VIBRATION ANALYSIS
AND VIBROACOUSTICS

VIBRATION ANALYSIS

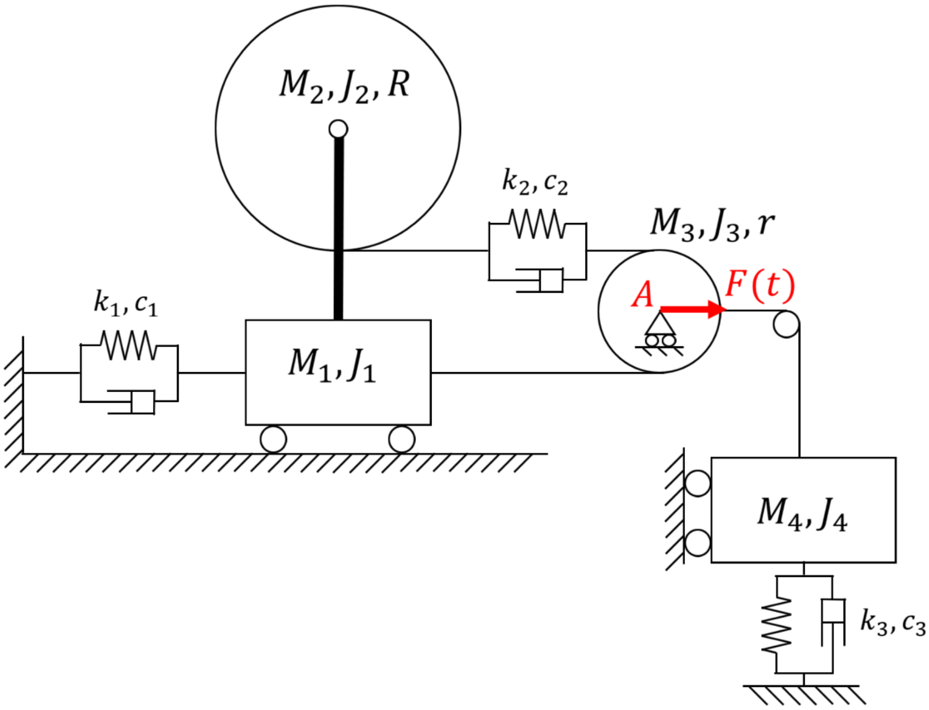
Assignment 2 - A.Y. 2023/24

ERNEST OUALI
242311
FRANCESCO PANETTIERI
250266
GIULIANO DI LORENZO
242712

POLITECNICO
MILANO 1863

1 Equations of Motion and system matrices

In order to describe the mechanical system in figure, a reference system convention is defined: all displacements along the positive x and positive y directions are positive, all counterclockwise rotations are positive and all springs/dampers elongation are positive.



The number of degrees of freedom (dof) has to be determined as:

$$n_{dof} = 3n_b - n_{con}$$

Where n_b is the total number of rigid bodies (4 in this system) and n_{con} the number of constraints. An equation of motion (EoM) for each dof allow to completely describe the mechanical system. Each EoM is the result of the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{x}}\right) - \frac{\partial E_K}{\partial x} + \frac{\partial D}{\partial \dot{x}} + \frac{\partial V}{\partial x} = Q_x$$

In which x represents the single independent variable of a mechanical system.
By combining all contributions in a matrix form:

$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

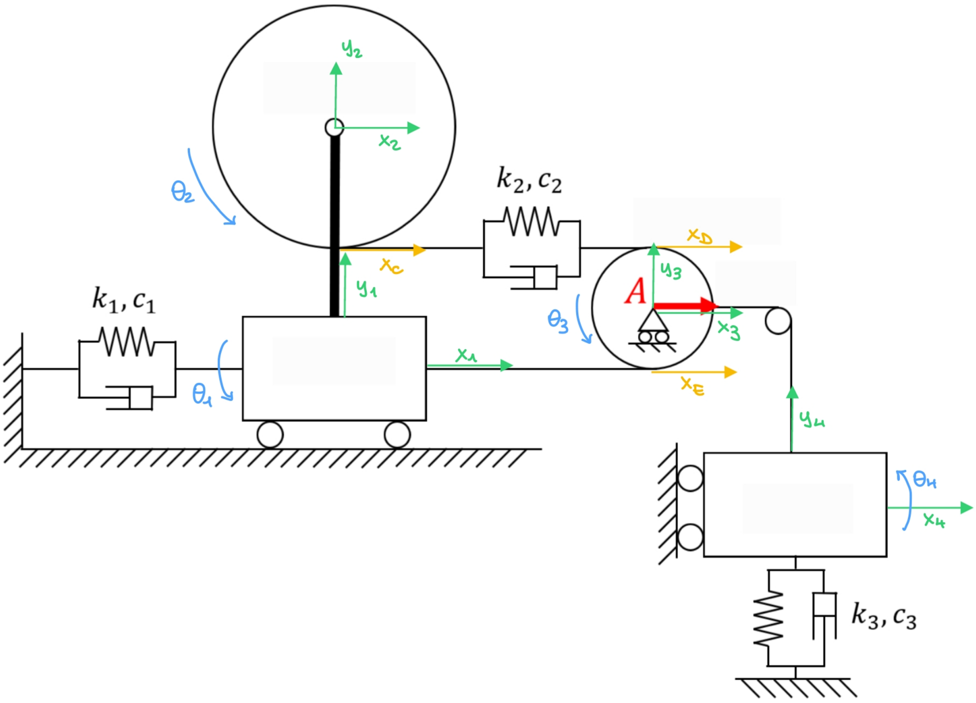
Where all independent variables have been gathered in the vector $\underline{x}$, as well as their first time derivatives in $\underline{\dot{x}}$.

1.1 Equations of Motion around the equilibrium position

The first goal is to determine the number of dof by evaluating all constraints:

- M_2 is constrained (through a hinge) to a mass-less vertical beam, rigidly connected to the mass M_1
  - hinge $\longrightarrow y_2 = y_1$
  - rigid connection $\longrightarrow \dot{x}_2 = \dot{x}_1$
- M_1 can only slide horizontally
  - slider $\longrightarrow y_1 = 0,\ \theta_1 = 0$

- M_3 can slide horizontally and rotate through an inextensible rope that connects the disk to both M_2 and M_1
  - slider and hinge $\longrightarrow y_3 = 0$
  - inextensible rope $\longrightarrow \dot{x}_1 = \dot{x}_E$
- M_4 can slide vertically and is rigidly connected to M_3 through an inextensible rope
  - slider $\longrightarrow x_4 = 0$, $\theta_4 = 0$
  - inextensible rope $\longrightarrow y_4 = -x_3$



To conclude, the number of constraints n_{con} is equal to $2+2+2+3$, so the number of dof is:

$$n_{dof} = 3n_b - n_{con} = 12 - 9 = 3$$

After that, it is necessary to describe all existing relations between independent variables (3 in this system) and physical variables:

$$\begin{cases} \dot{x}_2 = \dot{x}_1 \\ \dot{x}_1 = \dot{x}_E \end{cases} \Longrightarrow x_1 = x_2 = x_E$$

$$\dot{x}_C = \dot{x}_2 + R\dot{\theta}_2, \quad \dot{x}_D = \dot{x}_1 - 2r\dot{\theta}_3, \quad y_4 = -x_3 = -\left(x_E - r\theta_3\right)$$

$$\Longrightarrow \begin{cases} x_1 = x_2 = x_E \\ \dot{x}_C = \dot{x}_2 + R\dot{\theta}_2 \\ \dot{x}_D = \dot{x}_1 - 2r\dot{\theta}_3 \\ y_4 = -x_1 + r\theta_3 \end{cases}$$

Springs and dampers appear in the Lagrange equations by means of elongations Δl and their first time derivative $\dot{\Delta l}$:

$$\begin{cases} \dot{\Delta l}_1 = \dot{x}_1 \\ \dot{\Delta l}_2 = \dot{x}_D - \dot{x}_C = -R\dot{\theta}_2 - 2r\dot{\theta}_3 \\ \dot{\Delta l}_3 = \dot{y}_4 = -\dot{x}_1 + r\dot{\theta}_3 \end{cases} \Longrightarrow \begin{cases} \Delta l_1 = x_1 \\ \Delta l_2 = -R\theta_2 - 2r\theta_3 \\ \Delta l_3 = -x_1 + r\theta_3 \end{cases}$$

Being the system in static equilibrium position, all springs pre-load are neglected.

1.1.1 Kinetic energy E_K

It is now possible to compute, in matrix form, all energies of the Lagrange equation.
Regarding the kinetic energy, four rigid bodies may contribute to the total E_K with four translations and four rotations. In this system, all bodies can translate but not rotate (only M_2 and M_3):

$$E_K = \frac{1}{2}M_1 v_1^2 + \frac{1}{2}M_2 v_2^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}M_3 v_3^2 + \frac{1}{2}J_3\omega_3^2 + \frac{1}{2}M_4 v_4^2$$

Let's gather all physical coordinates (and their first time derivatives) in two column vectors $\underline{z}$ (and $\dot{\underline{z}}$ respectively):

$$\underline{z}_{6\times1} = (x_1, x_2, \theta_2, x_3, \theta_3, y_4)^T\,, \quad \dot{\underline{z}}_{6\times1} = (v_1, v_2, \omega_2, v_3, \omega_3, v_4)^T$$

By defining the following physical mass matrix $[M]$ as:

$$[M]_{6\times6} = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 \end{bmatrix}$$

It is now possible to express the total kinetic energy E_K in matrix form as:

$$E_K = \frac{1}{2}\dot{\underline{z}}^T[M]\dot{\underline{z}}$$

It is of interest to get the energy in function of the independent variables:

$$\underline{x}_{3\times1} = (x_1, \theta_2, \theta_3)^T$$

expressed respectively in m, rad and rad. Let's introduce a relation between physical variables $\underline{z}$ and independent ones $\underline{x}$ by using their first time derivative:

$$\dot{\underline{z}} = \left(\frac{\partial \underline{z}}{\partial \underline{x}}\right)\dot{\underline{x}} = [\Lambda_M]\,\dot{\underline{x}}$$

Where $[\Lambda_M]$ is the Jacobian matrix and describes the relation between velocities v and angular velocities ω with respect to the independent variables.
Its entries are obtained by looking at the following table:

	$\dot{x}_1$	$\dot{\theta}_2$	$\dot{\theta}_3$
v_1	1	0	0
v_2	1	0	0
ω_2	0	1	0
v_3	1	0	-r
ω_3	0	0	1
v_4	-1	0	r

So the Jacobian matrix is defined as:

$$[\Lambda_M]_{6\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -r \\ 0 & 0 & 1 \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_M]$ in the kinetic energy formulation:

$$E_K = \frac{1}{2}\dot{\underline{z}}^T[M]\dot{\underline{z}} = \frac{1}{2}\dot{\underline{x}}^T[\Lambda_M]^T[M][\Lambda_M]\dot{\underline{x}} = \frac{1}{2}\dot{\underline{x}}^T[M^*]\dot{\underline{x}}$$

Where the mass matrix $[M^*]$ is defined as:

$$[M^*]_{3\times3} = [\Lambda_M]^T[M][\Lambda_M] = \begin{bmatrix} M_1+M_2+M_3+M_4 & 0 & -r\,[M_3+M_4] \\ 0 & J_2 & 0 \\ -r\,(M_3+M_4) & 0 & J_3 + r^2\,(M_3+M_4) \end{bmatrix}$$

1.1.2 Potential energy V

Regarding the potential energy, the springs give a contribution in terms of elastic energy, while the mass M_4 gives a contribution in terms of gravitational energy:

$$V = V_{el} + V_g = \frac{1}{2}k_1\Delta l_1^2 + \frac{1}{2}k_2\Delta l_2^2 + \frac{1}{2}k_3\Delta l_3^2 + \frac{1}{2}M_4 g y_4$$

The gravitational term can be neglected due to the linearity of the system, so it does not affect the Lagrange equations, i.e. : $V_g = 0$.
Similarly to the kinetic case, a matrix approach is needed and the Jacobian matrix $[\Lambda_k]$ describes the relation between elongations Δl and independent variables.
By defining the physical stiffness matrix $[k]$ and the elongation vector $\underline{\Delta l}$ as:

$$[k]_{3\times 3} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad \underline{\Delta l}_{3\times 1} = \{\Delta l_1, \Delta l_2, \Delta l_3\}^T$$

It is possible to express the elastic potential energy in matrix form as:

$$V = V_{el} = \frac{1}{2}\underline{\Delta l}^T[k]\underline{\Delta l}$$

The Jacobian matrix $[\Lambda_k]$ entries are obtained by looking at the following table:

	x_1	θ_2	θ_3
Δl_1	1	0	0
Δl_2	0	-R	-2r
Δl_3	-1	0	r

So the Jacobian matrix is defined as:

$$[\Lambda_k]_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R & -2r \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_k]$ in the potential energy formulation:

$$V = V_{el} = \frac{1}{2}\underline{\dot{z}}^T[k]\underline{\dot{z}} = \frac{1}{2}\underline{\dot{x}}^T[\Lambda_k]^T[k][\Lambda_k]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[k^*]\underline{\dot{x}}$$

Where the stiffness matrix $[k^*]$ is defined as:

$$[k^*]_{3\times 3} = [\Lambda_k]^T[k][\Lambda_k] = \begin{bmatrix} k_1 + k_3 & 0 & -rk_3 \\ 0 & R^2k_2 & 2Rk_2r \\ -rk_3 & 2Rk_2r & 4k_2r^2 + k_3r^2 \end{bmatrix}$$

1.1.3 Dissipative energy D

Regarding the dissipative energy, the dampers give a contribution in terms of first time derivative of the elongations:

$$D = \frac{1}{2}c_1\dot{\Delta l_1}^2 + \frac{1}{2}c_2\dot{\Delta l_2}^2 + \frac{1}{2}c_3\dot{\Delta l_3}^2$$

Similarly to the previous cases, a matrix approach is needed and the Jacobian matrix $[\Lambda_c]$ describes the relation between the elongations derivatives $\dot{\Delta l}$ and independent variables.
By defining the physical damping matrix $[c]$ and the elongation derivatives vector $\underline{\dot{\Delta l}}$ as:

$$[c]_{3\times 3} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}, \quad \underline{\dot{\Delta l}}_{3\times 1} = \{\dot{\Delta l_1}, \dot{\Delta l_2}, \dot{\Delta l_3}\}^T$$

It is possible to express the dissipative energy in matrix form as:

$$D = \frac{1}{2}\underline{\dot{\Delta l}}^T[k]\underline{\dot{\Delta l}}$$

The Jacobian matrix $[\Lambda_c]$ entries are the very same of $[\Lambda_k]$:

$$[\Lambda_c]_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R & -2r \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_c]$ in the potential energy formulation:

$$D = \frac{1}{2}\underline{\dot{z}}^T[c]\underline{\dot{z}} = \frac{1}{2}\underline{\dot{x}}^T[\Lambda_c]^T[c][\Lambda_c]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[c^*]\underline{\dot{x}}$$

Where the damping matrix $[c^*]$ is defined as:

$$[c^*]_{3\times3} = [\Lambda_c]^T[c][\Lambda_c] = \begin{bmatrix} c_1 + c_3 & 0 & -rc_3 \\ 0 & R^2c_2 & 2Rc_2r \\ -rc_3 & 2Rc_2r & 4c_2r^2 + c_3r^2 \end{bmatrix}$$

1.1.4 Virtual works δW

The principle of virtual works declares that an external force, applied at a given force point generates an infinitesimal work δW proportional to an infinitesimal displacement δx. In this system, the only existing external force $F(t)$ is applied horizontally at the point A, resulting in an infinitesimal displacement δx_3 and an infinitesimal work:

$$\delta W = F \cdot \delta x_3$$

Similarly to the previous cases, it is of interest to express the infinitesimal displacement δx_3 as function of the infinitesimal independent variables $\underline{\delta x}$ by means of Jacobian vector $[\Lambda_F]$:

$$\delta x_3 = \delta x_1 - \delta\theta_3 r = (1, 0, -r)\begin{pmatrix} \delta x_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{pmatrix} = [\Lambda_F]_{1\times3} \cdot \underline{\delta x}_{3\times1}$$

$$\implies \delta W = F\delta x_1 - F\delta\theta_3 r = F \cdot [\Lambda_F] \cdot \underline{\delta x}$$

For the Lagrange equation, it is possible to identify the Lagrangian vector $\underline{Q}$ as:

$$\underline{Q}_{3\times1} = \begin{pmatrix} F \\ 0 \\ -rF \end{pmatrix} = [\Lambda_F]^T F$$

1.1.5 Lagrange equations

It is now possible to express the Lagrange equations in matrix form:

$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

$$\left\{\frac{d}{dt}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T = [M^*] \cdot \underline{\ddot{x}}\,, \quad \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T = \underline{0}$$

$$\left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T = [c^*] \cdot \underline{\dot{x}}, \quad \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = [k^*] \cdot \underline{x}$$

The total matrix expression of all Equations of Motion is:

$$[M^*] \cdot \underline{\ddot{x}} + [c^*] \cdot \underline{\dot{x}} + [k^*] \cdot \underline{x} = \underline{Q}$$

1.2 Eigenfrequencies and eigenvectors

1.2.1 Undamped system

The eigenfrequencies and eigenvectors computation problem consists in analyzing the system as undamped and in free motion, which results in setting $[c^*] = [0]$ and $\underline{Q} = \underline{0}$ in the matrix expression of the Equations of Motion:

$$[M^*] \cdot \ddot{\underline{x}} + [k^*] \cdot \underline{x} = \underline{0}$$

By assuming as a solution $\underline{x} = \underline{X} e^{\lambda t}$, the equation becomes:

$$\left(\lambda^2 [M^*] + [k^*]\right) \underline{X} = \underline{0}$$

The non-trivial solution consists in solving the characteristic equation:

$$\det\left(\lambda^2 [M^*] + [k^*]\right) = 0 \Longrightarrow \lambda^2 = -[M^*]^{-1} [k^*]$$

By solving the above expression for λ, six imaginary conjugate values are obtained:

$$\lambda_{1,4} = \pm j\omega_{0_1} = \pm j0.8017, \quad \lambda_{2,5} = \pm j\omega_{0_2} = \pm j4.6974, \quad \lambda_{3,6} = \pm j\omega_{0_3} = \pm j10.1326$$

It is of interest to consider real frequencies (positive values), so it is possible to identify the three natural angular frequencies of the system as:

$$\omega_{0_1} = \mathrm{Im}\{\lambda_1\} = 0.8017 \text{ rad/s}, \quad \omega_{0_2} = \mathrm{Im}\{\lambda_2\} = 4.6974 \text{ rad/s} \quad \omega_{0_3} = \mathrm{Im}\{\lambda_3\} = 10.1326 \text{ rad/s}$$

By substituting the solutions λ_i, λ_{i+3} (for $i = 1, 2, 3$) to the equation to solve, it is possible to find the eigenvectors $\underline{X}^U_{\omega_{0_i}}$, responses of each independent variable for the i-th natural frequency ω_{0_i}:

$$\left({\lambda_i}^2 [M^*] + [k^*]\right) \underline{X}^U_{\omega_{0_i}} = \underline{0}$$

Where:

$$\underline{X}^U_{\omega_{0_i}} = \begin{pmatrix} X^U_{1,\omega_{0_i}} \\ \Theta^U_{2,\omega_{0_i}} \\ \Theta^U_{3,\omega_{0_i}} \end{pmatrix}$$

The eigenvectors for each ω_{0_i}, normalized with respect to the first component $x_1 = 1$, are:

$$\underline{X}^U_{\omega_{0_1}} = \begin{pmatrix} X^U_{1,\omega_{0_1}} \\ \Theta^U_{2,\omega_{0_1}} \\ \Theta^U_{3,\omega_{0_1}} \end{pmatrix} = \begin{pmatrix} 1 \\ -19.0479 \\ 12.4436 \end{pmatrix}, \quad \underline{X}^U_{\omega_{0_2}} = \begin{pmatrix} X^U_{1,\omega_{0_2}} \\ \Theta^U_{2,\omega_{0_2}} \\ \Theta^U_{3,\omega_{0_2}} \end{pmatrix} = \begin{pmatrix} 1 \\ -0.2348 \\ 0.0486 \end{pmatrix}, \quad \underline{X}^U_{\omega_{0_3}} = \begin{pmatrix} X^U_{1,\omega_{0_3}} \\ \Theta^U_{2,\omega_{0_3}} \\ \Theta^U_{3,\omega_{0_3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 2.1885 \\ 3.2200 \end{pmatrix}$$

1.2.2 Damped system

By considering the damped case ($[c^*] \neq [0]$) the matrix expression of the Equations of Motion (in free motion) becomes:

$$[M^*] \ddot{\underline{x}} + [c^*] \dot{\underline{x}} + [k^*] \underline{x} = \underline{0}$$

By assuming as a solution $\underline{x} = \underline{X} e^{\lambda t}$, the equation becomes:

$$\left(\lambda^2 [M^*] + \lambda [c^*] + [k^*]\right) \underline{X} = \underline{0}$$

By adding the trivial equation $[M^*] \dot{\underline{x}} = [M^*] \dot{\underline{x}}$ to the matrix form of the Equations of Motion, the problem can be expressed as:

$$\begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \begin{pmatrix} \ddot{\underline{x}} \\ \dot{\underline{x}} \end{pmatrix} + \begin{bmatrix} [c^*] & [M^*] \\ -[M^*] & [0] \end{bmatrix} \begin{pmatrix} \dot{\underline{x}} \\ \underline{x} \end{pmatrix} = \underline{0}_{6\times 1}$$

By setting the vector of state variables $\underline{z}$ as:

$$\underline{z}_{6\times1} = \begin{pmatrix} \underline{\dot{x}} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda\underline{x} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda\underline{X} \\ \underline{X} \end{pmatrix} e^{\lambda t} = \underline{Z}_{6\times1} e^{\lambda t}$$

The problem can be expressed as:

$$\begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \underline{\dot{z}} + \begin{bmatrix} [c^*] & [M^*] \\ -[M^*] & [0] \end{bmatrix} \underline{z} = \underline{0} \Longrightarrow [B]\,\underline{\dot{z}} + [D]\,\underline{z} = \underline{0}$$

In which:

$$[B]_{6\times6} = \begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix}, \quad [D]_{6\times6} = \begin{bmatrix} [c^*] & [k^*] \\ -[M^*] & [0] \end{bmatrix}$$

The problem to solve is in the form:

$$\underline{\dot{z}} = -[B]^{-1}[D]\,\underline{z}$$

In which $\det([B]) \neq 0$.
The matrix product $[A]_{6\times6} = -[B]^{-1}[D]$ is called state matrix of the system. Let's consider:

$$\lambda\underline{Z} = -[B]^{-1}[D]\,\underline{Z} = [A]\,\underline{Z} \Longrightarrow \left(\lambda\,[I]_{6\times6} - [A]\right)\underline{Z} = \underline{0}$$

From the above equation, it is possible to analyze the last three rows of $\underline{z}$, equal to $\underline{x}$, which contain the mode shapes for the damped case.
The matrix for the eigenvalues and eigenvectors problem computation is the state matrix:

$$[A] = -[B]^{-1}[D] = \begin{bmatrix} -[M^*]^{-1}[c^*] & -[M^*]^{-1}[k^*] \\ [I]_{3\times3} & [0] \end{bmatrix}$$

The solutions $\lambda^d_{i,i+3} = -\alpha_i \pm j\omega_{d_i}$ (for $i = 1,2,3$) are complex conjugate, as in the undamped case, but have non-zero real part:

$$\lambda^d_{1,4} = -0.0557 \pm j0.8010, \quad \lambda^d_{2,5} = -0.3114 \pm j4.6816, \quad \lambda^d_{3,6} = -0.3257 \pm j10.1234$$

The real part α_i describes a decay behaviour in the time responses and is the contribution of the dampers. The imaginary part ω_{d_i} refers to the resonance frequencies in the damped system.
More precisely, three natural "damped" frequencies are identified as:

$$\omega_{d_1} = \mathrm{Im}\left\{\lambda^d_1\right\} = 0.8010 \text{ rad/s}, \quad \omega_{d_2} = \mathrm{Im}\left\{\lambda^d_2\right\} = 4.6816 \text{ rad/s}, \quad \omega_{d_3} = \mathrm{Im}\left\{\lambda^d_3\right\} = 10.1234 \text{ rad/s}$$

By analyzing the "damped" resonance frequencies, it is possible to see how close they are to the ones of the undamped case. This observation allows to define the system as lightly damped.
The confirmation of that comes from the adimensional damping ratios computation:

$$\underline{h}_{3\times1} = \begin{pmatrix} h_1 = \frac{\alpha_1}{\omega_{0_1}} \\ h_2 = \frac{\alpha_2}{\omega_{0_2}} \\ h_3 = \frac{\alpha_3}{\omega_{0_3}} \end{pmatrix} = \begin{pmatrix} 0.0695 \\ 0.0665 \\ 0.0322 \end{pmatrix}$$

In which h_i is the adimensional damping ratio for the i-th eigenfrequency (sorted with increasing frequency).
Similarly to the undamped scenario, it is now possible to define the eigenvectors $\underline{X}^D_{\omega_{d_i}}$ for each natural frequency, normalized with respect to the first component $x_1 = 1$:

$$\underline{X}^D_{\omega_{d_1}} = \begin{pmatrix} X^D_{1,\omega_{d_1}} \\ \Theta^D_{2,\omega_{d_1}} \\ \Theta^D_{3,\omega_{d_1}} \end{pmatrix} = \begin{pmatrix} 1 \\ -18.8050 + j2.6391 \\ 12.2816 - j1.7581 \end{pmatrix}$$

$$\underline{X}^D_{\omega_{d_2}} = \begin{pmatrix} X^D_{1,\omega_{d_2}} \\ \Theta^D_{2,\omega_{d_2}} \\ \Theta^D_{3,\omega_{d_2}} \end{pmatrix} = \begin{pmatrix} 1 \\ -0.2323 + j0.0809 \\ 0.0448 - j0.0283 \end{pmatrix}$$

$$\underline{X}^D_{\omega_{d_3}} = \begin{pmatrix} X^D_{1,\omega_{d_3}} \\ \Theta^D_{2,\omega_{d_3}} \\ \Theta^D_{3,\omega_{d_3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 2.1712 + j0.1567 \\ 3.1917 + j0.2604 \end{pmatrix}$$

1.3 Rayleigh damping

The Rayleigh damping assumption states that two constants α and β exist so that:

$$[c^*] = \alpha\,[M^*] + \beta\,[k^*]$$

The adimensional damping ratios h_i, computed in the previous paragraph, are equal to:

$$h_i = \frac{c^*_{ii}}{2m^*_{ii}\omega_{0_i}} = \frac{\alpha m^*_{ii} + \beta k^*_{ii}}{2m^*_{ii}\omega_{0,i}}, \quad i = 1, 2, 3$$

$$\Longrightarrow h_i = \frac{\alpha}{2\omega_{0,i}} + \frac{\beta\omega_{0,i}}{2} = \left(\frac{1}{2\omega_{0,i}}, \frac{\omega_{0,i}}{2}\right)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

In order to find values of α and β that best approximate the Rayleigh damping equation, it is necessary to solve the following equation:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\frac{1}{2\omega_{0,i}}\\ \frac{\omega_{0,i}}{2}\end{pmatrix}^{-1} h_i$$

In matrix form it becomes:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{bmatrix}\frac{1}{2\omega_{0,1}} & \frac{1}{2\omega_{0,2}} & \frac{1}{2\omega_{0,3}}\\ \frac{\omega_{0,1}}{2} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,3}}{2}\end{bmatrix}^{-1}\begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix} = \begin{bmatrix}\frac{1}{2\omega_{0,1}} & \frac{1}{2\omega_{0,2}} & \frac{1}{2\omega_{0,3}}\\ \frac{\omega_{0,1}}{2} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,3}}{2}\end{bmatrix}^{-1} \underline{h}$$

The results are:

$$\alpha = 0.1134, \quad \beta = 0.0083$$

2 Free motion

The analysis of the free motion system will be done by assuming Rayleigh damping, so the damping matrix $[c^*]$ is given by:

$$[c^*] = \alpha\,[M^*] + \beta\,[k^*]$$

2.1 Time responses

In the previous section, the time response solution was defined as $\underline{x} = \underline{X}e^{\lambda t}$, in which $\underline{X}$ collects amplitude and phase response from each system natural frequency to the independent variables.
The eigenvector $\underline{X}^D_{\omega_{d_1}}$ ($\underline{X}^D_{\omega_{d_2}}$ and $\underline{X}^D_{\omega_{d_3}}$ respectively) describes the amplitude and phase components of ω_{d_1} (ω_{d_2} and ω_{d_3} respectively) in the independent variables x_1, θ_2 and θ_3.
In particular:

$$x_1(t) = \sum_{i=1}^{3} e^{-\alpha_i t}\left|X^D_{1,\omega_{d_i}}\right| C_i \cos\left(\omega_{d_i}t + \Phi_i + \angle X^D_{1,\omega_{d_i}}\right) = \sum_{i=1}^{3} e^{-\alpha_i t} C_i \cos\left(\omega_{d_i}t + \Phi_i\right)$$

Due to the normalization of the eigenvectors $X^D_{1,\omega_{d_i}} = 1$, so $\left|X^D_{1,\omega_{d_i}}\right| = 1$ and $\angle X^D_{1,\omega_{d_i}} = 0$.

$$\theta_2(t) = \sum_{i=1}^{3} e^{-\alpha_i t}\left|\Theta^D_{2,\omega_{d_i}}\right| C_i \cos\left(\omega_{d_i}t + \Phi_i + \angle\Theta^D_{2,\omega_{d_i}}\right)$$

$$\theta_3(t) = \sum_{i=1}^{3} e^{-\alpha_i t}\left|\Theta^D_{3,\omega_{d_i}}\right| C_i \cos\left(\omega_{d_i}t + \Phi_i + \angle\Theta^D_{3,\omega_{d_i}}\right)$$

Each time response has three decaying sinusoids (with decay factor α_i and angular frequency ω_{d_i}), which manifest in different ways, depending on the values of $\left|X^D_{1,\omega_{d_i}}\right|$, $\left|\Theta^D_{2,\omega_{d_i}}\right|$ and $\left|\Theta^D_{3,\omega_{d_i}}\right|$.

2.1.1 Initial Conditions

All unknowns C_i and Φ_i are obtained by imposing the initial conditions:

$$x_{1_0} = 0.1 \text{ m}, \quad \theta_{2_0} = \frac{\pi}{12} \text{ rad}, \quad \theta_{3_0} = -\frac{\pi}{12} \text{ rad}$$

$$\dot{x}_{1_0} = 1 \text{ m/s}, \quad \dot{\theta}_{2_0} = 0.5 \text{ rad/s}, \quad \dot{\theta}_{3_0} = 2 \text{ rad/s}$$

Since the displacements expressions are highly similar, expressing the initial conditions by substituting $t = 0$ in the equations of $x_1(t)$ gives us the general formulation of the initial condition problem in which we are interested in finding the 6 unknowns $C_i, \Phi_i, \quad \forall i \in [1, 3]$:

$$\left. x_1(t)\right|_{t=0} = x_{1_0} \Leftrightarrow \sum_{i=1}^{3} C_i \cdot \left|X^D_{1,\omega_{d_i}}\right| \cdot \cos\left(\Phi_i + \angle X^D_{1,\omega_{d_i}}\right) = x_{1_0}$$

$$\left.\dot{x}_1\right|_{t=0} = v_{1_0} \Leftrightarrow \sum_{i=1}^{3}\left[C_i \cdot \left|X^D_{1,\omega_{d_i}}\right| \cdot \cos\left(\Phi_i + \angle X^D_{1,\omega_{d_i}} - \arctan\left(\omega_{di}/\alpha_{x_1}\right)\right)\right] = -v_{1_0}$$

The resulting six unknowns are:

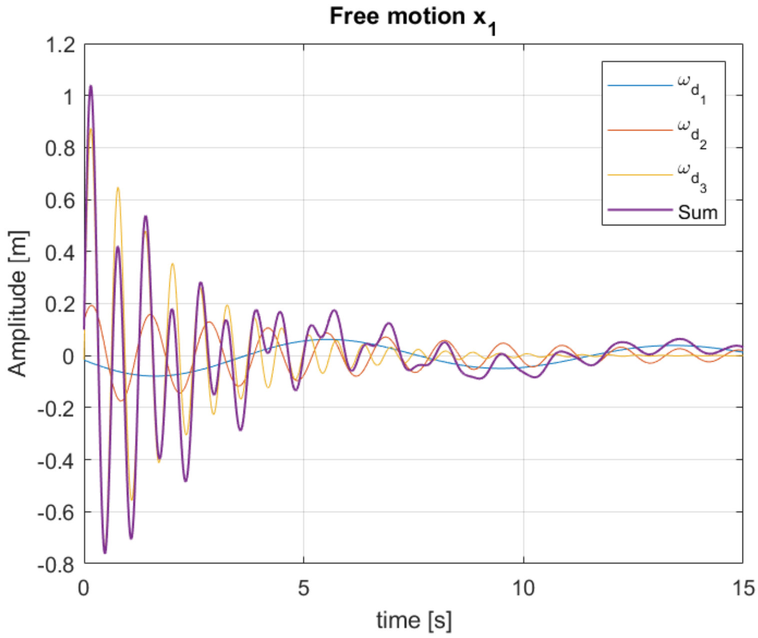
$$C_1 = 0.0870, \quad C_2 = 0.1981, \quad C_3 = -0.9423$$

$$\Phi_1 = 1.7709 \text{ rad}, \quad \Phi_2 = -0.8294 \text{ rad}, \quad \Phi_3 = 7.8365 \text{ rad}$$

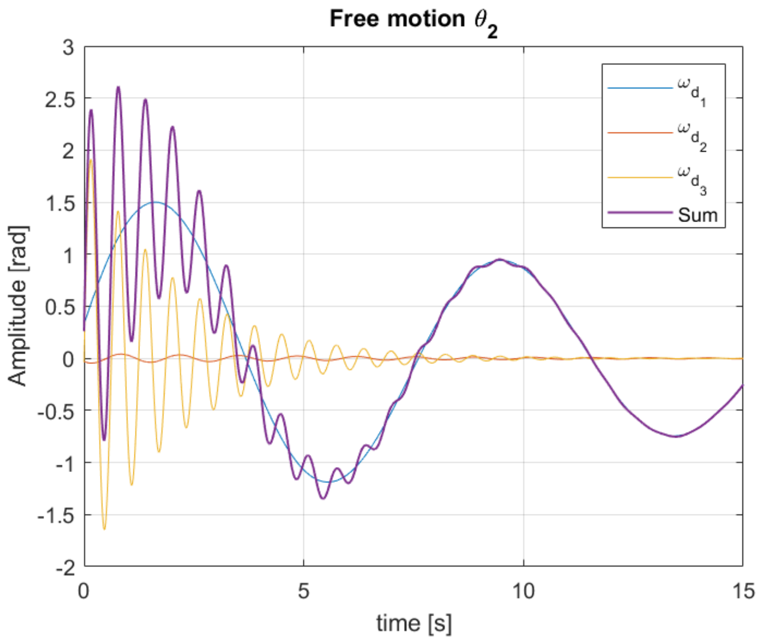
It is possible to check that the initial conditions are met by looking at the first element of the displacement vectors (in MATLAB).
It is now possible to fully define and plot the free motion response of each independent variable. By taking into account the products between the constants C_i and the amplitude factors $\left|X^D_{\omega_{d_i}}\right|$ or $\left|\Theta^D_{\omega_{d_i}}\right|$, it is possible to give a quick explanation of how much a certain frequency manifests in the time response of $x_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ using the following table:

	ω_{d_1}	ω_{d_2}	ω_{d_3}
$\lvert x_1 \rvert$	0.0870	0.1981	0.9423
$\lvert \theta_2 \rvert$	1.6566	0.0465	2.0623
$\lvert \theta_3 \rvert$	1.0822	0.0096	3.0362

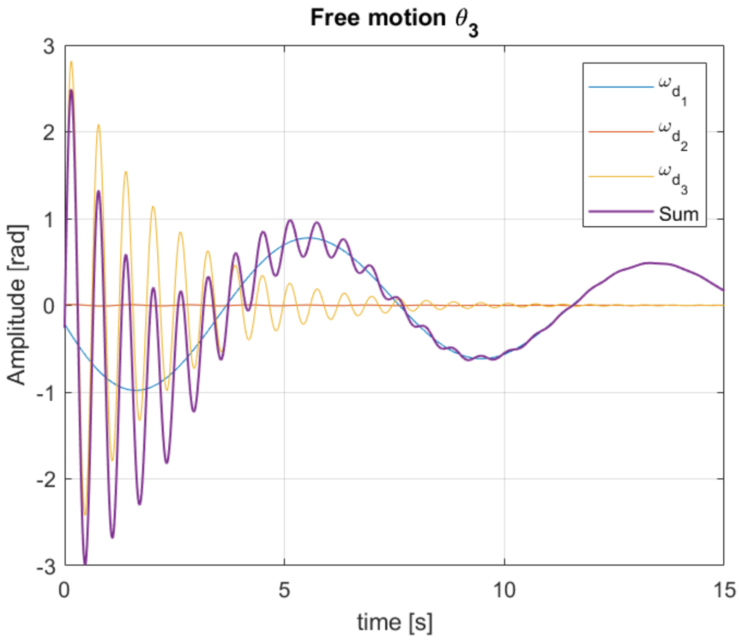
The element in position (i, j) represents the amplitude of the sinusoids of angular frequency ω_{d_j} in the time response of the independent variable x_i or θ_i.
The time response plot of $x_1(t)$ is mainly given by ω_{d_2} and ω_{d_3}:



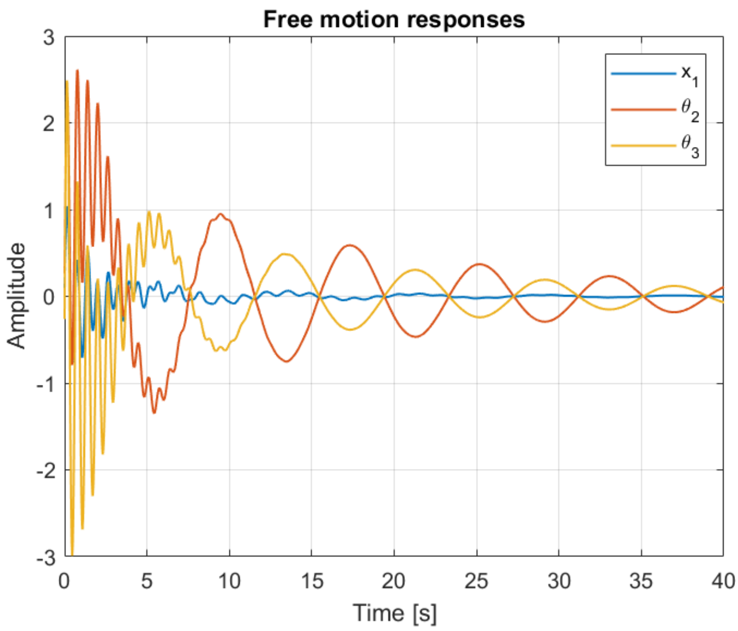
The time response plot of $\theta_2(t)$ is mainly given by ω_{d_1} and ω_{d_3}:



The time response plot of $\theta_3(t)$ is mainly given by ω_{d_1} and ω_{d_3}:



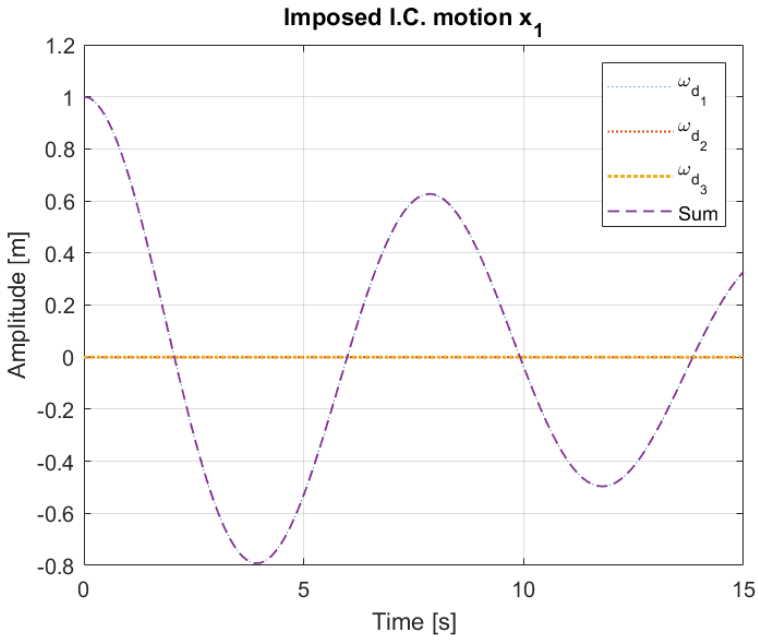
A comparison between all three time responses is given:



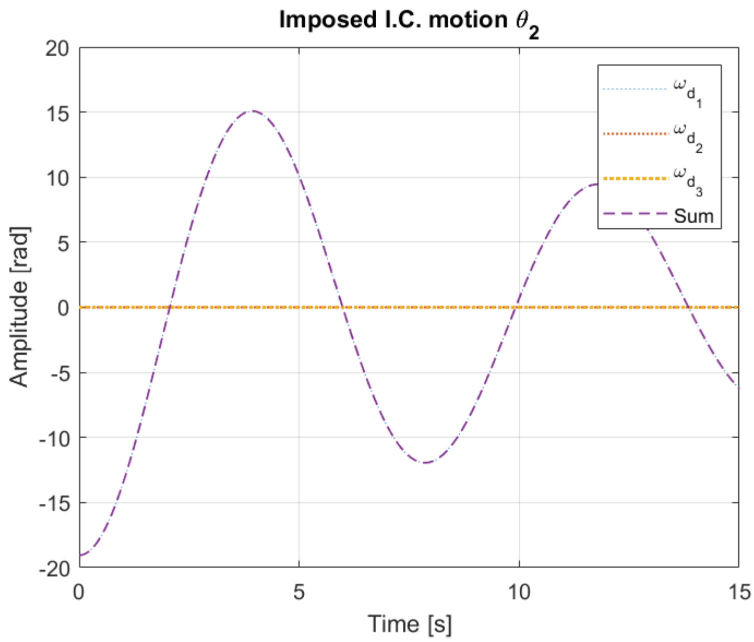
2.2 Eigenmode isolation

A way to impose the resulting time responses with a single mode is to put initial displacements equal to the eigenvector entries of the natural frequency of interest, while the initial velocities are all set to zero. In the following plots, the first mode is selected, so only the first natural frequency ω_{d_1} is going to be present.

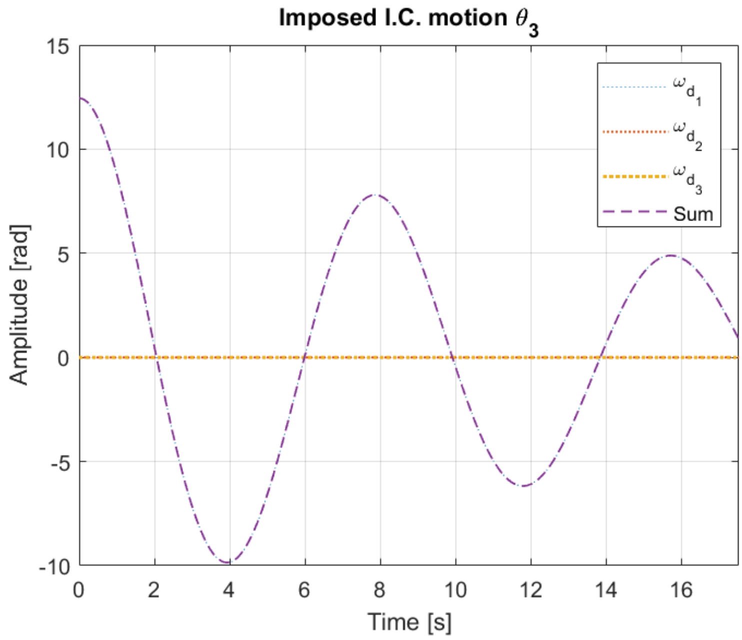
The time response plot of $x_1(t)$ is given:



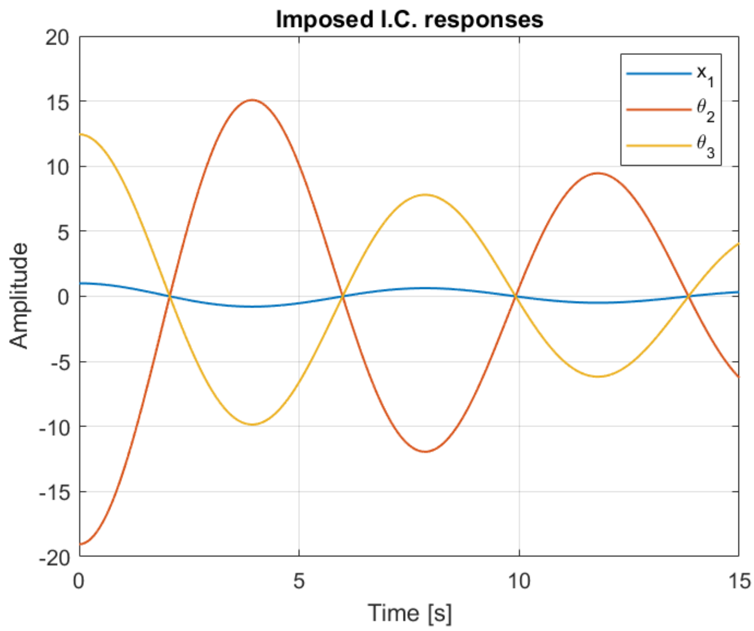
The time response plot of $\theta_2(t)$ is given:



The time response plot of $\theta_3(t)$ is given:



A comparison between all time responses is given:



3 Forced motion

As already explained at the beginning of the previous section, Rayleigh damping is assumed to occur as we have a lightly damped system:

$$[c^*] = \alpha\,[M^*] + \beta\,[k^*]$$

3.1 Frequency response matrix

By introducing the external force contribution, the equation to study becomes:

$$[M^*] \cdot \underline{\ddot{x}} + [c^*] \cdot \underline{\dot{x}} + [k^*] \cdot \underline{x} = \underline{Q} = \underline{Q_0} \cos(\Omega t)$$

Where:

$$\underline{Q} = \begin{pmatrix} F(t) \\ 0 \\ -rF(t) \end{pmatrix}, \quad F(t) = F_0 \cos(\Omega t)\,, \quad \underline{Q_0} = \begin{pmatrix} F_0 \\ 0 \\ -rF_0 \end{pmatrix} = [\Lambda_F]^T F_0$$

A complex function $\underline{\tilde{x}_p}(t)$ can be defined, so that its real part $\underline{x_p}(t)$ is a particular solution of the equation above:

$$\underline{\tilde{x}_p}(t) = \underline{\tilde{X}_0} e^{j\Omega t} = \begin{pmatrix} \tilde{X}_{1_0} \\ \tilde{\Theta}_{2_0} \\ \tilde{\Theta}_{3_0} \end{pmatrix} e^{j\Omega t} \Longrightarrow \underline{x_p}(t) = \mathrm{Re}\{\underline{\tilde{x}_p}(t)\}$$

The complex function $\underline{\tilde{x}_p}(t)$, on the other hand, is a particular solution of the following equation:

$$[M^*] \cdot \underline{\ddot{\tilde{x}}_p} + [c^*] \cdot \underline{\dot{\tilde{x}}_p} + [k^*] \cdot \underline{\tilde{x}_p} = \underline{Q_0} e^{j\Omega t}$$

By evaluating the first and second time derivative of $\underline{\tilde{x}_p}(t)$, it is possible to define a relation between $\underline{\tilde{X}_0}$ and $\underline{Q_0}$:

$$\underline{\tilde{X}_0} = \left[-\Omega^2\,[M^*] + j\Omega\,[c^*] + [k^*]\right]^{-1} \underline{Q_0} = [D(\Omega)]^{-1}_{3\times 3}\,\underline{Q_0} = [H(\Omega)]_{3\times 3}\,\underline{Q_0}$$

The matrix $[H(\Omega)]$ is the frequency response matrix and its entries are the frequency response functions $H_{i,j}(\Omega)$, given by:

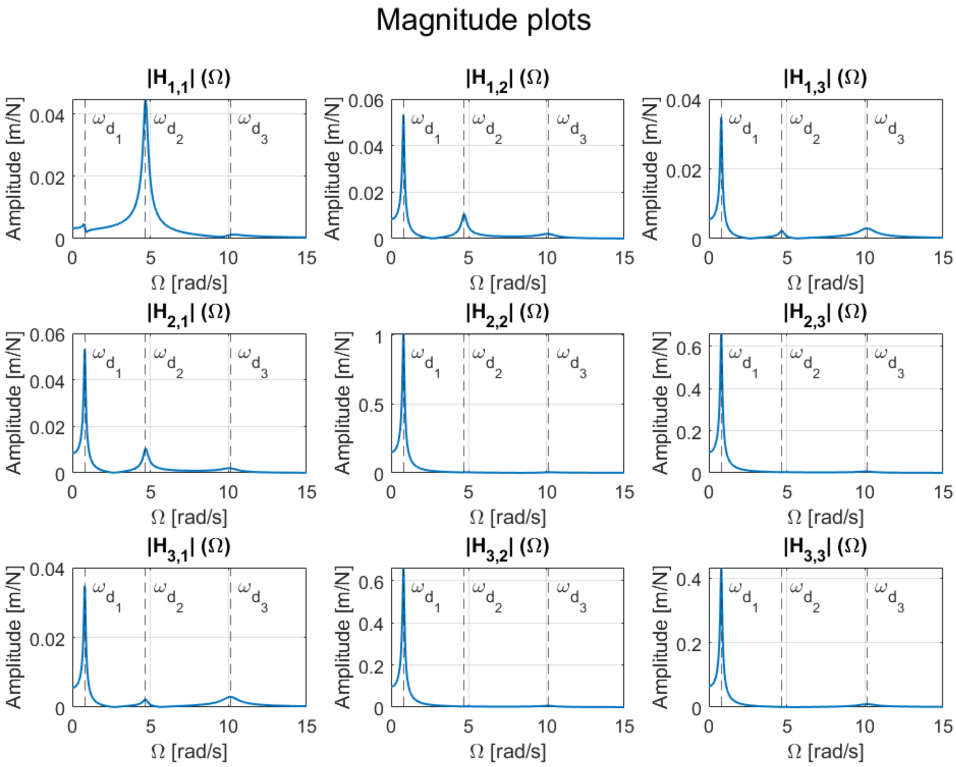
$$H_{i,j} = [D(\Omega)]^{-1}_{i,j} = \frac{1}{\det([D(\Omega)])}\,[C_D]^T_{i,j}$$

Where the cofactor matrix $[C_D]$ is defined as:

$$[C_D]_{i,j} = (-1)^{i+j} \det\left(\left[M_{D_{i,j}}\right]\right)$$

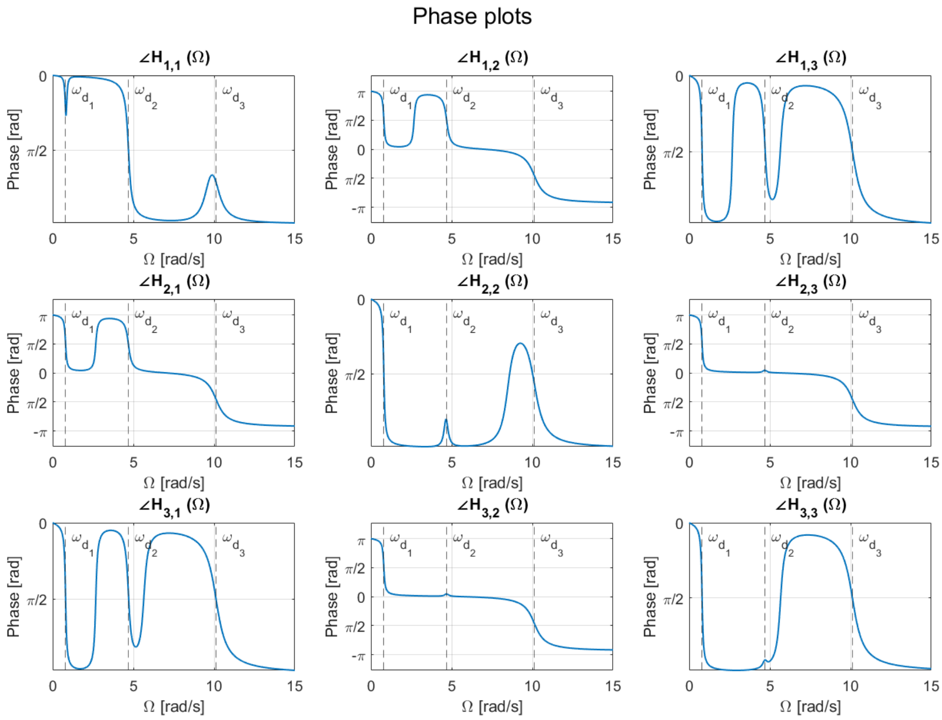
$\left[M_{D_{i,j}}\right]$ is the matrix obtained from $[D]$ after removing the *j-th* column and the *i-th* row.

The plot of every FRF amplitude $|H_{i,j}|$ is given:



Each frequency response function $H_{i,j}$ describes the system response of the *i-th* independent displacement/rotation in case of a force applied to the *j-th* body.
Whenever in a frequency response function one of the natural frequencies ω_{d_i} is not present, a node of vibration occurs.

The plot of every FRF phase $\angle H_{i,j}$ is given:

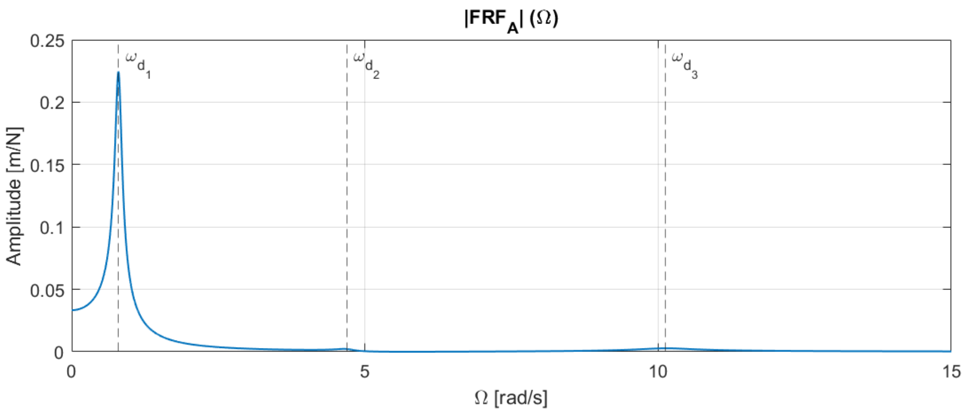


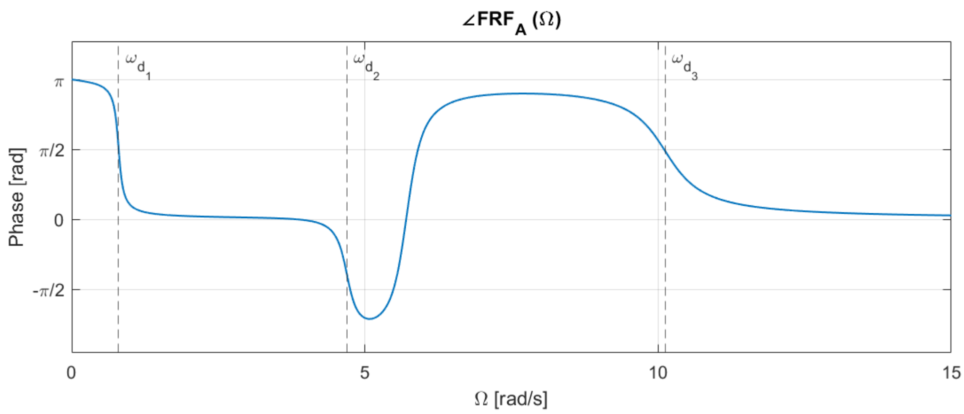
Whenever a $-\pi$ shift in the phase response occurs, the magnitude response presents a resonance (pole in the FRF), which is where a local maximum is present. On the other hand, a node of vibration (zero in the FRF) in the magnitude response translates into a $+\pi$ phase shift.
In all the other cases, a small phase shift can be described in terms of poles and zeros being closely spaced.
For example, in the phase response of $H_{1,1}$ a sudden phase shift occurs around ω_{d_1} and $\omega_{d,3}$, less than $\pm\pi$.

3.2 Co-located FRF in A

The co-located FRF of the point A (at the center of the disk M_3) is defined as the frequency response function that describes the displacement in A (called $x_3 = x_1 - r\theta_3$ in the system) and the force F applied in that point.

$$\text{FRF}_A(\Omega) = \begin{bmatrix} 1 & 0 & -r \end{bmatrix} \begin{bmatrix} H_{1,1} & H_{1,2} & H_{1,3} \\ H_{2,1} & H_{2,2} & H_{2,3} \\ H_{3,1} & H_{3,2} & H_{3,3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -r \end{bmatrix} \begin{bmatrix} H_{1,3} \\ H_{2,3} \\ H_{3,3} \end{bmatrix} = H_{1,3}(\Omega) - rH_{3,3}(\Omega)$$



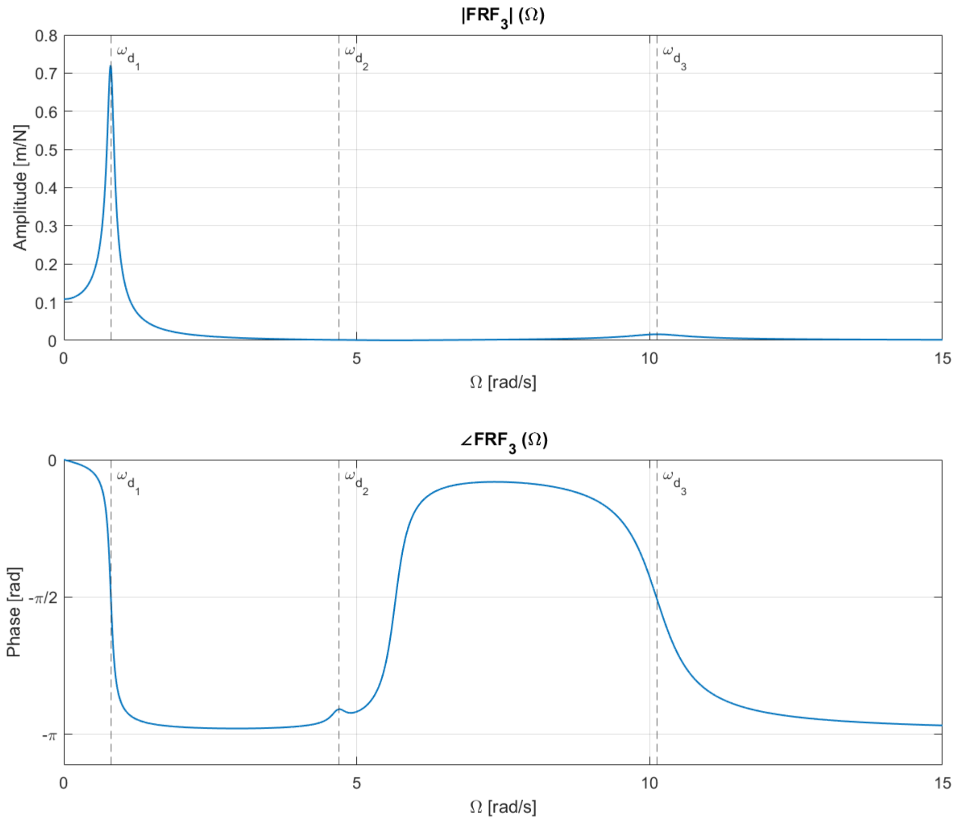


Considering that $\text{FRF}_A = H_{1,3} - rH_{3,3}$, it is expected that it presents a single resonance in ω_{d_1} similarly to $H_{1,3}$ and $H_{3,3}$.
The comments made in the previous section, related to the FRF plots, are still valid.

3.3 Co-located FRF for disk 3

The frequency response $H_{3,3}$ describes the system response in terms of θ_3 to a force applied at the center of the disk M_3. In order to get a response to a torque:

$$\mathrm{FRF}_3\left(\Omega\right) = \frac{\Theta_3}{Fr} = \frac{\Theta_3}{F}\frac{1}{r} = \frac{H_{3,3}\left(\Omega\right)}{r}$$

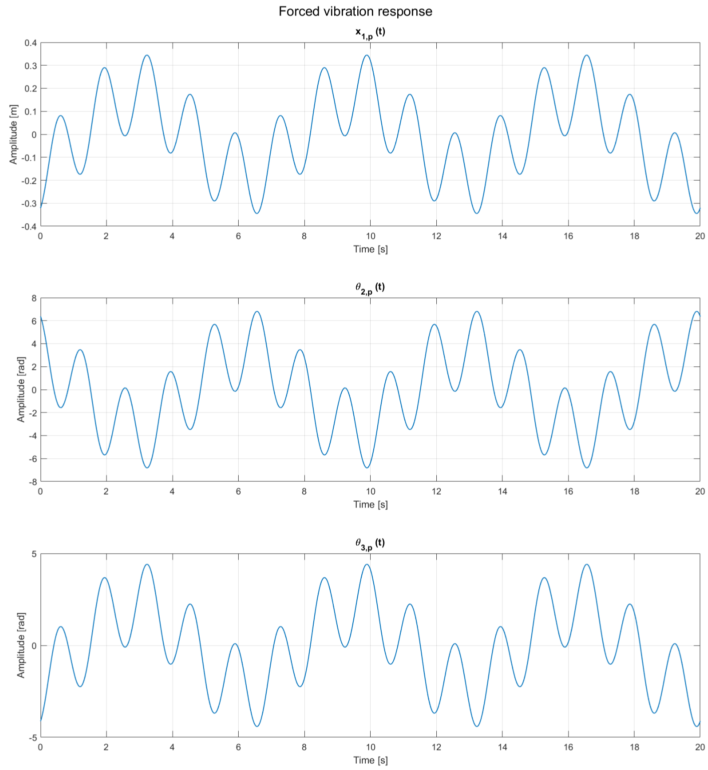


3.4 Harmonic force response

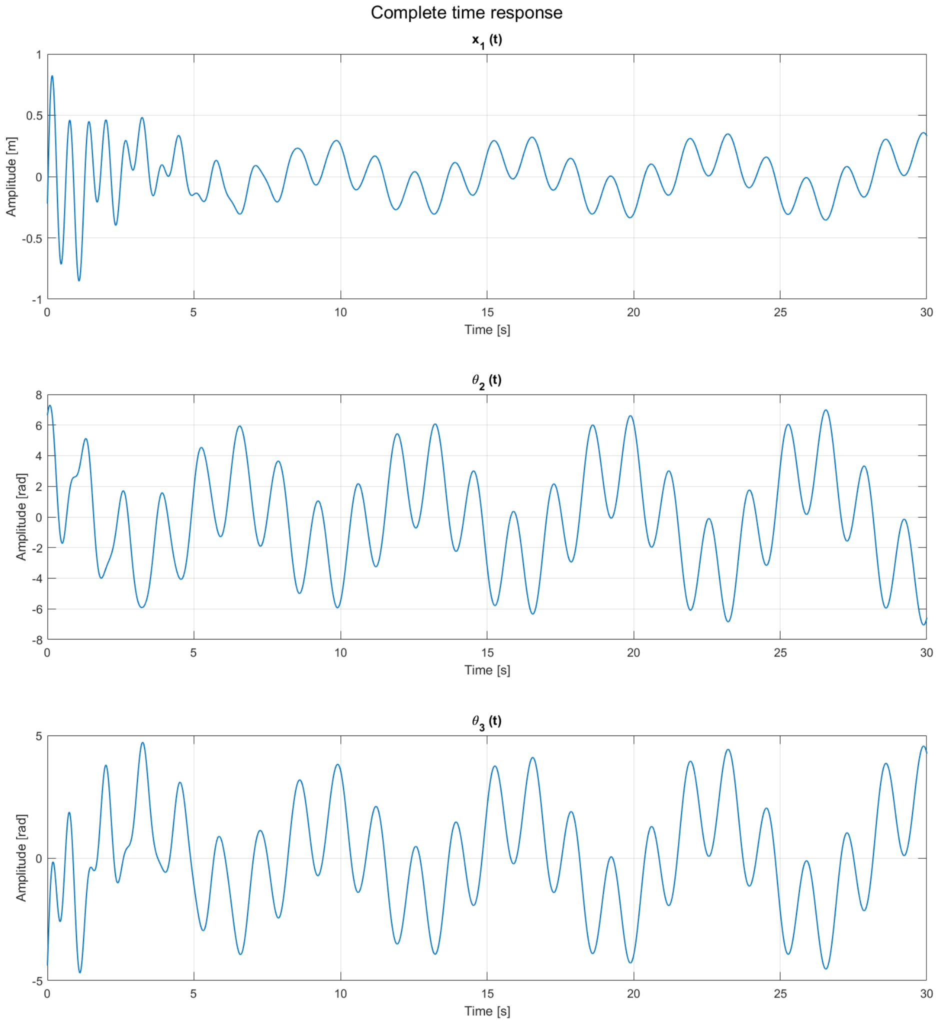
Considering the harmonic input:

$$F\left(t\right) = F_1\left(t\right) + F_2\left(t\right) = A_1\ \cos\left(2\pi\ f_1 t\right) + A_2\ \cos\left(2\pi\ f_2 t\right)$$

As we are studying a linear system, we know that its response will be a linear combination of the system's responses of each individual harmonic input with its free vibration Below, the plot with only the forced vibration response of the system with the considered input:



When considering the full response of the system, we need to consider the free vibration of the system. This will introduce a transient in the response of the system, which will go back to the forced vibration state as illustrated below:



OLD

As a very first thing we defined our convention on signs, assuming as positive the elongation for springs and dampers, the positive directions of x and y axes for displacements, and the counterclockwise direction for rotations.
Then we need to determine the degrees of freedom (DOF) of the system. In order to do this, we will subtract the degrees of constraint (DOC) in the system to the total amount of possible DOF. Since the system is composed by four rigid bodies, the maximum number of DOF is 12.
In order to get the equation of motion (EOM) of the system, we take into account the Lagrange equation in matrix form:

$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial E_k}{\partial \dot{\underline{x}}}\right)\right\}^T - \left\{\frac{\partial E_k}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \dot{\underline{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q} \tag{1}$$

Now we have to compute each component of this equation.

Equation of Motion and system matrices

Equation of Motion

Freedom and constraints

4 rigid bodies $\{B_1, D_2, D_3, B_4\}$:

- A first disk D_2 constrained through a hinge to a mass-less vertical beam which is in turn rigidly connected to a mass B_1
  - Slider and hinge : 1 constraint (no movement on y from the hinge)
  - Rigid connection $\rightarrow$ 1 constraint : $\dot{x}_2 = \dot{x}_1$
- B_1 slides in the horizontal plane
  - 1 slider $\rightarrow$ 2 constraints $\theta_1 = 0$ and $y_1 = 0$
- An in-extensible rope connects the periphery of the disk D_2 to another disk D_3 that can translate horizontally and can be rolled back and forth by the rope, which then connects it to the translating mass B_4.
  - 1 in-extensible rope $\rightarrow$ 1 constraint $\dot{x}_1 = \dot{x}_E$
  - Slider and hinge $\rightarrow$ 1 constraint : $y_3 = 0$
- The centre of D_3 is rigidly connected to a mass B_4 whose horizontal motion and rotation are constrained.
  - 1 slider $\rightarrow$ 2 constraints : $\theta_4 = 0$ and $x_4 = 0$
  - Rigid connection $\rightarrow$ 1 constraint : $x_3 = -y_4$

Overall, we obtain 9 degrees of constraints, for a total number of 12 degrees of freedom (as we have 4 bodies each having 3 DOF). Thus, we have a **system with 3 degrees of freedom**.

Kinematic Relationships

From the considered system, we extract the following kinematic relationships: ¨

$$\begin{cases} x_1 = x_2 = x_E \\ \dot{x}_C = \dot{x}_2 + R\dot{\theta}_2 \\ \dot{x}_D = \dot{x}_1 - 2r\dot{\theta}_3 \\ y_4 = -x_3 = -x_E + r\theta_3 = -x_1 + r\theta_3 \end{cases} \tag{2}$$

When it comes to the springs and dampers related kinematic relationships, we are interesting in expressing the elongation of each spring, and the elongation speed of each damper. As we assume being in the Static Equilibrium Position, we neglect the pre-load. We obtain the following relations :

$$\begin{cases} \dot{\Delta l}_1 = \dot{x}_1 \\ \dot{\Delta l}_2 = \dot{x}_D - \dot{x}_C = -R\dot{\theta}_2 - 2r\theta_3 \\ \dot{\Delta l}_3 = \dot{y}_4 = -\dot{x}_1 + r\dot{\theta}_3 \end{cases} \Leftrightarrow \begin{cases} \Delta l_1 = x_1 \\ \Delta l_2 = -R\theta_2 - 2r\theta_3 \\ \Delta l_3 = -x_1 + r\theta_3 \end{cases} \tag{3}$$

Kinetic Energy

Given that the system is composed by 3 rigid bodies, the kinetic energy expression could contain up to 6 contributions. In this particular case we only have 5 contributions, because M_3 can only translate and not rotate. Thus, we have:

$$E_k = \frac{1}{2}M_1v_1^2 + \frac{1}{2}M_2v_2^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}M_3v_3^2 + \frac{1}{2}J_3\omega_3^2 + \frac{1}{2}M_4v_4^2 \tag{4}$$

In order to be able to write this expression in function of the independent variables we chose, we need to introduce the column vector of physical coordinates $\underline{y}$ and its derivative $\underline{\dot{z}}$.

$$\underline{z} = (x_1, x_2, \theta_2, x_3, \theta_3, \theta_4)^T, \underline{\dot{z}} = \left(\dot{x}_1, \dot{x}_2, \dot{\theta}_2, \dot{x}_3, \dot{\theta}_3, \dot{\theta}_4\right)^T \tag{5}$$

The kinetic energy can be expressed also as follows:

$$E_k = \frac{1}{2}\underline{\dot{z}}^T[M]\underline{\dot{z}} \tag{6}$$

where [M] is the matrix:

$$[M] = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 \end{bmatrix} \tag{7}$$

Furthermore, we can express the $\dot{\underline{z}}$ vector as follows:

$$\dot{\underline{z}} = \left(\frac{\partial \underline{z}}{\partial \underline{x}}\right) \dot{\underline{x}} = [\Lambda_m]\dot{\underline{x}} \tag{8}$$

where $[\Lambda_m]$ is the jacobian matrix, which contains the partial derivatives of the vector $\underline{y}$ with respect to the vector of independent variables $\underline{x}$ of the independent variables.
In our case, we have $\dot{\underline{x}} = \left(\dot{x}_1, \dot{\theta}_2, \dot{\theta}_3\right)^T \Leftrightarrow \underline{x} = (x_1, \theta_2, \theta_3)^T$. To know the value of these derivatives we fill the following table:

	$\dot{x}_1$	$\dot{\theta}_2$	$\dot{\theta}_3$
v_1	1	0	0
v_2	1	0	0
ω_2	0	1	0
v_3	1	0	-r
ω_3	0	0	1
v_4	-1	0	r

We obtain that the jacobian matrix is:

$$[\Lambda_m] = \begin{bmatrix} 1 & 0 & R_2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{9}$$

Ernest's proposition :

$$[\Lambda_m]_{6\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -r \\ 0 & 0 & 1 \\ -1 & 0 & r \end{bmatrix} \tag{10}$$

From (6) and (8) we can express the kinetic energy as a function of the independent variables:

$$E_k = \frac{1}{2}\dot{\underline{x}}^T[\Lambda_m]^T[M][\Lambda_m]\dot{\underline{x}}$$

We can now define the generalized mass matrix as:

$$[M^*] = [\Lambda_m]^T[M][\Lambda_m] \tag{11}$$

$$[M^*]_{3\times 3} = \begin{pmatrix} M_1 + M_2 + M_3 + M_4 & 0 & -r\,(M_3 + M_4) \\ 0 & J_2 & 0 \\ -r\,(M_3 + M_4) & 0 & J_3 + r^2\,(M_3 + M_4) \end{pmatrix} \tag{12}$$

We observe that this matrix is symmetric. We obtain the final expression for the kinetic energy:

$$E_k = \frac{1}{2}\dot{\underline{x}}^T[M^*]\dot{\underline{x}} \tag{13}$$

Potential Energy

The potential energy of the system is given by the sum of the potential energy contributions for each rigid body in the system. In this case we only have 1 gravitational contribution, due to the fact that the body B_4 is able to move in the vertical direction. Therefore we have 4 contributions, related to the 3 springs in the system, and the body B_4:

$$V = V_{el} + V_g = \frac{1}{2}K_1\Delta l_1^2 + \frac{1}{2}K_2\Delta l_2^2 + \frac{1}{2}K_3\Delta l_3^2 + \frac{1}{2}M_4 g y_4 \tag{14}$$

In order to express the elastic potential energy as a function on the independent variables, we perform the same passages we just showed for the kinetic energy. As a first thing we define the matrix [K] and the vector $\underline{\Delta l}$:

$$[K]_{3\times3} = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_1 & 0 \\ 0 & 0 & K_3 \end{bmatrix} \tag{15}$$

$$\underline{\Delta l}_{3\times1} = \{\Delta l_1,\ \Delta l_2,\ \Delta l_3\}^T \tag{16}$$

Elastic potential energy can be expressed as:

$$V_{el} = \frac{1}{2}\underline{\Delta l}^T[K]\underline{\Delta l} \tag{17}$$

As we know that :

$$\underline{\Delta l} = [\Lambda_k]\cdot\underline{x} \text{ such as } [\Lambda_k] = \frac{\partial\underline{\Delta l}}{\partial\underline{x}} \tag{18}$$

Where $[\Lambda_k]$ is the jacobian matrix of the elongation. Thanks to the established kinematic relations, we fill the following table to find the elements of $[\Lambda_k]$

	x_1	θ_2	θ_3
Δl_1	1	0	0
Δl_2	0	-R	-2r
Δl_3	-1	0	r

We obtain that the jacobian matrix is:

$$[\Lambda_k]_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R & -2r \\ -1 & 0 & r \end{bmatrix} \tag{19}$$

By inserting this jacobian matrix in the elastic potential energy expression we obtain:

$$V_k = \frac{1}{2}\underline{x}^T[\Lambda_k]^T[K][\Lambda_k]\underline{x} \tag{20}$$

We define the generalized matrix $[K^*]$ as

$$[K^*] = [\Lambda_k]^T[K][\Lambda_k] \tag{21}$$

$$[K^*]_{3\times3} = \begin{bmatrix} k_1 + k_3 & 0 & -rk_3 \\ 0 & R^2k_2 & 2Rk_2r \\ -k_3r & 2Rk_2r & 4k_2r^2 + k_3r^2 \end{bmatrix} \tag{22}$$

We observe that this matrix is symmetric. We obtain the final expression for the potential energy:

$$V_{el} = \frac{1}{2}\underline{x}^T[K^*]\underline{x} \tag{23}$$

Gravitational potential energy can be expressed as:

$$V_g = \frac{1}{2}\underline{P}^T\cdot\underline{y} \tag{24}$$

Where we have the weight vector $\underline{P} = g\cdot\underline{M} = g\cdot(M_1,\ M_2,\ M_3,\ M_4)^T$ and the vertical displacement vector $\underline{y} = (y_1,\ y_2\ ,\ y_3,\ y_4)^T = (0,\ 0,\ 0,\ -x_1 + r\theta_3)^T$

We observe that $\underline{y} = [\Lambda_P] \cdot \underline{x}$ where

$$[\Lambda_P]_{4\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & r \end{pmatrix} \tag{25}$$

And we can rephrase the gravitational potential energy

$$V_g = \frac{1}{2}M_4 g y_4 = \frac{1}{2}\underline{P} \cdot [\Lambda_P] \cdot \underline{x} = \frac{1}{2}\underline{P}^* \cdot \underline{x} \tag{26}$$

Where $\underline{P}^* = \underline{P} \cdot [\Lambda_P] = (-gM_4,\, 0,\, grM_4)^T$ is the weight vector (**to rephrase**).

Thus, the total potential energy can be written

$$V = V_{el} + V_g = \frac{1}{2}\underline{x}^T\,[K^*]\,\underline{x} + \frac{1}{2}P^*\underline{x} \tag{27}$$

Dissipative Energy

The total dissipation of the system is given by:

$$D = \frac{1}{2}c_1\dot{\Delta}l_1^2 + \frac{1}{2}c_2\dot{\Delta}l_2^2 + \frac{1}{2}c_3\dot{\Delta}l_3^2 \tag{28}$$

In the same way we did for kinetic and potential energy, we want to express the dissipation as a function of the independent variables. To reach our goal we introduce the matrix $[C]$ and the vector $\underline{\dot{\Delta}l}$:

$$[C]_{3\times3} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \tag{29}$$

$$\underline{\dot{\Delta}l}_{3\times1} = \left\{\dot{\Delta}l\,_1, \dot{\Delta}l\,_2, \dot{\Delta}l\,_3\right\}^T \tag{30}$$

As far as the jacobian matrix is concerned, being all the dampers of the system in parallel with the springs, we can state that:

$$[\Lambda_c] = [\Lambda_k] \tag{31}$$

Therefore, following the same procedure that we saw for the potential energy, we express the dissipation as follows:

$$D = \frac{1}{2}\underline{\dot{\Delta}l}^T[C]\underline{\dot{\Delta}l} = \frac{1}{2}\underline{\dot{x}}^T[\Lambda_c]^T[C][\Lambda_c]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[C^*]\underline{\dot{x}} \tag{32}$$

Where our Damping matrix is diagonal, and:

$$[C^*] = [\Lambda_c]^T[C][\Lambda_c] \tag{33}$$

$$[C^*]_{3\times3} = \begin{bmatrix} c_1 + c_3 & 0 & -rc_3 \\ 0 & R^2c_2 & 2Rc_2r \\ -rc_3 & 2Rc_2r & 4c_2r^2 + c_3r^2 \end{bmatrix} \tag{34}$$

Principle of Virtual Works

From the principle of Virtual Works, considering that we have a single external force $F(t)$ applied at the point A along the horizontal direction, we have :

$$\delta W = F \cdot \delta x_3$$

And, as we have

$$\delta x_3 = \delta\,(x_1 - r \cdot \theta_3) = \delta x_1 - r \cdot \delta\theta_3$$

Thus, we obtain :

$$\delta W = F \cdot \delta x_1 + 0 \cdot \delta\theta_2 - rF \cdot \delta\theta_3 = (F, 0, -rF) \cdot \begin{pmatrix} \delta x_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{pmatrix} = \underline{F} \cdot \underline{\delta x}$$

Where we identify :

$$\underline{Q}_{3\times1} = (F, 0, -rF)^T \tag{35}$$

Matrix formulation of the Equation of Motion

From (1), we develop each terms based on their new exopression, leading us to :

$$\left\{\frac{d}{dt}\left(\frac{\partial E_k}{\partial \underline{\dot{x}}}\right)\right\}^T = [M^*]\cdot\underline{\ddot{x}}\,,\quad \left\{\frac{\partial E_k}{\partial \underline{x}}\right\}^T = \underline{0}$$

$$\left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T = [C^*]\cdot\underline{\dot{x}},\quad \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = [K^*]\cdot\underline{x} + \frac{1}{2}\cdot\underline{P}^*$$

Thus, we can write the Matrix Formulation of our Equation of Motion as follows :

$$[M^*]\cdot\underline{\ddot{x}} + [C^*]\cdot\underline{\dot{x}} + [K^*]\cdot\underline{x} = \underline{Q} = \begin{Bmatrix} F \\ 0 \\ -rF \end{Bmatrix} \tag{36}$$

For readability purposes, we refer to the content of each matrix or vector as defined earlier in this report.

Eigenfrequencies and eigenvectors of the system

Undamped system

In the case of an undamped system, we have to set the damping matrix $[C^*] = [0]$, and by setting $\underline{x} = \underline{X}e^{\lambda t}$ we obtain the equation

$$\left(\lambda^2\,[M^*] + [K^*]\right)\underline{X} = \underline{0}$$

Which is the same of solving:

$$\lambda^2 = -[M^*]^{-1}\,[k^*]$$

Computing it in Matlab with the provided numerical values, we obtain the following eigen frequencies in rad/s :

$$\lambda_{1,4} = \pm 0.8017,\quad \lambda_{2,5} = \pm 4.6974,\quad \lambda_{3,6} = \pm 10.1326$$

As we're actually looking for the roots of these values, we identify the 3 natural frequencies of the system :

$$\omega_1 = 0.8017 \text{ rad/s},\quad \omega_2 = 4.6974 \text{ rad/s}\quad \omega_3 = 10.1326 \text{ rad/s}$$

And the corresponding mode shapes, normalized with respect to the component $x_1 = 1$:

$$\underline{X}^D_{\omega_1} = \begin{pmatrix} X^D_{1,\omega_1} \\ \Theta^D_{2,\omega_1} \\ \Theta^D_{3,\omega_1} \end{pmatrix} = \begin{pmatrix} 1 \\ 18.9893 \\ 12.4068 \end{pmatrix},\quad \underline{X}^D_{\omega_2} = \begin{pmatrix} X^D_{1,\omega_2} \\ \Theta^D_{2,\omega_2} \\ \Theta^D_{3,\omega_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.2460 \\ 0.0530 \end{pmatrix},\quad \underline{X}^D_{\omega_3} = \begin{pmatrix} X^D_{1,\omega_3} \\ \Theta^D_{2,\omega_3} \\ \Theta^D_{3,\omega_3} \end{pmatrix} = \begin{pmatrix} 1 \\ 2.1769 \\ 3.2023 \end{pmatrix}$$

Damped system

In the case of a damped system, we consider the whole equation of motion

$$[M^*]\,\underline{\ddot{x}} + [C^*]\,\underline{\dot{x}} + [K^*]\,\underline{x} = \underline{0}$$

And by setting $\underline{x} = \underline{X}e^{\lambda t}$ we obtain the equation

$$\left(\lambda^2\,[M^*] + \lambda\,[C^*] + [K^*]\right)\underline{X} = \underline{0} \tag{37}$$

Adding the trivial equation $[M^*]\,\underline{\dot{x}} = [M^*]\,\underline{\dot{x}}$ to this system, we can rephrase both matrix problems as :

$$\begin{bmatrix} [M] & [0] \\ [0] & [M] \end{bmatrix}\begin{pmatrix} \underline{\ddot{x}} \\ \underline{\dot{x}} \end{pmatrix} + \begin{bmatrix} [C] & [M] \\ -[M] & [0] \end{bmatrix}\begin{pmatrix} \underline{\dot{x}} \\ \underline{x} \end{pmatrix} = \underline{0}_{6\times 1} \tag{38}$$

By setting

$$\underline{z} = \begin{pmatrix} \underline{\dot{x}} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda\underline{x} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda\underline{X} \\ \underline{X} \end{pmatrix} e^{\lambda t}$$

Thus, we identify that the 3 last lines of $\underline{Z}$ will be of matter, containing the modes shapes. And we obtain the following matrix problem :

$$[B]\,\dot{\underline{z}} + [D]\,\underline{z} = \underline{0} \quad \text{such as} \begin{cases} [B] = \begin{bmatrix} [M] & [0] \\ [0] & [M] \end{bmatrix} \\ [D] = \begin{bmatrix} [C] & [K] \\ -[M] & [0] \end{bmatrix} \end{cases}$$

Therefore, we now want to resolve the problem : $\dot{\underline{z}} \ = [B]^{-1}\,[D]\,\underline{z}$
As we can rewrite $\underline{z} \ = \underline{Z}\, e^{\lambda t}$, $\dot{\underline{z}} = \lambda \underline{z}$, the previous equation is reduced to :

$$\lambda \underline{Z} = [B]^{-1}\,[D]\,\underline{Z} \Leftrightarrow \Big(\lambda\,[1] - [B]^{-1}\,[D]\Big)\,\underline{Z} = \underline{0} \tag{39}$$

Where the matrix whose eigenvalues and eigenvectors we are looking for is :

$$[B]^{-1}\,[D] = \begin{bmatrix} -[M]^{-1}\,[C] & -[M]^{-1}\,[K] \\ [1] & [0] \end{bmatrix}_{6\times 6}$$

Solving this eigenvalue problem, will give us the appropriate eigenvalues $\lambda^{d}_{2r-1,2r}$, $\forall r \in [1;3]$ and eigenvectors $\underline{X}^{d}_{2r-1,2r}$ $\ \forall r \in [1;3]$

Computing it in Matlab with the provided numerical values, we obtain :

$$\lambda_{1,4} = -0.0557 \pm i0.8010, \quad \lambda_{2,5} = -0.3114 \pm i4.6816 \quad \lambda_{3,6} = -0.3257 \pm i10.1234$$

The real part of each eigenvalue is due to the manifestation of dampers, while the imaginary part refers to the new resonance frequency. More precisely, we identify the 3 natural frequencies of the system :

$$\omega_1 = 0.8010 \text{ rad/s}, \quad \omega_2 = 4.6816 \text{ rad/s} \quad \omega_3 = 10.1234 \text{ rad/s}$$

We observe that those damped resonance frequencies are really close to the undamped scenario, leading us to think that this system is a lightly damped one. Indeed, when computing the adimensional damping ratio, and by ordering it accordingly to the increasing order of the eigenvalues, we obtain :

$$\underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0.0693 \\ 0.0664 \\ 0.0322 \end{pmatrix}$$

The corresponding mode shapes, normalized with respect to the component $x_1 = 1$:

$$\underline{X}^{D}_{\omega_1} = \begin{pmatrix} X^{D}_{1,\omega_1} = 1 \text{ m} \\ \Theta^{D}_{2,\omega_2} = 18.9893\,\text{rad} \\ \Theta^{D}_{3,\omega_2} = 12.4068\,\text{rad} \end{pmatrix}, \quad \underline{X}^{D}_{\omega_2} = \begin{pmatrix} X^{D}_{1,\omega_2} = 1 \text{ m} \\ \Theta^{D}_{2,\omega_2} = 0.2460 \text{ rad} \\ \Theta^{D}_{3,\omega_2} = 0.0530 \text{ rad} \end{pmatrix}, \quad \underline{X}^{D}_{\omega_3} = \begin{pmatrix} X^{D}_{1,\omega_3} = 1 \text{ m} \\ \Theta_{2,\omega_3} = 2.1769 \text{ rad} \\ \Theta_{3,\omega_3} = 3.2023 \text{ rad} \end{pmatrix}$$

Rayleigh Damping Estimation

Starting point :

$$\begin{bmatrix} c^*_{11} & c^*_{12} & c^*_{13} \\ c^*_{12} & c^*_{22} & c^*_{23} \\ c^*_{13} & c^*_{23} & c^*_{33} \end{bmatrix} = \alpha \cdot \begin{bmatrix} M^*_{11} & M^*_{12} & M^*_{13} \\ M^*_{12} & M^*_{22} & M^*_{23} \\ M^*_{13} & M^*_{23} & M^*_{33} \end{bmatrix} + \beta \cdot \begin{bmatrix} k^*_{11} & k^*_{12} & k^*_{13} \\ k^*_{12} & k^*_{22} & k^*_{23} \\ k^*_{13} & k^*_{23} & k^*_{33} \end{bmatrix}$$

Mid point :

$$\begin{pmatrix} c^*_{11} \\ c^*_{12} \\ c^*_{13} \\ c^*_{21} \\ c^*_{22} \\ c^*_{23} \\ c^*_{31} \\ c^*_{32} \\ c^*_{33} \end{pmatrix} = \begin{pmatrix} \alpha M^*_{11} + \beta k^*_{11} \\ \alpha M^*_{12} + \beta k^*_{12} \\ \alpha M^*_{13} + \beta k^*_{13} \\ \alpha M^*_{21} + \beta k^*_{21} \\ \alpha M^*_{22} + \beta k^*_{22} \\ \alpha M^*_{23} + \beta k^*_{23} \\ \alpha M^*_{31} + \beta k^*_{31} \\ \alpha M^*_{32} + \beta k^*_{32} \\ \alpha M^*_{33} + \beta k^*_{33} \end{pmatrix}$$

End point:

$$\begin{pmatrix} c_{11}^* \\ c_{12}^* \\ c_{13}^* \\ c_{21}^* \\ c_{22}^* \\ c_{23}^* \\ c_{31}^* \\ c_{32}^* \\ c_{33}^* \end{pmatrix} = \begin{bmatrix} M_{11}^* & k_{11}^* \\ M_{12}^* & k_{12}^* \\ M_{13}^* & k_{13}^* \\ M_{21}^* & k_{21}^* \\ M_{22}^* & k_{22}^* \\ M_{23}^* & k_{23}^* \\ M_{31}^* & k_{31}^* \\ M_{32}^* & k_{32}^* \\ M_{33}^* & k_{33}^* \end{bmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = [G] \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Free Motion Analysis

Forced Motion Analysis

Modal approach