Restricted CSPs and F-free Digraph Algorithmics*

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Abstract

In recent years, much attention has been placed on the complexity of graph homomorphism problems when the input is restricted to $\mathbb{P}_k$-free and $\mathbb{P}_k$-subgraph-free graphs. We consider the directed version of this research line, by addressing the questions *is it true that digraph homomorphism problems* CSP($\mathbb{H}$) *have a* P *versus* NP*-complete dichotomy when the input is restricted to* $\vec{\mathbb{P}}_k$*-free (resp.* $\vec{\mathbb{P}}_k$*-subgraph-free) digraphs?* Our main contribution in this direction shows that if CSP($\mathbb{H}$) is NP-complete, then there is a positive integer N such that CSP($\mathbb{H}$) remains NP-hard even for $\vec{\mathbb{P}}_N$-subgraph-free digraphs. Moreover, it remains NP-hard for *acyclic* $\vec{\mathbb{P}}_N$-subgraph-free digraphs, and becomes polynomial-time solvable for $\vec{\mathbb{P}}_{N-1}$-subgraph-free *acyclic* digraphs. We then verify the questions above for digraphs on three vertices and a family of smooth tournaments. We prove these results by establishing a connection between $\mathbb{F}$-(subgraph)-free algorithmics and constraint satisfaction theory. On the way, we introduce *restricted CSPs*, i.e., problems of the form CSP($\mathbb{H}$) restricted to yes-instances of CSP($\mathbb{H}'$) — these were called restricted homomorphism problems by Hell and Nešetřil. Another main result of this paper presents a P versus NP-complete dichotomy for these problems. Moreover, this complexity dichotomy is accompanied by an algebraic dichotomy in the spirit of the finite domain CSP dichotomy.

As little as a few years ago, most graph theorists, while passively aware of a few classical results on graph homomorphisms, would not include homomorphisms among the topics of central interest in graph theory. We believe that this perception is changing, principally because of the usefulness of the homomorphism perspective... At the same time, the homomorphism framework strengthens the link between graph theory and other parts of mathematics, making graph theory more attractive, and understandable, to other mathematicians. Pavol Hell and Jaroslav Nesetril, *Graphs and Homomorphisms*, 2004 [31].

*Barnaby Martin has been supported by EPSRC grant EP/X03190X/1. Santiago Guzmán-Pro has been funded by the European Research Council (Project POCOCOP, ERC Synergy Grant 101071674). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

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1 Introduction

Story of the main questions

The Hell-Nešetřil theorem asserts that if $\mathbb{H}$ is a finite undirected graph, then CSP($\mathbb{H}$) is polynomial-time solvable whenever $\mathbb{H}$ is either a bipartite graph or contains a loop, and otherwise CSP($\mathbb{H}$) is NP-complete. In recent years, much attention has been placed on the complexity of graph homomorphism problems when the input is restricted to $\mathbb{F}$-free and $\mathbb{F}$-subgraph-free graphs, i.e., to graphs avoiding $\mathbb{F}$ as an induced subgraph, and as a subgraph, respectively. In [26], the authors show that CSP($\mathbb{C}_5$) is polynomial-time solvable when the input is restricted to $\mathbb{P}_8$-free graphs. Moreover, they show that there are finitely many $\mathbb{P}_8$-free obstructions to CSP($\mathbb{C}_5$). Also, in [15] the authors prove that CSP($\mathbb{K}_3$) becomes tractable if the input is restricted to $\mathbb{P}_7$-free graphs, while Huang [32] proved that CSP($\mathbb{K}_4$) remains NP-hard even for $\mathbb{P}_7$-free graphs. In general, several complexity classifications are known for $\mathbb{K}_n$-Colouring with input restriction to $\mathbb{P}_k$-free graphs (see, e.g., [27, Theorem 7]), however, a complete complexity classification of graph colouring problems with input restriction fo $\mathbb{F}$-(subgraph)-free graphs remains wide open.

The finite domain CSP dichotomy [38] (announced independently by Bulatov [18] and Zhuk [39]) asserts that digraph colouring problems also exhibit a P versus NP-complete dichotomy. In this paper we consider the directed version of the research line described above, where we consider the following to be the long-term question of this variant: Is there a P versus NP-complete dichotomy of CSP($\mathbb{H}$) where the input is restricted (1) to $\mathbb{F}$-free digraphs? and (2) to $\mathbb{F}$-subgraph-free digraphs?[1]

The Sparse Incomparability Lemma asserts that if CSP($\mathbb{H}$) is NP-hard, then it remains NP-hard even for high-girth digraphs. Hence, both questions above have a positive answer whenever $\mathbb{F}$ is not an oriented forest. Motivated by the literature on graph colouring problems restricted to $\mathbb{P}_k$-free and $\mathbb{P}_k$-subgraph-free digraphs, we consider the restriction of these questions to the case when F is a directed path $\vec{\mathbb{P}}_k$.

Question 1. *Is there a* P *versus* NP*-complete dichotomy of* CSP($\mathbb{H}$) *where the input is restricted*

1. *to $\vec{\mathbb{P}}_k$-free digraphs?*
2. *to $\vec{\mathbb{P}}_k$-subgraph-free digraphs?*

In our effort to settle Question 1 we stumble into three more questions which we also address in this paper. Allow us to elaborate. Clearly, if CSP($\mathbb{H}$) if NP-hard even for $\vec{\mathbb{P}}_k$-subgraph-free digraphs, then CSP($\mathbb{H}$) is NP-hard for $\vec{\mathbb{P}}_k$-free digraphs. In turn, if CSP($\mathbb{H}$) restricted to $\vec{\mathbb{P}}_k$-homomorphism-free digraphs is NP-hard, then CSP($\mathbb{H}$) restricted to $\vec{\mathbb{P}}_k$-subgraph-free digraphs. It is well-known that a digraph $\mathbb{D}$ is $\vec{\mathbb{P}}_k$-homomorphism-free if and only if $\mathbb{D}$ homomorphically maps to the transitive tournament in $k-1$ vertices $\mathbb{TT}_{k-1}$ (see, e.g, Observation 4). Hence, a simple way to find complexity upperbounds to the problems in Question 1 is to consider the complexity of CSP($\mathbb{H}$) restricted to CSP($\mathbb{TT}_k$).

The last problems have the following natural generalization: decide CSP($\mathbb{H}$) restricted to input digraphs $\mathbb{D}$ in CSP($\mathbb{H}'$). We denote this problem by RCSP($\mathbb{H}, \mathbb{H}'$). These problems have been called restricted homomorphism problems [16, 17] and it was conjectured by Hell and Nešetřil that

[1]Note that a negative answer to any of these questions implies that, if P $\neq$ NP, then there are finite digraphs $\mathbb{F}$ and $\mathbb{H}$ such that CSP($\mathbb{H}$) restricted to $\mathbb{F}$-(subgraph)-free digraphs is NP-intermediate — which we believe to be a more natural problem than the current NP-intermediate problems constructed in the literature. At this point we conjecture neither a positive nor a negative answer to the previous questions.

when $\mathbb{H}$ and $\mathbb{H}'$ are undirected graphs, then $\mathbb{H}' \to \mathbb{H}$, or $\mathbb{H}$ is bipartite, and in these cases CSP($\mathbb{H}$) restricted to CSP($\mathbb{H}'$) is in P; and otherwise it is NP-hard. This was later confirmed by Brewster and Graves [16], where they actually propose a hardness condition for a broader family of digraphs $\mathbb{H}'$ (see Theorem 3 below). It is then natural to ask our second main question.

Question 2. *Is there a* P *versus* NP*-hard dichotomy of problems* RCSP($\mathbb{H}, \mathbb{H}'$) *parametrized by finite digraphs (structures)* $\mathbb{H}$ *and* $\mathbb{H}'$*?*

Clearly, every digraph $\mathbb{D}$ that admits a homomorphism to some transitive tournament must be an acyclic digraph. A natural question of a digraph $\mathbb{H}$, for which CSP($\mathbb{H}$) is NP-complete, is to ask if this problem remains NP-complete on acyclic instances. Notice that this is the same problem as RCSP($\mathbb{H}, (\mathbb{Q}, <)$). If $\mathbb{H}$ is an undirected graph, then of course this will be true (choose any total order of the vertex set and orient the edges according to this ordering). However, when $\mathbb{H}$ is not undirected, the situation is not a priori clear. Indeed, it is addressed for some small digraphs by Hell and Mishra in [30]. They prove, for example, that CSP($\vec{\mathbb{C}}_3^+$) – where $\vec{\mathbb{C}}_3^+$ is drawn in the forthcoming Figure 4 – does indeed remain NP-complete on acyclic inputs. The general question is not posed in [30], but it is perfectly natural, and we address it here.

Question 3. *If* CSP($\mathbb{H}$) *is* NP*-hard for a finite graph* $\mathbb{H}$*, does* CSP($\mathbb{H}$) *remain* NP*-hard for acyclic instances? Equivalently, is* RCSP($\mathbb{H}, (\mathbb{Q}, <)$) NP*-hard whenever* CSP($\mathbb{H}$) *is* NP*-hard?*

Some readers might have already noticed that there is a third natural question motivated by the previous paragraph: is there a P versus NP-complete dichotomy of CSP($\mathbb{H}$) restricted to $\vec{\mathbb{P}}_k$-homomorphism-free digraphs? Or more generally: is there a P versus NP-hard dichotomy of CSP($\mathbb{H}$) where the input is restricted to $\mathcal{F}$-homomorphism-free digraphs? (where $\mathcal{F}$ is a fixed finite set of digraphs). This question can already be settled from results in the literature: every such problem can be coded into *monotone monadic strict NP* (MMSNP), and Feder and Vardi [23] proved that if finite domain CSPs have a P versus NP-complete dichotomy, then MMSNP exhibits the same dichotomy. However, we provide an alternative prove of this fact in Section 5.

Story of the paper

Our paper splits into two parts. The first part focuses on relational structures, of which digraphs are somewhat canonical examples, while the second part focuses on digraphs specifically. In Figure 1 we depict the flow of ideas and results of this paper.

Within the first part (Sections 3– 5), we begin by introducing *restricted* CSPs (RCSPs) which have been studied as restricted homomorphism problems in [16, 17]. In particular, we elaborate a connection with promise CSPs (PCSPs) that enables one to view RCSPs as PCSPs.

Our first main result shows that, for every pair of finite digraphs (structures) $\mathbb{A}$ and $\mathbb{B}$, the problem CSP($\mathbb{B}$) with input restricted to CSP($\mathbb{A}$) is either in P or NP-hard (settling Question 2). Moreover, this complexity classification is accompanied with an algebraic dichotomy (Theorem 21) which stems from the finite domain CSP dichotomy result. We then push the previous complexity dichotomy to finite domain CSPs with restrictions in GMSNP (Theorem 23). Another contribution of this work builds again on the finite domain CSP dichotomy to present a new proof of the complexity dichotomy for finite domain CSP restricted to $\mathcal{F}$-homomorphism-free digraphs (structures): we avoid going via MMSNP to the infinite domain CSP setting, and stay in the finite domain world via Lemma 27 (Section 5).

In the second part of the paper (Sections 6–9), we leverage results from the first part, in order to study digraph CSPs, where the ultimate focus will be on $\mathbb{F}$-free and $\mathbb{F}$-subgraph-free algorithmics.

Our second main result shows that, if $\mathbb{H}$ is a digraph and CSP($\mathbb{H}$) is NP-complete, then there is a positive integer N such that CSP($\mathbb{H}$) remains NP-complete even for $\vec{\mathbb{P}}_N$-subgraph-free acyclic inputs (settling Question 3). Moreover, N can be chosen so that CSP($\mathbb{H}$) is polynomial-time solvable for $\vec{\mathbb{P}}_{N-1}$-subgraph-free acyclic inputs. This also yields a partial answer to Question 1: for every digraph $\mathbb{H}$ there is a positive integer $N \leq 4^{|H|}$ such that Question 1 has a positive answer restricted to $k \geq N$. We complement this general partial answer by settling Question 1 for digraphs $\mathbb{H}$ on three vertices (Theorems 42 and 46), and for a family of smooth tournaments $\mathbb{TC}_n$ (Theorems 49 and Theorem 53). We note that eventual hardness on $\vec{\mathbb{P}}_N$-subgraph-free instances does not hold in general for CSP($\mathbb{H}$), if $\mathbb{H}$ is an infinite digraph. We provide a counterexample (Example 36) which is otherwise well-behaved (for example, by being ω-categorical). As byproducts of our work we see that there are finitely many minimal $\vec{\mathbb{P}}_3$-obstructions to CSP($\mathbb{TC}_n$) for each positive integer n (Theorem 55), and if $\mathbb{F}$ is not an oriented path, then CSP($\mathbb{TC}_n$) (and CSP($\vec{\mathbb{C}}_3^+$)) is NP-hard even for $\mathbb{F}$-subgraph-free instances (Theorem 57).

2 Preliminaries

2.1 Relational structures and digraphs

For a finite relational signature $\tau = (R_1, \ldots, R_k)$, a *relational ($\tau$-)structure* $\mathbb{A}$ on domain A consists of k relations $R_1 \subseteq A^{a_1}, \ldots, R_k \subseteq A^{a_k}$, where a_i is the arity of R_i. We denote the cardinality of A as $|A|$. We tend to conflate the relation symbol and actual relation since this will not introduce confusion.

For some signature τ, the *loop* $\mathbb{L}$ is the structure on one element a all of whose relations are maximally full, that is, contains one tuple $(a, \ldots, a)$ — the structure $\mathbb{L}$ clearly depends on the signature τ, but in this work τ will always be clear from context.

Directed graphs (digraphs) $\mathbb{D}$ are relational structures on the signature $\{E\}$ where E is a binary relation. A digraph is a *graph* if E is symmetric, i.e. $(x, y) \in E$ iff $(y, x) \in E$. We will use the same blackboard font notation for digraphs as we do for relational structures.

Given a positive integer k we denote by $\vec{\mathbb{C}}_k$ the directed cycle on k vertices, by $\vec{\mathbb{P}}_k$ the directed path on k vertices, by $\mathbb{T}_k$ the transitive tournament on k vertices. Similarly we denote by K_k the complete graph on k vertices, and we think of it as a digraph with edges (i, j) for every $i \neq j$.

A *(directed) walk* on a (directed) graph $\mathbb{D}$ is a sequence of vertices $x_1, \ldots, x_k$ such that for every $i \in [k-1]$ there is a (directed) edge $(x_i, x_{i+1}) \in E$. An *oriented graph* is a digraph with no pair of symmetric edges, i.e,. a loopless $\mathbb{K}_2$-free digraph. A *tree* is an oriented digraph whose underlying graph has no cycle (equivalently, is an orientation of a traditional undirected tree). It follows that trees are $\mathbb{K}_2$-free. A *forest* is a disjoint union of trees.

2.2 Constraint satisfaction problems

Given a pair of digraphs (structures) $\mathbb{D}$ and $\mathbb{H}$ a *homomorphism* $f \colon \mathbb{D} \to \mathbb{H}$ is a vertex mapping such that, for every (u, v) that is an edge of $\mathbb{D}$, the image $(f(u), f(v))$ is an edge of $\mathbb{H}$. If such a homomorphism exists, we write $\mathbb{D} \to \mathbb{H}$, and otherwise $\mathbb{D} \not\to \mathbb{H}$. We follow notation of constraint satisfaction theory, and denote by CSP($\mathbb{H}$) the class of finite digraphs such that $\mathbb{D} \to \mathbb{H}$. We also denote by CSP($\mathbb{H}$) the computational problem of deciding if an input digraph $\mathbb{D}$ belongs to CSP($\mathbb{H}$).

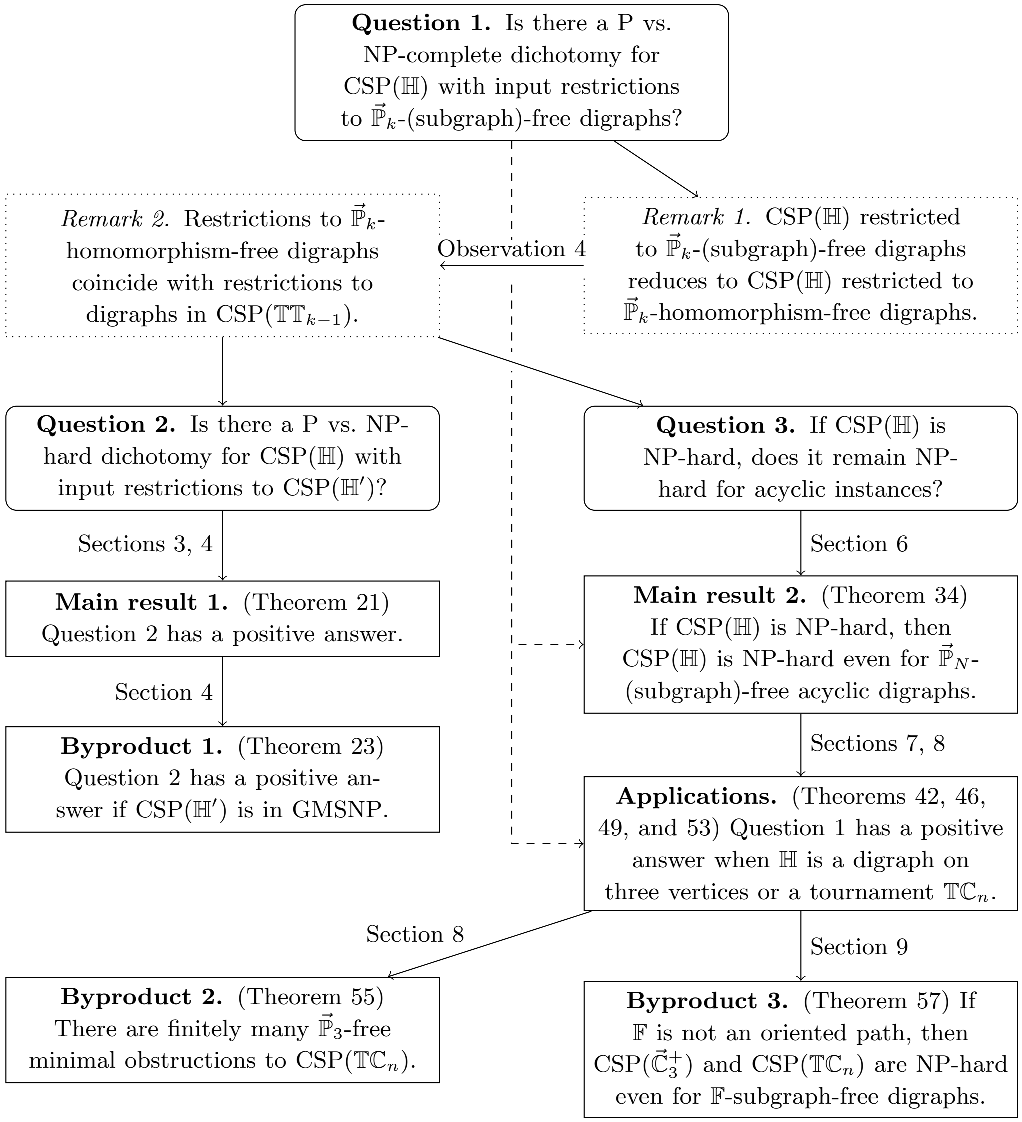


Figure 1: A depiction of the flow between main results and main questions addressed in this paper. Rectangles with rounded corners mark the main questions considered here. Dotted squares indicate the (simple) remarks connecting the $\mathcal{F}$-subgraph-free CSPs to the theory of digraph homomorphisms. Solid rectangles with straight corners indicate the main results of the present paper. A solid (resp. dashed) edge between a question and a result indicates that the result provides an answer (resp. a partial answer) to the corresponding question. Finally, edges between results represent that the result at the head is proved using tools introduced while proving the result at the tail of the corresponding edge.

An *endomorphism* is a homomorphism $f\colon \mathbb{H} \to \mathbb{H}$. A digraph (structure) $\mathbb{H}$ is a *core* if all of its endomorphisms are self-embeddings. If $\mathbb{H}$ is finite then this is equivalent to all endomorphisms being automorphisms.

2.3 Smooth digraphs

A digraph $\mathbb{D}$ is *smooth* if it has no sources nor sinks. The following statement was conjectured in [1] and proved in [7].

Theorem 1 (Conjecture 6.1 in [1] proved in [7])**.** *For every smooth digraph $\mathbb{H}$ one of the following holds.*

- *The core of $\mathbb{H}$ is a disjoint union of cycles, and in this case* $\mathrm{CSP}(\mathbb{H})$ *is polynomial-time solvable.*
- *Otherwise,* $\mathrm{CSP}(\mathbb{H})$ *is* NP*-complete.*

A digraph $\mathbb{H}$ is *hereditarily hard* if $\mathrm{CSP}(\mathbb{H}')$ is NP-complete for every loopless digraph $\mathbb{H}'$ such that $\mathbb{H} \to \mathbb{H}'$. Bang-Jensen, Hell, and Niven conjectured that a smooth digraph $\mathbb{H}$ is hereditarily hard whenever $\mathbb{H}$ does not homomorphically map to a disjoint union of directed cycles. Moreover, they showed that this conjecture is implied by the statement in Theorem 1 (which was only a conjecture at that time).

Theorem 2 (Conjecture 2.5 in [3] proved in [7])**.** *A digraph $\mathbb{H}$ is hereditarily hard whenever the digraph $R(\mathbb{H})$ obtained from $\mathbb{H}$ by iteratively removing sources and sinks does not admit a homomorphism to a disjoint unions of directed cycles.*

As far as we are aware, the most general result regarding hardness of restricted CSP problems is Theorem 3 in [16].

Theorem 3 (Theorem 3 in [16])**.** *If $\mathbb{H}$ is a hereditarily hard digraph and $\mathbb{H}'$ is a finite digraph such that $\mathbb{H}' \not\to \mathbb{H}$ then* $\mathrm{RCSP}(\mathbb{H}, \mathbb{H}')$ *is* NP*-hard.*

2.4 Duality pairs

A pair of digraphs (relational structure) $(\mathbb{T}, \mathbb{D}_{\mathbb{T}})$ are called a *duality pair* if for every digraph it is the case that $\mathbb{D} \to \mathbb{D}_{\mathbb{T}}$ if and only if $\mathbb{T} \not\to \mathbb{D}$. It was proved in [35] that for every tree $\mathbb{T}$ there is a digraph $\mathbb{D}_{\mathbb{T}}$ such that $(\mathbb{T}, \mathbb{D}_{\mathbb{T}})$ is a duality pair. Moreover, they also proved that if $(\mathbb{T}, \mathbb{D}_{\mathbb{T}})$ is a duality pair, then $\mathbb{T}$ is homomorphically equivalent to a tree. A well-known example of a family of duality pairs is the following one.

Observation 4. *For every positive integer k the directed path $\vec{\mathbb{P}}_{k+1}$ together with the transitive tournament $\mathbb{T}_k$ are duality pair.*

Given a set of digraphs $\mathcal{F}$, we denote by $\mathrm{Forb}(\mathcal{F})$ the set of digraphs $\mathbb{D}$ such that $\mathbb{F} \not\to \mathbb{D}$ for every $\mathbb{F} \in \mathcal{F}$. It is straightforward to observe that for any such set $\mathcal{F}$, there is a (possibly infinite) digraph $\mathbb{D}$ such that $\mathrm{Forb}(\mathcal{F}) = \mathrm{CSP}(\mathbb{D})$. A *generalized duality* is a pair $(\mathcal{F}, \mathcal{D})$ of finite sets of digraphs such that

$$\mathrm{Forb}(\mathcal{F}) = \bigcup_{\mathbb{D} \in \mathcal{D}} \mathrm{CSP}(\mathbb{D}).$$

In particular, when $\mathcal{D} = \{\mathbb{D}\}$ we simply write $(\mathcal{F}, \mathbb{D})$. Generalized dualities have a similar characterization to duality pairs.

Theorem 5 (Theorems 2 and 11 in [25])**.** *For every finite set of forests $\mathcal{F}$ there is a finite set of digraphs $\mathcal{D}$ such that $(\mathcal{D}, \mathcal{F})$ is a generalized duality pair. Moreover, if $\mathcal{F}$ is a finite set of trees, then there is a digraph $\mathbb{D}$ such that $(\mathcal{F}, \mathbb{D})$ is a generalized duality.*

2.5 Large girth

A well-known result from Erdős [22] about k-colourabilty states that for every pair of positive integers l, k there is a graph G with girth strictly larger than l and such that G does not admit a proper k-colouring. This result generalizes to arbitrary relational structures, and it is known as the Sparse Incomprability Lemma [34] — in order to stay within the scope of this paper, we state it for digraphs.

Given a digraph (structure) $\mathbb{D}$, the *incidence graph* of $\mathbb{D}$ is the undirected bipartite graph $\mathbb{I}(\mathbb{D})$ with vertex set $V \cup E$, for $v \in D$ a vertex of D and $e = (x, y) \in E$ an edge of D. There is an (undirected) edge (v, e) in $\mathbb{I}(\mathbb{D})$ if and only if $v \in \{x, y\}$. The *girth* of $\mathbb{D}$ is half the length of the shortest cycle in $\mathbb{I}(\mathbb{D})$. Notice that if $\mathbb{D}$ has a pair of symmetric arcs, its girth is 2, and otherwise, it is the graph theoretic girth of the underlying graph of $\mathbb{D}$.

Theorem 6 (Sparse Incomparability Lemma [34])**.** *For every digraph $\mathbb{D}$ and every pair of positive integers k and ℓ there is a digraph $\mathbb{D}'$ with the following properties:*

- $\mathbb{D}' \to \mathbb{D}$,
- *the girth of $\mathbb{D}'$ is larger than ℓ,*
- $\mathbb{D} \to \mathbb{H}$ *if and only if* $\mathbb{D}' \to \mathbb{H}$ *for every digraph* $\mathbb{H}$ *on at most k vertices,*
- $\mathbb{D}'$ *can be constructed in polynomial time (from $\mathbb{D}$).*

Corollary 7. *For ever finite digraph $\mathbb{H}$ and every positive integer ℓ,* CSP($\mathbb{H}$) *is polynomial-time equivalent to* CSP($\mathbb{H}$) *restricted to input digraphs of girth strictly larger that ℓ.*

3 Restricted constraint satisfaction problems

Promise problems (not to be confused with Promise CSPs) can be thought as decision problems with input restrictions. Formally [37], a *promise problem* is a pair $(\mathcal{P}, \mathcal{C})$ of decidable sets. A solution to $(\mathcal{P}, \mathcal{C})$ is a decidable set $\mathcal{S}$ such that $\mathcal{S} \cap \mathcal{P} = \mathcal{C} \cap \mathcal{P}$. We say that the promise problem $(\mathcal{P}, \mathcal{C})$ is polynomial-time solvable if it has a solution in P, and if every solution is NP-hard, we say that $(\mathcal{P}, \mathcal{C})$ is NP-hard.

Given a pair of (possibly infinite) structures $\mathbb{A}$ and $\mathbb{B}$ (with the same finite signature) whose CSPs are decidable, the *restricted CSP* RCSP($\mathbb{A}, \mathbb{B}$) is the promise problem (CSP($\mathbb{B}$), CSP($\mathbb{A}$)). In this case, we call $(\mathbb{A}, \mathbb{B})$ the *template* of the restricted CSP, $\mathbb{A}$ is called the *domain* and $\mathbb{B}$ the *restriction.* In particular, if $\mathbb{A}$ is finite we say that RCSP($\mathbb{A}, \mathbb{B}$) is a finite domain RCSP, and if $\mathbb{B}$ is finite, we say that RCSP($\mathbb{A}, \mathbb{B}$) is an RCSP with finite restriction.

Informally, the promise problem RCSP($\mathbb{A}, \mathbb{B}$) is CSP($\mathbb{A}$) where the input is promised to belong to CSP($\mathbb{B}$). For instance, RCSP($\mathbb{K}_3, \mathbb{K}_4$) is essentially the problem of deciding whether an input 4-colourable graph is 3-colourable.

Notice that for any digraph (structure) $\mathbb{A}$ the problems CSP($\mathbb{A}$) and RCSP($\mathbb{A}, \mathbb{L}$) are the same problems where $\mathbb{L}$ is the loop. So every decidable CSP is captured by an RCSP with finite restriction. One of the main results of this work shows that every RCSP with finite restriction is log-space equivalent to a CSP. It follows from the proof that every finite domain RCSP with finite restriction is log-space equivalent to a finite domain CSP, and thus, finite domain restricted CSPs with finite restrictions have a P versus NP-hard dichotomy. Moreover, the reduction mentioned in this paragraph are obtained by *restricted primitive positive construction (rpp-constructions)*, which are the natural cousins of pp-constructions for CSPs [9] and for PCSPs [5].

3.1 Restricted primitive positive constructions

Several reductions in graph algorithmics arise from gadget reductions. For instance, a standard way of proving that CSP($\mathbb{C}_5$) is NP-complete can be done with the following gadget reduction from CSP($\mathbb{K}_5$). On input $\mathbb{G}$ to CSP($\mathbb{K}_5$), consider the graph obtained from $\mathbb{G}$ by replacing every edge $e := xy$ by a path x, u_e, v_e, y (see, e.g., [31] where this is called an indicator construction). So in this case, the path on four vertices is the gadget associated to this reduction). The algebraic approach to CSPs proposes a general framework encompassing gadget reductions between constraint satisfaction problems. In this section we introduce primitive positive constructions, and *restricted* primitive positive constructions.

Primitive positive constructions

Given a pair of finite relational signature τ and σ, a *primitive positive definition* (of τ in σ)[2] of dimension $d \in \mathbb{Z}^+$ is a finite set Δ of primitive positive formulas $\delta_R(\bar{x})$ indexed by σ, and for each $R \in \sigma$ of arity r the formula $\delta_R(\bar{x})$ has $r \cdot d$ free variables.

For every primitive positive definition of σ in τ we associate a mapping Π_Δ from τ-structures to σ-structures as follows. Given a τ-structure $\mathbb{A}$ the *pp power* $\Pi_\Delta(\mathbb{A})$ of $\mathbb{A}$ is the structure

- with domain of $\Pi_\Delta(\mathbb{A})$ is A^d, and
- for each $R \in \sigma$ of arity r the interpretation of R in $\Pi_\Delta(\mathbb{A})$ consists of the tuples $(\bar{a}_1, \dots, \bar{a}_r) \in (A^d)^r$ such that $\mathbb{A} \models \delta_R(\bar{a}_1, \dots, \bar{a}_r)$.

We say that a structure $\mathbb{A}$ *pp-constructs* a structure $\mathbb{B}$ is there is a primitive positive definition Δ such that $\Pi_\Delta(\mathbb{A}) \to \mathbb{B} \to \Pi_\Delta(\mathbb{A})$, i.e., $\Pi_\Delta(\mathbb{A})$ is homomorphically equivalent to $\mathbb{B}$. For instance, if Δ is the 1-dimensional primitive positive definition consisting of

$$\delta_E(x, y) := \exists z_1, z_2.\ E(x, z_1) \wedge E(z_1, z_2) \wedge E(z_2, y),$$

then the pp power $\Pi_\Delta(\mathbb{C}_5)$ of the 5-cycle is the complete graph $\mathbb{K}_5$ (see Figure 2).

Remark 8. It is well-known that for every fixed primitive positive definition Δ, the pp-power Π_Δ is a monotone construction with respect to the homomorphism order, i.e., if $\mathbb{A} \to \mathbb{B}$, then $\Pi_\Delta(\mathbb{A}) \to \Pi_\Delta(\mathbb{B})$ (this follows from the fact that existential positive formulas are preserved by homomorphism [11, Theorem 2.5.2]).

Lemma 9. *For every primitive positive definitions Δ_1 of σ in τ, and Δ_3 of τ in ρ, there is a primitive positive definition of Δ_3 of σ in ρ such that for every ρ-structure $\mathbb{A}$*

$$\Pi_{\Delta_3}(\mathbb{A}) = \Pi_{\Delta_1}(\Pi_{\Delta_2}(\mathbb{A})).$$

[2] When the signatures are clear from context we will simply say talk about a primitive positive definition.

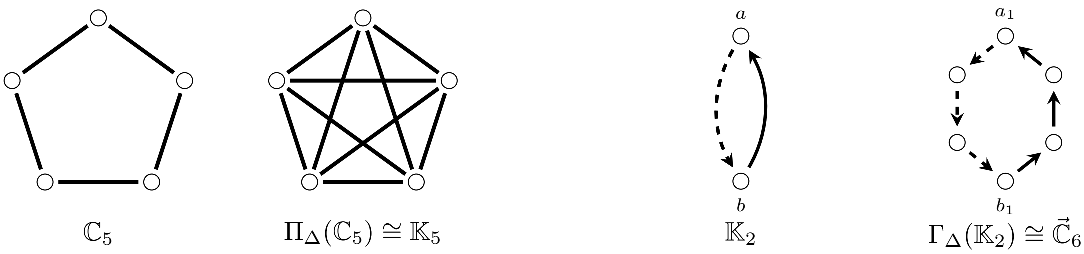


Figure 2: Let Δ be the primitive positive definition (of $\{E\}$ in $\{E\}$) where $\delta_E(x, y) := \exists z_1, z_2.\ E(x, z_1) \wedge E(z_1, z_2) \wedge E(z_2, y)$. On the left, we depict $\mathbb{C}_5$ and its pp power $\Pi_\Delta(\mathbb{C}_5) \cong \mathbb{K}_5$ (an undirected edge xy represents (x, y) and (y, x)), and on the right, we depict $\mathbb{K}_2$ and its gadget replacement $\Gamma_\Delta(\mathbb{K}_2) \cong \vec{\mathbb{C}}_6$ (dashed edges and solid edges indicate the respective edge replacements).

Proof. TOPROVE 0 $\square$

Primitive positive constructions yield a canonical class of log-space reductions between constraint satisfaction problems.

Lemma 10 (Corollary 3.5 in [9])**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be (possibly infinite) structures with finite relational signature. If $\mathbb{A}$ pp-constructs $\mathbb{B}$, then* $\mathrm{CSP}(\mathbb{B})$ *reduces in logarithmic space to* $\mathrm{CSP}(\mathbb{A})$.

Gadget replacements

For every primitive positive formula $\delta(\bar{x})$ without equalities[3] we construct a structure $\mathbb{D}_\delta(\bar{d})$ with a distinguished tuples of vertices d as follows.

- The domain D of $\mathbb{D}$ consists of a vertex v_y for every (free or bounded) variable y of δ.
- The distinguished vertices $d_1, \ldots, d_m$ are the vertices $v_{x_1}, \ldots, v_{x_m}$.
- For every relation symbol R there is a tuple $\bar{v} \in R^{\mathbb{D}}$ if and only if δ contains the conjunct $R(\bar{y})$ where $\bar{v}$ is the tuple of vertices corresponding to the tuple of variables $\bar{y}$.

The structure $\mathbb{D}(\bar{d})$ is sometimes called the *canonical database* of δ. For instance, if δ_E is the formula considered above, the the canonical database of δ_E is the directed with vertices $1, 2, 3, 4$, and the distinguished vertices are 1 and 4.

The canonical database $\mathbb{D}_\delta(\bar{d})$ and the primitive positive formula δ are closely related: for every structure $\mathbb{A}$ and a tuple $\bar{a}$ of elements of A

$$\mathbb{A} \models \delta(\bar{a}) \text{ if and only if there is a homomorphism } f \colon \mathbb{D} \to \mathbb{A} \text{ such that } f(\bar{d}) = f(\bar{a})$$

(see, e.g., [11, Proposition 1.2.5]).

Building on canonical databases, for every d-dimensional primitive positive definition Δ of σ in τ we associate a mapping Γ_Δ from σ-structures to τ-structures as follows. Given a σ-structure $\mathbb{B}$ the *gadget replacement* $\Gamma_\phi(\mathbb{B})$ is the τ-structure obtained from $\mathbb{B}$ where

[3] If a primitive positive formula contains a conjunct $x = y$, we obtain an equivalent formula δ' by deleting the conjunct $x = y$ and replacing each occurrence of the variable y by the variable x.

- for each vertex $b \in B$ we introduce d vertices $b_1, \ldots, b_d$, and
- for each $R \in \sigma$ of arity r, and every tuple $(b^1, \ldots, b^r) \in R^{\mathbb{B}}$ we introduce a fresh copy of $\mathbb{D}_{\delta_R}(\bar{d}^1, \ldots, \bar{d}^r)$ and we identify each d-tuple $\bar{d}^i$ with $(b^i_1, \ldots, b^i_d)$.

Going back to our on going example $\Delta := \{\delta_E\}$ and taking $\mathbb{B} := \mathbb{K}_2$, the gadget replacement $\Gamma_\Delta(\mathbb{B})$ is isomorphic to the directed 6-cycle $\vec{\mathbb{C}}_6$ (see Figure 2 for a depiction).

It is straightforward to observe that, in general, for every primitive positive definition Δ, the gadget replacement $\Gamma_\Delta(\mathbb{B})$ can be constructed in logarithmic space from $\mathbb{B}$. This fact and the following observation can be used to prove Lemma 10.

Observation 11 (Observation 4.4 in [33])**.** *The following statement holds for every primitive positive formula ϕ, and every pair of digraphs (structures) $\mathbb{A}$ and $\mathbb{A}'$*

$$\mathbb{A}' \to \Pi_\Delta(\mathbb{A}) \textit{ if and only if } \Gamma_\Delta(\mathbb{A}') \to \mathbb{A}.$$

Log-space reductions and rpp-constructions

We say that a (not necessarily finite) restricted CSP template $(\mathbb{A}, \mathbb{B})$ *rpp-constructs* a restricted CSP template $(\mathbb{A}', \mathbb{B}')$ if there is a primitive positive definition Δ such that

$$(\Pi_\Delta(\mathbb{A}) \times \mathbb{B}') \leftrightarrow (\mathbb{A}' \times \mathbb{B}') \text{ and } \mathbb{B}' \to \Pi_\Delta(\mathbb{B}).$$

Remark 12. Every primitive positive formula is satisfiable in some finite structure, e.g., in the loop $\mathbb{L}$. This implies that for every primitive positive definition of σ in τ the pp-power $\Pi_\Delta(\mathbb{L}_\tau)$ of the (τ-)loop is homomorphically equivalent to the (σ-)loop $\mathbb{L}_\sigma$. In particular, this implies that the following statement are equivalent for every τ-structure $\mathbb{A}$ and every σ-structure $\mathbb{A}'$.

- $\mathbb{A}$ pp-constructs $\mathbb{A}'$.
- $(\mathbb{A}, \mathbb{L}_\tau)$ rpp-constructs $(\mathbb{A}', \mathbb{L}_\sigma)$.
- $(\mathbb{A}, \mathbb{L}_\tau)$ rpp-constructs $(\mathbb{A}', \mathbb{B}')$ for ever structure $\mathbb{B}'$.

It is well-known that pp-constructions compose, and building on this fact, it is straightforward to observe that rpp-constructions compose.

Lemma 13. *Consider three restricted CSP templates $(\mathbb{A}_1, \mathbb{B}_1)$, $(\mathbb{A}_2, \mathbb{B}_2)$, and $(\mathbb{A}_3, \mathbb{B}_3)$. If $(\mathbb{A}_1, \mathbb{B}_1)$ rpp-constructs $(\mathbb{A}_2, \mathbb{A}_2)$, and $(\mathbb{A}_2, \mathbb{B}_2)$ rpp-constructs $(\mathbb{A}_3, \mathbb{B}_3)$, then $(\mathbb{A}_1, \mathbb{B}_1)$ rpp-constructs $(\mathbb{A}_3, \mathbb{B}_3)$.*

Proof. TOPROVE 1 □

Lemma 14. *Consider two (not necessarily finite) restricted CSP templates $(\mathbb{A}, \mathbb{B})$ and $(\mathbb{A}', \mathbb{B}')$. If $(\mathbb{A}, \mathbb{B})$ rpp-constructs $(\mathbb{A}', \mathbb{B}')$, then there is a log-space reduction from $\mathrm{RCSP}(\mathbb{A}', \mathbb{B}')$ to $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$.*

Proof. TOPROVE 2 □

Example 15. Observe that the restricted CSP template $(\mathbb{C}_5, \mathbb{K}_3)$ rpp-constructs the template $(\mathbb{K}_5, \mathbb{L})$ via the formula

$$\delta_E(x, y) := \exists z_1, z_2.\ E(x, z_1) \land E(z_1, z_2) \land E(z_2, y).$$

Indeed, we already noticed that the pp power $\Pi_\Delta(\mathbb{C}_5)$ is isomorphic to $\mathbb{K}_5$. It is also not hard to observe that the pp power $\Pi_\Delta(\mathbb{K}_5)$ is homomorphically equivalent to the loop $\mathbb{L}$. Hence, by Lemma 14 we conclude that $\mathrm{RCSP}(\mathbb{C}_5, \mathbb{K}_3)$ is NP-hard, i.e., deciding if a 3-colourable graph $\mathbb{G}$ admits a homomorphism to $\mathbb{C}_5$ is NP-hard.

4 Finite domain restrictions

In this section we show that for every restricted CSP template $(\mathbb{A}, \mathbb{B})$ with finite restriction, there is a structure $\mathbb{C}$ such that $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ and $\mathrm{CSP}(\mathbb{C})$ are log-space equivalent. Moreover, if $\mathbb{A}$ is also a finite structure, then $\mathbb{C}$ is a finite structure. It thus follows from the finite domain dichotomy [18, 39] that restricted CSPs with finite domain and finite restriction, have a P versus NP-complete dichotomy.

4.1 Exponential structures

Given a pair of τ-structure $\mathbb{A}$ and $\mathbb{B}$ the *exponential* $\mathbb{A}^{\mathbb{B}}$ is the τ-structure

- with vertex set all functions $f\colon B \to A$, and
- for each $R \in \tau$ of arity r there is an r-tuple $(f_1, \dots, f_r)$ belongs to the interpretation of R in $\mathbb{A}^{\mathbb{B}}$ if and only if $(f_1(b_1), \dots, f_r(b_r)) \in R^{\mathbb{A}}$ whenever $(b_1, \dots, b_r) \in R^{\mathbb{B}}$.

In Figure 3 we depict an exponential digraph construction. This image is also found in [31], were the reader can also find further discussion and properties of exponential digraphs. For this paper we state the following properties of general exponential structures.

Lemma 16. *The following statements hold for all τ-structures $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$.*

- $\mathbb{A}^{\mathbb{L}}$ *is isomorphic to* $\mathbb{A}$.
- $\mathbb{C} \to \mathbb{A}^{\mathbb{B}}$ *if and only if* $\mathbb{C} \times \mathbb{B} \to \mathbb{A}$.

Proof. TOPROVE 3 □

Corollary 17. *For every pair of (possibly infinite) τ-structures $\mathbb{A}$ and $\mathbb{B}$, the restricted CSP template $(\mathbb{A}^{\mathbb{B}}, \mathbb{L})$ rpp-constructs the template* $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$.

Proof. TOPROVE 4 □

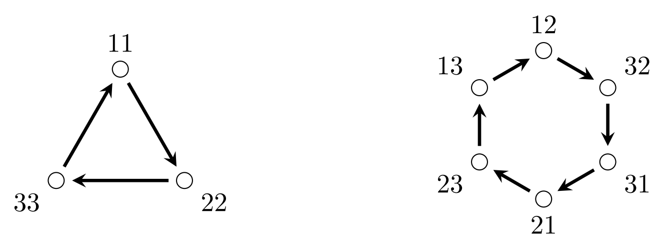


Figure 3: The exponential $\vec{\mathbb{C}}_3^{\mathbb{K}_2}$ where a label ij represents the function $f\colon \{1,2\} \to \{1,2,3\}$ defined by $1 \mapsto i$ and $2 \mapsto j$.

Lemma 18. *Let τ be a finite relational signature, and $\mathbb{A}$ be a (possible infinite) τ-structure. If $\mathbb{B}$ is a finite τ-structure, then the restricted* RCSP *template* $(\mathbb{A}, \mathbb{B})$ *rpp-constructs the* RCSP *template* $(\mathbb{A}^{\mathbb{B}}, \mathbb{L})$.

Proof. TOPROVE 5 □

We discuss two applications of this lemma. The first one being that rpp-constructions between RCSP templates with finite restrictions are captured by (standard) pp-constructions.

Theorem 19. *The following statements are equivalent for every pair of restricted* CSP *templates* $(\mathbb{A}_1, \mathbb{B}_1)$ *and* $(\mathbb{A}_2, \mathbb{B}_2)$ *with finite restrictions.*

- $(\mathbb{A}_1, \mathbb{B}_1)$ *rpp-constructs* $(\mathbb{A}_2, \mathbb{B}_2)$ *and,*
- $\mathbb{A}_1^{\mathbb{B}_1}$ *pp-constructs* $\mathbb{A}_2^{\mathbb{B}_2}$*.*

In particular, $(\mathbb{A}_1, \mathbb{B}_1)$ *and* $(\mathbb{A}_1^{\mathbb{B}_1}, \mathbb{L})$ *are mutually rpp-constructible (and so* $\mathrm{RCSP}(\mathbb{A}_1, \mathbb{B}_1)$ *and* $\mathrm{CSP}(\mathbb{A}_1^{\mathbb{B}_2})$ *are log-space Turing-equivalent).*

Proof. TOPROVE 6 □

The second application of Lemma 18 is the following statement which is analogous to the case of CSPs and pp-constructions (see, e.g., [11, Theorem 3.2.2]).

Theorem 20. *The following statements are equivalent for every (possibly infinite) structure* $\mathbb{A}$ *and a finite structure* $\mathbb{B}$ *with a finite signature.*

- *The restricted* CSP *template* $(\mathbb{A}, \mathbb{B})$ *rpp-constructs* $(\mathbb{K}_3, \mathbb{L})$*.*
- *The restricted* CSP *template* $(\mathbb{A}, \mathbb{B})$ *rpp-constructs every restricted* CSP *template* $(\mathbb{A}', \mathbb{B}')$ *where* $\mathbb{A}'$ *is a finite structure (and* $\mathbb{B}'$ *a possibly infinite structure).*
- *The structure* $\mathbb{A}^{\mathbb{B}}$ *pp-constructs* $\mathbb{K}_3$*.*

If any of these equivalent statement hold, then $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ *is* NP*-hard.*

Proof. TOPROVE 7 □

4.2 A dichotomy for finite domain RCSPs with finite restrictions

The dichotomy for finite domain CSPs asserts that if $\mathbb{A}$ is a finite structure, then either $\mathbb{A}$ pp-constructs $\mathbb{K}_3$, and in this case $\mathrm{CSP}(\mathbb{A})$ is NP-complete; or otherwise, $\mathrm{CSP}(\mathbb{A})$ is polynomial-time solvable. We apply the results of this section to obtain an analogous (actually, equivalent) statement for finite domain CSPs with finite restrictions.

Theorem 21. *For every pair of finite structures* $\mathbb{A}, \mathbb{B}$ *one of the following statement hold.*

- *Either* $(\mathbb{A}, \mathbb{B})$ *rpp-constructs* $(\mathbb{K}_3, \mathbb{L})$ *(and consequently,* $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ *is* NP*-hard), or*
- $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ *is polynomial-time solvable.*

Proof. TOPROVE 8 □

4.3 Finite-domain restrictions up to high girth

Given a structure $\mathbb{B}$ and a positive integer ℓ we denote by $\mathrm{CSP}_{>\ell}(\mathbb{B})$ the subclass of $\mathrm{CSP}(\mathbb{B})$ consisting of structure $\mathbb{A} \in \mathrm{CSP}(\mathbb{B})$ with girth strictly larger than ℓ. The CSP of a structure $\mathbb{B}$ is *finite-domain up to hight girth* if there is a positive integer ℓ and a finite structure $\mathbb{B}'$ such that $\mathrm{CSP}_{>\ell}(\mathbb{B}) = \mathrm{CSP}_{>\ell}(\mathbb{B}')$. It was proved in [28] that for every such structure $\mathbb{B}$, there is a finite structure $\mathbb{B}_S$ (unique up to homomorphic equivalence) such that for every finite structure $\mathbb{A}$ there is a homomorphism $\mathbb{B} \to \mathbb{A}$ if and only if $\mathbb{B}_S \to \mathbb{A}$. We call $\mathbb{B}_S$ the *smallest finite factor* of $\mathbb{B}$ — notice that in particular, $\mathbb{B} \to \mathbb{B}_S$. Moreover, if $\mathbb{B}$ is finite domain up to high girth and $\mathbb{B}_S$ is its smallest finite factor, then there is a positive integer ℓ such that $\mathrm{CSP}_{>\ell}(\mathbb{B}) = \mathrm{CSP}_{>\ell}(\mathbb{B}_S)$ [28, Corollary 10].

Lemma 22. *Let $\mathbb{B}$ be a structure such that $\mathrm{CSP}(\mathbb{B})$ is finite-domain up to high girth. If $\mathbb{B}_S$ is the smallest finite factor of $\mathbb{B}$, then $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ and $\mathrm{RCSP}(\mathbb{A}, \mathbb{B}_S)$ are polynomial-time equivalent.*

Proof. TOPROVE 9 □

4.4 A dichotomy for finite domain RCSPs with GMSNP restrictions

Forbidden pattern problems (for graphs) are parametrized by a finite set $\mathcal{F}$ of vertex and edge coloured connected graphs, and the task is to decide is an input graph $\mathbb{G}$ admits a colouring $\mathbb{G}'$ (with the same colours used in $\mathcal{F}$) such that $\mathbb{G}' \in \mathrm{Forb}(\mathcal{F})$, i.e., there is no homomorphism $\mathbb{F} \to \mathbb{G}'$ for any $\mathbb{F} \in \mathcal{F}$. A standard example of the problem of deciding if an input graph $\mathbb{G}$ admits a 2-edge-colouring with no monochromatic triangles.

There is a natural fragment of existential second order logic called *guarded monotone strict NP* (GMSNP) such that CSPs expressible in GMSNP capture forbidden pattern problems. We refer the reader to [4] for further background on GMSNP.

Lemma 12 in [28] shows that every CSP expressible in GMSNP is finite-domain up to high girth. The following statement is an immediate consequence of this fact, and of Lemma 22.

Theorem 23. *For every finite structure $\mathbb{A}$, and every structure $\mathbb{B}$ such that $\mathrm{CSP}(\mathbb{B})$ is expressible in GMSNP one of the following statement holds.*

- *Either $(\mathbb{A}, \mathbb{B}_S)$ rpp-constructs $(\mathbb{K}_3, \mathbb{L})$, where $\mathbb{B}_S$ is the smallest finite factor of $\mathbb{B}$, (and consequently, $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ is NP-hard), or*
- *$\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ is polynomial-time solvable.*

4.5 The tractability conjecture and RCSPs

The tractability conjecture is a generalization of the Feder-Vardi conjecture to a broad class of "well-behaved" infinite structures, namely, to *reducts* of *finitely bounded homogeneous* structure. A structure $\mathbb{B}$ is homogeneous if for every isomorphism $f\colon \mathbb{A} \to \mathbb{A}'$ between finite substructures of $\mathbb{B}$, there is an automorphism $f'\colon \mathbb{B} \to \mathbb{B}$ that extends f, i.e., $f'(a) = f(a)$ for every $a \in A$. A structure $\mathbb{B}$ is called finitely bounded if there exists a finite set $\mathcal{F}$ of finite structures such that a finite structure $\mathbb{A}$ embeds into $\mathbb{B}$ if and only if no structure from $\mathcal{F}$ embeds into $\mathbb{A}$. A reduct of a structure $\mathbb{B}$ is a structure $\mathbb{A}$ obtained from $\mathbb{B}$ by forgetting some relations.

Conjecture 1 (Conjecture 3.7.1 in [11])**.** *Let $\mathbb{A}$ be a reduct of a finitely bounded homogeneous structure. If $\mathbb{A}$ does not pp-construct $\mathbb{K}_3$, then $\mathrm{CSP}(\mathbb{A})$ is polynomial-time solvable.*

This is a wide-open conjecture yielding a very active research line (e.g., [6, 9–13, 36]). Here we show that this conjecture is equivalent to the following conjecture for restricted CSPs templates with finite restriction and whose domain is a reduct of a finitely bounded homogeneous structure.

Conjecture 2. *Let $\mathbb{A}$ be a reduct of a finitely bounded homogeneous structure, and $\mathbb{B}$ be a finite structure. If the restricted* CSP*-template does not rpp-construct* $(\mathbb{K}_3, \mathbb{L})$*, then* $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ *is polynomial-time solvable.*

Theorem 24. *Conjecture 1 and Conjecture 2 are equivalent.*

Proof. TOPROVE 10 □

4.6 A small remark regarding PCSPs

Theorem 19 asserts in particular that $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ is log-space equivalent to $\mathrm{CSP}(\mathbb{A}^{\mathbb{B}})$. The following statement strengthens this connection by showing that these two problems are further log-space equivalent to $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}^{\mathbb{B}})$.

Lemma 25. *For every possibly infinite structure $\mathbb{A}$ and a (possibly infinite) structure and $\mathbb{B}$ a finite structure, the following problems are polynomial-time equivalent.*

- *the constraint satisfaction problem* $\mathrm{CSP}(\mathbb{A}^{\mathbb{B}})$*,*
- *the promise constraint satisfaction problem* $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}^{\mathbb{B}})$*, and*
- *the restricted constraint satisfaction problem* $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$*.*

Proof. TOPROVE 11 □

Corollary 26. *For every non-bipartite graph $\mathbb{H}$ and every finite digraph $\mathbb{D}$ one of the following holds.*

- *Either $\mathbb{D} \to \mathbb{H}$, and in this case* $\mathrm{PCSP}(\mathbb{H}, \mathbb{H}^{\mathbb{D}})$ *is polynomial-time solvable,*
- *or $\mathbb{D} \not\to \mathbb{H}$, and in this case* $\mathrm{PCSP}(\mathbb{H}, \mathbb{H}^{\mathbb{D}})$ *is* NP*-hard.*

Proof. TOPROVE 12 □

5 Homomorphism-free restrictions

As mentioned above, the reader familiar with monotone monadic strict NP (MMSNP) can notice that digraph (finite domain) CSPs with input restriction to $\mathcal{F}$-homomorphism-free digraphs (structures) can be solved by infinite domain CSPs expressible in MMSNP. Thus, these problems exhibit a P versus NP-complete dichotomy (see, e.g., [14, 23]). Feder and Vardi reduce problems in MMSNP to finite domain CSPs. Their reduction changes the input signature, and then uses the Sparse Incomparability Lemma. Here, we prove the following lemma which allows us to preserve the signature by using duality pairs, and then, as Feder and Vardi, we proceed via Sparse Incomparability.

Lemma 27. *For every finite set of structures $\mathcal{F}$ and every finite digraph (structure) $\mathbb{B}$, there is a finite set of structures $\mathcal{C}$ such that the following problems are polynomial-time equivalent:*

- *deciding if an input structure* $\mathbb{A}$ *belongs to* $\mathrm{CSP}(\mathbb{C})$ *for some* $\mathbb{C} \in \mathcal{C}$*, and*
- $\mathrm{CSP}(\mathbb{B})$ *restricted to* $\mathcal{F}$*-homomorphism-free structures.*

Proof. TOPROVE 13 □

The following statement is an immediate consequence of Lemma 27 and finite domain CSP dichotomy [18, 38].

Theorem 28. *For every finite digraph (structure)* $\mathbb{H}$ *and every finite set of digraphs (structures)* $\mathcal{F}$*,* $\mathrm{CSP}(\mathbb{H})$ *restricted to* $\mathcal{F}$*-homomorphism-free digraphs (structures) is either in* P *or it is* NP*-complete.*

Proof. TOPROVE 14 □

6 Acyclic digraphs and bounded paths

In this section we apply our results above, and the theory of constraint satisfaction to obtain results in the context of $\mathcal{F}$-(subgraph)-free algorithmics. The main result of this section settles Question 3 (Theorem 34). Moreover, we show that if $\mathrm{CSP}(\mathbb{H})$ is NP-complete, then there is a positive integer N such that $\mathrm{CSP}(\mathbb{H})$ remains NP-complete for acyclic digraphs with no directed path on N vertices, and $\mathrm{CSP}(\mathbb{H})$ can be solved in polynomial time if the input is an acyclic digraph with no directed path on $N-1$ vertices. We begin by stating the following corollary of (the proof of) Lemma 27.

Corollary 29. *For every finite set of trees* $\mathcal{F}$ *with dual* $\mathbb{D}$*, and every digraph* $\mathbb{H}$ *the following problems are polynomial-time equivalent.*

- $\mathrm{RCSP}(\mathbb{H}, \mathbb{D})$*.*
- $\mathrm{CSP}(\mathbb{H} \times \mathbb{D})$*.*
- $\mathrm{CSP}(\mathbb{H})$ *restricted to* $\mathcal{F}$*-homomorphism-free digraphs.*

A *polymorphism* of a structure $\mathbb{A}$ is a homomorphism $f\colon \mathbb{A}^n \to \mathbb{A}$, and in this case we say that f is an n*-ary* polymorphism. Under composition, polymorphisms form an algebraic structure called a *clone*, and this structure captures the computational complexity of $\mathrm{CSP}(\mathbb{A})$ (see, e.g., [8]). A 4-ary polymorphism $f\colon \mathbb{A}^4 \to \mathbb{A}$ satisfies the *Sigger's identity* if

$$f(x_1, x_2, x_3, x_1) = f(x_2, x_1, x_2, x_3) \text{ for every } x_1, x_2, x_3, x_4 \in A.$$

The Sigger's identity is one of several identities that equivalently describe the tractability frontier for finite domain CSPs.

Theorem 30 (Equivalent to Theorem 1.4 in [38])**.** *For every finite structure* $\mathbb{A}$ *one of the following holds.*

- *Either* $\mathbb{A}$ *has a polymorphism* $f\colon \mathbb{A}^4 \to \mathbb{A}$ *that satisfies the Sigger's identity, and in this case* $\mathrm{CSP}(\mathbb{A})$ *is polynomial-time solvable, or*
- *otherwise,* $\mathbb{A}$ *pp-constructs* $\mathbb{K}_3$*, and in this case* $\mathrm{CSP}(\mathbb{A})$ *is* NP*-complete.*

Consider a finite digraph $\mathbb{H}$, and let k be a positive integer. For a given function $f\colon (H \times [k])^n \to (H \times [k])$ and for each $i \in [k]$ we define a function

$$f_i\colon H^n \to H \text{ by } (h_1, \dots, h_n) \mapsto \pi_H(f((h_1, i), \dots, (h_n, i)),$$

where π_H is the projection of $H \times [k]$ onto H.

Lemma 31. *Let $\mathbb{H}$ be a digraph, k a positive integer, and $f\colon (\mathbb{H} \times \mathbb{T}_k)^n \to (\mathbb{H} \times \mathbb{T}_k)$ and n-ary polymorphism of $\mathbb{H} \times \mathbb{T}_k$. If there are $i, j \in [k]$ with $i < j$ such that $f_i = f_j$, then f_i is an n-ary polymorphism of $\mathbb{H}$, and if f satisfies the equalities*

$$f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_{\rho(1)}, \dots, x_{\rho(n)}) \text{ for all } x_1, \dots, x_m \in H \times [k]$$

for some $\sigma, \rho\colon [n] \to [m]$, then f_i satisfies the equalities

$$f_i(h_{\sigma(1)}, \dots, h_{\sigma(n)}) = f_i(h_{\rho(1)}, \dots, h_{\rho(n)}) \text{ for all } h_1, \dots, h_n \in H.$$

Proof. TOPROVE 15 □

Building on this lemma and Theorem 30 together with Corollary 29, we prove the main result of this section.

Theorem 32. *For every finite digraph $\mathbb{H}$ one of the following statements holds.*

- *Either $\mathbb{H}$ has a polymorphism satisfying the Sigger's identity, and* $\mathrm{RCSP}(\mathbb{H}, \mathbb{T}_k)$ *is in* P *for every positive integer k, or*
- *otherwise, there is a positive integer N such that* $\mathrm{RCSP}(\mathbb{H}, \mathbb{T}_k)$ *is* NP*-hard for every $k \geq N$.*

Proof. TOPROVE 16 □

The following is an immediate application of this theorem, of its proof, and of Corollary 29.

Corollary 33. *For every finite digraph $\mathbb{H}$, there is a positive integer $N \leq |H|^{4|H|} + 1$ such that* $\mathrm{CSP}(\mathbb{H})$ *is polynomial-time equivalent to* $\mathrm{CSP}(\mathbb{H})$ *restricted to acyclic digraphs with no directed walk on N vertices.*

As promised, we apply the framework of RCSPs to the context of $\mathcal{F}$-(subgraph)-free algorithmics.

Theorem 34. *For every digraph $\mathbb{H}$ such that* $\mathrm{CSP}(\mathbb{H})$ *is* NP*-hard, there is a positive integer N such that* $\mathrm{CSP}(\mathbb{H})$ *remains* NP*-complete even for $\vec{\mathbb{P}}_N$-subgraph-free acyclic digraphs. Moreover, there is such an N such that* $\mathrm{CSP}(\mathbb{H})$ *is polynomial-time solvable when the input is a $\vec{\mathbb{P}}_{N-1}$-subgraph-free digraph.*

Proof. TOPROVE 17 □

Corollary 35. *Let $\mathbb{H}$ be a digraph such that* $\mathrm{CSP}(\mathbb{H})$ *is* NP*-hard. Then, there are positive integers N and M such that*

- $\mathrm{CSP}(\mathbb{H})$ *is* NP*-hard for $\vec{\mathbb{P}}_k$-free digraphs whenever $k \geq N$, and*
- $\mathrm{CSP}(\mathbb{H})$ *is* NP*-hard for $\vec{\mathbb{P}}_k$-subgraph-free digraphs whenever $k \geq M$.*

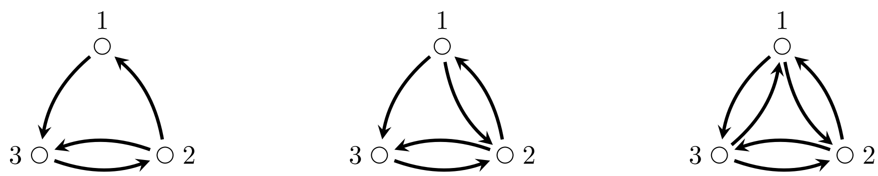
We conclude this section with a simple example showing that Theorem 34 does not hold when $\mathbb{H}$ is infinite: there are infinite graphs $\mathbb{H}$ such that CSP($\mathbb{H}$) is NP-complete, but CSP($\mathbb{H}$) becomes tractable on acyclic instances. Moreover, $\mathbb{H}$ can be chosen to be ω-*categorical*, i.e., for every positive integer k the automorphism group of $\mathbb{H}$ defines finitely many orbits of k-tuples.

Example 36. Let $\mathbb{H}$ be the disjoint union of $\mathbb{K}_3$ and the rational number with the strict linear order. It is straightforward to observe that 3-COLOURING reduces to CSP($\mathbb{H}$), and that every acyclic digraph $\mathbb{D}$ is a yes-instance of CSP($\mathbb{H}$). Hence, CSP($\mathbb{H}$) is NP-complete, but it is polynomial-time solvable on acyclic instance. The fact that $\mathbb{H}$ is ω-categorical follows from its definition — it is the disjoin union of two ω-categorical digraphs.

7 Digraphs on three vertices

In this section we answer Question 1 for digraphs on three vertices. A loopless digraph on three vertices either contains two directed cycles and its CSP if NP-complete, or otherwise its CSP is polynomial-time solvable. Hence, we focus on the three loopless digraphs on three vertices: $\vec{\mathbb{C}}_3^+$ (obtained from the directed cycle by adding one edge), $\vec{\mathbb{C}}_3^{++}$ (obtained from the directed cycle by adding two edges), and $\mathbb{K}_3$ — see also Figure 4.

Also note that these three digraphs are hereditarily hard (see Theorem 2), and it thus follows from Theorem 3 that RCSP($\mathbb{H}, \mathbb{T}_4$) is NP-hard for $\mathbb{H} \in \{\vec{\mathbb{C}}_3^+, \vec{\mathbb{C}}_3^{++}, \mathbb{K}_3\}$. In turn, this implies that for such digraphs $\mathbb{H}$ the problem CSP($\mathbb{H}$) is NP-hard even restricted to $\vec{\mathbb{P}}_5$-(subgraph)-free digraphs. We use this remark in both subsections below.



$\vec{\mathbb{C}}_3^+$ $\mathbb{K}_3$

Figure 4: The three digraphs on three vertices with at least two directed cycles, and whose CSP is NP-complete.

7.1 $\vec{\mathbb{P}}_k$-subgraph-free digraphs

It is straightforward to observe that any orientation of an odd cycle contains $\vec{\mathbb{P}}_3$ as a subgraph. Hence, if $\mathbb{D}$ is a $\vec{\mathbb{P}}_3$-subgraph-free digraph, then it is bipartite, i.e., $\mathbb{D} \to \mathbb{K}_2$. In particular, this implies that CSP($\vec{\mathbb{C}}_3^+$), CSP($\vec{\mathbb{C}}_3^{++}$), and CSP(K_3) are polynomial-time solvable for $\vec{\mathbb{P}}_3$-subgraph-free digraphs. As previously mentioned, each of these CSPs is NP-hard for $\vec{\mathbb{P}}_5$-subgraph-free digraphs. In this section, we study the complexity CSP($\vec{\mathbb{C}}_3^+$), CSP($\vec{\mathbb{C}}_3^{++}$), and CSP($\mathbb{K}_3$) restricted to $\vec{\mathbb{P}}_4$-subgraph-free digraphs. We begin with a remark which we use a couple times in the remaining of the paper.

Remark 37. If a $\vec{\mathbb{P}}_4$-subgraph-free weakly connected digraph $\mathbb{D}$ contains (as a subgraph) a directed 3-cycle v_1, v_2, v_3, then $D = \{v_1, v_2, v_3\}$. Indeed, since $\mathbb{D}$ is $\vec{\mathbb{P}}_4$-subgraph-free and $(v_1, v_2), (v_2, v_3), (v_3, v_1) \in E$, it must be the case that the out- and in-neighbourhood of each v_i is a subset of $\{v_1, v_2, v_3\}$ (and so, the claim follows because $\mathbb{D}$ is weakly connected).

Observation 38. *Every loopless $\vec{\mathbb{P}}_4$-subgraph-free digraph is* 3*-colourable.*

Proof. TOPROVE 18 □

Clearly, Observation 38 does not hold if we replace CSP($\mathbb{K}_3$) (i.e., 3-colourable) by CSP($\vec{\mathbb{C}}_3^{++}$): $\mathbb{K}_3$ is a simple counterexample. However, it turns out that $\mathbb{K}_3$ is the unique minimal counterexample. In the following proof we use the notion of a *leaf* of a digraph $\mathbb{D}$, i.e., a vertex $x \in \mathbb{D}$ with $|(N^+(x) \cup N^-(x)) \setminus \{x\}| = 1$.

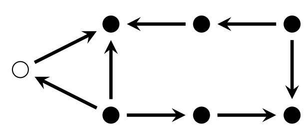
Lemma 39. *Every loopless $\{\mathbb{K}_3, \vec{\mathbb{P}}_4\}$-subgraph-free digraph $\mathbb{D}$ admits a homomorphism to $\vec{\mathbb{C}}_3^{++}$.*

Proof. TOPROVE 19 □

Now we show that CSP($\vec{\mathbb{C}}_3^{+}$) is polynomial-time solvable when the input is restricted to $\vec{\mathbb{P}}_4$-subgraph-free digraphs. To do so, we consider a unary predicate U, and we reduce the problem mentioned above to the CSP of the $\{E, U\}$-structure depicted in Figure 5. We first show that the underlying graph has a *conservative majority* polymorphism, i.e., a polymorphism $f\colon \mathbb{G}^3 \to \mathbb{G}$ such that $f(x,x,x) = f(x,x,y) = f(x,y,x) = f(y,x,x) = x$ and $f(x,y,z) \in \{x,y,z\}$.

Lemma 40. *The digraph $\mathbb{G}$ from Figure 5 has a conservative majority polymorphism. In particular,* CSP($\mathbb{G}, U$) *can be solved in polynomial time.*

Proof. TOPROVE 20 □



$(\mathbb{G}, U)$

Figure 5: A $\{E, U\}$-structure where E is a binary relation represented by arcs, and U is a unary relation represented by black vertices.

Lemma 41. CSP($\vec{\mathbb{C}}_3^{+}$) *can be solved in polynomial-time when the input is restricted to $\vec{\mathbb{P}}_4$-subgraph-free digraphs.*

Proof. TOPROVE 21 □

Theorem 42. *The following statements hold for every positive integer k.*

- *If $k \leq 4$, then* CSP($\vec{\mathbb{C}}_3^{+}$), CSP($\vec{\mathbb{C}}_3^{++}$), *and* CSP($\mathbb{K}_3$) *are solvable in polynomial-time when the input is restricted to $\vec{\mathbb{P}}_k$-subgraph-free digraphs.*
- *If $k \geq 5$, then* CSP($\vec{\mathbb{C}}_3^{+}$), CSP($\vec{\mathbb{C}}_3^{++}$), *and* CSP($\mathbb{K}_3$) *are* NP*-hard even if the input is restricted to $\vec{\mathbb{P}}_k$-subgraph-free digraphs.*

Proof. TOPROVE 22 □

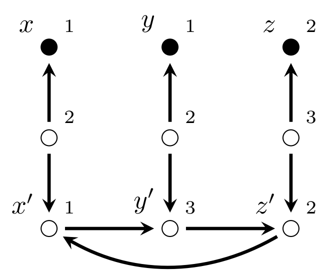
7.2 $\vec{\mathbb{P}}_k$-free digraphs

Notice that a digraph $\mathbb{D}$ is $\vec{\mathbb{P}}_2$-free if and only if it is a symmetric (undirected) graph. Hence, CSP($\mathbb{K}_3$) is NP-hard for $\vec{\mathbb{P}}_2$-free digraphs. Also note that a symmetric graph $\mathbb{D}$ maps to $\vec{\mathbb{C}}_3^+$ if and only if $\mathbb{D}$ is bipartite, and the same statement holds for $\vec{\mathbb{C}}_3^{++}$. Hence, in this section we study the complexity of CSP($\vec{\mathbb{C}}_3^+$) and CSP($\vec{\mathbb{C}}_3^{++}$) with input restrictions to $\vec{\mathbb{P}}_3$-free and to $\vec{\mathbb{P}}_4$-free digraphs.

Lemma 43. CSP($\vec{\mathbb{C}}_3^+$) *is* NP*-hard even when the input* $\mathbb{D}$ *satisfies the following conditions, where* $\mathbb{F}$ *is any given orientation of the claw* $\mathbb{K}_{1,3}$*:*

- $\mathbb{D}$ *is* $\{\mathbb{F}, \vec{\mathbb{P}}_5, \mathbb{P}_5^{\leftarrow\leftarrow\rightarrow\rightarrow}, \mathbb{P}_5^{\rightarrow\rightarrow\leftarrow\leftarrow}\}$*-subgraph-free,*
- $\mathbb{D}$ *is* $\vec{\mathbb{P}}_4$*-free, and*
- $d^+(v) + d^-(v) \leq 3$ *for every* $v \in D$.

Proof. TOPROVE 23 $\square$



$\mathbb{G}(x, y, z)$

Figure 6: A depiction of the gadget reduction $\mathbb{I} \mapsto \mathbb{D}$ from positive 1-IN-3 SAT to CSP($\vec{\mathbb{C}}_3^+$) applied to a clause $(x \vee y \vee z)$ of the instance $\mathbb{I}$ to 1-IN-3 SAT. The numbers indicate a function that defined a homomorphism $f \colon \mathbb{G} \to \vec{\mathbb{C}}_3^+$.

Now we show that CSP($\vec{\mathbb{C}}_3^+$) can be solved in polynomial time when the input digraph $\mathbb{D}$ contains no induced directed path on three vertices. On a given $\vec{\mathbb{P}}_3$-free digraph $\mathbb{D}$, the algorithm works as follows. At each step there are three sets $X_1, X_2, X_3 \subseteq D$ such that the mapping $x \mapsto i$ if $x \in X_i$ defines a partial homomorphism from $\mathbb{D}$ to $\vec{\mathbb{C}}_3^+$. We extend these sets to X_1', X_2', X_3' in such a way that the partial homomorphism defined by X_1, X_2, X_3 extends to a homomorphism from $\mathbb{D} \to \vec{\mathbb{C}}_3^+$ if and only if the partial homomorphism defined by X_1', X_2', X_3' extends to a homomorphism from $\mathbb{D} \to \vec{\mathbb{C}}_3^+$.

Lemma 44. CSP($\vec{\mathbb{C}}_3^+$) *can be solved in quadratic time when the input is restricted to* $\vec{\mathbb{P}}_3$*-free digraphs.*

Proof. TOPROVE 24 $\square$

Lemma 45. CSP($\vec{\mathbb{C}}_3^{++}$) *is* NP*-hard even for a* $\{\vec{\mathbb{P}}_3, \vec{\mathbb{P}}_3^{\leftarrow\rightarrow}, \vec{\mathbb{P}}_3^{\rightarrow\leftarrow}\}$*-free digraphs.*

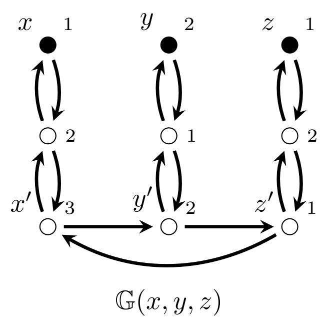


Figure 7: A depiction of the gadget reduction $\mathbb{I} \mapsto \mathbb{D}$ from positive 1-IN-3 SAT to CSP($\vec{\mathbb{C}}_3^{++}$) applied to a clause $(x \vee y \vee z)$ of the instance $\mathbb{I}$ to 1-IN-3 SAT. The numbers indicate a function that defined a homomorphism $f : \mathbb{G} \to \vec{\mathbb{C}}_3^{++}$.

Proof. TOPROVE 25 $\square$

The following statement now follows from Lemmas 43, 44, and 45.

Theorem 46. *The following statements hold for every positive integer $k \geq 2$.*

- CSP($\vec{\mathbb{C}}_3^{+}$) *is in solvable in quadratic time for $\vec{\mathbb{P}}_k$-free digraphs if $k \leq 3$, and* NP*-hard even for $\vec{\mathbb{P}}_k$-free digraphs if $k \geq 4$,*
- CSP($\vec{\mathbb{C}}_3^{++}$) *is solvable in linear time for $\vec{\mathbb{P}}_k$-free digraphs if $k = 2$, and* NP*-hard even for $\vec{\mathbb{P}}_k$-free digraphs if $k \geq 3$, and*
- CSP($\mathbb{K}_3$) *is* NP*-hard even for $\vec{\mathbb{P}}_k$-free digraphs.*

8 A family of smooth tournaments

In this section we answer Question 1 for a natural family of tournaments smooth tournaments $\mathbb{TC}_n$. Given a positive integer n, we denote by $\mathbb{TC}_n$ the tournament obtained from $\mathbb{T}_n$ be reversing the edge from the source to the sink (see Figure 8). In particular, $\mathbb{TC}_2 \cong \mathbb{T}_2$, and $\mathbb{TC}_3 \cong \vec{\mathbb{C}}_3$, so CSP($\mathbb{TC}_n$) is polynomial-time solvable for $n \leq 3$, and NP-complete for $n \geq 4$ (see, e.g., [2]).

Since $\mathbb{TC}_n$ is a hereditary hard digraph (Theorem 2) and $\mathbb{T}_n \not\to \mathbb{TC}_n$, it follows that RCSP($\mathbb{TC}_n, \mathbb{T}_n$) is NP-hard (Theorem 3). Equivalently, CSP($\mathbb{TC}_n$) is NP-hard for digraphs with no directed walk on $n+1$ vertices, and since $\mathbb{T}_{n-1} \to \mathbb{TC}_n$, CSP($\mathbb{TC}_n$) is polynomial-time solvable for digraphs with no directed walk on n vertices. However, is we only forbid $\vec{\mathbb{P}}_k$ as a subgraph (and not homomorphically), it turns out that CSP($\mathbb{TC}_n$) is NP-hard even for $\vec{\mathbb{P}}_5$-subgraph-free digraphs.

8.1 $\vec{\mathbb{P}}_k$-subgraph-free digraphs

For these hardness results we consider the gadget reduction depicted in Figure 9 and described in the proof of the following lemma.

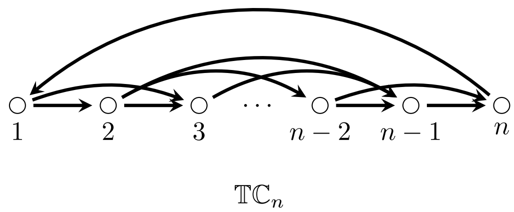


Figure 8: A depiction of the $\mathbb{TC}_n$ — for a cleaner picture we omit the edges $(1, n-2)$, $(1, n-1)$, $(2, n)$, and $(3, n)$.

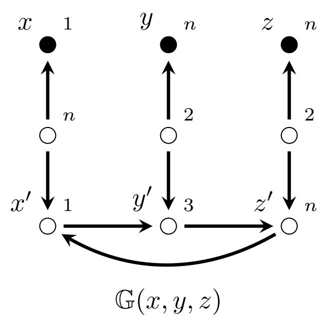


Figure 9: A depiction of the gadget reduction $\mathbb{I} \mapsto \mathbb{D}$ from positive 1-IN-3 SAT to CSP($\mathbb{TC}_n$) applied to a clause $(x \vee y \vee z)$ of the instance $\mathbb{I}$ to 1-IN-3 SAT. The numbers indicate a function that defined a homomorphism $f \colon \mathbb{G} \to \mathbb{TC}_n$ whenever $n \geq 4$.

Lemma 47. *For every positive integer $n \geq 4$, CSP($\mathbb{TC}_n$) is NP-hard even when the input $\mathbb{D}$ satisfies the following conditions:*

- *$\mathbb{D}$ is $\{\mathbb{F}, \vec{\mathbb{P}}_5, \mathbb{P}_5^{\leftarrow\leftarrow\rightarrow\rightarrow}, \mathbb{P}_5^{\rightarrow\rightarrow\leftarrow\leftarrow}\}$-subgraph-free,*
- *$\mathbb{D}$ is $\vec{\mathbb{P}}_4$-free, and*
- *$d^+(v) + d^-(v) \leq 3$ for every $v \in D$.*

Proof. TOPROVE 26 □

We now argue that CSP($\mathbb{TC}_n$) is polynomial-time solvable when the input is restricted to $\vec{\mathbb{P}}_4$-subgraph-free digraphs, and together with Lemma 47 we obtain a complexity classification for these CSPs restricted to $\vec{\mathbb{P}}_k$-subgraph-free digraphs.

Lemma 48. *Consider a (possibly infinite) digraph $\mathbb{H}$. If $\mathbb{T}_3 \to \mathbb{H}$, then CSP($\mathbb{H}$) is in P for $\vec{\mathbb{P}}_4$-subgraph-free oriented graphs. In particular, if $\mathbb{H}$ is an oriented graph and $\mathbb{T}_3 \to \mathbb{H}$, then CSP($\mathbb{H}$) is in P for $\vec{\mathbb{P}}_4$-subgraph-free digraphs.*

Proof. TOPROVE 27 □

Theorem 49. *For every pair of positive integers n and k the following statements hold.*

- *If $n \leq 3$ or $k \leq 4$, then* CSP($\mathbb{TC}_n$) *is in* P *for* $\vec{\mathbb{P}}_k$*-subgraph-free digraphs.*
- *If $n \geq 4$ and $k \geq 5$, then* CSP($\mathbb{TC}_n$) *is* NP*-hard even for* $\vec{\mathbb{P}}_k$*-subgraph-free digraphs.*

Proof. TOPROVE 28 □

8.2 $\vec{\mathbb{P}}_k$-free digraphs

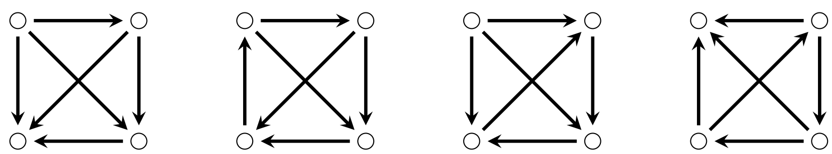
In this subsection we prove a structural result (Theorem 55) asserting that there is a finite set of digraphs $\mathcal{F}$ such that a $\vec{\mathbb{P}}_3$-free digraphs $\mathbb{D}$ belongs to CSP($\mathbb{TC}_n$) if and only if $\mathbb{D}$ is $\mathcal{F}$-free. We then use this result to propose a complexity classification for these CSPs restricted to $\vec{\mathbb{P}}_k$-free digraphs.

We begin by showing that there are finitely many minimal $\vec{\mathbb{P}}_3$-free digraphs that do not homomorphically map to CSP($\mathbb{TC}_4$). We depict the four non-isomorphic tournaments on four vertices in Figure 10.

Lemma 50. *The following statements are equivalent for a $\vec{\mathbb{P}}_3$-free digraph $\mathbb{D}$.*

- $\mathbb{D} \to \mathbb{TC}_4$*, and*
- $\mathbb{D}$ *is a* $\{\mathbb{T}_4, \mathbb{T}_4^a, \mathbb{T}_4^b\}$*-free loopless oriented graph with no tournament on five vertices.*

Proof. TOPROVE 29 □



$\mathbb{T}_4$ $\mathbb{TC}_4$ $\mathbb{T}_4^a$ $\mathbb{T}_4^b$

Figure 10: The four non-isomorphic oriented tournaments on 4 vertices

For the remaining of this section, for each positive integer $n \geq 4$ we fix the set $\mathcal{F}_n$ to be the (finite) set of tournaments on at most $n+1$ vertices that do not embed into $\mathbb{TC}_n$ (up to isomorphism).

Remark 51. Since every (induced subgraph) tournament on $n-1$ vertices in $\mathbb{TC}_n$ is either $\mathbb{TT}_{n-1}$ or $\mathbb{TC}_{n-1}$, the following equality holds

$$\mathcal{F}_{n-1} \setminus \mathcal{F}_n = \{\mathbb{TT}_{n-1}, \mathbb{TC}_n\}.$$

Building on Lemma 50 we prove the following statement.

Lemma 52. *For every $\vec{\mathbb{P}}_3$-free digraph $\mathbb{D}$ the following statements are equivalent*

- $\mathbb{D} \to \mathbb{TC}_n$*, and*
- $\mathbb{D}$ *is a* $\mathcal{F}_n$*-free loopless oriented graph.*

Proof. TOPROVE 30 □

Lemma 52 implies that for every positive integer n there is a polynomial-time algorithm that solves $\mathrm{CSP}(\mathbb{TC}_n)$ when the input is restricted to $\vec{\mathbb{P}}_3$-free digraphs. Hence, the following classification follows from this lemma and from Lemma 47.

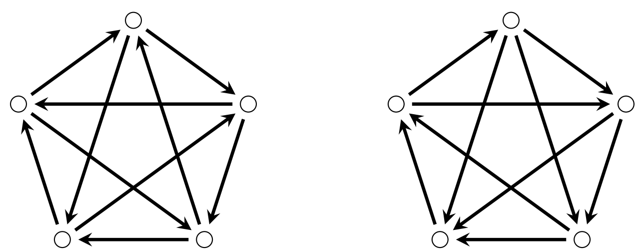
Theorem 53. *For every positive integer k and $n \geq 4$ one of the following holds*

- *$k \leq 3$ and in this case $\mathrm{CSP}(\mathbb{TC}_n)$ is tractable where the input is restricted to $\vec{\mathbb{P}}_k$-free digraphs, or*
- *$k \geq 4$ and in this case $\mathrm{CSP}(\mathbb{TC}_n)$ is NP-complete where the input is restricted to $\vec{\mathbb{P}}_k$-free digraphs.*

In the rest of this section we improve Lemma 52 by listing the (finitely many) *minimal* $\vec{\mathbb{P}}_3$-free obstructions to $\mathrm{CSP}(\mathbb{TC}_n)$. To do so, we introduce two new tournaments on five vertices depicted in Figure 11, and we prove the following lemma.

Lemma 54. *Let $\mathbb{D}$ be a $\{\mathbb{T}_4^a, \mathbb{T}_4^b\}$-free tournament. If $\mathbb{D}$ contains a pair of directed triangles with no common edge, then $\mathbb{D}$ contains a subtournament $\mathbb{D}'$ on five vertices that also contains a pair of directed triangles with no common edge.*

Proof. TOPROVE 31 □



$\mathbb{RT}_5$ $\qquad$ $\mathbb{T}_5^c$

Figure 11: The unique $\{\mathbb{T}_4^a, \mathbb{T}_4^b\}$-free tournaments on five vertices that contain a pair of directed triangles with no a common edge (up to isomorphism).

Theorem 55. *The following statements are equivalent for every positive integer $n \geq 4$ and every $\vec{\mathbb{P}}_3$-free digraph $\mathbb{D}$.*

- *$\mathbb{D} \to \mathbb{TC}_n$, and*
- *$\mathbb{D}$ is a $\{\mathbb{T}_4^a, \mathbb{T}_4^b, \mathbb{RT}_5, \mathbb{T}_5^c, \mathbb{TT}_n, \mathbb{TC}_{n+1}\}$-free loopless oriented graph.*

Proof. TOPROVE 32 □

9 Omitting a single digraph

As mentioned in the introduction, the long-term question of the research line introduced in this paper is the following.

Question 4. *Is there a* P *versus* NP*-complete dichotomy of* CSP($\mathbb{H}$) *where the input is restricted*

1. *to* $\mathbb{F}$*-free digraphs?*

2. *to* $\mathbb{F}$*-subgraph-free digraphs?*

Having settled Question 1 for the digraphs on three vertices and for the family of tournaments $\mathbb{TC}_n$, a natural next step is tackling Question 4 for these digraphs. Regarding digraphs $\mathbb{H}$ on three vertices, we leave Question 4 (2) open for $\mathbb{K}_3$ and $\vec{\mathbb{C}}_3^{++}$, and notice that (1) has a simple solution in these cases (and $\mathbb{F}$ connected).

Corollary 56. *For every connected digraph* $\mathbb{F}$ *the following statements hold.*

- *Either* $\mathbb{F} \cong \mathbb{TT}_2$*, then* CSP($\vec{\mathbb{C}}_3^{++}$) *is polynomial-time solvable for* $\mathbb{F}$*-free digraphs, and* CSP($\mathbb{K}_3$) *is* NP*-hard for* $\mathbb{F}$*-free digraphs, or*

- *otherwise,* CSP($\vec{\mathbb{C}}_3^{++}$) *and* CSP($\mathbb{K}_3$) *are* NP*-hard for* $\mathbb{F}$*-free digraphs.*

Proof. TOPROVE 33 $\square$

In the remaining of this section we see that some of our proof already yield the first steps for settling Question 4 for $\vec{\mathbb{C}}_3^{+}$ and the family of tournaments. The main result in this direction being the following one.

Theorem 57. *For every positive integer* n *and every digraph* $\mathbb{F}$ *which is not a disjoint union of oriented paths the following statements hold.*

- CSP($\vec{\mathbb{C}}_3^{+}$) *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs.*

- CSP($\mathbb{TC}_n$) *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs.*

Proof. TOPROVE 34 $\square$

The following statement asserts that the CSPs of $\mathbb{TC}_n$ and of $\vec{\mathbb{C}}_3^{+}$ remain NP-hard when restricted $\mathbb{P}$-subgraph-free digraphs whenever $\mathbb{P}$ is a path that contains two pairs of consecutive edges oriented in the same direction.

Proposition 58. *The following statements hold for every connected digraph* $\mathbb{F}$ *that contains* $\vec{\mathbb{P}}_3+\vec{\mathbb{P}}_3$ *as a subgraph.*

- CSP($\vec{\mathbb{C}}_3^{+}$) *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs.*

- CSP($\mathbb{TC}_n$) *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs.*

Proof. TOPROVE 35 $\square$

Consider a word $w \in \{\leftarrow, \rightarrow\}^*$, i.e., a sequence $w := w_1 \dots w_n$ where $w_i \in \{\leftarrow, \rightarrow\}$ for each $i \in [n]$. We denote by $\mathbb{P}^w_{n+1}$ the oriented path with vertex set $[n+1]$ where there is an edge $(i, i+i)$ if $w_i = \rightarrow$, and there is an edge $(i+1, i)$ if $w_i = \leftarrow$. In particular, $\vec{\mathbb{P}}_n = \mathbb{P}^w_n$ where w is the constant word on $n-1$ letters $\rightarrow$.

Corollary 59. *The following statements hold for every digraph $\mathbb{F}$ that contains (as a subgraph) the oriented path $\mathbb{P}_7^{(\leftarrow\rightarrow)^3}$, the path $\mathbb{P}_7^{(\rightarrow\leftarrow)^3}$.*

- $\mathrm{CSP}(\vec{\mathbb{C}}_3^+)$ *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs.*
- $\mathrm{CSP}(\mathbb{TC}_n)$ *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs for every* $n \geq 4$.

Proof. TOPROVE 36 □

Corollary 60. *The following statements hold for every connected digraph* $\mathbb{F}$ *on at least* 12 *vertices.*

- $\mathrm{CSP}(\vec{\mathbb{C}}_3^+)$ *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs.*
- $\mathrm{CSP}(\mathbb{TC}_n)$ *is* NP*-hard even when the input is restricted to* $\mathbb{F}$*-subgraph-free digraphs for every* $n \geq 4$.

Proof. TOPROVE 37 □

Forbidden paths on three vertices

Notice that if $\mathbb{D}$ is an oriented graph with no directed path on three vertices, then $\mathbb{D} \to \mathbb{TT}_2$ (Observation 4). Also, if $\mathbb{D}$ contains a symmetric pair of edges $(u, v), (v, u)$, then the subgraph with vertices u, v is a connected component of $\mathbb{D}$. With these simple arguments one can notice that for any digraph $\mathbb{H}$, the problem $\mathrm{CSP}(\mathbb{H})$ is in P when the input is restricted to $\vec{\mathbb{P}}_3$-subgraph-free digraphs.

Observation 61. *For every digraph* $\mathbb{H}$, $\mathrm{CSP}(\mathbb{H})$ *is in* P *for* $\vec{\mathbb{P}}_3$*-subgraph-free digraphs.*

Hell and Mishra [30] proved that, for any digraph $\mathbb{H}$, the problem $\mathrm{CSP}(\mathbb{H})$ is polynomial-time solvable where the input is a $\vec{\mathbb{P}}_3^{\leftarrow\rightarrow}$-subgraph-free or a $\vec{\mathbb{P}}_3^{\rightarrow\leftarrow}$-subgraph-free digraphs. Here we briefly argue that if $\mathbb{H}$ is an oriented graph, then $\mathrm{CSP}(\mathbb{H})$ is polynomial-time solvable when the input is a $\vec{\mathbb{P}}_3^{\leftarrow\rightarrow}$-free digraph.

A *tree decomposition* for a digraph $\mathbb{G} = (G, E)$ is a pair $(\mathbb{T}, X)$ where $\mathbb{T}$ is a tree and X consists of subsets of vertices from G which we call bags. Each node of $\mathbb{T}$ corresponds to a single bag of X. For each vertex $v \in G$ the nodes of $\mathbb{T}$ containing v must induce a non-empty connected subgraph of $\mathbb{T}$ and for each edge $(u, v) \in E$, there must be at least one bag containing both u and v. We can then define the *width* of $(\mathbb{T}, X)$ to be one less than the size of the largest bag. From this, the *treewidth* of a digraph, $tw(G)$, is the minimum width of any *tree decomposition*.

Lemma 62. *If* $\mathbb{H}$ *is a finite oriented graph, then* $\mathrm{CSP}(\mathbb{H})$ *is in* P *for both the class of* $\vec{\mathbb{P}}_3^{\leftarrow\rightarrow}$*-free digraphs and the class of* $\vec{\mathbb{P}}_3^{\rightarrow\leftarrow}$*-free digraphs.*

Proof. TOPROVE 38 □

Constraint (e.g. $\mathbb{F} :=$)	$\mathrm{CSP}(\mathbb{TC}_n)$ on $\mathbb{F}$-subgraph-free	$\mathrm{CSP}(\mathbb{TC}_n)$ on $\mathbb{F}$-free
$\mathbb{F}$ is not an oriented tree	NP-complete Sparse Incomparability (Corollary 7)	
$\mathbb{F}$ is not an oriented path	NP-complete (Theorem 57)	
$\mathbb{F}$ contains $\vec{\mathbb{P}}_3 + \vec{\mathbb{P}}_3$ as a subgraph	NP-complete (Proposition 58)	
$\|F\| \geq 12$	NP-complete (Corollary 60)	
$\leftrightarrow$	NP-complete [2]	NP-complete [2]
$\leftarrow\rightarrow$	P [30, Lemma 1]	P (Lemma 62)
$\rightarrow\leftarrow$	P [30, Lemma 1]	P (Lemma 62)
$\rightarrow\rightarrow$	P (Theorem 49)	P (Theorem 55)
$\rightarrow\rightarrow\rightarrow$	P (Theorem 49)	NP-complete (Theorem 55)
$\rightarrow\rightarrow\rightarrow\rightarrow$	NP-complete (Theorem 49)	NP-complete (Theorem 55)
$\rightarrow\rightarrow\leftarrow\leftarrow$	NP-complete (Lemma 47)	
$\leftarrow\leftarrow\rightarrow\rightarrow$	NP-complete (Lemma 47)	
$(\leftarrow\rightarrow)^3$	NP-complete (Corollary 59)	
$(\rightarrow\leftarrow)^3$	NP-complete (Corollary 59)	

Table 1: Complexity landscape for $\mathrm{CSP}(\mathbb{TC}_n)$ under the omission of single connected subgraph or induced connected subgraph.

10 Conclusion and outlook

In this paper we have brought together homomorphisms, digraphs and $\mathbb{H}$-(subgraph-)free algorithmics. In doing so, we have uncovered a series of results concerning not only restricted CSPs, but also hardness of digraph CSPs under natural restrictions such as acyclicity. Our work raises numerous open problems, which we believe deserve attention in the future. Besides Questions 1 and 4, we ask the following.

- Is it true that for every finite structure $\mathbb{A}$ and every (possibly infinite) $\mathbb{B}$, if $(\mathbb{A}, \mathbb{B})$ does not rpp-construct $(\mathbb{K}_3, \mathbb{L})$, then $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ is polynomial-time solvable? (Compare to Theorem 21).
- Is it true that for every finite structure $\mathbb{A}$ and every (possibly infinite) $\mathbb{B}$ the problem $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ is either in P or NP-hard? (Compare to Theorem 21).
- Let $\mathbb{A}$ be a finite structure and $\mathbb{B}$ a structure whose CSP is in GMSNP. Is it true that if $(\mathbb{A}, \mathbb{B})$ does not rpp-construct $(\mathbb{K}_3, \mathbb{L})$, then $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ is polynomial-time solvable? (Compare to Theorem 23).
- Since $\mathrm{CSP}(\mathbb{H})$ reduces to $\mathrm{CSP}(\mathbb{H})$ restricted to acyclic digraphs (Theorem 34), and the latter

Constraint (e.g. $\mathbb{F} :=$)	CSP($\vec{\mathbb{C}}_3^+$) on $\mathbb{F}$-subgraph-free	CSP($\vec{\mathbb{C}}_3^+$) on $\mathbb{F}$-free
$\mathbb{F}$ is not an oriented tree	NP-complete Sparse Incomparability (Corollary 7)	
$\mathbb{F}$ is not an oriented path	NP-complete (Theorem 57)	
$\mathbb{F}$ contains $\vec{\mathbb{P}}_3 + \vec{\mathbb{P}}_3$ as a subgraph	NP-complete (Proposition 58)	
$\lvert F\rvert \geq 12$	NP-complete (Corollary 60)	
$\leftarrow\rightarrow$	P [30, Lemma 1]	Open
$\rightarrow\leftarrow$	P [30, Lemma 1]	Open
$\rightarrow\rightarrow$	P (Theorem 42)	P (Theorem 46)
$\rightarrow\rightarrow\rightarrow$	P (Theorem 42)	NP-complete (Theorem 46)
$\rightarrow\rightarrow\rightarrow\rightarrow$	NP-complete (Theorem 42)	NP-complete (Theorem 46)
$\rightarrow\rightarrow\leftarrow\leftarrow$	NP-complete (Lemma 43)	
$\leftarrow\leftarrow\rightarrow\rightarrow$	NP-complete (Lemma 43)	
$(\leftarrow\rightarrow)^3$	NP-complete (Corollary 59)	
$(\leftarrow\rightarrow)^3$	NP-complete (Corollary 59)	

Table 2: Complexity landscape for CSP($\vec{\mathbb{C}}_3^+$) under the omission of single connected subgraph or induced connected subgraph.

is polynomial-time equivalent to CSP($\mathbb{H} \times \mathbb{Q}$) we ask: is it true that for every finite digraph $\mathbb{H}$ the (infinite) digraph $\mathbb{Q} \times \mathbb{H}$ pp-constructs $\mathbb{H}$?

- Is is true that for every oriented graph $\mathbb{H}$ there a finitely many $\vec{\mathbb{P}}_3$-free minimal obstructions to CSP($\mathbb{H}$)? (Compare to Theorem 55).
- Is is true that for every tournament $\mathbb{T}$ there a finitely many $\vec{\mathbb{P}}_3$-free minimal obstructions to CSP($\mathbb{T}$)? (Compare to Theorem 55).
- Settle Question 4 for digraphs on three vertices (see also Table 2).
- Settle Question 4 for tournaments, in particular, for the family of tournaments $\mathbb{TC}_n$ (see also Table 1).

Persistent structures

A natural question arising from restricted CSPs is if there are structures $\mathbb{A}$ such that RCSP($\mathbb{A}, \mathbb{B}$) is NP-hard whenever $\mathbb{B} \not\to \mathbb{A}$. We say that a structure $\mathbb{A}$ *persistently constructs* $\mathbb{K}_3$ if for every (possibly infinite) $\mathbb{B}$ such that $\mathbb{B} \not\to \mathbb{A}$ the restricted CSP template ($\mathbb{A}, \mathbb{B}$) rpp-constructs ($\mathbb{K}_3, \mathbb{L}$). Notice that in this case RCSP($\mathbb{A}, \mathbb{B}$) is NP-hard whenever $\mathbb{B} \not\to \mathbb{A}$.

Observation 63. *For a finite structure $\mathbb{A}$ the following statements are equivalent.*

- $(\mathbb{A}, \mathbb{B})$ *rpp-constructs* $(\mathbb{K}_3, \mathbb{L})$ *for every finite structure* $\mathbb{B} \not\to \mathbb{A}$.
- $(\mathbb{A}, \mathbb{B})$ *rpp-constructs* $(\mathbb{K}_3, \mathbb{L})$ *for every (possibly infinite) structure* $\mathbb{B} \not\to \mathbb{A}$.
- $\mathrm{RCSP}(\mathbb{A}, \mathbb{B})$ *is* NP*-hard for every finite structure* $\mathbb{B} \to \mathbb{A}$ *(assuming* $\mathrm{P} \neq \mathrm{NP}$*).*

Proof. TOPROVE 39 □

Remark 64. A similar compactness as in the proof of Observation 63 applies to *ω-categorical structures* (see, e.g., [11, Lemma 4.1.7]), i.e., to structures $\mathbb{A}$ whose automorphism group has finitely many orbits of k-tuples for each positive integer k. Hence, if $\mathbb{A}$ is an ω-categorical structure, then $\mathbb{A}$ persistently constructs $\mathbb{K}_3$ if and only if $(\mathbb{A}, \mathbb{B})$ rpp-constructs $(\mathbb{K}_3, \mathbb{L})$ for every finite structure $\mathbb{B} \not\to \mathbb{A}$.

Problem 5. *Characterize the class of finite digraphs (structures) that persistently construct* $\mathbb{K}_3$.

Theorem 3 asserts that for every hereditarily hard digraph $\mathrm{RCSP}(\mathbb{H}, \mathbb{H}')$ is NP-hard whenever $\mathbb{H}'$ is a finite digraph and $\mathbb{H}' \not\to \mathbb{H}$. Hence, assuming $\mathrm{P} \neq \mathrm{NP}$, this implies that every hereditarily hard digraph persistently constructs $\mathbb{K}_3$ (Observation 63).

Problem 6. *Characterize the class of finite digraphs (structures)* $\mathbb{H}$ *such that* $\mathrm{RCSP}(\mathbb{H}, \mathbb{H}')$ *is* NP*-hard whenever* $\mathbb{H}' \not\to \mathbb{H}$ *(assuming* $\mathrm{P} \neq \mathrm{NP}$*).*

Acknowledgments

We are grateful to Martin Milanič for the proof of Lemma 62.

References

[1] J. Bang-Jensen and P. Hell. The effect of two cycles on the complexity of colourings by directed graphs. *Discrete Applied Mathematics*, 26(1):1–23, 1990.

[2] J. Bang-Jensen, P. Hell, and G. MacGillivray. The complexity of colouring by semicomplete digraphs. *SIAM Journal on Discrete Mathematics*, 1(3):281–298, 1988.

[3] J. Bang-Jensen, P. Hell, and G. MacGillivray. Hereditarily hard H-colouring problems. *Discrete Mathematics*, 138:75–92, 1995.

[4] A. Barsukov. On dichotomy above Feder and Vardi's logic, 2022. PhD thesis, École Doctorale Science pour l'Engénieur, Clermont-Ferrand.

[5] L. Barto, J. Bulín, A. A. Krokhin, and J. Opršal. Algebraic approach to promise constraint satisfaction. *J. ACM*, 68(4):28:1–28:66, 2021.

[6] L. Barto, M. Kompatscher, M. Olšák, T. V. Pham, and M. Pinsker. The equivalence of two dichotomy conjectures for infinite domain constraint satisfaction problems. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science – LICS'17*, 2017. Preprint arXiv:1612.07551.

[7] L. Barto, M. Kozik, and T. Niven. The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell). *SIAM Journal on Computing*, 38(5), 2009.

[8] L. Barto, A. A. Krokhin, and R. Willard. Polymorphisms, and how to use them. In *The Constraint Satisfaction Problem: Complexity and Approximability*, pages 1–44. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.

[9] L. Barto, J. Opršal, and M. Pinsker. The wonderland of reflections. *Israel Journal of Mathematics*, 223(1):363–398, 2018.

[10] L. Barto and M. Pinsker. The algebraic dichotomy conjecture for infinite domain constraint satisfaction problems. In *Proceedings of the 31th Annual IEEE Symposium on Logic in Computer Science – LICS'16*, pages 615–622, 2016. Preprint arXiv:1602.04353.

[11] M. Bodirsky. *Complexity of Infinite-Domain Constraint Satisfaction*. Lecture Notes in Logic (52). Cambridge University Press, Cambridge, United Kingdom; New York, NY, 2021.

[12] M. Bodirsky and B. Bodor. Canonical polymorphisms of Ramsey structures and the unique interpolation property. In *Proceedings of the Symposium on Logic in Computer Science (LICS)*, 2021.

[13] M. Bodirsky and V. Dalmau. Datalog and constraint satisfaction with infinite templates. *Journal on Computer and System Sciences*, 79:79–100, 2013. A preliminary version appeared in the proceedings of the Symposium on Theoretical Aspects of Computer Science (STACS'05).

[14] M. Bodirsky, F. R. Madelaine, and A. Mottet. A proof of the algebraic tractability conjecture for monotone monadic SNP. *SIAM J. Comput.*, 50(4):1359–1409, 2021.

[15] F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein, and M. Zhong. Three-Coloring and List Three-Coloring of Graphs Without Induced Paths on Seven Vertices. *Combinatorica*, 38:779–801, 2018.

[16] R. C. Brewster and T. Graves. On the restricted homomorphism problem. *Discrete Applied Mathematics*, 156(14):2823–2826, 2008.

[17] R. C. Brewster, P. Hell, and G. MacGillivray. The complexity of restricted graph homomorphisms. *Discrete Mathematics*, 167-168:145–154, 1997. Selected Papers 15th British Combinatorial Conference.

[18] A. A. Bulatov. A dichotomy theorem for nonuniform CSPs. In *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17*, pages 319–330, 2017.

[19] C. Dallard, M. Milanič, and K. Štorgel. Treewidth versus clique number. i. graph classes with a forbidden structure. *SIAM J. Discret. Math.*, 35(4):2618–2646, Jan. 2021.

[20] V. Dalmau and A. A. Krokhin. Majority constraints have bounded pathwidth duality. *Eur. J. Comb.*, 29(4):821–837, 2008.

[21] R. Dechter. From local to global consistency. *Artificial Intelligence*, 55(1):87–108, 1992.

[22] P. Erdős. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

[23] T. Feder and M. Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. *SIAM Journal on Computing*, 28:57–104, 1999.

[24] J. Foniok. Homomorphisms and structural properties of relational systems. Doctoral Thesis, Charles University Prague, Faculty of Mathematics and Physics, 2007.

[25] J. Foniok, J. Nešetřil, and C. Tardif. Generalised dualities and maximal finite antichains in the homomorphism order of relational structures. *European J. Combin.*, 29(4):881–899, 2008.

[26] J. Goedgebeur, J. Jooken, K. Okrasa, P. Rzażewski, and O. Schaudt. Minimal Obstructions to C_5-Coloring in Hereditary Graph Classes. In R. Královič and A. Kučera, editors, *49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024)*, volume 306 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 55:1–55:15, Dagstuhl, Germany, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[27] P. A. Golovach, M. Johnson, D. Paulusma, and J. Song. A survey on the computational complexity of coloring graphs with forbidden subgraphs. *J. Graph Theory*, 84(4):331–363, 2017.

[28] S. Guzmán-Pro. GMSNP and finite structures. *arXiv:2406.13529*, 2024.

[29] T. R. Hartinger and M. Milanič. Partial characterizations of 1-perfectly orientable graphs. *Journal of Graph Theory*, 85(2):378–394, 2017.

[30] P. Hell and A. Mishra. H-coloring degree-bounded (acyclic) digraphs. *Theoretical Computer Science*, 554:40–49, 2014.

[31] P. Hell and J. Nešetřil. On the complexity of H-coloring. *Journal of Combinatorial Theory, Series B*, 48:92–110, 1990.

[32] S. Huang. Improved complexity results on k-coloring. *European J. Combin.*, 51:336–346, 2016.

[33] A. Krokhin, J. Opršal, M. Wrochna, and S. Živný. Topology and adjunction in promise constraint satisfaction. *SIAM J. Comput.*, 52(1):38–79, 2023.

[34] G. Kun. Constraints, MMSNP, and expander relational structures. *Combinatorica*, 33(3):335–347, 2013.

[35] J. Nešetřil and C. Tardif. Duality theorems for finite structures. *Journal of Combinatorial Theory, Series B*, 80:80–97, 2000.

[36] M. Pinsker. Current challenges in infinite-domain constraint satisfaction: Dilemmas of the infinite sheep. In *52nd IEEE International Symposium on Multiple-Valued Logic, ISMVL 2022, Dallas, TX, USA, May 18-20, 2022*, pages 80–87, 2022.

[37] A. L. Selman. Promise problems for complexity classes. *Information and Computation*, 78:87–98, 1988.

[38] D. Zhuk. A proof of the CSP dichotomy conjecture. *J. ACM*, 67(5):30:1–30:78, 2020.

[39] D. N. Zhuk. A proof of CSP dichotomy conjecture. In *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17*, pages 331–342, 2017. https://arxiv.org/abs/1704.01914.