VIBRATION ANALYSIS
AND VIBROACOUSTICS

VIBRATION ANALYSIS

Assignment 1 - A.Y. 2023/24

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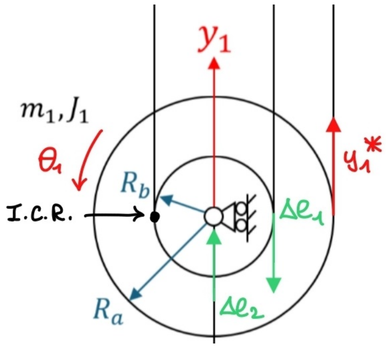
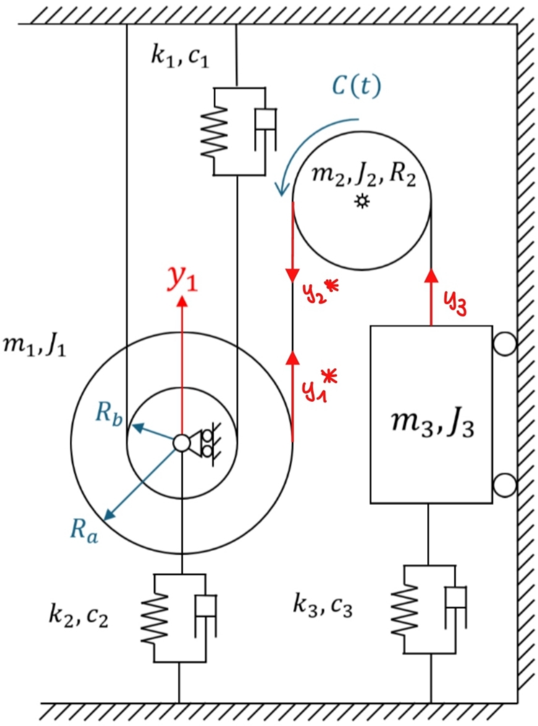
1 Question 1

1.1 Equation of Motion

The mechanical system in figure presents different physical variables and one independent variable. The motion of the mass m_1 is both translational and rotational, so both displacement y_1 and angle θ_1 are used to describe its energy forms. The presence of an inextensible string at the left of the disk of radius R_b determines an instantaneous center of rotation (I.C.R.) at the conjunction between string and disk.



The constraints on the disk of mass m_2 force it to rotate only, so the angle θ_2 is sufficient to describe its motion.
The mass m_3, on the other hand, can only translate vertically, so the displacement y_3 describes completely its motion.



By considering the displacements y_i positive upwards and the rotations θ_i positive counter-clockwise, the physical variables are defined in function of the independent variable y_1 as:

$$y_1 = R_b\theta_1 \implies \theta_1 = \frac{y_1}{R_b}, \quad \omega_1 = \dot{\theta}_1 = \frac{v_1}{R_b} = \frac{\dot{y_1}}{R_b}$$

$$y_1^* = -y_2^* \implies (R_a + R_b)\theta_1 = -R_2\theta_2$$

$$\theta_2 = \frac{R_a + R_b}{R_2}\frac{y_1}{R_b} = -\frac{R_a + R_b}{R_2 R_b}y_1, \quad \omega_2 = \dot{\theta}_2 = -\frac{R_a + R_b}{R_2 R_b}\dot{y_1}$$

$$y_3 = -y_2^* = -R_2\theta_2 = \frac{R_a + R_b}{R_b} y_1, \quad v_3 = \dot{y_3} = \frac{R_a + R_b}{R_b} \dot{y_1}$$

Regarding the elongations Δl_i of the springs and the altitudes h_i of the masses:

$$\Delta l_1 = -2y_1, \quad \Delta l_2 = y_1, \quad \Delta l_3 = y_3 = \frac{R_a + R_b}{R_b} y_1$$

$$h_1 = y_1, \quad h_3 = y_3 = \frac{R_a + R_b}{R_b} y_1$$

Let's consider the energy forms of the Lagrange equation one by one. The kinetic energy is given by two translational components and two rotational components:

$$E_K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_3 v_3^2 + \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} \omega_2^2 =$$

$$= \frac{1}{2} m_1 \dot{y_1}^2 + \frac{1}{2} m_3 \left(\frac{R_a + R_b}{R_b} \right)^2 \dot{y_1}^2 + \frac{1}{2} J_1 \left(\frac{1}{R_b} \right)^2 \dot{y_1}^2 + \frac{1}{2} J_2 \left(-\frac{R_a + R_b}{R_2 R_b} \right)^2 \dot{y_1}^2 =$$

$$= \frac{1}{2} \left(m_1 + \left(\frac{R_a + R_b}{R_b} \right)^2 m_3 + \frac{J_1}{R_b^2} + J_2 \left(\frac{R_a + R_b}{R_2 R_b} \right)^2 \right) \dot{y_1}^2 = \frac{1}{2} M_{eq} \dot{y_1}^2$$

The potential energy has three elastic components and two gravitational components:

$$V = \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2 + \frac{1}{2} k_3 \Delta l_3^2 + m_1 g h_1 + m_3 g h_3 =$$

$$= \frac{1}{2} k_1 \left(-2y_1 \right)^2 + \frac{1}{2} k_2 y_1^2 + \frac{1}{2} k_3 \left(\frac{R_a + R_b}{R_b} y_1 \right)^2 + m_1 g y_1 + m_3 g \left(\frac{R_a + R_b}{R_b} \right) y_1 =$$

$$= \frac{1}{2} \left(4k_1 + k_2 + \left(\frac{R_a + R_b}{R_b} \right)^2 k_3 \right) y_1^2 + \left(m_1 + \frac{R_a + R_b}{R_b} m_3 \right) g y_1 = \frac{1}{2} k_{eq} y_1^2 + m_{eq} g y_1$$

The dissipative function has three terms:

$$D = \frac{1}{2} c_1 \dot{\Delta l_1}^2 + \frac{1}{2} c_2 \dot{\Delta l_2}^2 + \frac{1}{2} c_3 \dot{\Delta l_3}^2 = \frac{1}{2} c_1 \left(-2\dot{y_1} \right)^2 + \frac{1}{2} c_2 \dot{y_1}^2 + \frac{1}{2} c_3 \left(\frac{R_a + R_b}{R_b} \dot{y_1} \right)^2 =$$

$$= \frac{1}{2} \left(4c_1 + c_2 + \left(\frac{R_a + R_b}{R_b} \right)^2 c_3 \right) \dot{y_1}^2 = \frac{1}{2} c_{eq} \dot{y_1}^2$$

The only work contribution is given by the torque $C(t)$ applied on the disk m_2:

$$\delta W = C(t) \delta\theta_2 = C(t) \left(-\frac{R_a + R_b}{R_2 R_b} \right) \delta y_1 = Q_{y_1} \delta y_1$$

It is now possible to apply the energies into the Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial E_K}{\partial \dot{y_1}} \right) = \frac{d}{dt} \left(\frac{\partial}{\partial \dot{y_1}} \left(\frac{1}{2} M_{eq} \dot{y_1}^2 \right) \right) = \frac{d}{dt} \left(M_{eq} \dot{y_1} \right) = M_{eq} \ddot{y_1}$$

$$\frac{\partial E_K}{\partial y_1} = 0$$

$$\frac{\partial D}{\partial \dot{y_1}} = \frac{\partial}{\partial \dot{y_1}} \left(\frac{1}{2} c_{eq} \dot{y_1}^2 \right) = c_{eq} \dot{y_1}$$

$$\frac{\partial V}{\partial y_1} = \frac{\partial}{\partial y_1} \left(\frac{1}{2} k_{eq} y_1^2 + m_{eq} g y_1 \right) = k_{eq} y_1 + m_{eq} g$$

Giving the result:

$$\frac{d}{dt} \left(\frac{\partial E_K}{\partial \dot{y_1}} \right) + \frac{\partial E_K}{\partial y_1} + \frac{\partial D}{\partial \dot{y_1}} + \frac{\partial V}{\partial y_1} = Q_{y_1} \Longrightarrow M_{eq} \ddot{y_1} + c_{eq} \dot{y_1} + k_{eq} y_1 = \left(-\frac{R_a + R_b}{R_2 R_b} \right) C(t) - m_{eq} g$$

1.2 Natural frequency

The natural frequency is a parameter of the system (in free motion) in the undamped case:

$$M_{eq}\ddot{y}_1 + k_{eq} y_1 = 0 \longrightarrow M_{eq}\lambda^2 + k_{eq} = 0 \Longrightarrow \lambda_{1,2} = \pm j\sqrt{\frac{k_{eq}}{M_{eq}}} = \pm j\omega_0 \Longrightarrow \omega_0 = \sqrt{\frac{k_{eq}}{M_{eq}}} = 9.8554 \text{ rad/s}$$

1.3 Adimensional damping factor and damped frequency

By considering also the effect of the dampers, the adimensional damping factor is defined as:

$$h = \frac{c_{eq}}{2M_{eq}\omega_0} = \frac{\alpha}{\omega_0} = 0.1057$$

together with the damped frequency:

$$\omega_d = \omega_0\sqrt{1-h^2} = 9.8002 \text{ rad/s} \ \approx \omega_0$$

2 Question 2

2.1 Free motion response

A damped system in free motion is described by the homogeneous equation of motion:

$$M_{eq}\ddot{y} + c_{eq}\dot{y} + k_{eq}y = 0$$

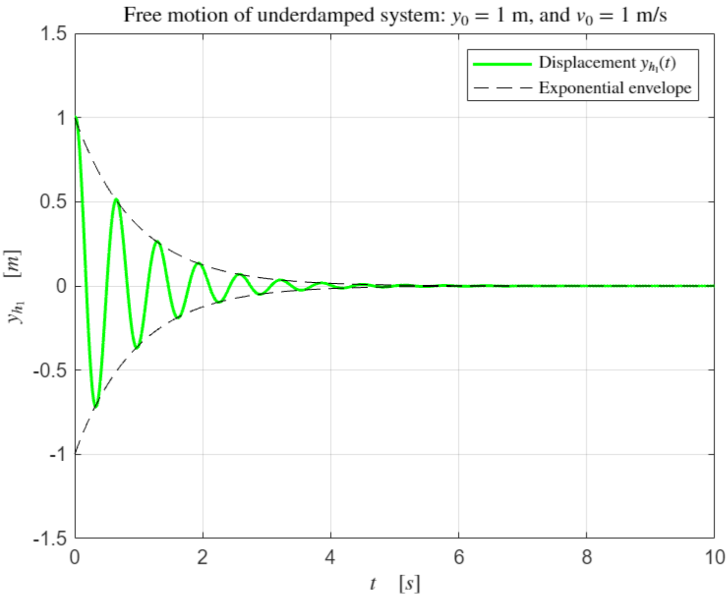
Its solution $y_{1_h}(t)$ is obtained by solving the characteristic equation:

$$M_{eq}\lambda^2 + c_{eq}\lambda + k_{eq} = 0$$

For the underdamped case (given by the small value of $h \approx 10\%$), the homogeneous solution $y_{1_h}(t)$ (in second form) is given by decaying sinusoids:

$$M_{eq}\lambda^2 + c_{eq}\lambda + k_{eq} = 0 \implies \lambda_{1,2} = -\alpha \pm j\omega_0\sqrt{1-h^2} = -\alpha \pm j\omega_d$$

$$\implies y_{h_1}(t) = e^{-\alpha t}\left[y_0\cos(\omega_d t) + \frac{v_0 + \alpha y_0}{\omega_d}\sin(\omega_d t)\right]$$

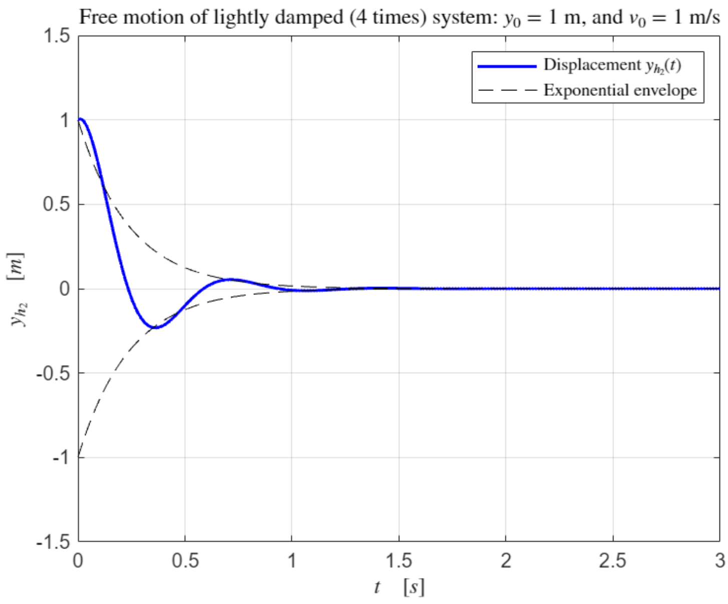


The initial conditions y_0 and v_0 represent the displacement and velocity values at $t = 0$ and they determine the coefficients of the two sinusoidal components. Considering the small value of $h \approx 10\%$, the system is underdamped, so the damped frequency $\omega_d \approx \omega_0$ as said in section 1.3. The decaying time is given by $\frac{5}{\alpha} \approx 4.80$ seconds, which gives the "slowly" decaying trend to the plot. The steady-state of the system is zero because of the free motion scenario.

2.2 Free motion response ($h \to 4h$)

The new value of $h_2 = 8h = 0.4228$, which means the system is lightly damped. The form of the homogeneous solution is the same, only the coefficients change. With respect to the previous point, $\alpha_2 = 4\alpha$ and $\omega_{d_2} = \omega_0\sqrt{1-h_2^2} = 8.9311$ rad/s:

$$y_{h_2}(t) = e^{-\alpha_2 t}\left[y_0\cos(\omega_{d_2} t) + \frac{v_0 + \alpha_2 y_0}{\omega_{d_2}}\sin(\omega_{d_2} t)\right]$$



The decaying time is given by $\frac{5}{\alpha_2} \approx 1.20$ seconds, which gives a faster (with respect to the previous case) decaying trend to the plot. The steady-state of the system is zero because of the free motion scenario.

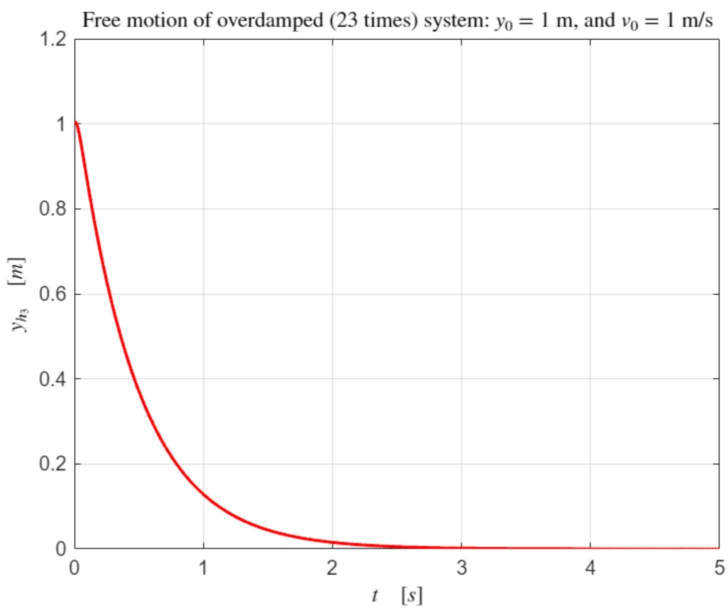
2.3 Free motion response ($h \to 23h$)

The new value of $h_3 = 23h = 2.4312$, which means the system is overdamped. In this case, the solutions λ_1 and λ_2 of the characteristic equation are real:

$$\lambda_1 = -\alpha_3 - \sqrt{\alpha_3^2 - \omega_0^2} = -a_1, \quad \lambda_2 = -\alpha_3 + \sqrt{\alpha_3^2 - \omega_0^2} = -a_2$$

The homogeneous solution $y_{1_h}(t)$ does not oscillate anymore:

$$y_{h_3}(t) = -\left(\frac{v_0 + a_2 y_0}{a_1 - a_2}\right) e^{-a_1 t} + \left(\frac{v_0 + a_1 y_0}{a_1 - a_2}\right) e^{-a_2 t}$$



The decaying time is mainly given by $\frac{5}{a_2} \approx 2.36$ seconds. The steady-state of the system is zero because of the free motion scenario.

3 Question 3

3.1 Forced motion: frequency response function

The forced motion adds the external forces and torques contribution to the equation of the free vibrations:

$$M_{eq}\ddot{y} + c_{eq}\dot{y} + k_{eq}y = F(t) = F_0 \cos(\Omega t + \varphi)$$

The complete time solution is the sum of the homogeneous solution $y_{1_h}(t)$ (from section 2.1) and a particular solution given by:

$$y_{1_p}(t) = Y_0 \cos(\Omega t + \varphi + \beta)$$

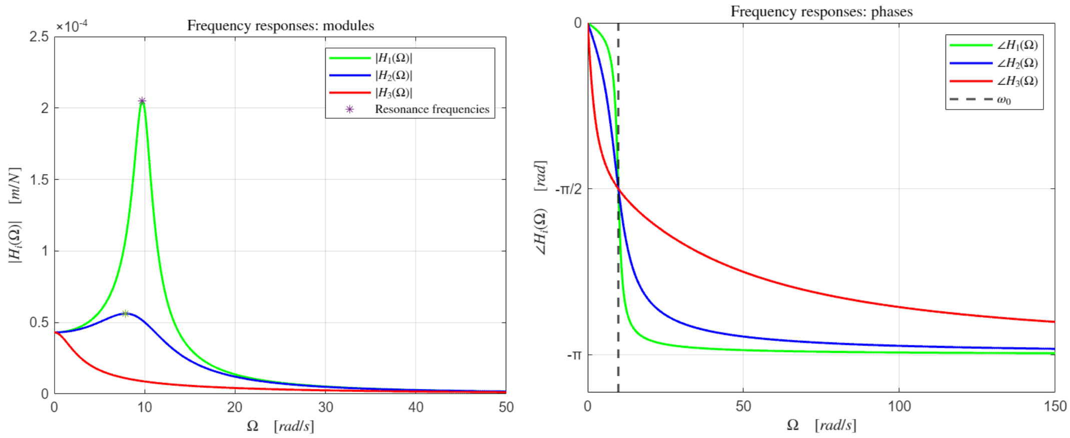
Or equivalently in a complex form:

$$\tilde{y}_{1_p}(t) = \tilde{Y}_0 e^{j(\Omega t + \varphi)}, \quad \tilde{Y}_0 = |\tilde{Y}_0| e^{j\beta} \in \mathbb{C}$$

The frequency response function $H(\Omega)$ compares the system result $\tilde{Y}_0$ and the external force contribution F_0: in general, an amplification or attenuation $|\tilde{Y}_0|$ and a phase shift β will be introduced by the mechanical system.

$$H(\Omega) = \frac{\tilde{Y}_0}{F_0} = \frac{\frac{1}{M_{eq}}}{\omega_0^2 - \Omega^2 + j2\alpha\Omega}$$

Where ω_0 is the natural frequency and $\alpha = \frac{c_{eq}}{2M_{eq}}$ represents the decaying contribution of the dampers.



In the figures above, the three different scenarios described in question 2 are plotted. The system normally presents (first case in green) an underdamped frequency response: an high quality factor describes the narrow bell centered in $\omega_d \approx \omega_0$, where the module response is almost 8 times higher than the static response ($H(\Omega = 0) = 0.43 \cdot 10^{-4}$ m/N), though the system works like an attenuator for all frequencies (module < 1). The phase response is given by a steep curve which tends to the one of the undamped case, in which an abrupt counterphase occurs in ω_0 (dotted black line).
In the second case (in blue) $h_2 = 4h$, so the damping effect is more present. The module response is more flat, the quality factor and the damped frequency are lower, and the phase response curve is less steep.
In the third case ($h_3 = 23h$, in red) the damping effect is dominant: there is no natural frequency (the system does not oscillate) and phase shift tends to reach $-\frac{\pi}{2}$ more steeply and $-\pi$ way slower.

3.2 Forced motion: complete time response

The complete time response $y_1(t)$ of a forced system is the sum of the homogeneous solution $y_{h_2}(t)$ (defined in section 2.2) and the particular solution $y_{p_2}(t)$ (using the FRF defined in section 3.1):

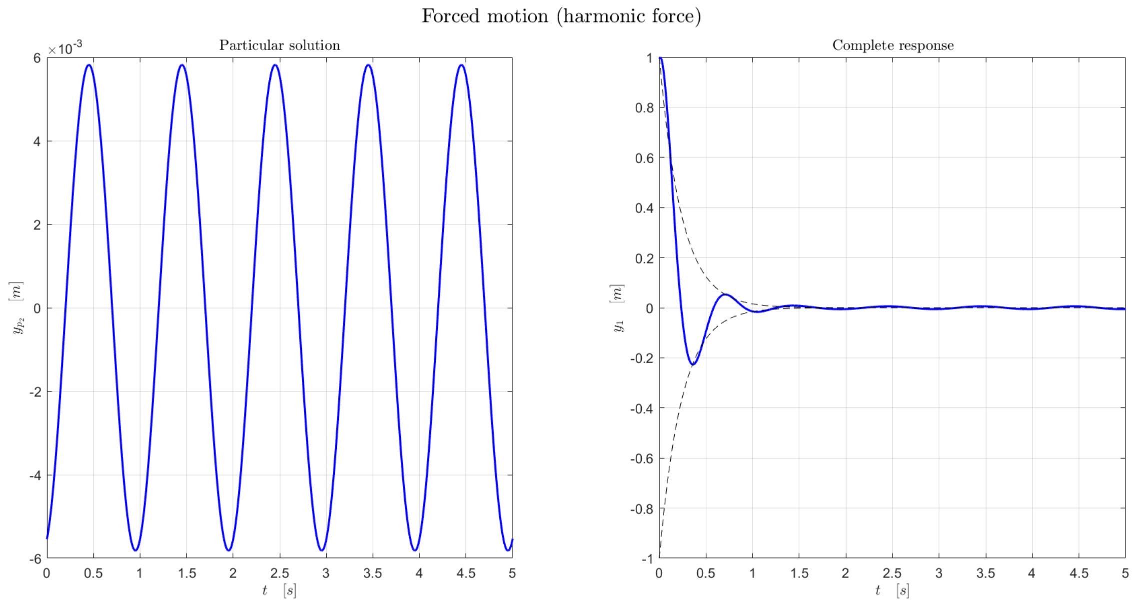
$$y_{h_2}(t) = e^{-\alpha_2 t}\left[y_0\cos(\omega_{d_2}t) + \frac{v_0+\alpha_2 y_0}{\omega_{d_2}}\sin(\omega_{d_2}t)\right]$$

$$y_{p_2}(t) = Y_0\cos(\Omega t+\varphi+\beta) = |H_2(\Omega)|F_0\cos\left(\Omega t+\varphi+\angle H_2(\Omega)\right)$$

By considering a simple harmonic torque ($A = 25$ N·m, $f = 1$ Hz, $\varphi = \pi/3$ rad):

$$C(t) = A\cos(2\pi f t+\varphi) \Longrightarrow F(t) = F_0\cos(2\pi f t+\varphi) = -\frac{R_a+R_b}{R_2R_b}\cdot A\cos(2\pi f t+\varphi)$$

$$\Longrightarrow y_{p_2}(t) = |H_2(2\pi f)|F_0\cos\left(2\pi f t+\varphi+\angle H_2(2\pi f)\right)$$



3.3 Steady-state response for two frequencies

The steady-state response of the system consists in the form that $y_1(t)$ has for $t \to \infty$. Considering that the homogeneous solution $y_{h_1}(t) \to 0$, the steady-state response is given only by the particular solution $y_{p_1}(t)$.

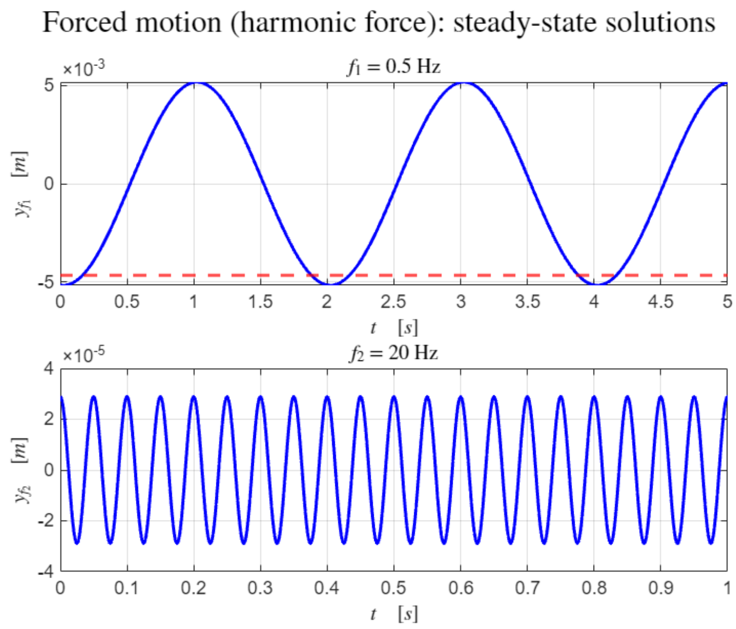
For two different harmonic torques (by considering the first scenario with $h = 0.1057$):

$$C(t) = 25\cos(2\pi 0.5t+0) \Longrightarrow y_{p_1}(t) = |H_1(2\pi 0.5)|F_0\cos\left(2\pi 0.5t+\angle H_1(2\pi 0.5)\right)$$

$$C(t) = 25\cos(2\pi 20t+0) \Longrightarrow y_{p_1}(t) = |H_1(2\pi 20)|F_0\cos\left(2\pi 20t+\angle H_1(2\pi 20)\right)$$

For a torque statically applied, instead:

$$C(t) = 25\cos(2\pi 0t+0) = 25 \Longrightarrow y_{p_1}(t) = |H_1(0)|F_0\cos\left(2\pi 0t+\angle H_1(2\pi 0)\right) = |H_1(0)|F_0$$

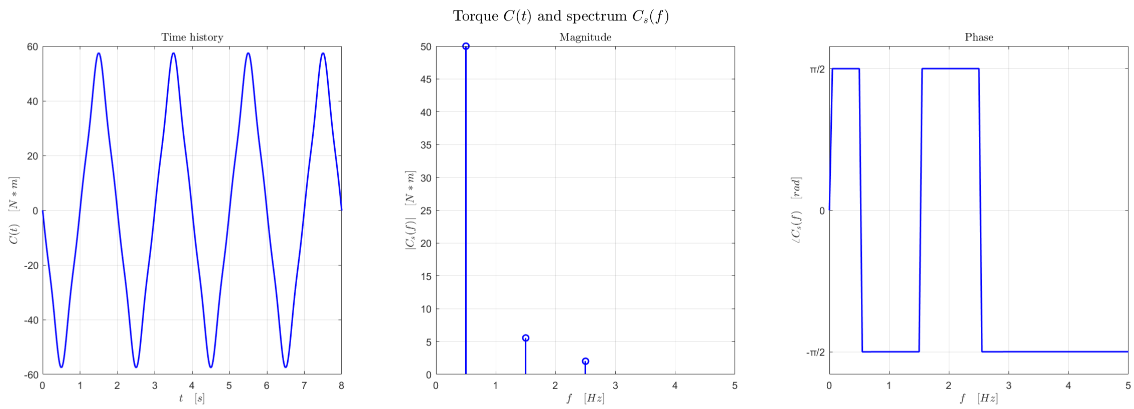


From the frequency response module $|H_1(\Omega)|$ graph, obtained in section 3.1, it is possible to compare the amplitudes of the different steady-state solutions with the static response. For the first frequency $f_1 = 0.5$ Hz (equivalent to π rad/s), the amplitude of the steady-state solution (in blue) is close to the static response value $y_{1_{st}} = -4.7 \cdot 10^{-3}$ m (in red): this frequency is located in the quasi-static zone of the frequency response function, where the magnitude is almost constant. For the second frequency $f_2 = 20$ Hz (equivalent to 125.66 rad/s), the amplitude of the steady-state solution is way smaller than the static response: this frequency is located in the seismographic zone of the frequency response function, where the magnitude is way smaller compared to the quasi-static zone and tends to zero.

3.4 Superposition of three harmonic torques

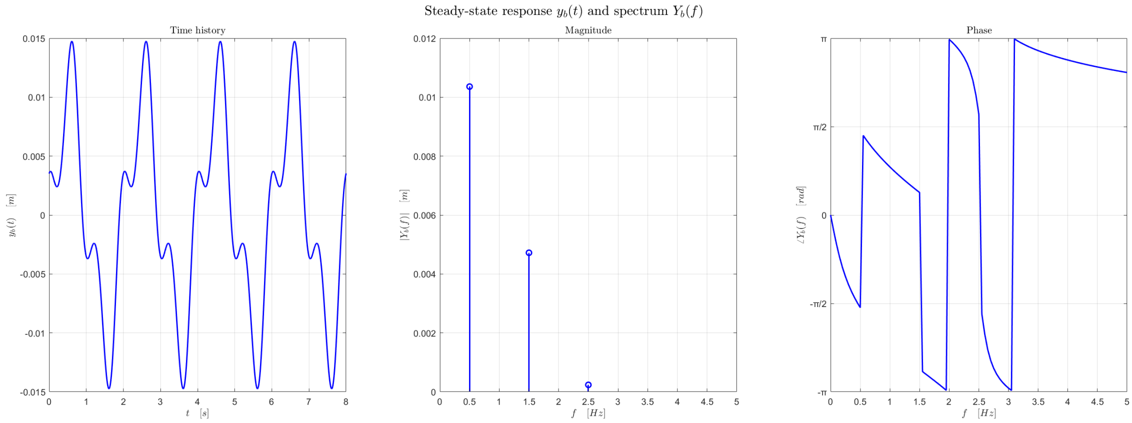
The linearity of the system allows to obtain the output displacement as sum of different input contributions. In case of a torque given by the superposition of three harmonics:

$$C(t) = \sum_{k=1}^{3} B_k \cos(2\pi f_k t + \varphi_k) = 50 \cos\left(2\pi 0.5t + \frac{\pi}{2}\right) + 5.55 \cos\left(2\pi 1.5t - \frac{\pi}{2}\right) + 2 \cos\left(2\pi 2.5t + \frac{\pi}{2}\right)$$



The overall steady-state response is the superposition of the three particular solutions:

$$y_{p_1}(t) = 50\left(-\frac{R_a+R_b}{R_2R_b}\right)|H_1(2\pi 0.5)|\cos\left(2\pi 0.5t+\frac{\pi}{2}+\angle H_1(2\pi 0.5)\right)+$$
$$+5.55\left(-\frac{R_a+R_b}{R_2R_b}\right)|H_1(2\pi 1.5)|\cos\left(2\pi 1.5t-\frac{\pi}{2}+\angle H_1(2\pi 1.5)\right)+$$
$$+2\left(-\frac{R_a+R_b}{R_2R_b}\right)|H_1(2\pi 2.5)|\cos\left(2\pi 2.5t+\frac{\pi}{2}+\angle H_1(2\pi 2.5)\right)$$



The torque waveform is mainly given by the first harmonic (frequency at 0.5 Hz), considering the amplitude coefficients (coherently represented in the magnitude plot).
Regarding the steady-state solution, the frequency response module $|H_1(\Omega)|$ modifies each harmonic contribution of the torque, giving a dominant first harmonic and a more present second harmonic. The relative amplitude difference between the harmonics f_1 and f_2 is higher in the input torque spectrum (f_1 almost 10 times f_2) than in the steady-state spectrum (f_1 at least 2 times f_2). The third harmonic is negligible in the particular solution (amplitude approximately 19 times lower than the second harmonic amplitude and 43 times lower than the first harmonic amplitude).
The frequency response phase graph is coherent with the model, considering that each harmonic contribution results in sudden phase shift from the frequency response phase graph. In particular, two phase wraps occur around 2 Hz and 3 Hz.

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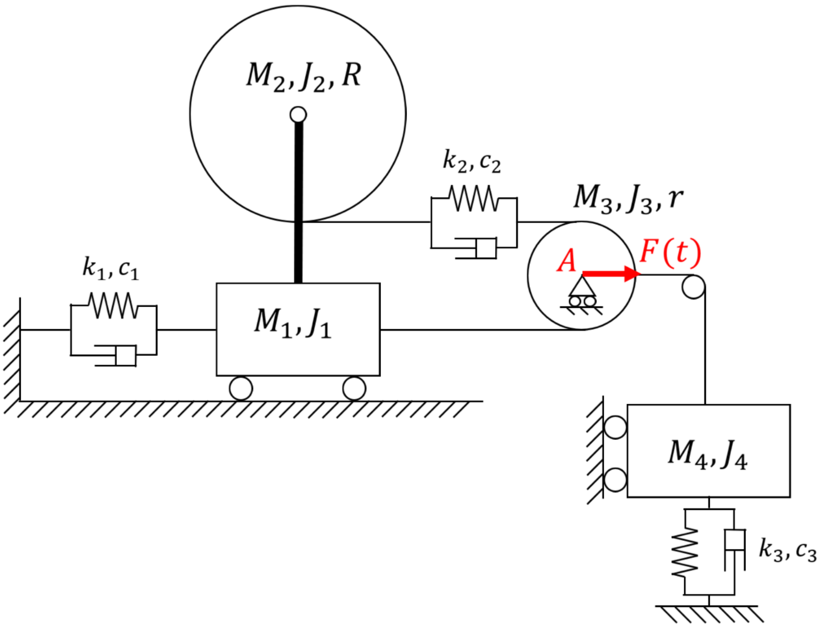
1 Equations of Motion and system matrices

In order to describe the mechanical system in figure, a reference system convention is defined: all displacements along the positive x and positive y directions are positive, all counterclockwise rotations are positive and all springs/dampers elongation are positive.



The number of degrees of freedom (dof) has to be determined as:

$$n_{dof} = 3n_b - n_{con}$$

Where n_b is the total number of rigid bodies (4 in this system) and n_{con} the number of constraints. An equation of motion (EoM) for each dof allow to completely describe the mechanical system. Each EoM is the result of the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{x}}\right) - \frac{\partial E_K}{\partial x} + \frac{\partial D}{\partial \dot{x}} + \frac{\partial V}{\partial x} = Q_x$$

In which x represents the single independent variable of a mechanical system.
By combining all contributions in a matrix form:

$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

Where all independent variables have been gathered in the vector $\underline{x}$, as well as their first time derivatives in $\underline{\dot{x}}$.

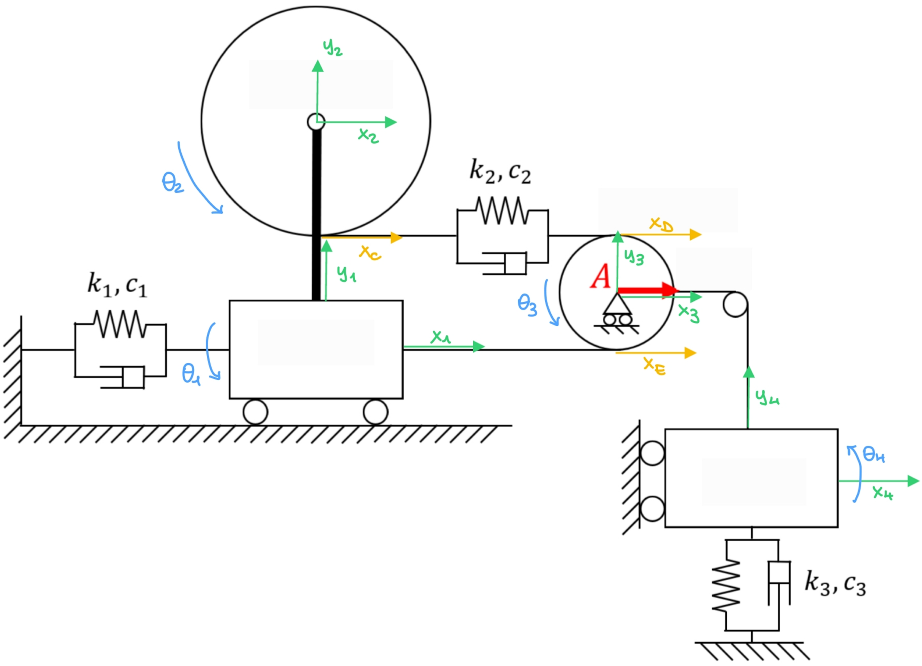
1.1 Equations of Motion around the equilibrium position

The first goal is to determine the number of dof by evaluating all constraints:

- M_2 is constrained (through a hinge) to a mass-less vertical beam, rigidly connected to the mass M_1
  - hinge $\longrightarrow y_2 = y_1$
  - rigid connection $\longrightarrow \dot{x}_2 = \dot{x}_1$
- M_1 can only slide horizontally
  - slider $\longrightarrow y_1 = 0,\ \theta_1 = 0$
- M_3 can slide horizontally and rotate through an inextensible rope that connects the disk to both M_2 and M_1
  - slider and hinge $\longrightarrow y_3 = 0$

- inextensible rope $\longrightarrow \dot{x}_1 = \dot{x}_E$

- M_4 can slide vertically and is rigidly connected to M_3 through an inextensible rope
  - slider $\longrightarrow x_4 = 0,\ \theta_4 = 0$
  - inextensible rope $\longrightarrow y_4 = -x_3$



To conclude, the number of constraints n_{con} is equal to $2 + 2 + 2 + 3$, so the number of dof is:

$$n_{dof} = 3n_b - n_{con} = 12 - 9 = 3$$

After that, it is necessary to describe all existing relations between independent variables (3 in this system) and physical variables:

$$\begin{cases} \dot{x}_2 = \dot{x}_1 \\ \dot{x}_1 = \dot{x}_E \end{cases} \Longrightarrow x_1 = x_2 = x_E$$

$$\dot{x}_C = \dot{x}_2 + R\dot{\theta}_2, \quad \dot{x}_D = \dot{x}_1 - 2r\dot{\theta}_3, \quad y_4 = -x_3 = -(x_E - r\theta_3)$$

$$\Longrightarrow \begin{cases} x_1 = x_2 = x_E \\ \dot{x}_C = \dot{x}_2 + R\dot{\theta}_2 \\ \dot{x}_D = \dot{x}_1 - 2r\dot{\theta}_3 \\ y_4 = -x_1 + r\theta_3 \end{cases}$$

Springs and dampers appear in the Lagrange equations by means of elongations Δl and their first time derivative $\dot{\Delta l}$:

$$\begin{cases} \dot{\Delta l}_1 = \dot{x}_1 \\ \dot{\Delta l}_2 = \dot{x}_D - \dot{x}_C = -R\dot{\theta}_2 - 2r\dot{\theta}_3 \\ \dot{\Delta l}_3 = \dot{y}_4 = -\dot{x}_1 + r\dot{\theta}_3 \end{cases} \Longrightarrow \begin{cases} \Delta l_1 = x_1 \\ \Delta l_2 = -R\theta_2 - 2r\theta_3 \\ \Delta l_3 = -x_1 + r\theta_3 \end{cases}$$

Being the system in static equilibrium position, all springs pre-load are neglected.

1.1.1 Kinetic energy E_K

It is now possible to compute, in matrix form, all energies of the Lagrange equation.
Regarding the kinetic energy, four rigid bodies may contribute to the total E_K with four translations and four rotations. In this system, all bodies can translate but not rotate (only M_2 and M_3):

$$E_K = \frac{1}{2}M_1 v_1^2 + \frac{1}{2}M_2 v_2^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}M_3 v_3^2 + \frac{1}{2}J_3\omega_3^2 + \frac{1}{2}M_4 v_4^2$$

Let's gather all physical coordinates (and their first time derivatives) in two column vectors $\underline{z}$ (and $\underline{\dot{z}}$ respectively):

$$\underline{z}_{6\times 1} = (x_1, x_2, \theta_2, x_3, \theta_3, y_4)^T\ , \quad \underline{\dot{z}}_{6\times 1} = (v_1, v_2, \omega_2, v_3, \omega_3, v_4)^T$$

By defining the following physical mass matrix $[M]$ as:

$$[M]_{6\times 6} = \begin{bmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_4 \end{bmatrix}$$

It is now possible to express the total kinetic energy E_K in matrix form as:

$$E_K = \frac{1}{2}\underline{\dot{z}}^T [M] \underline{\dot{z}}$$

It is of interest to get the energy in function of the independent variables:

$$\underline{x}_{3\times 1} = (x_1, \theta_2, \theta_3)^T$$

expressed respectively in m, rad and rad. Let's introduce a relation between physical variables $\underline{z}$ and independent ones $\underline{x}$ by using their first time derivative:

$$\underline{\dot{z}} = \left(\frac{\partial \underline{z}}{\partial \underline{x}}\right)\underline{\dot{x}} = [\Lambda_M]\,\underline{\dot{x}}$$

Where $[\Lambda_M]$ is the Jacobian matrix and describes the relation between velocities v and angular velocities ω with respect to the independent variables.
Its entries are obtained by looking at the following table:

	$\dot{x}_1$	$\dot{\theta}_2$	$\dot{\theta}_3$
v_1	1	0	0
v_2	1	0	0
ω_2	0	1	0
v_3	1	0	-r
ω_3	0	0	1
v_4	-1	0	r

So the Jacobian matrix is defined as:

$$[\Lambda_M]_{6\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -r \\ 0 & 0 & 1 \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_M]$ in the kinetic energy formulation:

$$E_K = \frac{1}{2}\underline{\dot{z}}^T [M] \underline{\dot{z}} = \frac{1}{2}\underline{\dot{x}}^T [\Lambda_M]^T [M][\Lambda_M]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T [M^*]\underline{\dot{x}}$$

Where the mass matrix $[M^*]$ is defined as:

$$[M^*]_{3\times 3} = [\Lambda_M]^T [M][\Lambda_M] = \begin{bmatrix} M_1 + M_2 + M_3 + M_4 & 0 & -r\,[M_3 + M_4] \\ 0 & J_2 & 0 \\ -r\,(M_3 + M_4) & 0 & J_3 + r^2\,(M_3 + M_4) \end{bmatrix}$$

1.1.2 Potential energy V

Regarding the potential energy, the springs give a contribution in terms of elastic energy, while the mass M_4 gives a contribution in terms of gravitational energy:

$$V = V_{el} + V_g = \frac{1}{2}k_1\Delta l_1^2 + \frac{1}{2}k_2\Delta l_2^2 + \frac{1}{2}k_3\Delta l_3^2 + \frac{1}{2}M_4 g y_4$$

The gravitational term can be neglected due to the linearity of the system, so it does not affect the Lagrange equations, i.e. : $V_g = 0$.
Similarly to the kinetic case, a matrix approach is needed and the Jacobian matrix $[\Lambda_k]$ describes the relation between elongations Δl and independent variables.
By defining the physical stiffness matrix $[k]$ and the elongation vector $\underline{\Delta l}$ as:

$$[k]_{3\times3} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad \underline{\Delta l}_{3\times1} = \{\Delta l_1, \Delta l_2, \Delta l_3\}^T$$

It is possible to express the elastic potential energy in matrix form as:

$$V = V_{el} = \frac{1}{2}\underline{\Delta l}^T[k]\underline{\Delta l}$$

The Jacobian matrix $[\Lambda_k]$ entries are obtained by looking at the following table:

	x_1	θ_2	θ_3
Δl_1	1	0	0
Δl_2	0	-R	-2r
Δl_3	-1	0	r

So the Jacobian matrix is defined as:

$$[\Lambda_k]_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R & -2r \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_k]$ in the potential energy formulation:

$$V = V_{el} = \frac{1}{2}\underline{\dot{z}}^T[k]\underline{\dot{z}} = \frac{1}{2}\underline{\dot{x}}^T[\Lambda_k]^T[k][\Lambda_k]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[k^*]\underline{\dot{x}}$$

Where the stiffness matrix $[k^*]$ is defined as:

$$[k^*]_{3\times3} = [\Lambda_k]^T[k][\Lambda_k] = \begin{bmatrix} k_1+k_3 & 0 & -rk_3 \\ 0 & R^2k_2 & 2Rk_2r \\ -rk_3 & 2Rk_2r & 4k_2r^2+k_3r^2 \end{bmatrix}$$

1.1.3 Dissipative energy D

Regarding the dissipative energy, the dampers give a contribution in terms of first time derivative of the elongations:

$$D = \frac{1}{2}c_1\dot{\Delta l_1}^2 + \frac{1}{2}c_2\dot{\Delta l_2}^2 + \frac{1}{2}c_3\dot{\Delta l_3}^2$$

Similarly to the previous cases, a matrix approach is needed and the Jacobian matrix $[\Lambda_c]$ describes the relation between the elongations derivatives $\dot{\Delta l}$ and independent variables.
By defining the physical damping matrix $[c]$ and the elongation derivatives vector $\underline{\dot{\Delta l}}$ as:

$$[c]_{3\times3} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}, \quad \underline{\dot{\Delta l}}_{3\times1} = \{\dot{\Delta l_1}, \dot{\Delta l_2}, \dot{\Delta l_3}\}^T$$

It is possible to express the dissipative energy in matrix form as:

$$D = \frac{1}{2}\underline{\dot{\Delta l}}^T[k]\underline{\dot{\Delta l}}$$

The Jacobian matrix $[\Lambda_c]$ entries are the very same of $[\Lambda_k]$:

$$[\Lambda_c]_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R & -2r \\ -1 & 0 & r \end{bmatrix}$$

By using the relation described by the Jacobian matrix $[\Lambda_c]$ in the potential energy formulation:

$$D = \frac{1}{2}\underline{\dot{z}}^T[c]\underline{\dot{z}} = \frac{1}{2}\underline{\dot{x}}^T[\Lambda_c]^T[c][\Lambda_c]\underline{\dot{x}} = \frac{1}{2}\underline{\dot{x}}^T[c^*]\underline{\dot{x}}$$

Where the damping matrix $[c^*]$ is defined as:

$$[c^*]_{3\times3} = [\Lambda_c]^T[c][\Lambda_c] = \begin{bmatrix} c_1+c_3 & 0 & -rc_3 \\ 0 & R^2c_2 & 2Rc_2r \\ -rc_3 & 2Rc_2r & 4c_2r^2+c_3r^2 \end{bmatrix}$$

1.1.4 Virtual works δW

The principle of virtual works declares that an external force, applied at a given force point generates an infinitesimal work δW proportional to an infinitesimal displacement δx. In this system, the only existing external force $F(t)$ is applied horizontally at the point A, resulting in an infinitesimal displacement δx_3 and an infinitesimal work:

$$\delta W = F \cdot \delta x_3$$

Similarly to the previous cases, it is of interest to express the infinitesimal displacement δx_3 as function of the infinitesimal independent variables $\underline{\delta x}$ by means of Jacobian vector $[\Lambda_F]$:

$$\delta x_3 = \delta x_1 - \delta\theta_3 r = (1, 0, -r)\begin{pmatrix} \delta x_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{pmatrix} = [\Lambda_F]_{1\times3} \cdot \underline{\delta x}_{3\times1}$$

$$\Longrightarrow \delta W = F\delta x_1 - F\delta\theta_3 r = F \cdot [\Lambda_F] \cdot \underline{\delta x}$$

For the Lagrange equation, it is possible to identify the Lagrangian vector $\underline{Q}$ as:

$$\underline{Q}_{3\times1} = \begin{pmatrix} F \\ 0 \\ -rF \end{pmatrix} = [\Lambda_F]^T F$$

1.1.5 Lagrange equations

It is now possible to express the Lagrange equations in matrix form:

$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T - \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T + \left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T + \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = \underline{Q}$$

$$\left\{\frac{d}{dt}\left(\frac{\partial E_K}{\partial \underline{\dot{x}}}\right)\right\}^T = [M^*] \cdot \underline{\ddot{x}}\,, \quad \left\{\frac{\partial E_K}{\partial \underline{x}}\right\}^T = \underline{0}$$

$$\left\{\frac{\partial D}{\partial \underline{\dot{x}}}\right\}^T = [c^*] \cdot \underline{\dot{x}}, \quad \left\{\frac{\partial V}{\partial \underline{x}}\right\}^T = [k^*] \cdot \underline{x}$$

The total matrix expression of all Equations of Motion is:

$$[M^*] \cdot \underline{\ddot{x}} + [c^*] \cdot \underline{\dot{x}} + [k^*] \cdot \underline{x} = \underline{Q}$$

1.2 Eigenfrequencies and eigenvectors

1.2.1 Undamped system

The eigenfrequencies and eigenvectors computation problem consists in analyzing the system as undamped and in free motion, which results in setting $[c^*] = [0]$ and $\underline{Q} = \underline{0}$ in the matrix expression of the Equations of Motion:

$$[M^*] \cdot \underline{\ddot{x}} + [k^*] \cdot \underline{x} = \underline{0}$$

By assuming as a solution $\underline{x} = \underline{X}e^{\lambda t}$, the equation becomes:

$$\left(\lambda^2 [M^*] + [k^*]\right) \underline{X} = \underline{0}$$

The non-trivial solution consists in solving the characteristic equation:

$$\det\left(\lambda^2 [M^*] + [k^*]\right) = 0 \Longrightarrow \lambda^2 = -[M^*]^{-1} [k^*]$$

By solving the above expression for λ, six imaginary conjugate values are obtained:

$$\lambda_{1,4} = \pm j\omega_{0_1} = \pm j0.8017, \quad \lambda_{2,5} = \pm j\omega_{0_2} = \pm j4.6974, \quad \lambda_{3,6} = \pm j\omega_{0_3} = \pm j10.1326$$

It is of interest to consider real frequencies (positive values), so it is possible to identify the three natural angular frequencies of the system as:

$$\omega_{0_1} = \mathrm{Im}\{\lambda_1\} = 0.8017 \text{ rad/s}, \quad \omega_{0_2} = \mathrm{Im}\{\lambda_2\} = 4.6974 \text{ rad/s} \quad \omega_{0_3} = \mathrm{Im}\{\lambda_3\} = 10.1326 \text{ rad/s}$$

By substituting the solutions λ_i, λ_{i+3} (for $i = 1, 2, 3$) to the equation to solve, it is possible to find the eigenvectors $\underline{X}_U^{(i)}$, responses of each independent variable for the *i-th* natural frequency ω_{0_i}:

$$\left({\lambda_i}^2 [M^*] + [k^*]\right) \underline{X}_U^{(i)} = \underline{0}$$

Where:

$$\underline{X}_U^{(i)} = \begin{pmatrix} X_{1,U}^{(i)} \\ \Theta_{2,U}^{(i)} \\ \Theta_{3,U}^{(i)} \end{pmatrix}$$

The eigenvectors for each ω_{0_i}, normalized with respect to the first component $X_{1,U}^{(i)} = 1$, are:

$$\underline{X}_U^{(1)} = \begin{pmatrix} X_{1,U}^{(1)} \\ \Theta_{2,U}^{(1)} \\ \Theta_{3,U}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -19.0479 \\ 12.4436 \end{pmatrix}, \quad \underline{X}_U^{(2)} = \begin{pmatrix} X_{1,U}^{(2)} \\ \Theta_{2,U}^{(2)} \\ \Theta_{3,U}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -0.2348 \\ 0.0486 \end{pmatrix}, \quad \underline{X}_U^{(3)} = \begin{pmatrix} X_{1,U}^{(3)} \\ \Theta_{2,U}^{(3)} \\ \Theta_{3,U}^{(3)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2.1885 \\ 3.2200 \end{pmatrix}$$

1.2.2 Damped system

By considering the damped case ($[c^*] \neq [0]$) the matrix expression of the equations of motion (in free vibration) becomes:

$$[M^*] \underline{\ddot{x}} + [c^*] \underline{\dot{x}} + [k^*] \underline{x} = \underline{0}$$

By assuming as a solution $\underline{x} = \underline{X}e^{\lambda t}$, the equation becomes:

$$\left(\lambda^2 [M^*] + \lambda [c^*] + [k^*]\right) \underline{X} = \underline{0}$$

By adding the trivial equation $[M^*] \underline{\dot{x}} = [M^*] \underline{\dot{x}}$ to the matrix form of the equations of motion, the problem can be expressed as:

$$\begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \begin{pmatrix} \underline{\ddot{x}} \\ \underline{\dot{x}} \end{pmatrix} + \begin{bmatrix} [c^*] & [M^*] \\ -[M^*] & [0] \end{bmatrix} \begin{pmatrix} \underline{\dot{x}} \\ \underline{x} \end{pmatrix} = \underline{0}_{6\times 1}$$

By setting the vector of state variables $\underline{z}$ as:

$$\underline{z}_{6\times 1} = \begin{pmatrix} \underline{\dot{x}} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda \underline{x} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \lambda \underline{X} \\ \underline{X} \end{pmatrix} e^{\lambda t} = \underline{Z}_{6\times 1} e^{\lambda t}$$

The problem can be expressed as:

$$\begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix} \underline{\dot{z}} + \begin{bmatrix} [c^*] & [M^*] \\ -[M^*] & [0] \end{bmatrix} \underline{z} = \underline{0} \Longrightarrow [B]\,\underline{\dot{z}} + [D]\,\underline{z} = \underline{0}$$

In which:

$$[B]_{6\times 6} = \begin{bmatrix} [M^*] & [0] \\ [0] & [M^*] \end{bmatrix}, \quad [D]_{6\times 6} = \begin{bmatrix} [c^*] & [k^*] \\ -[M^*] & [0] \end{bmatrix}$$

The problem to solve is in the form:

$$\underline{\dot{z}} = -[B]^{-1}[D]\,\underline{z}$$

In which $\det([B]) \neq 0$. The matrix product $[A]_{6\times 6} = -[B]^{-1}[D]$ is called state matrix of the system. Let's consider:

$$\lambda \underline{Z} = -[B]^{-1}[D]\,\underline{Z} = [A]\,\underline{Z} \Longrightarrow \left(\lambda [I]_{6\times 6} - [A]\right)\underline{Z} = \underline{0}$$

From the above equation, it is possible to analyze the last three rows of $\underline{z}$, equal to $\underline{x}$, which contain the mode shapes for the damped case.
The matrix for the eigenvalues and eigenvectors problem computation is the state matrix:

$$[A] = -[B]^{-1}[D] = \begin{bmatrix} -[M^*]^{-1}[c^*] & -[M^*]^{-1}[k^*] \\ [I]_{3\times 3} & [0] \end{bmatrix}$$

The solutions $\lambda^d_{i,i+3} = -\alpha_i \pm j\omega_{d_i}$ (for $i = 1, 2, 3$) are complex conjugate, as in the undamped case, but have non-zero real part:

$$\lambda^d_{1,4} = -0.0557 \pm j0.8010, \quad \lambda^d_{2,5} = -0.3114 \pm j4.6816, \quad \lambda^d_{3,6} = -0.3257 \pm j10.1234$$

The real part α_i describes a decay behaviour in the time responses and is the contribution of the dampers. The imaginary part ω_{d_i} refers to the resonance frequencies in the damped system.
More precisely, three natural "damped" frequencies are identified as:

$$\omega_{d_1} = \mathrm{Im}\left\{\lambda^d_1\right\} = 0.8010 \text{ rad/s}, \quad \omega_{d_2} = \mathrm{Im}\left\{\lambda^d_2\right\} = 4.6816 \text{ rad/s}, \quad \omega_{d_3} = \mathrm{Im}\left\{\lambda^d_3\right\} = 10.1234 \text{ rad/s}$$

By analyzing the "damped" resonance frequencies, it is possible to see how close they are to the ones of the undamped case. This observation allows to define the system as lightly damped.
The confirmation of that comes from the adimensional damping ratios computation:

$$\underline{h}_{3\times 1} = \begin{pmatrix} h_1 = \frac{\alpha_1}{\omega_{0_1}} \\ h_2 = \frac{\alpha_2}{\omega_{0_2}} \\ h_3 = \frac{\alpha_3}{\omega_{0_3}} \end{pmatrix} = \begin{pmatrix} 0.0695 \\ 0.0665 \\ 0.0322 \end{pmatrix}$$

In which h_i is the adimensional damping ratio for the *i-th* eigenfrequency (sorted with increasing frequency).
Similarly to the undamped scenario, it is now possible to define the eigenvectors $\underline{X}^{(i)}$ for each natural frequency, normalized with respect to the first component $x_1 = 1$:

$$\underline{X}^{(1)} = \begin{pmatrix} X_1^{(1)} \\ \Theta_2^{(1)} \\ \Theta_3^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -18.8050 + j2.6391 \\ 12.2816 - j1.7581 \end{pmatrix}$$

$$\underline{X}^{(2)} = \begin{pmatrix} X_1^{(2)} \\ \Theta_2^{(2)} \\ \Theta_3^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -0.2323 + j0.0809 \\ 0.0448 - j0.0283 \end{pmatrix}$$

$$\underline{X}^{(3)} = \begin{pmatrix} X_1^{(3)} \\ \Theta_2^{(3)} \\ \Theta_3^{(3)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2.1712 + j0.1567 \\ 3.1917 + j0.2604 \end{pmatrix}$$

1.3 Rayleigh damping

The Rayleigh damping assumption states that two constants α and β exist so that:

$$[c^*] = \alpha\,[M^*] + \beta\,[k^*]$$

The adimensional damping ratios h_i, computed in the previous paragraph, are equal to:

$$h_i = \frac{c^*_{ii}}{2m^*_{ii}\omega_{0_i}} = \frac{\alpha m^*_{ii} + \beta k^*_{ii}}{2m^*_{ii}\omega_{0,i}}, \quad i = 1, 2, 3$$

$$\Longrightarrow h_i = \frac{\alpha}{2\omega_{0,i}} + \frac{\beta\omega_{0,i}}{2} = \left(\frac{1}{2\omega_{0,i}}, \frac{\omega_{0,i}}{2}\right)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

In order to find values of α and β that best approximate the Rayleigh damping equation, it is necessary to solve the following equation:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\frac{1}{2\omega_{0,i}}\\ \frac{\omega_{0,i}}{2}\end{pmatrix}^{-1} h_i$$

In matrix form it becomes:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{bmatrix}\frac{1}{2\omega_{0,1}} & \frac{1}{2\omega_{0,2}} & \frac{1}{2\omega_{0,3}}\\ \frac{\omega_{0,1}}{2} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,3}}{2}\end{bmatrix}^{-1}\begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix} = \begin{bmatrix}\frac{1}{2\omega_{0,1}} & \frac{1}{2\omega_{0,2}} & \frac{1}{2\omega_{0,3}}\\ \frac{\omega_{0,1}}{2} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,3}}{2}\end{bmatrix}^{-1}\underline{h}$$

The results are:

$$\alpha = 0.1134, \quad \beta = 0.0083$$

2 Free motion

The analysis of the free motion system will be done by assuming Rayleigh damping, so the damping matrix $[c^*]$ is given by:

$$[c^*] = \alpha [M^*] + \beta [k^*]$$

2.1 Time responses

In the previous section, the time response solution was defined as $\underline{x} = \underline{X} e^{\lambda t}$, in which $\underline{X}$ collects amplitude and phase response from each system natural frequency to the independent variables. The eigenvector $\underline{X}^{(1)}$ ($\underline{X}^{(2)}$ and $\underline{X}^{(3)}$ respectively) describes the amplitude and phase components of ω_{d_1} (ω_{d_2} and ω_{d_3} respectively) in the independent variables x_1, θ_2 and θ_3.
In particular:

$$x_1(t) = \sum_{i=1}^{3} e^{-\alpha_i t} \left| X_1^{(i)} \right| C_i \cos\left(\omega_{d_i} t + \Phi_i + \angle X_1^{(i)}\right) = \sum_{i=1}^{3} e^{-\alpha_i t} C_i \cos\left(\omega_{d_i} t + \Phi_i\right)$$

Due to the normalization of the eigenvectors $X_1^{(i)} = 1$, so $\left| X_1^{(i)} \right| = 1$ and $\angle X_1^{(i)} = 0$.

$$\theta_2(t) = \sum_{i=1}^{3} e^{-\alpha_i t} \left| \Theta_2^{(i)} \right| C_i \cos\left(\omega_{d_i} t + \Phi_i + \angle \Theta_2^{(i)}\right)$$

$$\theta_3(t) = \sum_{i=1}^{3} e^{-\alpha_i t} \left| \Theta_3^{(i)} \right| C_i \cos\left(\omega_{d_i} t + \Phi_i + \angle \Theta_3^{(i)}\right)$$

Each time response has three decaying sinusoids (with decay factor α_i and angular frequency ω_{d_i}), which manifest in different ways, depending on the values of $\left| X_1^{(i)} \right|$, $\left| \Theta_2^{(i)} \right|$ and $\left| \Theta_3^{(i)} \right|$.
All unknowns C_i and Φ_i are obtained by imposing the initial conditions:

$$x_{1_0} = 0.1 \text{ m}, \quad \theta_{2_0} = \frac{\pi}{12} \text{ rad}, \quad \theta_{3_0} = -\frac{\pi}{12} \text{ rad}$$
$$\dot{x}_{1_0} = 1 \text{ m/s}, \quad \dot{\theta}_{2_0} = 0.5 \text{ rad/s}, \quad \dot{\theta}_{3_0} = 2 \text{ rad/s}$$

Since the displacements expressions are highly similar, expressing the initial conditions by substituting $t = 0$ in the equations of $x_1(t)$ gives us the general formulation of the initial condition problem in which we are interested in finding the six unknowns $C_i, \Phi_i, \quad \forall i \in [1, 3]$:

$$x_1(t)|_{t=0} = x_{1_0} \Leftrightarrow \sum_{i=1}^{3} C_i \cdot \left| X_1^{(i)} \right| \cdot \cos\left(\Phi_i + \angle X_1^{(i)}\right) = x_{1_0}$$

$$\dot{x}_1(t)|_{t=0} = v_{1_0} \Leftrightarrow \sum_{i=1}^{3} \left[C_i \cdot \left| X_1^{(i)} \right| \cdot \cos\left(\Phi_i + \angle X_1^{(i)} - \arctan\left(\omega_{d_i}/\alpha_i\right)\right) \right] = -v_{1_0}$$

The resulting six unknowns are:

$$C_1 = 0.0870, \quad C_2 = 0.1981, \quad C_3 = -0.9423$$

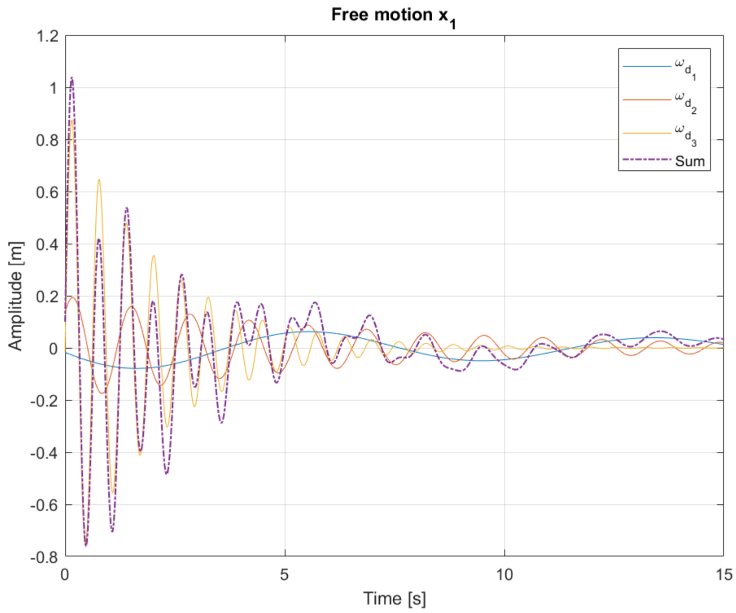
$$\Phi_1 = 1.7709 \text{ rad}, \quad \Phi_2 = -0.8294 \text{ rad}, \quad \Phi_3 = 7.8365 \text{ rad}$$

It is possible to check that the initial conditions are met by looking at the first element of the displacement vectors (in MATLAB).
It is now possible to fully define and plot the free motion response of each independent variable. By taking into account the products between the constants C_i and the amplitude factors $\left| X_i^{(j)} \right|$, it is possible to give a quick explanation of how much a certain frequency manifests in the time response of $x_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ using the following matrix:

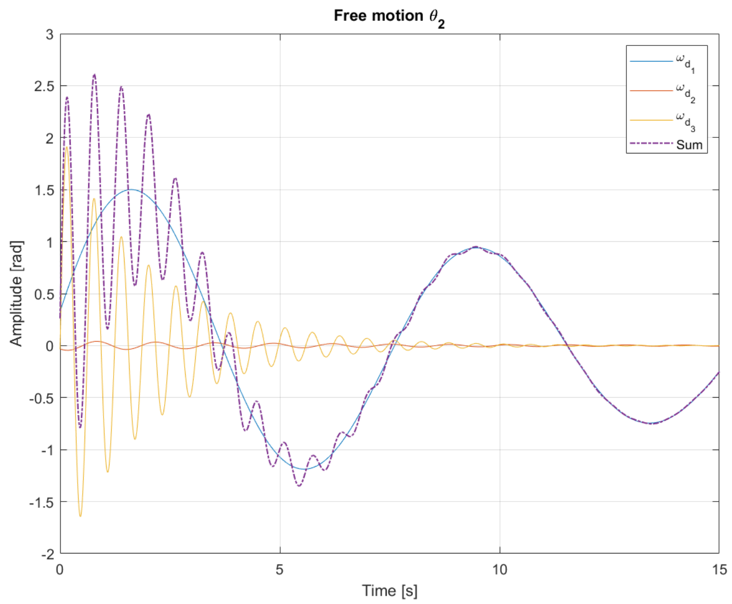
$$\left[C_1 \cdot \underline{X}^{(1)} \quad C_2 \cdot \underline{X}^{(2)} \quad C_3 \cdot \underline{X}^{(3)} \right] = \begin{bmatrix} 0.0870 & 0.1981 & 0.9423 \\ 1.6566 & 0.0465 & 2.0623 \\ 1.0822 & 0.0096 & 3.0362 \end{bmatrix}$$

The element in position (i, j) represents the amplitude of the sinusoids of angular frequency ω_{d_j} in the time response of the independent variable x_i or θ_i.

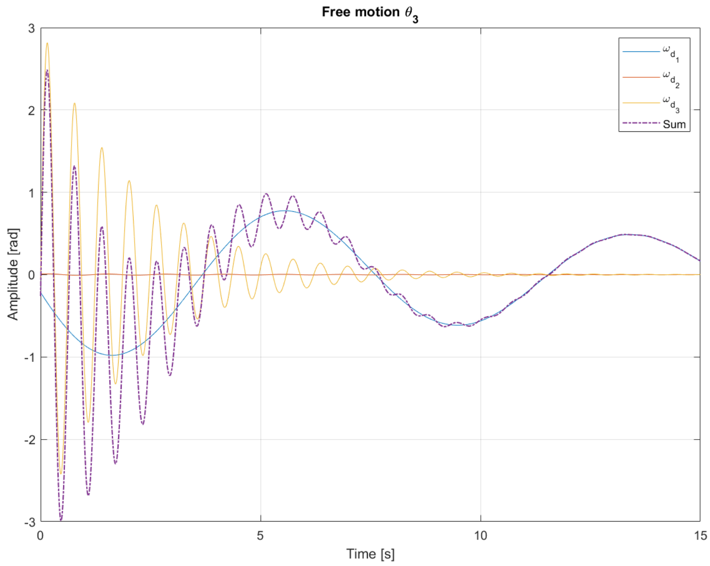
The time response plot of $x_1(t)$ is mainly given by ω_{d_2} and ω_{d_3}:



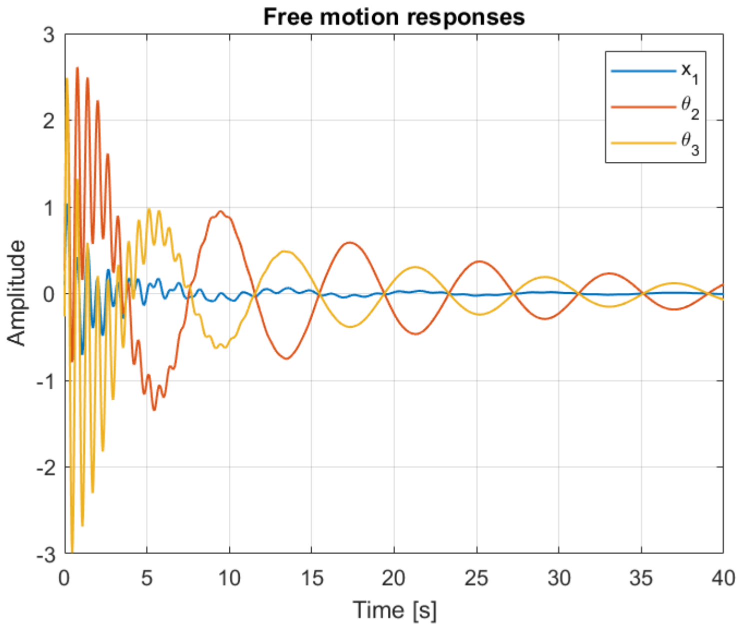
The time response plot of $\theta_2(t)$ is mainly given by ω_{d_1} and ω_{d_3}:



The time response plot of $\theta_3(t)$ is mainly given by ω_{d_1} and ω_{d_3}:

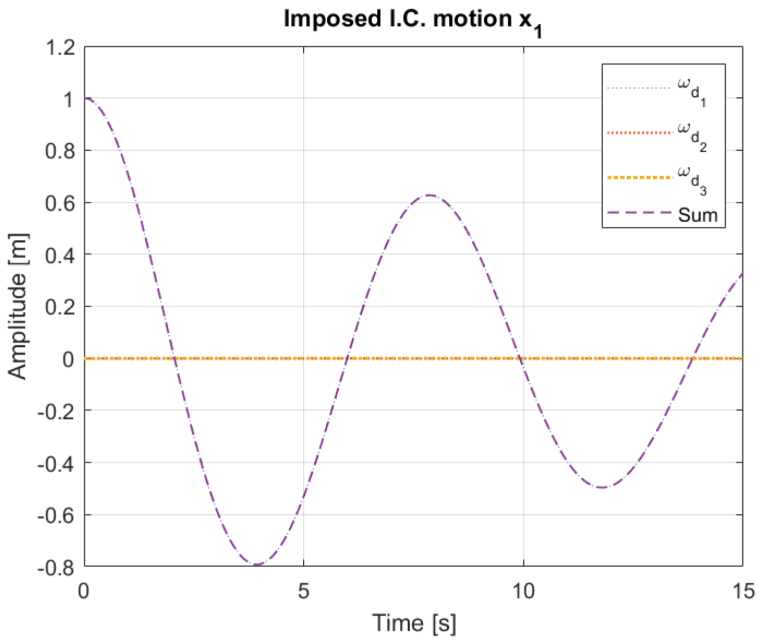


A comparison between all three time responses is given:

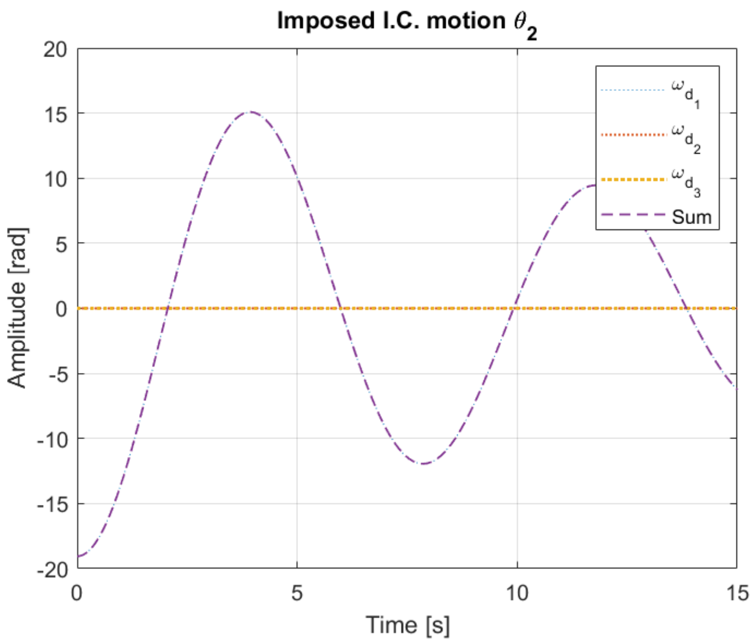


2.2 Eigenmode isolation

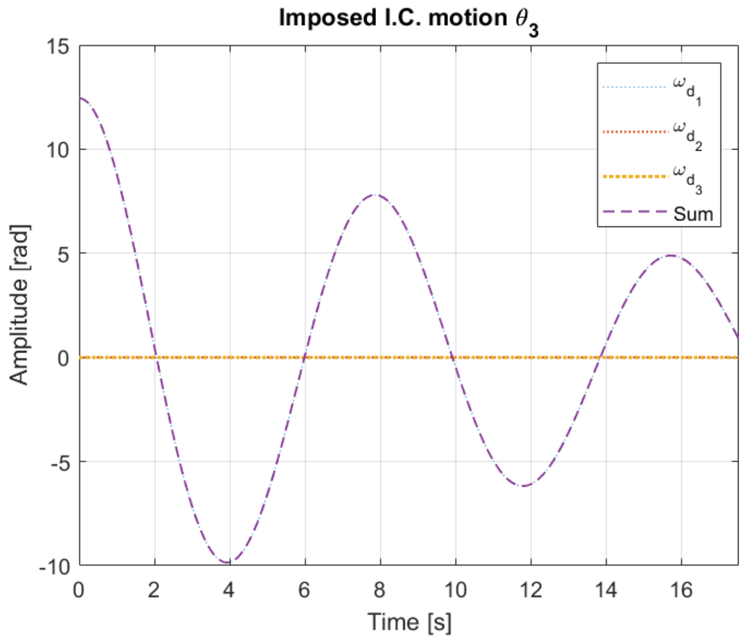
A way to impose the resulting time responses with a single mode is to put initial displacements equal to the eigenvector entries of the natural frequency of interest, while the initial velocities are all set to zero. In the following plots, the first mode is selected, so only the first natural frequency ω_{d_1} is going to be present.
The time response plot of $x_1(t)$ is given:



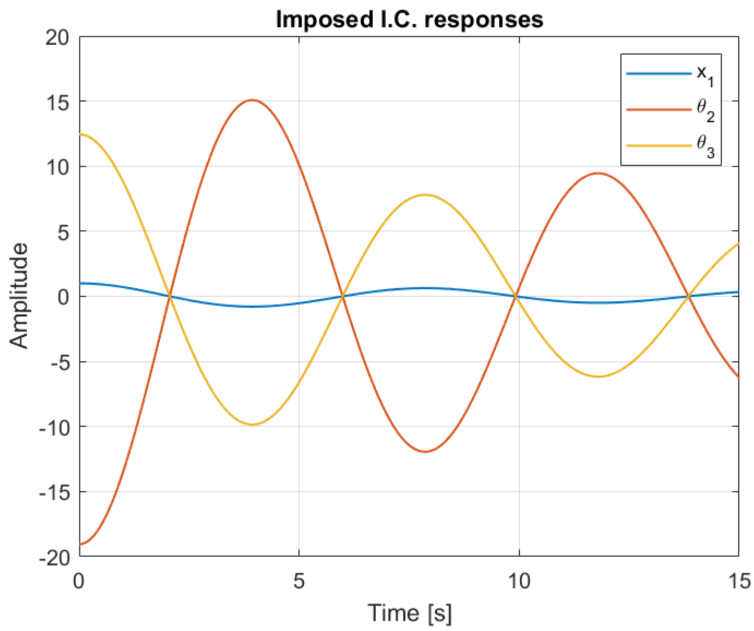
The time response plot of $\theta_2(t)$ is given:



The time response plot of $\theta_3(t)$ is given:



A comparison between all time responses is given:



3 Forced motion

As already explained at the beginning of the previous section, Rayleigh damping is assumed to occur as we have a lightly damped system:

$$[c^*] = \alpha\,[M^*] + \beta\,[k^*]$$

3.1 Frequency response matrix

By introducing the external force contribution, the equation to study becomes:

$$[M^*]\cdot\underline{\ddot{x}} + [c^*]\cdot\underline{\dot{x}} + [k^*]\cdot\underline{x} = \underline{Q} = \underline{Q_0}\cos(\Omega t)$$

Where:

$$\underline{Q} = \begin{pmatrix} F(t) \\ 0 \\ -rF(t) \end{pmatrix}, \quad F(t) = F_0\cos(\Omega t)\,, \quad \underline{Q_0} = \begin{pmatrix} F_0 \\ 0 \\ -rF_0 \end{pmatrix} = [\Lambda_F]^T F_0$$

A complex function $\underline{\tilde{x}_p}(t)$ can be defined, so that its real part $\underline{x_p}(t)$ is a particular solution of the equation above:

$$\underline{\tilde{x}_p}(t) = \underline{\tilde{X}_0}e^{j\Omega t} = \begin{pmatrix} \tilde{X}_{1_0} \\ \tilde{\Theta}_{2_0} \\ \tilde{\Theta}_{3_0} \end{pmatrix} e^{j\Omega t} \Longrightarrow \underline{x_p}(t) = \mathrm{Re}\{\underline{\tilde{x}_p}(t)\}$$

The complex function $\underline{\tilde{x}_p}(t)$, on the other hand, is a particular solution of the following equation:

$$[M^*]\cdot\underline{\ddot{\tilde{x}}_p} + [c^*]\cdot\underline{\dot{\tilde{x}}_p} + [k^*]\cdot\underline{\tilde{x}_p} = \underline{Q_0}e^{j\Omega t}$$

By evaluating the first and second time derivative of $\underline{\tilde{x}_p}(t)$, it is possible to define a relation between $\underline{\tilde{X}_0}$ and $\underline{Q_0}$:

$$\underline{\tilde{X}_0} = \left[-\Omega^2\,[M^*] + j\Omega\,[c^*] + [k^*]\right]^{-1}\underline{Q_0} = [D(\Omega)]^{-1}_{3\times 3}\,\underline{Q_0} = [H(\Omega)]_{3\times 3}\,\underline{Q_0}$$

The matrix $[H(\Omega)]$ is the frequency response matrix and its entries are the frequency response functions $H_{i,j}(\Omega)$, given by:

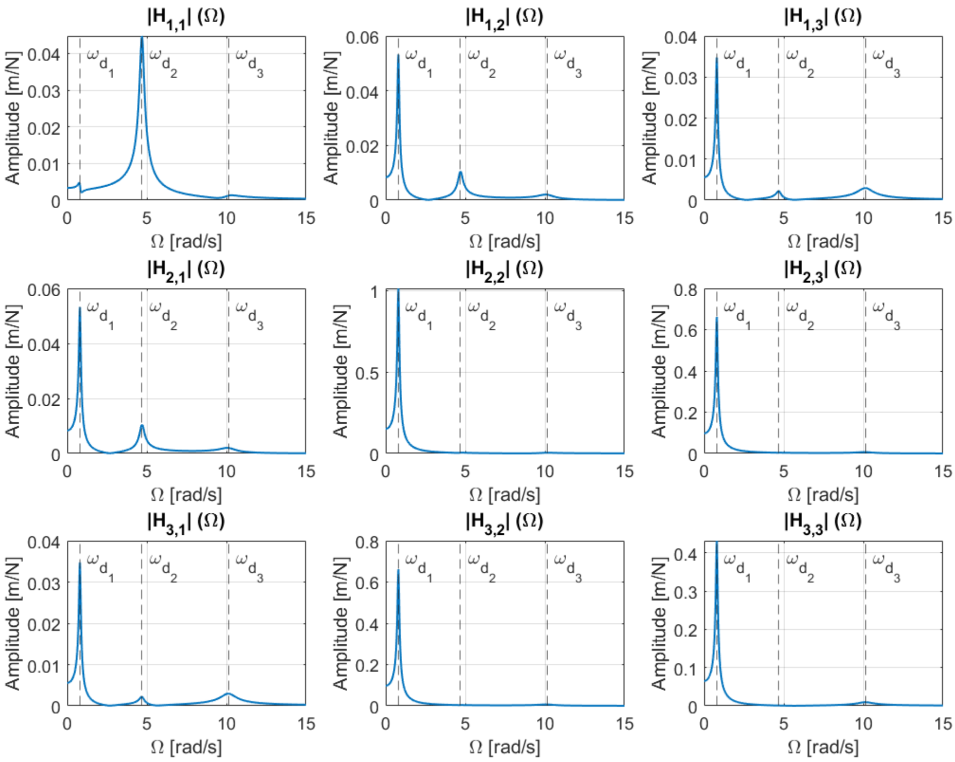
$$H_{i,j} = [D(\Omega)]^{-1}_{i,j} = \frac{1}{\det([D(\Omega)])}\,[C_D]^T_{i,j}$$

Where the cofactor matrix $[C_D]$ is defined as:

$$[C_D]_{i,j} = (-1)^{i+j}\det\left(\left[M_{D_{i,j}}\right]\right)$$

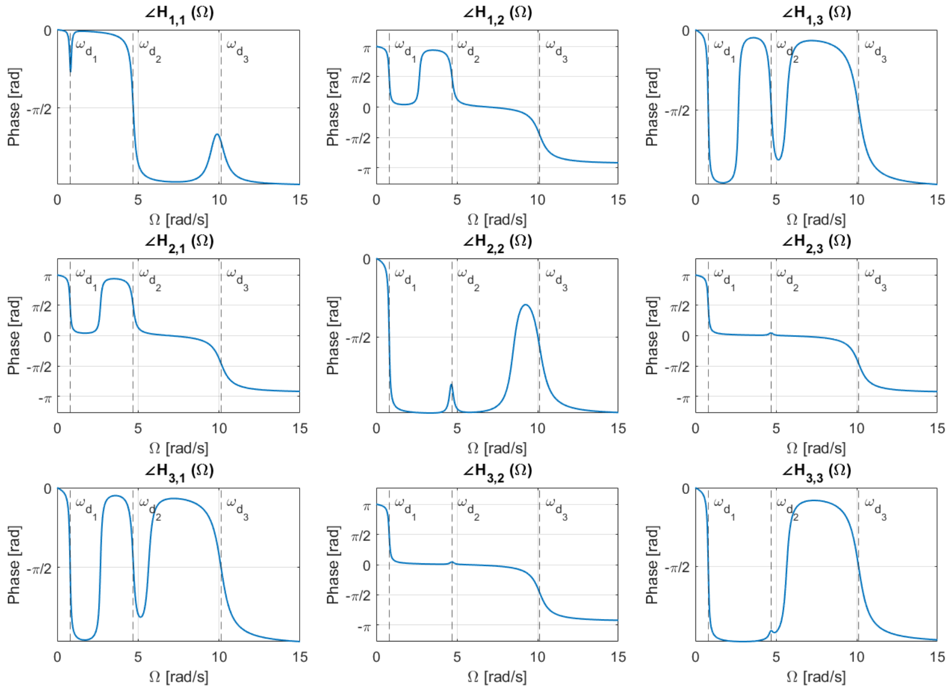
$\left[M_{D_{i,j}}\right]$ is the matrix obtained from $[D]$ after removing the *j-th* column and the *i-th* row.

The plot of every FRF amplitude $|H_{i,j}|$ is given:



Each frequency response function $H_{i,j}$ describes the system response of the *i-th* independent displacement/rotation in case of a force applied to the *j-th* body.
Whenever in a frequency response function one of the natural frequencies ω_{d_i} is not present, a node of vibration occurs.

The plot of every FRF phase $\angle H_{i,j}$ is given:

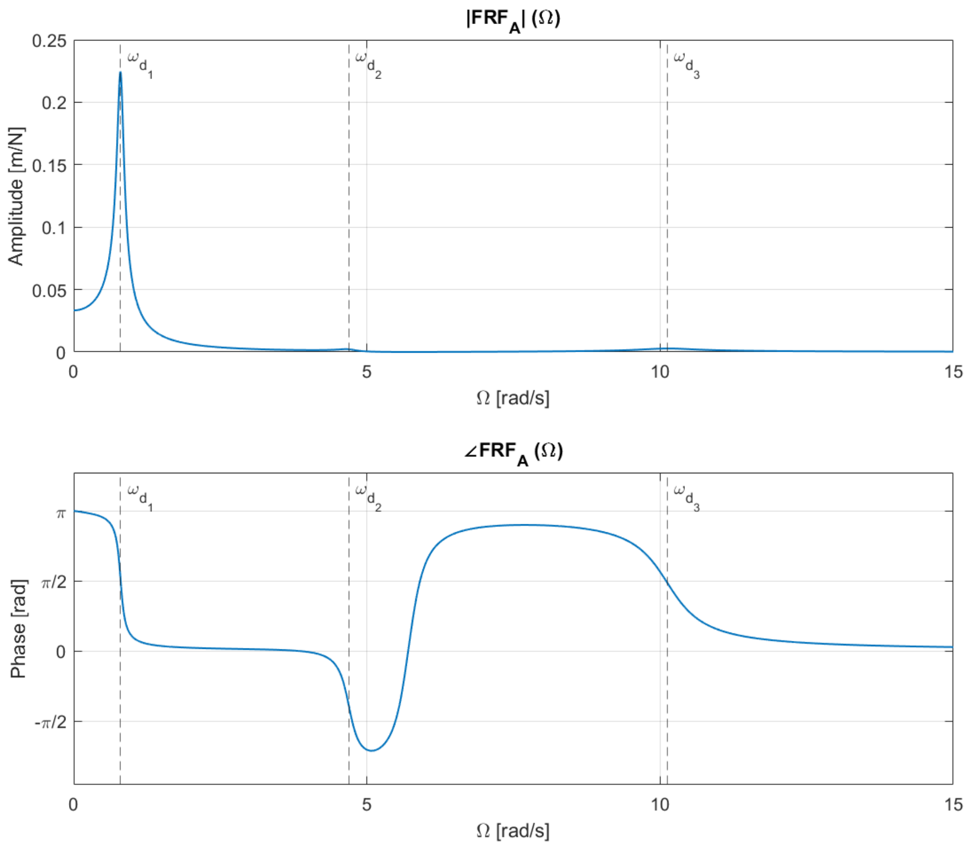


Whenever a $-\pi$ shift in the phase response occurs, the magnitude response presents a resonance (pole in the FRF), which is where a local maximum is present. On the other hand, a node of vibration (zero in the FRF) in the magnitude response translates into a $+\pi$ phase shift.
In all the other cases, a small phase shift can be described in terms of poles and zeros being closely spaced.
For example, in the phase response of $H_{1,1}$ a sudden phase shift occurs around ω_{d_1} and $\omega_{d,3}$, less than $\pm\pi$.

3.2 Co-located FRF in A

The co-located FRF of the point A (at the center of the disk M_3) is defined as the frequency response function that describes the displacement in A (called $x_3 = x_1 - r\theta_3$ in the system) and the force F applied in that point.

$$\mathrm{FRF}_A(\Omega) = [\Lambda_F][H(\Omega)]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & -r\end{bmatrix}\begin{bmatrix}H_{1,3}\\H_{2,3}\\H_{3,3}\end{bmatrix} = H_{1,3}(\Omega) - rH_{3,3}(\Omega)$$

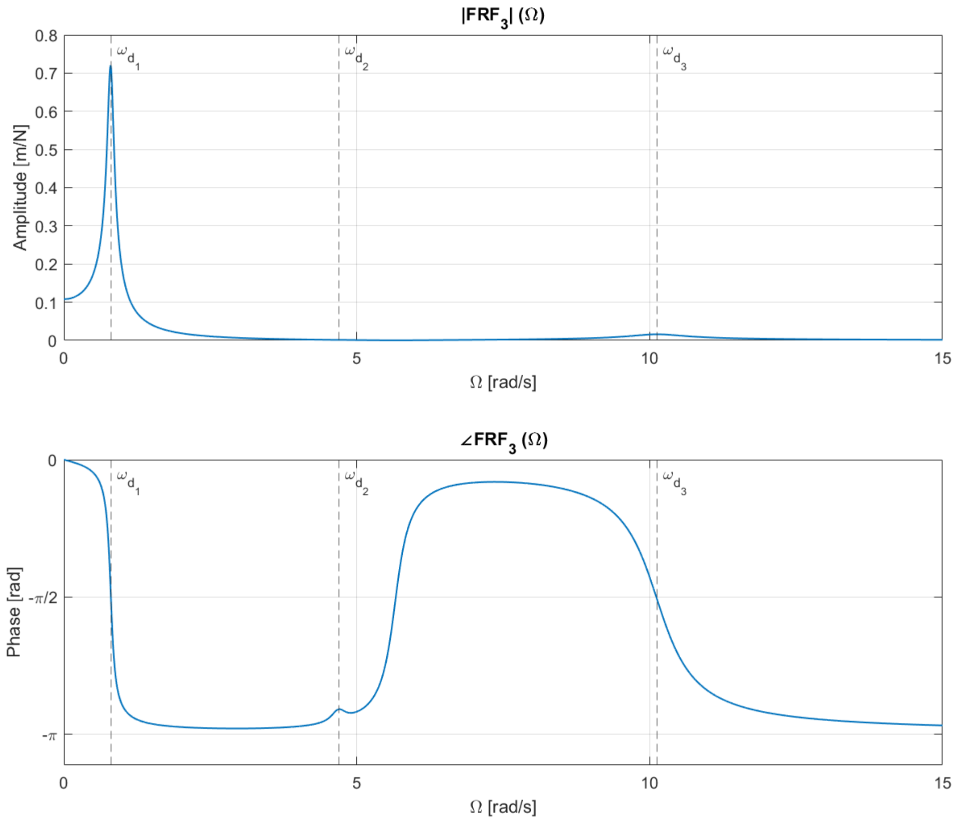


Considering that $\mathrm{FRF}_A = H_{1,3} - rH_{3,3}$, it is expected that a single resonance in ω_{d_1} is presents, similarly to $H_{1,3}$ and $H_{3,3}$.
The comments made in the previous section, related to the FRF plots, are still valid.

3.3 Co-located FRF for disk 3

The frequency response $H_{3,3}$ describes the system response in terms of θ_3 to a force applied at the center of the disk M_3. In order to get a response to a torque:

$$\mathrm{FRF}_3(\Omega) = \frac{\Theta_3}{Fr} = \frac{\Theta_3}{F}\frac{1}{r} = \frac{H_{3,3}(\Omega)}{r}$$



3.4 Harmonic force response

Considering an harmonic force applied to the center of the disk M_3:

$$F(t) = F_1(t) + F_2(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)$$

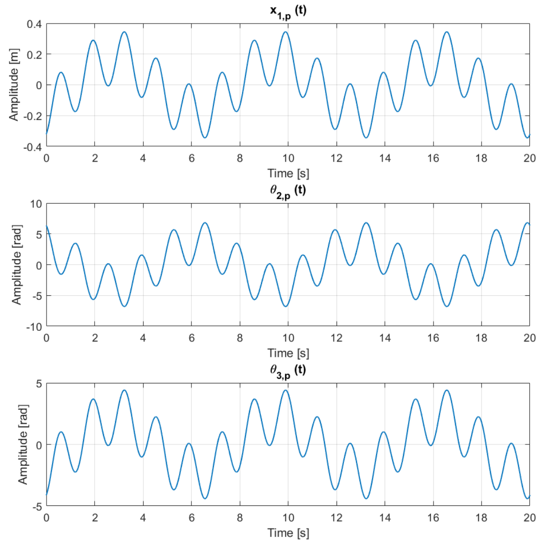
It is possible to study the system response in terms of linear combination of responses to F_1 and F_2, given the linearity of the system. From each force component, it is possible to compute the particular solutions for $x_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ as:

$$x_{1_p}(t) = |H_{1,3}(\Omega = 2\pi f_1)| A_1 \cos(2\pi f_1 t + \angle H_{1,3}(\Omega = 2\pi f_1)) + \\ + |H_{1,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{1,3}(\Omega = 2\pi f_2))$$

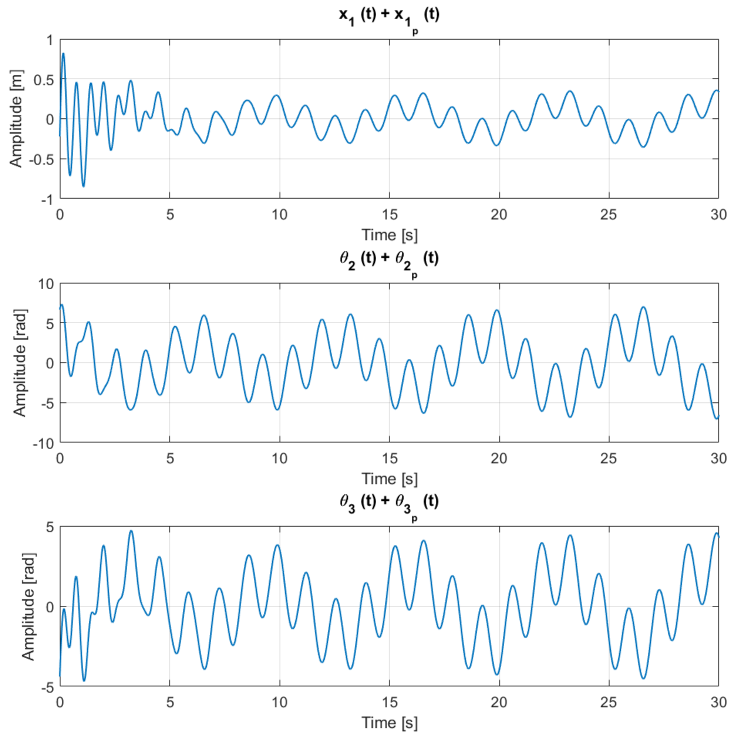
$$\theta_{2_p}(t) = |H_{2,3}(\Omega = 2\pi f_1)| A_1 \cos(2\pi f_1 t + \angle H_{2,3}(\Omega = 2\pi f_1)) + \\ + |H_{2,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{2,3}(\Omega = 2\pi f_2))$$

$$\theta_{3_p}(t) = |H_{3,3}(\Omega = 2\pi f_1)| A_1 \cos(2\pi f_1 t + \angle H_{3,3}(\Omega = 2\pi f_1)) + \\ + |H_{3,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{3,3}(\Omega = 2\pi f_2))$$

Their plot is given:



In order to obtain the complete time response of all the independent variables, it is necessary to sum the particular solutions just obtained to the free motion time (homogeneous) responses in the section 2.1:



Given the decaying nature of the homogeneous solutions $x_1(t)$, $\theta_2(t)$ and $\theta_3(t)$, a transient is present. After that, the particular solution represents the steady state of the response ($t \to \infty$).

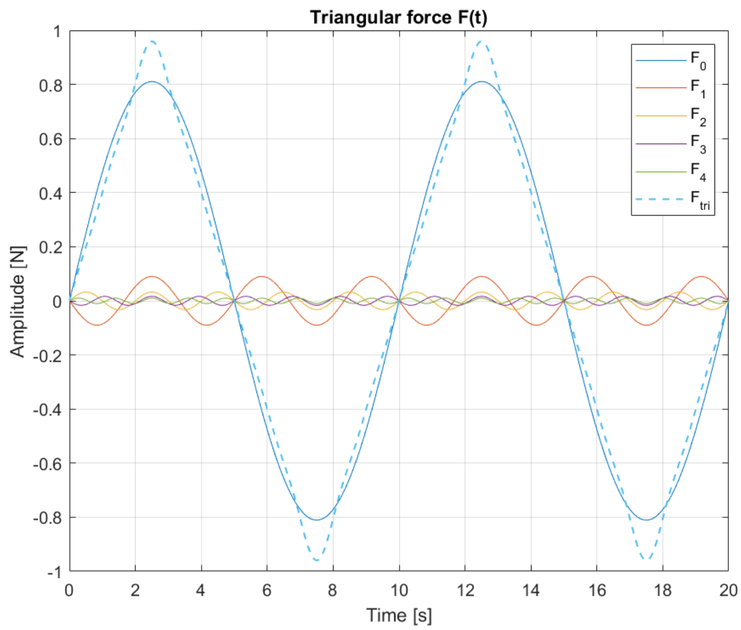
3.5 Triangular force response

Similarly to the previous paragraph, the system response of a linear combination of forces can be obtained as linear combination of the single responses.
Considering a periodic force like:

$$F(t) = \sum_{k=0}^{4} F_k(t) = \sum_{k=0}^{4} \frac{8}{\pi^2}(-1)^k \frac{\sin(\Omega_k t)}{(2k+1)^2}, \quad f_0 = 0.10 \text{ Hz}, \quad \Omega_k = (2k+1)2\pi f_0$$

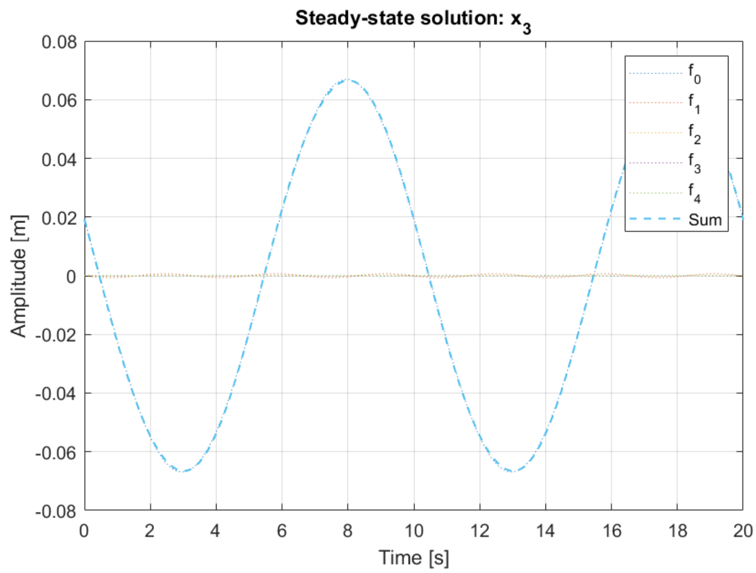
The angular frequency (rad/s) of each component of the input force is:

$$\Omega_0 = 0.6283, \quad \Omega_1 = 1.8850, \quad \Omega_2 = 3.1416, \quad \Omega_3 = 4.3982, \quad \Omega_4 = 5.6549$$



The steady-state response for $x_3(t)$ is given by:

$$x_{3_{ss}}(t) = \sum_{k=0}^{4} x_{3_{ss_k}}(t) = \sum_{k=0}^{4} |\text{FRF}_A(\Omega_k)| \frac{8}{\pi^2}(-1)^k \frac{\sin(\Omega_k t + \angle \text{FRF}_A(\Omega_k))}{(2k+1)^2}$$



The mechanical system filters out all frequencies but f_0, which is coherent with the magnitude plot of $\text{FRF}_A(\Omega)$, in which only $\omega_{d_1} = 0.7995$ rad/s resonates.

4 Modal approach

The modal approach for the mechanical system analysis starts from a linear transformation of coordinates, from the independent variables $\underline{x}(t)$ to the modal coordinates $\underline{q}(t)$:

$$\underline{x}(t) = \begin{pmatrix} x_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = [\Phi] \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = [\Phi]\, \underline{q}(t)$$

Where $[\Phi]$ is the matrix of the mode shapes (computed in the undamped case):

$$[\Phi]_{3\times 3} = \left[\underline{X}_U^{(1)} \quad \underline{X}_U^{(2)} \quad \underline{X}_U^{(3)} \right] = \begin{bmatrix} X_{1,U}^{(1)} & X_{1,U}^{(2)} & X_{1,U}^{(3)} \\ \Theta_{2,U}^{(1)} & \Theta_{2,U}^{(2)} & \Theta_{2,U}^{(3)} \\ \Theta_{3,U}^{(1)} & \Theta_{3,U}^{(2)} & \Theta_{3,U}^{(3)} \end{bmatrix}$$

4.1 Equations of Motion and Frequency Response Matrix

It is possible to obtain a matrix EoM in terms of modal coordinates as:

$$[M^*] \cdot \underline{\ddot{x}} + [c^*] \cdot \underline{\dot{x}} + [k^*] \cdot \underline{x} = \underline{Q} \Longrightarrow [M^*] \cdot [\Phi]\, \underline{\ddot{q}} + [c^*] \cdot [\Phi]\, \underline{\dot{q}} + [k^*] \cdot [\Phi]\, \underline{q} = \underline{Q}$$

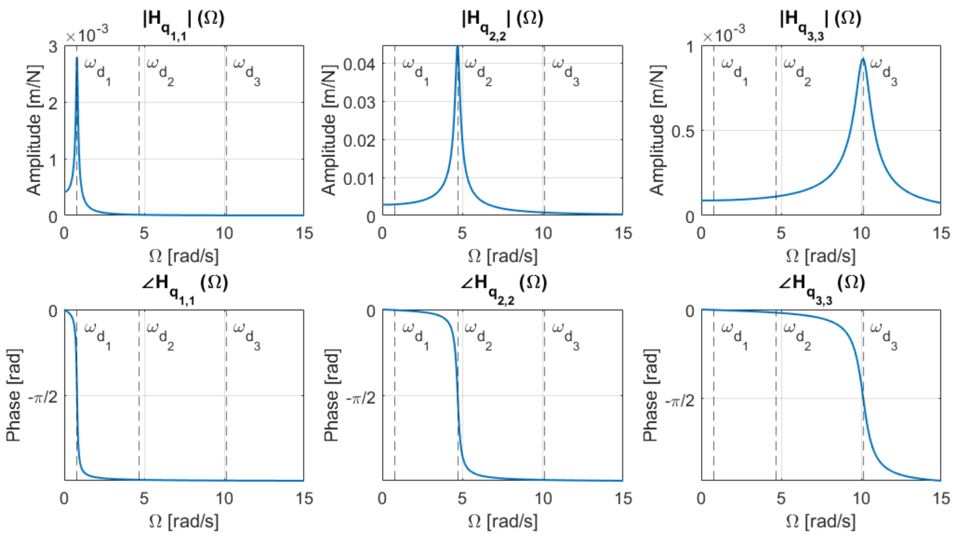
$$[\Phi]^T [M^*] [\Phi] \cdot \underline{\ddot{q}} + [\Phi]^T [c^*] [\Phi] \cdot \underline{\dot{q}} + [\Phi]^T [k^*] [\Phi] \cdot \underline{q} = [\Phi]^T \underline{Q}$$

$$[M_q] \cdot \underline{\ddot{q}} + [c_q] \cdot \underline{\dot{q}} + [k_q] \cdot \underline{x} = \underline{Q_q}$$

By construction, $[M_q]$ and $[k_q]$ are always diagonal, while $[c_q]$ can be non-diagonal. However, given the Rayleigh damping assumption, where $[c^*] = \alpha [M^*] + \beta [k^*]$, we observe that $[c_q]$ is diagonal. Similar to the FRM definition with independent variables, the frequency response matrix in modal coordinates $H_q(\Omega)$ is defined as:

$$H_q(\Omega) = \left[-\Omega^2 [M_q] + j\Omega [c_q] + [k_q] \right]^{-1} = \begin{bmatrix} H_{q_{1,1}}(\Omega) & 0 & 0 \\ 0 & H_{q_{2,2}}(\Omega) & 0 \\ 0 & 0 & H_{q_{3,3}}(\Omega) \end{bmatrix}$$

The frequency response matrix is diagonal, and its entries $H_{q_{i,i}}(\Omega)$ are the modal frequency responses:



4.2 Co-located modal FRF in A

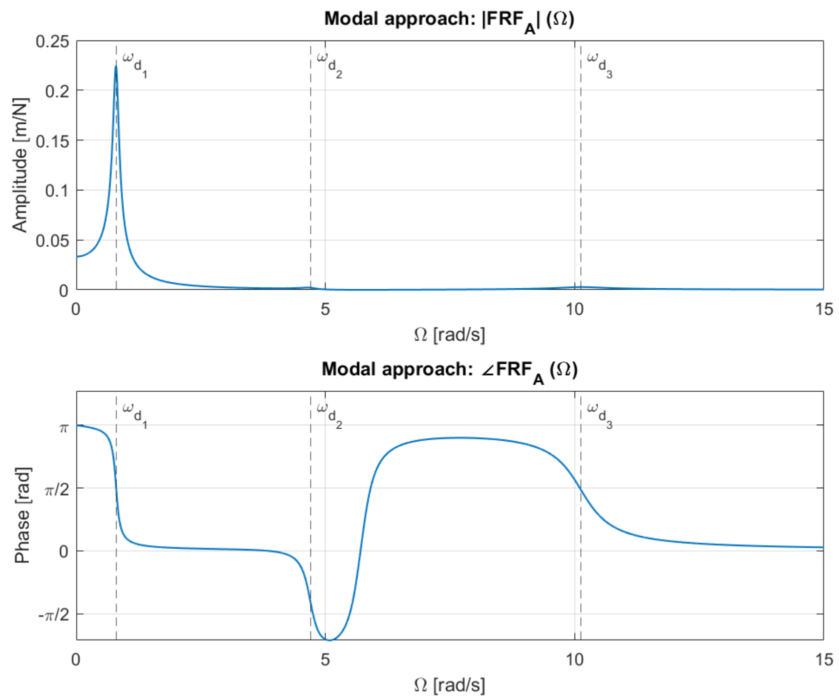
The co-located FRF of the point A, in independent variables, has been defined as:

$$\mathrm{FRF}_A(\Omega) = H_{1,3}(\Omega) - rH_{3,3}(\Omega)$$

Considering that the generic FRF $H_{j,k}(\Omega)$ is described by means of modal FRFs as:

$$H_{j,k}(\Omega) = \sum_{i=1}^{3} X_{j,U}^{(i)} X_{k,U}^{(i)} H_{q_{i,i}} \Longrightarrow \mathrm{FRF}_A(\Omega) = \sum_{i=1}^{3} X_{1,U}^{(i)} \Theta_{3,U}^{(i)} H_{q_{i,i}} - r\sum_{i=1}^{3} \Theta_{3,U}^{(i)} \Theta_{3,U}^{(i)} H_{q_{i,i}}$$

The reconstruction with modal approach is identical to the original FRF:



4.3 Co-located FRF for disk 3

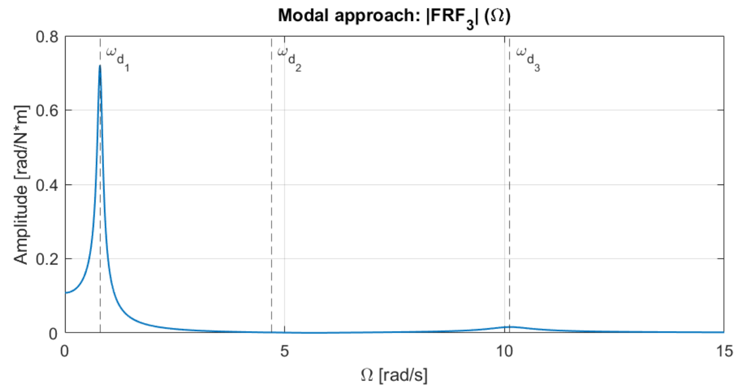
The co-located FRF for the disk M_3, in independent variables, has been defined as:

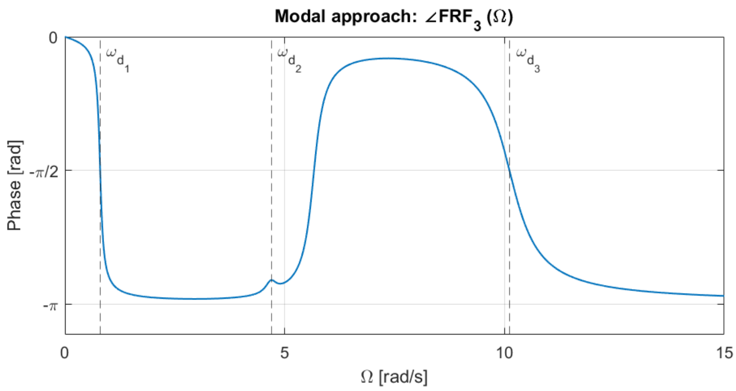
$$\mathrm{FRF}_3(\Omega) = \frac{H_{3,3}(\Omega)}{r}$$

The only FRF in independent variables is described by means of modal FRFs as:

$$H_{3,3}(\Omega) = \sum_{i=1}^{3} \Theta_{3,U}^{(i)} \Theta_{3,U}^{(i)} H_{q_{i,i}}$$

The reconstruction with modal approach is identical to the original FRF:





4.4 Forced vibrations

The steady-state responses of a mechanical system to a single harmonic force are given by harmonics of same frequency, scaled in amplitude and shifted in phase:

$$F(t) = A_1 \cos(2\pi f_1 t) \Longrightarrow \begin{cases} x_{1_{ss}}(t) = |H_{1,3}(\Omega = 2\pi f_1)| \, A_1 \cos(2\pi f_1 t + \angle H_{1,3}(\Omega = 2\pi f_1)) \\ \theta_{2_{ss}}(t) = |H_{2,3}(\Omega = 2\pi f_1)| \, A_1 \cos(2\pi f_1 t + \angle H_{2,3}(\Omega = 2\pi f_1)) \\ \theta_{3_{ss}}(t) = |H_{3,3}(\Omega = 2\pi f_1)| \, A_1 \cos(2\pi f_1 t + \angle H_{3,3}(\Omega = 2\pi f_1)) \end{cases}$$

In which the frequency response functions $H_{1,3}(\Omega)$, $H_{2,3}(\Omega)$ and $H_{3,3}(\Omega)$ have been evaluated in terms of modal FRFs as:

$$H_{j,k}(\Omega) = \sum_{i=1}^{3} X_{j,U}^{(i)} X_{k,U}^{(i)} H_{q_{i,i}}$$

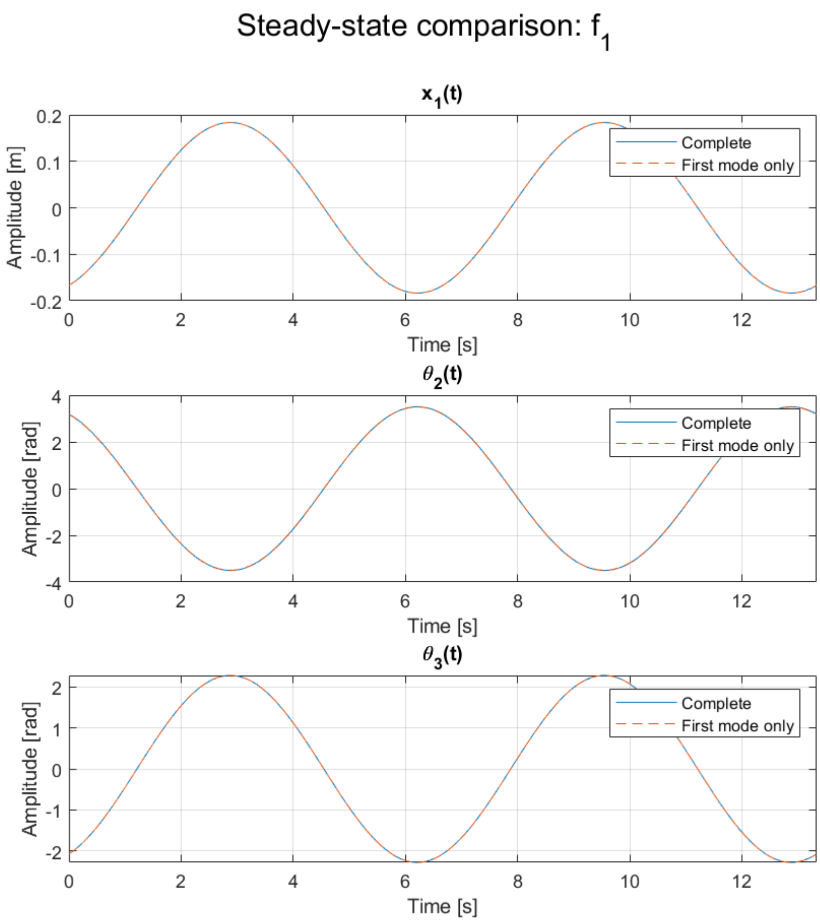
It is possible to define the system response considering only the first mode of vibration as:

$$\begin{cases} x_{1_{var}}(t) = \left|\tilde{H}_{1,3}(\Omega = 2\pi f_1)\right| A_1 \cos\left(2\pi f_1 t + \angle\tilde{H}_{1,3}(\Omega = 2\pi f_1)\right) \\ \theta_{2_{var}}(t) = \left|\tilde{H}_{2,3}(\Omega = 2\pi f_1)\right| A_1 \cos\left(2\pi f_1 t + \angle\tilde{H}_{2,3}(\Omega = 2\pi f_1)\right) \\ \theta_{3_{var}}(t) = \left|\tilde{H}_{3,3}(\Omega = 2\pi f_1)\right| A_1 \cos\left(2\pi f_1 t + \angle\tilde{H}_{3,3}(\Omega = 2\pi f_1)\right) \end{cases}$$

Where:

$$\tilde{H}_{j,k}(\Omega) = X_{j,U}^{(1)} X_{k,U}^{(1)} H_{q_{1,1}}$$

A comparison between the complete responses and the first mode only ones is given:



In this case, given $\omega_1 = 2\pi f_1 = 0.9425$ rad/s, the input force excites the first mode of vibration $\omega_{d_1} = 0.7995$ rad/s. The plot confirms the fact that the steady-state response is equal to the first mode contribution.

Considering another harmonic force:

$$F(t) = A_2 \cos(2\pi f_2 t) \implies \begin{cases} x_{1_{ss}}(t) = |H_{1,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{1,3}(\Omega = 2\pi f_2)) \\ \theta_{2_{ss}}(t) = |H_{2,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{2,3}(\Omega = 2\pi f_2)) \\ \theta_{3_{ss}}(t) = |H_{3,3}(\Omega = 2\pi f_2)| A_2 \cos(2\pi f_2 t + \angle H_{3,3}(\Omega = 2\pi f_2)) \end{cases}$$

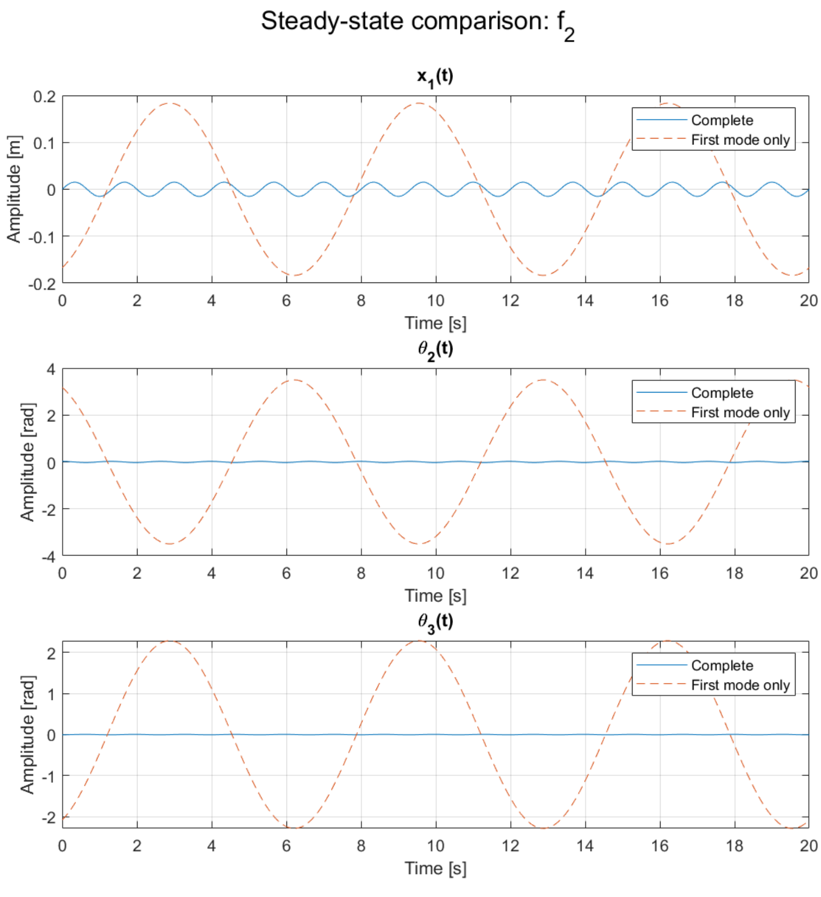
It is possible to define the system response considering only the first mode of vibration as:

$$\begin{cases} x_{1_{var}}(t) = \left|\tilde{H}_{1,3}(\Omega = 2\pi f_2)\right| A_2 \cos\left(2\pi f_2 t + \angle \tilde{H}_{1,3}(\Omega = 2\pi f_2)\right) \\ \theta_{2_{var}}(t) = \left|\tilde{H}_{2,3}(\Omega = 2\pi f_2)\right| A_2 \cos\left(2\pi f_2 t + \angle \tilde{H}_{2,3}(\Omega = 2\pi f_2)\right) \\ \theta_{3_{var}}(t) = \left|\tilde{H}_{3,3}(\Omega = 2\pi f_2)\right| A_2 \cos\left(2\pi f_2 t + \angle \tilde{H}_{3,3}(\Omega = 2\pi f_2)\right) \end{cases}$$

Where:

$$\tilde{H}_{j,k}(\Omega) = X_{j,U}^{(1)} X_{k,U}^{(1)} H_{q_{1,1}}$$

A comparison between the complete responses and the first mode only ones is given:

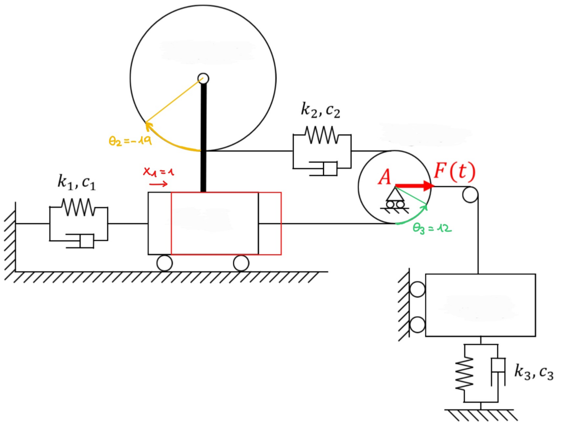


In this case, given $\omega_2 = 2\pi f_2 = 4.7124$ rad/s, the input force is near the second mode of vibration $\omega_{d_2} = 4.6950$ rad/s and is in the seismographic zone of $H_{q_{1,1}}$. Thus, the input force does not involve the first mode contribution, which is confirmed by the plot above.

5 Vibration modes graphical representation

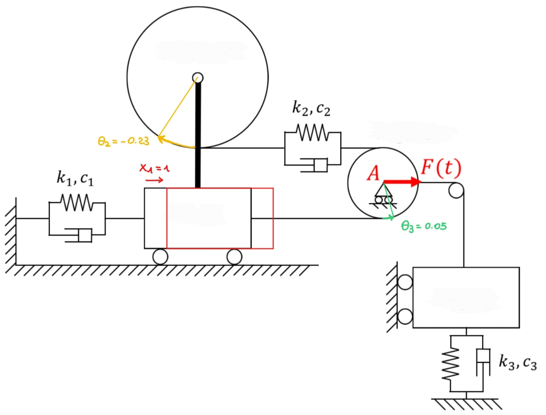
5.1 Mode 1

The visual representation of the system response to the first mode is given:



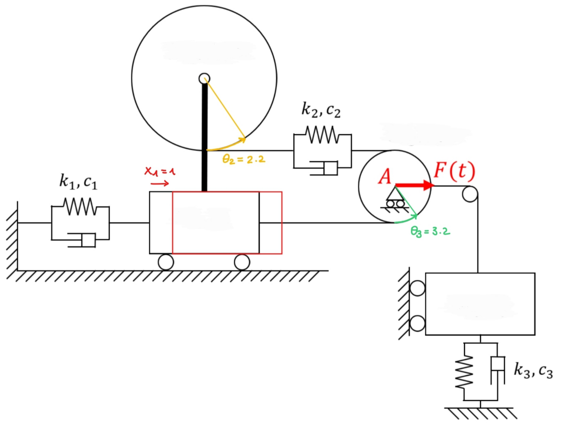
5.2 Mode 2

The visual representation of the system response to the second mode is given:



5.3 Mode 3

The visual representation of the system response to the third mode is given:



VIBRATION ANALYSIS
AND VIBROACOUSTICS

VIBRATION ANALYSIS

Assignment 3 - A.Y. 2023/24

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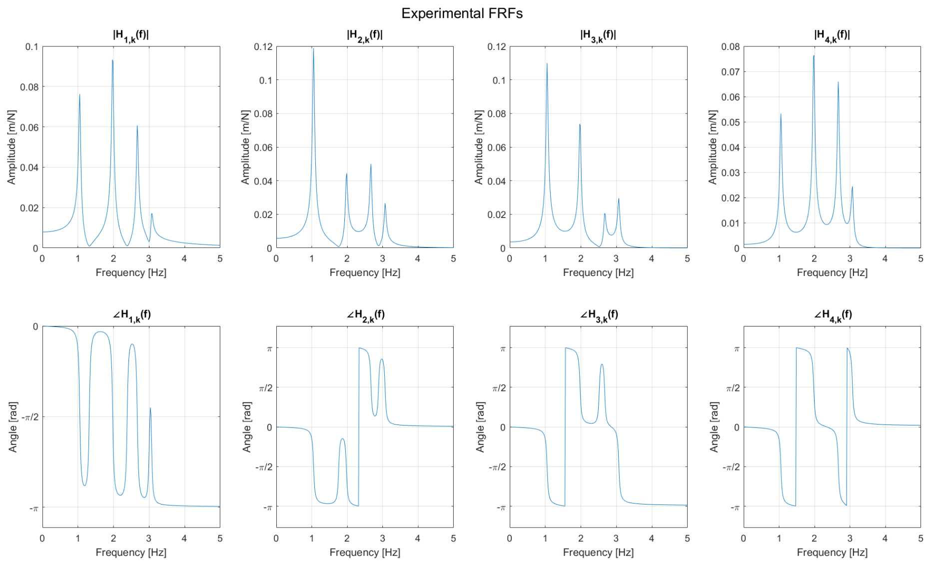
1 Experimental FRFs

In order to identify the modal parameters of a mechanical system, an approach consists in exciting the system with an external force (i.e. impulsive force) and measuring the resulting displacements. The definition of a FRF (Frequency Response Function), for a force $F_k(t)$ applied at the point k and a displacement $x_j(t)$ for the body j, is given by:

$$H_{j,k}^{(\exp)}(\Omega_f) = |H_{j,k}(\Omega_f)|\, e^{j\angle H_{j,k}(\Omega_f)} = \frac{|X_{j_0}(\Omega_f)|}{|F_{k_0}(\Omega_f)|} e^{j\left(\varphi_{x_j}(\Omega_f) - \varphi_{F_k}(\Omega_f)\right)}$$

The amplitude components $|X_{j_0}(\Omega_f)|$ and $|F_{k_0}(\Omega_f)|$,in addition to the phase components $\varphi_{x_j}(\Omega_f)$ and $\varphi_{F_k}(\Omega_f)$, are the result of the DFT (Discrete Fourier Transform) applied to both force and displacement. This procedure can be extended to all displacements $x_j(t)$.
The plots for magnitude and phase of the FRFs $H_{j,k}^{(\exp)}$ are given:



These plots describe a 4-dof mechanical system. There are four angular frequencies (resonances), where a magnitude maximum (magnitude plot) and a $-\pi$ rad phase shift (phase plot) occur. The presence of nodes of vibration in the magnitude plots results in a $+\pi$ rad phase shift in the phase plots.
In addition to that, phase wraps occur when there is a 2π rad phase shift. In the end, sudden phase shifts (less than $\pm\pi$) occur whenever poles and zeros are closely-spaced.
Based on these observations, magnitude and phase plots are coherent between each other.

2 Modes and damping estimation

2.1 Estimation procedure

Starting from all the FRFs plots, it is possible to estimate parameters of the system, such as natural frequencies, mode shapes and damping ratios.
This procedure starts with defining a frequency range $[\Omega_A, \Omega_B]$ of arbitrary choice, in which a single resonance occurs (at ω_{d_i}), in order to locally describe the response as a 1-dof system.

It is now possible to find the local peak (maximum) of the FRF magnitude, which occurs at the damped angular frequency ω_{d_i} of resonance ($i = 1, 2, 3, 4$).

The adimensional damping ratio h_i can be computed from the phase response of the system for angular frequencies Ω_f around each detected resonance. From the expression of a 1-dof system phase response:

$$\angle H_{j,k}(\Omega_f) = -\arctan\left(\frac{2h_i\frac{\Omega_f}{\omega_{d_i}}}{1-\frac{{\Omega_f}^2}{\omega_{d_i}^2}}\right) = -\arctan\left(\frac{2h_i\omega_{d_i}\cdot\Omega_f}{\omega_{d_i}^2-{\Omega_f}^2}\right) = -\arctan\left(f(\Omega_f)\right)$$

$$\phi(\Omega_f) = \frac{d\left[\angle H_{j,k}(\Omega_f)\right]}{d\Omega_f} = -\frac{d\left[\arctan(f)\right]}{df}\frac{df}{d\Omega_f} = -\frac{1}{1+f^2(\Omega_f)}\cdot\frac{df}{d\Omega_f}$$

$$\Rightarrow \phi(\Omega_f) = -\frac{1}{\left(\omega_{d_i}^2-{\Omega_f}^2\right)^2+\left(2h_i\omega_{d_i}\cdot\Omega_f\right)^2}\cdot 2h_i\omega_{d_i}\left(\omega_{d_i}^2+{\Omega_f}^2\right) \quad \Rightarrow \phi(\omega_{d_i}) = -\frac{1}{h_i\omega_{d_i}}$$

Thus, for each individual mode, the adimensional damping ratio h_i is computed thanks to the phase response derivative ϕ of $H_{j,k}$ around each resonance frequency ω_{d_i} :

$$h_i = -\frac{1}{\omega_{d_i}\cdot\phi(\omega_{d_i})}$$

The mode shape can be evaluated from the generic FRF expression for the *j-th* measurement:

$$H_{j,k}(\Omega_f) = \frac{A_j + iB_j}{-\Omega_f^2 m_{q_{ii}} + i\Omega_f c_{q_{ii}} + k_{q_{ii}}}, \quad \Omega_f \in [\Omega_A, \Omega_B]$$

Considering that, in the frequency range of choice, only a single FRF is present and a resonance occurs at ω_{d_i}.
It is possible to fix the modal normalization with $m_{q_{ii}} = 1$, in which case:

$$k_{q_{ii}} = m_{q_{ii}}\omega_{d_i}^2 = \omega_{d_i}^2, \quad c_{q_{ii}} = 2m_{q_{ii}}\omega_{d_i}h_i = 2\omega_{d_i}h_i$$

$$\Rightarrow H_{j,k}(\Omega_f) = \frac{A_j + iB_j}{-\Omega_f^2 + i\Omega_f 2\omega_{d_i}h_i + \omega_{d_i}^2}$$

It is of interest to evaluate it at the resonance frequency:

$$H_{j,k}(\omega_{d_i}) = \frac{A_j + iB_j}{-\omega_{d_i}^2 + i\omega_{d_i}2\omega_{d_i}h_i + \omega_{d_i}^2} = \frac{A_j + iB_j}{i\omega_{d_i}2\omega_{d_i}h_i} = -i\frac{A_j + iB_j}{2\omega_{d_i}^2 h_i} = \frac{B_j}{2\omega_{d_i}^2 h_i} - i\frac{A_j}{2\omega_{d_i}^2 h_i}$$

The real part of a 1-dof frequency response function is equal to zero at ω_{d_i}:

$$\mathrm{Re}\{H_{j,k}(\Omega)\} = \frac{1}{k_{q_{ii}}}\frac{1-\frac{\Omega^2}{\omega_{d_i}^2}}{\left(1-\frac{\Omega^2}{\omega_{d_i}^2}\right)+i\left(2h_i\frac{\Omega}{\omega_{d_i}}\right)}$$

So $B_j = 0$.

$$\mathrm{Im}\{H_{j,k}(\omega_{d_i})\} = -\frac{A_j}{2\omega_{d_i}^2 h_i} \Longrightarrow A_j = -2\omega_{d_i}^2 h_i\,\mathrm{Im}\{H_{j,k}(\omega_{d_i})\}$$

The value A_j represents the mode shape $X_j^{(i)}$ of the *i-th* mode for the *j-th* measurement.

2.2 Results

After computation from the Matlab script, we obtain 16 different estimations of the adimensional damping ratios h_i:

	Mode 1	Mode 2	Mode 3	Mode 4
h_{FRF_1}	0.0286	0.0161	0.0113	0.0144
h_{FRF_2}	0.0283	0.0160	0.0113	0.0099
h_{FRF_3}	0.0283	0.0160	0.0115	0.0097
h_{FRF_4}	0.0282	0.0158	0.0111	0.0096

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
h_{mean}	0.0283	0.0160	0.0113	0.0109
h_{std}	0.0002	0.0002	0.0002	0.0023

Considering the very small values of standard deviation, we can keep the averages of each adimensional damping ratio for each mode as the estimated value. This leads us to an underdamped system as we have:

$$h_1 = 0.0283, \quad h_2 = 0.0160, \quad h_3 = 0.0113, \quad h_4 = 0.0109$$

Doing the same for the estimated resonance frequencies (Hz), we obtain the following tables:

	Mode 1	Mode 2	Mode 3	Mode 4
f_{d,FRF_1}	1.0500	1.9666	2.6666	3.0833
f_{d,FRF_2}	1.0500	1.9833	2.6666	3.0666
f_{d,FRF_3}	1.0500	1.9666	2.6666	3.0666
f_{d,FRF_4}	1.0500	1.9833	2.6666	3.0666

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$f_{d,mean}$	1.0500	1.9750	2.6666	3.0708
$f_{d,std}$	0	0.0096	0	0.0083

Considering the very small values of standard deviation, we can keep the averages of each frequency of resonance for each mode as the estimated value. This leads us to values highly coherent with the considered FRFs plots as we have:

$$f_{d,1} = 1.0500 \text{ Hz}, \quad f_{d,2} = 1.9750 \text{ Hz}, \quad f_{d,3} = 2.6666 \text{ Hz}, \quad f_{d,4} = 3.0708 \text{ Hz}$$

Finally, for the mode shapes (m), we obtain:

	Mode 1	Mode 2	Mode 3	Mode 4
$X_1^{(i)}$	0.1891	0.4335	0.3852	0.1209
$X_2^{(i)}$	0.2928	0.2031	-0.3121	-0.1943
$X_3^{(i)}$	0.2704	-0.3393	-0.1325	0.2123
$X_4^{(i)}$	0.1308	-0.3565	0.4094	-0.1736

We remind that each estimated mode shape is the parameter used to express the approximated FRF for each mode (i.e. considering a finite segment of the overall FRF), for each different point of measurement. There is no need in computing the mean or standard deviation here as each mode shape helps to characterize a different section of the FRFs.

3 Residual minimization

3.1 Estimation procedure

The generic FRF $H_{j,k}(\Omega)$ is expressed as:

$$H_{j,k}(\Omega) = \sum_{i=1}^{n=4} \frac{X_j^{(i)} X_k^{(i)}}{-\Omega^2 m_{q_{ii}} + i\Omega c_{q_{ii}} + k_{q_{ii}}}$$

The experimental FRF, evaluated in the frequency range of interest, is given by a resonating contribution, a quasi-static contribution and a seismographic contribution as:

$$H_{j,k}^{\mathrm{exp}}(\Omega_f) = \frac{A_j + iB_j}{-\Omega_f^2 m_{q_{ii}} + i\Omega_f c_{q_{ii}} + k_{q_{ii}}} + (C_j + iD_j) + \frac{E_j + iF_j}{\Omega_f^2}, \quad \Omega_f \in [\Omega_A, \Omega_B]$$

The angular frequencies Ω_A and Ω_B are chosen, as said before, so that the resonating contribution is predominant with respect to the others.
By fixing the modal normalization with $m_{q_{ii}} = 1$, it is possible to obtain the values of $\overline{A}_j$ and $\overline{B}_j$:

$$H_{j,k}^{\mathrm{exp}}(\Omega_f) = \frac{\overline{A}_j + i\overline{B}_j}{-\Omega_f^2 + i\Omega_f c_{q_{ii}} + k_{q_{ii}}} + (C_j + iD_j) + \frac{E_j + iF_j}{\Omega_f^2}$$

The modal parameters identification consists in a problem of $n_f \times n_m$, where n_f is the number of frequency bins (between Ω_A and Ω_B) and n_m is the number of measurements (equal to 4, one for each displacement).
The unknowns of this problem are $c_{q_{ii}}$, $k_{q_{ii}}$, $\overline{A}_j$, $\overline{B}_j$, C_j, D_j, E_j and F_j. So, for each *i-th* mode, the number of unknowns is $2 + 6n_m$ (because $j = 1, 2, 3, 4$).

In order to have an over-determined system, n_f should be sufficiently large. It is now possible to solve the minimization problem with the mean square method.
The minimization problem consists in finding the parameters vector $\underline{x}$ as:

$$\underline{x}_{(2+6n_m)\times 1} = \left(c_{q_{ii}}, k_{q_{ii}}, \bar{A}_1, \bar{B}_1, ..., F_1, \bar{A}_2, \bar{B}_2, ..., F_2 ... F_{n_m}\right)^T$$

So that the energy J of ϵ the error of the approximation is minimum:

$$\epsilon_r = H_{j,k}(\Omega_f, \underline{x}) - H_{j,k}^{(\mathrm{exp})}(\Omega_f) \Longrightarrow J = \sum_{r=1}^{n_f \times n_m} \left(\mathrm{Re}\{\epsilon_r\}^2 + \mathrm{Im}\{\epsilon_r\}^2\right)$$

So, to goal is to find the parameters vector $\underline{x}^*$ that best satisfies the Mean-Square Error (MSE) problem also known as minimization problem:

$$\underline{x}^* = \arg\left\{\min_{\underline{x}} (J)\right\}$$

The values of $c_{q_{ii}}$ and $k_{q_{ii}}$ are obtained directly, while the other coefficients depend on the chosen resonance zone of the experimental FRF.
In order to check the validity of the results, it is possible to observe that B_j should be small, so that $A_j + iB_j = X_j^{(i)} X_k^{(i)}$. In addition to that, the other four coefficients should also be small with respect to A_j, since the resonating contribution is the predominant one.

The parameters vector search is implemented in MATLAB by means of the `fminsearch` function, which is a built-in optimization function that aims to find the minimum of an unconstrained multi-variable function. More specifically, this function is called as follows in the code:

```
xpar = fminsearch(@(xpar) errHjki_cw(xpar, rfHjki, Hjkiexp(:, jj)), xpar0, options);
```

The instruction `errHjki_cw(xpar, rfHjki, Hjkiexp(:, jj))` calls the function `errHjki_cw`, which computes the energy of the approximation error given the approximation parameters contained in the `xpar` array.
The minimization algorithm is initialised with the `xpar0` array, obtained from the section 2 results. Specifying `@xpar` informs the algorithm that the energy is studied when varying the values of `xpar`.

Morever, the `options` argument is declared as follows:

```
options = optimset('fminsearch');
options = optimset(options, 'TolFun', 1e-8, 'TolX', 1e-8);
```

Those commands initialize the `options` structure with the default settings for the `fminsearch` function and set the termination tolerance on the function value and solution vector: i.e. the minimization algorithm will stop running when the changes in both the function and solution are $\leq 10^{-8}$.
When the minimization problem is solved, it is possible to reconstruct the experimental FRF defined earlier thanks to the `funHjk` function. Note that the function reconstruction occurs in the arbitrary selected frequency range.

3.2 Results

Solving the minimization problem with the Matlab script, we obtain 16 different estimations of the adimensional damping ratios h_i:

	Mode 1	Mode 2	Mode 3	Mode 4
$h^{MSE}_{FRF_1}$	0.0286	0.0283	0.0283	0.0282
$h^{MSE}_{FRF_2}$	0.0127	0.0134	0.0136	0.0137
$h^{MSE}_{FRF_3}$	0.0099	0.0097	0.0102	0.0103
$h^{MSE}_{FRF_4}$	0.0086	0.0081	0.0092	0.0087

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
h^{MSE}_{mean}	0.0283	0.0134	0.0100	0.0087
h^{MSE}_{std}	$0.1581 \cdot 10^{-3}$	$0.4280 \cdot 10^{-3}$	$0.3020 \cdot 10^{-3}$	$0.4246 \cdot 10^{-3}$

Considering the very small values of standard deviation, we can keep the averages of each adimensional damping ratio for each mode as the estimated value. This leads us to an underdamped system as we have:

$$h^{MSE}_1 = 0.0283, \quad h^{MSE}_2 = 0.0134, \quad h^{MSE}_3 = 0.0100, \quad h^{MSE}_4 = 0.0087$$

Doing the same for the estimated resonance frequencies (Hz), we obtain the following tables:

	Mode 1	Mode 2	Mode 3	Mode 4
f^{MSE}_{d,FRF_1}	1.0500	1.9757	2.6689	3.0693
f^{MSE}_{d,FRF_2}	1.0500	1.9752	2.6697	3.0673
f^{MSE}_{d,FRF_3}	1.0500	1.9753	2.6675	3.0652
f^{MSE}_{d,FRF_4}	1.0500	1.9751	2.6686	3.0635

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$f^{MSE}_{d,mean}$	1.0500	1.9753	2.6687	3.0663
$f^{MSE}_{d,std}$	0	0.0003	0.0009	0.0025

Considering the very small values of standard deviation, we can keep the averages of each frequency of resonance for each mode as the estimated value. This leads us to values highly coherent with the considered FRFs plots as we have:

$$f^{MSE}_{d,1} = 1.0500 \text{ Hz}, \quad f^{MSE}_{d,2} = 1.9753 \text{ Hz}, \quad f^{MSE}_{d,3} = 2.6687 \text{ Hz}, \quad f^{MSE}_{d,4} = 3.0663 \text{ Hz}$$

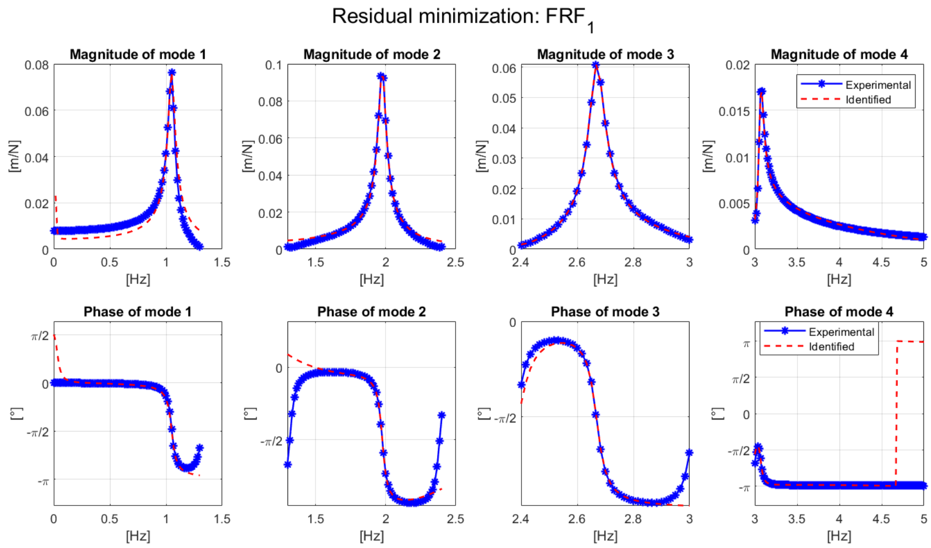
Finally, for the mode shapes (m), we obtain:

	Mode 1	Mode 2	Mode 3	Mode 4
$X_1^{(i)MSE}$	0.1891	0.3795	0.3396	0.0958
$X_2^{(i)MSE}$	0.2928	0.1841	-0.2685	-0.1569
$X_3^{(i)MSE}$	0.2704	-0.3217	-0.1193	0.2033
$X_4^{(i)MSE}$	0.1308	-0.3366	0.3872	-0.1589

Once again, there is no need in computing the mean or standard deviation here as each mode shape helps to characterize a different section of the FRFs.

3.3 Experimental and Approximated FRFs comparison

The identification plots for the first FRF is given:



We can see that both the approximated and experimental FRFs are very similar both in amplitude and phase. However, we also see that the FRF approximation becomes less accurate as $f \to 0$. Indeed, the approximated FRF is based on the assumption that we study the response of the system around frequencies for which the resonating mode is the prevalent one, which becomes less and less relevant as $f \to 0$.

The phase response of the mode 4 for the experimental and identified FRFs differ for $f \geq 4$ Hz (roughly). Indeed, this is due to the fact that we are in the seismographic zone (experimental FRF) of the system while still considering that the resonating mode is the prevalent one (identified FRF).

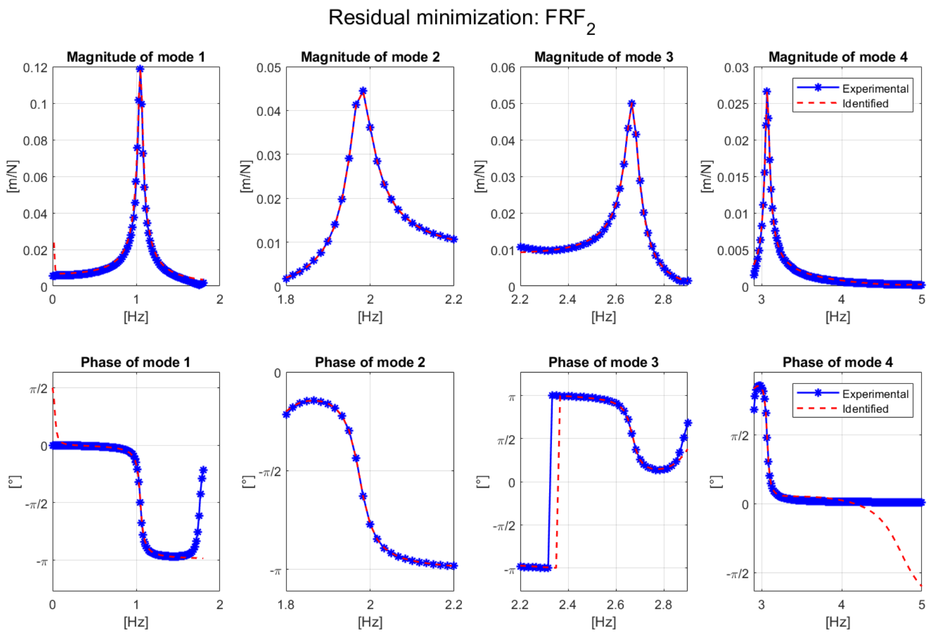
Between the first and second modes, the experimental FRF shows a node of vibration around 1.3 Hz, while the identified one does not show it.

The plot of the phase response for the mode 4 presents a phase wrap around 4.6 Hz.

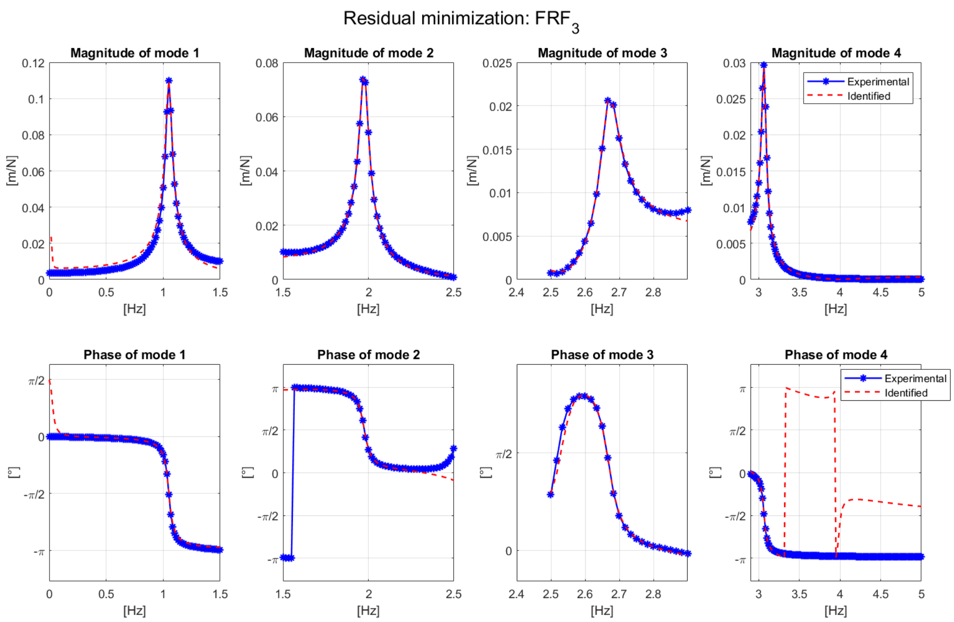
Comments about the similarity of the experimental and identified FRFs still stand for the next graphs : overall similarity, and difference for $f \to 0$ and $f \geq 4$ Hz.

Differences in phase responses between the 2 plotted FRFs around the phase wrappings should not be accounted.

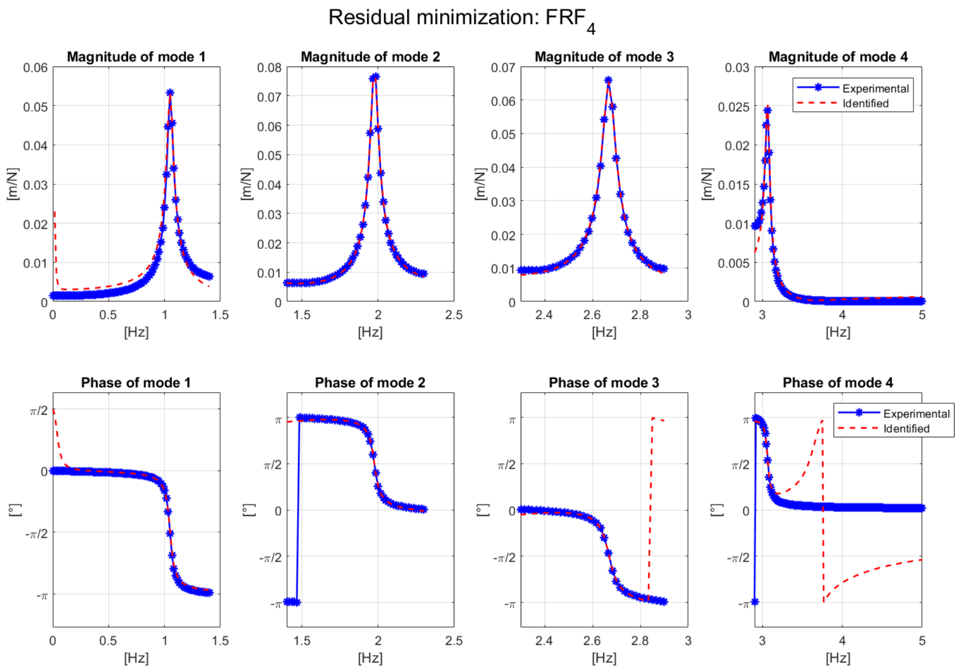
The identification plots for the second FRF is given:



The identification plots for the third FRF is given:



The identification plots for the fourth FRF is given:



4 Modal parameters comparison

4.1 Simplified method

Using a simplified method for estimating the different relevant parameters of our FRFs (section 2), and using the following formulas

$$k_{q_{ii}} = m_{q_{ii}}\omega_{d_i}^2 = \omega_{d_i}^2, \quad c_{q_{ii}} = 2m_{q_{ii}}\omega_{d_i}h_i = 2\omega_{d_i}h_i, \quad \text{such that } m_{q_{ii}} = 1$$

we obtain the following numerical values for the modal stiffness (N/m) from our script :

	Mode 1	Mode 2	Mode 3	Mode 4
$k_{q_{ii}}^{FRF_1}$	43.5235	152.6887	280.7261	375.3066
$k_{q_{ii}}^{FRF_2}$	43.5235	155.2876	280.7261	371.2602
$k_{q_{ii}}^{FRF_3}$	43.5235	152.6887	280.7261	371.2602
$k_{q_{ii}}^{FRF_4}$	43.5235	155.2876	280.7261	371.2602

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$k_{q_{ii},mean}$	43.5235	153.9881	280.7261	372.2718
$k_{q_{ii},std}$	0	1.5005	0	2.0232

Considering the very small values of standard deviation, we can keep the averages of each modal stiffness for each mode as the estimated value:

$$k_{q_{11}} = 43.5235 \text{ N/m}, \quad k_{q_{22}} = 153.9881 \text{ N/m}, \quad k_{q_{33}} = 280.7261 \text{ N/m}, \quad k_{q_{44}} = 372.2718 \text{ N/m}$$

We obtain the following numerical values for the modal damping ($N.s/m$) from our script :

	Mode 1	Mode 2	Mode 3	Mode 4
$c_{q_{ii}}^{FRF_1}$	0.1886	0.1989	0.1894	0.2787
$c_{q_{ii}}^{FRF_2}$	0.1867	0.1997	0.1895	0.1900
$c_{q_{ii}}^{FRF_3}$	0.1864	0.1979	0.1928	0.1861
$c_{q_{ii}}^{FRF_4}$	0.1863	0.1963	0.1862	0.1856

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$c_{q_{ii},mean}$	0.1870	0.1982	0.1895	0.2101
$c_{q_{ii},std}$	0.0010	0.0015	0.0027	0.0458

Considering the very small values of standard deviation, we can keep the averages of each modal stiffness for each mode as the estimated value:

$$c_{q_{11}} = 0.1870 \text{ Ns/m}, \quad c_{q_{22}} = 0.1982 \text{ Ns/m}, \quad c_{q_{33}} = 0.1895 \text{ Ns/m}, \quad c_{q_{44}} = 0.2101 \text{ Ns/m}$$

Due to the normalization we used, we have that $m_{q_{ii}} = 1 \quad \forall i \in [1, 4]$.

4.2 Minimization method

From the MSE algorithm, we simply extract the values for each modal parameters and their mean value and standard deviation, after computation, give us:

	Mode 1	Mode 2	Mode 3	Mode 4
$m_{q_{ii},mean}^{MSE}$	1	1	1	1
$m_{q_{ii},std}^{MSE}$	0	0	0	0
$k_{q_{ii},mean}^{MSE}$	43.5235	154.0408	281.1602	371.1879
$k_{q_{ii},std}^{MSE}$	0	0.0431	0.1931	0.6156
$c_{q_{ii},mean}^{MSE}$	0.3740	0.3314	0.3367	0.3336
$c_{q_{ii},std}^{MSE}$	0.0021	0.0106	0.0100	0.0163

Considering the very small values of standard deviation, we can keep the averages of each modal stiffness for each mode as the estimated value:

$$k_{q_{11}}^{MSE} = 43.5235 \text{ N/m}, \quad k_{q_{22}}^{MSE} = 154.0408 \text{ N/m}, \quad k_{q_{33}}^{MSE} = 281.1602 \text{ N/m}, \quad k_{q_{44}}^{MSE} = 371.1879 \text{ N/m}$$

$$c_{q_{11}}^{MSE} = 0.3740 \text{ Ns/m}, \quad c_{q_{22}}^{MSE} = 0.3314 \text{ Ns/m}, \quad c_{q_{33}}^{MSE} = 0.3367 \text{ Ns/m}, \quad c_{q_{44}}^{MSE} = 0.3336 \text{ Ns/m}$$

4.3 Comparison

We see that both methods provide satisfying (very low standard deviation) yet different results. Indeed, values of the estimated modal damping factors $c_{q_{ii}}$ are significantly different from the simplified method compared to the MSE. This will be clear in the modal FRF plots in next section.
The values of $k_{q_{ii}}$ and $c_{q_{ii}}$, obtained with the two methods, are given:

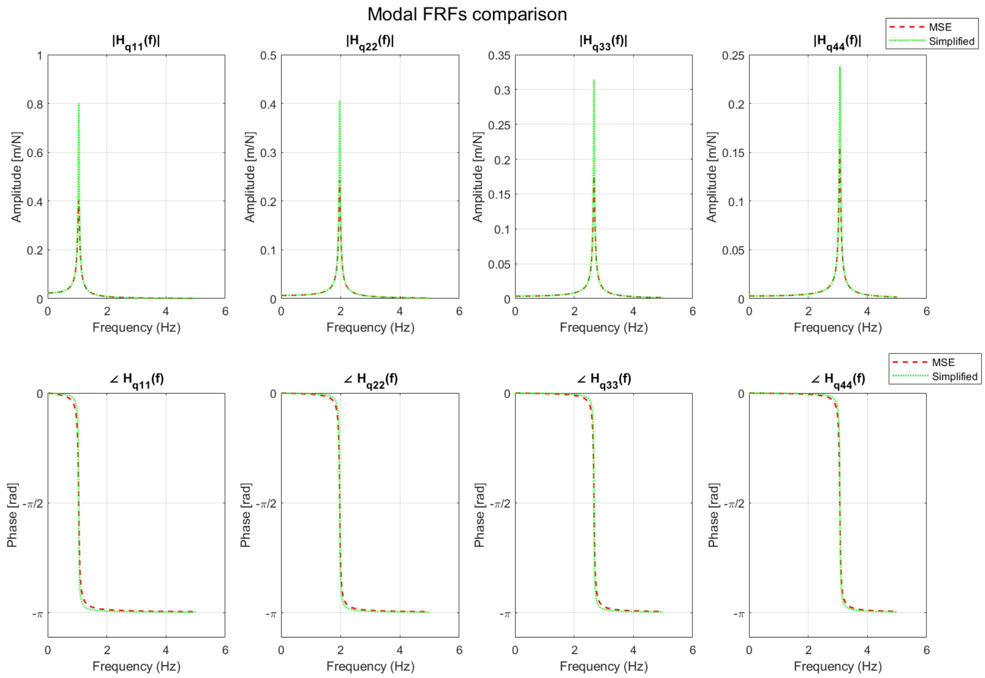
	Mode 1	Mode 2	Mode 3	Mode 4
$c_{q_{ii}}^{MSE}$	0.3740	0.3314	0.3367	0.3336
$c_{q_{ii}}^{simp}$	0.1870	0.1982	0.1895	0.2101
$k_{q_{ii}}^{MSE}$	43.5235	154.0408	281.1602	371.1879
$k_{q_{ii}}^{simp}$	43.5235	153.9881	280.7261	372.2718

Moreover, the MSE algorithm also tries to provide a parametric representation of the different experimental FRFs to best fit the measurements. One might want to compare the reconstructed FRFs of each method to better see which one gives the most satisfying results.

5 FRF reconstruction

In order to reconstruct the experimental FRFs, it is possible to apply the modal approach. Each mode is described by a modal FRF as:

$$H_{q_{ii}}(\Omega) = \frac{1}{-\Omega^2 m_{q_{ii}} + i\Omega c_{q_{ii}} + k_{q_{ii}}}, \quad m_{q_{ii}} = 1$$



It is possible to notice that the different $c_{q_{ii}}$ values, between the two different methods, result in different characteristics: higher dampings result in lower amplitude peaks and smoother phase shift around resonance.

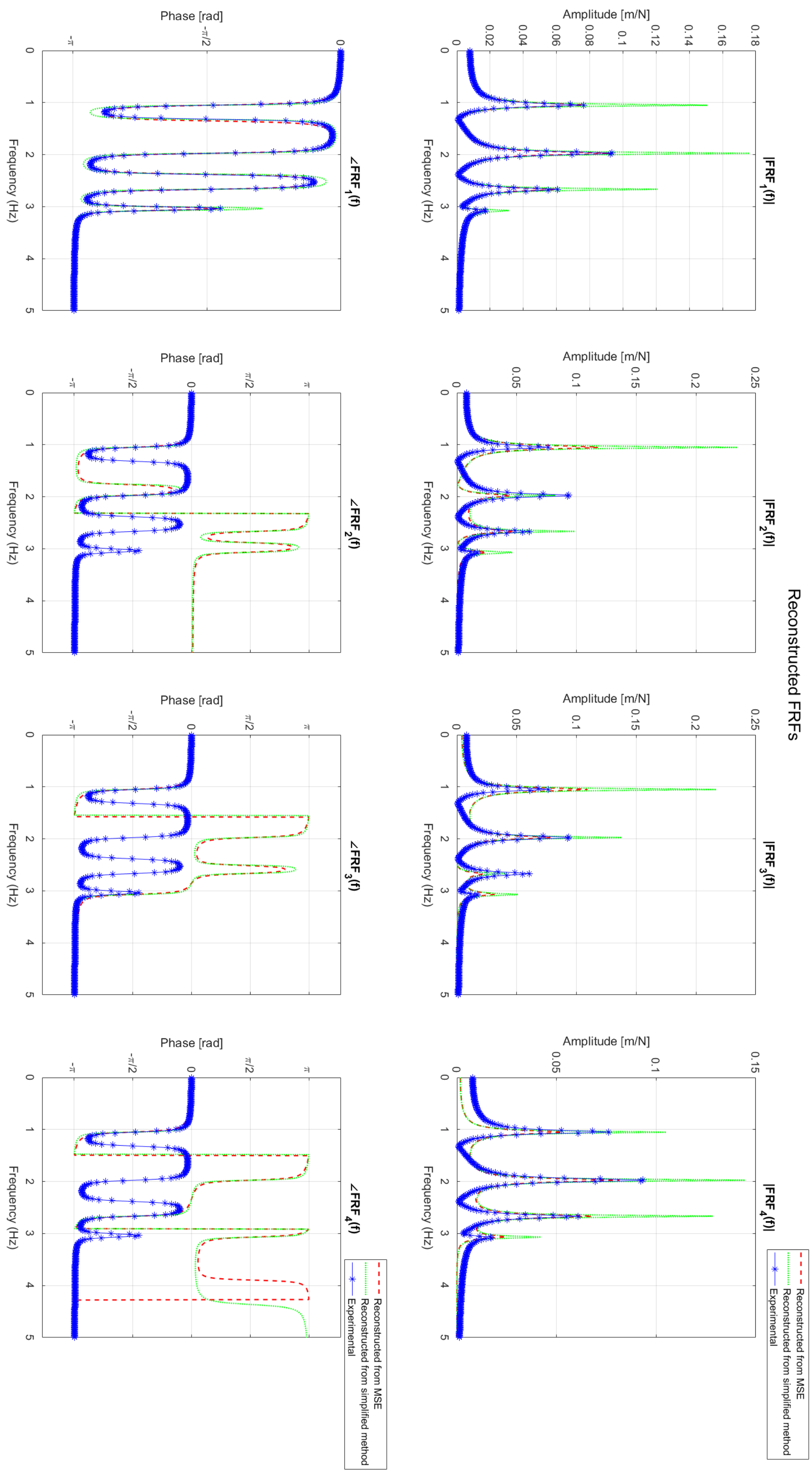
The generic FRF $H_{j,k}$ is given by the product between modal FRF and mode shapes:

$$H_{j,k}(\Omega) = \sum_{i=1}^{4} X_j^{(i)} X_k^{(i)} H_{q_{ii}}(\Omega) = \sum_{i=1}^{4} \frac{A_j + iB_j}{-\Omega^2 m_{q_{ii}} + i\Omega c_{q_{ii}} + k_{q_{ii}}}, \quad B_j << A_j$$

The reconstructed FRF plots (FRF$_i$ in the figure, with $i = 1, 2, 3, 4$) are given in the next page.

Reconstructing the FRFs for each point of measurement allow us to see that the minimization algorithm gives way better results than the simplified method. Indeed, the MSE's amplitude response is way closer to the one obtained from experimental results, as for the phase response. However, it doesn't perfectly fit the experimental FRFs.
The nodes of the experimental FRFs and the ones obtained through reconstruction do not perfectly match. Once again, differences due to phase wraps should not be accounted.

Note that each FRF $H_{j,k}$ describes the system response to a force $F_k(t)$ (applied at the k-*th* point force) at the measurement $x_j(t)$. So, the co-located FRF at the location of the displacement $x_2(t)$ would be equal to $H_{2,2}$.


Reconstructed FRFs
|FRF1(f)|
|FRF2(f)|
|FRF3(f)|
|FRF4(f)|
∠FRF1(f)
∠FRF2(f)
∠FRF3(f)
∠FRF4(f)
Amplitude [m/N]
Phase [rad]
Frequency (Hz)
Reconstructed from MSE
Reconstructed from simplified method
Experimental