Convolution as Matrix Multiplication

$$f(x) \longrightarrow \boxed{h(x)} \longrightarrow g(x)$$

$$g(x) = h(x) * f(x)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \\ a_{-4} & a_{-3} & a_{-2} & a_{-1} \end{bmatrix}$$

<u>Toeplitz matrix</u> – constant along every diagonal

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$$g(x) = h(x) * f(x)$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ -1 \end{bmatrix}$$

<u>Circulant matrix</u> – each row is a circular shift of the previous row

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{bmatrix}$$

Another Example

$$H = \begin{bmatrix} 3 & 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \end{bmatrix}$$

2-d Convolution as Matrix Multiplication

$$H = \begin{bmatrix} H_0 & H_1 & H_2 & H_3 \\ H_3 & H_0 & H_2 & H_3 \\ H_2 & H_3 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_0 \end{bmatrix}$$

Each block is a circulant matrix.

H is a <u>doubly block circulant</u> matrix.

2-d Convolution as Matrix Multiplication

- For a 512 by 512 image:
  - $\underline{f}$ and $\underline{g}$ are 262144 by 1 $\quad \underline{g} = H\,\underline{f}$
  - H is 262144 by 262144
- Computing $(H^T H)^{-1} H^T$ directly is not feasible
- Solution is subject to numerical instability
- Looking ahead: we will convert a "bad" problem to a "good" problem by looking for a "well behaved" or "smooth" solution

A Little Matrix Calculus

$$\frac{\partial}{\partial \underline{x}} f(\underline{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(\underline{x}) \\ \frac{\partial}{\partial x_2} f(\underline{x}) \\ \vdots \\ \frac{\partial}{\partial x_N} f(\underline{x}) \end{bmatrix}$$

$$\frac{\partial}{\partial \underline{x}}\left(\underline{x}^T \underline{y}\right) = \underline{y}$$

$$\frac{\partial}{\partial \underline{x}}\left(\underline{y}^T \underline{x}\right) = \underline{y}$$

$$\frac{\partial}{\partial \underline{x}}\left(\underline{x}^T P\, \underline{x}\right) = P\,\underline{x} + P^T\,\underline{x}$$

$$\frac{\partial}{\partial \underline{x}}\left(\underline{x}^T P^T P\, \underline{x}\right) = 2P^T P\,\underline{x}$$

Lagrange Multiplier Method

Minimize $f(\underline{x})$ subject to $\underline{g}(\underline{x}) = 0$.

define: $\quad \phi(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \underline{\lambda}^T \underline{g}(\underline{x})$

Then optimization problem can be solved by solving

$$\frac{\partial}{\partial \underline{x}} \phi(\underline{x}, \underline{\lambda}) = 0$$

$$\underline{g}(\underline{x}) = 0$$

for $\underline{x}$ and $\underline{\lambda}$

Constrained Least Squares Algorithm

1. Choose an initial guess for γ
2. Calculate the reconstruction

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2}\right] G(u,v)$$

3. Calculate the residual

$$R(u,v) = G(u,v) - H(u,v)\,\hat{F}(u,v)$$

$$\phi(\gamma) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} |R(u,v)|^2$$

4. Compare residual to target value, and either
   a. Choose a better value for γ, and go to 2 above, or
   b. We are done

Blind Deconvolution

- D. Kundur and D. Hatzinakos, “Blind image deconvolution,” IEEE Signal Processing Magazine, May 1996.
- Reconstruct image with imperfect knowledge of degradation function $H(u,v)$
- If there is *no noise*, then blind deconvolution in 2 or more dimensions can be solved using polynomial factorization
- In general, problem is ill-conditioned, solution may not be unique
- Need to use a priori assumptions about the undegraded image, the degradation function, or both:
  - Assume a form for the degradation function such as motion blur or lens defocus, then estimate model parameters by examining zeros in the frequency plane
  - Assume undegraded image is an autoregressive (AR) process and that the degradation function is a moving average (MA) process, then fit data to a ARMA model

Blind Deconvolution

- Constraints for the image:
  - The image data is a non-negative
  - The image has a limited region of support
  - The image contains sharp edges or point features
- Constraints for the degradation function:
  - The degradation function is circularly symmetric
  - The degradation function is non-negative
  - The degradation function has a limited region of support

Iterative Blind Deconvolution

$$\tilde{F}_k(u,v) = \frac{\hat{H}^*_{k-1}(u,v)G(u,v)}{|\hat{H}_{k-1}(u,v)|^2 + \dfrac{\alpha}{|\hat{F}_{k-1}(u,v)|^2}}$$

$$\tilde{H}_k(u,v) = \frac{\hat{F}^*_{k-1}(u,v)G(u,v)}{|\hat{F}_{k-1}(u,v)|^2 + \dfrac{\alpha}{|\hat{H}_{k-1}(u,v)|^2}}$$

Iterative Blind Deconvolution