VIBRATION ANALYSIS
AND VIBROACOUSTICS

VIBRATION ANALYSIS

Assignment 1 - A.Y. 2023/24

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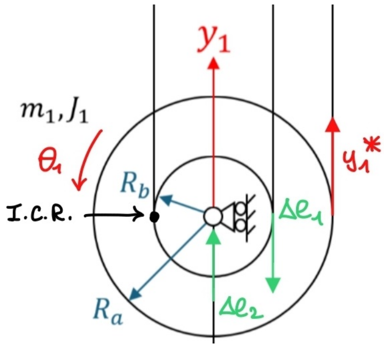
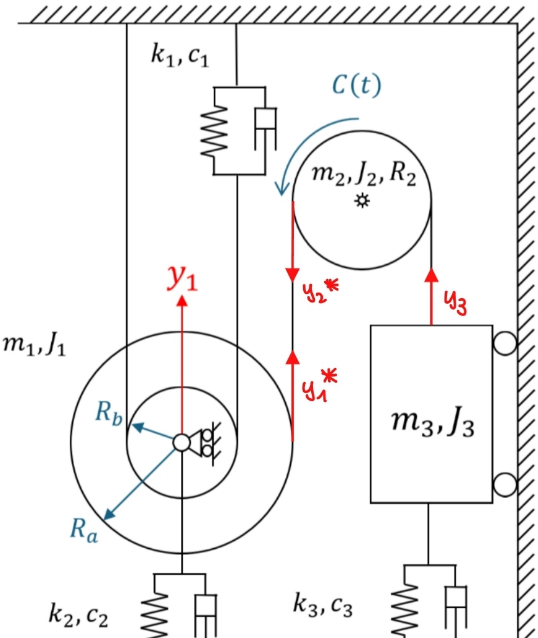
1 Question 1

1.1 Equation of Motion

The mechanical system in figure presents different physical variables and one independent variable. The motion of the mass m_1 is both translational and rotational, so both displacement y_1 and angle θ_1 are used to describe its energy forms. The presence of an inextensible string at the left of the disk of radius R_b determines an instantaneous center of rotation (I.C.R.) at the conjunction between string and disk.



The constraints on the disk of mass m_2 force it to rotate only, so the angle θ_2 is sufficient to describe its motion.
The mass m_3, on the other hand, can only translate vertically, so the displacement y_3 describes completely its motion.



By considering the displacements y_i positive upwards and the rotations θ_i positive counter-clockwise, the physical variables are defined in function of the independent variable y_1 as:

$$y_1 = R_b\theta_1 \Longrightarrow \theta_1 = \frac{y_1}{R_b}, \quad \omega_1 = \dot{\theta}_1 = \frac{v_1}{R_b} = \frac{\dot{y_1}}{R_b}$$

$$y_1^* = -y_2^* \Longrightarrow (R_a + R_b)\theta_1 = -R_2\theta_2$$

$$\theta_2 = \frac{R_a + R_b}{R_2}\frac{y_1}{R_b} = -\frac{R_a + R_b}{R_2 R_b}y_1, \quad \omega_2 = \dot{\theta}_2 = -\frac{R_a + R_b}{R_2 R_b}\dot{y_1}$$

$$y_3 = -y_2^* = -R_2\theta_2 = \frac{R_a+R_b}{R_b}y_1, \quad v_3 = \dot{y_3} = \frac{R_a+R_b}{R_b}\dot{y_1}$$

Regarding the elongations Δl_i of the springs and the altitudes h_i of the masses:

$$\Delta l_1 = -2y_1, \quad \Delta l_2 = y_1, \quad \Delta l_3 = y_3 = \frac{R_a+R_b}{R_b}y_1$$

$$h_1 = y_1, \quad h_3 = y_3 = \frac{R_a+R_b}{R_b}y_1$$

Let's consider the energy forms of the Lagrange equation one by one. The kinetic energy is given by two translational components and two rotational components:

$$E_K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_3v_3^2 + \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}\omega_2^2 =$$

$$= \frac{1}{2}m_1\dot{y_1}^2 + \frac{1}{2}m_3\left(\frac{R_a+R_b}{R_b}\right)^2\dot{y_1}^2 + \frac{1}{2}J_1\left(\frac{1}{R_b}\right)^2\dot{y_1}^2 + \frac{1}{2}J_2\left(-\frac{R_a+R_b}{R_2R_b}\right)^2\dot{y_1}^2 =$$

$$= \frac{1}{2}\left(m_1 + \left(\frac{R_a+R_b}{R_b}\right)^2 m_3 + \frac{J_1}{R_b^2} + J_2\left(\frac{R_a+R_b}{R_2R_b}\right)^2\right)\dot{y_1}^2 = \frac{1}{2}M_{eq}\dot{y_1}^2$$

The potential energy has three elastic components and two gravitational components:

$$V = \frac{1}{2}k_1\Delta l_1^2 + \frac{1}{2}k_2\Delta l_2^2 + \frac{1}{2}k_3\Delta l_3^2 + m_1gh_1 + m_3gh_3 =$$

$$= \frac{1}{2}k_1(-2y_1)^2 + \frac{1}{2}k_2y_1^2 + \frac{1}{2}k_3\left(\frac{R_a+R_b}{R_b}y_1\right)^2 + m_1gy_1 + m_3g\left(\frac{R_a+R_b}{R_b}\right)y_1 =$$

$$= \frac{1}{2}\left(4k_1 + k_2 + \left(\frac{R_a+R_b}{R_b}\right)^2 k_3\right)y_1^2 + \left(m_1 + \frac{R_a+R_b}{R_b}m_3\right)gy_1 = \frac{1}{2}k_{eq}y_1^2 + m_{eq}gy_1$$

The dissipative function has three terms:

$$D = \frac{1}{2}c_1\dot{\Delta l_1}^2 + \frac{1}{2}c_2\dot{\Delta l_2}^2 + \frac{1}{2}c_3\dot{\Delta l_3}^2 = \frac{1}{2}c_1(-2\dot{y_1})^2 + \frac{1}{2}c_2\dot{y_1}^2 + \frac{1}{2}c_3\left(\frac{R_a+R_b}{R_b}\dot{y_1}\right)^2 =$$

$$= \frac{1}{2}\left(4c_1 + c_2 + \left(\frac{R_a+R_b}{R_b}\right)^2 c_3\right)\dot{y_1}^2 = \frac{1}{2}c_{eq}\dot{y_1}^2$$

The only work contribution is given by the torque $C(t)$ applied on the disk m_2:

$$\delta W = C(t)\delta\theta_2 = C(t)\left(-\frac{R_a+R_b}{R_2R_b}\right)\delta y_1 = Q_{y_1}\delta y_1$$

It is now possible to apply the energies into the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{y_1}}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y_1}}\left(\frac{1}{2}M_{eq}\dot{y_1}^2\right)\right) = \frac{d}{dt}\left(M_{eq}\dot{y_1}\right) = M_{eq}\ddot{y_1}$$

$$\frac{\partial E_K}{\partial y_1} = 0$$

$$\frac{\partial D}{\partial \dot{y_1}} = \frac{\partial}{\partial \dot{y_1}}\left(\frac{1}{2}c_{eq}\dot{y_1}^2\right) = c_{eq}\dot{y_1}$$

$$\frac{\partial V}{\partial y_1} = \frac{\partial}{\partial y_1}\left(\frac{1}{2}k_{eq}y_1^2 + m_{eq}gy_1\right) = k_{eq}y_1 + m_{eq}g$$

Giving the result:

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{y_1}}\right) + \frac{\partial E_K}{\partial y_1} + \frac{\partial D}{\partial \dot{y_1}} + \frac{\partial V}{\partial y_1} = Q_{y_1} \Longrightarrow M_{eq}\ddot{y_1} + c_{eq}\dot{y_1} + k_{eq}y_1 = \left(-\frac{R_a+R_b}{R_2R_b}\right)C(t) - m_{eq}g$$

1.2 Natural frequency

The natural frequency is a parameter of the system (in free motion) in the undamped case:

$$M_{eq}\ddot{y}_1 + k_{eq}y_1 = 0 \longrightarrow M_{eq}\lambda^2 + k_{eq} = 0 \Longrightarrow \lambda_{1,2} = \pm j\sqrt{\frac{k_{eq}}{M_{eq}}} = \pm j\omega_0 \Longrightarrow \omega_0 = \sqrt{\frac{k_{eq}}{M_{eq}}} = 9.8554 \text{ rad/s}$$

1.3 Adimensional damping factor and damped frequency

By considering also the effect of the dampers, the adimensional damping factor is defined as:

$$h = \frac{c_{eq}}{2M_{eq}\omega_0} = \frac{\alpha}{\omega_0} = 0.1057$$

together with the damped frequency:

$$\omega_d = \omega_0\sqrt{1-h^2} = 9.8002 \text{ rad/s} \ \approx \omega_0$$

2 Question 2

2.1 Free motion response

A damped system in free motion is described by the homogeneous equation of motion:

$$M_{eq}\ddot{y} + c_{eq}\dot{y} + k_{eq}y = 0$$

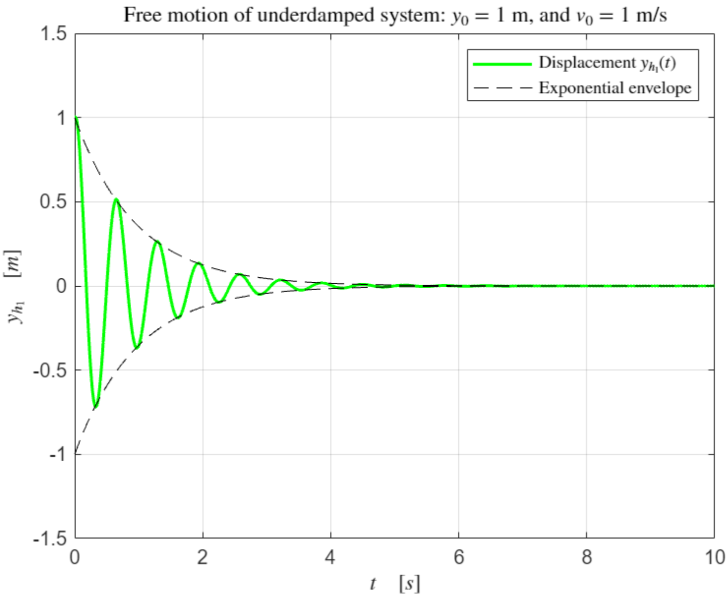
Its solution $y_{1_h}(t)$ is obtained by solving the characteristic equation:

$$M_{eq}\lambda^2 + c_{eq}\lambda + k_{eq} = 0$$

For the underdamped case (given by the small value of $h \approx 10\%$), the homogeneous solution $y_{1_h}(t)$ (in second form) is given by decaying sinusoids:

$$M_{eq}\lambda^2 + c_{eq}\lambda + k_{eq} = 0 \implies \lambda_{1,2} = -\alpha \pm j\omega_0\sqrt{1-h^2} = -\alpha \pm j\omega_d$$

$$\implies y_{h_1}(t) = e^{-\alpha t}\left[y_0\cos(\omega_d t) + \frac{v_0 + \alpha y_0}{\omega_d}\sin(\omega_d t)\right]$$

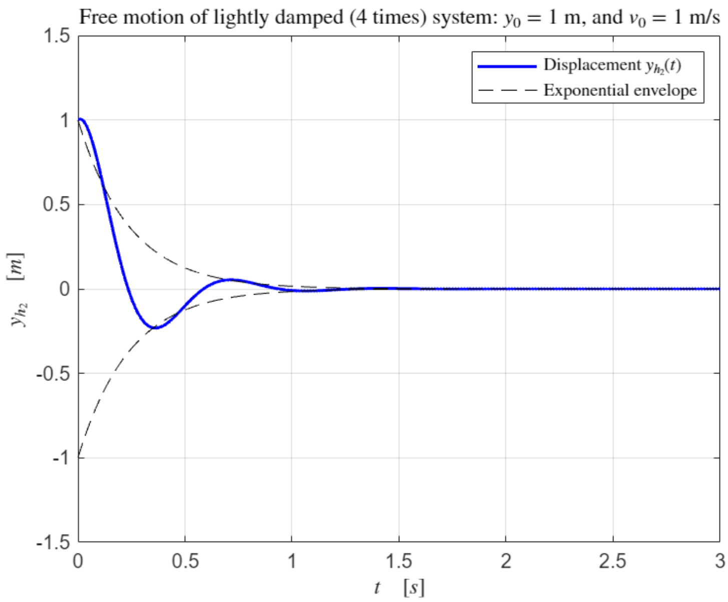


The initial conditions y_0 and v_0 represent the displacement and velocity values at $t = 0$ and they determine the coefficients of the two sinusoidal components. Considering the small value of $h \approx 10\%$, the system is underdamped, so the damped frequency $\omega_d \approx \omega_0$ as said in section 1.3. The decaying time is given by $\frac{5}{\alpha} \approx 4.80$ seconds, which gives the "slowly" decaying trend to the plot. The steady-state of the system is zero because of the free motion scenario.

2.2 Free motion response $(h \to 4h)$

The new value of $h_2 = 8h = 0.4228$, which means the system is lightly damped. The form of the homogeneous solution is the same, only the coefficients change. With respect to the previous point, $\alpha_2 = 4\alpha$ and $\omega_{d_2} = \omega_0\sqrt{1-h_2^2} = 8.9311$ rad/s:

$$y_{h_2}(t) = e^{-\alpha_2 t}\left[y_0\cos(\omega_{d_2} t) + \frac{v_0 + \alpha_2 y_0}{\omega_{d_2}}\sin(\omega_{d_2} t)\right]$$



The decaying time is given by $\frac{5}{\alpha_2} \approx 1.20$ seconds, which gives a faster (with respect to the previous case) decaying trend to the plot. The steady-state of the system is zero because of the free motion scenario.

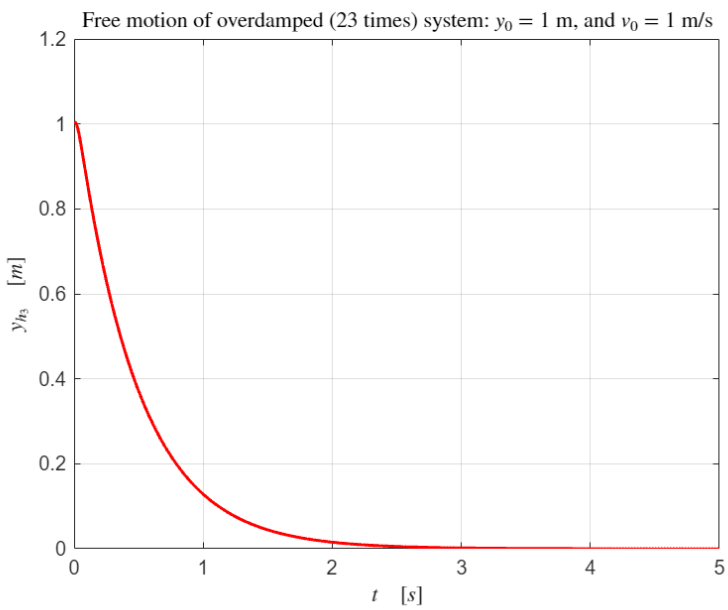
2.3 Free motion response ($h \to 23h$)

The new value of $h_3 = 23h = 2.4312$, which means the system is overdamped. In this case, the solutions λ_1 and λ_2 of the characteristic equation are real:

$$\lambda_1 = -\alpha_3 - \sqrt{\alpha_3^2 - \omega_0^2} = -a_1, \quad \lambda_2 = -\alpha_3 + \sqrt{\alpha_3^2 - \omega_0^2} = -a_2$$

The homogeneous solution $y_{1_h}(t)$ does not oscillate anymore:

$$y_{h_3}(t) = -\left(\frac{v_0 + a_2 y_0}{a_1 - a_2}\right) e^{-a_1 t} + \left(\frac{v_0 + a_1 y_0}{a_1 - a_2}\right) e^{-a_2 t}$$



The decaying time is mainly given by $\frac{5}{a_2} \approx 2.36$ seconds. The steady-state of the system is zero because of the free motion scenario.

3 Question 3

3.1 Forced motion: frequency response function

The forced motion adds the external forces and torques contribution to the equation of the free vibrations:

$$M_{eq}\ddot{y} + c_{eq}\dot{y} + k_{eq}y = F(t) = F_0 \cos(\Omega t + \varphi)$$

The complete time solution is the sum of the homogeneous solution $y_{1_h}(t)$ (from section 2.1) and a particular solution given by:

$$y_{1_p}(t) = Y_0 \cos(\Omega t + \varphi + \beta)$$

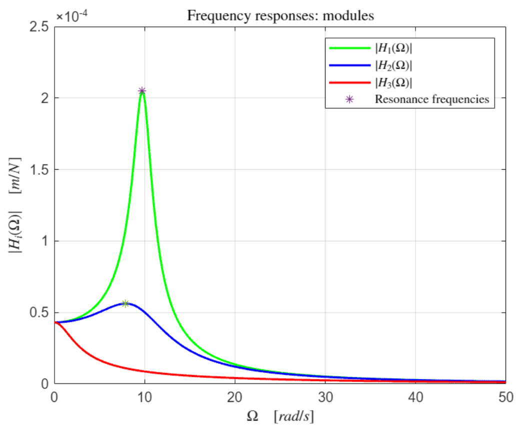
Or equivalently in a complex form:

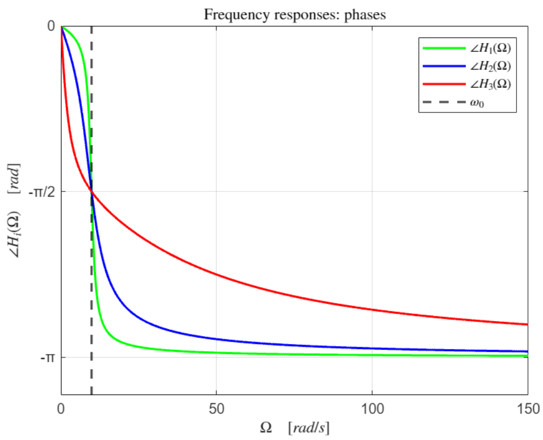
$$\tilde{y}_{1_p}(t) = \tilde{Y}_0 e^{j(\Omega t + \varphi)}, \quad \tilde{Y}_0 = |\tilde{Y}_0| e^{j\beta} \in \mathbb{C}$$

The frequency response function $H(\Omega)$ compares the system result $\tilde{Y}_0$ and the external force contribution F_0: in general, an amplification or attenuation $|\tilde{Y}_0|$ and a phase shift β will be introduced by the mechanical system.

$$H(\Omega) = \frac{\tilde{Y}_0}{F_0} = \frac{\frac{1}{M_{eq}}}{\omega_0^2 - \Omega^2 + j2\alpha\Omega}$$

Where ω_0 is the natural frequency and $\alpha = \frac{c_{eq}}{2M_{eq}}$ represents the decaying contribution of the dampers.





In the figures above, the three different scenarios described in question 2 are plotted. The system normally presents (first case in green) an underdamped frequency response: an high quality factor describes the narrow bell centered in $\omega_d \approx \omega_0$, where the module response is almost 8 times higher than the static response ($H(\Omega = 0) = 0.43 \cdot 10^{-4}$ m/N), though the system works like an attenuator for all frequencies (module < 1). The phase response is given by a steep curve which tends to the one of the undamped case, in which an abrupt counterphase occurs in ω_0 (dotted black line).
In the second case (in blue) $h_2 = 4h$, so the damping effect is more present. The module response is more flat, the quality factor and the damped frequency are lower, and the phase response curve is less steep.
In the third case ($h_3 = 23h$, in red) the damping effect is dominant: there is no natural frequency (the system does not oscillate) and phase shift tends to reach $-\frac{\pi}{2}$ more steeply and $-\pi$ way slower.

3.2 Forced motion: complete time response

The complete time response $y_1(t)$ of a forced system is the sum of the homogeneous solution $y_{h_2}(t)$ (defined in section 2.2) and the particular solution $y_{p_2}(t)$ (using the FRF defined in section 3.1):

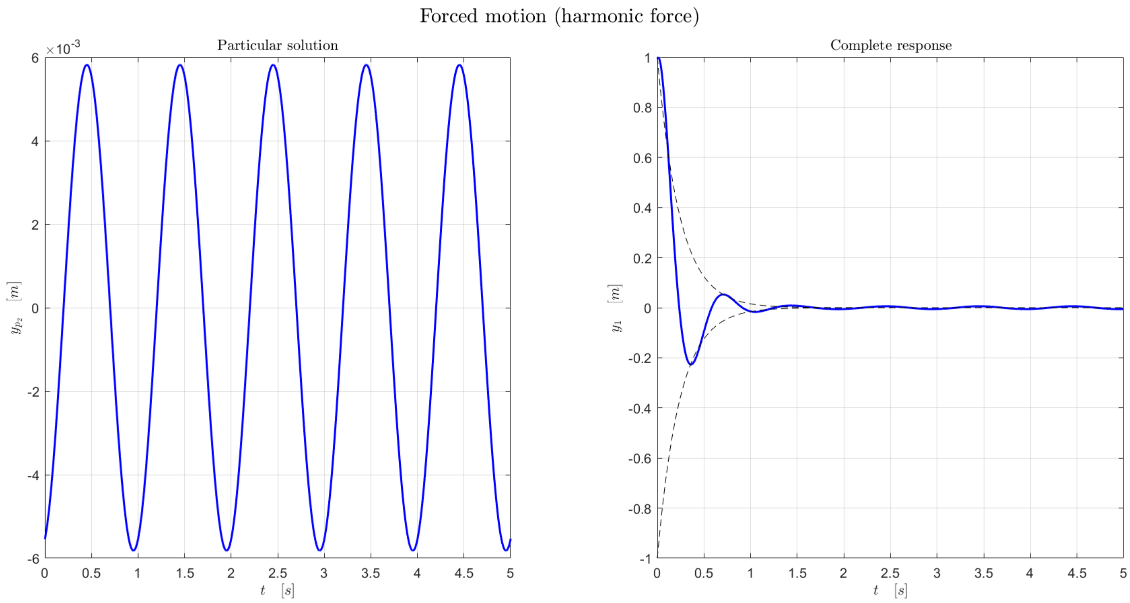
$$y_{h_2}(t) = e^{-\alpha_2 t}\left[y_0 \cos(\omega_{d_2} t) + \frac{v_0 + \alpha_2 y_0}{\omega_{d_2}} \sin(\omega_{d_2} t)\right]$$

$$y_{p_2}(t) = Y_0 \cos(\Omega t + \varphi + \beta) = |H(\Omega)| F_0 \cos\left(\Omega t + \varphi + \angle H(\Omega)\right)$$

By considering a simple harmonic torque ($A = 25$ N·m, $f = 1$ Hz, $\varphi = \pi/3$ rad):

$$C(t) = A\cos(2\pi f t + \varphi) \Longrightarrow F(t) = F_0 \cos(2\pi f t + \varphi) = -\frac{R_a + R_b}{R_2 R_b} \cdot A\cos(2\pi f t + \varphi)$$

$$\Longrightarrow y_{p_2}(t) = |H(2\pi f)| F_0 \cos\left(2\pi f t + \varphi + \angle H(2\pi f)\right)$$



3.3 Steady-state response for two frequencies

The steady-state response of the system consists in the form that $y_1(t)$ has for $t \to \infty$. Considering that the homogeneous solution $y_{h_2}(t) \to 0$, the steady-state response is given only by the particular solution $y_{p_2}(t)$.

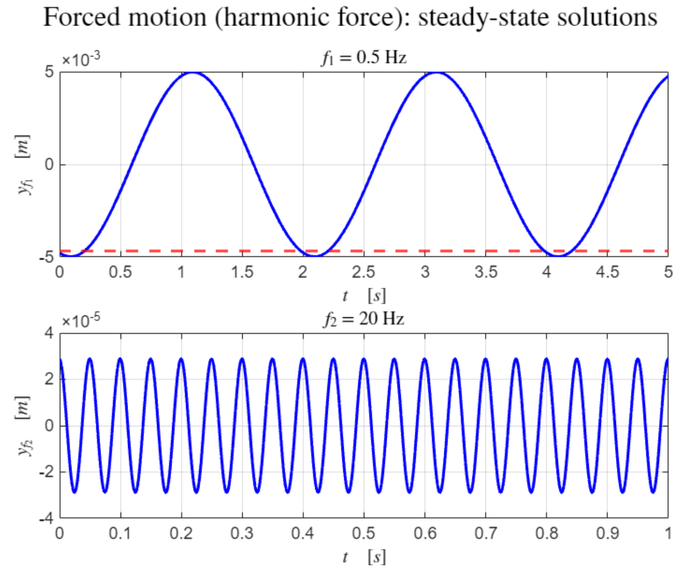
For two different harmonic torques:

$$C(t) = 25\cos(2\pi 0.5t + 0) \Longrightarrow y_{p_2}(t) = |H(2\pi 0.5)| F_0 \cos\left(2\pi 0.5t + \angle H(2\pi 0.5)\right)$$

$$C(t) = 25\cos(2\pi 20t + 0) \Longrightarrow y_{p_2}(t) = |H(2\pi 20)| F_0 \cos\left(2\pi 20t + \angle H(2\pi 20)\right)$$

For a torque statically applied, instead:

$$C(t) = 25\cos(2\pi 0t + 0) = 25 \Longrightarrow y_{p_2}(t) = |H(0)| F_0 \cos\left(2\pi 0t + \angle H(2\pi 0)\right) = |H(0)| F_0$$

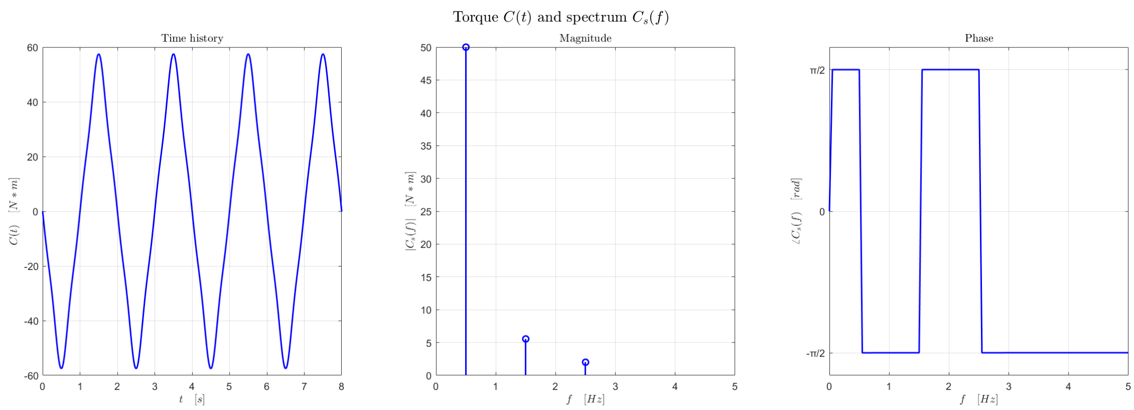


From the frequency response module $|H(\Omega)|$ graph, obtained in section 3.1, it is possible to compare the amplitudes of the different steady-state solutions with the static response. For the first frequency $f_1 = 0.5$ Hz, the amplitude of the steady-state solution (in blue) is close to the static response value $y_{1_{st}} = -4.7 \cdot 10^{-3}$ m (in red): this frequency is located in the quasi-static zone of the frequency response function, where the magnitude is almost constant. For the second frequency $f_2 = 20$ Hz, the amplitude of the steady-state solution is way smaller than the static response: this frequency is located in the seismographic zone of the frequency response function, where the magnitude is way smaller compared to the quasi-static zone and tends to zero.

3.4 Superposition of three harmonic torques

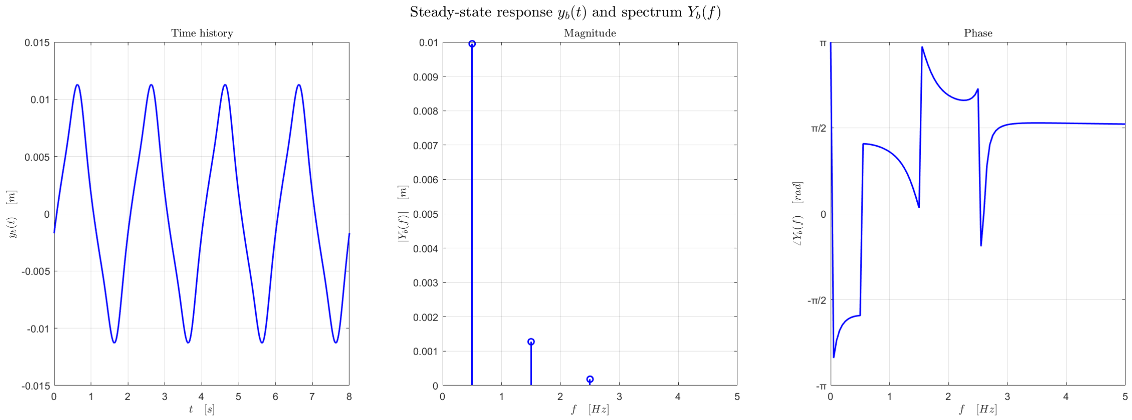
The linearity of the system allows to obtain the output displacement as sum of different input contributions. In case of a torque given by the superposition of three harmonics:

$$C(t) = \sum_{k=1}^{3} B_k \cos\left(2\pi f_k t + \varphi_k\right) = 50\cos\left(2\pi 0.5t + \frac{\pi}{2}\right) + 5.55\cos\left(2\pi 1.5t - \frac{\pi}{2}\right) + 2\cos\left(2\pi 2.5t + \frac{\pi}{2}\right)$$



The overall steady-state response is the superposition of the three particular solutions:

$$y_{1_p}(t) = 50\left(-\frac{R_a+R_b}{R_2R_b}\right)|H(2\pi0.5)|\cos\left(2\pi0.5t+\frac{\pi}{2}+\angle H(2\pi0.5)\right)+$$
$$+5.55\left(-\frac{R_a+R_b}{R_2R_b}\right)|H(2\pi1.5)|\cos\left(2\pi1.5t-\frac{\pi}{2}+\angle H(2\pi1.5)\right)+$$
$$+2\left(-\frac{R_a+R_b}{R_2R_b}\right)|H(2\pi2.5)|\cos\left(2\pi2.5t+\frac{\pi}{2}+\angle H(2\pi2.5)\right)$$



The torque waveform is mainly given by the first harmonic (frequency at 0.5 Hz), considering the amplitude coefficients (coherently represented in the magnitude plot.)
Regarding the steady-state solution, the frequency response module $|H(\Omega)|$ modifies each harmonic contribution of the torque, giving a dominant first harmonic and a more present second harmonic. The relative amplitude difference between the harmonics f_1 and f_2 is higher in the input torque spectrum (almost 10 times) than in the steady-state spectrum (almost 8 times). The third harmonic is negligible in the particular solution (amplitude approximately 10 times lower than the second harmonic amplitude and 84 times lower than the first harmonic amplitude).
The frequency response phase graph is coherent with the model, considering that each harmonic contribution results in sudden phase shift.