# Rigidity expander graphs

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**Abstract**

Jordán and Tanigawa recently introduced the $d$-dimensional algebraic connectivity $a_d(G)$ of a graph $G$. This is a quantitative measure of the $d$-dimensional rigidity of $G$ which generalizes the well-studied notion of spectral expansion of graphs. We present a new lower bound for $a_d(G)$ defined in terms of the spectral expansion of certain subgraphs of $G$ associated with a partition of its vertices into $d$ parts. In particular, we obtain a new sufficient condition for the rigidity of a graph $G$. As a first application, we prove the existence of an infinite family of $k$-regular $d$-rigidity-expander graphs for every $d \geq 2$ and $k \geq 2d+1$. Conjecturally, no such family of $2d$-regular graphs exists. Second, we show that $a_d(K_n) \geq \frac{1}{2}\left\lfloor \frac{n}{d} \right\rfloor$, which we conjecture to be essentially tight. In addition, we study the extremal values $a_d(G)$ attained if $G$ is a minimally $d$-rigid graph.

## 1 Introduction

Graph expansion is one of the most influential concepts in modern graph theory, with numerous applications in discrete mathematics and computer science (see [13, 18]). Intuitively speaking, an expander is a "highly-connected" graph, and a standard way to quantitatively measure the connectivity, or expansion, of a graph uses the spectral gap in its Laplacian matrix. A main theme in the study of expander graphs deals with the construction of sparse expanders. In particular, bounded-degree regular expander graphs have been studied extensively in various areas of mathematics [8, 19, 23, 7, 20]. This paper studies a generalization of spectral graph expansion that was

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recently introduced by Jordán and Tanigawa via the theory of graph rigidity [14].

A $d$-dimensional framework is a pair $(G, p)$ consisting of a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$. The framework is called $d$-rigid if every continuous motion of the vertices starting from $p$ that preserves the distance between every two adjacent vertices in $G$, also preserves the distance between *every pair* of vertices; see e.g. [4, 9] for background on framework rigidity.

Asimow and Roth showed in [1] that if the map $p$ is generic (e.g. if the $d|V|$ coordinates of $p$ are algebraically independent over the rationals), then the framework rigidity of $(G, p)$ does not depend on the map $p$. Moreover, they showed that for a generic $p$, rigidity coincides with the following stronger linear-algebraic notion of infinitesimal rigidity.

For every $u, v \in V$ we define $d_{uv} \in \mathbb{R}^d$ by

$$d_{uv} = \begin{cases} \frac{p(u)-p(v)}{\|p(u)-p(v)\|} & \text{if } p(u) \neq p(v), \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbf{v}_{u,v} := (1_u - 1_v) \otimes d_{uv} \in \mathbb{R}^{d|V|}$, where $\{1_u\}_{u \in V}$ is the standard basis of $\mathbb{R}^{|V|}$ and $\otimes$ denotes the Kronecker product. Equivalently,

$$\mathbf{v}_{u,v}^T = \begin{array}{c} \begin{array}{ccccccccccc} & & & u & & & & v & & & \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & d_{uv}^T & 0 & \dots & 0 & d_{vu}^T & 0 & \dots & 0 \end{pmatrix} \end{array}.$$

The (normalized) *rigidity matrix* $R(G, p) \in \mathbb{R}^{d|V| \times |E|}$ is the matrix whose columns are the vectors $\mathbf{v}_{u,v}$ for all $\{u, v\} \in E$. We always assume that the image $p(V)$ does not lie on any affine hyperplane in $\mathbb{R}^d$. In such a case, it is possible to show (see [1]) that $\operatorname{rank}(R(G, p)) \leq d|V| - \binom{d+1}{2}$. The framework $(G, p)$ is called *infinitesimally rigid* if this bound is attained, that is, if $\operatorname{rank}(R(G, p)) = d|V| - \binom{d+1}{2}$.

A graph $G$ is called *rigid in* $\mathbb{R}^d$, or *$d$-rigid*, if it is infinitesimally rigid with respect to some map $p$ (or, equivalently, if it is infinitesimally rigid for all generic maps [1]).

For $d = 1$ and an injective map $p : V \to \mathbb{R}^d$, the rigidity matrix $R(G, p)$ is equal to the incidence matrix of $G$, hence both notions of rigidity coincide with graph connectivity. One can extend this analogy and define a higher dimensional version of the graph's Laplacian matrix, that is called the *stiffness matrix* of $(G, p)$, and is defined by

$$L(G, p) = R(G, p)R(G, p)^T \in \mathbb{R}^{d|V| \times d|V|}.$$

We denote by $\lambda_i(A)$ the $i$-th smallest eigenvalue of a symmetric matrix $A$. Since $\operatorname{rank}(L(G, p)) = \operatorname{rank}(R(G, p)) \leq d|V| - \binom{d+1}{2}$, the kernel of $L(G, p)$ is of dimension at least $\binom{d+1}{2}$. Therefore, $\lambda_{\binom{d+1}{2}+1}(L(G, p)) \neq 0$ if and only if $(G, p)$ is infinitesimally rigid.

In [14], Jordán and Tanigawa defined the *d-dimensional algebraic connectivity* of $G$, $a_d(G)$, as

$$a_d(G) = \sup\left\{\lambda_{\binom{d+1}{2}+1}(L(G,p))\,\middle|\, p: V \to \mathbb{R}^d\right\}.$$

For $d = 1$, $L(G,p)$ coincides with the graph Laplacian matrix $L(G)$, and $a_1(G) = a(G)$ is the usual algebraic connectivity, or Laplacian spectral gap, of $G$, introduced by Fiedler in [6]. For every $d \geq 1$, $a_d(G) \geq 0$ since $L(G,p)$ is positive semi-definite, and $a_d(G) > 0$ if and only if $G$ is $d$-rigid.

The following notion of *rigidity expander graphs* extends the classical notion of (spectral) expander graphs, corresponding to the $d = 1$ case:

**Definition 1.1.** Let $d \geq 1$. A family of graphs $\{G_i\}_{i\in\mathbb{N}}$ of increasing size is called a *family of $d$-rigidity expander graphs* if there exists $\epsilon > 0$ such that $a_d(G_i) \geq \epsilon$ for all $i \in \mathbb{N}$.

It is well known that, for every $k \geq 3$, there exist families of $k$-regular (1-dimensional) expander graphs (see e.g. [13]). Our main result is an extension of this fact to general $d$:

**Theorem 1.2.** *Let $d \geq 1$ and $k \geq 2d+1$. Then, there exists an infinite family of $k$-regular $d$-rigidity expander graphs.*

It was conjectured by Jordán and Tanigawa that families of $2d$-regular $d$-rigidity expanders do not exist (see [14, Conj. 2] for the statement in the $d = 2$ case, and see [16, Conj. 6.2] for the general case), and clearly families of $k$-regular $d$-rigidity expanders do not exist for $k < 2d$ since, for $n$ large enough, such graphs have less than $dn - \binom{d+1}{2}$ edges, and are therefore not even $d$-rigid. Therefore, assuming this conjecture, our result is sharp.

Our main tool for the proof of Theorem 1.2 is a new lower bound on $a_d(G)$, given in terms of the (1-dimensional) algebraic connectivity of certain subgraphs of $G$ associated with a partition of its vertex set into $d$ parts. For convenience, we let $a(G) = \infty$ if $G$ consists of a single vertex.

Let $G = (V,E)$ be a graph, and let $A, B \subset V$ be two disjoint sets. We denote by $G[A]$ the subgraph of $G$ induced on $A$, and by $G(A,B)$ the subgraph of $G$ with vertex set $A \cup B$ and edge set $E(A,B) = \{e \in E : |e\cap A| = |e \cap B| = 1\}$. In addition, we say that a partition $V = A_1 \cup\cdots\cup A_d$ is *non-trivial* if $A_i \neq \emptyset$ for all $i = 1,\ldots,d$.

**Theorem 1.3.** *Let $d \geq 2$. For every graph $G = (V,E)$ and a non-trivial partition $V = A_1 \cup \cdots \cup A_d$ there holds*

$$a_d(G) \geq \min\left(\left\{a(G[A_i])\right\}_{1\leq i\leq d} \bigcup \left\{\frac{1}{2}a(G(A_i,A_j))\right\}_{1\leq i<j\leq d}\right).$$

*In particular, if $G[A_i]$ is connected for all $i \in [d]$ and $G(A_i,A_j)$ is connected for all $1 \leq i < j \leq d$, then $G$ is $d$-rigid.*

**Remark 1.4.** *In the $d = 2$ dimensional case, the statement in Theorem 1.3 can be slightly improved (by removing the constant $1/2$) to*

$$a_2(G) \geq \min\{a(G[A_1]), a(G[A_2]), a(G(A_1, A_2))\},$$

*for every non-trivial partition $A_1, A_2$ of $V$.*

In the case $d = 2$, we can think of Theorem 1.3 as a quantitative version of (a special case of) a theorem of Crapo [5, Theorem 7]. For $d \geq 3$, Theorem 1.3 seems to give, in addition to a lower bound on $a_d(G)$, a new sufficient condition for $d$-rigidity, which we believe to be of independent interest (this sufficient condition could also be derived from [17, Theorem 5.5]).

To derive Theorem 1.2 from Theorem 1.3 we consider a balanced partition of the vertex set, and construct each of the $\binom{d+1}{2}$ subgraphs induced by the partition in separate. In the main case $k = 2d + 1$, our "building blocks" are (1-dimensional) expander graphs with maximum degree 3 and a large proportion of vertices of degree 2. Such graphs are constructed by subdividing edges in classical constructions of 3-regular expander graphs. In Theorem 4.1 below, we hedge the effect of edge subdivision on the algebraic connectivity of the graph.

For another application of Theorem 1.3, we derive a slight improvement of the previously known lower bound for $a_d(K_n)$ from [16, Theorem 1.5].

**Corollary 1.5.** *Let $d \geq 3$ and $n \geq d + 1$. Then*

$$a_d(K_n) \geq \frac{1}{2} \left\lfloor \frac{n}{d} \right\rfloor.$$

In addition, we establish the following upper bound on $a_d(G)$, generalizing the case $d = 2$ proved by Jordán and Tanigawa in [14, Theorem 4.2].

**Theorem 1.6.** *Let $d \geq 2$, and let $G$ be a graph. Then,*

$$a_d(G) \leq a(G).$$

Theorem 1.6 was proved recently and independently in [22]. Our proof is different, using the probabilistic method, and we believe it to be of independent interest.

Finally, we study how small and how large can $a_d(G)$ be provided that $G$ is a minimally $d$-rigid graph. A graph $G$ is called *minimally $d$-rigid* if it is $d$-rigid, but $G \setminus e$ is not $d$-rigid for every edge $e \in E$. For $d = 1$, these are exactly spanning trees. This question is related to the aforementioned conjecture that no $2d$-regular $d$-rigidity expanders exist (see Conjecture 8.2).

Among the minimally $d$-rigid graphs, $a_d(G)$ is maximized by a $d$-analog of the star graph. For every $d \geq 1$ and $n \geq d + 1$, let $S_{n,d}$ be the graph consisting of a clique of size $d$, and $n - d$ additional vertices, each adjacent to all of the vertices of the clique, and not adjacent to any other vertex. It is easy to check that $S_{n,d}$ is minimally $d$-rigid.

**Theorem 1.7.** *For every $d \geq 1$ and $T \neq K_2, K_3$ a minimally $d$-rigid graph there holds*

$$a_d(T) \leq 1,$$

*and equality holds if $T = S_{n,d}$.*

This extends a result of Fiedler (see [6, 4.1], more explicitly stated by Merris in [21, Cor. 2]) corresponding to the case $d = 1$. Note that for $T = K_2$ (which is a minimally 1-rigid graph), we have $a_1(K_2) = 2$, and for $T = K_3$ (which is a minimally 2-rigid graph), we have $a_2(K_3) = \frac{3}{2}$ (see [14, Theorem 4.4]).

Considering the other extreme, of minimizers of $a_d$ among all $n$-vertex $d$-rigid graphs, it was shown by Fiedler in [6] that $a(G) \geq a(P_n) = 2(1 - \cos(\pi/n))$ for every connected graph $G$, where $P_n$ is the $n$-vertex path (see [11] for an explicit statement). We conjecture that a similar situation holds in higher dimensions: in Subsection 7.1 we define *generalized path graphs* $P_{n,d}$, which are certain $n$-vertex minimally $d$-rigid graphs, and provide in Proposition 7.6 bounds on their $d$-dimensional algebraic connectivity implying that $a_d(P_{n,d}) = \Theta_d(1/n^2)$. We conjecture that these graphs are extremal:

**Conjecture 1.8.** *Let $G$ be a $d$-rigid graph on $n$ vertices. Then,*

$$a_d(G) \geq a_d(P_{n,d}).$$

The paper is organized as follows: In Section 2 we present some results about stiffness matrices that are used later. In particular, we recall the definition of the *lower stiffness matrix* $L^-(G,p)$ introduced in [16]. In Section 3 we prove Theorem 1.3. In Section 4 we study the effects of edge subdivisions on the spectral gap of a graph. Section 5 contains the proof of our main result, Theorem 1.2, showing the existence of $k$-regular $d$-rigidity expanders for $k \geq 2d+1$. In Section 6 we give a proof of the upper bound $a_d(G) \leq a(G)$ (Theorem 1.6). In Section 7 we study the $d$-dimensional algebraic connectivity of minimally $d$-rigid graphs. We conclude in Section 8 with several open problems and directions for further research.

# 2 Preliminaries

Occasionally, it is simpler to work with the *lower stiffness matrix* of the framework $(G,p)$, defined by

$$L^-(G,p) = R(G,p)^T R(G,p) \in \mathbb{R}^{|E|\times|E|}.$$

By standard linear algebra, we have that $\text{rank}(L(G,p)) = \text{rank}(L^-(G,p)) = \text{rank}(R(G,p))$ and that the non-zero eigenvalues of $L(G,p)$, with multiplicities, coincide with those of $L^-(G,p)$. In particular, assuming that

$|E| \geq d|V| - \binom{d+1}{2}$, we have

$$\lambda_k(L(G,p)) = \lambda_{|E|-d|V|+k}(L^-(G,p)), \tag{1}$$

for every $k \geq \binom{d+1}{2} + 1$. In addition, the entries of $L^-(G,p)$ are given explicitly by the following lemma.

**Lemma 2.1** ([16, Lemma 2.1])**.** *Let $(G,p)$ be a $d$-dimensional framework. Then, for every $e, e' \in E(G)$,*

$$L^-(G,p)_{e,e'} = \begin{cases} 2 & \text{if } e = e' = \{u,v\} \text{ and } p(u) \neq p(v), \\ d_{uv} \cdot d_{uw} & \text{if } e = \{u,v\},\ e' = \{u,w\} \\ 0 & \text{otherwise,} \end{cases}$$

where $d_{uv} \cdot d_{uw}$ denotes the dot product. In the case that $e = \{u,v\}$ and $e' = \{u,w\}$, we denote by $\theta(e,e')$ the angle between $d_{uv}$ and $d_{uw}$. Hence, $L^-(G,p)_{e,e'} = \cos(\theta(e,e'))$ (by convention, $\cos(\theta(e,e')) = 0$ if $d_{uv} = 0$ or $d_{uw} = 0$).

## 3 A lower bound on $a_d(G)$

We turn to the proof of Theorem 1.3, starting with the following very simple lemma about the eigenvalues of a block diagonal matrix.

For convenience, given a "$0 \times 0$" matrix $M$, we define $\lambda_1(M) = \infty$.

**Lemma 3.1.** *Let $M \in \mathbb{R}^{n \times n}$ be a block diagonal matrix, with blocks $M_1, \ldots, M_k$, where $M_i \in \mathbb{R}^{n_i \times n_i}$ is symmetric for every $1 \leq i \leq k$. Then, for every $1 \leq m \leq n$ and $r_1, \ldots, r_k$ satisfying $m = 1 - k + \sum_{i=1}^{k} r_i$ there holds*

$$\lambda_m(M) \geq \min\{\lambda_{r_i}(M_i) : 1 \leq i \leq k\}.$$

*Proof.* TOPROVE 0 □

**Remark 3.2.** *Note that, under the convention $\lambda_1(M_i) = \infty$ for $M_i \in \mathbb{R}^{0 \times 0}$, Lemma 3.1 holds also if we allow values $n_i = 0$ and $r_i = 1$ for one or more $i \in [k]$.*

*Proof.* TOPROVE 1 □

To derive the stronger bound in the case $d = 2$ mentioned in Remark 1.4, we note that in this case $L^-$ itself is a block diagonal matrix with 3 blocks which are the 1-dimensional lower stiffness matrices $L^-(G[A_1], q_1)$, $L^-(G[A_2], q_2)$ and $L^-(G(A_1, A_2), q_{12})$ (that is, there is no need for the "correction" term $Q$). Therefore, by the same reasoning we applied to $M$ in the general case, we find that

$$a_2(G) \geq \lambda_m(L^-) \geq \min(\{a_2(G[A_1]), a_2(G[A_2]), a_2(G(A_1, A_2))\}),$$

where $m = |E| - 2|V| + \binom{2+1}{2} + 1$.

**Remark 3.3.** *The criterion for $d$-rigidity given by Theorem 1.3 is minimal in terms of the edge count. Namely, the assumption that all the $\binom{d+1}{2}$ graphs in the partition are connected implies that there are at least $|A_i| - 1$ edges in $G[A_i]$ for $i \in [d]$, and at least $|A_i| + |A_j| - 1$ edges in $G(A_i, A_j)$, for $1 \le i < j \le d$. In total, there needs be at least*

$$\sum_{i=1}^{d}(|A_i| - 1) + \sum_{1 \le i < j \le d}(|A_i| + |A_j| - 1) = d|V| - \binom{d+1}{2}$$

*edges in $G$ — precisely the number of edges in a minimally $d$-rigid graph.*

As a consequence of Theorem 1.3, we obtain a simple proof of Corollary 1.5, giving a lower bound on the $d$-dimensional algebraic connectivity of $K_n$.

*Proof.* TOPROVE 2 □

**Remark 3.4.** *For $d = 2$, it was shown by Jordán and Tanigawa in [14], relying on a result by Zhu ([25]), that $a_2(K_n) = n/2$. Dividing the vertex set into two parts of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively, we obtain a bound of $a_2(K_n) \ge \lfloor \frac{n}{2} \rfloor$. This gives a simple proof of the sharp lower bound in the case that $n$ is even.*

We conjecture that the lower bound we obtained in Corollary 1.5 is almost tight:

**Conjecture 3.5.** *Let $d \ge 3$ and $n \ge d + 1$. Then,*

$$a_d(K_n) = \begin{cases} 1 & \text{if } d + 1 \le n \le 2d, \\ \frac{n}{2d} & \text{if } n \ge 2d. \end{cases}$$

Note that this is a strong version of Conjecture 6.1 in [16].

## 4 Expansion under edge subdivisions

The goal of this section is to prove the following theorem regarding the effect of edge subdivision on the algebraic connectivity of a graph. Let $G = (V, E)$ be a graph without isolated vertices. Given an edge $e$ in $G$, replacing $e$ with an induced path containing $m \ge 0$ new internal vertices is called a subdivision of $e$ with $m$ vertices.

**Theorem 4.1.** *Let $G$ be a connected graph with minimum degree at least 2 and maximum degree $\Delta$, and let $G'$ be obtained from $G$ by a subdivision of each edge of $G$ with at most $m$ vertices. Then,*

$$a(G') \ge \frac{\min\left\{\frac{1}{\Delta}a(G), 4\right\}}{2(m+1)^2}.$$

Let $D(G)$ be the diagonal matrix with $D(G)_{i,i} = \deg_G(i)$, and $\mathcal{L}(G) = D(G)^{-\frac{1}{2}} L(G) D(G)^{-\frac{1}{2}}$ be the normalized Laplacian of $G$. The effect of edge subdivision on the normalized Laplacian was studied by Xie, Zhang and Comellas in [24]. Denote by $s(G)$ the subdivision of $G$, that is, the graph obtained from $G$ subdividing each edge of $G$ with 1 vertex, thus subdividing each edge into two edges. Furthermore, let $s^k(G)$ be the $k$-th iterated subdivision. That is, $s^k(G) = s(s^{k-1}(G))$ (where $s^0(G) = G$).

**Lemma 4.2** ([24, Lemma 3.1])**.** *If $\lambda \neq 1$ is an eigenvalue of $\mathcal{L}(s(G))$ then $2\lambda(2-\lambda)$ is an eigenvalue of $\mathcal{L}(G)$.*

In order to relate the spectral gap of the normalized Laplacian to the one of the unnormalized Laplacian, we will use the following result due to Higham and Cheng [12].

**Lemma 4.3** ([12, Theorem 3.2])**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and let $X \in \mathbb{R}^{n\times m}$, for some $m \leq n$. Then, for every $1 \leq i \leq m$,*

$$\lambda_i(X^T A X) = \theta_i \mu_i,$$

*where*

$$\lambda_i(A) \leq \mu_i \leq \lambda_{i+n-m}(A)$$

*and*

$$\lambda_1(X^T X) \leq \theta_i \leq \lambda_m(X^T X).$$

**Lemma 4.4.** *Let $G$ be a graph on $n$ vertices, with minimum degree $\delta > 0$ and maximum degree $\Delta$. Then, for all $2 \leq i \leq n$,*

$$\delta \leq \frac{\lambda_i(L(G))}{\lambda_i(\mathcal{L}(\mathcal{G}))} \leq \Delta.$$

*Proof.* TOPROVE 3 □

**Proposition 4.5.** *Let $G$ be a connected graph with minimum degree at least 2 and maximum degree $\Delta$. Then,*

$$a(s^k(G)) \geq \frac{\min\left\{\frac{2}{\Delta}a(G), 8\right\}}{4^k}.$$

*Proof.* TOPROVE 4 □

The next lemma establishes that algebraic connectivity is monotone with respect to edge subdivision.

**Lemma 4.6.** *Suppose that $G' = (V', E')$ is obtained from $G = (V, E)$ by a subdivision of an edge $e = uv$ of $G$ with 1 new vertex $w$. Then, $a(G') \leq a(G)$.*

*Proof.* TOPROVE 5 □

*Proof.* TOPROVE 6 □

Next, we apply Theorem 4.1 to show the existence of (1-dimensional) expander graphs with a desired degree sequence that are used subsequently as building blocks in the construction of $d$-dimensional rigidity expanders in the proof of Theorem 1.2.

**Corollary 4.7.** *For every $d \geq 1$ there exists $c = c(d) > 0$ and an infinite family of $2dn$-vertex bipartite graphs $(H_n)_{n=3}^{\infty}$ such that $a(H_n) \geq c$ and each part consists of $n$ vertices of degree 3 and $(d-1)n$ vertices of degree 2.*

*Proof.* TOPROVE 7 □

## 5 Existence of rigidity expander graphs

We turn to combine the results in the previous sections to establish the existence of a family of $k$-regular $d$-rigidity expanders for every $k \geq 2d+1$.

*Proof.* TOPROVE 8 □

## 6 An upper bound on $a_d(G)$

In this section we give a proof of Theorem 1.6, stating that for every graph $G$, $a_d(G) \leq a(G)$. In Section 8 below we discuss whether this theorem is tight.

We will need the following simple result about stiffness matrices.

**Lemma 6.1** (Jordán-Tanigawa [14, 3.2])**.** *Let $G = (V, E)$ be a graph, and let $p : V \to \mathbb{R}^d$ and $x \in \mathbb{R}^{d|V|}$. Then*

$$x^T L(G,p)x = \sum_{\{u,v\} \in E} \langle x(u) - x(v), d_{uv} \rangle^2 ,$$

*where $x(u) \in \mathbb{R}^d$ consists of the $d$ coordinates of $x$ corresponding to the vertex $u$.*

We will also need the following lemma from [16], that states that when computing $a_d(G)$, it is enough to consider maps $p : V \to \mathbb{R}^d$ that are embeddings (that is, injective).

**Lemma 6.2** ([16, Lemma 2.4])**.** *Let $G = (V, E)$ be a graph, and $d \geq 1$. Then*

$$a_d(G) = \sup \left\{ \lambda_{\binom{d+1}{2}+1}(L(G,p)) \,\middle|\, p : V \to \mathbb{R}^d, \;\; p \text{ is injective} \right\}.$$

*Proof.* TOPROVE 9 □

## 7 Minimally rigid graphs

In this section we study the extremal values of the $d$-dimensional algebraic connectivity of minimally $d$-rigid graphs. In Proposition 7.1 we prove the upper bound $a_d(T) \leq 1$ for minimally $d$-rigid graphs, and in Proposition 7.5 we show that the upper bound is attained for "generalized star" graphs. This establishes the proof of Theorem 1.7. The section is concluded with a discussion about generalized path graphs and their algebraic connectivity.

**Proposition 7.1.** *Let $d \geq 1$, and let $T \neq K_2, K_3$ be a minimally $d$-rigid graph. Then,*

$$a_d(T) \leq 1.$$

For the proof of Proposition 7.1 we will need the following results.

**Lemma 7.2** (Jordán-Tanigawa [14, Lemma 4.5])**.** *Let $d \geq 1$. Let $G = (V, E)$ and $v \in V$. Then,*

$$a_d(G \setminus v) \geq a_d(G) - 1.$$

**Theorem 7.3** ([16, Theorem 1.2])**.** *Let $d \geq 3$. Then*

$$a_d(K_{d+1}) = 1.$$

*Proof.* TOPROVE 10 □

Let $d \geq 1$ and $n \geq d+1$. Let $S_{n,d}$ be the graph on vertex set $[n]$ with edge set

$$E(S_{n,d}) = \{\{i,j\} : i \in [d], j \in [n] \setminus \{i\}\}.$$

It is easy to check that $S_{n,d}$ is minimally $d$-rigid.

We consider the following mapping of the vertices of $S_{n,d}$ to $\mathbb{R}^d$: Let $e_1, \ldots, e_d \in \mathbb{R}^d$ be the standard basis vectors. We define $p^* : [n] \to \mathbb{R}^d$ by

$$p^*(i) = \begin{cases} e_i & \text{if } 1 \leq i \leq d, \\ 0 & \text{if } d < i \leq n. \end{cases}$$

**Proposition 7.4.** *The spectrum of $L(S_{n,d}, p^*)$ is*

$$\left\{ 0^{(\binom{d+1}{2})}, 1^{(dn - \binom{d+1}{2} - d)}, (n - d/2)^{(d-1)}, n^{(1)} \right\}$$

*(where the superscript $(m)$ indicates multiplicity $m$ of the corresponding eigenvalue). In particular, $\lambda_{\binom{d+1}{2}+1}(L(S_{n,d}, p^*)) = 1$.*

*Proof.* TOPROVE 11 □

**Proposition 7.5.** *Let $d \geq 1$ and $n \geq d+1$. Then, unless $d = 2$ and $n = 3$, we have*

$$a_d(S_{n,d}) = 1.$$

*Proof.* TOPROVE 12 □

It would be interesting to determine whether the graphs $S_{n,d}$ are the only extremal cases in Theorem 1.7 (for $d = 1$ this is a result of Merris, [21, Cor. 2]).

### 7.1 Generalized path graphs

Let $n \geq d+1$. Let $P_{n,d}$ be the graph on vertex set $[n]$ with edges

$$\{\{i,j\} : 1 \leq i < j \leq n,\, j-i \leq d\}.$$

Note that, for $d = 1$, $P_n = P_{n,1}$ is just the path with $n$ vertices. It is not hard to check that, for $n \geq d+1$, $P_{n,d}$ is minimally rigid in $\mathbb{R}^d$.

As mentioned in the introduction, Fiedler [6] showed that $a_1(G) \geq a_1(P_n) = 2(1-\cos(\pi/n))$ for every connected graph $G$. For $d > 1$, we do not know the exact value of $a_d(P_{n,d})$, but the following result gives us its order of magnitude:

**Proposition 7.6.** *Let $d \geq 2$ and $n \geq d+1$. Then*

$$1-\cos\left(\frac{\pi}{2}\left\lceil\frac{n}{d}\right\rceil^{-1}\right) \leq a_d(P_{n,d}) \leq 2d - 2\sum_{k=1}^{d}\cos(2k\pi/n).$$

*For $d = 2$ we have a slightly better lower bound, $a_2(P_{n,2}) \geq 2(1-\cos(\pi/n))$.*

For large $n$, we have $1-\cos\left(\frac{\pi}{2}\left\lceil\frac{n}{d}\right\rceil^{-1}\right) \approx 1-\cos\left(\frac{d\pi}{2n}\right) \approx \frac{d^2\pi^2}{8n^2}$. Moreover, $2d - 2\sum_{k=1}^{d}\cos(2k\pi/n) \leq \frac{2\pi^2 d(d+1)(2d+1)}{3n^2}$. Therefore, $a_d(P_{n,d}) = \Theta_d(1/n^2)$.

*Proof.* TOPROVE 13 □

## 8 Concluding remarks

Many fascinating open problems suggest themselves. In this paper, we showed that families of $k$-regular $d$-rigidity expanders exist for $k > 2d$, and it is natural to seek for the best possible construction.

**Problem 8.1.** *Let $d \geq 2$ and $k > 2d$ be integers. What is*

$$c_d(k) := \sup_{(G_n)_{n\in\mathbb{N}}} \liminf_{n} a_d(G_n),$$

*where $(G_n)_{n\in\mathbb{N}}$ runs over families of $k$-regular graphs of increasing size?*

The 1-dimensional case of this problem is perhaps the most important question in the theory of expander graphs. The Alon-Boppana Theorem asserts an upper bound of $c_1(k) \leq k - 2\sqrt{k-1}$, and constructions attaining this bound are known as (one-sided) Ramanujan graphs [19, 20]. For $d \geq 2$, the proof of Theorem 1.2 gives a lower bound for $c_d(k)$ which applies to all $k \geq 2d+1$, and whose rate of decay is in the order of $1/d^2$ as $d \to \infty$. If $k \geq td$ for some $t \geq 3$, one can easily adapt our methods and attain a lower bound for $c_d(k) \geq c_1(t)/2$ that is independent of $d$. That is, by using $t$-regular bipartite Ramanujan graphs in Section 5 instead of the subdivided graphs from Corollary 4.7. The question whether $c_d(2d+1) \to 0$ as $d \to \infty$ remains open.

It is known that $a(T) = O(1/n)$ if $T$ is a bounded-degree tree [15], and we conjecture that this phenomenon extends to higher dimensions.

**Conjecture 8.2.** *Fix integers $d, b \geq 1$. Then,*

$$\max_{G_n} a_d(G_n) \to 0 \quad \text{as } n \to \infty,$$

*where $G_n$ runs over all minimally $d$-rigid $n$-vertex graphs of max-degree $b$.*

A stronger but still plausible conjecture — that

$$\sup_{(G_n, p_n)} \lambda_{d(d+1)+1}(L(G_n, p_n)) \to 0 \quad \text{as } n \to \infty,$$

where $(G_n, p_n)$ runs over all $d$-frameworks of minimally $d$-rigid $n$-vertex graphs of maximum degree $b$ — would imply that no $2d$-regular $d$-rigidity expanders exist, via interlacing of spectra under adding an edge to a graph (see [16, Theorem 2.3]).

Regarding the relations between the values $a_d(G)$, for $G$ fixed and $d$ that varies, we propose the following strengthening of Theorem 1.6:

**Conjecture 8.3.** *(Monotonicity) Let $1 \leq d' < d$ be integers and $G$ a graph on $n$ vertices. Then, $a_d(G) \leq a_{d'}(G)$.*

In addition, the fact that we do not know if the bound in Theorem 1.6 is tight raises the following problem:

**Problem 8.4.** *What is $\sup_G(a_d(G)/a(G))$ over all connected graphs $G$?*

The best lower bound that we currently have for this problem that applies to every $d$ is $1/d$ which is given by the generalized star graph $S_{n,d}$. Indeed, $a_d(S_{n,d}) = 1$ and $a(S_{n,d}) = d$. For the special case $d = 2$ we obtained by computer calculations $a_2(P_{12,2}) \geq 0.667 \cdot a(P_{12,2})$ (where $P_{n,d}$ is the generalized path graph). It remains a possibility that Theorem 1.6 is tight, and we suspect that generalized paths might be the extremal examples.

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