FAT MINORS IN FINITELY PRESENTED GROUPS

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Abstract. We show that a finitely presented group virtually admits a planar Cayley graph if and only if it is asymptotically minor-excluded, partially answering a conjecture of Georgakopoulos and Papasoglu in the affirmative.

1. Introduction

Given graphs Γ and H, recall that Γ is said to contain an *H-minor* if H can be obtained by contracting edges of a subgraph of Γ. The idea of a minor is one of the most fundamental and important concepts in all of graph theory, the highlight of their study being the famous Robertson–Seymour 'Graph Minors Project' [18].

In the realm of geometric group theory, there has been some recent work in the direction of understanding when the minors present in a Cayley graph affect the structure of the group. For example, an infinite graph Γ is said to be *minor-excluded* if there exists a finite graph which is not a minor of Γ, and a theorem of Khukhro [13] states that **every** Cayley graph of a finitely generated group G is minor-excluded if and only if G is virtually free. In a similar vein, theorems of Esperet, Giocanti, Legrand-Duchesne and the author [8, 7, 14] combine to show that if G admits **some** minor-excluded Cayley graph then G is virtually a free product of free and surface groups.

Since 'the' Cayley graph of a finitely generated group is only well-defined up to quasi-isometry, it is likely hard to say much more than the above about minors in Cayley graphs. However, recently interest has grown in a new topic known as 'coarse graph theory' where one studies the 'large-scale' features of graphs. Popularised by Georgakopoulos and Papasoglu in their seminal paper [10], the analogue of a graph minor in this field is something called an 'asymptotic minor'. Roughly speaking, the idea is that a graph H is called an asymptotic minor of a metric space X if we see arbitrarily 'fat' copies of H within X. See Section 2.2 below for a precise definition. The benefit here is that the set of asymptotic minors of a graph is easily seen to be a quasi-isometry invariant. Originally, Georgakopoulos and Papasoglu posed the following conjecture.

False Conjecture ([10, 1.1])**.** *Let X, H be connected graphs with H finite. Then H is not an asymptotic minor of X if and only if X is quasi-isometric to some graph Y which contains no H-minor.*

The 'if' direction of the above is an easy exercise. Regarding the converse, several positive results are known for certain small H [15, 3, 9, 10]. Recently, a counterexample was found by Davies, Hickingbotham, Illingworth, and McCarty [4].

Georgakopoulos and Papasoglu also posed several related questions. In particular, the following conjecture is still open.

Conjecture 1.1 ([10, 9.3])**.** *A connected, locally finite, quasi-transitive graph X is asymptotically minor-excluded if and only if it is quasi-isometric to a planar graph.*

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In this paper, we partially answer Conjecture 1.1 in the affirmative, under the additional assumption that X is a Cayley graph of some finitely presented group.

Theorem A. *A finitely* ***presented*** *group is asymptotically minor-excluded if and only if some finite index subgroup of G admits a planar Cayley graph.*

Since the asymptotic minors of a graph are a quasi-isometry invariant, we immediately obtain the following corollary.

Corollary B. *Let G be a finitely presented group which is quasi-isometric to some minor-excluded graph. Then some finite-index subgroup of G admits a planar Cayley graph.*

This can be seen as a strengthening for finitely presented groups of [14, Cor. D], which records the same conclusion for those finitely generated groups which are quasi-isometric to planar graphs.

Dropping the hypothesis of finite presentability would require a different approach to that presented in this paper, as the geometry of an arbitrary finitely generated group, or more generally a quasi-transitive graph, has the potential to be far more pathological than that of any finitely presented group. For example, one such challenge comes from the existence of inaccessible groups [6].

Outline. In order to prove Theorem A, we may first note that Dunwoody's accessibility theorem [5] allows us to immediately restrict to the one-ended case. Then, known structure results imply that given a one-ended finitely presented group G which is not a virtual surface group, at least one of the following holds:

(1) G contains a infinite descending chain of one-ended subgroups
$$G \geq G_1 \geq G_2 \dots,$$
where each has infinite index in the last.

(2) G contains an infinite-index surface subgroup.

(3) G contains a one-ended, finitely presented subgroup which is not virtually a surface group, and does not split over a two-ended subgroup.

To prove Theorem A we thus proceed by considering each of these cases individually, making use of the hypothesised geometry present to directly construct fat minors in our Cayley graph.

The first two cases are fairly elementary and self-contained, and do not require the hypothesis of finite presentability. However, the proof in the third case heavily relies on Papasoglu's geometric characterisation of two-ended splittings [16], which fails for arbitrary one-ended groups [17].

2. Preliminaries

2.1. **Notation and terminology.** Given a metric space X, we denote by d_X its metric. If there is no risk of ambiguity then we may write $\mathrm{d} = \mathrm{d}_X$. Given $x \in X$ and $R > 0$ we denote by $N_R(x)$ the closed R-ball around x. If further clarity is needed then we may instead write $B_X(x; R)$ to make the choice of ambient space clear. We define similarly notation for the closed R-neighbourhood of a subset of X.

If $A, B \subset X$ are subsets of X then we write $\mathrm{d}_X(A, B)$ to mean their *infimal distance*. That is, we define
$$\mathrm{d}_X(A, B) := \inf\{\mathrm{d}_X(a, b) : a \in A,\ b \in B\}$$
Given $x \in X$ and $A \subset X$ we will simplify notation by writing $\mathrm{d}_X(x, A) = \mathrm{d}_X(\{x\}, A)$. We denote the *Hausdorff distance* of A and B in X as
$$\mathrm{Haus}_X(A, B) := \inf\{R > 0 : A \subset N_R(B) \text{ and } B \subset N_R(A)\}.$$

Given $R, \varepsilon > 0$ and $Z \subset X$, we will write

$$A_{R\pm\varepsilon}(Z) := \{x \in X : \mathrm{d}_X(x, Z) \in [R-\varepsilon, R+\varepsilon]\}.$$

We will also write

$$S_R(Z) := \{x \in X : \mathrm{d}_X(x, Z) = R\}.$$

If Y is another metric space and $\lambda \geq 1$ is a constant, then a map $\varphi : X \to Y$ is called a *λ-quasi-isometric embedding* if

$$\tfrac{1}{\lambda}\,\mathrm{d}_X(x, y) - \lambda \leq \mathrm{d}_Y(\varphi(x), \varphi(y)) \leq \lambda\,\mathrm{d}_X(x, y) + \lambda,$$

for all $x, y \in X$. Furthermore, if φ satisfies that for every $y \in Y$ there exists $x \in X$ such that

$$\mathrm{d}_Y(y, \varphi(x)) \leq \lambda,$$

then we call φ a λ-quasi-isometry. Given two quasi-isometries $\varphi : X \to Y$, $\psi : Y \to X$, and $\lambda \geq 1$, we say that ψ *is a λ-quasi-inverse to* φ if for all $x \in X$ we have that $\mathrm{d}_X(\psi \circ \varphi(x), x) \leq \lambda$. It is easy to check that every quasi-isometry has a quasi-inverse.

We will also need the following weaker notion. We say that a map $F : X \to Y$ is a *coarse embedding* if there exists $\xi_-, \xi_+ : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} \xi_\pm(t) = \infty$ and

$$\xi_-(\mathrm{d}_X(x, y)) \leq \mathrm{d}_Y(F(x), F(y)) \leq \xi_+(\mathrm{d}_X(x, y))$$

for all $x, y \in X$. It is easy to see that we may assume without loss of generality that ξ_- is increasing, surjective, and every value in $[0, \infty)$ except for 0 has exactly one preimage under ξ_-. The canonical example of a coarse embedding is that if X, Y are Cayley graphs of finitely generated groups H and G respectively where $H \leq G$, then X coarsely embeds into Y.

Now, suppose that X is a (connected, locally finite) graph. For us, a *graph* shall always be taken to mean a 1-dimensional simplicial complex. We write VX and EX for the vertex set and edge set of X, respectively. For the purposes of this paper, every graph we consider will be assumed to be simplicial. We denote by $\Delta(X)$ the maximal degree of any vertex of X. We will normally abuse notation and identify X with its geometric realisation as a CW-complex. We then metrise VX via the natural path metric and extend this metric naturally to all of X by identifying each edge with a copy of the unit interval.

Throughout this paper we will abuse notation relating to paths. Sometimes we may view path a path p in a graph X as subgraph, and other times we will parameterise p as a map $p : I \to X$, where I is some interval. In the latter case, unless otherwise stated we will usually take $I = [0, \ell]$ for some $\ell \geq 0$, and p will be taken to be parameterised at unit speed. That is, $\ell = \mathrm{length}(p)$.

2.2. **Fat and asymptotic minors.** Let (X, d) be a length space, H a connected simplicial graph, and $K \geq 1$. Then a *K-fat H-minor* is a subspace $M \subset X$ equipped with a decomposition into pieces

$$M = \bigcup_{v \in VH} U_v \cup \bigcup_{e \in EH} P_e,$$

satisfying the following:

(1) each U_v is path connected,
(2) if $e \in EH$ is an edge with endpoints $u, v \in VH$, then P_e is a path connecting U_u to U_v,
(3) For all distinct $A, B \in \{U_v : v \in VH\} \cup \{P_e : e \in EH\}$ we have that $\mathrm{d}(A, B) > K$, unless $A = P_e$, $B = U_v$, or vice versa, and v is an endpoint of e.

The U_v are called the *branch sets* of M and the P_e are called the *edge paths*.

If X contains a K-fat H-minor for every $K \geq 1$, then we say that H is an *asymptotic minor* of X. If there exists some finite H such that H is not an asymptotic minor of X, then we say that X is *asymptotically minor-excluded*.

It is immediate that if X and Y are length spaces and X coarsely embeds into Y, then every asymptotic minor of X is also an asymptotic minor of Y.

As a warm-up to constructing fat minors, we record the following easy example of groups which are not asymptotically minor-excluded. Given integers $n, m \neq 0$, the *Baumslag–Solitar group* $BS(n,m)$ is given by the one-relator presentation

$$BS(n,m) := \langle x, t \,;\; x^n = t^{-1}x^m t\rangle.$$

For example, $BS(1,1) \cong \mathbb{Z}^2$ and $BS(1,-1)$ is the fundamental group of a Klein bottle. Outside of these basic examples, we note the following.

Proposition 2.1. *Given $n > 1$, $m \neq 0$, we have that the group $BS(n,m)$ is not asymptotically minor-excluded.*

Proof. This follows very quickly from inspecting the standard Cayley graphs of these groups. We leave this as an exercise to the reader. $\square$

3. Descending chains of one-ended subgroups

In this section we prove that groups which contain an infinite descending chain of one-ended, infinite-index subgroups cannot be asymptotically minor-excluded.

Throughout this paper, we will use the following terminology.

Definition 3.1 (Normal geodesic)**.** Let X be a graph and $Y \subset X$ a subgraph. Let $\rho : I \to X$ be a (finite or one-way infinite) geodesic with one endpoint y_0 lying on Y. We say that ρ is a *normal to* Y if $\mathrm{d}_X(\rho(t), Y) = t$ for all $t \in I$.

We first note the following standard application of the Arzela–Ascoli theorem, which says that infinite normals often exist.

Lemma 3.2. *Let X be a connected, locally finite graph. Let $Y \subset X$ be a subgraph such that the setwise satibliser of Y in $\mathrm{Aut}(X)$ acts coboundedly on Y, and $\mathrm{Haus}_X(Y,X) = \infty$. Then there exists a one-way infinite geodesic ray $\rho : [0,\infty) \to X$ which is normal to Y.*

Proof. Since $\mathrm{Haus}_X(Y,X) = \infty$ we have that for all $k > 0$ there exists a geodesic segment $\rho_k : [0,k] \to X$ of length k which is normal to Y. Using the action of Γ_Y, we may assume that there exists some compact subgraph $K \subset Y$ such that $\rho_k(0) \in K$ for all $k \geq 0$.

By the Arzela–Ascoli theorem, the sequence (ρ_k) contains a subsequence (ρ_{n_k}) which converges uniformly on compact subsets to an infinite geodesic ray ρ. In particular, for all $N \geq 1$ there exists $M \geq 1$ such that for all $k \geq M$ we have

$$\rho|_{[0,N]} = \rho_{n_k}|_{[0,N]}.$$

From this property, it is clear that ρ is normal to Y. $\square$

Our proof of Theorem 3.4 will mostly follow from the next lemma.

Lemma 3.3. *Let G be finitely generated, one-ended. Fix $m \geq 1$. Suppose there exists some infinite-index, finitely generated $H \leq G$ such that H contains K_m as an asymptotic minor. Then G contains K_{m+1} as an asymptotic minor.*

Proof. Let X be some fixed Cayley graph of G. Fix $K > 0$, $m \geq 0$, and suppose that H contains K_m as an asymptotic minor. Let $Y \subset X$ be a connected tubular neighbourhood of H, so that $H \curvearrowright Y$ freely and cocompactly. Note that Y therefore also contains K_m as an asymptotic minor by assumption, and also the inclusion

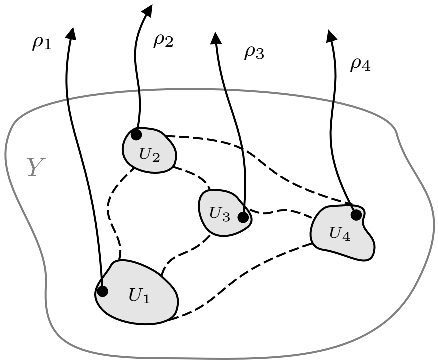

FIGURE 1. Extending a fat K_m-minor in Y to a fat K_{m+1}-minor in X.

$Y \hookrightarrow X$ is a coarse embedding. Fix $B \geq 1$ such that the action of H on Y is B-cobounded (with respect to the ambient metric on X).

Using Lemma 3.2, let $\rho : [0, \infty) \to X$ be a one-way infinite geodesic ray based in H such that $\mathrm{d}_X(\rho(t), H) = t$ for every $t \geq 0$. Since Y contains K_m as an asymptotic minor and the inclusion $Y \hookrightarrow X$ is a coarse embedding, we have that there exists arbitrarily fat K_m-minors in X which are contained in Y. Let $K' = 10(K + B)$. Let $M \subset Y$ be a K'-fat (with respect to the ambient metric on X) K_m-minor. Let $U_1, \ldots, U_m$ be the branch sets of M.

Using the action of H on Y, we place a copy ρ_i of ρ near each U_i, so $\mathrm{d}_X(\rho_i(0), U_i) \leq B$ and $\mathrm{d}_X(\rho_i(0), U_j) > K' - B$ for $j \neq i$. Since X is one-ended, let U_{m+1} denote the unique unbounded component of $X \setminus N_{5K}(M)$. We have that all the ρ_i are eventually contained in U_{m+1}. In particular, it is clear that

$$M' := M \cup U_{m+1} \cup \bigcup_i \rho_i,$$

can be decomposed as a K-fat K_{m+1}-minor. See Figure 1 for a cartoon of this construction. □

From Lemma 3.3, we may easily deduce the following.

Theorem 3.4. *Let G be a finitely generated group. Suppose there exists an infinite descending chain of subgroups*

$$G \geq G_0 \geq G_1 \geq \ldots \geq G_i \geq \ldots$$

such that for all $i \geq 0$ we have that G_i is one-ended and $|G_i : G_{i+1}| = \infty$. Then G is not asymptotically minor-excluded.

Proof. Given $n \geq 0$ we have that G_n trivially contains K_0 as an asymptotic minor. Applying Lemma 3.3 a total of n times, we deduce that G contains K_n as an asymptotic minor. Since n was arbitrary, the result follows. □

4. GROUPS WITH SURFACE SUBGROUPS

In this section we prove that a one-ended, finitely generated group with an infinite-index surface subgroup is not asymptotically minor-excluded.

4.1. **Idea of proof.** Given any finite graph F, we can consider a particular drawing of this graph, where we allow edges to cross each other. We can then view this drawing itself as a planar graph P, where the 'crossings' correspond to degree-four vertices, which we will call *marked* vertices. Given a one-ended finitely generated group G with Cayley graph X and an infinite-index surface subgroup $H \leq G$, our strategy to build fat minors of an arbitrary F is thus as follows:

(1) Consider a particular 'drawing' of F and its corresponding planar graph P.
(2) Draw a very large, very controlled copy of P in the (hyperbolic or Euclidean) plane.
(3) Embed this copy of P into a neighbourhood of H in X, and use the aforementioned control together with the action of H to turn the marked vertices back into 'fat crossings'.

See Figure 5 for an illustration of this strategy.

4.2. **Controlled fat minors in the plane.** It is obvious that the (hyperbolic or Euclidean) plane contains every finite planar graph as an asymptotic minor. Our immediate goal is to gain a bit more control over the drawings of these fat minors.

We first state the following standard definition.

Definition 4.1 (Coarse cover)**.** Let X be a metric space, $\varepsilon > 0$. Then an *ε-cover* of X is a subset $N \subset X$ such that for all $x \in X$ there exists $a \in N$ such that $\mathrm{d}(a, x) \leq \varepsilon$.

If N is an ε-cover for some $\varepsilon > 0$, we may suppress ε from our notation and simply call N a *coarse cover* of X.

We now have the following key lemma, dubbed the 'carefully-drawn lemma'.

Lemma 4.2 (Carefully-drawn lemma)**.** *Consider the* (*Euclidean or hyperbolic*) *plane. Let N be a coarse cover of the plane. Let P be a finite simplicial planar graph such that $\Delta(P) \leq 4$, equipped with a fixed drawing in the plane. Let $r, K > 0$ such that $K > 10r$.*

Then there exists a K-fat P-minor M in the plane with the following properties:

(1) *For every vertex $v \in VP$, the branch set U_v is a solid circular disk of radius r based at a point in N.*
(2) *Suppose that for every $x \in N$ we are given a list of four evenly spaced angles $\theta_1, \dots, \theta_4$. If U_v is a branch set centred at x then we may assume that any edge path terminating at U_v does so at one of the θ_i.*
(3) *The cyclic order in which the edge paths leave the branch sets agrees with the cyclic order with which the edges leave the vertices in the given drawing of P.*

Remark 4.3. Note that we only consider the case where $\Delta(P) \leq 4$ for simplicity. This lemma should be true without this restriction, but a proof in this generality would be a bit more involved and is not necessary for our purposes.

Intuitively, Lemma 4.2 says that we may build a fat P-minor M for any finite planar graph P, where the vertices of P are circular 'dials' centered at prescribed points in the plane which can be rotated without destroying the fatness of M. Condition (3) essentially says that the constructed drawing of M and the given drawing of P 'look the same'.

We encourage the reader to try to convince themselves that this lemma is intuitively true. For the sake of completeness we now sketch a proof of Lemma 4.2, though the exact details of this construction are not particularly enlightening.

Sketch of Lemma 4.2. Let N be an ε-coarse cover of the plane, and fix $r, K > 0$ such that $K > 10r$. We will build a K-fat P-minor in the plane such that the branch sets are uniform disks of radius r centred at points in N, with the property that the edge paths meet these disks at prescribed angles.

We first note that both the Euclidean and hyperbolic planes contain arbitrarily 'fat' copies of the square half-grid. Indeed, in the Euclidean plane this is a wholly trivial observation, and in the hyperbolic plane we can construct examples easily using the Poincaré half-plane model. See Figure 2.

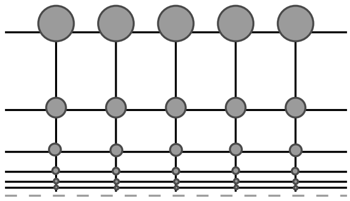

FIGURE 2. Example of a fat half-grid in the hyperbolic plane with a uniform disk at every lattice point, shown here in the Poincaré half-plane model.

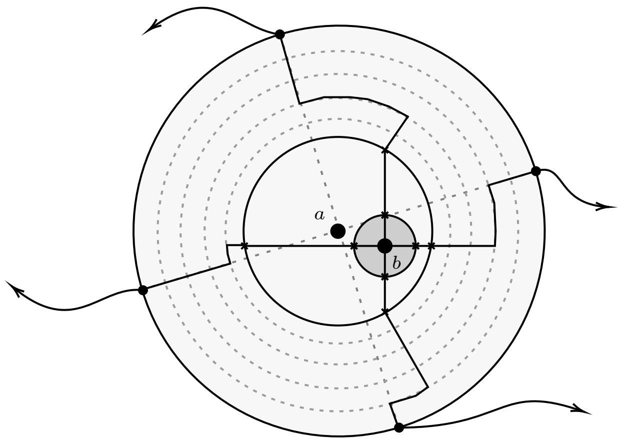

FIGURE 3. An example of a construction to modify the centres of the disks in Lemma 4.2 to align with points of a coarse cover, while also letting us 'rotate' the edge paths so that they arrive at prescribed angles.

It is clear that any finite planar graph P satisfying $\Delta(P) \leq 4$ embeds topologically into the half-grid, in such a way that will 'agree' with any given drawing of P. We thus obtain arbitrarily fat P-minors in the plane where the branch sets are uniform solid disks of any given size and the edge-paths meet these disks are right angles. All that remains is to modify these disks so that they are centred on elements of a coarse cover, and the edge-paths meet these disks are prescribed angles. This can be achieved as follows. We choose our half-grid to be such that the branch-set disks have radius $R := \varepsilon + r + 5K + 1$. If a lies at the centre of some branch set of the constructed fat P-minor, then there is some $b \in N$ which lies at most ε away from a. It is now easy to construct four pairwise distance-K paths inside the disk of radius R at a which terminate at on the circle of radius r based at b at prescribed angles, while preserving the cyclic order these paths arrive in. One such construction is shown in Figure 3. Note that this figure depicts the construction in the Euclidean plane, but it is easy to check that an analogous construction is possible in the hyperbolic plane. □

4.3. **Building fat minors using surface subgroups.** We now apply Lemma 4.2 and prove Theorem 4.4 of the introduction.

Theorem 4.4. *Let G be a one-ended finitely generated group. Suppose G contains an infinite-index surface subgroup H. Then G is not asymptotically minor-excluded.*

Proof. Let X be a Cayley graph of G. Let F be an arbitrary connected finite simplicial graph. We assume without loss of generality that $\Delta(F) \leq 4$, as any graph certainly embeds as a minor into a graph with this property. Fix $K > 0$. We will construct a K-fat F-minor in X. Throughout our construction, note that we will make no attempt to optimise our choice of constants.

Let $Y \subset X$ be a connected tubular neighbourhood of H, so that $H \curvearrowright Y$ freely and cocompactly. Recall that the inclusion of $Y \hookrightarrow X$ is a coarse embedding. Therefore, we can fix a choice of function $\xi_- : [0, \infty) \to [0, \infty)$ such that

$$\xi_-(\mathrm{d}_Y(x, y)) \leq \mathrm{d}_X(x, y),$$

for all $x, y \in Y$ (we will not need the upper bound function ξ_+ here). We assume without loss of generality that ξ_- is increasing, surjective, and every value in $[0, \infty)$ except for 0 has exactly one preimage under ξ_-. Since H acts on Y properly discontinuously and cocompactly, we have by the Švarc–Milnor lemma that Y is quasi-isometric to H, and thus to either the Euclidean plane $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}^2$ as H is a surface group. Let us assume without loss of generality that we have a quasi-isometry $\varphi : Y \to \mathbb{R}^2$ with quasi-inverse $\psi : \mathbb{R}^2 \to Y$, as the argument is identical in the hyperbolic case. Let $\lambda \geq 1$ now be fixed so that φ and ψ are both λ-quasi-isometries, and that ψ is a λ-quasi-inverse to φ (and vice-versa).

Our first goal is to construct a certain gadget which will allow us to build 'coarse crossings'. Using Lemma 3.2, let $\rho : [0, \infty) \to X$ be a one-way infinite geodesic ray based in Y such that $\mathrm{d}_X(\rho(t), Y) = t$ for every $t \geq 0$. That is, ρ is normal to Y. Let $C = 10K$. Fix $D > 0$ to be some large constant compared to C, λ. For example, it is sufficient to take

$$D = 2\lambda^2(10\lambda + \xi_-^{-1}(C))$$

Since X is one-ended, it is clear that we can find a path $\alpha : [0, \ell] \to X$ with the following properties:

(1) $\alpha(\ell) \in \rho$ and $\alpha(0) \in Y$.
(2) α is disjoint from $N_D(\rho(0))$.
(3) The initial segment $\alpha|_{[0,C]}$ is a geodesic in X which is normal to Y.
(4) Otherwise, the terminal segment $\alpha|_{[C+1,\ell]}$ is disjoint from $N_C(Y)$.

Indeed, this can be achieved by taking a large neighbourhood N of $\rho(0)$ and taking a shortest path from some $\rho(t)$ to Y through $X \setminus N$, then replacing the final segment of this path with a normal. In particular, the union $\alpha \cup \rho$ contains a path $p : [0, s] \to X$ such that

(1) $\mathrm{d}_X(p(0), p(s)) > D$,
(2) The initial and terminal segments $p|_{[0,C]}$ and $p|_{[s-C,s]}$ are disjoint geodesics which are normal to Y, and p is otherwise disjoint from the C-neighbourhood of Y.

Let $R = \mathrm{diam}_X(p)$. Intuitively, the goal now is to use copies of p to build 'bridges' which allow us to take fat 'planar drawings' of non-planar graphs in Y and turn them into fat minors in X. Indeed, from now on we shall refer to each translate of p as a *bridge*.

For each $h \in H$ write $p_h := h \cdot p$. Let

$$x_h = \varphi(p_h(0)), \ \ y_h = \varphi(p_h(s)).$$

Note that $\frac{1}{\lambda}D - \lambda < \mathrm{d}_{\mathbb{R}^2}(x_h, y_h) < \lambda D + \lambda$ for all $h \in H$. Let L_h the line segment in $\mathbb{R}^2$ connecting x_h to y_h, and let $m_h \in L_h$ denote its midpoint. Since the action of H on Y is cobounded, it is clear that $N := \{m_h : h \in H\}$ is a coarse cover of $\mathbb{R}^2$.

Consider some planar, straight-line drawing of F, where edges are allowed to cross other edges. We view this drawing itself as a finite simplicial planar graph P, where 'crossings' in the aforementioned drawing become degree-four vertices of P. Note that $\Delta(P) \leq 4$, and P comes naturally equipped with a fixed drawing. If $v \in VP$ corresponds to a crossing in F then we call v a *marked vertex*, and denote the set of marked vertices by $\widehat{V}P \subset VP$.

Let $r = 3(\lambda D + \lambda)$. Let $K' > 0$ be some fixed constant which is chosen to be sufficiently large compared to C, λ, r, and R. For example, it is sufficient to take

$$K' = 4\lambda(10\lambda + \xi_-^{-1}(C + 4R)) + 10r.$$

Using the carefully-drawn lemma (4.2), let $M \subset \mathbb{R}^2$ be a K'-fat P-minor where

(1) For every vertex $v \in VP$, the branch set U_v is a solid circular disk of radius r based at a point in N.
(2) The cyclic order in which the edge paths leave the branch sets agrees with the cyclic order with which the edges leave the vertices in the given drawing of P.
(3) Let $v \in \widehat{V}P$ be a marked vertex of P, and say U_v is centred on $m_h \in N$. Then the edge paths which meet U_v do so at four evenly spaced points. Moreover, if we extend L_h in both directions so that it forms a diameter of U_v, then we also assume that two of edge paths in M which meet U_v do so at exactly where this diameter intersects ∂U_v.

We now need to set up some more notation. If $x_h, y_h, m_h \in U_v$ then let us write $x_v := x_h$, and so on and also write $L_v := L_h$. We also relabel the corresponding bridge $p_v := p_h$. Given $e = uv \in EP$, let $P_e = P_{uv}$ denote the corresponding edge path in M. For each marked vertex $v \in \widehat{V}P$, let $a_v^1, \ldots, a_v^4 \in \partial U_v$ be where the edge paths of M meet U_v, cyclically ordered so that a_v^1, x_v, y_v, and a_v^3 appear on a common diameter in this given order. If P_{uw} is an edge path such that that u is a marked vertex, and this path terminates at either a_u^1 (or a_u^3), then we extend P_{uw} by adjoining the line segment connecting a_u^1 to x_u (resp. a_u^3 to y_u). If w is also a marked vertex then we extend P_{uw} similarly at the other end. Call the resulting extended path P'_{uw}. If neither u nor w are marked then just set $P'_{uw} = P_{uw}$. Finally, given marked $v \in \widehat{V}P$ we let Q_v denote the line segment connecting a_v^2 to a_v^4.

We now consider the figure

$$M' = \bigcup_{v \in VP \setminus \widehat{V}P} U_v \cup \bigcup_{v \in \widehat{V}P} Q_v \cup \bigcup_{e \in EP} P'_e.$$

In other words, we delete each U_v where $v \in \widehat{V}P$ is a marked vertex, then add back line segments connecting a_v^2 to a_v^4, a_v^1 to x_v, and a_v^3 to y_v. See Figure 4 for a cartoon of the construction of M'.

Our goal now is to push M' through the quasi-isometry $\psi : \mathbb{R}^2 \to Y$ and use this to construct a fat copy of F in X.

For each $e \in EP$ let $C_e \subset P'_e$ be a 1-coarse cover of P'_e which contains the endpoints of P'_e. For a fixed e, enumerate $C_e = \{b_1, \ldots, b_l\}$ in the order that they appear. By joining $\psi(b_i)$ with a geodesic in Y to $\psi(b_{i+1})$ for each i, we form a path A_e in Y connecting the ψ-images of P'_e, which is contained in the 2λ-neighbourhood of $\psi(C_e)$, with respect to the intrinsic metric d_Y of Y. We similarly repeat this construction with each Q_v where $v \in \widehat{V}P$, and obtain a similar path B_v.

Given $v \in \widehat{V}P$, let us write $a_v = p_v(0)$, $b_v = p_v(s)$, $a'_v = \psi(x_v)$ $b'_v = \psi(y_v)$. Note that

$$\mathrm{d}_Y(a_v, a'_v) \leq 2\lambda, \ \ \mathrm{d}_Y(b_v, b'_v) \leq 2\lambda.$$

Thus, we extend any path A_{uv} $u \in VP$ terminating at a'_v (resp. b'_v) with a geodesic of length at most 2λ so that it terminates at a_v (resp. b_v). If $v \in VP \setminus \widehat{V}P$ is not marked, then let U'_v denote the 2λ-neighbourhood of $\psi(U_v)$ in Y, so U'_v is connected and contains the endpoints of all the paths A_e where e abuts v in P.

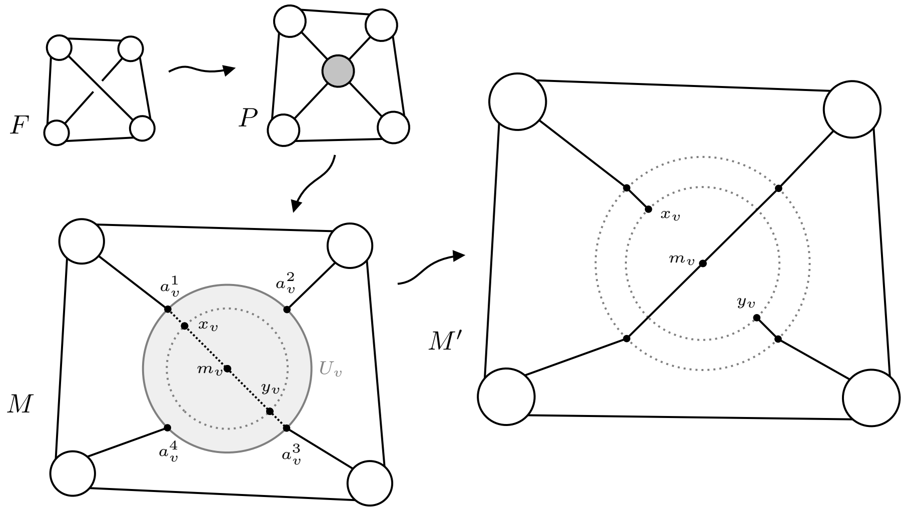

FIGURE 4. The construction of $M' \subset \mathbb{R}^2$. In this example, $F = K_4$ and P is constructed from the shown non-planar drawing of F.

We now consider the subset of X given by

$$M'' = \bigcup_{v \in VP \setminus \widehat{V}P} U'_v \cup \bigcup_{e \in EP} A_e \cup \bigcup_{v \in \widehat{V}P} B_v \cup \bigcup_{v \in \widehat{V}P} p_v.$$

By comparing M'' with the non-planar drawing of F we started with, it is now straightforward to verify that M'' decomposes as a K-fat F-minor. We need only verify a few inequalities. For example, consider $v, u \in VF$. These correspond to unmarked vertices in $VP \setminus \widehat{V}P$, and the corresponding branch sets are thus $U'_v, U'_u \subset M''$. We may lower bound the infimal distance $\mathrm{d}_X(U'_v, U'_u)$ via:

$$\begin{aligned}
\mathrm{d}_X(U'_v, U'_u) &\geq \xi_-(\mathrm{d}_Y(U'_v, U'_u)) \\
&\geq \xi_-(\mathrm{d}_Y(\psi(U_v), \psi(U_u)) - 4\lambda) \\
&\geq \xi_-(\tfrac{1}{\lambda}\, \mathrm{d}_{\mathbb{R}^2}(U_v, U_u) - \lambda - 4\lambda) \\
&\geq \xi_-(\tfrac{1}{\lambda} K' - \lambda - 4\lambda) \\
&= \xi_-(\tfrac{1}{\lambda}(10r + 4\lambda(10\lambda + \xi_-^{-1}(C + 4R))) - \lambda - 4\lambda) \\
&> \xi_-(\xi_-^{-1}(C + 4R)) \\
&= C + 4R \\
&> K.
\end{aligned}$$

Thus, our branch sets are all sufficiently spaced-out. We should also check that our bridges stay far away from everything they are supposed to. For example, let $v \in \widehat{V}P$ be a marked vertex. One must verify that B_v and p_v are sufficiently far apart.

Recall that the bridge p_v has the form of two geodesics which are normal to Y, based at a_v and b_v, connected at the top by a path which lies at distance at least C from Y everywhere. We thus may assume without loss of generality that $\mathrm{d}_X(b_v, B_v) \geq \mathrm{d}_X(a_v, B_v)$, as if a vertex in Y is close to the bridge B_v then its closest point is either a_v, b_v, or a point on the 'top' of the bridge. We then bound $\mathrm{d}_X(B_v, p_v)$

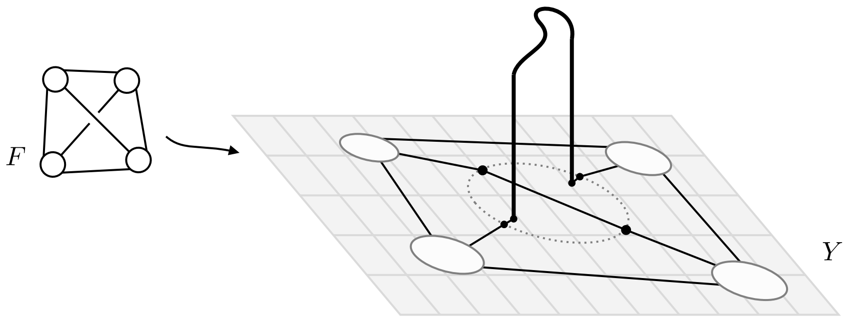

FIGURE 5. Cartoon of the fat F-minor M''. Here $F = K_4$ and the construction proceeds using the shown non-planar drawing.

as follows:

$$\begin{aligned} \mathrm{d}_X(a_v, B_v) &\geq \xi_-(\mathrm{d}_Y(a_v, B_v)) \\ &\geq \xi_-(\mathrm{d}_Y(\psi(x_h), \psi(Q_v)) - 4\lambda) \\ &\geq \xi_-(\tfrac{1}{\lambda}\, \mathrm{d}_{\mathbb{R}^2}(x_h), Q_v) - \lambda - 4\lambda) \\ &\geq \xi_-(\tfrac{1}{\lambda}(\tfrac{1}{\lambda}\tfrac{D}{2} - \lambda) - \lambda - 4\lambda) \\ &= \xi_-(\tfrac{D}{2\lambda^2} - 1 - 5\lambda) \\ &> \xi_-(\xi_-^{-1}(C)) \\ &> K. \end{aligned}$$

This implies that $\mathrm{d}_X(p_v, B_v) > K$.

There are, of course, several other inequalities to check here, all of which follow similarly to the above. For example, one can check that the bridges p_v are pairwise far apart from each other, due to the dependence of K' on R. One should also check that the A_e are pairwise far apart from each other, and so on. We leave these as an exercise to the keen reader.

See Figure 5 for a cartoon of this completed construction. □

5. Finitely presented groups

In this section we construct fat minors in finitely presented groups. Throughout this section, G will denote a one-ended finitely presented group which is not a surface group. Let X denote some choice of Cayley graph of G.

5.1. **Idea of proof.** Given a one-ended finitely presented group G, we will roughly proceed as follows. By applying known structure results for finitely presented groups together with Theorems 3.4 and 4.4, we can restrict to the case where G does not split over a two-ended subgroup. Under this assumption, we proceed roughly as follows:

(1) By a theorem of Papasoglu [16], we know that no neighbourhood of a bi-infinite geodesic can separate our Cayley graph into deep pieces.
(2) By the Arzela–Ascoli theorem, we see that for every $K > 0$ there exists some $R > 0$ such that the K-neighbourhood of any long geodesic segment cannot 'nicely' separate the R-ball about its midpoint.
(3) We use this to build a fat K_m, where the vertices are 'long geodesics'. The previous observation allows us to redirect the edge paths so that they avoid any branch sets they aren't meant to meet.

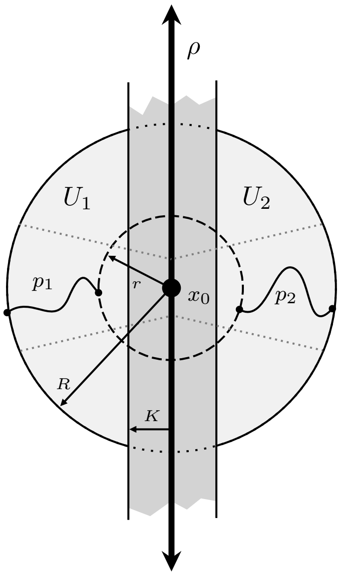

FIGURE 6. Local coarse separation.

See Figure 7 below for a cartoon of the final construction.

We need to be a bit careful about what we mean for a geodesic to 'nicely' separate a ball. This is the topic of the next subsection.

5.2. **Local coarse separation.** We begin with the following definition, which is useful for shorthand.

Definition 5.1 (Coarsely perpendicular)**.** Let $A, B \subset X$ be subgraphs, $x_0 \in A$, and $L \geq 1$. We say that B is *L-coarsely perpendicular to A at x_0* if

$$\mathrm{d}(b, x_0) \leq L\,\mathrm{d}(b, A) + L,$$

for all $b \in B$.

Intuitively, B is coarsely perpendicular to A at x_0 if x_0 is approximately the nearest point in A to every $b \in B$, up to some linear error. We now have the following key definition.

Definition 5.2 (Local coarse separation)**.** Let $R > r > K > 0$ and $L \geq 0$. Let $\rho \subset X$ be a geodesic, and let $x_0 \in \rho$ be a vertex which lies at distance at least $5R$ from the endpoints of ρ.

Then ρ is said to be *(R, r, L, K)-locally coarsely separating at x_0* if the complement

$$N_R(x_0) \setminus N_K(\rho)$$

contains two distinct connected components U_1, U_2 such that for $i = 1, 2$ we have:

(1) the subgraph U_i contains a path p_i connecting $S_r(x_0)$ to $S_R(x_0)$, and
(2) the path p_i is L-coarsely perpendicular to ρ.

See Figure 6 for a cartoon of this definition.

We now prove some basic lemmas about local coarse separation. We begin with the following lemma which says that we can decrease the value of R in the above definition without destroying the local coarse separation property.

Lemma 5.3. *Let $R > r > K > 0$ and $L \geq 0$. Fix $x_0 \in X$. Let ρ be a geodesic segment which (R, r, L, K)-locally coarsely separates at x_0. Let R' be such that $r < R' \leq R$. Then ρ also (R', r, L, K)-locally coarsely separates at x_0.*

Proof. Let $U_1, U_2 \subset N_R(x_0) \setminus N_K(\rho)$ be connected components, such that U_i contains a path p_i connecting $S_r(x_0)$ to $S_R(x_0)$ which is L-coarsely perpendicular to ρ at x_0. Now, it follows easily from the definition that a subpath of an L-coarsely perpendicular path is also L-coarsely perpendicular. Note that

$$U_1' := U_1 \cap N_{R'}(x_0), \quad U_2' := U_2 \cap N_{R'}(x_0)$$

do not intersect a common connected component of $N_{R'}(x_0) \setminus N_K(\rho)$. Also, each U_i' contains a subpath p_i' of p_i which connects $S_r(x_0)$ to $S_{R'}(x_0)$. As noted earlier, this subpath is L-coarsely perpendicular to ρ at x_0. The lemma follows. □

The following two lemmas are easy exercises in the triangle inequality, but helpful to note.

Lemma 5.4. *Let $\rho \subset X$ be a geodesic, let $x_0 \in \rho$ be a vertex which lies at distance at least $5R$ from the endpoints of ρ, and let $K \geq 0$. Let $\rho' \subset X$ be another geodesic such that $\rho \subset \rho'$. Then $N_R(x_0) \setminus N_K(\rho) = N_R(x_0) \setminus N_K(\rho')$.*

Lemma 5.5. *Let $R > r > K > 0$ and $L \geq 0$. Fix $x_0 \in X$. Let ρ be a geodesic segment such that $x_0 \in \rho$, and x_0 lies at distance at least $5R$ from the endpoints of ρ. Let ρ' be another geodesic segment such that $\rho \subset \rho'$. Then ρ is (R, r, L, K)-locally coarsely separating at x_0 if and only if ρ' is.*

The latter of the previous two lemmas essentially says that if ρ locally coarsely separating then we may 'extend' or 'trim' ρ while preserving this property.

We now state the following standard definition.

Definition 5.6 ((Global) coarse separation)**.** Given $K \geq 0$, a subset $A \subset X$ is said to *K-coarsely separate X into deep components* if $X \setminus N_K(A)$ contains two distinct connected components U_1, U_2 and neither of the U_i is contained in $N_{K'}(A)$ for any $K' \geq 0$.

We may omit mention of the constant K and simply say that the subset *A coarsely separates X*.

Lemma 5.7. *Let $r > K > 0$ and $L > 0$. Suppose that for all $R > r$ we have that there is a geodesic segment ρ which (R, r, L, K)-locally coarsely separates. Then X contains a bi-infinite geodesic γ which coarsely separates X.*

Proof. Fix $x_0 \in X$. Since X is a transitive graph, the hypotheses imply that there exists a sequence $(\rho_k)_{k\geq 1}$ of geodesic segments and a sequence of integers $(R_k)_{k\geq 1}$ with the following properties:

(1) we have that $R_k \to \infty$ as $k \to \infty$,
(2) for all $k \geq 1$, we have that $x_0 \in \rho_k$ and lies at distance at least $5R_k$ from the endpoints of ρ_k, and
(3) the segment ρ_k is (R_k, r, L, K)-locally coarsely separating at x_0.

Contrary to our usual convention in this paper, we will parametrise each ρ_k at unit speed on a *symmetric* interval, so $\rho_k : [-\frac{\ell_k}{2}, \frac{\ell_k}{2}] \to X$ where $\ell_k = \mathrm{length}(\rho_k)$. We also assume that $\rho_k(0) = x_0$ for all $k > 0$.

By the Arzela–Ascoli theorem, the sequence (ρ_k) contains a subsequence (ρ_{n_k}) which converges uniformly on compact subsets to a bi-infinite geodesic γ. In particular, for all $N \geq 1$ there exists $M \geq 1$ such that for all $k \geq M$ we have

$$\gamma|_{[-N,N]} = \rho_{n_k}|_{[-N,N]}.$$

We will abuse notation and simply write ρ_k and R_k for ρ_{n_k} and R_{n_k}.

By Lemmas 5.3 and 5.5 we may again pass to a subsequence, decrease the values of R_k, and 'trim' our geodesics so that we may assume without loss of generality that

$$\rho_1 \subset \rho_2 \subset \rho_3 \subset \ldots \subset \bigcup_{k\geq 1} \rho_k = \gamma.$$

By Lemma 5.4, we have that

$$N_{R_k}(x_0) \setminus N_K(\rho_k) = N_{R_k}(x_0) \setminus N_K(\gamma).$$

For each $k \geq 1$, let $U_1^{(k)}, U_2^{(k)}$ denote two distinct connected components of $N_{R_k}(x_0) \setminus N_K(\rho_k)$, and for each $i = 1, 2$ let $p_i^{(k)} \subset U_i^{(k)}$ be a path which connects $N_r(x_0)$ to $S_{R_k}(x_0)$, and diverges L-linearly from ρ_k. By Lemma 5.5, $p_i^{(k)}$ also L-linearly diverges from γ. In particular, each $U_i^{(k)}$ contains vertices which are arbitrarily far from γ as $k \to \infty$.

We would now like to say that $U_i^{(k)} \subset U_i^{(m)}$ for all $k < m$, $i = 1, 2$. As it stands, this is not currently true as there are potentially many choices for the $U_i^{(k)}$. In order to ensure this, we will need to modify our choices of the $U_i^{(k)}$. Given $m \geq k$, let $q_i^{(k,m)}$ be a subpath of $p_i^{(m)}$ contained in $N_{R_k}(x_0) \setminus N_K(\gamma)$ connecting $N_r(x_0)$ to $S_{R_k}(x_0)$, and let $W_i^{(k,m)}$ be the connected component of $N_{R_k}(x_0) \setminus N_K(\gamma)$ which contains $q_i^{(k,m)}$. It is clear that the $q_i^{(k,m)}$ and $W_i^{(k,m)}$ also witness the fact that ρ_k is (R_k, r, L, K)-locally coarsely separating at x_0, for any $m \geq k$. In particular, the previous claims about the $U_i^{(k)}$ also apply to the $W_i^{(k,m)}$. Note that

$$W_i^{(k,m)} \subset W_i^{(k',m)}$$

for all $k \leq k' \leq m$, by construction.

Now, if we fix k and vary $m > k$, then $W_i^{(k,m)}$ can only take a bounded number of possibilities, as every $W_i^{(k,m)}$ is a connected component of $N_{R_k}(x_0) \setminus N_K(\gamma)$ which intersects $N_r(x_0)$. In particular, there are only at most $|N_r(x_0)|$ many possibilities. Fix $k = 1$, $i = 1$, and consider the sequence

$$W_1^{(1,2)}, W_1^{(1,3)}, W_1^{(1,4)}, \ldots.$$

By passing to a subsequence and relabelling our indices, we may assume that

$$W_1^{(1,2)} = W_1^{(1,3)} = W_1^{(1,4)} = \ldots.$$

Similarly, now considering $i = 2$ we may also assume that

$$W_2^{(1,2)} = W_2^{(1,3)} = W_2^{(1,4)} = \ldots$$

by passing to another subsequence and again relabelling. We now define $V_i^{(1)} := W_i^{(1,2)}$ as above. Note that the $V_i^{(1)}$ still witness the fact that ρ_1 is (R_1, r, L, K)-locally coarsely separating.

We now choose $V_i^{(k)}$ for $k > 1$. Here, the choice is basically made for us. Indeed, let $k > 1$ and $i \in \{1, 2\}$ and consider the sequence $W_i^{(k,k)}, W_i^{(k,k+1)}, W_i^{(k,k+2)}, \ldots$. This sequence is now already constant, as by construction we have that each $W_i^{(k,m)}$ is a connected component of $N_{R_k}(x_0) \setminus N_K(\gamma)$. Moreover, we have for any $m \geq k$ that

$$V_i^{(1)} = W_i^{(1,2)} = W_i^{(1,m)} \subset W_i^{(k,m)},$$

for $i = 1, 2$. Since $V_i^{(1)}$ is connected subgraph of $N_{R_k}(x_0) \setminus N_K(\gamma)$, this completely determines $W_i^{(k,m)}$. We thus define $V_i^{(k)} = W_i^{(k,k)}$ inductively for all $k \in \mathbb{N}$, $i = 1, 2$.

By replacing the $U_i^{(k)}$ with the newly chosen $V_i^{(k)}$, we have the new property that

$$U_i^{(1)} \subset U_i^{(2)} \subset U_i^{(3)} \subset \ldots \subset \bigcup_{n\geq 1} U_i^{(n)} =: U_i$$

for each $i = 1, 2$. Now, we certainly have that U_1 and U_2 are disjoint and connected. Moreover, each U_i contains points arbitrarily far away from γ. We claim that there is no path p from U_1 to U_2 avoiding $N_K(\gamma)$. Indeed, if one existed then there would exist $R \geq 0$ such that $p \subset N_R(x_0) \setminus N_K(\gamma)$. Choose $k \geq 1$ such that $R_k \geq R$. Then we have that $U_1^{(k)}$ and $U_2^{(k)}$ are in the same connected component of $N_R(x_0) \setminus N_K(\rho_k)$. This is a contradiction. It follows that γ coarsely separates X. $\square$

We will need the following theorem of Papasoglu. In what follows, a *quasi-line* in X is the image of a uniformly proper map $\mathbb{R} \to X$. See [16] for a precise definition. Examples of uniformly proper maps include quasi-isometries and coarse embeddings, and this is sufficient for our purposes.

Theorem 5.8 (Papasoglu [16])**.** *Suppose that there is some quasi-line γ which coarsely separates X. Then G splits non-trivially over a two-ended subgroup.*

In all that follows, **we now additionally assume that G does not split over a two-ended subgroup**. In particular, the above theorem tells us that no bi-infinite geodesic coarsely separates X into multiple deep components. Combining Theorem 5.8 with Lemma 5.7, we get the following corollary.

Corollary 5.9. *For all $r > K > 0$, $L > 0$, there exists $R > 0$ such that there is no geodesic segment in X which is (R, r, L, K)-locally coarsely separating.*

By simply rephrasing this corollary in more detail, we arrive at what we call the 'diversion lemma'.

Lemma 5.10 (Diversion lemma)**.** *Given $K, L > 0$ there exists $R = R(L, K) > K$ such that the following holds:*

Let $\rho \subset X$ be a geodesic, and fix $x_0 \in \rho$ which is at distance at least $5R$ from the endpoints of ρ. Let $q_1, q_2 \subset N_R(x_0)$ be paths which are L-coarsely perpendicular to ρ, originating in $S_R(x_0)$ and terminating in $N_K(\rho)$. Then there exists a path from a vertex in q_1 to a vertex in q_2 contained entirely in $N_R(x_0) \setminus N_K(\rho)$.

Remark 5.11. Note that a similar statement does not hold for arbitrary finitely generated groups. Indeed, the lamplighter group contains a separating quasi-line but does not split over a two-ended subgroup [17].

5.3. **Constructing minors in groups which don't split.** We now begin constructing fat minors in X. First, note the following standard result about vertex transitive graphs.

Proposition 5.12. *Let X be a infinite, locally finite, vertex-transitive graph. Then X contains a bi-infinite geodesic.*

Proof. This follows easily from the Arzela–Ascoli theorem, together the fact that X contains arbitrarily long geodesic segments whose midpoints all coincide thanks to vertex-transitivity. $\square$

For the remainder of this section, let $\gamma \subset X$ denote some fixed bi-infinite geodesic.

Lemma 5.13 (Constant height paths)**.** *There exists $\varepsilon_0 > 0$ depending only on X such that the following holds:*

Let $r \geq 0$. Let $a, b \in X$ be such that $\mathrm{d}(a, \gamma) = \mathrm{d}(b, \gamma) = r$. Assume further that there exists a path connecting a to b disjoint from $N_{r-1}(\gamma)$. Then there exists a path p from a to b contained in $A_{r \pm \varepsilon_0}(\gamma)$.

Proof. This is an easy consequence of G being finitely presented. Take $\varepsilon_0 > 0$ to be the length of the longest defining relator in a presentation corresponding to X. $\square$

Recall the definition of a geodesic which is normal to a given subgraph (Definition 3.1). From now on, we will refer to a geodesic ρ which is normal to γ as simply a *normal*. The following lemma is an easy consequence of the fact that G is one-ended.

Lemma 5.14. *Fix $m > r \geq 0$. Let $U \subset X \setminus N_r(\gamma)$ denote the unique deep component. Then there exists infinitely many distinct normals of length m which terminate in U.*

In particular, we can find normals which are arbitrarily far apart from each other. Next, we note that almost constant-height paths are coarsely perpendicular to normals, with very controlled constants.

Lemma 5.15. *Let $r, \varepsilon > 0$. Let $\rho \subset X$ be a normal to γ and let $x_0 \in \rho$ be such that $\mathrm{d}(x_0, \gamma) = r$. Given $q \subset A_{r\pm\varepsilon}(\gamma)$, we have that q is L-coarsely perpendicular to ρ at x_0, where $L = \max\{2, \varepsilon\}$.*

Proof. This follows immediately from the triangle inequality. □

We now proceed with our general construction.

Lemma 5.16. *Let G be a one-ended finitely presented group which is not a virtual surface group, and suppose that G does not split over a two-ended subgroup. Then G is not asymptotically minor-excluded.*

Proof. Fix integers, $n, K > 0$. Fix $\varepsilon_0 > 0$ as in Lemma 5.13. Let $L = \max\{2, \varepsilon_0\}$. Choose $R = R(L, K) > 0$ as in the Diversion Lemma (5.10). Let $\gamma \subset X$ be some choice of bi-infinite geodesic.

Write $m = \binom{n}{2}$. Let $h = 3mR$. Given $i \in \{1, \ldots, m\}$, let $h_i = (3i - \frac{3}{2})R$. Note that $R < h_i < h - R$ for all i, and $|h_i - h_j| > 2R$ for all $i \neq j$. Let U denote the unique deep component of $X \setminus N_h(\gamma)$. By Lemma 5.14, let $\rho_1, \ldots, \rho_n$ be a collection of normals of γ of length h, all terminating in U. Since we can choose the ρ_i to be arbitrarily far apart, we may ensure that $N_{2R}(\rho_i) \cap N_{2R}(\rho_j) = \emptyset$ for all $i \neq j$. Without loss of generality, assume that $\rho_i(0) \in \gamma$ for all i.

Choose some ordering of the set of pairs

$$S = \{(i, j) : 1 \leq i < j \leq m\},$$

say $S = \{P_1, \ldots, P_m\}$, where $P_i = (x_i, y_i)$.

We will now attempt to construct a K_n-minor naïvely, before modifying it into a K-fat minor. For each $i \in \{1, \ldots, m\}$, apply Lemma 5.13 and let q_i be a path connecting $\rho_{x_i}(h_i)$ to $\rho_{y_i}(h_i)$ such that

$$q_i \subset A_{h_i \pm \varepsilon_0}(\gamma).$$

We will modify each q_i such that it avoids the K-neighbourhoods of all ρ_j where $j \notin \{x_i, y_i\}$. Once this is achieved, the union of the ρ_i and the q_j will form a K-fat K_n-minor, where the ρ_i are the branch sets and the q_j are the edge paths.

To ease notation, let us fix i and write $q = q_i$. Write $\rho_+ = \rho_{x_i}$, and $\rho_- = \rho_{y_i}$. Suppose there exists some ρ_j distinct from ρ_+, ρ_- such that q intersects $N_K(\rho_j)$. Write $x_0 = \rho_j(h_i)$. Parameterise q such that it begins at ρ_+ and terminates at ρ_-. Let $q_+ \subset q$ denote the very first contiguous subpath of q which is contained in $N_R(x_0)$, begins in $S_R(x_0)$ and terminates in $N_K(\rho_j)$. Similarly, let $q_- \subset q$ be the very last contiguous subpath of q which is contained in $N_R(x_0)$, begins in $N_K(\rho_j)$ and terminates in $S_R(x_0)$. Let $a \in S_R(x_0)$ be the initial vertex of q_+ and $b \in S_R(x_0)$ be the terminal vertex of q_-. By the diversion lemma (5.10), there is a path through $N_R(x_0)$ connecting a to b which avoids $N_K(\rho_j)$. We replace the midsection of q with this new path.

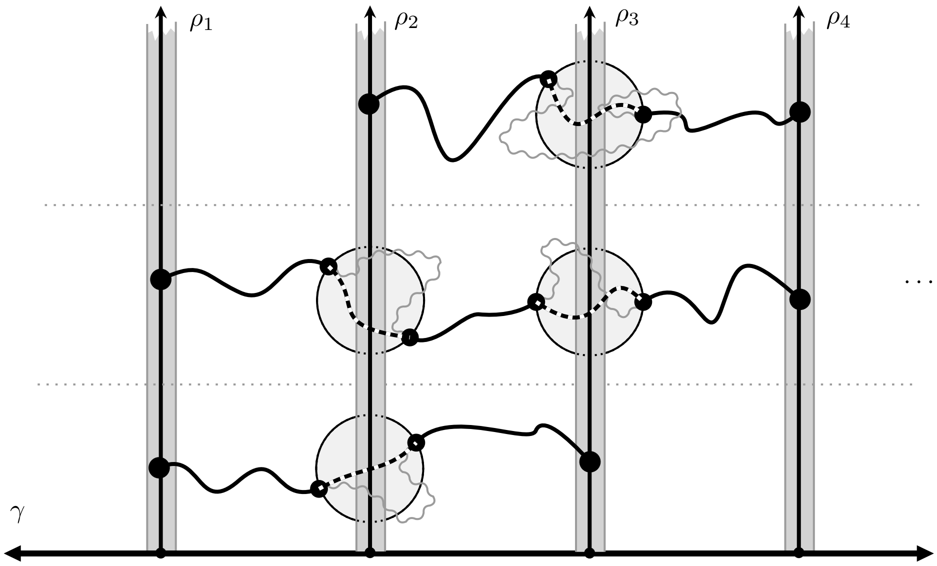

FIGURE 7. Constructing a fat K_n in a group which doesn't split. Using the diversion lemma (5.10), the 'wiggly' paths which break the fatness of the minor are replaced with the striped paths which avoid the K-neighbourhoods of the ρ_i.

If we repeat this process at every one of the 'bad intersections', it is clear that the resulting figure is a K-fat K_n-minor. See Figure 7 for a cartoon of this construction. The proposition follows. □

Finally, we can deduce our main result.

Theorem A. *A finitely* ***presented*** *group is asymptotically minor-excluded if and only if some finite index subgroup of G admits a planar Cayley graph.*

Proof. If a finite index subgroup of G admits a planar Cayley graph, then clearly G is quasi-isometric to a planar graph. Since asymptotic minors are invariant under quasi-isometry, we conclude that G is asymptotically minor-excluded.

Now, suppose that no finite index subgroup of G admits a planar Cayley graph. Let X be a Cayley graph of G. We first assume that G is one-ended, and so G is not a virtual surface group. If G does not split over a two-ended subgroup, then by Lemma 5.16 we are done. Thus, let us assume that G splits as an amalgam $G \cong A *_C B$ or an HNN extension $G \cong A*_C$ where C is two-ended. For the sake of clarity, we will assume that G is an HNN extension; the amalgam case is identical.

First, note that A is itself necessarily finitely presented [11]. By Dunwoody's accessibility theorem [5], either A is virtually free or A contains a one-ended, finitely presented subgroup H. If H is a virtual surface group then we are done by Theorem 4.4. Otherwise, we may repeat the preceding argument with H in place of G. We will return to this inductive step shortly.

Suppose now that A is virtually free. By a corollary of the combination theorem of Bestvina–Feighn [1, 2], if G is not hyperbolic then G contains some Baumslag–Solitar subgroup. Since G is not virtually $\mathbb{Z}^2$, it follows from Proposition 2.1 and Theorem 4.4 that G is not asymptotically minor-excluded. Thus we assume that G is hyperbolic. By a theorem of Wise [20, Thm. 4.19] we have that G is residually finite and virtually torsion-free, and thus G contains a finite index subgroup K isomorphic to a one-ended, hyperbolic graph of free groups with cyclic edge groups.

By a theorem of Wilton [19], we have that K contains a one-ended subgroup H of infinite index. A theorem of Karrass and Solitar [12] implies that K is coherent[1] and so H is finitely presented. If H a virtual surface group, then we are once again done by Theorem 4.4, otherwise we repeat this argument with H in place of G.

In each of the above cases considered, we have either concluded that G is not asymptotically minor-excluded, or we have found an infinite-index, one-ended, finitely presented subgroup H which is not a virtual surface group. We now repeat this process by iteratively passing to this subgroup at each stage. If this process never terminates then we find a descending chain

$$G \geq H_1 \geq H_2 \geq \ldots \geq H_i \geq \ldots$$

of one-ended subgroups, where $|H_i : H_{i+1}| = \infty$ for every $i \geq 1$. By Theorem 3.4 we deduce that G is not asymptotically minor-excluded.

Now, let us assume that G is infinite-ended and does not virtually admit a planar Cayley graph. By Dunwoody's accessibility theorem [5], G splits as a graph of groups with finite edge groups and vertex groups with at most one end. Let $H_1, \ldots, H_n$ be the one-ended vertex groups of this splitting. There must be at least one such vertex group lest G is virtually free. Since G is not virtually free we have that $n \geq 1$. If each H_i contains a finite index subgroup K_i with a planar Cayley graph, then each H_i is residually finite and virtually torsion-free. It is an easy exercise to deduce that G itself is also virtually torsion-free and so G is virtually a free product of free and surface groups. It follows that G virtually admits a planar Cayley graph, a contradiction. Thus, some H_i must not virtually admit a planar Cayley graph, and the result follows from the one-ended case discussed above. □

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[1] Recall that a finitely presented group is said to be *coherent* if every finitely generated subgroup is finitely presented.

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