1 is partitioned into singletons $p(I_{P,Q}(+,-))$, $p(I_{P,Q}(-,+))$ and non-empty open intervals $p(I_{P,Q}(+))$, $p(I_{P,Q}(-))$.*

(III) *If (P,Q) is of hyperbolic type, S^1 is partitioned into half-open intervals $p(I_{P,Q}(+))$, $p(I_{P,Q}(-))$.*

(IV) *If (P,Q) is of $\frac{k}{r}$-O elliptic type, where r and k are coprime positive integers, let*

$$s_j = \left(\operatorname{sgn}\sin\frac{j-ik+0.5}{r}\pi\right)_{i=0,\dots,2r-1}, \qquad t_j = \left(\operatorname{sgn}\sin\frac{j-ik}{r}\pi\right)_{i=0,\dots,2r-1}$$

for each $j = 0, \dots, 2r-1$.

- *If $\frac{Q(x)}{P(x)P(x-1)}$ is constant, S^1 is partitioned into $p(I_{P,Q}(t_0))$, $p(I_{P,Q}(s_0)), \dots$, $p(I_{P,Q}(t_{2r-1}))$, $p(I_{P,Q}(s_{2r-1}))$, arranged in this order (clockwise or anticlockwise), of which $p(I_{P,Q}(t_j))$ are singletons and $p(I_{P,Q}(s_j))$ are non-empty open intervals.*
- *Otherwise, S^1 is partitioned into non-empty half-open intervals $p(I_{P,Q}(s_0)), \dots$, $p(I_{P,Q}(s_{2r-1}))$, arranged in this order, where for each $j = 0, \dots, 2r-1$, the intersection of the closures of $p(I_{P,Q}(s_j))$ and $p(I_{P,Q}(s_{j+1}))$ (where $s_{2r} = s_0$) belongs to $p(I_{P,Q}(s_{j+1}))$ if $\frac{Q(x)}{P(x)P(x-1)}$ is eventually increasing (i.e., increasing for sufficiently large x), and to $p(I_{P,Q}(s_j))$ if it is eventually decreasing.*

(V) *If (P,Q) is of $\mathbb{Q}$-Ω elliptic type, then no non-zero (P,Q)-holonomic sequence has an ultimate sign.*

In Part (IV), the value 0.5 can be replaced by any value between 0 and 1. If (P,Q) is of $\frac{1}{2}$-O elliptic type, then $\frac{Q(x)}{P(x)P(x-1)}$ necessarily decreases eventually.

In Parts (I), (II), (III) and (IV), the union of the boundaries of the sets $I(s)$ is a finite union of lines. Following [16], which handles restricted cases of (II) and (III) with $\deg \frac{Q(x)}{P(x)P(x-1)} \le -1$, we call these lines the *critical lines*.

▶ **Example 5.** Let $P(x) = \frac{x+2}{x+1}$ and $Q(x) = -\frac{x+3}{x+1}$, so that $\frac{Q(x)}{P(x)P(x-1)} = -1 + \frac{2}{x^2+3x+2}$ is decreasing and (P,Q) is $\frac{1}{3}$-O elliptic. Part (IV) of Theorem 4 states that non-zero (P,Q)-holonomic sequences f in this case have ultimate signs

$$\begin{aligned} &s_0 = (+,-,-,-,+,+), \qquad s_1 = (+,+,-,-,-,+), \qquad s_2 = (+,+,+,-,-,-), \\ &s_3 = (-,+,+,+,-,-), \qquad s_4 = (-,-,+,+,+,-), \qquad \text{or } s_5 = (-,-,-,+,+,+), \end{aligned} \tag{6}$$

and that the set $I_{P,Q}(s_j)$ of initial values that result in each ultimate sign s_j is the area between two halves of critical lines and includes the boundary facing $I_{P,Q}(s_{j+1})$ (where we

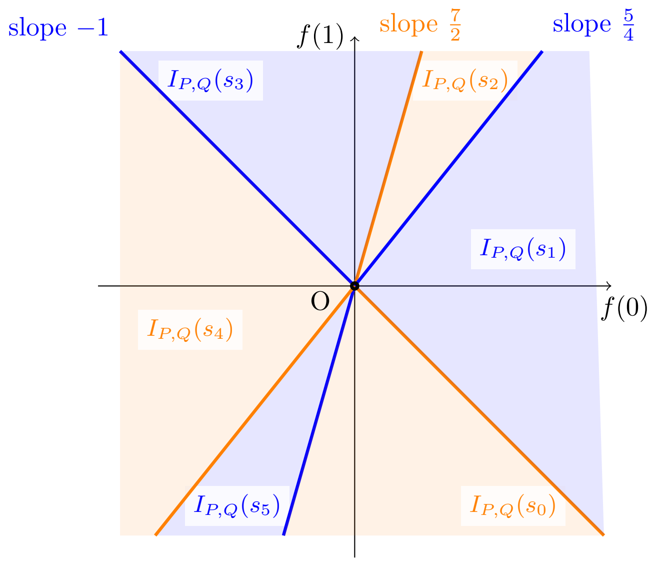

Figure 1 The set of initial values $(f(0), f(1))$ of $\left(\frac{x+2}{x+1}, -\frac{x+3}{x+1}\right)$-holonomic sequences f having each of the ultimate signs in (6).

write $s_6 = s_0$). For this particular example, we can verify this by finding $I_{P,Q}(s_j)$ explicitly, as we see by induction n that the solution of (2) is

$$f(n) = \begin{cases} (-1)^m \left(\left(\frac{7}{2}m+1\right) f(0) - mf(1)\right) & \text{if } n = 3m, \\ (-1)^m \left(mf(0) + (m+1)f(1)\right) & \text{if } n = 3m+1, \\ (-1)^{m+1} \left(\left(\frac{5}{2}m+3\right) f(0) - 2(m+1)f(1)\right) & \text{if } n = 3m+2, \end{cases} \tag{7}$$

so that $I_{P,Q}(s_0), \ldots, I_{P,Q}(s_5)$ are as depicted in Figure 1.

Note that the solution (7) is a normal form of a hypergeometric sequence in the sense of [26] and can be found algorithmically.

Restricting Theorem 4 to rational-coefficient (P, Q), we obtain the following:

▶ **Corollary 6.** *Suppose that P, $Q \in \mathbb{Q}(x)$ have no zeros or poles in $\mathbb{N}$. Then every (P, Q)-holonomic sequence has an ultimate sign of length 1, 2, 3, 4, 6, 8 or 12, if it has an ultimate sign at all.*

Proof. TOPROVE 0 ◀

We can derive from Theorem 4 another corollary (Corollary 20 in Section 3.2). Appropriate subsequences of second-order holonomic sequences are again second-order holonomic sequences. That corollary describes the types of the coefficients of the recurrence which the subsequences satisfy.

2.1.1 Connection to continued fractions

In this section, we discuss the connection between Theorem 4 and convergence theorems of continued fractions

$$\operatorname*{K}_{k=0}^{n} \frac{Q(k)}{P(k)} = \cfrac{Q(0)}{P(0) + \cfrac{Q(1)}{P(1) + \ddots + \cfrac{Q(n)}{P(n)}}}.$$

Note that continued fractions take values in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with $x/\infty = 0$ for $x \in \mathbb{R}$ and $x/0 = \infty$ for $x \in \mathbb{R} \setminus \{0\}$. See [14] about their deep theory and application. Continued fractions are closely related to second-order holonomic sequences through the next proposition, which can be verified by induction on n (simultaneously for all P and Q):

▶ **Proposition 7.** *Let $P, Q \in \mathbb{R}(x)$ have no poles in $\mathbb{N}$ and A and B be the (P, Q)-holonomic sequences with initial values $(1, 0)$ and $(0, 1)$ respectively. Then*

$$\operatorname*{K}_{k=0}^{n} \frac{Q(k)}{P(k)} = \frac{A(n+2)}{B(n+2)} \tag{8}$$

in $\hat{\mathbb{R}}$ for all $n \in \mathbb{N}$.

For this reason, $A(n)$ and $B(n)$ are called the nth canonical numerator and denominator, respectively. We can interpret Theorem 4 to a convergence theorem of subsequences $\{p(A(n), B(n))\}_{n \equiv i \pmod{\tau}}$, $i = 0, \ldots, \tau - 1$, of $p(A(n), B(n))$ where p is the projection (5) and $\tau \geq 1$ is a suitable integer below.

Let τ be 2, 1, 1, $2r$ in Theorem 4 (I), (II), (III), (IV), respectively. Then the set $I_i(+)$ of initial values of (P, Q)-holonomic sequence f such that $\{f(n)\}_{n \equiv i \pmod{\tau}}$ has the ultimate sign $(+)$ is a half-plane on $\mathbb{R}^2$. Since f satisfies

$$f(n) = A(n)f(0) + B(n)f(1) = \sqrt{A(n)^2 + B(n)^2}\; p(A(n), B(n)) \cdot (f(0), f(1)), \tag{9}$$

$\{p(A(n), B(n))\}_{n \equiv i \pmod{\tau}}$ converges to the midpoint of the interval $p(I_i(+))$. Similarly, it can be derived that, for any $\tau \geq 1$, one of $\{p(A(n), B(n))\}_{n \equiv i \pmod{\tau}}$ must diverge in the case of Theorem 4 (V). In this sense, Theorem 4 is a convergence theorem of the subsequences of $p(A(n), B(n))$.

By the discussion above, the slopes of the critical lines in Theorem 4 (I), (II), (III), (IV) can be represented by the limits of $\left\{-\operatorname{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \equiv i \pmod{\tau}} = \left\{-\frac{A(n)}{B(n)}\right\}_{n \equiv i \pmod{\tau}}$, $i = 0, \ldots, \tau - 1$, and thus the convergence of subsequences of $\operatorname{K}_{k=0}^{n} \frac{Q(k)}{P(k)}$ follows.

▶ **Theorem 8.** *Let P, $Q \in \mathbb{R}(x)$ be rational functions without zeros or poles in $\mathbb{N}$. First, in (I), (II), (III) and (IV) of Theorem 4, the slopes of the critical lines are exactly the accumulation points of the continued fraction $\left\{-\operatorname{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \in \mathbb{N}}$. Second, the accumulation of the continued fraction is as follows:*

(1) *If (P, Q) is of ∞-O loxodromic type, the subsequences $\left\{\operatorname{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \equiv i \pmod{2}}$, $i = 0$, 1, converge in $\hat{\mathbb{R}}$ to distinct values.*

(2) *If (P, Q) is of ∞-Ω loxodromic or hyperbolic type, the sequence $\left\{\operatorname{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \in \mathbb{N}}$ converges in $\hat{\mathbb{R}}$.*

(3) *If (P, Q) is of $\frac{k}{r}$-O elliptic type, where r and k are coprime positive integers, the sequences* $\left\{\mathrm{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \equiv i \pmod r}$, $i = 0, \ldots, r-1$, *converge in $\hat{\mathbb{R}}$ to distinct values.*

(4) *If (P, Q) is of $\mathbb{Q}$-Ω elliptic type, then for no positive integer τ and no $i \in \{0, \ldots, \tau-1\}$ does the sequence* $\left\{\mathrm{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \equiv i \pmod \tau}$ *converge in $\hat{\mathbb{R}}$.*

We consider the "gap-r subsequences" $\left\{\mathrm{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \equiv i \pmod r}$ instead of the gap-$2r$ subsequences in (3) because the limit of $\{p(A(n), B(n))\}_{n \equiv i \pmod{2r}}$ is equal to the limit of $\{p(A(n), B(n))\}_{n \equiv i+\tau \pmod{2r}}$ except for multiplication by ± 1.

Part (1) of this theorem is included in [14, Theorems 3.12 and 3.13]. Part (3) is similar to [14, Lemma 4.28]. Part (2) can be derived from the following well-known convergence theorem. Although Parts (1), (2) and (3) follow from Theorem 4, Part (4) does not follow from Theorem 4 alone since it states divergence instead of convergence. We prove (4) in Section 4.1 using the convergence theorem below and Corollary 20 (2) (in Section 3.2), which is derived from Theorem 4.

▶ **Theorem 9** ([12, Theorem 7.1]). *Let $P, Q \in \mathbb{R}(x)$ be rational functions without zeros or poles in $\mathbb{N}$. The continued fraction* $\left\{\mathrm{K}_{k=0}^{n} \frac{Q(k)}{P(k)}\right\}_{n \in \mathbb{N}}$ *converges in $\hat{\mathbb{R}}$ if and only if (P, Q) is of ∞-Ω loxodromic or hyperbolic type.*

2.1.2 Connection to monotonic convergence of continued fractions

If we identify the ultimate sign of B, we can extend the convergence of subsequences of $\frac{A(n)}{B(n)}$ to that of $p(A(n), B(n))$. But this is not enough to prove each part of Theorem 4; we need monotonic convergence theorems. This is because Theorem 4 even describes the ultimate signs of holonomic sequences with initial values on the critical lines, and therefore figures out not only the convergence of subsequences of $p(A(n), B(n))$, but also the direction in which the subsequences of $p(A(n), B(n))$ converge to their limits.

[14, Theorems 3.12 and 3.13] and [11, Lemma 3.4] are monotonic convergence theorems for (P, Q) of ∞-O, -Ω loxodromic type and of hyperbolic type, respectively, and both literature identify the ultimate sign of B in their cases. Hence Theorem 4 (I) and (II) can be derived from the former literature, and (III) can be derived from the latter.

2.2 Computing the ultimate sign

The partial algorithm in the following theorem tells us, for given $(P, Q) \in \mathbb{Q}(x)^2$ and $f_0 \in \mathbb{Q}^2$, the index $N \in \mathbb{N}$ at which the (P, Q)-holonomic sequence with initial value f_0, whenever it terminates. Note that once we get N, we can obtain the ultimate sign itself by looking at the signs of a finite number of terms $f(N)$, $f(N+1)$, ... according to Theorem 4.

▶ **Theorem 10.** *There exists a partial algorithm that,*

- *given $P, Q \in \mathbb{Q}(x)$ without zeros or poles in $\mathbb{N}$, together with a pair $f_0 \in \mathbb{Q}^2$,*
- *terminates if and only if the (P, Q)-holonomic sequence f with initial value f_0 has an ultimate sign and it is stable in the sense that there is a neighbourhood $\mathcal{N} \subseteq \mathbb{Q}^2$ of f_0 such that all (P, Q)-holonomic sequences with initial value in $\mathcal{N}$ have the same ultimate sign, and*
- *whenever it terminates, outputs an index at which f has its ultimate sign.*

Note that the type of (P, Q) can be computed from P and Q, and hence, although the partial algorithm does not terminate when $f_0 = (0, 0)$ or when (P, Q) is $\mathbb{Q}$-Ω elliptic (because of Theorem 4 (V)), we could make it terminate also on these inputs and declare the non-existence of an ultimate sign in the latter case.

This partial algorithm terminates on "most" inputs since, for (P, Q) of ∞-O, -Ω loxodromic, hyperbolic and θ-O type, the (P, Q)-holonomic sequence f with initial value f_0 has an unstable ultimate sign if and only if f_0 is on the finitely many critical lines delimiting the areas $I_{P,Q}(s)$ in Theorem 4. For a small but substantial class of (P, Q), it is known that all $f_0 \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ lead f to a stable ultimate sign, or in other words, the slopes of the critical lines are irrational, which is the main topic of Section 2.3. However there is no known general method to determine the stability, and it is a wide-open problem whether we can make the algorithm terminate on all inputs [11, 8, 16].

Theorem 10 is stated for rational-coefficient P, Q and rational-valued f_0, so that the problem is computationally meaningful. By studying the proofs in some detail we could, however, modify the statement appropriately so that the partial algorithm accepts inputs involving real numbers represented as infinite sequences of approximations, in a way analogous to the discussion in [15] about signs of C-finite sequences.

▶ **Example 11.** Let us compare the values of the sums

$$\sum_{0 \le k \le \frac{n+1}{2},\ k \in 2\mathbb{Z}} k\binom{n+1-k}{k} \quad \text{and} \quad \sum_{0 \le k \le \frac{n+1}{2},\ k \in 2\mathbb{Z}+1} k\binom{n+1-k}{k}$$

using the partial algorithm in Theorem 10. It suffices to identify the ultimate sign of the difference $f(n) := \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^k k \binom{n+1-k}{k}$ of the two sums and at which index f has it. By creative telescoping [21, Chapter 6], we find that f is the (P, Q)-holonomic sequence with initial value $(0, -1)$, where (P, Q) is as in Example 5. Now we can input this into our partial algorithm and it tells that f has the ultimate sign $(+, -, -, -, +, +)$ at 1, i.e., when $n \ge 1$, the former sum is greater than and less than the latter sum if $n \equiv 0, 4, 5 \pmod 6$ and if $n \equiv 1, 2, 3 \pmod 6$, respectively.

Unlike the above case, for the same (P, Q), the partial algorithm never terminates with initial values on the critical lines in Figure 1, such as $(1, -1)$, $(4, 5)$ and $(2, 7)$.

Let us consider another example: compare

$$\sum_{0 \le k \le n,\ k \in 2\mathbb{Z}} k\binom{n}{k}^3 \quad \text{and} \quad \sum_{0 \le k \le n,\ k \in 2\mathbb{Z}+1} k\binom{n}{k}^3.$$

Taking a similar process, we find that the difference $g(n) := \sum_{k=0}^{n} (-1)^k k \binom{n}{k}^3$ is the (R, S)-holonomic sequence with initial value $(0, -1)$, where

$$(R(x), S(x)) = \left(\frac{18x^2 + 36x + 12}{(x+1)(x+2)(6x^2 + 4x + 1)}, -\frac{3(3x+2)(3x+1)(6x^2 + 16x + 11)}{(x+1)(x+2)(6x^2 + 4x + 1)} \right).$$

Note that (R, S) is of $\frac{1}{2}$-O elliptic type. Our partial algorithm proves that g has the ultimate sign $(+, -, -, +)$ at 1, i.e., when $n \ge 1$, the former sum is greater than and less than the latter sum if $n \equiv 0, 3 \pmod 4$ and if $n \equiv 1, 2 \pmod 4$, respectively.

However, for this (R, S), we do not know how to identify the critical lines. Algorithm Hyper [21, Chapter 8] declared that there is no explicit formula like (7) (more precisely, "closed form"), so the above discussion for (P, Q) does not work for (R, S). By numerical analysis using our partial algorithm, we find that the slope of the critical line between $I_{R,S}(+, -, -, +)$ and $I_{R,S}(+, +, -, -)$ is in the interval $(-2.452, -2.434)$, and the one between $I_{R,S}(+, +, -, -)$ and $I_{R,S}(-, +, +, -)$ is in $(4.8094, 4.816)$.

Theorem 10 can be described in a reduction form that is an extension of [11, Theorem 3.1]:

▶ **Theorem 12.** *For second-order holonomic sequences, the Ultimate Sign Problem Turing-reduces to the Minimality Problem.*

2.3 Input set admitting a total algorithm

The main predecessor to our work [16, Theorem 1, 3 and 7] relies on [16, Lemma 14] whose proof contained an error in the calculation of an inverse image. Their classification and the partial algorithm [16, Theorem 1 and 3] analogous to our Theorems 4 and 10 are correct after all, as our theorems imply. In this section, we state Theorem 13, an amendment of [16, Theorem 7]. Note that our theorem is slightly weaker than the original one due to one more gap in the proof. We mention this in detail after proving Theorem 13 in Section 4.2.

Theorem 13 gives a sufficient condition on $P, Q \in \mathbb{Q}(x)$ for all non-zero (P,Q)-holonomic sequences $f \in \mathbb{Q}^{\mathbb{N}} \setminus \{0\}$ to have stable ultimate signs. This gives a nontrivial input set on which the Ultimate Sign Problem is solvable by the partial algorithm in Theorem 10.

The restriction of P and Q to $\mathbb{Z}[x]$ instead of $\mathbb{Q}(x)$ is no essential loss of generality: For $P,\ Q \in \mathbb{Q}(x)$, let $P_1, P_2, Q_1, Q_2 \in \mathbb{Z}[x]$ satisfy $P = \frac{P_1}{P_2}$ and $Q = \frac{Q_1}{Q_2}$. Then we can apply the theorem on P' and Q', where $P'(x) = P_1(x+1)Q_2(x+1)$ and $Q'(x) = Q_1(x+1)Q_2(x)P_2(x+1)P_2(x)$. The ultimate sign of a (P,Q)-holonomic sequence f is stable if and only if that of the (P',Q')-holonomic sequence $\{f(n+1)\prod_{k=0}^{n-1} P_2(k)Q_2(k)\}_{n\in\mathbb{N}}$ is stable.

▶ **Theorem 13.** *Let $P(x) = p_0x^d + p_1x^{d-1} + \cdots + p_d \in \mathbb{Z}[x]$ and $Q(x) = q_0x^d + q_1x^{d-1} + \cdots + q_d \in \mathbb{Z}[x]$ be polynomials without zeros in $\mathbb{N}$. Suppose that $p_0 > 0$ and $d \geq 1$ (where q_0 might be zero). Then, if P and Q satisfy either of the following conditions, any (P,Q)-holonomic sequence $f \in \mathbb{Q}^{\mathbb{N}} \setminus \{0\}$ has a stable ultimate sign.*

(1) $|q_0| < p_0$

(2) $|q_0| = p_0$ *and the two conditions below hold for $s := \operatorname{sgn} q_0 \in \{1, -1\}$:*

- $Q(x) - sP(x) \neq 1$ *in* $\mathbb{Z}[x]$,
- $\begin{cases} sq_1 - p_1 - s < p_0 & \text{if } d = 1, \\ sq_1 - p_1 < p_0 & \text{if } d \geq 2. \end{cases}$

3 Proof of the Main Results

In this section, we prove Theorems 4, 10, and 12. All the proofs of the lemmas in the following Sections 3.1 and 3.2 are postponed to Section 3.3.

3.1 Proof of Theorem 4

Let us first focus on identifying the lengths of the ultimate signs that (P,Q)-holonomic sequences can have and get an overview of the proof of Theorem 4. Lemmas 14 and 15 below, by types of (P,Q), characterize (P,Q) admitting (P,Q)-holonomic sequences with ultimate signs of lengths 1 and 2, respectively. Then only lengths $\tau \geq 3$ are left. For each $\tau \geq 3$, we will introduce a special recurrence such that we can decide if $F \in \mathbb{R}^{\mathbb{N}}$ satisfying the recurrence has a (shortest) ultimate sign of length τ (Lemma 16). Next, by types of (P,Q), we characterize (P,Q) and τ that allow all (P,Q)-holonomic sequences f to be transformed to F satisfying the special recurrence and having the same ultimate sign as f (Lemma 18). Finally we show that, for the other (P,Q) and $\tau \geq 3$, no non-zero (P,Q)-holonomic sequences have the shortest ultimate sign of length τ in the proof of Theorem 4 (V). Note that some lemmas below are superfluous for identifying the lengths of ultimate signs, but required to identify the ultimate signs themselves and how they partition the space of the initial values.

▶ **Lemma 14.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and P have the ultimate sign $(+)$.*

(1) *$I_{P,Q}(+) \neq \varnothing \iff (P,Q)$ is of loxodromic type or hyperbolic type.*
(2) *If (P,Q) is of hyperbolic type, then $I_{P,Q}(+) \cup I_{P,Q}(-) = \mathbb{R}^2 \setminus \{(0,0)\}$.*

Similar results to the above lemma appear in, e.g., [11].

The following lemma is relatively easy and similar propositions appear in context of continued fractions (e.g., [14, Theorem 3.12]).

▶ **Lemma 15.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and P have the ultimate sign $(+)$.*
(1) *$I_{P,Q}(+,-) \neq \varnothing \iff (P,Q)$ is of loxodromic type.*
(2) *$p(I_{P,Q}(+,-))$ is a closed interval.*
(3) *If (P,Q) is of loxodromic type, then $I_{P,Q}(+) \cup I_{P,Q}(-) \cup I_{P,Q}(+,-) \cup I_{P,Q}(-,+) = \mathbb{R}^2 \setminus \{(0,0)\}$.*

Now we introduce the special recurrence mentioned in the first paragraph of this section. For a (not necessarily holonomic) sequence $F \in \mathbb{R}^{\mathbb{N}}$, consider a *single-term-feedback recurrence*

$$F(n+\tau) - F(n) = R(n)F(n+1), \tag{10}$$

where τ is an integer ≥ 2 and $R \in \mathbb{R}^{\mathbb{N}}$. This recurrence expresses the difference between two neighbouring terms in the gap-τ subsequences $\{F(n)\}_{n\equiv i \pmod{\tau}}$, $i = 0, \dots, \tau-1$, as a single term in the next subsequence $\{F(n)\}_{n\equiv i+1 \pmod{\tau}}$ multiplied by the coefficient R. In the following lemma, we treat the case where $|R(n)|$ rapidly decreases in (1) and the case where $|R(n)|$ does not rapidly decrease in (2).

▶ **Lemma 16.** *Let $F \in \mathbb{R}^{\mathbb{N}}$ satisfy the single-term-feedback recurrence (10) for a coefficient $R \in \mathbb{R}^{\mathbb{N}}$ and an integer $\tau \geq 2$.*
(1) *(restricted case of [12, Theorem 6]) Suppose $R(n) = O(n^{-1-\varepsilon})$ for some $\varepsilon > 0$.*
  (1a) *Each of the gap-τ subsequences $\{F(n)\}_{n\equiv i \pmod{\tau}}$, $i = 0, \dots, \tau-1$, converges.*
  (1b) *If $F \neq 0$, then there is $i \in \{0, \dots, \tau-1\}$ for which $\{F(n)\}_{n\equiv i \pmod{\tau}}$ does not converge to 0.*
(2) *Suppose that $|R(n)| = \Omega(n^{-1})$ and R has an ultimate sign $(+)$ or $(-)$. If F has an ultimate sign of length τ, then F also has an ultimate sign of length ≤ 2.*
(3) *Suppose that R has an ultimate sign (q), $q \in \{+,-,0\}$. Let $i \in \{0, \dots, \tau-1\}$. If a subsequence $\{F(n)\}_{n\equiv i+1 \pmod{\tau}}$ of F has the ultimate sign (s), $s \in \{+,-,0\}$ and $\{F(n)\}_{n\equiv i \pmod{\tau}}$ converges to 0, then $\{F(n)\}_{n\equiv i \pmod{\tau}}$ has the ultimate sign $(-qs)$.*

In the situation of (1), F has an ultimate sign of length τ as follows. If $F = 0$, it is obvious. If $F \neq 0$, then by (1a) and (1b), there is i such that $\{F(n)\}_{n\equiv i \pmod{\tau}}$ has the ultimate sign $(+)$ or $(-)$. Then $\{F(n)\}_{n\equiv i-1 \pmod{\tau}}$ also has $(+)$ or $(-)$ if it converges to a non-zero real number. It has $(+)$, $(-)$ or (0) even if it converges to zero by (3). Thus, by induction, every gap-τ subsequence of F has ultimate sign of length 1, meaning that F has an ultimate sign of length τ. On the other hand, in the situation of (2), F does not have the shortest ultimate sign of length $\tau \geq 3$.

Part (1) of Lemma 16 is known for a larger class of recurrences [12, Theorem 6]. Our restriction to the single-term-feedback recurrence allows (2) and (3) to hold.

Now we want to find sequences $T, R \in \mathbb{R}^{\mathbb{N}}$ such that for each (P,Q)-holonomic sequence f, the transformed sequence $F(n) := T(n)f(n)$ has the same ultimate sign as f and satisfies the recurrence (10). F and f have the same ultimate sign if and only if T has the ultimate sign $(+)$. To find the condition on T and R for F to satisfy the recurrence (10), we use $A^{(\tau)}, B^{(\tau)} \in \mathbb{R}(x)$ below.

▶ **Definition 17.** *For $P, Q \in \mathbb{R}(x)$ without zeros or poles in $\mathbb{N}$, there uniquely exist $A^{(\tau)}, B^{(\tau)} \in \mathbb{R}(x)$ such that any (P, Q)-holonomic sequence f satisfies the recurrence*

$$f(n+\tau) = B^{(\tau)}(n) f(n+1) + A^{(\tau)}(n) f(n) \tag{11}$$

for all $n \in \mathbb{N}$. Let us call $A^{(\tau)}$ and $B^{(\tau)}$ the generalized τth canonical numerator *and* denominator *(of (P, Q)) respectively.*

These are generalizations of the notions of τth canonical numerator A and denominator B in Proposition 7 since $(A^{(\tau)}(0), B^{(\tau)}(0)) = (A(\tau), B(\tau))$. We can generalize Equation (8) to $\mathrm{K}_{k=n}^{n+\tau} \frac{Q(k)}{P(k)} = \frac{A^{(\tau+2)}(n)}{B^{(\tau+2)}(n)}$. Equation (11) is a generalization of the equation $f(\tau) = B(\tau)f(1) + A(\tau)f(0)$ that A and B satisfy for any (P, Q)-holonomic sequence f.

Let $\tau \geq 2$ and $T, R \in \mathbb{R}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, by Equation (11), $F(n) = T(n)f(n)$ satisfy Equation (10) for all (P, Q)-holonomic sequences f if and only if

$$T(n+\tau)A^{(\tau)}(n) = T(n), \quad R(n)T(n+1) = B^{(\tau)}(n)T(n+\tau). \tag{12}$$

To allow T to have the ultimate sign $(+)$, we want $A^{(\tau)}$ to have $(+)$. In addition, to apply Lemma 16 (1) for $F(n) = T(n)f(n)$, the absolute value of the coefficient $|R(n)|$ has to decrease rapidly. The next lemma shows that there exists τ satisfying these conditions if and only if (P, Q) is of O type.

▶ **Lemma 18.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$, and P have the ultimate sign $(+)$. Let $\tau \geq 2$ be an integer and $A^{(\tau)}$ and $B^{(\tau)}$ be the τth generalized canonical numerator and denominator, respectively.*

(1) *Assume that $T, R \in \mathbb{R}^{\mathbb{N}}$ satisfy (12) and $T(n) \neq 0$ for all sufficiently large n. Then $\left|\frac{T(n+1)}{T(n)}\right| = \Theta(|A^{(\tau)}(n)|^{-1/\tau})$. Especially, $|R(n)| = \Theta\left(|B^{(\tau)}(n)||A^{(\tau)}(n)|^{-1+1/\tau}\right)$.*

(2) *The following are equivalent.*

- **(2a)** *$A^{(\tau)}$ has the ultimate sign $(+)$ and $|B^{(\tau)}(n)|A^{(\tau)}(n)^{-1+1/\tau} = O(n^{-1-\varepsilon})$ for some $\varepsilon > 0$.*
- **(2b)** *(P, Q) is of θ-O elliptic type and $\tau\theta \in 2\mathbb{Z}$, or (P, Q) is of ∞-O loxodromic type and $\tau \in 2\mathbb{Z}$.*

Now we are ready to show Theorem 4.

Proof. TOPROVE 1 ◀

Proof. TOPROVE 2 ◀

Proof. TOPROVE 3 ◀

It remains to show (IV). Let (P, Q) be of θ-O elliptic type. As already mentioned, for τ such that $\tau\theta \in 2\mathbb{Z}$, all (P, Q)-holonomic sequences f have ultimate signs of length τ. Now we need to determine which ultimate signs (of length τ) f can have. This will be derived from the following lemma.

▶ **Lemma 19.** *Take (P, Q) as in Lemma 18 and assume that it is of $\frac{k}{r}$-O elliptic type.*

(1) *The generalized $2r$th canonical denominator $B^{(2r)}$ has the ultimate sign $(+)$, $(-)$ and (0) if $\frac{Q(x)}{P(x)P(x-1)}$ is eventually increasing, if it is eventually decreasing and if it is constant, respectively.*

(2) *By Lemma 18 (2), we can choose $T \in \mathbb{R}^{\mathbb{N}}$ such that $T(n) > 0$ and the relation (12) for $\tau = 2r$ hold for all sufficiently large n. Then, for each $j = 0, \ldots, \tau - 1$, there exists a (P, Q)-holonomic $f^{(j)}$ such that for all $i \in \{0, \ldots, \tau - 1\}$, $\{T(n)f^{(j)}(n)\}_{n \equiv i \pmod{2r}}$ converges to a real number of sign $\operatorname{sgn} \sin \frac{j-ik}{r}\pi$.*

Proof. TOPROVE 4 ◀

3.2 Proof of Theorems 10 and 12

Theorems 10 and 12 are algorithmic claims stating that the ultimate signs can be partially computed in each sense. We could prove them by analyzing the proof of Theorem 4 quantitatively. But instead of carrying out such analysis for each case of Theorem 4 separately, we choose to do so just for the hyperbolic type (Lemma 21 below), and argue that all other types (having ultimate signs) reduce to it in the sense of the following Corollary 20.

From the original recurrence (2), we can obtain, for each positive integer τ, a "gap-τ recurrence"

$$f(n+2\tau) = P_\tau(n)f(n+\tau) + Q_\tau(n)f(n), \tag{13}$$

where P_τ and Q_τ are rational functions. Specifically, they can be written as

$$P_\tau = \frac{B^{(2\tau)}}{B^{(\tau)}}, \qquad Q_\tau = A^{(2\tau)} - \frac{B^{(2\tau)}}{B^{(\tau)}}A^{(\tau)} \tag{14}$$

using the generalized canonical numerators $A^{(0)}$, $A^{(1)}$, …and denominators $B^{(0)}$, $B^{(1)}$, …of (P,Q) (see Definition 17), assuming that $B^{(\tau)}$ is non-zero. (Note that if $B^{(\tau)} = 0$, we have $f(n+\tau) = A^{(\tau)}(n)f(n)$, in which case the ultimate sign of f can be found easily.) Thus, the subsequence $\{f(\tau n+N)\}_{n\in\mathbb{N}}$ of f, for any number $N \in \mathbb{N}$ greater than all zeros of $B^{(\tau)}$, is the $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$-holonomic sequence with initial value $(f(N), f(N+\tau))$. The following corollary to Theorem 4 says that with a right choice of τ, this $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of hyperbolic type, unless (P,Q) is of $\mathbb{Q}$-Ω elliptic type.

▶ **Corollary 20.** *Suppose that P, $Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$. Let $A^{(0)}$, $A^{(1)}$, …and $B^{(0)}$, $B^{(1)}$, …be the generalized canonical numerators and denominators, respectively.*

(1) *Suppose that (P,Q) is either of loxodromic type or of $\frac{k}{r}$-O elliptic type for some coprime positive integers r and k. Let $\tau = 2$ in the former case, and $\tau = 2r$ in the latter case. Suppose that $B^{(\tau)}$ and $B^{(2\tau)}$ are non-zero. Then P_τ and Q_τ defined by (14) are non-zero, and $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of hyperbolic type for all $N \in \mathbb{N}$.*

(2) *Suppose that (P,Q) is of $\mathbb{Q}$-Ω elliptic type. Then $B^{(\tau)}$ is non-zero, P_τ and Q_τ defined by (14) are also non-zero, and $(P_\tau(\tau x+N), Q_\tau(\tau x+N))$ is of $\mathbb{Q}$-Ω elliptic type for all $N \in \mathbb{N}$ and $\tau \geq 1$.*

Proof. TOPROVE 5 ◀

▶ **Lemma 21** (A quantitative version of Lemma 14)**.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$.*

(1) *The following are equivalent.*

- **(1a)** *(P,Q) is of loxodromic or hyperbolic type.*
- **(1b)** *There exists $q \in \mathbb{R}^{\mathbb{N}}$ with ultimate sign $(+)$ that satisfies*

$$q(n)(1-q(n+1)) \geq -\frac{Q(n)}{P(n)P(n-1)} \tag{15}$$

for all sufficiently large $n \in \mathbb{N}$.

(2) *If (1b) holds, then it holds for the sequence q defined by $q(0) = q(1) = 1$ and $q(n) = \frac{1}{2} + \frac{1}{4n} + \frac{1}{4n\log n}$, $n \geq 2$.*

(3) *Let (P,Q) be of hyperbolic type and P have the ultimate sign $(+)$. Take any q in (1b). Take $N \in \mathbb{N}$ such that P, q, Q have their ultimate signs at N and the condition (15) is satisfied for any $n \geq N$. Let f be a (P,Q)-holonomic sequence. Then if*

$$f(n) \neq 0 \text{ and } \frac{f(n+1)}{f(n)} > q(n)P(n-1) \tag{16}$$

holds for some $n \geq N$, this condition also holds for $n+1, n+2, \ldots$. In particular, f has an ultimate sign $(+)$ or $(-)$ at n.

The sequence q in Lemma 21 (2) is what appears in the proof of [11, Lemma 3.4].

Proof. TOPROVE 6 ◀

Proof. TOPROVE 7 ◀

3.3 Proof of the lemmas

Proof. TOPROVE 8 ◀

Proof. TOPROVE 9 ◀

Proof. TOPROVE 10 ◀

Proof of Lemma 16

Proof. TOPROVE 11 ◀

Proof. TOPROVE 12 ◀

Proof. TOPROVE 13 ◀

Proof of Lemmas 18 and 19

Proof. TOPROVE 14 ◀

To prove Lemma 18 (2) and Lemma 19, let us study the properties of the generalized τth canonical numerator and denominator.

▶ **Lemma 22.** *Let $P, Q \in \mathbb{R}(x)^2$ have no zeros or poles in $\mathbb{N}$. The generalized ith canonical denominators $B^{(i)} \in \mathbb{R}(x)$ of (P, Q) satisfy the recurrence*

$$B^{(i+2)}(x) = P(x)B^{(i+1)}(x+1) + Q(x+1)B^{(i)}(x+2), \quad (B^{(0)}, B^{(1)}) = (0, 1). \tag{17}$$

The generalized ith canonical numerator ($i \geq 1$) is $A^{(i)}(x) = Q(x)B^{(i-1)}(x+1)$.

Proof. TOPROVE 15 ◀

Let us calculate the ultimate sign of $B^{(i)}$ and $\deg B^{(i)}$. Let $\deg 0 := -\infty$.

▶ **Lemma 23.** *Let $P, Q \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and P have the ultimate sign $(+)$. Let $i \geq 1$ be an integer and $B^{(i)}$ be the generalized ith canonical denominator. Let $L := \lim_{x\to\infty} \frac{Q(x)}{P(x)P(x-1)} \in [-\infty, \infty]$. Then $\deg B^{(i)}$ is:*

$$\begin{cases} (i-1)\deg P + \deg\left(\frac{Q(x)}{P(x)P(x-1)} + \frac{1}{4\cos^2\theta\pi}\right) & \text{if } L = -\frac{1}{4\cos^2\theta\pi} \in (-\infty, -\frac{1}{4}) \text{ and } i\theta \in \mathbb{Z}, \\ (i-1)\deg P + \lfloor\frac{i-1}{2}\rfloor \max\left\{0, \deg\frac{Q(x)}{P(x)P(x-1)}\right\} & \text{Otherwise.} \end{cases}$$

Let $q \in \{+, -, 0\}$ be $+$ if $\frac{Q(x)}{P(x)P(x-1)}$ is eventually increasing, $-$ if eventually decreasing, and 0 if constant. Then the ultimate sign of $B^{(i)}$ is:

$$\begin{cases} (+) & \text{if } L \geq -\frac{1}{4}, \\ (\operatorname{sgn}\sin i\theta) & \text{if } L = -\frac{1}{4\cos^2\theta\pi} \in [-\infty, -\frac{1}{4}) \text{ and } i\theta \notin \mathbb{Z}, \\ (\operatorname{sgn}(-1)^{i\theta}q) & \text{if } L = -\frac{1}{4\cos^2\theta\pi} \in [-\infty, -\frac{1}{4}) \text{ and } i\theta \in \mathbb{Z}. \end{cases}$$

Proof. TOPROVE 16 ◀

Proof. TOPROVE 17 ◀

Proof. TOPROVE 18 ◀

4 Proof of the Other Results

In this section, we prove Theorems 8 and 13.

4.1 Proof of Theorem 8

As we pointed out in Section 2.1.1, the first half and Parts (1), (2) and (3) of the second half of Theorem 8 follow from Theorem 4. We will prove Part (4) here.

Proof. TOPROVE 19 ◀

4.2 Proof of Theorem 13

By the assumption of the theorem, we have $\deg \frac{Q(x)}{P(x)P(x-1)} \leq -1$, so (P, Q) is of ∞-Ω loxodromic type or hyperbolic type. Then, by Theorem 4, (P, Q)-holonomic sequences $g \in \mathbb{R}^{\mathbb{N}}$ with unstable ultimate signs form a one-dimensional linear subspace in the linear space of all (P, Q)-holonomic sequences. Therefore, $g(n+1)$ and $g(n)$ must satisfy a linear relation as shown below. To keep the statement simple, let $R(x) := \frac{Q(x)}{P(x)P(x-1)}$ and consider the $(1, R)$-holonomic sequence $f(n) = \frac{g(n)}{P(n-1)\cdots P(-1)}$ with an unstable ultimate sign instead of g.

▶ **Lemma 24.** *Let $R \in \mathbb{R}(x)$ have no zeros or poles in $\mathbb{N}$ and satisfy $\deg R \leq -1$. Then, for all sufficiently large $n \in \mathbb{N}$, there exists $h(n) \in [1-R(n+1)-3R(n+1)^2, 1-R(n+1)+3R(n+1)^2]$ such that any $(1, R)$-holonomic sequence f whose ultimate sign is unstable satisfies the relation*

$$f(n+1) = -R(n)h(n)f(n). \tag{18}$$

The relation (18) corresponds to the equation (6) in [16]. Instead of using [16, Lemma 14], whose proof contains a gap, we use Theorem 4 and Lemma 22 to prove this lemma.

Proof. TOPROVE 20 ◀

We are now ready to prove Theorem 13.

Proof. TOPROVE 21 ◀

We changed the assumption of the theorem from the original $sq_1 - p_1 - s < 3p_0$ (if $d = 1$) and $sq_1 - p_1 < (d+2)p_0$ (if $d \geq 2$) to our stronger one to fill in the gap at the top of page 13 in [16] as shown in the last paragraph of the proof above. We did not make any other changes to the original proof in [16, § 3.3].

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