VIBRATION ANALYSIS
AND VIBROACOUSTICS

VIBRATION ANALYSIS

Assignment 2 - A.Y. 2023/24

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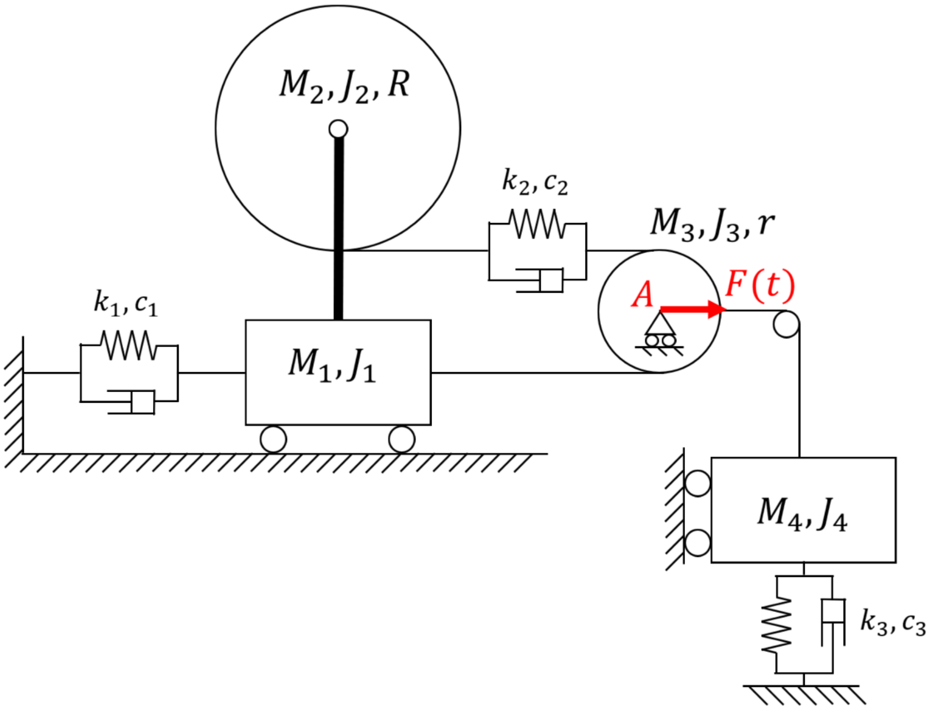
1 Equations of Motion and system matrices

In order to describe the mechanical system in figure, a reference system convention is defined: all displacements along the positive x and positive y directions are positive, all counterclockwise rotations are positive and all springs/dampers elongation are positive.



The number of degrees of freedom (dof) has to be determined as:

$$n_{dof} = 3n_b - n_{con}$$

Where n_b is the total number of rigid bodies (4 in this system) and n_{con} the number of constraints. An equation of motion (EoM) for each dof allow to completely describe the mechanical system. Each EoM is the result of the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{x}}\right) - \frac{\partial E_K}{\partial x} + \frac{\partial D}{\partial \dot{x}} + \frac{\partial V}{\partial x} = Q_x$$

In which x represents the single independent variable of a mechanical system.
By combining all contributions in a matrix form:

$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

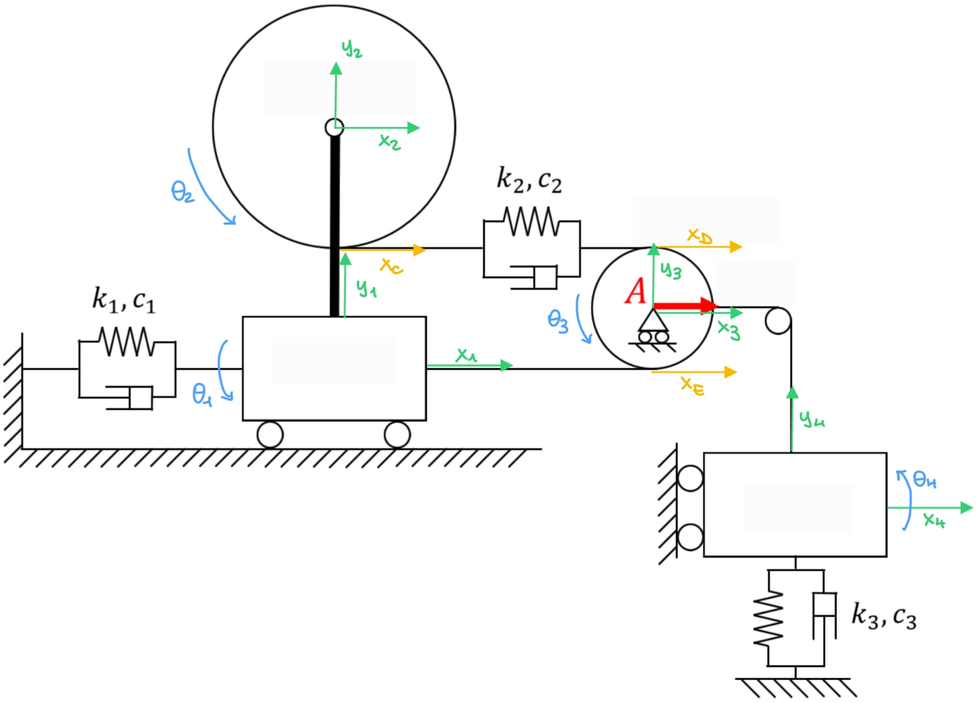
Where all independent variables have been gathered in the vector $\underline{x}$, as well as their first time derivatives in $\underline{\dot{x}}$.

1.1 Equations of Motion around the equilibrium position

The first goal is to determine the number of dof by evaluating all constraints:

- M_2 is constrained (through a hinge) to a mass-less vertical beam, rigidly connected to the mass M_1
  - hinge $\longrightarrow y_2 = y_1$
  - rigid connection $\longrightarrow \dot{x}_2 = \dot{x}_1$
- M_1 can only slide horizontally
  - slider $\longrightarrow y_1 = 0, \theta_1 = 0$

- M_3 can slide horizontally and rotate through an inextensible rope that connects the disk to both M_2 and M_1
  - slider and hinge $\longrightarrow y_3 = 0$
  - inextensible rope $\longrightarrow \dot{x}_1 = \dot{x}_E$
- M_4 can slide vertically and is rigidly connected to M_3 through an inextensible rope
  - slider $\longrightarrow x_4 = 0$, $\theta_4 = 0$
  - inextensible rope $\longrightarrow y_4 = -x_3$



To conclude, the number of constraints n_{con} is equal to $2+2+2+3$, so the number of dof is:

$$n_{dof} = 3n_b - n_{con} = 12 - 9 = 3$$

After that, it is necessary to describe all existing relations between independent variables (3 in this system) and physical variables:

$$\begin{cases} \dot{x}_2 = \dot{x}_1 \\ \dot{x}_1 = \dot{x}_E \end{cases} \implies x_1 = x_2 = x_E$$

$$\dot{x}_C = \dot{x}_2 + R\dot{\theta}_2, \quad \dot{x}_D = \dot{x}_1 - 2r\dot{\theta}_3, \quad y_4 = -x_3 = -\left(x_E - r\theta_3\right)$$

$$\implies \begin{cases} x_1 = x_2 = x_E \\ \dot{x}_C = \dot{x}_2 + R\dot{\theta}_2 \\ \dot{x}_D = \dot{x}_1 - 2r\dot{\theta}_3 \\ y_4 = -x_1 + r\theta_3 \end{cases}$$

Springs and dampers appear in the Lagrange equations by means of elongations Δl and their first time derivative $\dot{\Delta l}$:

$$\begin{cases} \dot{\Delta l}_1 = \dot{x}_1 \\ \dot{\Delta l}_2 = \dot{x}_D - \dot{x}_C = -R\dot{\theta}_2 - 2r\dot{\theta}_3 \\ \dot{\Delta l}_3 = \dot{y}_4 = -\dot{x}_1 + r\dot{\theta}_3 \end{cases} \implies \begin{cases} \Delta l_1 = x_1 \\ \Delta l_2 = -R\theta_2 - 2r\theta_3 \\ \Delta l_3 = -x_1 + r\theta_3 \end{cases}$$

Being the system in static equilibrium position, all springs pre-load are neglected.

1.1.1 Kinetic energy E_K

It is now possible to compute, in matrix form, all energies of the Lagrange equation.
Regarding the kinetic energy, four rigid bodies may contribute to the total E_K with four translations and four rotations. In this system, all bodies can translate but not rotate (only M_2 and M_3):

$$E_K = \frac{1}{2}M_1 v_1^2 + \frac{1}{2}M_2 v_2^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}M_3 v_3^2 + \frac{1}{2}J_3\omega_3^2 + \frac{1}{2}M_4 v_4^2$$

Let's gather all physical coordinates (and their first time derivatives) in two column vectors $\underline{z}$ (and $\dot{\underline{z}}$ respectively):

$$\underline{z}_{6\times1} = (x_1, x_2, \theta_2, x_3, \theta_3, y_4)^T \ , \quad \dot{\underline{z}}_{6\times1} = (v_1, v_2, \omega_2, v_3, \omega_3, v_4)^T$$

By defining the following physical mass matrix $[M]$ as:

$$[M]_{6\times6} = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 \end{bmatrix}$$

It is now possible to express the total kinetic energy E_K in matrix form as:

$$E_K = \frac{1}{2}\dot{\underline{z}}^T[M]\dot{\underline{z}}$$

It is of interest to get the energy in function of the independent variables:

$$\underline{x}_{3\times1} = (x_1, \theta_2, \theta_3)^T$$

expressed respectively in m, rad and rad. Let's introduce a relation between physical variables $\underline{z}$ and independent ones $\underline{x}$ by using their first time derivative:

$$\dot{\underline{z}} = \left(\frac{\partial \underline{z}}{\partial \underline{x}}\right)\dot{\underline{x}} = [\Lambda_M]\,\dot{\underline{x}}$$

Where $[\Lambda_M]$ is the Jacobian matrix and describes the relation between velocities v and angular velocities ω with respect to the independent variables.
Its entries are obtained by looking at the following table:

	$\dot{x}_1$	$\dot{\theta}_2$	$\dot{\theta}_3$
v_1	1	0	0
v_2	1	0	0
ω_2	0	1	0
v_3	1	0	-r
ω_3	0	0	1
v_4	-1	0	r

So the Jacobian matrix is defined as:

$$[\Lambda_M]_{6\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -r \\ 0 & 0 & 1 \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_M]$ in the kinetic energy formulation:

$$E_K = \frac{1}{2}\dot{\underline{z}}^T[M]\dot{\underline{z}} = \frac{1}{2}\dot{\underline{x}}^T[\Lambda_M]^T[M][\Lambda_M]\dot{\underline{x}} = \frac{1}{2}\dot{\underline{x}}^T[M^*]\dot{\underline{x}}$$

Where the mass matrix $[M^*]$ is defined as:

$$[M^*]_{3\times3} = [\Lambda_M]^T[M][\Lambda_M] = \begin{bmatrix} M_1+M_2+M_3+M_4 & 0 & -r\,[M_3+M_4] \\ 0 & J_2 & 0 \\ -r\,(M_3+M_4) & 0 & J_3+r^2\,(M_3+M_4) \end{bmatrix}$$

1.1.2 Potential energy V

Regarding the potential energy, the springs give a contribution in terms of elastic energy, while the mass M_4 gives a contribution in terms of gravitational energy:

$$V = V_{el} + V_g = \frac{1}{2}k_1\Delta l_1^2 + \frac{1}{2}k_2\Delta l_2^2 + \frac{1}{2}k_3\Delta l_3^2 + \frac{1}{2}M_4 g y_4$$

The gravitational term can be neglected due to the linearity of the system, so it does not affect the Lagrange equations, i.e. : $V_g = 0$.
Similarly to the kinetic case, a matrix approach is needed and the Jacobian matrix $[\Lambda_k]$ describes the relation between elongations Δl and independent variables.
By defining the physical stiffness matrix $[k]$ and the elongation vector $\underline{\Delta l}$ as:

$$[k]_{3\times3} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad \underline{\Delta l}_{3\times1} = \{\Delta l_1, \Delta l_2, \Delta l_3\}^T$$

It is possible to express the elastic potential energy in matrix form as:

$$V = V_{el} = \frac{1}{2}\underline{\Delta l}^T[k]\underline{\Delta l}$$

The Jacobian matrix $[\Lambda_k]$ entries are obtained by looking at the following table:

	x_1	θ_2	θ_3
Δl_1	1	0	0
Δl_2	0	-R	-2r
Δl_3	-1	0	r

So the Jacobian matrix is defined as:

$$[\Lambda_k]_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R & -2r \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_k]$ in the potential energy formulation:

$$V = V_{el} = \frac{1}{2}\underline{\dot{z}}^T[k]\underline{\dot{z}} = \frac{1}{2}\underline{\dot{x}}^T[\Lambda_k]^T[k][\Lambda_k]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[k^*]\underline{\dot{x}}$$

Where the stiffness matrix $[k^*]$ is defined as:

$$[k^*]_{3\times3} = [\Lambda_k]^T[k][\Lambda_k] = \begin{bmatrix} k_1 + k_3 & 0 & -rk_3 \\ 0 & R^2k_2 & 2Rk_2r \\ -rk_3 & 2Rk_2r & 4k_2r^2 + k_3r^2 \end{bmatrix}$$

1.1.3 Dissipative energy D

Regarding the dissipative energy, the dampers give a contribution in terms of first time derivative of the elongations:

$$D = \frac{1}{2}c_1\dot{\Delta l_1}^2 + \frac{1}{2}c_2\dot{\Delta l_2}^2 + \frac{1}{2}c_3\dot{\Delta l_3}^2$$

Similarly to the previous cases, a matrix approach is needed and the Jacobian matrix $[\Lambda_c]$ describes the relation between the elongations derivatives $\dot{\Delta l}$ and independent variables.
By defining the physical damping matrix $[c]$ and the elongation derivatives vector $\underline{\dot{\Delta l}}$ as:

$$[c]_{3\times3} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}, \quad \underline{\dot{\Delta l}}_{3\times1} = \{\dot{\Delta l_1}, \dot{\Delta l_2}, \dot{\Delta l_3}\}^T$$

It is possible to express the dissipative energy in matrix form as:

$$D = \frac{1}{2}\underline{\dot{\Delta l}}^T[k]\underline{\dot{\Delta l}}$$

The Jacobian matrix $[\Lambda_c]$ entries are the very same of $[\Lambda_k]$:

$$[\Lambda_c]_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R & -2r \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_c]$ in the potential energy formulation:

$$D = \frac{1}{2}\underline{\dot{z}}^T[c]\underline{\dot{z}} = \frac{1}{2}\underline{\dot{x}}^T[\Lambda_c]^T[c][\Lambda_c]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[c^*]\underline{\dot{x}}$$

Where the damping matrix $[c^*]$ is defined as:

$$[c^*]_{3\times 3} = [\Lambda_c]^T[c][\Lambda_c] = \begin{bmatrix} c_1 + c_3 & 0 & -rc_3 \\ 0 & R^2 c_2 & 2Rc_2 r \\ -rc_3 & 2Rc_2 r & 4c_2 r^2 + c_3 r^2 \end{bmatrix}$$

1.1.4 Virtual works δW

The principle of virtual works declares that an external force, applied at a given force point generates an infinitesimal work δW proportional to an infinitesimal displacement δx. In this system, the only existing external force $F(t)$ is applied horizontally at the point A, resulting in an infinitesimal displacement δx_3 and an infinitesimal work:

$$\delta W = F \cdot \delta x_3$$

Similarly to the previous cases, it is of interest to express the infinitesimal displacement δx_3 as function of the infinitesimal independent variables $\underline{\delta x}$ by means of Jacobian vector $[\Lambda_F]$:

$$\delta x_3 = \delta x_1 - \delta\theta_3 r = (1, 0, -r)\begin{pmatrix} \delta x_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{pmatrix} = [\Lambda_F]_{1\times 3} \cdot \underline{\delta x}_{3\times 1}$$

$$\implies \delta W = F\delta x_1 - F\delta\theta_3 r = F \cdot [\Lambda_F] \cdot \underline{\delta x}$$

For the Lagrange equation, it is possible to identify the Lagrangian vector $\underline{Q}$ as:

$$\underline{Q}_{3\times 1} = \begin{pmatrix} F \\ 0 \\ -rF \end{pmatrix} = [\Lambda_F]^T F$$

1.1.5 Lagrange equations

It is now possible to express the Lagrange equations in matrix form:

$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

$$\left\{\frac{d}{dt}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T = [M^*] \cdot \underline{\ddot{x}}\,, \quad \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T = \underline{0}$$

$$\left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T = [c^*] \cdot \underline{\dot{x}}, \quad \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = [k^*] \cdot \underline{x}$$

The total matrix expression of all Equations of Motion is:

$$[M^*] \cdot \underline{\ddot{x}} + [c^*] \cdot \underline{\dot{x}} + [k^*] \cdot \underline{x} = \underline{Q}$$

1.2 Eigenfrequencies and eigenvectors

1.2.1 Undamped system

The eigenfrequencies and eigenvectors computation problem consists in analyzing the system as undamped and in free motion, which results in setting $[c^*] = [0]$ and $\underline{Q} = \underline{0}$ in the matrix expression of the Equations of Motion:

$$[M^*] \cdot \underline{\ddot{x}} + [k^*] \cdot \underline{x} = \underline{0}$$

By assuming as a solution $\underline{x} = \underline{X} e^{\lambda t}$, the equation becomes:

$$\left(\lambda^2 [M^*] + [k^*]\right) \underline{X} = \underline{0}$$

The non-trivial solution consists in solving the characteristic equation:

$$\det\left(\lambda^2 [M^*] + [k^*]\right) = 0 \Longrightarrow \lambda^2 = -[M^*]^{-1} [k^*]$$

By solving the above expression for λ, six imaginary conjugate values are obtained:

$$\lambda_{1,4} = \pm j\omega_{0_1} = \pm j0.8017, \quad \lambda_{2,5} = \pm j\omega_{0_2} = \pm j4.6974, \quad \lambda_{3,6} = \pm j\omega_{0_3} = \pm j10.1326$$

It is of interest to consider real frequencies (positive values), so it is possible to identify the three natural angular frequencies of the system as:

$$\omega_{0_1} = \mathrm{Im}\{\lambda_1\} = 0.8017 \text{ rad/s}, \quad \omega_{0_2} = \mathrm{Im}\{\lambda_2\} = 4.6974 \text{ rad/s} \quad \omega_{0_3} = \mathrm{Im}\{\lambda_3\} = 10.1326 \text{ rad/s}$$

By substituting the solutions λ_i, λ_{i+3} (for $i = 1, 2, 3$) to the equation to solve, it is possible to find the eigenvectors $\underline{X}^U_{\omega_{0_i}}$, responses of each independent variable for the i-th natural frequency ω_{0_i}:

$$\left({\lambda_i}^2 [M^*] + [k^*]\right) \underline{X}_U^{(i)} = \underline{0}$$

Where:

$$\underline{X}_U^{(i)} = \begin{pmatrix} X_{1,U}^{(i)} \\ \Theta_{2,U}^{(i)} \\ \Theta_{3,U}^{(i)} \end{pmatrix}$$

The eigenvectors for each ω_{0_i}, normalized with respect to the first component $x_1 = 1$, are:

$$\underline{X}_U^{(1)} = \begin{pmatrix} X_{1,U}^{(1)} \\ \Theta_{2,U}^{(1)} \\ \Theta_{3,U}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -19.0479 \\ 12.4436 \end{pmatrix}, \quad \underline{X}_U^{(2)} = \begin{pmatrix} X_{1,U}^{(2)} \\ \Theta_{2,U}^{(2)} \\ \Theta_{3,U}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -0.2348 \\ 0.0486 \end{pmatrix}, \quad \underline{X}_U^{(3)} = \begin{pmatrix} X_{1,U}^{(3)} \\ \Theta_{2,U}^{(3)} \\ \Theta_{3,U}^{(3)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2.1885 \\ 3.2200 \end{pmatrix}$$

1.2.2 Damped system

By considering the damped case ($[c^*] \neq [0]$) the matrix expression of the Equations of Motion (in free motion) becomes:

$$[M^*] \underline{\ddot{x}} + [c^*] \underline{\dot{x}} + [k^*] \underline{x} = \underline{0}$$

By assuming as a solution $\underline{x} = \underline{X} e^{\lambda t}$, the equation becomes:

$$\left(\lambda^2 [M^*] + \lambda [c^*] + [k^*]\right) \underline{X} = \underline{0}$$

By adding the trivial equation $[M^*] \underline{\dot{x}} = [M^*] \underline{\dot{x}}$ to the matrix form of the Equations of Motion, the problem can be expressed as:

$$\begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \begin{pmatrix} \underline{\ddot{x}} \\ \underline{\dot{x}} \end{pmatrix} + \begin{bmatrix} [c^*] & [M^*] \\ -[M^*] & [0] \end{bmatrix} \begin{pmatrix} \underline{\dot{x}} \\ \underline{x} \end{pmatrix} = \underline{0}_{6\times1}$$

By setting the vector of state variables $\underline{z}$ as:

$$\underline{z}_{6\times1} = \begin{pmatrix} \underline{\dot{x}} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda \underline{x} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda \underline{X} \\ \underline{X} \end{pmatrix} e^{\lambda t} = \underline{Z}_{6\times1} e^{\lambda t}$$

The problem can be expressed as:

$$\begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \underline{\dot{z}} + \begin{bmatrix} [c^*] & [M^*] \\ -[M^*] & [0] \end{bmatrix} \underline{z} = \underline{0} \Longrightarrow [B]\, \underline{\dot{z}} + [D]\, \underline{z} = \underline{0}$$

In which:

$$[B]_{6\times 6} = \begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix}, \quad [D]_{6\times 6} = \begin{bmatrix} [c^*] & [k^*] \\ -[M^*] & [0] \end{bmatrix}$$

The problem to solve is in the form:

$$\underline{\dot{z}} = -[B]^{-1}[D]\, \underline{z}$$

In which $\det([B]) \neq 0$.
The matrix product $[A]_{6\times 6} = -[B]^{-1}[D]$ is called state matrix of the system. Let's consider:

$$\lambda \underline{Z} = -[B]^{-1}[D]\, \underline{Z} = [A]\, \underline{Z} \Longrightarrow \left(\lambda [I]_{6\times 6} - [A]\right) \underline{Z} = \underline{0}$$

From the above equation, it is possible to analyze the last three rows of $\underline{z}$, equal to $\underline{x}$, which contain the mode shapes for the damped case.
The matrix for the eigenvalues and eigenvectors problem computation is the state matrix:

$$[A] = -[B]^{-1}[D] = \begin{bmatrix} -[M^*]^{-1}[c^*] & -[M^*]^{-1}[k^*] \\ [I]_{3\times 3} & [0] \end{bmatrix}$$

The solutions $\lambda^d_{i,i+3} = -\alpha_i \pm j\omega_{d_i}$ (for $i = 1, 2, 3$) are complex conjugate, as in the undamped case, but have non-zero real part:

$$\lambda^d_{1,4} = -0.0557 \pm j0.8010, \quad \lambda^d_{2,5} = -0.3114 \pm j4.6816, \quad \lambda^d_{3,6} = -0.3257 \pm j10.1234$$

The real part α_i describes a decay behaviour in the time responses and is the contribution of the dampers. The imaginary part ω_{d_i} refers to the resonance frequencies in the damped system.
More precisely, three natural "damped" frequencies are identified as:

$$\omega_{d_1} = \mathrm{Im}\left\{\lambda^d_1\right\} = 0.8010 \text{ rad/s}, \quad \omega_{d_2} = \mathrm{Im}\left\{\lambda^d_2\right\} = 4.6816 \text{ rad/s}, \quad \omega_{d_3} = \mathrm{Im}\left\{\lambda^d_3\right\} = 10.1234 \text{ rad/s}$$

By analyzing the "damped" resonance frequencies, it is possible to see how close they are to the ones of the undamped case. This observation allows to define the system as lightly damped.
The confirmation of that comes from the adimensional damping ratios computation:

$$\underline{h}_{3\times 1} = \begin{pmatrix} h_1 = \frac{\alpha_1}{\omega_{0_1}} \\ h_2 = \frac{\alpha_2}{\omega_{0_2}} \\ h_3 = \frac{\alpha_3}{\omega_{0_3}} \end{pmatrix} = \begin{pmatrix} 0.0695 \\ 0.0665 \\ 0.0322 \end{pmatrix}$$

In which h_i is the adimensional damping ratio for the *i-th* eigenfrequency (sorted with increasing frequency).
Similarly to the undamped scenario, it is now possible to define the eigenvectors $\underline{X}^{(i)}$ for each natural frequency, normalized with respect to the first component $x_1 = 1$:

$$\underline{X}^{(1)} = \begin{pmatrix} X_1^{(1)} \\ \Theta_2^{(1)} \\ \Theta_3^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -18.8050 + j2.6391 \\ 12.2816 - j1.7581 \end{pmatrix}$$

$$\underline{X}^{(2)} = \begin{pmatrix} X_1^{(2)} \\ \Theta_2^{(2)} \\ \Theta_3^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -0.2323 + j0.0809 \\ 0.0448 - j0.0283 \end{pmatrix}$$

$$\underline{X}^{(3)} = \begin{pmatrix} X_1^{(3)} \\ \Theta_2^{(3)} \\ \Theta_3^{(3)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2.1712 + j0.1567 \\ 3.1917 + j0.2604 \end{pmatrix}$$

1.3 Rayleigh damping

The Rayleigh damping assumption states that two constants α and β exist so that:

$$[c^*] = \alpha\,[M^*] + \beta\,[k^*]$$

The adimensional damping ratios h_i, computed in the previous paragraph, are equal to:

$$h_i = \frac{c^*_{ii}}{2m^*_{ii}\omega_{0_i}} = \frac{\alpha m^*_{ii} + \beta k^*_{ii}}{2m^*_{ii}\omega_{0,i}}, \quad i = 1, 2, 3$$

$$\Longrightarrow h_i = \frac{\alpha}{2\omega_{0,i}} + \frac{\beta\omega_{0,i}}{2} = \left(\frac{1}{2\omega_{0,i}}, \frac{\omega_{0,i}}{2}\right)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

In order to find values of α and β that best approximate the Rayleigh damping equation, it is necessary to solve the following equation:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\frac{1}{2\omega_{0,i}}\\ \frac{\omega_{0,i}}{2}\end{pmatrix}^{-1} h_i$$

In matrix form it becomes:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{bmatrix}\frac{1}{2\omega_{0,1}} & \frac{1}{2\omega_{0,2}} & \frac{1}{2\omega_{0,3}}\\ \frac{\omega_{0,1}}{2} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,3}}{2}\end{bmatrix}^{-1}\begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix} = \begin{bmatrix}\frac{1}{2\omega_{0,1}} & \frac{1}{2\omega_{0,2}} & \frac{1}{2\omega_{0,3}}\\ \frac{\omega_{0,1}}{2} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,3}}{2}\end{bmatrix}^{-1}\underline{h}$$

The results are:

$$\alpha = 0.1134, \quad \beta = 0.0083$$

2 Free motion

The analysis of the free motion system will be done by assuming Rayleigh damping, so the damping matrix $[c^*]$ is given by:

$$[c^*] = \alpha [M^*] + \beta [k^*]$$

2.1 Time responses

In the previous section, the time response solution was defined as $\underline{x} = \underline{X} e^{\lambda t}$, in which $\underline{X}$ collects amplitude and phase response from each system natural frequency to the independent variables. The eigenvector $\underline{X}^{(1)}$ ($\underline{X}^{(2)}$ and $\underline{X}^{(3)}$ respectively) describes the amplitude and phase components of ω_{d_1} (ω_{d_2} and ω_{d_3} respectively) in the independent variables x_1, θ_2 and θ_3.
In particular:

$$x_1(t) = \sum_{i=1}^{3} e^{-\alpha_i t} \left| X_1^{(i)} \right| C_i \cos\left(\omega_{d_i} t + \Phi_i + \angle X_1^{(i)}\right) = \sum_{i=1}^{3} e^{-\alpha_i t} C_i \cos\left(\omega_{d_i} t + \Phi_i\right)$$

Due to the normalization of the eigenvectors $X_1^{(i)} = 1$, so $\left| X_1^{(i)} \right| = 1$ and $\angle X_1^{(i)} = 0$.

$$\theta_2(t) = \sum_{i=1}^{3} e^{-\alpha_i t} \left| \Theta_2^{(i)} \right| C_i \cos\left(\omega_{d_i} t + \Phi_i + \angle \Theta_2^{(i)}\right)$$

$$\theta_3(t) = \sum_{i=1}^{3} e^{-\alpha_i t} \left| \Theta_3^{(i)} \right| C_i \cos\left(\omega_{d_i} t + \Phi_i + \angle \Theta_3^{(i)}\right)$$

Each time response has three decaying sinusoids (with decay factor α_i and angular frequency ω_{d_i}), which manifest in different ways, depending on the values of $\left| X_1^{(i)} \right|$, $\left| \Theta_2^{(i)} \right|$ and $\left| \Theta_3^{(i)} \right|$.

2.1.1 Initial Conditions

All unknowns C_i and Φ_i are obtained by imposing the initial conditions:

$$x_{1_0} = 0.1 \text{ m}, \quad \theta_{2_0} = \frac{\pi}{12} \text{ rad}, \quad \theta_{3_0} = -\frac{\pi}{12} \text{ rad}$$

$$\dot{x}_{1_0} = 1 \text{ m/s}, \quad \dot{\theta}_{2_0} = 0.5 \text{ rad/s}, \quad \dot{\theta}_{3_0} = 2 \text{ rad/s}$$

Since the displacements expressions are highly similar, expressing the initial conditions by substituting $t = 0$ in the equations of $x_1(t)$ gives us the general formulation of the initial condition problem in which we are interested in finding the six unknowns $C_i, \Phi_i, \quad \forall i \in [1, 3]$:

$$\left. x_1(t) \right|_{t=0} = x_{1_0} \Leftrightarrow \sum_{i=1}^{3} C_i \cdot \left| X_1^{(i)} \right| \cdot \cos\left(\Phi_i + \angle X_1^{(i)}\right) = x_{1_0}$$

$$\left. \dot{x}_1(t) \right|_{t=0} = v_{1_0} \Leftrightarrow \sum_{i=1}^{3} \left[C_i \cdot \left| X_1^{(i)} \right| \cdot \cos\left(\Phi_i + \angle X_1^{(i)} - \arctan\left(\omega_{d_i} / \alpha_{x_1}\right)\right) \right] = -v_{1_0}$$

WHAT IS α_{x_1} ??

The resulting six unknowns are:

$$C_1 = 0.0870, \quad C_2 = 0.1981, \quad C_3 = -0.9423$$

$$\Phi_1 = 1.7709 \text{ rad}, \quad \Phi_2 = -0.8294 \text{ rad}, \quad \Phi_3 = 7.8365 \text{ rad}$$

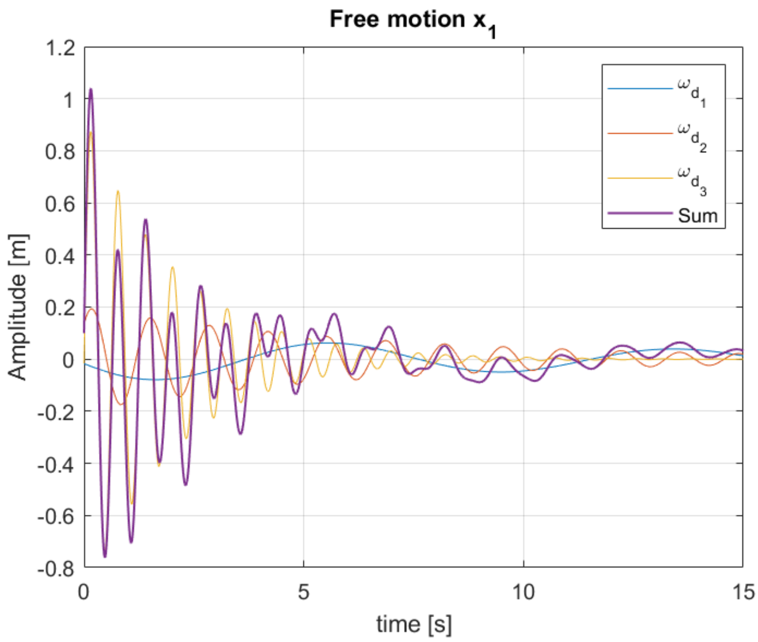
It is possible to check that the initial conditions are met by looking at the first element of the displacement vectors (in MATLAB).
It is now possible to fully define and plot the free motion response of each independent variable. By taking into account the products between the constants C_i and the amplitude factors $\left| X_i^{(j)} \right|$, it is possible to give a quick explanation of how much a certain frequency manifests in the time response of $x_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ using the following table:

$$\left[C_1 \cdot \underline{X}^{(1)} \quad C_2 \cdot \underline{X}^{(2)} \quad C_3 \cdot \underline{X}^{(3)}\right] = \begin{bmatrix} 0.0870 & 0.1981 & 0.9423 \\ 1.6566 & 0.0465 & 2.0623 \\ 1.0822 & 0.0096 & 3.0362 \end{bmatrix}$$

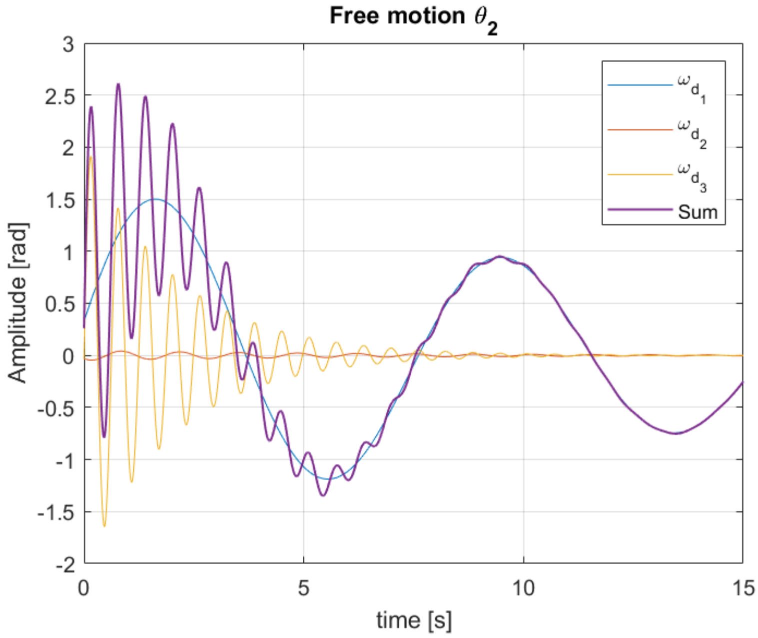
CHANGED TABLE INTO MATRIX

	ω_{d_1}	ω_{d_2}	ω_{d_3}
$\lvert x_1\rvert$	0.0870	0.1981	0.9423
$\lvert\theta_2\rvert$	1.6566	0.0465	2.0623
$\lvert\theta_3\rvert$	1.0822	0.0096	3.0362

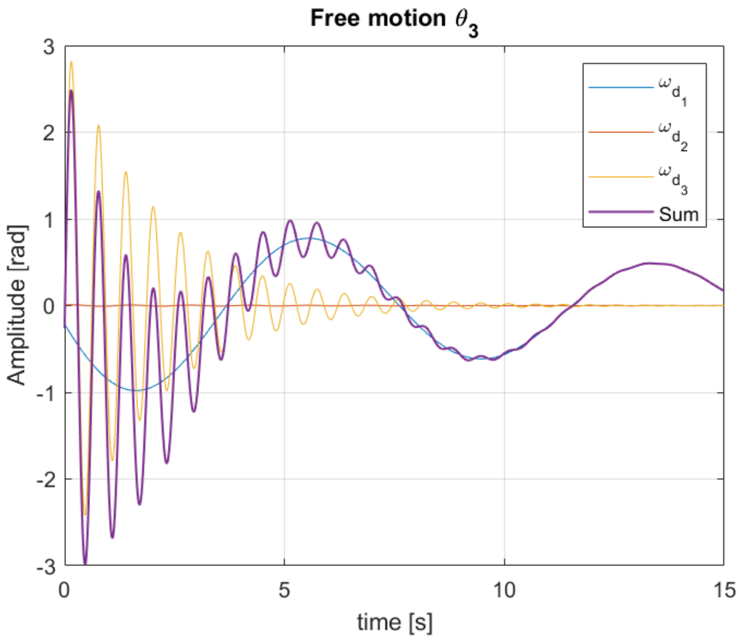
The element in position (i, j) represents the amplitude of the sinusoids of angular frequency ω_{d_j} in the time response of the independent variable x_i or θ_i.
The time response plot of $x_1(t)$ is mainly given by ω_{d_2} and ω_{d_3}:



The time response plot of $\theta_2(t)$ is mainly given by ω_{d_1} and ω_{d_3}:



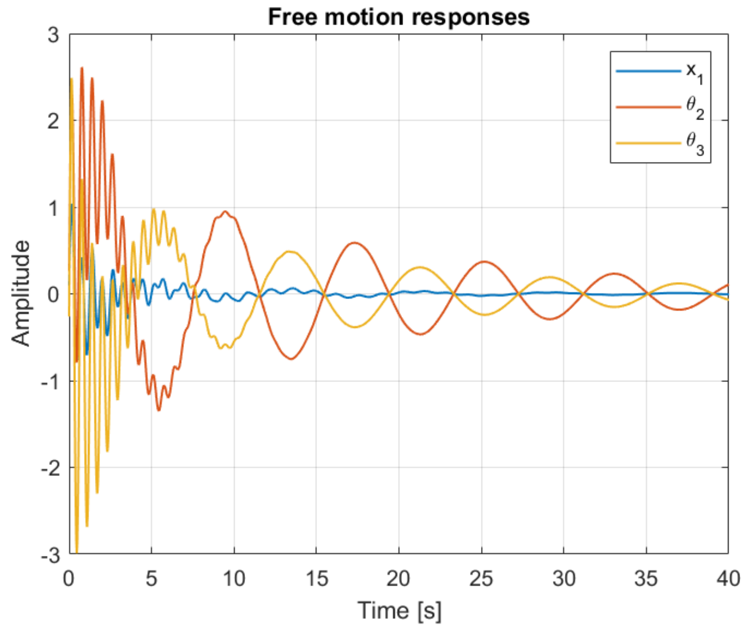
The time response plot of $\theta_3(t)$ is mainly given by ω_{d_1} and ω_{d_3}:



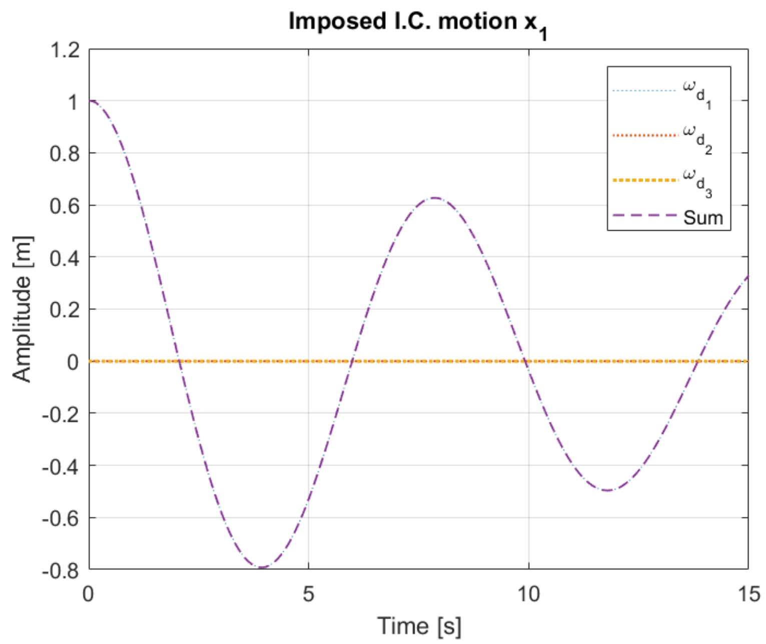
A comparison between all three time responses is given:

2.2 Eigenmode isolation

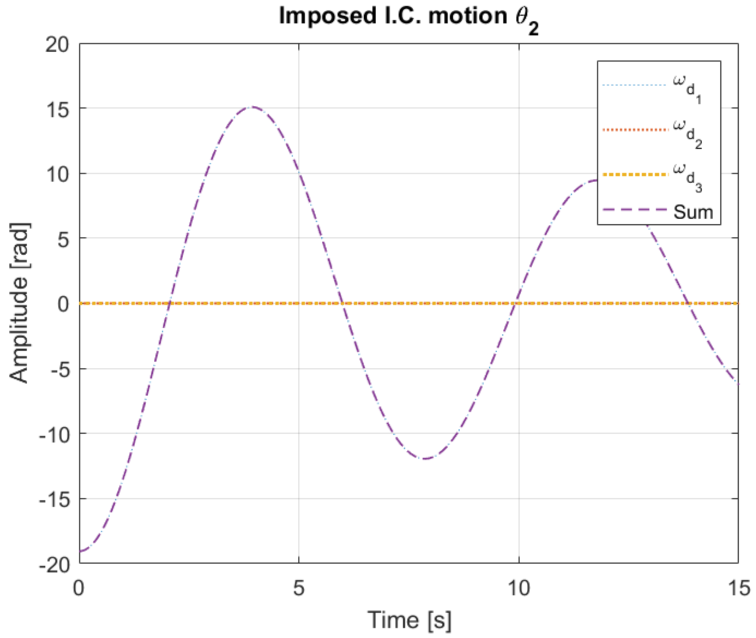
A way to impose the resulting time responses with a single mode is to put initial displacements equal to the eigenvector entries of the natural frequency of interest, while the initial velocities are all set to zero. In the following plots, the first mode is selected, so only the first natural frequency ω_{d_1} is going to be present.



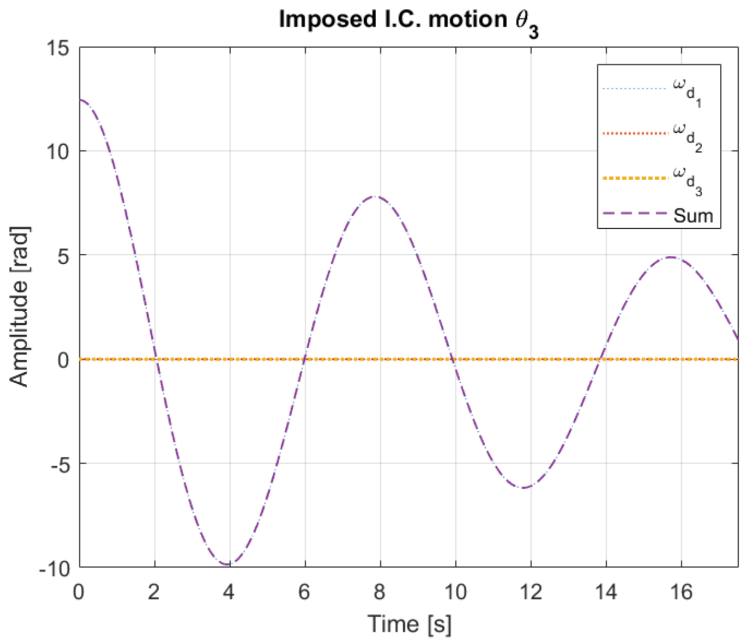
The time response plot of $x_1(t)$ is given:



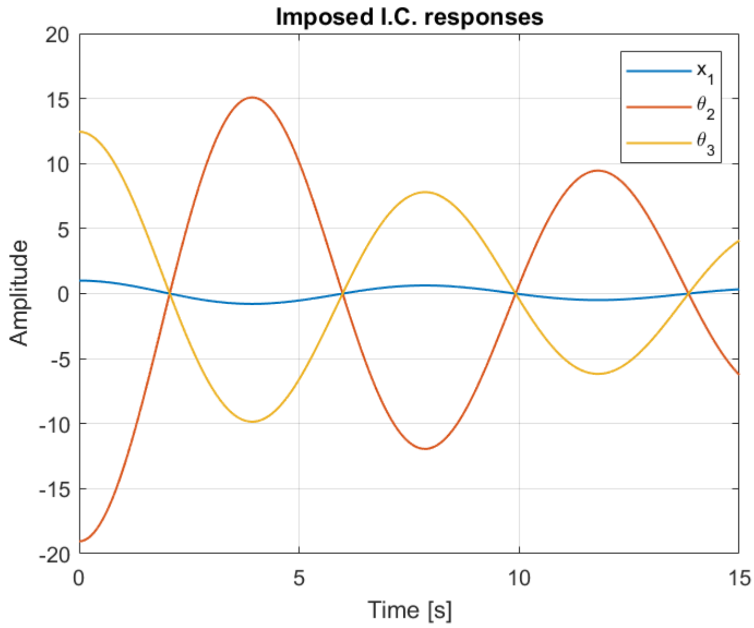
The time response plot of $\theta_2(t)$ is given:



The time response plot of $\theta_3(t)$ is given:



A comparison between all time responses is given:



3 Forced motion

As already explained at the beginning of the previous section, Rayleigh damping is assumed to occur as we have a lightly damped system:

$$[c^*] = \alpha [M^*] + \beta [k^*]$$

3.1 Frequency response matrix

By introducing the external force contribution, the equation to study becomes:

$$[M^*] \cdot \underline{\ddot{x}} + [c^*] \cdot \underline{\dot{x}} + [k^*] \cdot \underline{x} = \underline{Q} = \underline{Q_0} \cos(\Omega t)$$

Where:

$$\underline{Q} = \begin{pmatrix} F(t) \\ 0 \\ -rF(t) \end{pmatrix}, \quad F(t) = F_0 \cos(\Omega t), \quad \underline{Q_0} = \begin{pmatrix} F_0 \\ 0 \\ -rF_0 \end{pmatrix} = [\Lambda_F]^T F_0$$

A complex function $\underline{\tilde{x}_p}(t)$ can be defined, so that its real part $\underline{x_p}(t)$ is a particular solution of the equation above:

$$\underline{\tilde{x}_p}(t) = \underline{\tilde{X}_0} e^{j\Omega t} = \begin{pmatrix} \tilde{X}_{1_0} \\ \tilde{\Theta}_{2_0} \\ \tilde{\Theta}_{3_0} \end{pmatrix} e^{j\Omega t} \Longrightarrow \underline{x_p}(t) = \mathrm{Re}\{\underline{\tilde{x}_p}(t)\}$$

The complex function $\underline{\tilde{x}_p}(t)$, on the other hand, is a particular solution of the following equation:

$$[M^*] \cdot \underline{\ddot{\tilde{x}}_p} + [c^*] \cdot \underline{\dot{\tilde{x}}_p} + [k^*] \cdot \underline{\tilde{x}_p} = \underline{Q_0} e^{j\Omega t}$$

By evaluating the first and second time derivative of $\underline{\tilde{x}_p}(t)$, it is possible to define a relation between $\underline{\tilde{X}_0}$ and $\underline{Q_0}$:

$$\underline{\tilde{X}_0} = \left[-\Omega^2 [M^*] + j\Omega [c^*] + [k^*]\right]^{-1} \underline{Q_0} = [D(\Omega)]^{-1}_{3\times3} \underline{Q_0} = [H(\Omega)]_{3\times3} \underline{Q_0}$$

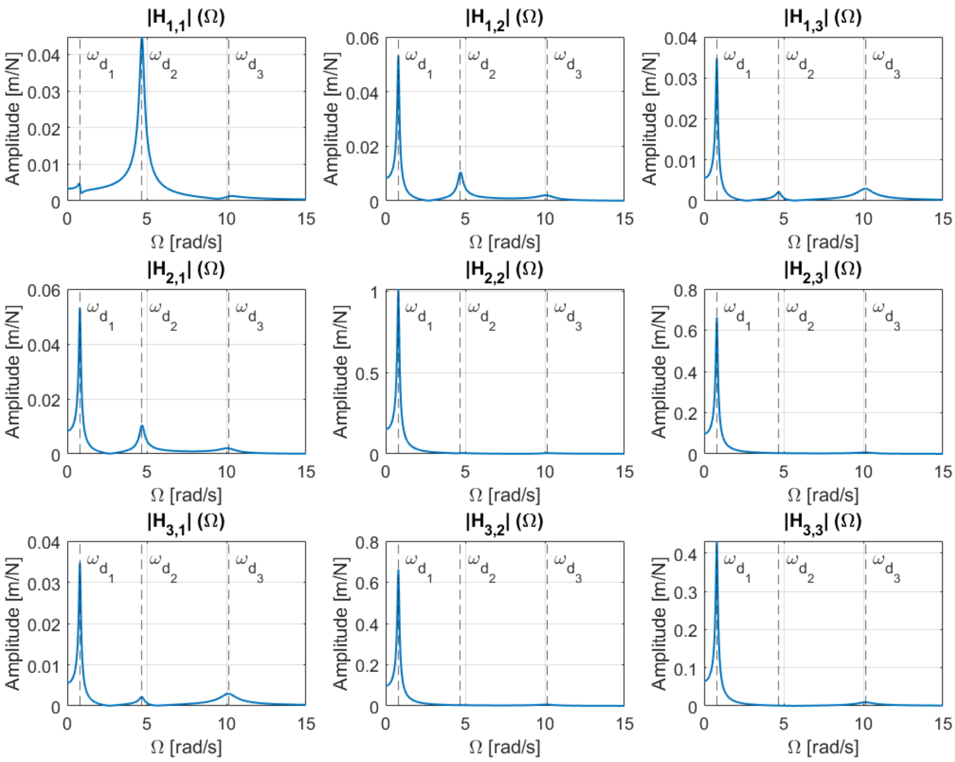
The matrix $[H(\Omega)]$ is the frequency response matrix and its entries are the frequency response functions $H_{i,j}(\Omega)$, given by:

$$H_{i,j} = [D(\Omega)]^{-1}_{i,j} = \frac{1}{\det([D(\Omega)])} [C_D]^T_{i,j}$$

Where the cofactor matrix $[C_D]$ is defined as:

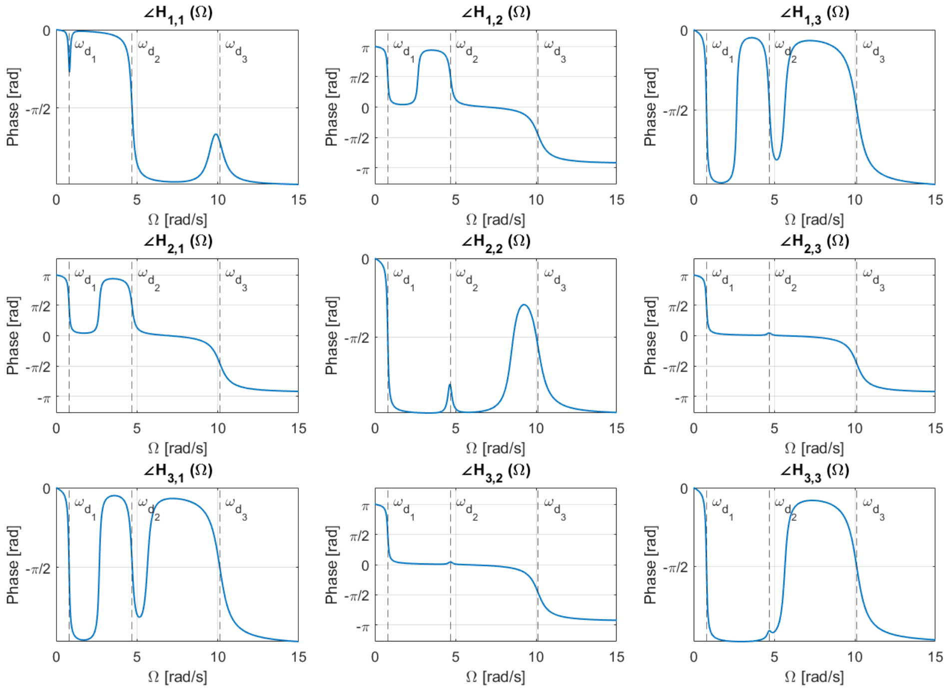
$$[C_D]_{i,j} = (-1)^{i+j} \det\left(\left[M_{D_{i,j}}\right]\right)$$

$\left[M_{D_{i,j}}\right]$ is the matrix obtained from $[D]$ after removing the *j-th* column and the *i-th* row. The plot of every FRF amplitude $|H_{i,j}|$ is given:



Each frequency response function $H_{i,j}$ describes the system response of the *i-th* independent displacement/rotation in case of a force applied to the *j-th* body.
Whenever in a frequency response function one of the natural frequencies ω_{d_i} is not present, a node of vibration occurs.

The plot of every FRF phase $\angle H_{i,j}$ is given:

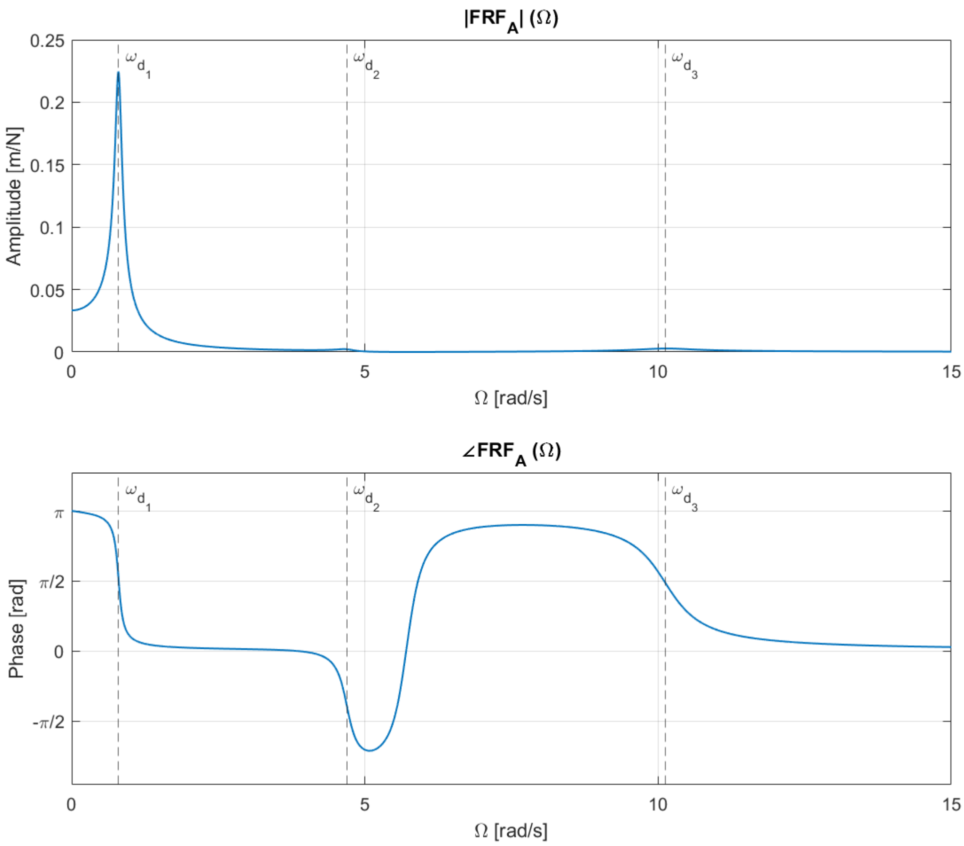


Whenever a $-\pi$ shift in the phase response occurs, the magnitude response presents a resonance (pole in the FRF), which is where a local maximum is present. On the other hand, a node of vibration (zero in the FRF) in the magnitude response translates into a $+\pi$ phase shift.
In all the other cases, a small phase shift can be described in terms of poles and zeros being closely spaced.
For example, in the phase response of $H_{1,1}$ a sudden phase shift occurs around ω_{d_1} and $\omega_{d,3}$, less than $\pm\pi$.

3.2 Co-located FRF in A

The co-located FRF of the point A (at the center of the disk M_3) is defined as the frequency response function that describes the displacement in A (called $x_3 = x_1 - r\theta_3$ in the system) and the force F applied in that point.

$$\text{FRF}_A(\Omega) = [\Lambda_F][H(\Omega)]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & -r\end{bmatrix}\begin{bmatrix}H_{1,3}\\H_{2,3}\\H_{3,3}\end{bmatrix} = H_{1,3}(\Omega) - rH_{3,3}(\Omega)$$



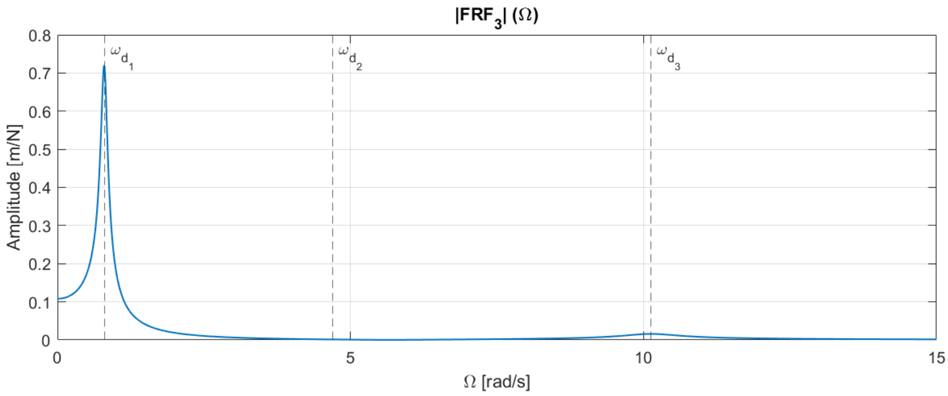
Considering that $\text{FRF}_A = H_{1,3} - rH_{3,3}$, it is expected that a single resonance in ω_{d_1} is presents, similarly to $H_{1,3}$ and $H_{3,3}$.

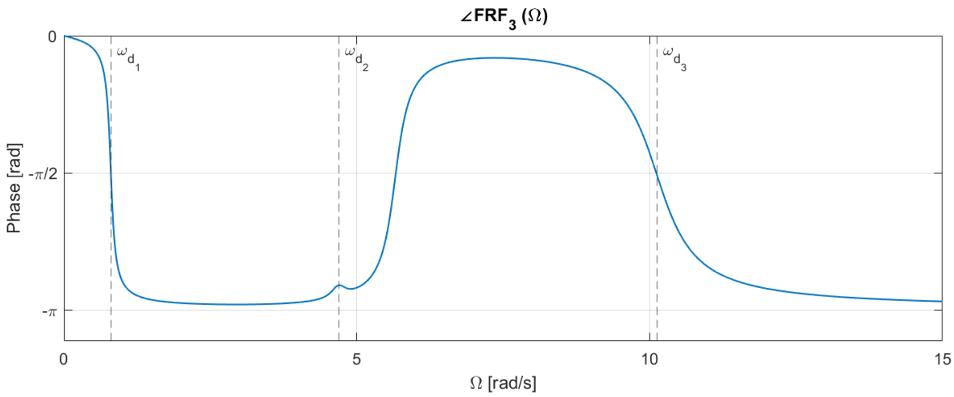
The comments made in the previous section, related to the FRF plots, are still valid.

3.3 Co-located FRF for disk 3

The frequency response $H_{3,3}$ describes the system response in terms of θ_3 to a force applied at the center of the disk M_3. In order to get a response to a torque:

$$\mathrm{FRF}_3(\Omega) = \frac{\Theta_3}{Fr} = \frac{\Theta_3}{F}\frac{1}{r} = \frac{H_{3,3}(\Omega)}{r}$$





3.4 Harmonic force response

Considering an harmonic force applied to the center of the disk M_3:

$$F(t) = F_1(t) + F_2(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)$$

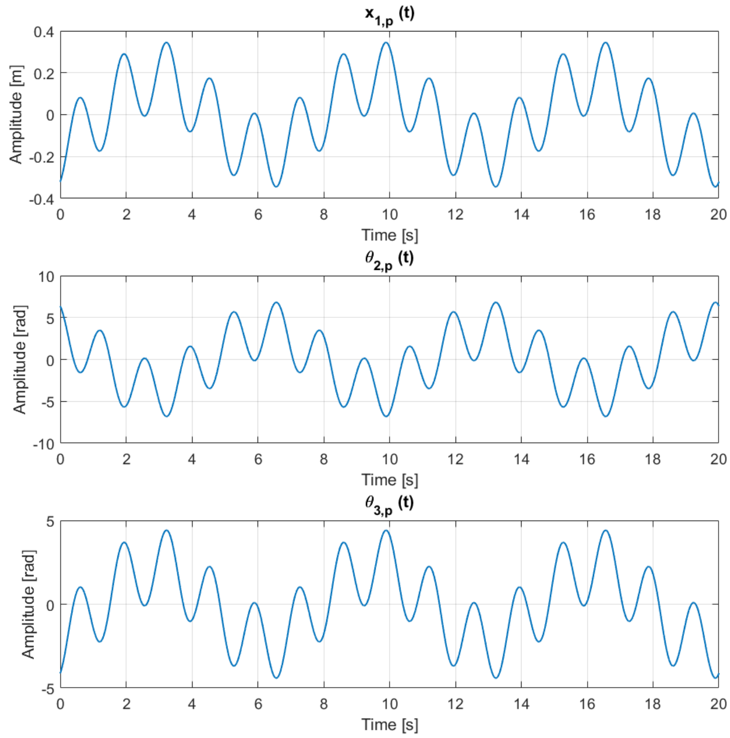
It is possible to study the system response in terms of linear combination of responses to F_1 and F_2, given the linearity of the system. From each force component, it is possible to compute the particular solutions for $x_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ as:

$$x_{1_p}(t) = |H_{1,3}(\Omega = 2\pi f_1)| A_1 \cos(2\pi f_1 t + \angle H_{1,3}(\Omega = 2\pi f_1)) + \\ + |H_{1,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{1,3}(\Omega = 2\pi f_2))$$

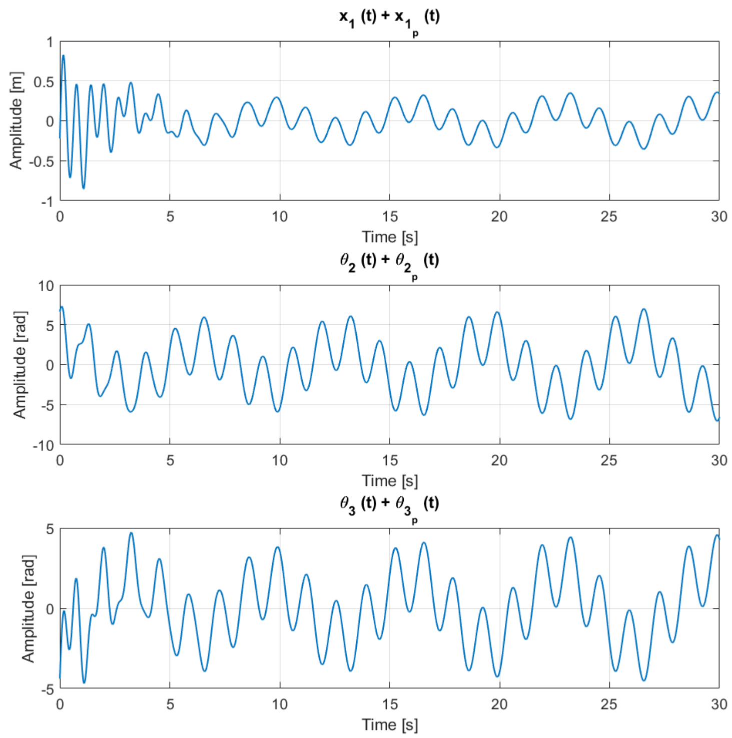
$$\theta_{2_p}(t) = |H_{2,3}(\Omega = 2\pi f_1)| A_1 \cos(2\pi f_1 t + \angle H_{2,3}(\Omega = 2\pi f_1)) + \\ + |H_{2,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{2,3}(\Omega = 2\pi f_2))$$

$$\theta_{3_p}(t) = |H_{3,3}(\Omega = 2\pi f_1)| A_1 \cos(2\pi f_1 t + \angle H_{3,3}(\Omega = 2\pi f_1)) + \\ + |H_{3,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{3,3}(\Omega = 2\pi f_2))$$

Their plot is given:



In order to obtain the complete time response of all the independent variables, it is necessary to sum the particular solutions just obtained to the free motion time (homogeneous) responses in the section 2.1:

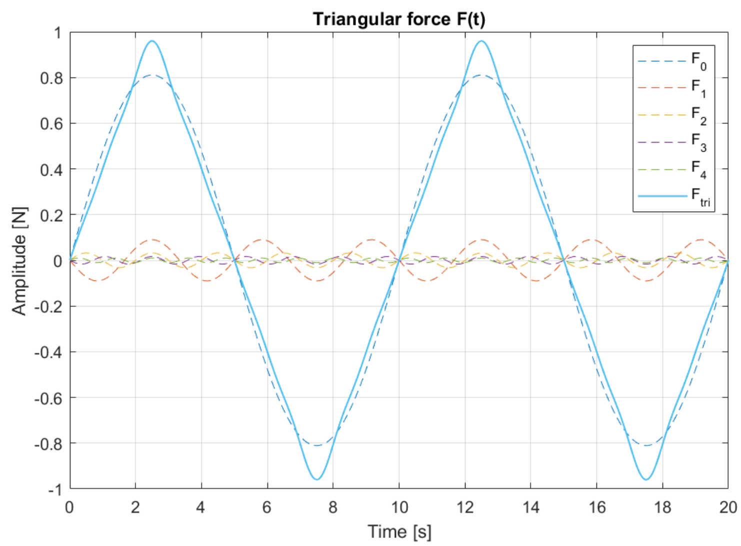


Given the decaying nature of the homogeneous solutions $x_1(t)$, $\theta_2(t)$ and $\theta_3(t)$, an initial transient is present. After that, the particular solution represents the steady state of the response ($t \to \infty$).

3.5 Triangular force response

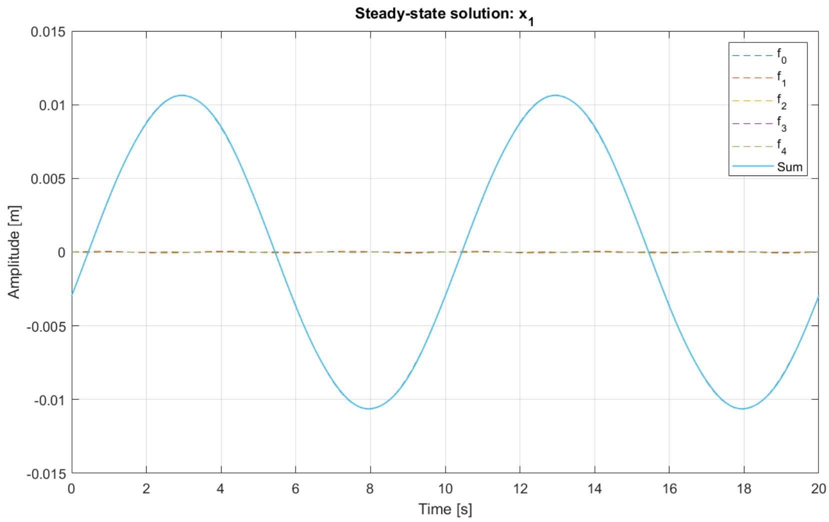
Similarly to the previous paragraph, the system response of a linear combination of forces can be obtained as linear combination of the single responses.
Considering a periodic force like:

$$F(t) = \sum_{k=0}^{4} F_k(t) = \sum_{k=0}^{4} \frac{8}{\pi^2}(-1)^k \frac{\sin(\Omega_k t)}{(2k+1)^2}, \quad f_0 = 0.10 \text{ Hz}, \quad \Omega_k = (2k+1)2\pi f_0$$



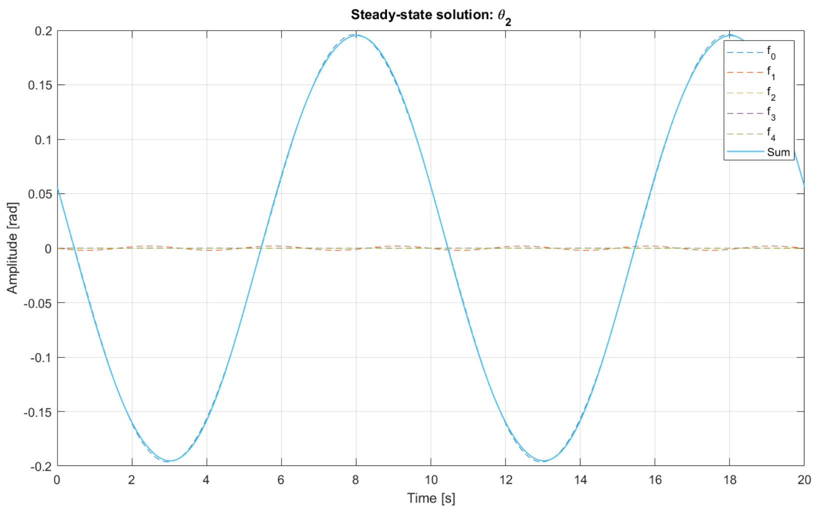
The steady-state response for $x_1(t)$ is given by:

$$x_{1_{ss}}(t) = \sum_{k=0}^{4} x_{1_{ss_k}}(t) = \sum_{k=0}^{4} |H_{1,3}(\Omega_k)| \frac{8}{\pi^2}(-1)^k \frac{\sin(\Omega_k t + \angle H_{1,3}(\Omega_k))}{(2k+1)^2}$$



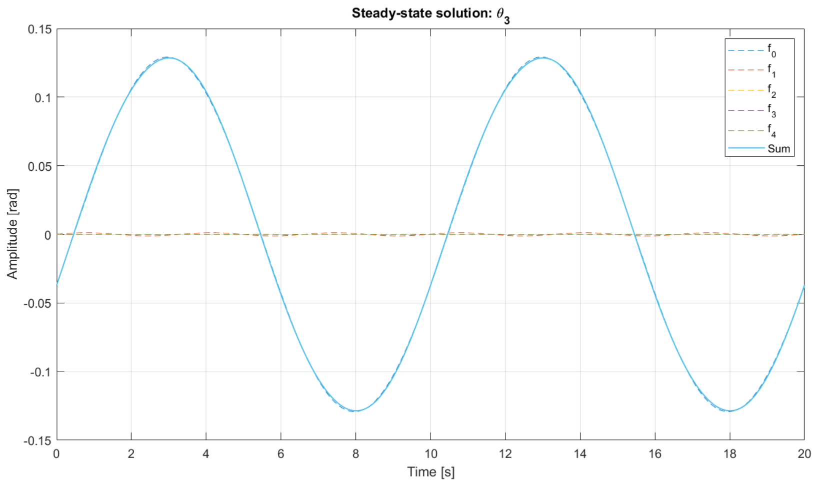
The steady-state response for $\theta_2(t)$ is given by:

$$\theta_{2_{ss}}(t) = \sum_{k=0}^{4} \theta_{2_{ss_k}}(t) = \sum_{k=0}^{4} |H_{2,3}(\Omega_k)| \frac{8}{\pi^2}(-1)^k \frac{\sin(\Omega_k t + \angle H_{2,3}(\Omega_k))}{(2k+1)^2}$$

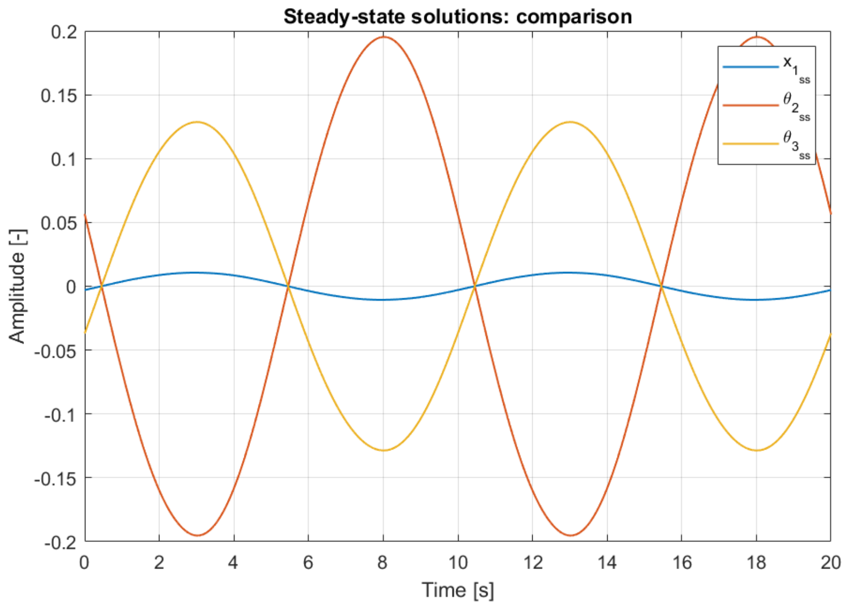


The steady-state response for $\theta_3(t)$ is given by:

$$\theta_{3_{ss}}(t) = \sum_{k=0}^{4} \theta_{3_{ss_k}}(t) = \sum_{k=0}^{4} |H_{3,3}(\Omega_k)| \frac{8}{\pi^2}(-1)^k \frac{\sin(\Omega_k t + \angle H_{3,3}(\Omega_k))}{(2k+1)^2}$$



The mechanical system works as a low-pass filter, judging by the fact that all steady-state solutions are given only by the lowest frequency component at f_0.
A comparison between all steady-state solutions is given:



4 Modal approach

The modal approach for the mechanical system analysis starts from a linear transformation of coordinates, from the independent variables $\underline{x}(t)$ to the modal coordinates $\underline{q}(t)$:

$$\underline{x}(t) = \begin{pmatrix} x_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = [\Phi] \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = [\Phi]\, \underline{q}(t)$$

Where $[\Phi]$ is the matrix of the mode shapes (computed in the undamped case):

$$[\Phi]_{3\times 3} = \left[\underline{X}_U^{(1)} \quad \underline{X}_U^{(2)} \quad \underline{X}_U^{(3)}\right] = \begin{bmatrix} X_{1,U}^{(1)} & X_{1,U}^{(2)} & X_{1,U}^{(3)} \\ \Theta_{2,U}^{(1)} & \Theta_{2,U}^{(2)} & \Theta_{2,U}^{(3)} \\ \Theta_{3,U}^{(1)} & \Theta_{3,U}^{(2)} & \Theta_{3,U}^{(3)} \end{bmatrix}$$

4.1 Equations of Motion and Frequency Response Matrix

It is possible to obtain a matrix EoM in terms of modal coordinates as:

$$[M^*] \cdot \underline{\ddot{x}} + [c^*] \cdot \underline{\dot{x}} + [k^*] \cdot \underline{x} = \underline{Q} \Longrightarrow [M^*] \cdot [\Phi]\, \underline{\ddot{q}} + [c^*] \cdot [\Phi]\, \underline{\dot{q}} + [k^*] \cdot [\Phi]\, \underline{q} = \underline{Q}$$

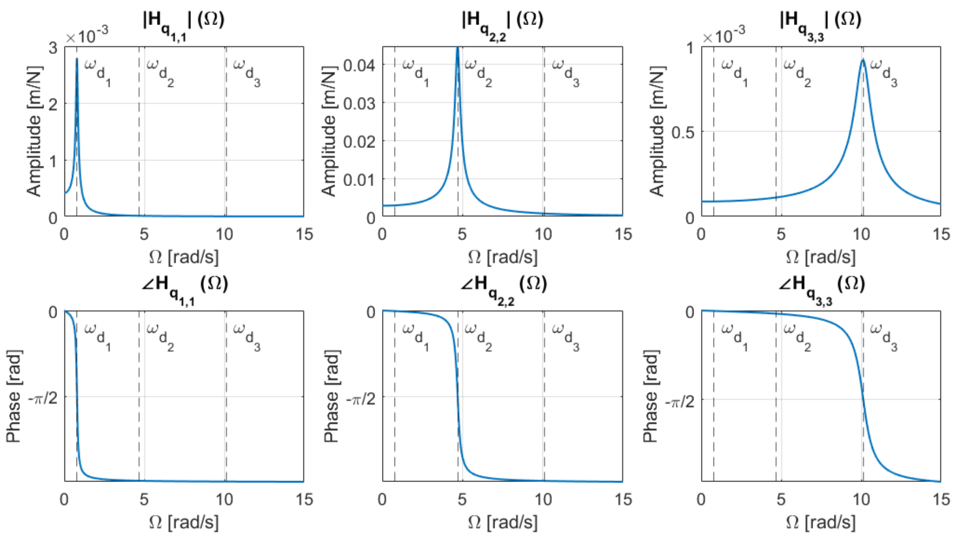
$$[\Phi]^T [M^*] [\Phi] \cdot \underline{\ddot{q}} + [\Phi]^T [c^*] [\Phi] \cdot \underline{\dot{q}} + [\Phi]^T [k^*] [\Phi] \cdot \underline{q} = [\Phi]^T \underline{Q}$$

$$[M_q] \cdot \underline{\ddot{q}} + [c_q] \cdot \underline{\dot{q}} + [k_q] \cdot \underline{x} = \underline{Q_q}$$

Similar to the FRM definition with independent variables, the frequency response matrix in modal coordinates $H_q(\Omega)$ is defined as:

$$H_q(\Omega) = \left[-\Omega^2 [M_q] + j\Omega [c_q] + [k_q]\right]^{-1} = \begin{bmatrix} H_{q_{1,1}}(\Omega) & 0 & 0 \\ 0 & H_{q_{2,2}}(\Omega) & 0 \\ 0 & 0 & H_{q_{3,3}}(\Omega) \end{bmatrix}$$

The frequency response matrix is diagonal, and its entries $H_{q_{i,i}}(\Omega)$ are the modal frequency responses:



4.2 Co-located modal FRF in A

The co-located FRF of the point A, in independent variables, has been obtained as:

$$\mathrm{FRF}_A(\Omega) = H_{1,3}(\Omega) - rH_{3,3}(\Omega)$$

Considering that the generic FRF $H_{j,k}(\Omega)$ is obtained from the modal FRFs as:

$$H_{j,k}\left(\Omega\right)=\sum_{i=1}^{3}X_{j,U}^{(i)}X_{k,U}^{(i)}H_{q_{i,i}}\Longrightarrow \mathrm{FRF}_A\left(\Omega\right)=\sum_{i=1}^{3}X_{1,U}^{(i)}\Theta_{3,U}^{(i)}H_{q_{i,i}}-r\sum_{i=1}^{3}\Theta_{3,U}^{(i)}\Theta_{3,U}^{(i)}H_{q_{i,i}}$$

The reconstruction with modal approach is identical to the original FRF: