Improved Integrality Gap in Max-Min Allocation, or, Topology at the North Pole*

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Abstract

In the max-min allocation problem a set P of players are to be allocated disjoint subsets of a set R of indivisible resources, such that the minimum utility among all players is maximized. We study the restricted variant, also known as the Santa Claus problem, where each resource has an intrinsic positive value, and each player covets a subset of the resources. Bezáková and Dani [15] showed that this problem is NP-hard to approximate within a factor less than 2, consequently a great deal of work has focused on approximate solutions. The principal approach for obtaining approximation algorithms has been via the Configuration LP (CLP) of Bansal and Sviridenko [12]. Accordingly, there has been much interest in bounding the integrality gap of this CLP. The existing algorithms and integrality gap estimations are all based one way or another on the combinatorial augmenting tree argument of Haxell [26] for finding perfect matchings in certain hypergraphs.

Our main innovation in this paper is to introduce the use of topological methods, to replace the combinatorial argument of [26] for the restricted max-min allocation problem. This approach yields substantial improvements in the integrality gap of the CLP. In particular we improve the previously best known bound of 3.808 to 3.534. We also study the $(1, \varepsilon)$-restricted version, in which resources can take only two values, and improve the integrality gap in most cases. Our approach applies a criterion of Aharoni and Haxell, and Meshulam, for the existence of independent transversals in graphs, which involves the connectedness of the independence complex. This is complemented by a graph process of Meshulam that decreases the connectedness of the independence complex in a controlled fashion and hence, tailored appropriately to the problem, can verify the criterion. In our applications we aim to establish the flexibility of the approach and hence argue for it to be a potential asset in other optimization problems involving hypergraph matchings.

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1 Introduction

In this paper we consider the *restricted max-min allocation* problem. An instance $\mathcal{I} = (P, R, v, \{L_p : p \in P\})$ of the problem consists of a set P of players, a set R of indivisible resources, where each resource $r \in R$ has an intrinsic positive value $v_r > 0$, and each $p \in P$ covets a set $L_p \subseteq R$ of resources. An *allocation* of the resources is a function $a : P \to 2^R$, with $a(p) \subseteq L_p$ for each $p \in P$, such that every resource is allocated to (at most) one player, that is $a(p) \cap a(q) = \emptyset$ for every $p \neq q$. The *min-value of allocation* a is $\min_{p \in P} v(a(p))$, where for a set $S \subseteq R$ of resources $v(S) = \sum_{r \in S} v_r$ represents the total value of S. The objective is to maximize the min-value over all allocations of resources. This value will be denoted by $OPT = OPT(\mathcal{I})$.

The choice of a max-min objective function is arguably a good one for achieving overall individual "Fairness" in the distribution of a set of indivisible resources that are considered desirable by the players.[1] Since the seminal paper of Bansal and Sviridenko [12], the restricted max-min allocation problem often goes under the name *Santa Claus Problem*, where the players represent children, and the resources are presents to be distributed by Santa Claus. One imagines each present r having a "catalogue" value v_r, but some presents may not be interesting to some children.[2] To be fair[3], Santa might wish to distribute the presents so that the smallest total value received by any child is as large as possible.

The problem of how to find an optimal solution efficiently was studied first in the special case when $L_p = R$ for every player $p \in P$. In this case Woeginger [40] and Epstein and Sgall [22] gave polynomial time approximation schemes (PTAS), and Woeginger [41] gave an FPTAS when the number of players is constant. For the general case however, Bezáková and Dani [15] showed that the problem is hard to approximate up to any factor < 2. On the positive side, there has been a great deal of progress towards finding good approximations. In [15] an approximation ratio of $|R| - |P| + 1$ is achieved, as well as an additive approximation algorithm using the standard assignment LP relaxation of the problem. This finds a solution of value at least $T_{ALP} - \max_{r \in R} v_r$, where T_{ALP} is the optimal value of the assignment LP. This algorithm however does not offer any approximation factor guarantee when $\max_{r \in R} v_r$ is large.

To address the fact that the assignment LP can have arbitrarily large integrality gap in general, Bansal and Sviridenko [12] introduced the important innovation of using a stronger LP, called the configuration LP for the problem, which we now describe. Given a problem instance $\mathcal{I}$ and $T \geq 0$, for each player $p \in P$ we define the family $\mathcal{C}_p(T) = \{C \subseteq L_p : v(C) \geq T\}$ of *configurations for p*. The *configuration LP* for $\mathcal{I}$ with *target* T has a variable $x_{p,S} \geq 0$ for every player $p \in P$ and configuration $S \in \mathcal{C}_p(T)$, and a constraint

$$\sum_{S \in \mathcal{C}_p(T)} x_{p,S} \geq 1$$

for every player $p \in P$ and a constraint

$$\sum_{p \in P} \sum_{S \in \mathcal{C}_p(T), S \ni r} x_{p,S} \leq 1$$

for every resource $r \in R$.

[1] This is in contrast with the situation where resources are considered rather "chores", when one would usually aim to minimize the maximum values of the subsets of resources allocated to each player. That would be the setup for example in the classical makespan minimization problem, where various jobs have to be allocated to a set of machines.

[2] ... since perhaps they already secured the latest edition of their favorite smartphone for their birthday.

[3] ... and to avoid criticism from jealous parents

We will refer to this LP as CLP(T) for $\mathcal{I}$. Formally we minimize the objective function 0, but the main point is whether CLP(T) is feasible. In the language of discrete optimization, to say that CLP(T) is feasible means that the union $\bigcup_{p\in P}\mathcal{C}_p(T)$ of the $|P|$ hypergraphs $\mathcal{C}_p(T)$ has a fractional matching $x : \bigcup_{p\in P}\mathcal{C}_p(T) \to [0,1]$ that has total value at least 1 on each $\mathcal{C}_p(T)$.

For a given instance $\mathcal{I}$, let $T^* = T^*(\mathcal{I})$ be the maximum T for which CLP(T) is feasible. It is a striking fact from [12] that even though CLP(T) has exponentially many variables, T^* can be approximated up to any desired accuracy in polynomial time. Note that any allocation for $\mathcal{I}$ of min-value T' gives an (integer) feasible solution to CLP(T'). Hence $OPT \le T^*$. We will refer to T^*/OPT as the *integrality gap*. Thus to prove the upper bound $1/\alpha$ on the integrality gap is to prove that, given any T and fractional matching x as described above, there exist $|P|$ disjoint sets $\{e^p \subseteq L_p : p \in P\}$ with $v(e^p) \ge \alpha T$ for each $p \in P$.

Using their configuration LP, Bansal and Sviridenko [12] obtained an $O(\log\log|P|/\log\log\log|P|)$-approximation algorithm for the Santa Claus problem. They also formulated a combinatorial conjecture and connected it to the problem of finding an allocation with large min-value given a feasible solution of CLP(T). Feige [23] proved this conjecture via repeated applications of the Lovász Local Lemma and hence established a constant integrality gap for the CLP. This was later made algorithmic by Haeupler, Saha, and Srinivasan [25] using Local Lemma algorithmization, which provided the first (huge, but) constant factor approximation algorithm for the Santa Claus problem.

Asadpour, Feige, and Saberi [10] formulated the problem in terms of hypergraph matching and proved an upper bound of 4 on the integrality gap of the CLP. Via the machinery of [12] this result implies an efficient algorithm to estimate the value of OPT up to a factor $(4+\delta)$. The approach of [10] is based on a local search technique introduced by Haxell [26], where the corresponding procedure is not known to be efficient. Polacek and Svensson [37] modified the local search of [10] and were able to prove a quasi-polynomial running time for a $(4+\delta)$-approximation algorithm. Finally, Annamalai, Kalaitzis, and Svensson [9] managed to adapt the local search procedure to terminate in polynomial time, introducing several influential novel ideas, which resulted in a polynomial time 12.33-approximation algorithm. Subsequently Cheng and Mao [18] altered the algorithm to establish a $(6+\delta)$-approximation guarantee, improving further in [20] to obtain a $(4+\delta)$-approximation algorithm. Davies, Rothvoss, and Zhang [21] also gave an $(4+\delta)$-approximation algorithm, working in a more general setting, where a matroid structure is imposed on the players. The integrality gap of the configuration LP was further improved by Cheng and Mao [19] and Jansen and Rohwedder [32] to 3.833 and then to 3.808 by Cheng and Mao [20] by better and better analysis of the procedure of [10].

A special case of the problem, that already captures much of its difficulty, comes from limiting the number of distinct values taken by resources to two. In the $(1,\varepsilon)$-restricted allocation problem resources can take only two values 1 or ε, where $0 < \varepsilon \le 1$. The relevance of this case is also underlined by the fact that a key reduction step in the foundational result of [12] required an approximation algorithm for the $(1,\varepsilon)$-restricted allocation problem for arbitrarily small $\varepsilon > 0$.

Chan, Tang, and Wu [17], extending work of Golovin [24] and Bezáková and Dani [15], show that approximating OPT up to a factor less than 2 is already NP-hard for the $(1,\varepsilon)$-restricted problem, for any fixed $\varepsilon \le 1/2$. Note that when $\varepsilon = 1$, so each resource has the same value, the problem can be solved exactly and easily via applications of a bipartite matching algorithm. This algorithm can also be used to give a $1/\varepsilon$-approximation, which is better than 2-approximation for $\varepsilon > 1/2$. In [17] it was proved that the integrality gap of the CLP for the $(1,\varepsilon)$-restricted allocation problem is at most 3, for every ε. The paper also gives a quasipolynomial-time algorithm that finds a $(3+4\varepsilon)$-approximation.

1.1 Our contributions

The existing algorithms and integrality gap estimation for the Santa Claus problem are, one way or another, based on the combinatorial augmenting tree argument of [26] for finding perfect matchings in certain hypergraphs. Many of them are sophisticated variants of the local search technique of [10] and its efficient algorithmic realization in [9].

Our main innovation in this work is to introduce the use of topological methods for the Santa Claus problem, and replace the combinatorial argument of [26]. This approach yields substantial improvements in the integrality gap of the CLP.

Our first main result improves the integrality gap from 3.808 to 3.534.

Theorem 1.1. *The integrality gap of the CLP is at most* $\frac{53}{15}$*.*

For our approach we make use of a criterion of Aharoni and Haxell [7] and Meshulam [36] for the existence of independent transversals in graphs, using the (topological) connectedness of the independence complex. In our application we apply this to an appropriately modified line graph of the multihypergraph of all those subsets that are valuable enough to be potentially allocated to the players. In order to show that the connectedness of the independence complex is large enough, we run a graph theoretic process, which is based on a theorem of Meshulam [36]. In the process we dismantle our line graph, but control the topological connectedness of the independence complex throughout, to make sure that the process runs for long enough. This necessitates that we choose our dismantling process with care and apply intricate analysis of the underlying structures, carefully tailored to the specifics of the problem. We employ the dual of the CLP to certify the length of the process.

Our approach is conceptually different from that of all previous work on the Santa Claus problem. The topological theorems in the background provide an incredibly rich family of independent sets in the modified line graph, that is geometrically highly structured via a triangulation of a high-dimensional simplex. In this setting, good allocations of disjoint sets of resources to players correspond to multicolored simplices in the triangulation, and the existence of such an allocation is guaranteed by Sperner's Lemma. This is in sharp contrast to the much simpler sparse spanning tree-like structure at the heart of the combinatorial approach, and where a solution is found via a direct step-by-step augmentation process.

Our general strategy to show the existence of a solution of large minimum utility seems quite flexible and we expect it to be a useful asset for other algorithmic problems of interest involving hypergraph matchings.

The machinery developed for the proof of Theorem 1.1 can also be used to improve significantly the known results on the integrality gap of the CLP for the $(1, \varepsilon)$-restricted allocation problem. In the next theorem we highlight some of the main consequences of this aspect of our work.

Theorem 1.2. *Let* $\varepsilon < \frac{1}{2}$ *and let* $\mathcal{I}$ *be an instance of the* $(1, \varepsilon)$*-restricted Santa Claus problem with maximum CLP-target* $T^* := T^*(\mathcal{I})$*. Then the integrality gap of* $\mathcal{I}$ *is at most* $f(\frac{\varepsilon}{T^*})$*, where* $f : (0, 1] \to \mathbb{R}^+$ *is a function satisfying*

- $f(x) < 3$ *unless* $x = \frac{1}{6}$ *or* $x = \frac{1}{3}$*,*
- $f(x) \leq 2.75$ *for all* $x \in (0, \frac{1}{6}) \cup [\frac{2}{11}, \frac{1}{3}) \cup [\frac{4}{11}, 1]$*, and*
- $\lim_{x \to 0} f(x) < 2.479$*.*

One important message of this theorem is the identification of a couple of specific instances that seem especially hard to crack. For example, we would be delighted to see a $(1, 1/3)$ instance with an optimal CLP target of 1 and no allocation of min-value $2/3$. Furthermore, we see that as long as $\frac{\varepsilon}{T^*}$ is not too close to either of the two problematic values, the integrality gap is substantially below 3.

As observed in [17] (and also explained in the proof of Theorem 1.2), the assumption $1 \leq T^* < 2$ captures the challenging case of the problem. Under this assumption, the last part tells us that the integrality gap is less than 2.479 when $\varepsilon \to 0$. This estimate compares favorably with an instance of the problem given in [17], that has integrality gap 2 for arbitrarily small ε.

We remark that the restriction on ε in the theorem is not crucial since, as mentioned earlier, there is a simple $\frac{1}{\varepsilon}$-approximation algorithm based on bipartite matchings, which gives an approximation ratio ≤ 2 if $\varepsilon \geq \frac{1}{2}$. Moreover the restriction $x \leq 1$ is also natural as $T^* \geq \varepsilon$ whenever T^* is positive.

Finally, we note that our proofs in this paper can be turned into an algorithmic procedure that constructs an allocation with the promised min-value, but at the moment we have no control over the running time. Thus our results are in the same spirit as those of [23, 10, 17, 32, 19, 20] in which the strongest estimate on the integrality gap did not come with a corresponding efficient algorithm to find an allocation. Nevertheless, together with the machinery of [12], our work can be used to efficiently estimate the min-value of an optimal allocation. As an application of such a theorem we can imagine a scenario where Santa Claus might be prone to favoritism. Having supernatural powers and plenty of summer leisure time at his traditional home at the North Pole, he can certainly calculate an optimal allocation, yet may choose a suboptimal one benefitting his favorites. Our Theorem 1.1 combined with [12] leads to a polynomial time algorithm that parents can use to uncover any bias Santa might have that is more blatant than $(\frac{15}{53} - \delta)$-times the optimum.

1.2 Related work

The max-min allocation problem is also widely studied in the more general case, where different players p might have different utility value v_{pr} for resource $r \in R$. The Santa Claus problem corresponds to the case when $v_{pr} \in \{0, v_r\}$. This scenario was first considered by Lipton, Markakis, Mossel, and Saberi [34]. The NP-hardness result of Bezáková and Dani [15] about approximating with a factor less than 2 is still the best known for the general case. Bansal and Sviridenko [12] showed that their CLP has an integrality gap of order $\Omega(\sqrt{|P|})$ for the general problem. Asadpour and Saberi [11] could match this with an $O(\sqrt{|P|}\log^3 |P|)$-approximation algorithm using the CLP. Chakrabarty, Chuzhoy, and Khanna [16] give an $|R|^\varepsilon$-approximation algorithm for any constant ε, that works in polynomial time, as well as a $O(\log^{10} |R|)$-approximation algorithm that works in quasipolynomial time.

The special case where each resource is coveted by only two players is interesting algorithmically. In this case Bateni, Charikar, and Guruswami [13] showed that the Santa Claus problem is NP-hard to approximate to within a factor smaller than 2. Complementing this, Chakrabarty, Chuzhoy, and Khanna [16] give a 2-approximation algorithm, even if the values are unrestricted. The case when resources can be coveted only by three players is shown to be equivalent to the general case [13].

For the classical dual scenario of min-max allocation Lenstra, Shmoys, and Tardos [33] gave a 2-approximation algorithm and showed that it is NP-hard to approximate within a factor of $3/2$. Using a configuration LP and a local search algorithm inspired by those developed for the Santa Claus problem, Svensson [39] managed to break the factor 2-barrier for the integrality gap of the restricted version of the min-max allocation problem. Once more, this result comes with an efficient algorithm to estimate the optimum value up to a factor arbitrarily close to $\frac{33}{17}$, but not with an efficient algorithm to find such an allocation. The approximation factor was subsequently improved to $\frac{11}{6}$ by Jansen and Rohwedder [30], who later [31] also provided an algorithm that finds such an allocation in quasipolynomial time.

Organization of the paper In Section 2 we present our topological tools and describe our proof strategy. In Section 3 we demonstrate how our method works by giving a clean proof of the known fact that the integrality gap is at most 4. In Section 4 we introduce the main innovation that makes our improvement on the integrality gap possible, and we use it in Section 5 to prove Theorem 1.1. In the subsequent Section 6 we give the proof of the main statement from Section 4. Finally, in Section 7 we prove Theorem 1.2 on the two-values problem. Background and intuition for the topological notions we use are provided for the interested reader in the Appendix. We also provide in the Appendix a guide to the notation and terminology used throughout Sections 1 to 7.

2 Topological tools and the proof strategy

2.1 The setup

Let $\mathcal{I} = (P, R, v, \{L_p : p \in P\})$ be an instance of the Santa Claus problem and let $T \in \mathbb{R}$ be a target such that CLP(T) is feasible. A subset $e \subseteq L_p$ of coveted resources of some player $p \in P$ with the property that $v(e) > \alpha T$ and $v(e') \leq \alpha T$ for every proper subset $e' \subset e$ is called an *α-hyperedge*. We say that p is the *owner* of e or e is an α-hyperedge of p. To indicate this we might write e^p if necessary. Note that the hypergraph consisting of all α-hyperedges is a multihypergraph, since the same subset e may be an α-hyperedge of several players p. For example if an α-hyperedge $e \subseteq L_p \cap L_q$ with $p \neq q$, we will have both e^p and e^q in the multihypergraph. An allocation with min-value greater than αT constitutes choosing for every player $p \in P$ an α-hyperedge of p, such that they are pairwise disjoint.[4]

For $\alpha \in \mathbb{R}$, the *α-approximation allocation graph* $H(\mathcal{I}, T, \alpha) = H(\alpha)$ is the auxiliary $|P|$-partite graph with vertex set

$$V(H(\alpha)) = \cup_{p \in P} V_p, \text{ where } V_p = \{e^p : \ e \subseteq R \text{ is an } \alpha\text{-hyperedge of } p\},$$

and edge set

$$E(H(\alpha)) = \{e^p f^q : p \neq q, e \cap f \neq \emptyset\}.$$

An *independent transversal* in a vertex-partitioned graph such as $H(\alpha)$ is an independent set (i.e. one that induces no edges) that is a *transversal*, i.e. it consists of exactly one vertex in each partition class. Thus a problem instance $\mathcal{I}$ with feasible CLP(T) has an allocation with min-value greater than αT for some $\alpha > 0$ if and only if the α-approximation allocation graph $H(\mathcal{I}, T, \alpha)$ has an independent transversal. Hence our Theorem 1.1 is implied by the following.

Theorem 2.1. *Let $(P, R, \{L_p : p \in P\}, v)$ be an instance of the Santa Claus problem and let $T \in \mathbb{R}$ be such that the CLP(T) is feasible. Then the corresponding α-approximation allocation graph $H(\alpha)$ has an independent transversal with $\alpha = \frac{15}{53}$.*

2.2 Topological tools

In this section we introduce the main topological tools needed and describe how we use them in our arguments.

For a given graph G, let $\mathcal{J}(G) = \{I \subseteq V(G) : I \text{ is independent}\}$ be its *independence complex*. Following Aharoni and Berger [2] we define $\eta(G)$ to be the (topological) connectedness of $\mathcal{J}(G)$ plus 2. An advantage of this shifting by 2 is that the formulas for the following simple properties

[4] We note that defining α-hyperedges to have value *at least* αT would capture more directly the integrality gap problem. However for our proof strategy the strict inequality turns out to be more natural.

of η simplify (see e.g. [1, 2, 6]). (In fact Part (2) is true in much greater generality, see e.g. [2], but this simple statement is all we require.)

Fact 1. *Let G be a graph.*

(1) $\eta(G) \geq 0$ with equality if and only if G is the empty graph (i.e. the graph with no vertices).

(2) If graph G is the disjoint union of G_1 and a non-empty graph G_2 then $\eta(G) \geq \eta(G_1) + 1$. Moreover, if G_2 is a single (isolated) vertex then $\eta(G) = \infty$.

Intuitively, $\eta(G)$ represents the smallest dimension of a "hole" in the geometric realization of the abstract simplicial complex $\mathcal{J}(G)$. For the purposes of this paper it suffices to regard η strictly as a graph parameter satisfying Fact 1 and the upcoming Theorems 2.2 and 2.3. However, for the interested reader we provide the formal definition, background and some intuition in the Appendix.

Our proof of Theorem 2.1 is based on two key theorems involving the parameter η. The first one provides a sufficient Hall-type condition for the existence of independent transversals. This result was implicit already in [7] and [35], and was first observed by Aharoni.[5] It was first stated explicitly in the form below in [36] (see also [2]). Let I be an index set and J be an $|I|$-partite graph with vertex partition $V_1, \dots, V_{|I|}$. For a subset $U \subseteq I$ we denote by $J|_U$ the induced subgraph $J[\cup_{i\in U} V_i]$ of J defined on the vertex set $\cup_{i\in U} V_i$.

Theorem 2.2. *Let I be an index set and J be an $|I|$-partite graph with vertex partition $V_1, \dots, V_{|I|}$. If for every subset $U \subset I$ we have $\eta\left(J|_U\right) \geq |U|$, then there is an independent transversal in J.*

The formal resemblance of Theorem 2.2 to Hall's Theorem for matchings in bipartite graphs is no coincidence: the latter is a consequence of the former. Indeed, for a bipartite graph $B = (X \cup Y, E)$ satisfying Hall's Condition we can define an $|X|$-partite (simple) graph $J(B)$, where for every $x \in X$ there is a part $V_x = \{y^x : y \in N_B(x)\}$ and $y_1^{x_1} y_2^{x_2}$ is an edge if and only if $y_1 = y_2$. Then a matching of B saturating X corresponds to an independent transversal in $J(B)$. For a subset $U \subseteq X$, the subgraph $J(B)|_U$ is the union of $|N(U)|$ disjoint cliques, so $\eta(J(B)|_U) \geq |N(U)| \geq |U|$ by Properties (1) and (2) in Fact 1 and Hall's condition.

Our second tool is a theorem of Meshulam [36], reformulated in a way that is particularly well-suited for our arguments. Let G be a graph, and let e be an edge of G. We denote by $G - e$ the graph obtained from G by deleting the edge e (but not its end vertices). We denote by $G \circledast e$ the graph obtained from G by removing both endpoints of e and all of their neighboring vertices. The graph $G \circledast e$ is called G with e *exploded*.

Theorem 2.3. *Let G be a graph and let $e \in E(G)$, such that $\eta(G - e) > \eta(G)$. Then we have that $\eta(G) \geq \eta(G \circledast e) + 1$.*

Inspired by Meshulam's Theorem we call an edge e of G *deletable* if $\eta(G - e) \leq \eta(G)$ and *explodable* if $\eta(G \circledast e) \leq \eta(G) - 1$. By the theorem, if an edge is not deletable then it is explodable. A *deletion/explosion sequence, or DE-sequence, starting with graph G_{start}* is a sequence of operations, which, starting with G_{start}, in each step either deletes a deletable edge or explodes an explodable edge in the current graph. The *length* $\ell(\sigma)$ of the sequence σ is the number of explosions in σ. A DE-sequence is called a *KO-sequence* if its outcome is a graph with an isolated vertex. The following are simple yet crucial properties of DE-sequences.

[5] According to [35], it was noted by Aharoni (via private communication) that the method of [7] implies Theorem 2.2 for line graphs of hypergraphs. However, the special properties of line graphs are not essential to the proof, so this version also captures the main essence of Theorem 2.2.

Observation 2.4. *Let G be the outcome of a DE-sequence σ of length ℓ, starting with G_{start}. Then the following are true.*

(i) $\eta(G_{start}) \geq \eta(G) + \ell$.

(ii) If σ is a KO-sequence then $\eta(G_{start}) = \infty$.

(iii) For any vertex $w \in V(G)$, there is a (possibly empty) sequence of deletions starting from G, after which w is either an isolated vertex or some edge of G incident to w can be exploded.

Proof. Part (i) follows since during performing the DE-sequence σ the deletion of a deletable edge does not increase the value of η and the explosion of an explodable edge decreases the value of η by at least 1. For (ii), by (i) and by Fact 1(2) we have $\eta(G_{start}) \geq \eta(G) = \infty$. For (iii) let us consider all edges of G incident to w, in some order. If we can delete all of them following the order, then w becomes isolated. Otherwise after some (possibly 0) number of deletions, the next incident edge in the order is not deletable. By Meshulam's theorem this edge is explodable. □

In Appendix 9.2 we give a small concrete example demonstrating how to use DE-sequences to obtain a lower bound on η.

2.3 The proof strategy

Let T be such that the $CLP(T)$ of instance $\mathcal{I}$ with the target T has a feasible solution. Our proof strategy is to take, for our chosen α, the graph $H(\alpha)$ defined in Section 2.1 and use Theorem 2.2 to derive the existence of an independent transversal in it.

Those α-hyperedges that contain a single resource will have a special status. A resource $r \in R$ is called *fat* if $v_r > \alpha T$, otherwise it is called *thin*. The set $F = F(\alpha) := \{r \in R : v_r > \alpha T\}$ is the set of fat resources. Any set $S \subseteq R$ of resources with $S \cap F = \emptyset$ is called thin. We will in particular be speaking of *thin α-hyperedges* and *thin configurations*. Note that an α-hyperedge is thin if and only if it contains at least two elements. The corresponding vertices of $H(\alpha)$ are also called thin. For a fat resource $r \in R$, the singleton $\{r\}$ is called a *fat α-hyperedge*, and if $r \in L_p$ then r^p is called a *fat vertex* of $H(\alpha)$. Each fat resource $r \in F$ corresponds to a clique $C_r := \{r^p : r \in L_p\}$ in $H(\alpha)$ which forms a component, since no other α-hyperedge contains r (due to their minimality).

As we show next, we can shift our main focus to the subgraph $J(\alpha) := H(\alpha) - \cup_{r\in F} C_r$ of $H(\alpha)$ induced by the set of thin vertices. To verify the condition of Theorem 2.2 we need to consider an arbitrary subset $U \subseteq P$ of the players and the corresponding induced subgraph $H(\alpha)|_U$ of $H(\alpha)$. By Fact 1(2) the disjoint clique components corresponding to fat vertices $r \in F_U := F \cap (\cup_{p\in U} L_p)$ each contribute at least one to the value of $\eta(H(\alpha)|_U)$. We thus need to prove that for the remaining graph we have $\eta(J(\alpha)|_U) \geq |U| - |F_U|$.

To that end, starting with $G_{start} = J(\alpha)|_U$ we will specify a DE-sequence σ and prove that either σ is a KO-sequence or $\ell(\sigma) \geq |U| - |F_U|$. In the former case Observation 2.4(ii) implies $\eta(J(\alpha)|_U) = \infty$. In the latter case, denoting by G_{end} the final graph of σ, Observation 2.4(i) and Fact 1(1) imply $\eta(J(\alpha)|_U) \geq \eta(G_{end}) + |U| - |F_U| \geq |U| - |F_U|$. In both cases we have that

$$\eta(H(\alpha)|_U) \geq \eta(J(\alpha)|_U) + |F_U| \geq |U|,$$

so the condition of Theorem 2.2 is verified. Hence there exists an independent transversal in $H(\alpha)|_U$ and we are done. We have just proved the following.

Theorem 2.5. *Let $\mathcal{I} = (P, R, v, \{L_p : p \in P\})$ be a problem instance and $T \in \mathbb{R}$ such that CLP(T) has a feasible solution. Suppose for every $U \subseteq P$ there exists a DE-sequence σ starting* *with $G_{start} = J(\alpha)|_U$ such that either σ is a KO-sequence, or $\ell(\sigma) \geq |U| - |F_U|$. Then $H(\alpha)$ has an independent transversal.*

We remark that this approach to proving the existence of an independent transversal using η was described in terms of a game in [6], and used in many settings, see e.g. [4, 5, 8, 27, 28, 29].

With Theorem 2.5 we have reduced our task to constructing, for every $U \subseteq P$, a DE-sequence σ starting with $G_{start} = J(\alpha)|_U$ such that either σ is a KO-sequence, or $\ell(\sigma) \geq |U| - |F_U|$. To prove lower bounds on the length of a DE-sequence σ that starts with $J(\alpha)|_U$, we will maintain a cover $W \subseteq R$ of all α-hyperedges that correspond to vertices of $J(\alpha)|_U$, that disappeared during explosions of σ, and control the size of W. If we are able to do this, then the complement of W is large, allowing us to find an α-hyperedge in it and hence extend the DE-sequence further. Note that deletions do not remove any vertices of $J(\alpha)|_U$.

More generally, we say W is a *cover of the DE-sequence* σ starting with a subgraph $G_{start} \subseteq J(\alpha)|_U$ and ending with G_{end} if

($\star$) every vertex e^p of G_{start} with $e \cap W = \emptyset$ is present in G_{end}.

The natural choice to cover the α-hyperedges corresponding to vertices that disappeared from $G_{start} \subseteq J(\alpha)|_U$ during the explosions in a DE-sequence σ is $\bigcup(e \cup f)$, where the union is over all edges $e^p f^q$ of G_{start} exploded in σ. This will be called the *basic cover* of σ. Note that for the basic cover W_σ, every vertex h^s of G_{start} with $h \cap W_\sigma = \emptyset$ is unaffected by each explosion that happened during σ and hence is still present in the graph G_{end}.

In the next subsection we will demonstrate how the simple accounting by adding up the values of the basic covers of the explosions of an arbitrary DE-sequence starting with $J(\alpha)|_U$ and ending with a graph with no edges is already sufficient to derive the existence of an allocation of min-value greater than $\frac{1}{4}T$. To achieve our improved bounds in Theorem 2.1, in Sections 4 and 5 we will choose our DE-sequences and account for their accompanying covers more carefully.

3 The demonstration of the method

In this section, we apply our method to verify Theorem 2.1 for the ratio $\alpha = \frac{1}{4}$. We emphasize that this can easily be proved by using instead the combinatorial method of [26]; indeed, as described in the Introduction, this approach and intricate refinements of it have been the basis of essentially all progress on the integrality gap of the CLP for this problem since the pivotal paper of [10]. The aim of this section is to re-prove the basic ratio of $\frac{1}{4}$ using our topology-based proof strategy, to establish the context for later refinements that we employ in the rest of the paper, and that lead to the improved ratio $\alpha = \frac{15}{53}$ in Theorem 2.1.

Our setup in this section is as follows.

Setup 3. Let $\mathcal{I} = (P, R, v, \{L_p : p \in P\})$ be an instance of the Santa Claus problem and let $T \in \mathbb{R}$ be a target such that CLP(T) is feasible. Fix $0 < \alpha < 1$ and let U be an arbitrary subset of P.

Our approach to proving Theorem 2.1 with ratio α will be as described in Section 2.3: for the arbitrarily chosen subset $U \subseteq P$, we will construct a DE-sequence σ starting with $G_{start} = J(\alpha)|_U$ such that either σ is a KO-sequence, or $\ell(\sigma) \geq |U| - |F_U|$. Then by Theorem 2.5 we will have proved Theorem 2.1 for this choice of α.

In this section, to prove Theorem 2.1 for $\alpha = \frac{1}{4}$, in fact it will suffice to choose an *arbitrary* DE-sequence σ starting with $G_{start} = J(\alpha)|_U$ and ending with a graph G_{end} with no edges. This is possible by repeated application of Observation 2.4(iii). If G_{end} contains a vertex then σ is a

KO-sequence and we are done, so we may assume that G_{end} has no vertices. We are left to show that $\ell(\sigma) \geq |U| - |F_U|$, provided that $\alpha = \frac{1}{4}$ (which we will assume only at the end of the argument).

To estimate the value of covers, the following definition will be useful. A subset $s \subseteq R$ is called a *block* if $v(s) \leq \alpha T$. Note then that any proper subset of an α-hyperedge is a block.

We estimate the value of the basic cover W_σ by simply adding up estimates for the basic covers of its individual explosions.

Observation 3.1. *With the assumptions of Setup 3 suppose $e^p f^q$ is an explodable edge in a subgraph G of $J(\alpha)|_U$. Then the value of its basic cover $e \cup f$ is at most $3\alpha T$.*

Proof. The cover $e \cup f$ is a subset of the union of three blocks: $(e \setminus \{x\}) \cup \{x\} \cup (f \setminus e)$, where $x \in e$ is arbitrary. Indeed, $f \setminus e$ is a block since it is a proper subset of f, and both $e \setminus \{x\}$ and $\{x\}$ are blocks since they are proper subsets of e. For this recall that all α-hyperedges under consideration are thin. Consequently $v(e \cup f) \leq 3\alpha T$. □

Hence $v(W_\sigma) \leq 3\alpha T \ell(\sigma)$.

To give a lower bound on this value, we invoke the dual DCLP(T) of the configuration LP for instance $\mathcal{I}$ and target value T. In DCLP(T) there is a variable $y_p \geq 0$ for each player $p \in P$, a variable $z_r \geq 0$ for each resource $r \in R$, and for each configuration $S \in \mathcal{C}_p(T)$ there is a constraint

$$y_p \leq \sum_{r \in S} z_r.$$

The objective function, which is to be maximized, is

$$\sum_{p \in P} y_p - \sum_{r \in R} z_r.$$

We will use the dual as a convenient way to verify certain inequalities by checking the feasibility of well-chosen solutions to DCLP(T). Since CLP(T) is minimization problem with objective function 0, weak duality amounts to the following observation.

Observation 3.2. *Let $\mathcal{I}$ be an instance of the Santa Claus problem and let $T \in \mathbb{R}$ be such that the CLP(T) for $\mathcal{I}$ is feasible. If $y \in \mathbb{R}^P$ and $z \in \mathbb{R}^R$ represent a feasible solution of the DCLP(T) for $\mathcal{I}$ then*

$$\sum_{p \in P} y_p - \sum_{r \in R} z_r \leq 0.$$

The following consequence of Observation 3.2, as well as its more refined version (Proposition 5.1), will be applied repeatedly throughout our paper. Here it will provide a lower bound on the value of W_σ.

Proposition 3.3. *With the assumptions of Setup 3, let $c \geq 0$ and $Y \subseteq R \setminus F$ be such that $v(Y \cap S) \geq c$ for every thin configuration $S \in \mathcal{C}_p(T)$ for $p \in U$. Then*

$$v(Y) \geq c(|U| - |F_U|).$$

Proof. We define a feasible solution to DCLP(T) and then invoke Observation 3.2. Let

$$y_p = \begin{cases} 0 & p \notin U \\ c & p \in U \end{cases} \qquad z_r = \begin{cases} c & r \in F_U \\ v_r & r \in Y \\ 0 & \text{otherwise} \end{cases}$$

To check the feasibility of the solution, let $S \in \mathcal{C}_p(T)$ be an arbitrary configuration. If $p \notin U$, then $y_p = 0$ and the corresponding constraint holds by the non-negativity of the z_r. If $p \in U$, then $y_p = c$. If there is a fat resource $s \in S$, then $s \in F_U$, so $\sum_{r\in S} z_r \geq z_s = c = y_p$. Otherwise $S \cap F = \emptyset$ and hence $\sum_{r\in S} z_r \geq \sum_{r\in S\cap Y} v_r \geq c = y_p$. So the solution is feasible. Observation 3.2 then implies $0 \geq \sum_{p\in P} y_p - \sum_{r\in R} z_r = |U|c - (|F_U|c + \sum_{r\in Y} v_r)$ and the claim follows. $\square$

Now we assume $\alpha = \frac{1}{4}$. To obtain a lower bound on $v(W_\sigma)$ we apply Proposition 3.3 with U, $Y = W_\sigma$ and $c = 3\alpha T$. To that end we need to check $v(S \cap W_\sigma) \geq 3\alpha T$ for every thin configuration $S \in \mathcal{C}_p(T)$ with $p \in U$. Since $v(S) \geq T = 4\alpha T$ for every configuration S, it is enough to verify that $v(S \setminus W_\sigma) \leq \alpha T$. As G_{end} has no vertices, Property ($\star$) of W_σ implies that $R\setminus(F \cup W_\sigma)$ should contain no α-hyperedge of any $p \in U$. Consequently, for any thin configuration $S \in \mathcal{C}_p(T)$ with $p \in U$, the value of $S \setminus W_\sigma$ should not be large enough for an α-hyperedge. Hence $v(S \setminus W_\sigma) \leq \alpha T$ as needed. Proposition 3.3 then implies $v(W_\sigma) \geq 3\alpha T(|U| - |F_U|)$. Combining this with $v(W_\sigma) \leq 3\alpha T\ell(\sigma)$, we obtain $\ell(\sigma) \geq |U| - |F_U|$ and we are done by Theorem 2.5.

4 Economical DE-sequences

In this section we start our proof of Theorem 2.1 by introducing a couple of important definitions and our main tool. The key to improving the bound from the previous section is to improve upon Observation 3.1, whose proof amounts to saying that any explosion has a cover that is the union of three blocks. For example, if the intersection of α-hyperedges e and f happens to contain at least two resources, then their explosion has a cover that is the union of only two blocks, a clear savings over Observation 3.1. Thus the lack of such explosions introduces restrictions on the remaining graph G and helps in searching for not one, but perhaps a sequence of two explosions which has a cover that is the union of fewer than six blocks, the fewer the better, again a savings over Observation 3.1. The lack of such a sequence introduces further restrictions that we can exploit.

More generally, our improvement on Section 3 relies on finding DE-sequences whose accounting (through their covers) is done more economically when some of the explosions are packed together. We use two different approaches, one based on total value (in the form of "cheap DE-sequences") and the other based on total cardinality ("i/j-DE-sequences").

We say that a DE-sequence σ is *cheap* if there exists a cover of σ of value at most $2\alpha T\ell(\sigma)$. Note that any sequence of deletions is a cheap DE-sequence of length 0, hence by Meshulam's Theorem, if there is no cheap DE-sequence starting with graph G^*, every edge of G^* is explodable. The example above, of an explosion $e^p f^q$ with $|e \cap f| \geq 2$, is a cheap DE-sequence of length 1, since $e \cup f$ is the union of two blocks. In practice we often demonstrate that a DE-sequence σ is cheap by exhibiting a cover that is a subset of the union of at most $2\ell(\sigma)$ blocks.

Our second type of "economical" DE-sequence is based on cardinality. For integers $1 \leq j \leq i$, a DE-sequence σ is called an *i/j-DE-sequence* if it has length j and a cover of cardinality at most i. In our proofs j will be either 2 or 3, and i will be $2j + 1$. Since in particular an i/j sequence has j explosions and a cover of value at most $i\alpha T$,a $7/3$-sequence is "more economical" than a $5/2$-sequence, and both are "more economical" than the "full price" sequence used in Observation 3.1.

Our main technical theorem tells us that, during the execution of a DE-sequence σ, if σ cannot be extended by a cheap DE-sequence (or a KO-sequence), and if some thin configuration still has total value more than $j\alpha T$ outside the cover W of σ, then we can extend σ by a $(2j + 1)/j$-DE-sequence.

Theorem 4.1. *Assume Setup 3. Let $G^* \subseteq J(\alpha)|_U$ and $W \subseteq R\setminus F$ be a subset of resources such that ($\star$) holds with $G_{start} = J(\alpha)|_U$ and $G_{end} = G^*$. Let $j = 2$ or 3. Suppose there is no KO-sequence* *and no cheap DE-sequence starting with G^*. If there is a thin configuration $C \in \mathcal{C}_p(T)$ with $p \in U$ and $v(C \cap W) < (1 - j\alpha)T$, then there exists a $(2j+1)/j$-DE-sequence starting with G^*.*

Hence if we cannot continue σ with any step that improves upon Observation 3.1, every thin configuration has large intersection with the current cover W. This will be the key fact that allows us to complete our proof, which is given in the following section. The proof of Theorem 4.1 is quite intricate, and is postponed to Section 6.

5 Proof of Theorem 2.1

Our setup in this section is Setup 3.

Proof of Theorem 2.1. As we saw in Section 2.3 and again in Section 3, to prove Theorem 2.1 with the ratio α it is sufficient to verify that, given Setup 3, there exists a DE-sequence σ starting with $G_{start} = J(\alpha)|_U$ such that either σ is a KO-sequence, or $\ell(\sigma) \geq |U| - |F_U|$. Here we will show that this holds for $\alpha = \frac{15}{53}$, which again will only be used in the last step of the argument.

We define σ in four phases. Here G denotes the current graph of the sequence. A *maximal cheap DE-sequence* is one that is not a proper initial subsequence of another cheap DE-sequence.

- Phase 1. WHILE a KO-sequence or cheap DE-sequence τ exists in G, DO perform τ
- Phase 2. WHILE there exists a 7/3-DE-sequence τ in G, DO perform τ and then perform a maximal cheap DE-sequence.
- Phase 3. WHILE there exists a 5/2-DE-sequence τ in G, DO perform τ, and then perform a maximal cheap DE-sequence.
- Phase 4. WHILE G has an edge DO perform a deletion or an explosion in G.

The DE-sequence σ is the concatenation of all the sequences τ over all the phases, in order. When the procedure terminates, the final graph G_{end} has no edge. If G_{end} contains a vertex, then σ is a KO-sequence starting with G_{start}, as desired. Therefore we may assume that G_{end} has no vertices. Note that, in this case, at no time during our procedure did there exist a KO-sequence starting at the current graph G.

Let n_1 denote the total number of explosions performed in the cheap DE-sequences throughout Phases 1, 2, and 3, and W_1 be the union of all the covers associated to these cheap DE-sequences. By definition of cheap DE-sequence we know that

$$v(W_1) \leq 2\alpha T n_1. \tag{5.1}$$

For $j = 2, 3$, let n_j denote the number of explosions performed in 7/3-DE-sequences during Phase 2 and 5/2-DE-sequences during Phase 3, respectively, and let W_j be the union of their corresponding covers. For a $(2j+1)/j$-DE-sequence the number of resources in the cover is $2j+1$ and the number of explosions is j. Hence

$$|W_2| \leq \frac{7}{3}n_2 \text{ and } |W_3| \leq \frac{5}{2}n_3. \tag{5.2}$$

For these sets it will also be useful to estimate their values. For this, recall that each thin resource is of value at most αT. Therefore

$$v(W_2) \leq \frac{7}{3}\alpha T n_2 \text{ and } v(W_3) \leq \frac{5}{2}\alpha T n_3. \tag{5.3}$$

Let n_4 be the number of explosions performed in Phase 4 and W_4 the union of the basic covers corresponding to them. Then by Observation 3.1 we have

$$v(W_4) \le 3\alpha T n_4. \tag{5.4}$$

The next proposition is a more refined version of Proposition 3.3, and uses the dual DCLP(T) in a way similar to that of Section 3. We will employ it to take snapshots at various points during σ in order to derive lower bounds involving linear combinations of the quantities n_j, $j = 1, 2, 3, 4$. For each j, we will apply it with Y being a cover of the initial subsequence of σ defined up to the end of Phase j, and the lower bound on $v(Y_{\le d} \cap S)$ will be ensured by Theorem 4.1.

Proposition 5.1. *Assume the conditions of Setup 3. Let $0 \le d \le c \le 2d$ and $Y \subseteq R \setminus F$ be given, and let $Y_{\le d} := \{y \in Y : v_y \le d\}$ and $Y_{>d} := \{y \in Y : v_y > d\}$. Suppose that for every $p \in U$ and every thin configuration $S \in \mathcal{C}_p(T)$ with $|Y_{>d} \cap S| \le 1$, we have*

$$v(Y_{\le d} \cap S) \ge \begin{cases} c & \text{if } Y_{>d} \cap S = \emptyset \\ c - d & \text{if } |Y_{>d} \cap S| = 1. \end{cases}$$

Then for any partition of $Y = Y_1 \cup Y_2$ we have

$$c|U| - c|F_U| \le d|Y_1| + v(Y_2).$$

In fact we will apply this proposition only when $c = d$ and $c = 2d$. Note that we can recover Proposition 3.3 by setting $d = c$ and using that $Y = Y_{>d} \cup Y_{\le d}$ is a partition and $d|Y_{>d}| \le v(Y_{>d})$.

Proof. We define a feasible solution to DCLP(T) and then invoke Observation 3.2. Let

$$y_p = \begin{cases} 0 & p \notin U \\ c & p \in U \end{cases} \qquad z_r = \begin{cases} c & r \in F_U \\ d & r \in Y_{>d} \\ v_r & r \in Y_{\le d} \\ 0 & \text{otherwise.} \end{cases}$$

To verify the feasibility, let $S \in \mathcal{C}_p(T)$ be an arbitrary configuration. If $p \notin U$, then $y_p = 0$ and the corresponding constraint holds by the non-negativity of the z_r.

Otherwise $p \in U$ and we must check $\sum_{r \in S} z_r \ge y_p = c$.

If there is a fat resource $s \in F \cap S \subset F_U$, then $\sum_{r \in S} z_r \ge z_s = c = y_p$. Otherwise $S \cap F = \emptyset$ and we make a case distinction based on $|Y_{>d} \cap S|$. If $Y_{>d} \cap S = \emptyset$, then $\sum_{r \in S} z_r \ge \sum_{r \in S \cap Y_{\le d}} z_r = v(S \cap Y_{\le d}) \ge c$.

If $Y_{>d} \cap S = \{s\}$, then $\sum_{r \in S} z_r \ge \sum_{r \in S \cap Y_{>d}} z_r + \sum_{r \in S \cap Y_{\le d}} z_r = z_s + v(S \cap Y_{\le d}) \ge d + c - d = c$.

Finally, if $|Y_{>d} \cap S| \ge 2$, then $\sum_{r \in S} z_r \ge \sum_{r \in S \cap Y_{>d}} z_r \ge 2d \ge c$.

So in all cases the solution is feasible. Observation 3.2 then implies that

$$0 \ge \sum_{p \in P} y_p - \sum_{r \in R} z_r = c|U| - c|F_U| - d|Y_{>d}| - v(Y_{\le d}).$$

To derive our conclusion note that

$$\begin{aligned} c|U| - c|F_U| &\le d|Y_{>d}| + v(Y_{\le d}) \\ &= d|(Y_1)_{>d}| + v((Y_1)_{\le d}) + d|(Y_2)_{>d}| + v((Y_2)_{\le d}) \\ &\le d|(Y_1)_{>d}| + d|(Y_1)_{\le d}| + v((Y_2)_{>d}) + v((Y_2)_{\le d}) \\ &= d|Y_1| + v(Y_2). \end{aligned}$$

$\square$

Corresponding to the last step of our proof in Section 3 where we applied Proposition 3.3, here we derive our first inequality from Proposition 5.1, after Phase 2 is complete.

Lemma 5.2. *Assume Setup 3. Then*

$$|U| - |F_U| \leq \frac{2\alpha}{1-3\alpha} n_1 + \frac{7}{3} n_2. \tag{5.5}$$

Proof. Let G be the current graph after the end of Phase 2 and set $W = W_1' \cup W_2$, where W_1' is the union of the covers associated with cheap DE-sequences up to the end of Phase 2. We use Proposition 5.1 with U, $c = d = (1-3\alpha)T$, and $Y = W$. For this we only need to check for every thin configuration $S \in \mathcal{C}_p(T)$ with $p \in U$ and $W_{>(1-3\alpha)T} \cap S = \emptyset$ that $v(W_{\leq(1-3\alpha)T} \cap S) = v(W \cap S) \geq (1-3\alpha)T$. This follows from Theorem 4.1 applied with W, $j = 3$, $G^* = G$, and $C = S$, since after Phase 2 is complete there is no KO-sequence, cheap DE-sequence, or 7/3-DE-sequence starting with G. By Proposition 5.1 we conclude that

$$(1-3\alpha)T(|U| - |F_U|) \leq v(W_1') + (1-3\alpha)T|W_2|.$$

By (5.1) we have $v(W_1') \leq v(W_1) \leq 2\alpha T n_1$ and by (5.2) we have $|W_2| \leq \frac{7}{3} n_2$. This completes the proof. □

In our next lemma we take a snapshot after Phase 3 and derive two inequalities.

Lemma 5.3. *Assume Setup 3. If $\alpha \geq \frac{1}{4}$ then we have*

$$|U| - |F_U| \leq \frac{\alpha}{1-3\alpha} n_1 + \frac{7}{6} n_2 + \frac{5}{4} n_3 \tag{5.6}$$

$$|U| - |F_U| \leq \frac{2\alpha}{1-2\alpha} n_1 + \frac{7\alpha}{3(1-2\alpha)} n_2 + \frac{5\alpha}{2(1-2\alpha)} n_3. \tag{5.7}$$

Proof. Let G be the current graph at the end of Phase 3 and set $W = W_1 \cup W_2 \cup W_3$ to be the corresponding cover. As there is no KO-sequence, cheap DE-sequence, or 5/2-DE-sequence at the end of Phase 3, Theorem 4.1, applied with W, $j = 2$, $G^* = G$, and $C = S$, implies $v(W \cap S) \geq (1-2\alpha)T$ for any thin configuration $S \in \mathcal{C}_p(T)$ with $p \in U$.

For the first inequality we use Proposition 5.1 with U, $d = (1-3\alpha)T, c = 2d$, and $Y = W$. We need to check for every thin $S \in \mathcal{C}_p(T)$ with $p \in U$ and $|W_{>(1-3\alpha)T} \cap S| \leq 1$ that its value is large enough. If $W_{>(1-3\alpha)T} \cap S = \emptyset$ then $v(W_{\leq(1-3\alpha)T} \cap S) = v(W \cap S) \geq (1-2\alpha)T \geq 2(1-3\alpha)T = c$ by our assumption on α. If $W_{>(1-3\alpha)T} \cap S = \{s\}$ then $v(W_{\leq(1-3\alpha)T} \cap S) = v(W \cap S) - v_s \geq (1-2\alpha)T - \alpha T = c - d$, since s is thin.

By Proposition 5.1 we conclude that

$$2(1-3\alpha)T(|U| - |F_U|) \leq v(W_1) + (1-3\alpha)T|W_2 \cup W_3|.$$

Using (5.1) and (5.2), we are done.

For the second inequality we use Proposition 3.3 with U, $c = (1-2\alpha)T$, and $Y = W$. Recall that by the application of Theorem 4.1, we already know that $v(W \cap S) \geq (1-2\alpha)T$ for every thin configuration $S \in \mathcal{C}_p(T)$ with $p \in U$. Hence by Proposition 3.3 we conclude that $(1-2\alpha)T(|U| - |F_U|) \leq v(W) \leq v(W_1) + v(W_2) + v(W_3)$. Using (5.1) and (5.3), the inequality follows. □

Finally, after Phase 4, we also measure the covers.

Lemma 5.4. *Assume Setup 3. We then have*

$$|U| - |F_U| \leq \frac{2\alpha}{1-\alpha}n_1 + \frac{7\alpha}{3(1-\alpha)}n_2 + \frac{5\alpha}{2(1-\alpha)}n_3 + \frac{3\alpha}{1-\alpha}n_4. \tag{5.8}$$

Proof. Set $W = W_1 \cup W_2 \cup W_3 \cup W_4$ and apply Proposition 3.3 with U, $c = (1-\alpha)T$, and $Y = W$. This is possible as after Phase 4 is complete, there are no vertices left in the final subgraph G_{end} of $J(\alpha)|_U$. Consequently the value of resources in $S \setminus W$ is at most αT for any thin configuration $S \in \mathcal{C}_p(T)$ with $p \in U$. That means $v(W \cap S) \geq (1-\alpha)T$ holds.

By Proposition 3.3 we then find that $(1-\alpha)T(|U| - |F_U|) \leq v(W) \leq v(W_1) + v(W_2) + v(W_3) + v(W_4)$. Using (5.1), (5.3) and (5.4), the inequality follows. □

Now if $\alpha = \frac{15}{53} > \frac{1}{4}$, then $T = \frac{53}{15}m$ and each of the upper bounds (5.5), (5.6), (5.7), and (5.8) on $|U| - |F_U|$ is true. It is then straightforward to check that the convex combination of these with coefficients $\frac{1}{35}$, $\frac{26}{245}$, $\frac{46}{2205}$, and $\frac{38}{45}$, respectively, implies that

$$|U| - |F_U| \leq n_1 + n_2 + n_3 + n_4 = \ell(\sigma).$$

This completes the proof of Theorem 2.1. □

6 Proof of the existence of economical DE-sequences

The main goal of this section is the proof of Theorem 4.1. Before getting to it we prove two lemmas, the second of which represents the heart of our argument.

We begin with some notation and terminology in the setting of Setup 3, and for a subgraph G^* of $J(\alpha)|_U$. When our attention is focused on the multihypergraph of α-hyperedges, we often refer to a vertex e^p of G^* as an α-hyperedge e *of* p or *owned by* p. When the identity of the owner is irrelevant or already established, we often omit the reference to the owner. In particular we also sometimes say the pair of α-hyperedges e and f are explodable, without specifying their owners. We also say that an α-hyperedge g of $p \in U$ *survives* an explosion if the vertex g^p is still present in the current graph after the explosion. We say that α-hyperedges e and f are *explodable at resource* r if $e \cap f = \{r\}$ and the pair e and f is explodable. For an element x and set e we write $e - x$ and $e + x$ as shorthand for $e \setminus \{x\}$ and $e \cup \{x\}$, respectively. We use the term *α-edge* for an α-hyperedge with exactly two elements.

The setting in which we will apply our first lemma is as follows: we have begun to construct a DE-sequence starting from $J(\alpha)|_U$ and have reached a graph G^*. If we cannot continue our sequence "cheaply", then certain special conditions must hold.

Lemma 6.1. *Assume Setup 3 and let $G^* = (V, E)$ be a subgraph of $J(\alpha)|_U$. Suppose there is no KO-sequence or cheap DE-sequence starting with G^*. Let $f^q \in V$ be an α-hyperedge of player q. Then the following hold.*

(i) If for player $p \in U$ there exists an α-hyperedge $g^p \in V$ such that $g^p f^q \in E$ then for every α-hyperedge $e^p \in V$ of p we have $|f \cap e| \leq 1$.
In particular, for any $e^p \in V$ with $|e \cap f| \geq 2$, we have $e^p f^q \notin E$.
Even more in particular, if $e^p, e^q \in V$, then $e^p e^q \notin E$.

(ii) For every resource $r \in f$ there exists an α-hyperedge g in V that is explodable with f at r.

Proof. Recall that since there is no cheap DE-sequence starting with G^*, every edge of G^* is explodable.

To prove (i), first we show that if $e^p \in V$ with $|e \cap f| \geq 2$ then $e^p f^q \notin E$. Indeed, as mentioned in Section 4, otherwise $e^p f^q$ is explodable and $(e - x) \sqcup ((f \setminus e) + x))$, where $x \in e \cap f$, is a partition of the basic cover $e \cup f$ into two blocks. This demonstrates the explosion of the edge $e^p f^q$ is cheap, a contradiction. Here we use that $e - x$ is a proper subset of the α-hyperedge e since $x \in e$, and $(f \setminus e) + x$ is a proper subset of the α-hyperedge f, since $|e \cap f| \geq 2$.

Let now $g^p \in V$ be an α-hyperedge of p that is explodable with f^q. Suppose on the contrary that there exists an α-hyperedge e^p of p such that $|e \cap f| \geq 2$. By the above $e^p f^q \notin E$ and $e \neq g$. We define a DE-sequence of length two (starting with G^*) and show that it is cheap, giving a contradiction. First explode f^q with g^p. Note that e^p survives, since $e^p f^q$ is not in E, and g^p and e^p are owned by the same player. Then, since there is no KO-sequence isolating e^p, after possibly some deletions, we can explode e^p with some neighbor h. (See Observation 2.4(iii).) Then the basic cover $f \cup g \cup e \cup h$ of this DE-sequence of length two has a partition $(e - x) \sqcup ((f \setminus e) + x) \sqcup (g \setminus f) \sqcup (h \setminus e)$, where $x \in e \cap f$, into four blocks. Hence this DE-sequence is cheap, as claimed. This verifies the main statement of (i), which in turn implies the second statement.

For the last statement of (i) note that in our setting V contains only thin hyperedges, so $|e| \geq 2$.

For (ii) note that since there is no KO-sequence starting with G^*, f has some neighbor in G^*. Once again, every edge of G^* is explodable. For a contradiction assume that every α-hyperedge g with $f \cap g = \{r\}$ is not explodable with f. Explode f with an arbitrary neighbor h. By (i) we know that $f \cap h = \{s\}$ for some s. By our assumption $s \neq r$. The key observation here is that the set $W^* = (f \cup h) - r$, i.e. something less than the basic cover, is also a cover of the explosion of the edge fh. For this, let g be an α-hyperedge of G^* that did not survive the explosion of the edge fh. If g is a neighbor of h in G^* then $W^* \cap g \supseteq h \cap g \neq \emptyset$, since $r \notin h$. Otherwise g is a neighbor of f and is covered by W^* unless $g \cap f = \{r\}$. However our starting assumption was that f had no such neighbor in G^*. Then the cover $W^* = (f - r) \cup (h \setminus f)$ of the single explosion of the edge fh is the union of two blocks, which makes it cheap, a contradiction. $\square$

Our next lemma, which will be key to the proof of Theorem 4.1, applies in the same setting as that of Lemma 6.1. It provides much stronger consequences of the fact that a DE-sequence starting from $J(\alpha)|_U$ with cover W cannot be extended cheaply.

Lemma 6.2. *Assume Setup 3. Let $G^* = (V, E)$ be a subgraph of $J(\alpha)|_U$, and let $W \subseteq R \setminus F$ be a subset of resources such that $(\star)$ holds with $G_{start} = J(\alpha)|_U$ and $G_{end} = G^*$. Suppose there is no KO-sequence and no cheap DE-sequence starting with $G^* = (V, E)$.*

Let $C \in \mathcal{C}_p(T)$ be a configuration of player $p \in U$, such that $C \setminus W$ contains an α-hyperedge e with $|e| \geq 3$. Then $v(C \setminus W) \leq \frac{3}{2}\alpha T$.

Proof of Lemma 6.2. To begin we observe that from our setup it follows that if a subset $S \subseteq C \setminus W$ has value $v(S) > \alpha T$ then S contains some α-hyperedge h of p and by property $(\star)$ of W we have $h^p \in V$. Throughout this proof, when talking about α-hyperedges contained in $C \setminus W$, we mean those owned by p, unless otherwise specified.

To prove Theorem 6.2 we will verify that the assumptions imply the following stronger statement.

Claim. *Let s denote the second-most valuable element of e (where ties are broken arbitrarily). Then $C \setminus (W + s)$ contains no α-hyperedge.*

To derive the conclusion of Theorem 6.2 from the Claim: note first that the second-most valuable element s of e satisfies $v_s \le \frac{\alpha T}{2}$, otherwise the value of the two most valuable elements of e would exceed αT, contradicting that $|e| \ge 3$. Then $v(C \setminus W) = v(C \setminus (W+s)) + v_s \le \alpha T + \frac{\alpha T}{2}$.

The rest of the proof consists of proving the Claim. Suppose on the contrary that $C \setminus (W+s)$ contains an α-hyperedge. We claim that some such α-hyperedge g intersects $e - s$ nontrivially. Let $f \subseteq C \setminus (W+s)$ be an α-hyperedge and suppose it is disjoint from $e - s$. We know f contains at least two elements, so removing the least valuable element r from f results in $v(f-r) \ge \frac{v(f)}{2} > \frac{\alpha T}{2}$. Similarly, since s is not the unique most valuable element of e we find that $v(e-s) \ge \frac{v(e)}{2} > \frac{\alpha T}{2}$. Therefore $(f-r) \cup (e-s) \subseteq C \setminus (W+s)$ is valuable enough to contain an α-hyperedge g of p, and since $f - r$ is not valuable enough to contain it, we have $g \cap (e-s) \neq \emptyset$, as claimed.

Recall that since $g \subseteq C \setminus W$, we know $g^p \in V$. Let a be the most valuable element in $g \cap (e-s)$.

Case 1. $|g \cap (e-s)| \ge 2$. Let $b \neq a$ such that $b \in g \cap (e-s)$. Then $v_a \ge v_b$, so $v_s \ge v_b$ and therefore $v((g \cup e) - b) \ge v(g - b + s) \ge v(g) > \alpha T$. Hence $(g \cup e) - b$ contains an α-hyperedge h of p and $h^p \in V$.

We achieve a contradiction by defining a DE-sequence of length two starting with G^*, which turns out to be cheap. First explode g^p at b with some α-hyperedge d_1 in G^*, which exists by Lemma 6.1(ii). By part (i) $|d_1 \cap e| \le 1$ and hence $d_1 \cap e = \{b\}$. Consequently h^p survives the explosion of the edge gd_1, since $h \cap d_1 \subseteq ((g \cup e) - b) \cap d_1 = \emptyset$ and h has the same owner as g. Since there is no KO-sequence isolating h^p starting with G^*, after possible deletions now we can explode h^p with some α-hyperedge d_2. We claim that the basic cover of this DE-sequence of length two (starting with G^*) is a subset of the union of four blocks:

$$d_1 \cup g \cup h \cup d_2 \subseteq (d_1 \setminus g) \cup (e-b) \cup (g-a) \cup (d_2 \setminus h).$$

Here we used that $(h \setminus g) + a$ is contained in $e - b$. This contradiction completes Case 1.

Case 2. $g \cap (e-s) = \{a\}$.

Case 2.a. $v((g \cup e) - a) > \alpha T$.

Let us choose an α-hyperedge $f \subseteq (g \cup e) - a$ of p, as follows. If there is a resource $b \in e - a$ such that $v(g - a + b) > \alpha T$, choose $f \subseteq g - a + b \subseteq (g \cup e) - a$, and otherwise choose one arbitrarily.

By Lemma 6.1(ii) g^p has an explodable neighbour d at a in G^*. Since g and e are both α-hyperedges of p, by Lemma 6.1(i) we have that $|d \cap e| \le 1$ and consequently $d \cap e = \{a\}$.

We achieve a contradiction by defining a DE-sequence of length two starting with G^*, which turns out to be cheap. First we explode g^p with d. The α-hyperedge f^p survives this explosion since $f \cap d \subseteq ((g \cup e) - a) \cap d = \emptyset$ and f and g are both α-hyperedges of p. Secondly, (after some possible deletions) we explode f^p with some neighbour h. This is possible since there is no KO-sequence isolating f^p.

We claim that the basic cover $d \cup g \cup h \cup f$ of this DE-sequence of length two is the subset of the union of four blocks, providing the contradiction we seek. There is a slight difference in the accounting depending how f was chosen.

If b is such that $v(g - a + b) > \alpha T$ and $f \subseteq g - a + b$, we take the blocks $(d \setminus g) \cup (h \setminus f)) \cup (g - a) \cup \{a, b\}$. Note that $\{a, b\}$ is a block since it is a proper subset of the α-hyperedge e with at least three elements.

Otherwise $f \subseteq (g \cup e) - a$ and $g - a + b$ is a block for every $b \in e - a$. Then we take the blocks $(d \setminus g) \cup (h \setminus f)) \cup (g - a + b) \cup (e - b)$, where $b \in e - a$ is arbitrary. Here note that the union of the third and the fourth term is $g \cup e$, which in turn contains f. This completes Case 2.a.

Case 2.b. $v((g \cup e) - a) \le \alpha T$.

Let x denote the most valuable element in $g-a$ (here $\{x\}=g-a$ is possible). By Lemma 6.1(ii) there is an α-hyperedge h, that is explodable with g^p at x in G^*.

Case 2.b.i. $h\cap(e-a)=\emptyset$.

In this case we define a cheap DE-sequence starting with G^*, giving a contradiction. We start by exploding g^p with h. The α-hyperedge e^p survives this explosion, since $h\cap e=\emptyset$ (note that $a\notin h$ since $a\in g-x=g\setminus h$) and g and e are both α-hyperedges of p. Again, since there is no KO-sequence isolating e^p, after possibly some deletions, we explode e^p with some neighbour d. The basic cover $g\cup h\cup e\cup d$ of this DE-sequence is the subset of the union of four blocks: $(h\setminus g)\cup\{a\}\cup((g\cup e)-a)\cup(d\setminus e)$. This contradiction shows that this case cannot hold.

Case 2.b.ii. $h\cap(e-a)\neq\emptyset$.

Since e is an α-hyperedge of the owner of g^p, by Lemma 6.1(i) we have $|h\cap e|\leq 1$, and hence $h\cap e=\{b\}$ for some $b\neq a$. To complete the proof, we will argue that the explosion of the edge hg is cheap by establishing that the basic cover $h\cup g$ can be partitioned into two blocks: $(h-b)\cup(g-x+b)$. The rest of the proof is concerned with demonstrating that $g-x+b$ is indeed a block (the first term is clearly a block).

If $g-a=\{x\}$ is a singleton, then $g-x+b=\{a,b\}$ which is a proper subset of the α-hyperedge e of size at least three and hence is a block.

Otherwise fix a resource $y\in g-a-x$, and suppose on the contrary that $v(g-x+b)>\alpha T$. We will find an α-hyperedge d of p with $h\cap d\supseteq\{b,x\}$, which would contradict Lemma 6.1(i) since h is explodable with an α-hyperedge of p, namely g.

Since $y\in g$, we have $v_x\geq v_y$, and hence for $X:=g-y+b$ we have $v(X)\geq v(g-x+b)>\alpha T$. Note however, that both $X-a$ and $X-b$ are blocks, since $v(X-a)\leq v(g\cup e)-a)\leq\alpha T$, and $X-b=g-y$ is a proper subset of the α-hyperedge g. Hence $a,b\in f$ for any α-hyperedge $f\subseteq X$ of p.

If $x\in f$ as well, then we are done. Otherwise let us fix an α-hyperedge $f=\{a,b,u_1,\dots,u_t\}\subseteq X$ and modify it slightly to obtain the appropriate d.

Note that $t\geq 1$ since $\{a,b\}$ is a proper subset of e. Then $f'=f-u_1+x=\{a,b,x,u_2,\dots,u_t\}$ contains an α-hyperedge d of p because $v(x)\geq v(u_1)$ since $u_1\in f-a-b\subseteq g-y-a$. Note that since $d\subseteq f-u_1+x\subseteq X+x$ and $x\in g-a\subseteq X$ we find $d\subseteq X$, and so $\{a,b\}\subset d$ (since $X-a$ and $X-b$ are blocks). Furthermore d must also contain x, since otherwise $d\subseteq f'-x=\{a,b,u_2,\dots,u_t\}$ is a block (as a proper subset of f). This completes the proof of Case 2.b, and that of the theorem. $\square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Since $v(C\setminus W)=v(C)-v(C\cap W)>T-(1-j\alpha)T)=j\alpha T$ and C is thin, there are strictly more than j resources in $C\setminus W$; let $s,t_1,t_2,\dots,t_j$ be the $j+1$ most valuable ones, in increasing order of value. Then $v(C\setminus(W\cup\{t_2,\dots,t_j\}))>\alpha T$ and consequently $C\setminus(W\cup\{t_2,\dots,t_j\})$ must contain an α-hyperedge.

Since $v(C\setminus W)>j\alpha T>\frac{3}{2}\alpha T$, by Lemma 6.2 every α-hyperedge in $C\setminus W$ has cardinality two. So in particular the two most valuable elements s and t_1 of $C\setminus(W\cup\{t_2,\dots,t_j\})$ form an α-edge of p and $(st_1)^p\in V$. Then, since $v_{t_1}\leq v_{t_i}$ for $i\in\{2,\dots,j\}$, we also know that each st_i is an α-edge of p and $(st_i)^p\in V$.

Next we derive a couple of crucial observations about the explodable neighbors of st_i at t_i.

Claim. *For every $i=1,\dots,j$, if an α-hyperedge f^q (of some player $q\neq p$), with $t_i\in f$, is explodable with $(st_i)^p$ in G^* then*

- *f is an α-edge and*
- *$f\cap(C\setminus W)=\{t_i\}$.*

Proof. If f had at least 3 resources then the explosion would be cheap, contradicting our assumption. To see this choose $a \in f - t_i$, and observe that the basic cover $\{s, t_i\} \cup f$ can be partitioned into two blocks $\{a, t_i\}$ and $f \setminus \{a, t_i\} + s$. Indeed, $\{a, t_i\}$ is a proper subset of f and hence is a block, and using $v_s \leq v_{t_i}$ we see that $v(f \setminus \{a, t_i\} + s) \leq v(f \setminus \{a, t_i\} + t_i) = v(f - a) \leq \alpha T$. Thus every such f is an α-edge.

If $f \subseteq C \setminus W$ then by our setup $(\star)$ implies that $f^p \in V$. This contradicts Lemma 6.1(i) since f^q is explodable with an α-hyperedge owned by p (that is st_i) and $|f \cap f| \geq 2$. So f is not contained in $C \setminus W$, and since f is an α-edge the Claim follows. □

We now define a DE-sequence τ of length j starting with G^* that has basic cover W_τ of size $(2j+1)$.

We construct τ by finding explosions one by one. Let τ_0 be the empty DE-sequence. Suppose for some $i, 1 \leq i \leq j$ we have already found DE-sequence τ_{i-1} which performs explosions of the edges $(st_1)f_1, \dots, (st_{i-1})f_{i-1}$, in this order, such that for every $k = 1, \dots i-1$ we have

(a) f_k is an α-edge and

(b) $f_k \cap (C \setminus W) = \{t_k\}$.

Note that by (a) and (b) for the basic cover $W_{\tau_{i-1}} = \{s\} \cup \cup_{k=1}^{i-1} f_k$ we have $|W_{\tau_{i-1}}| = 2i - 1$.

Our first step in constructing τ_i from τ_{i-1} is to perform, iteratively, all possible deletions, so any remaining edge in the current graph G is explodable. Note that $(st_i)^p$ survived all the explosions of τ_{i-1} since it is disjoint from $f_1, \dots, f_{i-1}$ by (b) and has the same owner as each other $(st_j)^p$. To complete the definition of τ_i, our aim is to find an explodable neighbor f_i of $(st_i)^p$ at t_i in the current graph G, and explode it.

Note we cannot apply Lemma 6.1(ii) here directly to show the existence of such an f_i, since after executing τ_{i-1}, we do not know any more that there is no cheap DE-sequence starting with the current graph G. So we suppose that $(st_i)^p$ has no explodable neighbour at t_i in the current graph G and obtain a contradiction. Since there is no KO-sequence starting with G^* that isolates $(st_i)^p$, by Observation 2.4(iii) there must still exist an α-hyperedge e explodable with it, and this explosion by our assumption is at $s \in e$. If we now perform this explosion then we claim that the resulting DE-sequence τ' starting with G^* would be cheap, providing a contradiction. Indeed, the basic cover of τ_{i-1} together with e forms a cover of τ'. (By our assumption that $(st_i)^p$ has no explodable neighbour at t_i in G, we do not need to include t_i in the cover.) The value $v(W_{\tau_{i-1}}) + v(e - s) \leq (2i-1)\alpha T + \alpha T = 2i\alpha T$ shows that τ' is cheap. So for some player $q \neq p$ there exists an α-hyperedge $f_i^q \in V(G)$ which is explodable with $(st_i)^p$ at t_i in G.

Now we show that f_i satisfies properties (a) and (b). To see this observe that f_i^q was also explodable with $(st_i)^p$ in G^*. Otherwise, by Meshulam's Theorem, the edge $f_i^q(st_i)^p$ was deletable in G^*. But a deletable edge in G^* is a cheap DE-sequence of length zero, contradicting our assumption on G^*. Hence our Claim applies to f_i and so (a) and (b) hold for $k = i$.

For the basic cover of the ultimate DE-sequence τ_j we have $|W_{\tau_j}| = 2j + 1$, showing that τ_j is a $(2j+1)/j$-DE-sequence. □

7 Two values

Here we consider the $(1, \varepsilon)$-restricted version of the Santa Claus problem, in which the value of each resource in R is either 1 or ε, where $0 < \varepsilon < 1$. Our overall approach in this section conceptually parallels our work in the earlier sections.

We begin with a very high-level overview of our argument. Let an instance $\mathcal{I} = (P, R, v, \{L_p : p \in P)$ of the $(1, \varepsilon)$-restricted problem be given, and let T be such that the CLP(T) for $\mathcal{I}$ is feasible. Let $c = \lceil \frac{T}{\varepsilon} \rceil$. We will define an integer r_c and a real number α, as functions of T and ε. It will be straightforward to check (see the proof of Theorem 7.2) that we may assume with these definitions that the resources of value 1 are fat and those of value ε are thin. Hence for each $p \in P$, the set of thin configurations for p is precisely the set of subsets of L_p of cardinality at least c.

Our proof will consist of showing that there exists an allocation of disjoint sets of resources to the players in P, where each set is either a single resource of value 1 or an r_c-subset of resources of value ε. Hence the integrality gap for $\mathcal{I}$ is at most $\frac{T}{r_c \varepsilon} \leq \frac{c}{r_c}$ (since it will also be easy to show that we may assume $1 \geq r_c \varepsilon$).

Our main lemma for these purposes will apply in the following setting.

Setup 7. Let $\mathcal{I} = (P, R, v, \{L_p : p \in P\})$ be an instance of the $(1, \varepsilon)$-restricted Santa Claus problem and let $T \in \mathbb{R}$ be a target such that CLP(T) is feasible. Fix an integer r with $2 \leq r$ and suppose $\alpha \in \mathbb{R}$ satisfies

$$\min \{r\varepsilon, 1\} > \alpha T \geq (r - 1)\varepsilon.$$

Let U be an arbitrary subset of P, let $G^* \subseteq J(\alpha)|_U$ and $W \subseteq R \setminus F$ be a subset of resources such that ($\star$) holds with $G_{start} = J(\alpha)|_U$ and $G_{end} = G^*$. Suppose that there is no KO-sequence starting with G^* and every edge of G^* is explodable.

We remark that the conditions on α are simply to ensure that resources of value 1 are fat, resources of value ε are thin, and thin α-hyperedges have cardinality r and value $r\varepsilon$.

The idea of our proof is to use the same basic framework as that of Theorem 2.1. As before, we will define a DE-sequence starting with $J(\alpha)|_U$ by concatenating many shorter DE-sequences constructed in phases. Each phase lasts as long as there remains a thin configuration C, with value $v(C \setminus W)$ outside the current cover W, that exceeds a certain threshold associated with that phase. However, in our current $(1, \varepsilon)$-restricted setting, the value $v(C \setminus W)$ is determined entirely by $|C \setminus W|$. Instead of four phases, we will have one phase for each $X, c \geq X \geq r_c$, which lasts as long as there still exists a thin configuration C with $|C \setminus W| \geq X$. Our notion of "economical" DE-sequence will be one of low *average cost*, where the average cost of σ with (minimum) cover W_σ is $av(\sigma) = |W_\sigma|/\ell(\sigma)$. Each iteration in Phase X finds a DE-sequence of average cost bounded above by $a_{r_c}(X)$, where $a_r(X)$ is the function defined for all integers $X \geq r \geq 2$ as

$$a_r(X) := \begin{cases} 3r - X - 1 & r \leq X \leq \frac{3r-1}{2} \\ 2r - \frac{X+1}{3} & \frac{3r}{2} \leq X \leq 2r \\ \frac{4r-1}{3} & 2r + 1 \leq X. \end{cases}$$

The main tool that enables this is the following lemma (which corresponds to Theorem 4.1 in the proof of Theorem 2.1). We say that a DE-sequence σ starting with G^* is *based in* a configuration C if one α-hyperedge out of every pair of α-hyperedges exploded during σ is in $C \setminus W$ (and is owned by the owner of C).

Lemma 7.1. *Assume Setup 7. Let $X \geq r$ be an integer. For every thin configuration $C \in \mathcal{C}_p(T)$ with $p \in U$ and $|C \setminus W| \geq X$ there exists a DE-sequence σ starting with G^* based in C with $av(\sigma) \leq a_r(X)$.*

The technical definition of the function $a_r(X)$ is an artefact of our proof this lemma, presented in Subsection 7.2. A plot of $a_r(X)$ for some representative values of r appears in the Appendix.

In Section 7.1 we describe our procedure in which we execute the phases to construct a long DE-sequence. Analogously to Section 5, after each phase we define a feasible solution to DCLP(T), which gives a lower bound on the total length of the DE-sequences constructed during that phase. The result will be an overall lower bound on the length of the whole DE-sequence, which we then optimize to derive Theorem 7.2 below. In Subsection 7.3 we will complete the proof of Theorem 1.2 by deriving it from Theorem 7.2.

Theorem 7.2. *Let $\varepsilon < \frac{1}{2}$. Let $\mathcal{I}$ be an instance of the $(1,\varepsilon)$-restricted Santa Claus problem and let $T \in \mathbb{R}$ be such that CLP(T) has a feasible solution. Suppose that $1 \le T < 2$, and that $c := \lceil T/\varepsilon \rceil \ge 4$. Suppose $r \ge 2$ is an integer such that $\sum_{X=r}^{c} \frac{1}{a_r(X)} \ge 1$. Then there is an allocation with min-value at least $r\varepsilon$.*

Clearly this theorem is strongest when r is largest. We therefore choose r_c to be the largest integer $r \in \mathbb{N}$ satisfying $\sum_{X=r}^{c} \frac{1}{a_r(X)} \ge 1$. Hence Theorem 7.2 implies the upper bound of c/r_c on the integrality gap for all $c \ge 4$. In the proof of Theorem 1.2 we will verify directly that c/r_c is an upper bound on the integrality gap when $1 \le c \le 3$ as well.

For convenience, we provide a table showing the triples $(c, r_c, c/r_c)$ for $1 \le c \le 30$ (with c/r_c truncated to two decimal places).

c	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
r_c	1	1	1	2	2	2	3	3	4	4	4	5	5	6	6
c/r_c	1	2	3	2	2.5	3	2.33	2.66	2.25	2.5	2.75	2.4	2.6	2.33	2.5
c	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
r_c	6	7	7	8	8	8	9	9	10	10	11	11	11	12	12
c/r_c	2.66	2.42	2.57	2.37	2.5	2.62	2.44	2.55	2.4	2.5	2.36	2.45	2.54	2.4	2.5

In the following subsections we will need to refer to some simple properties of the pairs (c, r_c). These are spelled out in the following observation.

Observation 7.3. *(i) $r_c \ge \frac{c}{4}$ for every $c \ge 4$,*

(ii) $c \ge 2r_c + 1$ for every $c \ge 5$,

(iii) $c \ge 2r_c + 2$ for every $c \ge 10$.

Proof. The sum $A = \sum_{X=r}^{c} \frac{1}{a_r(X)}$ has $c - r + 1$ terms, that form a non-decreasing sequence. The smallest term is $\frac{1}{2r-1}$, implying that if $c - r + 1 \ge 2r - 1$ then $A \ge 1$. This is easily satisfied by $r = \lceil \frac{c}{4} \rceil$, implying (i).

The largest term in A is (at most) $\frac{3}{4r-1}$. Note that $\frac{3}{4r-1} < \frac{1}{r+1}$ when $r \ge 5$, so to reach 1 in this case there must be at least $c - r + 1 > r + 1$ terms. Hence $c \ge 2r_c + 1$ if $r \ge 5$. For values of $r \le 4$ see the table to complete the proof of (ii).

Similarly $\frac{3}{4r-1} < \frac{1}{r+2}$ when $r \ge 8$, so to reach 1 in this case there must be at least $c-r+1 > r+2$ terms. Hence $c \ge 2r + 2$ if $r \ge 8$, and again the table completes the proof of (iii) for the smaller values. □

7.1 Proof of Theorem 7.2

To prove Theorem 7.2 we will again apply Theorem 2.5 to infer the existence of the independent transversal in $H(\alpha)$ with an appropriate α, which in turn is equivalent to the existence of an allocation of min-value more than αT. Hence to infer Theorem 7.2 we need an α satisfying $r\varepsilon > \alpha T$, which is one of the conditions on α in Setup 7.

Since we would like to appeal to Lemma 7.1, we will start by verifying that an α satisfying all three conditions in Setup 7 exists. To that end it suffices to show that $r\varepsilon < 1$. Indeed, when

$c \geq 5$ then by Observation 7.3(ii) we know $c \geq 2r_c + 1$, so by the maximality of r_c we have $\frac{2}{\varepsilon} + 1 > \frac{T}{\varepsilon} + 1 > c \geq 2r + 1$. Otherwise, when $c = 4$ then $r = 2$ and the assumption $\varepsilon < 1/2$ implies $r\varepsilon < 1$.

Let us choose, say, $\alpha = (r - 0.5)\varepsilon/T$, which satisfies the conditions of Setup 7. Then with this α, resources of value 1 are fat since $1 > r\varepsilon > \alpha T$. Resources of value ε are thin because $r \geq 2$ implies $\varepsilon = \alpha T/(r - 0.5) < \alpha T$. Thin α-hyperedges have size exactly r since $r\varepsilon > \alpha T > (r - 1)\varepsilon$.

For the proof we fix a subset $U \subseteq P$, assume there is no KO-sequence starting with $J(\alpha)|_U$, and seek a DE-sequence τ of length at least $|U| - |F_U|$ starting with $J(\alpha)|_U$. Our strategy will be as follows.

> INITIALIZATION. Let τ be a sequence of deletions starting with $J(\alpha)|_U$ until no further deletion is possible and let G^* be the resulting subgraph. Let $W = \emptyset$.
>
> For each X, $c \geq X \geq r$, in decreasing order, execute the following Phase X;
>
> PHASE X: WHILE there is a configuration C with at least X resources remaining in $C \setminus W$, DO perform a DE-sequence σ starting with G^* based in C (as given by Lemma 7.1 corresponding to the value of X) and perform all possible deletions afterwards. Update G^* to be the resulting current graph. Append σ to the end of τ. Let W_σ denote the cover of σ and set $W := W \cup W_\sigma$.

Next we verify that this process is well-defined, that is, whenever Lemma 7.1 is called upon in some Phase X, the conditions of Setup 7 are satisfied.

The instance $\mathcal{I} = (P, R, v, \{L_p : p \in P\})$, target $T \in \mathbb{R}$ and integer $r \geq 2$ are given in the assumptions of Theorem 7.2. As indicated above, our choice of $\alpha = (r - 0.5)\varepsilon$ ensures that the conditions on α are satisfied. We have fixed a subset $U \subseteq P$.

Consider the graph $G^* \subseteq J(\alpha)|_U$ in Phase X to which we apply Lemma 7.1. Observe that each iteration of Phase X is immediately preceded by an iteration of Phase X, of Phase $X + 1$, or the initialization phase. To verify the condition on W in Setup 7, note that throughout the procedure, the $(\star)$ property is maintained after each execution of an iteration of Phase X or $X + 1$, as the cover W_σ of the new segment of τ is added to W. The $(\star)$ property holds trivially after initialization, since no explosions have yet occurred so $W = \emptyset$ is a cover. Hence W satisfies $(\star)$ with $G_{start} = J(\alpha)|_U$ and $G_{end} = G^*$. Since we have assumed there is no KO-sequence starting with $J(\alpha)|_U$, there cannot be a KO-sequence starting with G^*. Finally, we check that every edge of G^* is explodable. Since we end the initialization phase and each iteration of Phase X or $X + 1$ by performing deletions until no more were possible, we know that all edges of the graph G^* are explodable. Hence the conditions of Setup 7 hold.

Let W_X be the union of the covers and n_X be the number of explosions done in Phase X. By Lemma 7.1, for each X with $c \geq X \geq r$ we have $|W_X| \leq n_X a_r(X)$.

For $c \geq X \geq r$ we consider the moment after the last step in Phase X is executed and set $W = W_c \cup \ldots \cup W_X$. For each thin configuration $S \in \mathcal{C}_p(T)$ with $p \in U$, we know that $|S \cap W| \geq c - X + 1$, otherwise we could have continued with another step of Phase X. Recalling that ε is the common value of all thin resources, we conclude that $v(S \cap W) \geq \varepsilon(c - X + 1)$ for each such S. Hence we may apply Proposition 3.3 with W in place of Y and $\varepsilon(c - X + 1)$ in place of c to obtain

$$\varepsilon|W| = v(W) \geq \varepsilon(c - X + 1)(|U| - |F_U|).$$

Comparing the upper and lower bounds on $|\cup_{j=X}^{c} W_j|$ for each $X = c, c - 1, \ldots, r$, we obtain

$$\sum_{j=X}^{c} a_r(j) n_j \geq (c - X + 1)(|U| - |F_U|).$$

Since the coefficient function a_r is non-increasing in j, in order to minimize the objective function $\sum_{j=r}^{c} n_j$, we have to choose $n_c, n_{c-1}, \ldots, n_r$ in reverse order such that all the inequalities are equalities

$$\sum_{j=X}^{c} a_r(j)n_j = (c - X + 1)(|U| - |F_U|).$$

This implies that the length $\ell(\tau) = \sum_{j=r}^{c} n_j$ of the DE-sequence τ our process creates is minimized when $n_j = \frac{|U|-|F_U|}{a_r(j)}$. Consequently $\ell(\tau) \geq \sum_{j=r}^{c} \frac{1}{a_r(j)}(|U| - |F_U|)$, which is at least $|U| - |F_U|$, as required for Theorem 2.5. This completes the proof of Theorem 7.2.

7.2 Proof of Lemma 7.1

The main goal of this subsection is to prove Lemma 7.1. To start we describe two criteria that guarantee that a DE-sequence based in C can be continued (plus a consequence of the first one). Both here and in Lemma 7.1, the sets C and W do not play separate roles in the proofs, but appear only in the form $C \setminus W$. To emphasise this we will write $(C \setminus W)$.

Lemma 7.4. *Assume Setup 7. Let $C \in \mathcal{C}_p(T)$ be a thin configuration with $p \in U$ and let σ be a DE-sequence starting with G^*, with explosions of $e_1f_1, \ldots, e_\ell f_\ell$, in this order, where $e_i \subseteq (C \setminus W)$, $1 \leq i \leq \ell$, is owned by p.*

(a) If $e \subseteq (C \setminus W) \setminus \left(\bigcup_{i=1}^{\ell} f_i\right)$ is of size $|e| = r$, then σ can be extended to a longer DE-sequence with an explosion involving e^p.

(b) If σ cannot be extended to a longer DE-sequence based in C, then $av(\sigma) \leq r + \frac{r-1}{\ell(\sigma)}$.

(c) Let G be the current graph immediately before the ℓth explosion and suppose G has only explodable edges. If $e_\ell f_\ell \in E(G)$ was chosen such that $e_\ell \subseteq (C \setminus W) \setminus \bigcup_{j=1}^{\ell-1} f_j$ and $|e_\ell \cap f_\ell|$ is maximized with this property, and if $f_\ell \cap e_\ell \neq f_\ell \cap \left((C \setminus W) \setminus \bigcup_{j=1}^{\ell-1} f_j\right)$, then σ can be extended to a longer DE-sequence based in C.

Proof. For Part (a), note that being disjoint from all f_js, the α-hyperedge e^p survived all explosions so far. Since there is no KO-sequence starting with G^* isolating e^p, after possible deletions, there will be an explodable edge incident to e^p in the current graph. So we can extend σ with a further explosion based in C involving e^p.

For (b), if σ is maximal then $\left|(C \setminus W) \setminus \bigcup_{j=1}^{\ell} f_j\right| \leq r - 1$ and for the size of the basic cover we have

$$|W_\sigma| \leq \left|\left(\bigcup_{i=1}^{\ell} f_i\right) \cup (C \setminus W)\right| \leq r\ell(\sigma) + r - 1.$$

For Part (c) let us take $y \in (f_\ell \setminus e_\ell) \cap \left((C \setminus W) \setminus \bigcup_{j=1}^{\ell-1} f_j\right) \neq \emptyset$ and $w \in e_\ell \setminus f_\ell \neq \emptyset$. Then the α-hyperedge $g = e_\ell - w + y$ is contained in $(C \setminus W) \setminus \bigcup_{j=1}^{\ell-1} f_j$, and consequently by Part (a) g^p in particular is *present* in the current graph G immediately before the explosion of $e_\ell f_\ell$. Since $|f_\ell \cap g| > |f_\ell \cap e_\ell|$, the α-hyperedge g^p was not explodable with f_ℓ immediately before the explosion of $e_\ell f_\ell$, when G had only explodable edges. Hence $g^p f_\ell$ was not an edge of G (i.e. it must have been deleted earlier in the sequence σ or was already not present in G^*). Therefore g^p survives the ℓth explosion as well and since there is no KO-sequence isolating g^p, another explosion involving g^p is possible. □

We are now ready to prove Lemma 7.1.

Proof. First we assume that $X \leq \frac{3r-1}{2}$. If there is a DE-sequence of length two based in C, then by Lemma 7.4(b) it has average cost at most $r + \frac{r-1}{2} \leq 3r - X - 1$.

We may thus assume that there is no DE-sequence of length two based in C. Since $|C \setminus W| \geq X \geq r$ there exists $e^p \in V(G^*)$ with $e \subseteq (C \setminus W)$ and an explosion involving e^p by Lemma 7.4(a). Among all explodable pairs $e^p f \in E(G^*)$ with $e \subseteq (C \setminus W)$, let us choose one with $|e \cap f|$ largest. If $f \cap e \neq f \cap (C \setminus W)$, then by Lemma 7.4(c) there is a second explosion based in C, a contradiction. For this recall that G^* has only explodable edges.

Otherwise $f \cap e = f \cap (C \setminus W)$. Since no further explosion based in C is possible, $|(C \setminus W) \setminus f| \leq r - 1$ by Lemma 7.4(a). In other words $|f \cap (C \setminus W)| \geq X - r + 1$, in which case for the size of the basic cover $f \cup e$ we have $|e| + |f| - |f \cap (C \setminus W)| \leq 2r - (X - r + 1) = 3r - X - 1$, as desired.

We consider now the range $\frac{3r}{2} \leq X$. We aim to construct a DE-sequence based in C with average cost at most $2r - \frac{X+1}{3}$, or one of length at least 3, which has average cost at most $\frac{4r-1}{3}$ by Lemma 7.4(b). This shows that $av(\sigma) \leq \max\{2r - \frac{X+1}{3}, \frac{4r-1}{3}\}$. The bound on $av(\sigma)$ in the second and third ranges then follows since $2r - \frac{X+1}{3} \geq \frac{4r-1}{3}$ if and only if $2r \geq X$.

For the purposes of this proof we set $b = 2r - \frac{X+1}{3}$.

To begin, first suppose that there exist $g^p \in V(G^*)$ with $g \subseteq (C \setminus W)$ and $g^p h \in E(G^*)$ such that $|h \cup g| \leq b$. Then since every edge of G^* is explodable, the single explosion of $g^p h$ is a DE-sequence of length 1 of the type we seek. We may therefore assume from now on that G^* contains no such edge.

We now distinguish two cases.

Case 1. Suppose first that some e^p with $e \subseteq (C \setminus W)$ has an explodable neighbor f such that $|(C \setminus W) \cap f| \leq \lceil \frac{2X-1}{3} \rceil - r = 3r - 1 - \lfloor 2b \rfloor =: t_1$ (note that this expression is non-negative for X in our range).
We perform this explosion and then deletions until no more are possible. Then at least $X - t_1 \geq r$ resources remain in $(C \setminus W) \setminus f$, so by Lemma 7.4(a) there is a further explosion possible. Among all edges hg^p in the current graph, where $g \subseteq (C \setminus W) \setminus f$, we choose one with $|h \cap g|$ largest. Note that hg^p was an edge of G^* as well, and hence we can assume $|h \cap g| \leq 2r - 1 - \lfloor b \rfloor$. Indeed, otherwise we find $|h \cup g| \leq |h| + |g| - |h \cap g| \leq \lfloor b \rfloor \leq b$, giving a DE-sequence of length 1 that achieves our aim as noted above.

Now for our second explosion we explode hg^p, and follow with a sequence of deletions until no more are possible.

If $h \cap g = h \cap ((C \setminus W) \setminus f)$, then we still have $|(C \setminus W) \setminus f| - |h \cap g| \geq X - t_1 - (2r - 1 - \lfloor b \rfloor) \geq X - 5r + 2 + \lfloor b \rfloor + \lfloor 2b \rfloor \geq X - 5r + 2 + b + 2b - 1 = X - 5r + 1 + 6r - (X + 1) = r$ resources in $(C \setminus W) \setminus (f \cup h)$. Hence by Lemma 7.4(a) our sequence can be extended to a sequence of length 3.

Otherwise $h \cap g \neq h \cap ((C \setminus W) \setminus f)$, in which case Lemma 7.4(c) guarantees the extension of our sequence to a third explosion.

Case 2. Suppose now that $|(C \setminus W) \cap f| \geq t_1 + 1$ for every explodable neighbor f of an α-hyperedge in $C \setminus W$ owned by p.
For our first explosion we choose edge hg^p of G^* with $g \subseteq (C \setminus W)$ such that $|g \cap h|$ is largest. Again, if $|h \cup g| \leq b$ we have achieved our aim with this DE-sequence of length 1, hence we may assume that its basic cover $h \cup g$ satisfies $|h \cup g| \geq \lfloor b \rfloor + 1$.

If $h \cap g = h \cap (C \setminus W)$ then the set $(C \setminus W) \setminus h$ contains $X - |h \cap (C \setminus W)| = X - (|h| + |g| - |h \cup g|) \geq X - (2r - (\lfloor b \rfloor + 1)) = X - \lceil \frac{X+1}{3} \rceil + 1$ resources, which is at least r for $X \geq \frac{3r}{2}$. Consequently we can apply Lemma 7.4(a) to extend our DE-sequence with a second explosion involving some $g' \subseteq (C \setminus W)$ owned by p.

If $h \cap g \neq h \cap (C \setminus W)$, then by Lemma 7.4(c) we can also extend our DE-sequence with a second explosion involving such a g'.

Either way we have a DE-sequence σ of length two, based in C, with explosions gh and $g'h'$. We claim that the set $W_\sigma = h \cup h' \cup (C \setminus W) \setminus Y$, where $Y \subseteq (C \setminus W) \setminus (h \cup h')$ is an *arbitrary* subset of size $\min\{t_1, |(C \setminus W) \setminus (h \cup h')|\}$, is a cover. Indeed, let f be an α-hyperedge disjoint from W_σ, and let us show that f survived the explosions of σ. Since f is in particular disjoint from h and h', the only way f could have disappeared is if it had an edge of the current graph, and hence also of G^*, to g or g'. Recall that p is the owner of both g and g'. But then f is owned by some $q \neq p$. Since we are in Case 2, the condition $|f \cap (C \setminus W)| \geq t_1 + 1$ holds in particular for f. Since $f \cap W_\sigma = \emptyset$ we have that $f \cap (C \setminus W) \subseteq Y$, which is a contradiction since $|Y| \leq t_1$ and hence is too small to contain $f \cap (C \setminus W)$.

The DE-sequence σ is based in C and has length two, so unless its average cost is already small enough for our lemma, we have that $2b < |W_\sigma| = |h| + |h'| + |((C \setminus W) \setminus (h \cup h')) \setminus Y|$, which implies

$$|(C \setminus W) \setminus (h \cup h')| \geq \lfloor 2b \rfloor + 1 - 2r + |Y|.$$

We claim that this implies $|(C \setminus W) \setminus (h \cup h')| \geq r$ and therefore by Lemma 7.4(a) we can extend σ to a third explosion. Indeed, if $|Y| = t_1$, then this is immediate from $t_1 = 3r - 1 - \lfloor 2b \rfloor$. Otherwise we have $2r - 1 \geq \lfloor 2b \rfloor = 4r - \lceil \frac{2X+2}{3} \rceil$, or equivalently $\lceil \frac{2X+2}{3} \rceil \geq 2r + 1$, which implies $X \geq 3r$. But then $|(C \setminus W) \setminus (h \cup h')| \geq |(C \setminus W)| - |h| - |h'| = X - 2r \geq r$ as needed. □

7.3 Proof of Theorem 1.2

Proof of Theorem 1.2. We define f by

$$f(x) = \frac{1}{x r_{\lceil \frac{1}{x} \rceil}}.$$

Here, recall, that for an integer $c \in \mathbb{N}$, we denote by r_c the largest integer $r \in \mathbb{N}$ such that $\sum_{X=r}^{c} \frac{1}{a(X)} \geq 1$. We show that $f(x)$ bounds the integrality gap for $\mathcal{I}$, where $x = \frac{\varepsilon}{T^*}$.

First observe that for any instance with $T^* > 0$, it is easy to check Hall's condition on the bipartite graph of players and coveted resources to demonstrate that there is a valid allocation of one resource to each player. Hence, in the two-values case $OPT \geq \varepsilon$. Since $T^* \geq OPT$, we always have that the integrality gap is at most $\frac{T^*}{\varepsilon} \geq 1$. This shows that $f(x) = \frac{1}{x}$ is an appropriate choice to bound the integrality gap for every $x \in (0, 1]$. Since $r_1 = r_2 = r_3 = 1$, this verifies the statement when $x \geq \frac{1}{3}$.

We proceed now with the case $x := \frac{\varepsilon}{T^*} < \frac{1}{3}$.

Analogously to [17] we start by reducing the problem to the case when $1 \leq T^* < 2$. If $T^* \geq 2$ recall that the additive approximation result of Bezáková and Dani [15], mentioned in the Introduction, gives a polynomial-time algorithm to find an allocation with min-value $T_{ALP} - \max v_r$, where T_{ALP} is the optimum of the standard assignment LP. Hence $OPT \geq T_{ALP} - 1 \geq T^* - 1 \geq \frac{T^*}{2}$ as the CLP is stronger than the ALP and in our case $\max v_r = 1$. So the integrality gap is at most 2, which is less than $f(x)$ for every x.

If $T^* < 1$, then we create another instance $\mathcal{I}'$ where for each resource $r \in R$ with $v_r = \varepsilon$, we change the value of r to $v'_r = \varepsilon' := \frac{\varepsilon}{T^*}$, and keep $v'_r = v_r = 1$ otherwise. It is easy check that $T^*(\mathcal{I}') = 1$ and $OPT(\mathcal{I}) \geq OPT(\mathcal{I}')T^*$. Note that $\varepsilon' < \frac{1}{3}$. Hence, applying our theorem to the $(1, \varepsilon')$-instance $\mathcal{I}'$, there is an allocation with min-value at least $\frac{T^*(\mathcal{I}')}{f(\varepsilon'/T^*(\mathcal{I}'))}$. The very same allocation in $\mathcal{I}$ has min-value at least $\frac{T^*}{f(\varepsilon')}$ and hence exhibits an integrality gap of at most $f(\varepsilon/T^*)$ for $\mathcal{I}$.

From now on we assume $1 \leq T^* < 2$ and apply Theorem 7.2. For this we note that $c := \lceil \frac{T^*}{\varepsilon} \rceil = \lceil \frac{1}{x} \rceil > 3$ and $\varepsilon < \frac{1}{2}$. Theorem 7.2 then implies that we have an allocation for $\mathcal{I}$ with min-value at least $r_c \varepsilon$, thus $OPT \geq r_c \varepsilon$. Hence the integrality gap for $\mathcal{I}$ is at most

$$\frac{T^*}{r_c \varepsilon} = \frac{1}{x r_{\lceil \frac{1}{x} \rceil}} = f(x).$$

as promised.

To prove the assertions of Theorem 1.2 about the values, first note that when $x \geq \frac{1}{3}$ we have $f(x) = \frac{1}{x}$, which is less than 3 unless $x = \frac{1}{3}$, and at most 2.75 for $x \geq \frac{4}{11}$. For $x < \frac{1}{3}$ note that with $c = \lceil \frac{1}{x} \rceil$, we have $f(x) = \frac{1}{x r_c} \leq \frac{c}{r_c}$. It is easy to verify that for every $c \geq 4$ we have $c/r_c \leq 2.75$, unless $c = 6$. (In fact the ratio 2.75 is attained on the unique pair $c = 11, r = 4$.) That is, unless $\lceil \frac{1}{x} \rceil = 6$ we have $f(x) \leq 2.75$. When $\lceil \frac{1}{x} \rceil = 6$ we have $x \geq \frac{1}{6}$ and $f(x) = \frac{1}{r_6 x} = \frac{1}{2x}$, which is at most 2.75 for $x \geq \frac{2}{11}$ and strictly less than 3 unless $x = \frac{1}{6}$.

Finally, we deal with the case in which $x \to 0$. Then $c = \lceil \frac{1}{x} \rceil \to \infty$, and hence also $r_c \geq \frac{c}{4} \to \infty$ by Observation 7.3(i).

Let us assume that $x < \frac{1}{10}$, so $c \geq 10$ implying that $c \geq 2r_c + 2$ by Observation 7.3(iii). Setting $r := r_c$, we write the sum as $\sum_{X=r}^{c} \frac{1}{a(X)} = A_r + B_r + C_r$, where

$$A_r := \sum_{X=r}^{\lfloor (3r-1)/2 \rfloor} \frac{1}{3r - X - 1} = \sum_{k=\lceil (3r-1)/2 \rceil}^{2r-1} \frac{1}{k} = (H_{2r-1} - H_{\lceil (3r-1)/2 \rceil - 1}) \to \ln(4/3),$$

$$B_r := \sum_{\lceil 3r/2 \rceil}^{2r} \frac{3}{6r - X - 1} = \sum_{k=4r-1}^{\lfloor \frac{9r}{2} - 1 \rfloor} \frac{3}{k} = 3(H_{\lfloor \frac{9r}{2} - 1 \rfloor} - H_{4r-2}) \to 3\ln(9/8),$$

$$C_r := \sum_{X=2r+1}^{c} \frac{3}{4r-1} = \frac{3(c - 2r)}{4r - 1},$$

when $r \to \infty$. Here we use the well-known fact for the harmonic series $H_n = \sum_{k=1}^{n} \frac{1}{k}$, that $H_n - \ln n$ converges to a constant.

By the maximality of r_c and using $c \geq 2r_c + 2$, we have that

$$A_{r_c+1} + B_{r_c+1} + \frac{3(c - 2(r_c + 1))}{4(r_c + 1) - 1} < 1.$$

Recall that when $x \to 0$, we also have $c \to \infty$ and $r_c \to \infty$, so we obtain $\ln(4/3) + 3\ln(9/8) + \frac{3}{4} \lim_{x \to 0} f(x) - \frac{3}{2} \leq 1$, where we again use that $f(x) \leq \frac{c}{r_c}$. Hence

$$\lim_{x \to 0} f(x) \leq \frac{10}{3} - \frac{4}{3}\ln(4/3) - 4\ln(9/8) < 2.479,$$

as desired. $\square$

8 Conclusion

In this paper we give an entirely novel approach, based on topological notions, for bounding the integrality gap of the Santa Claus problem. This leads to significant improvements on the best known estimates. We believe that this approach will prove to be fruitful in addressing other algorithmic problems involving hypergraph matchings.

As mentioned in the introduction, our argument at the moment does not come with an efficient algorithm for finding an allocation with the promised min-value. This is primarily due to the fact that we do not have a good upper bound on the number of simplices in the triangulation described in the Appendix, which ultimately governs the running time of any algorithmic procedure based on our argument. It would be of great interest to develop methods to make the approach more efficient.

A possible ray of hope comes from recalling the eventual success of turning the initially highly ineffective combinatorial procedure of [10], based on [26], into an efficient algorithm with the same constant factor appriximation. This was achieved through a series of important contributions of several authors, as described in the introduction. Even a quasipolynomial-time algorithm based on our approach that provides *any* constant factor approximation would seem to require new ideas. Such an algorithm would be a first step towards an efficient approximation algorithm that breaks the factor 4 barrier.

Finally, we would like to recall from the introduction that our work on the $(1, \varepsilon)$-restricted case identifies certain parameter choices that seem to capture a key difficulty for the CLP-approach. Specifically, we would like to see a $(1, 1/3)$-restricted problem instance that has optimal CLP-target $T^* = 1$, and no allocation of min-value $2/3$.

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9 Appendix

9.1 The parameter η

As mentioned in Section 2.2 for our results we need only that there exists a graph parameter η that satisfies Fact 1 and Theorems 2.2 and 2.3. In fact such a parameter can be defined in a purely combinatorial way, without any explicit reference to topology. For completeness we begin with a precise definition of η following the treatment of [27], where the required properties are verified. However, the intuition behind the parameter η and how we use it in our proofs is very much topological, as we describe after the definition.

The definition. An *abstract simplicial complex* is a set $\mathcal{A}$ of subsets A of a finite set $V = V(\mathcal{A}) = \cup_{A\in\mathcal{A}} A$ with the property that if $A \in \mathcal{A}$ and $B \subset A$ then $B \in \mathcal{A}$. We call the sets A the *simplices* of $\mathcal{A}$, and the *dimension* of A is $|A| - 1$. The *dimension* of $\mathcal{A}$ is the maximum dimension of any $A \in \mathcal{A}$. Let $\mathcal{A}$ and Σ be abstract simplicial complexes. A function $f : V(\mathcal{A}) \to V(\Sigma)$ is called a *simplicial map from $\mathcal{A}$ to Σ* if $f(A) \in \Sigma$ for every $A \in \mathcal{A}$. We say that $\mathcal{A}$ is a *d-PSC*, i.e. a *pure* d-dimensional simplicial complex, if every maximal $A \in \mathcal{A}$ has the same dimension d. Note then that a d-PSC is the *closure* of the $(d+1)$-uniform hypergraph $\mathcal{A}^d$ consisting of the d-dimensional simplices of $\mathcal{A}$, that is, we form $\mathcal{A}$ from $\mathcal{A}^d$ by adding all subsets of the hyperedges of $\mathcal{A}^d$. For a d-PSC $\mathcal{A}$, the *boundary* $\partial\mathcal{A}$ of $\mathcal{A}$ is the $(d-1)$-PSC that is the closure of the d-uniform hypergraph

$$\{B : |B| = d, |\{A \in \mathcal{A}^d : B \subset A\}| \equiv 1 \mod 2\}.$$

If $\partial(\mathcal{A})$ is empty we say that $\mathcal{A}$ is a *d-dimensional Z_2-cycle*. The abstract simplicial complex Σ is said to be *k-connected* if for each d, $-1 \le d \le k$, for every d-dimensional Z_2-cycle $\mathcal{A}$ and every simplicial map $f : \mathcal{A} \to \Sigma$, there exists a $(d+1)$-PSC $\mathcal{B}$ and a simplicial map $f' : \mathcal{B} \to \Sigma$ such that $\partial\mathcal{B} = \mathcal{A}$ and the restriction $f'|_{\mathcal{A}}$ of f' to $\mathcal{A}$ satisfies $f'|_{\mathcal{A}} = f$.

The independence complex of a graph is an abstract simplicial complex and the value of $\eta(G)$ for a graph G is defined as the largest integer t such that the independence complex $\mathcal{J}(G)$ is $(t-2)$-connected. The parameter η is not explicitly defined in [27], but the main theorems about η are stated and proved there in terms of the above definition of k-connected. (Theorem 2.2 and 2.3 appear as Theorems 11 and 12, respectively.) We may verify Fact 1 as follows. Fact 1(1) follows directly from the definition of η, as saying that Σ is (-1)-connected is the same as saying that $V(\Sigma)$ is nonempty. For the second statement of Fact 1(2), suppose G contains an isolated vertex x. Let $\mathcal{A}$ be an arbitrary Z_2-cycle, with a simplicial map f from $\mathcal{A}$ to $\mathcal{J}(G)$. Then f can be extended to a simplicial map from the closure $\mathcal{B}$ of $\{A \cup \{w\} : A \in \mathcal{A}\}$, where $w \notin V(\mathcal{A})$ is a new vertex, by mapping w to x. Since the dimension of $\mathcal{A}$ is arbitrary, this implies that $\eta(G)$ is infinite. Otherwise G has no isolated vertices, and so G_2 contains an edge. Then by Theorem 2.3 we can keep deleting and/or exploding edges from G_2, one by one, until all edges of G_2 have disappeared. The resulting graph G_{end} still contains G_1. If G_{end} has an isolated vertex, then $\eta(G) \ge \eta(G_{end}) = \infty$ by the above. Otherwise at least one explosion was performed and $G_{end} = G_1$, hence $\eta(G) \ge \eta(G_1) + 1$ by Observation 2.4(i).

The intuition. In what follows, we describe the topological nature of our work at an intuitive level, without getting into precise details. The topological space X is said to be *k-connected* if for every d, $-1 \le d \le k$, every continuous map from the d-dimensional sphere to X extends to a continuous map from the $(d+1)$-dimensional ball to X. This property indicates that X lacks a $(d+1)$-dimensional "hole".

To get a better understanding for the topological core of our arguments, it helps to think of connectedness as defined in the following way, that provides a link between the notion of connectedness for a topological space and our earlier definition for abstract simplicial complexes. This link goes via triangulations of a simplex, which are *geometric simplicial complexes*, that can be viewed both as topological spaces and as abstract simplicial complexes. We say that an abstract simplicial complex Σ is *k-connected* if for every d, $-1 \le d \le k$, for every triangulation $\mathcal{T}$ of the boundary of the $(d+1)$-dimensional simplex τ, and every simplicial map f from $\mathcal{T}$ to Σ, there exists a triangulation $\mathcal{T}'$ of the whole of τ that extends $\mathcal{T}$, and a simplicial map f' from $\mathcal{T}'$ to Σ that extends f.

Our argument gives a process that, given an instance $\mathcal{I}$ of the Santa Claus problem with player set P, produces an allocation with the promised min-value. Very broadly speaking, the process has two main stages. Following the proofs of Theorems 2.2 and 2.3, the first stage constructs a triangulation $\mathcal{T}$ of the $(|P|-1)$-dimensional simplex τ, and a simplicial map f from $\mathcal{T}$ to the independence complex of the graph $H(\alpha)$ (defined in Section 2.1), such that the $|P|$-coloring of the points $v \in V(\mathcal{T})$, defined by the "owner" of the α-hyperedge $f(v)$, satisfies the conditions of Sperner's Lemma. The second stage applies Sperner's Lemma to find a multicolored simplex, which corresponds to an independent transversal of $H(\alpha)$, i.e. an allocation for instance $\mathcal{I}$ with min-value at least αT as promised.

Executing the first stage is the main aim of our paper and here is where topological connectedness helps us. The triangulation $\mathcal{T}$ and the map f are built on the faces of τ one by one, in increasing order of dimension. When $\mathcal{T}$ and f on a face σ of dimension d are to be defined, triangulations and maps of all the facets of σ are already in place, forming the boundary of σ, and these need to be extended to a triangulation and a map of the whole of σ. This notion of extending a map from the boundary of σ to the interior is captured by the parameter η, so if η is sufficiently large for each σ, then this extension is possible.

9.2 Demonstrating DE-sequences

Here we demonstrate how to use DE-sequences to show that $\eta(G) \ge 2$ for the cycle $G = C_5$ of length 5. (In fact $\eta(G) = 2$, since the independence complex itself is a 5-cycle, which has a 2-dimensional hole.) We do this without thinking about the underlying "topology" of the current graph (i.e. whether the next edge is deletable or explodable), but rather following through *all* sequences of deletions and explosions that are possible combinatorially (some of which might not actually represent a "topologically legal" DE-sequence) and arriving at a lower bound of at least 2 for each.

For an edge e of G, the graph $G \circledast e$ consists of a single isolated vertex, and hence $\eta(G \circledast e) = \infty$ by Observation 2.4(ii). Therefore if e is explodable we are done, so we may assume that e is deletable. Deleting e results in the path P_5 with 5 vertices, and by the definition of deletable edge $\eta(G) \ge \eta(P_5)$. Next consider an edge e' of P_5 joining two of its degree-2 vertices. Again $P_5 \circledast e'$ consists of a single isolated vertex, showing that we may assume e' is not explodable and hence deletable. The graph $P_5 - e'$ consists of two components, a P_2 and a P_3. Each of these has a positive value of η by Fact 1(1). Hence

$$\eta(G) \ge \eta(P_5) \ge \eta(P_5 - e') \ge 1 + \eta(P_3) \ge 2$$

by Fact 1(2).

9.3 The function a_r

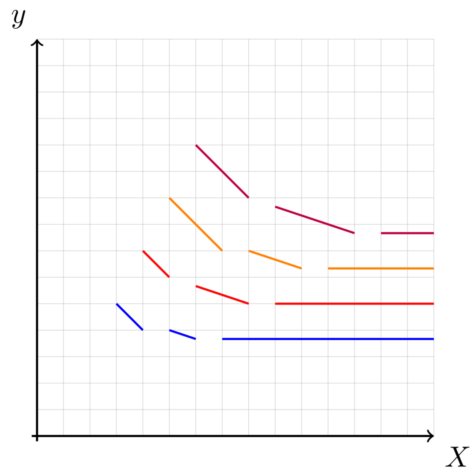

Figure 1: Plots of the function $a_r(X)$ for different values of r. The function for $r = 3$, $r = 4$, $r = 5$ and $r = 6$ is given in blue, red, orange and purple respectively

9.4 Notation Finder

For each term we indicate the section in which is defined. Most of the definitions appear very close to the beginning of the relevant section or subsection.

- $(1, \varepsilon)$-restricted problem (1)
- α-hyperedge of p (2.1)
- α-edge (6)
- allocation (1)
- $a_r(X)$ (7)
- $av(\sigma)$, the average cost of σ (7)
- block (3)
- CLP(T), the configuration LP with target T (1)
- $C_p(T)$, the set of configurations for p (1)
- cheap DE-sequence (4)
- cover (2.3)
- DCLP(T), the dual of the CLP(T) (3)
- deletable (2.2)
- DE-sequence (2.2)
- $e - x$, $e + x$ (6)
- explodable (2.2)