BC2410 : Prescriptive Analytics

Assignment 1

(Due: 1 Feb. 2022)

1. Let
$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
   (a) Determine $0.1\boldsymbol{a} + 0.9\boldsymbol{b}$, $\boldsymbol{a} - \boldsymbol{b}$, and $\boldsymbol{a}'\boldsymbol{b}$.

   (b) Let $\lambda \in [0,1]$, determine $\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}$. Try to change λ and plot all the results, what do you observe?

2. Let
$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & 7 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$$
   (a) Determine $\boldsymbol{AB}$ and $\boldsymbol{BA}$. What do you observe?

   (b) Determine $(\boldsymbol{AB})'$, $\boldsymbol{B}'$, $\boldsymbol{A}'$ and $\boldsymbol{B}'\boldsymbol{A}'$. What do you observe?

3. Define the set
$$X = \left\{ \boldsymbol{x} \in \Re^2 \ : \ \boldsymbol{Ax} \leq \boldsymbol{b} \right\}.$$
   (a) What are the possible size of matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$?

   (b) Draw the set X for
$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$
   (c) (Optional) Provide possible values of $\boldsymbol{A}$ and $\boldsymbol{b}$ to describe the feasible region of a square with corners at $(0,0), (1,-1), (1,1), (2,0)$.

4. Let
$$X_1 = \{\boldsymbol{x} \in \Re^n \ : \ \boldsymbol{Ax} \leq \boldsymbol{b}\}.$$
   and
$$X_2 = \{\boldsymbol{x} \in \Re^n \ : \ \boldsymbol{Ax} + \boldsymbol{s} = \boldsymbol{b}, \boldsymbol{s} \geq \boldsymbol{0} \ \text{ for some } \boldsymbol{s}\}.$$
   We want to show that $X_1 = X_2$, i.e., the two sets are equivalent. We can do so by proving $X_1 \subseteq X_2$ and $X_2 \subseteq X_1$.

(a) Show that $X_1 \subseteq X_2$. You need to show that if $\boldsymbol{x} \in X_1$, then $\boldsymbol{x} \in X_2$.

(b) Show that $X_2 \subseteq X_1$

5. The table in the following shows the data corresponding to a civilization with two types of grains (G1 and G2) and three types of nutrients (starch, proteins, vitamins):

	G1	G2
Starch	5	7
Proteins	4	2
Vitamins	2	1
Cost (\$/kg)	0.6	0.35

The requirement per day of starch, proteins and vitamins is 8, 15 and 3 respectively.

(a) Could you help to decide how much of each food to consume per day so as to get the required amount per day of each nutrient at minimal cost?

(b) Now suppose that there are n different foods and m different nutrients, and that we are given the following table with the nutritional content of a unit of each food:

	Food 1	$\cdots$	Food n
Nutrient 1	a_{11}	$\cdots$	a_{1n}
$\vdots$	$\vdots$		$\vdots$
Nutrient m	a_{m1}	$\vdots$	a_{mn}
Cost	c_1	$\vdots$	c_n

Please formulate this problem as a LOP and write it in a matrix form.

6. NBS Pte Ltd produces shoes and is opening a new franchise in JB. The company needs to hire works. Assume each worker can only produce 50 pairs of shoes per quarter, is paid \$500 per quarter, and works three consecutive quarters per year. The demand (in pairs of shoes) is 600 for the first quarter, 300 for the second quarter, 800 for the third quarter, and 100 for the fourth quarter. Excessive pairs of shoes may be carried over to the next quarter at a cost of \$50 per quarter per pair of shoes, and there will be no inventory at the end of quarter 4. Could you assist NBS Pte Ltd in lowering its costs?

7. (Optional) Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Determine $\boldsymbol{AB}$ and $\boldsymbol{BA}$. What do you observe?

8. (Optional) Let

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

   (a) Determine $\boldsymbol{A}^{-1}$.

   (b) Determine $\boldsymbol{A}'$, $(\boldsymbol{A}')^{-1}$ and $(\boldsymbol{A}^{-1})'$. What do you observe?

9. (Optional) Consider the following linear equation with unknown variables $x_1, x_2, \cdots, x_n$,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Express the equation in matrix formulation, i.e., find out $\boldsymbol{A}$, $\boldsymbol{x}$ and $\boldsymbol{b}$ such that $\boldsymbol{Ax} = \boldsymbol{b}$.

10. (Optional) A linear function, $f(\boldsymbol{x}) : \Re^n \mapsto \Re$ is one that satisfies the following properties:

   - Additivity: For any $\boldsymbol{x}, \boldsymbol{y} \in \Re^n$, $f(\boldsymbol{x} + \boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y})$
   - Homogeneity: For any $a \in \Re$, $f(a\boldsymbol{x}) = af(\boldsymbol{x})$

Given a vector, $\boldsymbol{a} \in \Re^n$. and define a function $g(\boldsymbol{x}) : \Re^n \mapsto \Re$ as

$$g(\boldsymbol{x}) = \boldsymbol{a}'\boldsymbol{x}.$$

   (a) Show that $g(\boldsymbol{x})$ is a linear function.

   (b) Show that $g(\boldsymbol{x}) + b$ is not a linear function if $b \neq 0$.

   (c) Show that any linear function can be expressed as a g function. Hint: Observe that

$$\boldsymbol{x} = \boldsymbol{Ix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} x_2 + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} x_n.$$

11. (Optional) Watch the video on Gaussian Elimination **(link here)**. Implement Gaussian elimination to solve a system of linear equations in Python.

    (a) Write a function that takes inputs: Matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, $\boldsymbol{A}$ invertible. Output the solution $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

    (b) Use numpy package to store array.

    (c) Compare how fast your code perform against the numpy function "linalg.solve".

    (d) Solution would not be posted, but you can easily find similar code online.

12. (Optional) Suppose that there are N available currencies, and assume that one unit of currency i can be exchanged for r_{ij} units of currency j. (Naturally, we assume that $r_{ij} > 0$.) There are also certain regulations that impose a limit u_i on the total amount of currency i that can be exchanged on any given day. Suppose that we start with B units of currency 1 and that we would like to maximize the number of units of currency N that we end up with at the end of the day, through a sequence of currency transactions. Provide a linear programming formulation of this problem. Assume that for any sequence $i_1, \cdots, i_k$ of currencies, we have $r_{i_1 i_2} r_{i_2 i_3} \ldots r_{i_{k-1} i_k} r_{i_k i_1} \leq 1$, which means that wealth cannot be multiplied by going through a cycle of currencies.