Light Spanners with Small Hop-Diameter

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Abstract

Lightness, *sparsity*, and *hop-diameter* are the fundamental parameters of geometric spanners. Arya et al. [STOC'95] showed in their seminal work that there exists a construction of Euclidean $(1+\varepsilon)$-spanners with hop-diameter $O(\log n)$ and lightness $O(\log n)$. They also gave a general tradeoff of hop-diameter k and sparsity $O(\alpha_k(n))$, where α_k is a very slowly growing inverse of an Ackermann-style function. The former combination of logarithmic hop-diameter and lightness is optimal due to the lower bound by Dinitz et al. [FOCS'08]. Later, Elkin and Solomon [STOC'13] generalized the light spanner construction to doubling metrics and extended the tradeoff for more values of hop-diameter k. In a recent line of work [SoCG'22, SoCG'23], Le et al. proved that the aforementioned tradeoff between the hop-diameter and sparsity is tight for every choice of hop-diameter k. A fundamental question remains: *What is the optimal tradeoff between the hop-diameter and lightness for every value of* k?

In this paper, we present a general framework for constructing light spanners with small hop-diameter. Our framework is based on *tree covers*. In particular, we show that if a metric admits a tree cover with γ trees, stretch t, and lightness L, then it also admits a t-spanner with hop-diameter k and lightness $O(kn^{2/k} \cdot \gamma L)$. Further, we note that the tradeoff for trees is tight due to a construction in uniform line metric, which is perhaps the simplest tree metric. As a direct consequence of this framework, we obtain a tight tradeoff between lightness and hop-diameter for doubling metrics in the entire regime of k.

1 Introduction

Let $M_X = (X, \delta_X)$ be a finite metric space, which can be viewed as a complete graph with vertex set X, where the weight of each edge $(u,v) \in \binom{X}{2}$ is equal to the metric distance between its endpoints, $\delta_X(u,v)$. Let $t \geq 1$ be a real parameter and let $H = (X, E)$ be a subgraph of M_X such that $E \subseteq \binom{X}{2}$. We say that H is a *t-spanner* for M_X, if for every two points u and v in X, it holds that $\delta_H(u,v) \leq t \cdot \delta_X(u,v)$, where $\delta_H(u,v)$ denotes the length of the shortest path between u and v in H. Such a path is called t-spanner path and parameter $t \geq 1$ is called the *stretch factor* (shortly, *stretch*) of H.

Spanners for Euclidean metric spaces (henceforth Euclidean spanners) are fundamental geometric structures with numerous applications, such as topology control in wireless networks [SVZ07], efficient regression in metric spaces [GKK17], approximate distance oracles [GLNS08], and more.

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Rao and Smith [RS98] showed the relevance of Euclidean spanners in the context of other geometric NP-hard problems, e.g., Euclidean traveling salesman problem and Euclidean minimum Steiner tree problem. Intensive ongoing research is dedicated to exploring diverse properties of Euclidean spanners; see [ADD+93, ADM+95, BKK+25, BT21a, BT21b, BT22, BCH+25, CG09, DNS95a, GLN02, KG92, ES15, RS98, LS19]. In fact, several distinct constructions have been developed for Euclidean spanners over the years, such as well-separated pair decomposition (WSPD) based spanners [Cal93, GLN02], skip-list spanners [AMS94], path-greedy and gap-greedy spanners [ADD+93, AS97], and many more; each such construction found further applications in various geometric optimization problems. For an excellent survey of results and techniques on Euclidean spanners, we refer to the book titled "Geometric Spanner Networks" by Narasimhan and Smid [NS07], and the references therein.

Besides having a small stretch, perhaps the most basic property of a spanner is its *sparsity*, defined as the number of edges in the spanner divided by $n-1$, which is the size of an MST of the underlying n-point metric. Chew [Che86] was the first to show that there exists an Euclidean spanner with constant sparsity and stretch $\sqrt{10}$. Later, Keil and Gutwin [KG92] showed that the Delaunay triangulation is in fact a 2.42-spanner with constant sparsity. Clarkson [Cla87] designed the first Euclidean $(1+\varepsilon)$-spanner for $\mathbb{R}^2$ with sparsity $O(1/\varepsilon)$, for an arbitrary small $\varepsilon > 0$; an alternative algorithm was presented by Keil [Kei88]. These two papers [Cla87, Kei88] introduced the so-called Θ-graph as a new tool for designing $(1+\varepsilon)$-spanners with sparsity $O(1/\varepsilon)$ in $\mathbb{R}^2$. Ruppert and Seidel [RS91] later generalized the Θ-graph to any constant dimension d, showing that one can construct a $(1+\varepsilon)$-spanner with sparsity $O(\varepsilon^{-d+1})$. Recently, Le and Solomon [LS19] showed that this bound is tight.

Another fundamental and extensively studied property of a spanner is its *lightness*, defined as the ratio of the sum of the edge weights of the spanner to the weight of the MST of the underlying metric. Das et al. [DHN93] showed that the "greedy spanner", introduced by Althöfer et al. [ADD+93], has a constant lightness and stretch $1+\varepsilon$ in $\mathbb{R}^3$, for any constant $\varepsilon > 0$. This was generalized later by Das et al. [DNS95b] to $\mathbb{R}^d$, for all $d \in \mathbb{N}$. Rao and Smith [RS98] showed in their seminal work that the greedy spanner with stretch $1+\varepsilon$ has lightness $\varepsilon^{-O(d)}$ in $\mathbb{R}^d$ for every constant ε and d. After a long line of work, finally in 2019, Le and Solomon [LS19] improved the lightness bound of the greedy spanner to $O(\varepsilon^{-d}\log\varepsilon^{-1})$ in $\mathbb{R}^d$. For metrics with doubling dimension[1] d, Borradaile et al. [BLWN19] showed that the greedy spanner with stretch $1+\varepsilon$ has lightness $\varepsilon^{-2\cdot d}$ (see also [Got15] for an earlier work).

Besides having small *stretch* and *sparsity*, a spanner often possesses additional valuable properties of the underlying metric. One such critical property is the *hop-diameter*: a t-spanner for M_X has hop-diameter of k if, for any two points $u, v \in X$, there is a t-spanner path between u and v with at most k edges (or *hops*). Already in 1994, Arya et al. [AMS94] proposed a construction of Euclidean $(1+\varepsilon)$-spanners with logarithmic hop-diameter and a constant sparsity. In a subsequent work, Arya et al. [ADM+95] showed that there exists a construction of Euclidean $(1+\varepsilon)$-spanners with hop-diameter $O(\log n)$ and lightness $O(\log n)$. The same paper gives a general tradeoff of hop-diameter k and sparsity $O(\alpha_k(n))$ for $(1+\varepsilon)$-spanners, where α_k is an extremely slowly growing inverse of an Ackermann-style function (see also [BTS94, Sol13]). The former tradeoff of hop-diameter versus lightness is optimal due to the lower bound by Dinitz et al. [DES10]. Later, in

[1] The doubling dimension of a metric space (X, δ_X) is the smallest value d such that every ball B in the metric space can be covered by at most 2^d balls of half the radius of B. A metric space is called doubling if its doubling dimension is constant.

2015, Elkin and Solomon [ES15] presented a light $(1+\varepsilon)$ spanner construction for doubling metrics and gave a tradeoff between the hop-diameter k and lightness for more values of k. Recent works by Le et al. [LMS22, LMS23] showed that the tradeoff of hop-diameter k and sparsity $O(\alpha_k(n))$ is asymptotically optimal, for every value of $k \geq 1$. However, despite a plethora of results on tradeoffs between *lightness* and *hop-diameter*, the following question remained open.

Question 1. *Given a set of points in $\mathbb{R}^d$, what is the optimal tradeoff between lightness and hop-diameter k, for every value of k?*

A notion arguably stronger than that of a *spanner* is a *tree cover*. For a metric space $M_X = (X, \delta_X)$ let $T = (V_T, E_T)$ be a tree with $X \subseteq V_T$. We say that the tree T is *dominating*, if for every $u, v \in X$, it holds that $\delta_T(u,v) \geq \delta_X(u,v)$. A *tree cover* with *stretch* t is a collection of dominating trees such that for every pair of vertices $u, v \in X$, there exists a tree T in the collection with $\delta_T(u,v) \leq t \cdot \delta_G(u,v)$. The *size* of a tree cover is the number of trees in it. The lightness of a tree in the cover is the ratio of its weight to the weight of an MST of the underlying metric space. The *lightness* of a tree cover is the maximum lightness among the trees in the cover. We use L-light (γ, t)-tree cover to denote a tree cover with lightness L, γ trees, and stretch t. Clearly, the union of all the trees in a L-light (γ, t)-tree cover constitutes a t-spanner with sparsity bounded by $O(\gamma)$ and lightness bounded by $O(\gamma L)$.

The aforementioned tradeoff between hop-diameter and sparsity for Euclidean $(1+\varepsilon)$-spanners by Arya et al. [ADM+95] was in fact achieved via tree covers. Their celebrated *"Dumbbell Theorem"* demonstrated that in $\mathbb{R}^d$, any set of points admits a tree cover of stretch $1+\varepsilon$ that uses only $O(\varepsilon^{-d} \cdot \log(1/\varepsilon))$ trees. Later, Bartal et el. [BFN22] generalized this theorem for doubling metrics. The bound on the tree cover size of [ADM+95] was recently improved by Chang et al. [CCL+24a] by a factor of $1/\varepsilon$. This is tight up to a logarithmic factor in $(1/\varepsilon)$ due to the lower bound on the sparse Euclidean spanners by Le and Solomon [LS19]. There are other tree cover constructions for doubling metrics [CGMZ16, BFN22], and for other metrics, such as planar and minor-free [BFN22, CCL+23, CCL+24b, GKR05, KLMN04] and general [BFN22, MN07, NT12].[2]

1.1 Our contributions

We present a general framework for constructing light spanners with small hop-diameter. Our starting point is the construction of a light 1-spanner with a bounded hop-diameter for tree metrics. The bounds are summarized in the following theorem, with the proof presented in Section 2.

Theorem 1.1. *For every $n \geq 1$, every $k \geq 1$, and every metric M_T induced by an n-vertex tree T, there is a 1-spanner for M_T with hop-diameter k and lightness $O(kn^{2/k})$.*

In order to go from tree metrics to arbitrary metrics, we rely on tree cover theorems. This reduction is summarized in the following corollary, with the proof provided in Section 4.

Corollary 1.2. *Let $n \geq 1$ be an arbitrary integer and let M_X be an arbitrary metric space with n points. If M_X admits an L-light (γ, t)-tree cover, then for any $k \geq 1$, the metric space M_X has a t-spanner with hop-diameter k and lightness $O(\gamma \cdot L \cdot k \cdot n^{2/k})$.*

[2] Metric induced by a graph $G = (V, E)$ is a metric space with point set V, where for every $u, v \in V$, their distance in the metric is equal to the shortest path distance between u and v in G, denoted by $\delta_G(u,v)$.

We note that the reduction from Corollary 1.2 holds for any metric space which admits a light tree cover. To exemplify the reduction, we focus on doubling metrics and provide a general tradeoff between hop-diameter and lightness for $(1+\varepsilon)$-spanners in the following theorem. (See Section 4 for the proof.)

Theorem 1.3. *For every $k \geq 1$ and $\varepsilon \in (0,1)$, every n-point metric space with doubling dimension d has a $(1+\varepsilon)$-spanner with hop-diameter k and lightness $O(\varepsilon^{-O(d)} \cdot kn^{2/k})$.*

Next, we compare our result with the aforementioned upper bound by Elkin and Solomon [ES15]. Since both constructions have a term $\varepsilon^{-O(d)}$ in lightness, we ignore those dependencies for clarity. The aforementioned construction of Elkin and Solomon [ES15] achieves hop-diameter $O(\log_\rho n + \alpha(\rho))$ and lightness $O(\rho \cdot \log_\rho n)$, for a parameter $\rho \geq 2$. In other words, the construction has hop-diameter $k' + O(\alpha(\rho))$ and lightness $k' n^{c/k'}$ for $k' \geq 1$, $\rho = n^{c/k'}$ and a constant c. Note that this tradeoff does not include values in the regime where hop-diameter is $o(\alpha(n))$. Namely, when $k' = O(\alpha(n))$, then $\rho = n^{\Omega(1/\alpha(n))}$ and the dominant term in hop-diameter is $\alpha(\rho) = \Theta(\alpha(n))$. In addition, the exponent $O(1/k)$ of n in the lightness is not asymptotically tight. Recall that the tradeoff we achieve is hop-diameter k versus lightness of $O(kn^{2/k})$, which holds for every value of $1 \leq k \leq n$. In other words, we give a fine-grained tradeoff for every value of k while nailing down the correct exponent of n, due to the lower bound we discuss next.

We complement our constructions with a lower bound for the n-point *uniform line metric*, which is a set of points on $[0,1]$ with coordinates i/n for $0 \leq i \leq n-1$. Refer to Section 3 for the proof.

Theorem 1.4. *For every $n \geq 1$, every k such that $1 \leq k \leq n$, any spanner with hop-diameter k for the n-point uniform line metric must have lightness $\Omega(kn^{2/k})$.*

Note that, the lower bound holds for any value of stretch $t \geq 1$. Moreover, Dinitz et al. [DES10] previously obtained similar bounds, using a more intricate analysis based on linear programming method and connections to the so-called minimum linear arrangement problem. On the other hand, our proof is based on a rather simple combinatorial argument described in less than two pages. We believe such a simple combinatorial argument will have further implications in designing lower bounds for other related problems. The purpose of the lower bound is two-fold. First, since the uniform line metric is conceivably the simplest tree metric, we conclude that our upper bound for tree metrics (Theorem 1.1) is tight. In other words, our reduction (Corollary 1.2) is lossless. Second, the uniform line metric is also a doubling metric, which means that the tradeoff for the doubling metric (Theorem 1.3) of hop-diameter k and lightness $O(kn^{2/k})$ is tight.

2 Upper bound for tree metrics

This section is dedicated to proving Theorem 1.1.

Theorem 1.1. *For every $n \geq 1$, every $k \geq 1$, and every metric M_T induced by an n-vertex tree T, there is a 1-spanner for M_T with hop-diameter k and lightness $O(kn^{2/k})$.*

We first describe the spanner construction in Algorithm 1. The construction uses the following well-known result.

Lemma 2.1 ([FL22]). *Given any integer parameter $\ell > 0$ and an n-vertex tree T, there is a subset X of at most $\frac{2n}{\ell+1} - 1$ vertices such that every connected component of $T \setminus X$ has at most ℓ vertices and at most two outgoing edges towards X.*

Let $n \leftarrow |V(T)|$. If $n \le k$, return the edge set of T. If $k = 1$, return the clique on $V(T)$. When $k = 2$, let c be the centroid of T, i.e., a vertex such that every component in $T \setminus \{c\}$ has size at most $n/2$. Let F be the returned set of edges, initialized to an empty set. For every vertex v in $V(T) \setminus \{c\}$, add to F the edge (c, v) of weight $\delta_T(c, u)$. Recurse with $k = 2$ on each of the subtrees of $T \setminus \{c\}$. When $k \ge 3$, let $\ell = k$ if $n \le 2k^2$, and $\ell = \lfloor 2n^{2/k} \rfloor$ otherwise. Let X be the subset of V guaranteed by Lemma 2.1 with parameter ℓ. Let $T_1, \ldots, T_g$ be the components of $T \setminus X$, each of which neighbors up to two vertices of X and each of which has size at most ℓ. For every connected component T_i of $T \setminus X$, let u_i and v_i be the vertices in X neighboring T_i. (It is possible that there is only one such vertex, but that case is handled analogously.) For each vertex w in T_i, add to F the edge (u_i, w) of weight $\delta_T(u_i, w)$ and (v_i, w) of weight $\delta_T(v_i, w)$. Recurse on T_i with parameter k. Consider a new tree T_X with a vertex set X and the edge set E_X consisting of: (1) all the edges in E with both endpoints in X and (line 25 in Algorithm 1); (2) edges (u_i, v_i) for every component T_i which has two neighbors in X (line 34 in Algorithm 1). Continue recursively for T_X and hop-diameter $k - 2$. This concludes the description of the algorithm.

Lemma 2.2 asserts that the constructed spanner has stretch 1 and hop-diameter k. Lemma 2.3 asserts that the constructed spanner has lightness $O(kn^{2/k})$.

Lemma 2.2. *For every $k \ge 1, n \ge 1$ and every metric M_T induced by an n-vertex tree T, procedure* SPANNER*(k, T) returns a 1-spanner of M_T with hop-diameter k.*

Proof. When $n \le k$, the tree T already has hop-diameter k. If $k = 1$, the procedure construct a clique on T and the lemma holds immediately. (Recall that every edge in F has weight equal to the distance of its endpoints in T.)

Next we analyze the case $k = 2$. Consider two vertices u and v in T and consider the last recursion level where both u and v were in the same tree, T'. The centroid vertex c' of T' is connected via an edge to both u and v. Vertex c' is on the shortest path in T' between u and v, because after the removal of c', vertices u and v are not in the same subtree anymore by the choice of T'. Hence, there is a 2-hop path between u and v, consisting of edges (u, c') and (c', v). By construction, the weight of this path is $\delta_T(u, c') + \delta_T(c', v) = \delta_T(u, v)$, where the equality holds because c' is on the shortest path between u and v.

It remains to analyze the case $k \ge 3$. Consider two vertices u and v in T and consider the last recursion level where both u and v were in the same tree, T'. Let X' be the subset of $V(T')$ that is used in the construction to split T into connected components. Let T_u and T_v be the connected components containing u and v, respectively. By the choice of T', the components T_u and T_v are different. Let u' and v' be the vertices in X' such that u' is neighboring T_u, v' is neighboring T_v and u' and v' lie on the shortest path between u and v. Such a vertex u' exists because all the shortest paths stemming from T_u and going outside of T_u contain one of the at most two vertices in X that neighbors T_u. The argument is analogous for v'. From the construction, we know that the constructed spanner contains edges (u, u') and (v', v). Recall that the vertices in X are connected recursively using a construction for hop-diameter $k - 2$. This means that there is a path between u and v with at most $k - 2 + 2 = k$ hops. (It is possible that $u' = v'$, but this case is handled similarly.) The stretch of the path between u' and v' is 1 by the induction hypothesis. The weights of edges (u, u') and (v, v') correspond to the underlying distances of their endpoints in T. Since u' and v' lie on the shortest path between u and v in T, the weight of the spanner path between u and v is equal to their distance in T. □

Lemma 2.3. *For every $k \geq 1, n \geq 1$ and every metric M_T induced by an n-vertex tree T of weight L, procedure* SPANNER*(k, T) returns a spanner H_k of M_T with weight $W_k(n, L) = O(kn^{2/k}L)$.*

The lemma is implied by the following claims.

Claim 2.4. *For every $1 \leq n \leq k$, $W_k(n, L) \leq L$.*

Proof. The claim is true because the algorithm returns the edge set of T in this case. □

Claim 2.5. *For every $n \geq 1$, $W_1(n, L) \leq \frac{n^2 L}{2}$.*

Proof. The claim is true because the algorithm returns the clique on the n vertices of V. Each edge in the clique has weight at most L. The total weight is thus at most $\binom{n}{2}L \leq \frac{n^2 L}{2}$. □

Claim 2.6. *For every $n \geq 1$, $W_2(n, L) \leq nL$.*

Proof. We use L_i to denote the weight of T_i plus the weight of the edge connecting T_i to c. Clearly, $L = \sum_{i=1}^{g} L_i$. From the construction we have that c is connected by an edge to each of the vertices of $T \setminus \{c\}$. The weight of these edges can be upper bounded by $\sum_{i=1}^{g} n_i L_i$, since for each $i \in [g]$, the edge between c and a vertex in T_i has weight of at most L_i. The construction proceeds recursively on each T_i, and the total weight incurred by recursion is at most $\sum_{i=1}^{g} W_2(n_i, L_i)$. We proceed to upper bound $W_2(n, L)$ inductively.

$$\begin{aligned}
W_2(n, L) &\leq \sum_{i=1}^{g} n_i L_i + \sum_{i=1}^{g} W_2(n_i, L_i) \\
&\leq \sum_{i=1}^{g} n_i L_i + \sum_{i=1}^{g} n_i L_i && \text{induction hypothesis} \\
&= 2\sum_{i=1}^{g} n_i L_i \\
&\leq 2\sum_{i=1}^{g} \frac{n}{2} L_i \\
&\leq nL
\end{aligned}$$

□

Claim 2.7. *Consider an invocation of* SPANNER*(k, T) for $k \geq 3$ and let L_i be the weight of T_i plus the weight of the (at most two) edges connecting T_i to X. Then, $W_k(n, L) \leq 2\ell L + W_{k-2}\left(\frac{2n}{\ell+1}, L\right) + \sum_{i=1}^{g} W_k(\ell, L_i)$.*

Proof. Clearly, $L = \sum_{i=1}^{g} L_i$. Recall that from Lemma 2.1, we have $|X| \leq \frac{2n}{\ell+1} - 1 < \frac{2n}{\ell+1}$. For each T_i, $1 \leq i \leq g$, the spanner construction adds an edge between each $v \in T_i$ and (at most two) vertices from X which neighbor T_i. The total weight of these edges is at most $2\sum_{i=1}^{g} n_i L_i \leq 2\ell \sum_{i=1}^{g} L_i = 2\ell L$. The first inequality holds because each subtree has at most $n_i \leq \ell$ vertices. In addition, the vertices in X are connected using a construction with hop-diameter $k-2$. The total weight of the edges used is at most $W_{k-2}(|X|, L) \leq W_{k-2}\left(\frac{2n}{\ell+1}, L\right)$. Finally, each of the components T_i is handled inductively and this contributes at most $\sum_{i=1}^{g} W_k(n_i, L_i) \leq \sum_{i=1}^{g} W_k(\ell, L_i)$ to the weight. □

Claim 2.8. *For every $n \geq 1$, $W_3(n, L) \leq 16n^{2/3}L$.*

Proof. When $k = 3$, parameter ℓ is set to $\lfloor n^{2/3} \rfloor$.

$$\begin{aligned}
W_3(n, L) &\le 2\ell L + W_1\left(\frac{2n}{\ell+1}, L\right) + \sum_{i=1}^{g} W_3(\ell, L_i) \\
&\le 2n^{2/3}L + W_1(2n^{1/3}, L) + \sum_{i=1}^{g} W_3(\ell, L_i) \\
&\le 4n^{2/3}L + \sum_{i=1}^{g} W_3(\ell, L_i) \\
&\le 4n^{2/3}L + 16\ell^{2/3}L && \text{induction hypothesis} \\
&\le 4n^{2/3}L + 16n^{4/9}L \\
&\le 16n^{2/3}L
\end{aligned}$$

The last inequality holds for every $n \ge 4$. □

Claim 2.9. *For every $1 \le n \le 16$, $W_k(n, L) \le 9nL$.*

Proof. When $k = 1$, we have $W_1(n, L) \le \frac{n^2}{2} \cdot L \le 8nL$. When $k = 2$, we have $W_2(n, L) \le nL$. For $k = 3$ and $1 \le n \le 3$, by Claim 2.4 we have $W_3(n, L) \le L$. It is straightforward to verify the bound for $k = 3$ and $n \in \{4, 5\}$. For $k = 3$ and $n \ge 6$, the bound is implied by Claim 2.8 because $16n^{2/3} \le 9n$. Finally, when $k \ge 4$, parameter ℓ is set to k and the upper bound on $W_k(n, L)$ is obtained as follows.

$$\begin{aligned}
W_k(n, L) &\le 2kL + W_{k-2}\left(\frac{2n}{k}, L\right) + \sum_{i=1}^{g} W_k(k, L_i) \\
&\le 2kL + 9 \cdot \frac{2n}{k} \cdot L + L \\
&\le 2nL + \frac{18n}{3} \cdot L + L \\
&\le 9nL
\end{aligned}$$

□

Claim 2.10. *For every $1 \le k < n \le 8k$, $W_k(n, L) \le 39kL$.*

Proof. When $k = 1$, we have $W_1(n, L) \le \frac{n^2}{2} \cdot L \le \frac{(8k)^2}{2} \cdot L \le 32kL$. When $k = 2$, we have $W_2(n, L) \le nL \le 8kL$. When $k = 3$, we have $W_3(n, L) \le 16n^{2/3}L \le 16 \cdot (8k)^{2/3}L \le 64kL$. When $k \ge 4$, we have that $8k \le 2k^2$ and so $\ell = k$.

$$\begin{aligned}
W_k(n, L) &\le 2kL + W_{k-2}\left(\frac{2n}{k}, L\right) + \sum_{i=1}^{g} W_k(k, L_i) \\
&\le 2kL + W_{k-2}(16, L) + L \\
&\le 2kL + 9 \cdot 16L + L \\
&\le 39kL
\end{aligned}$$

□

Claim 2.11. *For every $k \ge 1$ and $1 \le n \le 2k^2$, $W_k(n, L) \le 41kL$.*

Proof. When $k = 1$, we have $W_1(n, L) = \frac{n^2}{2} \cdot L \le \frac{(2k^2)^2}{2} \cdot L \le 2kL$. When $k = 2$, we have $W_2(n, L) \le nL \le 2k^2 L \le 4kL$. When $k = 3$, we have $W_3(n, L) \le 16n^{2/3}L \le 16 \cdot (2k^2)^{2/3}L \le 38kL$. When $k \ge 4$, we have $\ell = k$.

$$\begin{aligned} W_k(n, L) &\le 2kL + W_{k-2}\left(\frac{2n}{k}, L\right) + \sum_{i=1}^{g} W_k(k, L_i) \\ &\le 2kL + W_{k-2}(4k, L) + L \\ &\le 2kL + 39(k-2)L + L \\ &\le 41kL \end{aligned}$$

□

Claim 2.12. *For all $k \ge 4$ and $n \ge 2k^2$, $W_k(n, L) \le ckn^{2/k}$, for an absolute constant c.*

Proof. We have $\ell = \lfloor 2n^{2/k} \rfloor$.

$$\begin{aligned} W_k(n, L) &\le 2\ell L + W_{k-2}\left(\frac{2n}{\ell + 1}, L\right) + \sum_{i=1}^{g} W_k(\ell, L_i) \\ &\le 4n^{2/k}L + W_{k-2}\left(n^{\frac{k-2}{k}}, L\right) + \sum_{i=1}^{g} W_k\left(2n^{2/k}, L_i\right) \end{aligned}$$

Case 1: $n < (k/2)^{k/2}$. Rearranging, we have that $2n^{2/k} < k$. This means that $W_k\left(2n^{2/k}, L_i\right) \le L_i$.

$$\begin{aligned} W_k(n, L) &\le 4n^{2/k}L + W_{k-2}\left(n^{\frac{k-2}{k}}, L\right) + \sum_{i=1}^{g} W_k\left(2n^{2/k}, L_i\right) \\ &\le 4n^{2/k}L + c(k-2)n^{2/k}L + L \\ &\le ckn^{2/k}L \end{aligned}$$

The last inequality holds for any $c \ge 5/2$.
Case 2: $(k/2)^{k/2} \le n < k^k$. Rearranging, we have that $2n^{2/k} < k^2$.

$$\begin{aligned} W_k(n, L) &\le 4n^{2/k}L + W_{k-2}\left(n^{\frac{k-2}{k}}, L\right) + \sum_{i=1}^{g} W_k\left(2n^{2/k}, L_i\right) \\ &\le 4n^{2/k}L + c(k-2)n^{2/k}L + \sum_{i=1}^{g} 41kL_i \\ &\le ckn^{2/k}L \end{aligned}$$

The last inequality is true for any $c \geq 23$. *Case 3:* $k^k \leq n$

$$
\begin{aligned}
W_k(n, L) &\leq 4n^{2/k}L + W_{k-2}\left(n^{\frac{k-2}{k}}, L\right) + \sum_{i=1}^{g} W_k\left(2n^{2/k}, L_i\right) \\
&\leq 4n^{2/k}L + c \cdot (k-2) \cdot n^{2/k}L + \sum_{i=1}^{g} ck(2n^{2/k})^{2/k}L_i \\
&\leq 4n^{2/k}L + c \cdot (k-2) \cdot n^{2/k}L + ck \cdot 2^{2/k} \cdot n^{4/k^2}L \\
&= n^{2/k}L\left(4 + c(k-2) + ck \cdot 2^{2/k} \cdot n^{\frac{4-2k}{k^2}}\right) \\
&\leq n^{2/k}L\left(4 + c(k-2) + c\sqrt{2}\right) \\
&\leq ckn^{\frac{2}{k}}L
\end{aligned}
$$

The penultimate inequality holds because $k \cdot 2^{2/k} \cdot n^{\frac{4-2k}{k^2}} \leq \sqrt{2}$ for $k \geq 4$ and $n \geq k^k$. The last inequality holds for any $c \geq 7$. $\square$

3 Lower bound on the uniform line metric

This section is dedicated to proving Theorem 1.4, restated here for convenience.

Theorem 1.4. *For every $n \geq 1$, every k such that $1 \leq k \leq n$, any spanner with hop-diameter k for the n-point uniform line metric must have lightness $\Omega(kn^{2/k})$.*

Due to the inductive nature of the proofs, we consider a generalization of the uniform line metric.

Definition 3.1. *Let $n \geq 1$ and $1 \leq p \leq n$ be arbitrary integers. A (p, n) line metric is a set of p points on $[0, 1]$ such that for every $0 \leq i \leq n-1$, the interval $[i/n, (i+1)/n)$ contains at most one point.*

The uniform line metric with n points is an (n, n) line metric. The proof of Theorem 1.4 follows from Lemma 3.2 for $k = 1$, Lemma 3.3 for $k = 2$, and Lemma 3.4 for $k \geq 3$.

Lemma 3.2. *For every n and p such that $1 \leq p \leq n$, let M be an arbitrary (p, n) line metric. Then, any spanner with hop-diameter 1 for M has lightness at least $W_1(p, n) \geq \frac{1}{64}\left(\frac{p^2}{n}\right)^2$.*

Proof. Partition M into four consecutive parts, M_1, M_2, M_3, and M_4, each consisting of $p/4$ points. The distance between a point in M_1 and a point in M_4 is at least $\frac{p}{4} \cdot \frac{1}{n}$ and there is at least $(p/4)^2$ such pairs. Since every pair requires a direct edge, the total weight these edges incur is at least $\frac{p}{4n} \cdot \frac{p^2}{16} \geq \frac{1}{64}\left(\frac{p^2}{n}\right)^2$. $\square$

Lemma 3.3. *For every n and p such that $1 \leq p \leq n$, let M be an arbitrary (p, n) line metric. Then, any spanner with hop-diameter 2 for M has lightness at least $W_2(p, n) \geq \frac{1}{16} \cdot \frac{p^2}{n}$.*

Proof. Partition M into four consecutive parts, M_1, M_2, M_3, and M_4, each consisting of $p/4$ points. Consider two complementary cases. First, if every point in M_1 is incident on an edge that goes to either M_3 or M_4, the total weight of these edges is at least $\frac{p}{4} \cdot \frac{p}{4n}$. In the complementary case, there

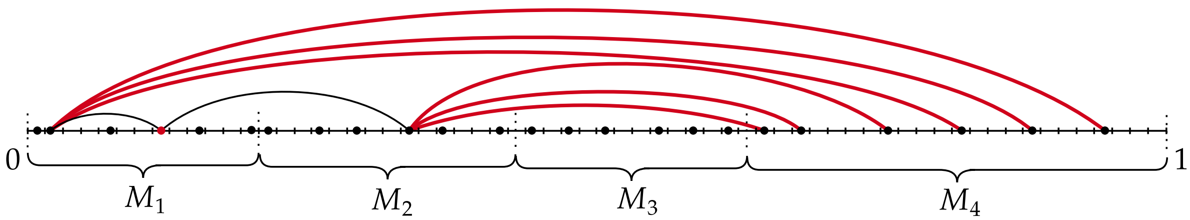

Figure 1: An illustration of the proof of the lower bound for $k = 2$ (Lemma 3.3). There is a vertex $p \in M_1$ (highlighted in red) which is not incident on any edge going to M_3 or M_4. For every point $q \in M_4$, a 2-hop path from p to q induces a long edge, highlighted in red.

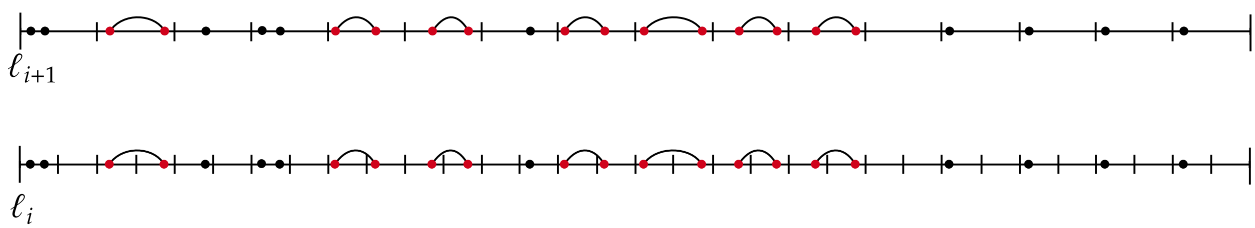

Figure 2: Monotonicity of the number of global points as we increase the interval size from ℓ_i to ℓ_{i+1}. The points highlighted in red were global with respect to ℓ_i and became non-global with respect to ℓ_{i+1}.

is a point a in M_1 that is not incident on any edge that goes to M_3 or M_4. Consider an arbitrary point b in M_4. A 2-hop path between a and b has to have the first edge between a and a point in $M_1 \cup M_2$ and the second edge between $M_1 \cup M_2$ and b. The weight of the second edge is at least $p/(4n)$. Since every point in M_4 induces a different edge, the total weight is at least $\frac{p}{4} \cdot \frac{p}{4n}$. (See Figure 1 for an illustration.) In conclusion, both of the cases require total weight of $\frac{p}{n} \cdot \frac{p}{16}$ and the lower bound follows. $\square$

Lemma 3.4. *For every n and p such that $1 \le p \le n$, let M be an arbitrary (p,n) line metric. Then, for any $k \ge 3$, any spanner with hop-diameter k for M has lightness at least $W_k(p,n) \ge ck\left(\frac{p^2}{n}\right)^{2/k}$, for $c = 1/73728$.*

Proof. Let H_k be an arbitrary spanner with hop-diameter k of M. Partition M into consecutive intervals, each containing ℓ points where the value of ℓ will be set later. We call an edge *global* if it has endpoints in two different intervals. We call a point *global* if it is incident on a global edge and *non-global* otherwise.

Claim 3.5. *If there are γn global points for $\gamma \in [0,1]$, then they contribute at least $\frac{\gamma^2 \ell}{16}$ to the total lightness of H_k.*

Proof. Consider an arbitrary interval and recall that it has length ℓ/n and consists of ℓ points. The interval contains at most $\frac{\gamma}{4} \cdot \ell$ points that are at distance at most $\frac{\gamma}{4} \cdot \frac{\ell}{n}$ from the left border of the interval. Hence, there are at most $2 \cdot \frac{\gamma}{4} \cdot \ell$ points that are at distance at most $\frac{\gamma}{4} \cdot \frac{\ell}{n}$ from either of the interval borders. Summing over all the n/ℓ intervals, we conclude that there is at most $2 \cdot \frac{\gamma}{4} \cdot \ell \cdot \frac{n}{\ell} = \frac{\gamma n}{2}$ points that are at distance at most $\frac{\gamma \ell}{4n}$ from the adjacent interval. The other $\frac{\gamma n}{2}$ points are at distance at least $\frac{\gamma \ell}{4n}$. These points induce edges of total weight of at least $\frac{1}{2} \cdot \frac{\gamma n}{2} \cdot \frac{\gamma \ell}{4n}$, where the factor $1/2$ comes from the fact that each edge might be counted twice. □

Let $\alpha_0 = \frac{1}{4e}$ and let $\ell_0 = \alpha_0 n^{2/k}$. We distinguish between two cases. Let γ_0 be the fraction of global points.

Case 0.1: $\gamma_0 > \frac{1}{2}$. Proceed with case analysis below for $i = 1$.

Case 0.2: $\gamma_0 \leq \frac{1}{2}$. In this case, the fraction of the non-global points is $1 - \gamma_0 \geq 1/2$. We construct a new line metric, M' by taking a non-global point from every interval containing a non-global point. There are at $n' = \frac{n}{\ell_0}$ intervals and at least $p' = (1 - \gamma_0)\frac{n}{\ell_0}$ of them contain a non-global point. (We ignore the rounding issues for simplicity of exposition.) Consider an interval A containing a non-global point a and an interval B containing a non-global point b. Any k-hop path between a and b has to have the first edge inside of the interval A, towards some point a' and the last edge inside of interval B towards some point b'. Hence, a' and b' have to be connected via a $(k-2)$-hop path. This is true for any pair of non-global points in M'. We use the induction hypothesis for $k-2$ to lower bound the total weight of H_k.

$$
\begin{aligned}
W_{k-2}(p', n') &\geq c(k-2)\left(\frac{\left(\frac{n}{2\ell_0}\right)^2}{\frac{n}{\ell_0}}\right)^{\frac{2}{k-2}} \\
&\geq c(k-2)\left(\frac{n}{4\alpha_0 n^{2/k}}\right)^{\frac{2}{k-2}} \\
&\geq c(k-2)\left(\frac{1}{4\alpha_0}\right)^{\frac{2}{k-2}} n^{2/k} \\
&\geq c(k-2)e^{\frac{2}{k-2}} n^{2/k} \\
&\geq ckn^{2/k} \\
&\geq ck\left(\frac{p^2}{n}\right)^{2/k}
\end{aligned}
$$

The following case analysis consists of cases $i.1$ and $i.2$ for $1 \leq i \leq \lfloor \log_{32}(ek) \rfloor = i'$. We let $\alpha_i = \frac{32^i}{4e}$ and $\ell_i = \alpha_i \cdot n^{2/k}$. We use γ_i to denote the number of global points with respect to ℓ_i. (See Figure 2 for an illustration of the monotonicity of the number of global points.)

Case i.1: $\gamma_i > \gamma_{i-1} - \frac{1}{4^i}$. Proceed with $i + 1$.

Case i.2: $\gamma_i \leq \gamma_{i-1} - \frac{1}{4^i}$. We have that $\gamma_{i-1} - \gamma_i \geq \frac{1}{4^i}$. This means that $\gamma = \gamma_{i-1} - \gamma_i$ fraction of the points are global with respect to ℓ_{i-1} and are not global with respect to ℓ_i. Their total contribution, by

Claim 3.5, is at least $\gamma^2\ell_{i-1}/16$. Similarly to Case 0.2 above, we employ the induction hypothesis for $(k-2)$ to lower bound the contribution of the non-global points. We construct a new line metric, M' by taking a non-global point from every interval containing a non-global point. There are at $n' = \frac{n}{\ell_i}$ intervals and at least $p' = (1-\gamma_i)\frac{n}{\ell_i}$ of them contain a non-global point. Interconnecting the points in M' contributes at least $W_{k-2}(p', n')$ to the total weight of H_k. The total weight of H_k can be lower bounded as follows.

$$\begin{aligned}
\frac{\gamma^2\ell_{i-1}}{16} + W_{k-2}(p', n') &\geq \left(\frac{1}{4^i}\right)^2 \cdot \frac{\ell_{i-1}}{16} + c(k-2)\left(\frac{(p')^2}{n'}\right)^{\frac{2}{k-2}} \\
&\geq \left(\frac{1}{4^i}\right)^2 \cdot \frac{\frac{32^{i-1}}{4e}n^{2/k}}{16} + c(k-2)\left((1-\gamma_i)^2 \cdot \frac{n}{\ell_i}\right)^{\frac{2}{k-2}} \\
&\geq \frac{8^{i-1}n^{2/k}}{256e} + c(k-2)\left(\frac{1}{4} \cdot \frac{n}{\alpha_i n^{2/k}}\right)^{\frac{2}{k-2}} \\
&\geq \frac{8^{i-1}n^{2/k}}{256e} + c(k-2)\left(\frac{e}{32^i}\right)^{\frac{2}{k-2}} n^{2/k} \\
&\geq n^{2/k}\left(\frac{8^i}{256e} + c(k-2)\left(1 + \frac{2}{k-2}\ln\frac{e}{32^i}\right)\right) \\
&\geq n^{2/k}\left(\frac{8^i}{256e} + c(k-2) + 2c\ln\frac{e}{32^i}\right) \\
&\geq ckn^{2/k} \qquad\qquad \text{for } c = 1/73728
\end{aligned}$$

Using that $p \leq n$, we have $ckn^{2/k} \geq ck\left(p^2/n\right)^{2/k}$, as required.

Finally, we consider the cases $i'.1$ and $i'.2$, where $i' = \lfloor \log_{32}(ek) \rfloor$.

Case i'.1: $\gamma_{i'} > \gamma_{i'-1} - \frac{1}{4^{i'}}$. Observe that $\gamma_{i'} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, since $\gamma_0 \geq \frac{1}{2}$ and for every $1 \leq i \leq i'$ it holds that $\gamma_i > \gamma_{i-1} - \frac{1}{4^i}$. By Claim 3.5, the contribution of the global points is at least $\gamma_{i'}^2\ell_{i'}/16$, which can be lower bounded as follows.

$$\begin{aligned}
&\frac{\gamma_{i'}^2\ell_{i'}}{16} \geq \frac{\ell_{i'}}{16 \cdot 36} = \frac{\alpha_{i'}}{16 \cdot 36} \cdot n^{2/k} \geq \frac{32^{i'}}{16 \cdot 36 \cdot 4e} \cdot n^{2/k} \geq \\
&\frac{32^{\log_{32}(ek)-1}}{16 \cdot 36 \cdot 4e} \cdot n^{2/k} \geq \frac{1}{73728} \cdot kn^{2/k} \geq ck\left(\frac{p^2}{n}\right)^{2/k}
\end{aligned}$$

Case i'.2: $\gamma_i' \leq \gamma_{i'-1} - \frac{1}{4^{i'}}$. Same as case i.2 above.

□

4 From light tree covers to light spanners

We start this section by proving Corollary 1.2, restated here for convenience.

Corollary 1.2. *Let $n \geq 1$ be an arbitrary integer and let M_X be an arbitrary metric space with n points. If M_X admits an L-light (γ, t)-tree cover, then for any $k \geq 1$, the metric space M_X has a t-spanner with hop-diameter k and lightness $O(\gamma \cdot L \cdot k \cdot n^{2/k})$.*

Proof. We construct the spanner H_X for M_X as follows. Let $\mathcal{T}$ be the L-light (γ, t)-tree cover for M_X as in the statement and let T be a tree in $\mathcal{T}$. From Theorem 1.1, we know that the metric M_T induced by T has a 1-spanner with hop-diameter k and lightness $O(kn^{2/k})$ with respect to M_T. Since M_T has lightness L with respect to M_X, the lightness of H_T with respect to M_X is $O(L \cdot kn^{2/k})$. Spanner H_X is obtained as the union of H_T for all T in $\mathcal{T}$. The lightness of H_X is $O(\gamma L \cdot kn^{2/k})$, because there are γ trees in $\mathcal{T}$ and each tree has lightness $O(L \cdot kn^{2/k})$ with respect to M_X. Consider two arbitrary points u and v in M_X. Since $\mathcal{T}$ is a tree cover with stretch t, there is a tree T in $\mathcal{T}$ such that $\delta_T(u, v) \leq t \cdot \delta_X(u, v)$. By construction, H_T is a 1-spanner with hop-diameter k, so there is a k-hop path in H_T (and hence in H_X) with length at most $t \cdot \delta_X(u, v)$. The corollary follows. $\square$

Next, we use the following construction of light tree cover from [CCL+25].

Theorem 4.1 (Cf. Theorem 1.2 in [CCL+25])**.** *Given a point set P in a metric of constant doubling dimension d and any parameter $\varepsilon \in (0,1)$, there exists an $\varepsilon^{-O(d)}$-light $O(\varepsilon^{-O(d)}, 1 + \varepsilon)$-tree cover for P.*

Corollary 1.2 and Theorem 4.1 immediately imply the following theorem.

Theorem 1.3. *For every $k \geq 1$ and $\varepsilon \in (0,1)$, every n-point metric space with doubling dimension d has a $(1 + \varepsilon)$-spanner with hop-diameter k and lightness $O(\varepsilon^{-O(d)} \cdot kn^{2/k})$.*

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Algorithm 1: Procedure for constructing a spanner of a tree metric induced by a given tree $T = (V, E)$. Parameter $k \geq 1$ is the required hop-diameter. The procedure returns the edge set F of a spanner. The weight of every edge in F is assigned to be equal to the distance of its endpoints in T.

procedure $\textsc{Spanner}(k, T = (V, E))$
    $n \leftarrow |V|$
    if $n \leq k$ **then**
        return E
    end if
    if $k = 1$ **then**
        return $\{(u, v) \mid u \in V, v \in V\}$ ▷ Clique on V
    end if
    if $k = 2$ **then**
        Let c be the centroid of T
        $F \leftarrow \{(c, v) \mid v \in V \setminus \{c\}\}$ ▷ Connect c to every vertex in $V \setminus \{c\}$
        Let $T_1, \ldots, T_g$ be the components of $T \setminus \{c\}$
        for $1 \leq i \leq g$ **do**
            $n_i \leftarrow |V(T_i)|$
            $F \leftarrow F \cup \textsc{Spanner}(2, T_i)$
        end for
        return F
    end if
    $\ell \leftarrow \lfloor 2n^{2/k} \rfloor$
    if $k = 3$ **then** $\ell \leftarrow \lfloor n^{2/3} \rfloor$
    end if
    if $k \geq 4$ **and** $n \leq 2k^2$ **then** $\ell \leftarrow k$
    end if
    Let X be the set as in Lemma 2.1 with parameter ℓ and let $T_1, \ldots, T_g$ be the components of $T \setminus X$
    $E_X \leftarrow \{(u, v) \mid (u, v) \in E, u \in X, v \in X\}$
    for $1 \leq i \leq g$ **do**
        $n_i \leftarrow |V(T_i)|$
        $F \leftarrow F \cup \textsc{Spanner}(k, T_i)$
        if T_i has one neighbor u_i in X **then** $F \leftarrow F \cup \{(u_i, w) \mid w \in V(T_i)\}$
        else
            Let u_i and v_i be the two neighbors of T_i in X
            $F \leftarrow F \cup \{(u_i, w) \mid w \in V(T_i)\}$
            $F \leftarrow F \cup \{(v_i, w) \mid w \in V(T_i)\}$
            $E_X \leftarrow (u_i, v_i)$
        end if
    end for
    $F \leftarrow F \cup \textsc{Spanner}(k - 2, T_X = (X, E_X))$
    return F
end procedure