VIBRATION ANALYSIS
AND VIBROACOUSTICS

VIBRATION ANALYSIS

Assignment 3 - A.Y. 2023/24

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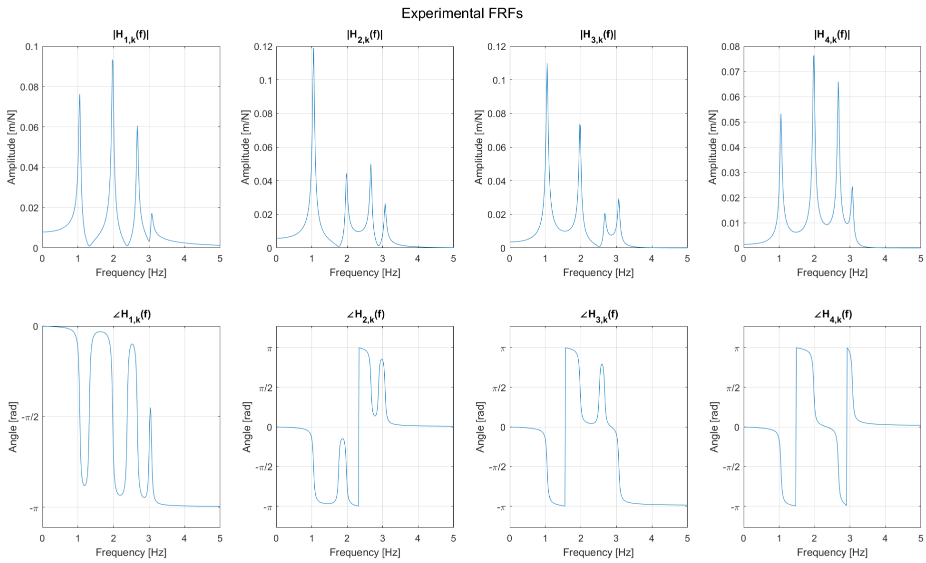
1 Experimental FRFs

In order to identify the modal parameters of a mechanical system, an approach consists in exciting the system with an external force (i.e. impulsive force) and measuring the resulting displacements. The definition of a FRF (Frequency Response Function), for a force $F_k(t)$ applied at the point k and a displacement $x_j(t)$ for the body j, is given by:

$$H_{j,k}^{(\exp)}(\Omega_f) = |H_{j,k}(\Omega_f)|\, e^{j\angle H_{j,k}(\Omega_f)} = \frac{|X_{j_0}(\Omega_f)|}{|F_{k_0}(\Omega_f)|} e^{j\left(\varphi_{x_j}(\Omega_f) - \varphi_{F_k}(\Omega_f)\right)}$$

The amplitude components $|X_{j_0}(\Omega_f)|$ and $|F_{k_0}(\Omega_f)|$,in addition to the phase components $\varphi_{x_j}(\Omega_f)$ and $\varphi_{F_k}(\Omega_f)$, are the result of the DFT (Discrete Fourier Transform) applied to both force and displacement. This procedure can be extended to all displacements $x_j(t)$.
The plots for magnitude and phase of the FRFs $H_{j,k}^{(\exp)}$ are given:



These plots describe a 4-dof mechanical system. There are four angular frequencies (resonances), where a magnitude maximum (magnitude plot) and a $-\pi$ rad phase shift (phase plot) occur. The presence of nodes of vibration in the magnitude plots results in a $+\pi$ rad phase shift in the phase plots.
In addition to that, phase wraps occur when there is a 2π rad phase shift. In the end, sudden phase shifts (less than $\pm\pi$) occur whenever poles and zeros are closely-spaced.
Based on these observations, magnitude and phase plots are coherent between each other.

2 Modes and damping estimation

2.1 Estimation procedure

Starting from all the FRFs plots, it is possible to estimate parameters of the system, such as natural frequencies, mode shapes and damping ratios.
This procedure starts with defining a frequency range $[\Omega_A, \Omega_B]$ of arbitrary choice, in which a single resonance occurs (at ω_{d_i}), in order to locally describe the response as a 1-dof system.

It is now possible to find the local peak (maximum) of the FRF magnitude, which occurs at the damped angular frequency ω_{d_i} of resonance ($i = 1, 2, 3, 4$).

The adimensional damping ratio h_i can be computed from the phase response of the system for angular frequencies Ω_f around each detected resonance. From the expression of a 1-dof system phase response:

$$\angle H_{j,k}(\Omega_f) = -\arctan\left(\frac{2h_i \frac{\Omega_f}{\omega_{d_i}}}{1-\frac{{\Omega_f}^2}{\omega_{d_i}^2}}\right) = -\arctan\left(\frac{2h_i\omega_{d_i}\cdot\Omega_f}{\omega_{d_i}^2-{\Omega_f}^2}\right) = -\arctan\left(f(\Omega_f)\right)$$

$$\phi(\Omega_f) = \frac{d\left[\angle H_{j,k}(\Omega_f)\right]}{d\Omega_f} = -\frac{d\left[\arctan(f)\right]}{df}\frac{df}{d\Omega_f} = -\frac{1}{1+f^2(\Omega_f)}\cdot\frac{df}{d\Omega_f}$$

$$\Rightarrow \phi(\Omega_f) = -\frac{1}{\left(\omega_{d_i}^2-{\Omega_f}^2\right)^2+\left(2h_i\omega_{d_i}\cdot\Omega_f\right)^2}\cdot 2h_i\omega_{d_i}\left(\omega_{d_i}^2+{\Omega_f}^2\right) \quad \Rightarrow \phi(\omega_{d_i}) = -\frac{1}{h_i\omega_{d_i}}$$

Thus, for each individual mode, the adimensional damping ratio h_i is computed thanks to the phase response derivative ϕ of $H_{j,k}$ around each resonance frequency ω_{d_i} :

$$h_i = -\frac{1}{\omega_{d_i}\cdot\phi(\omega_{d_i})}$$

The mode shape can be evaluated from the generic FRF expression for the *j-th* measurement:

$$H_{j,k}(\Omega_f) = \frac{A_j+iB_j}{-\Omega_f^2 m_{q_{ii}}+i\Omega_f c_{q_{ii}}+k_{q_{ii}}}, \quad \Omega_f \in [\Omega_A, \Omega_B]$$

Considering that, in the frequency range of choice, only a single FRF is present and a resonance occurs at ω_{d_i}.
It is possible to fix the modal normalization with $m_{q_{ii}} = 1$, in which case:

$$k_{q_{ii}} = m_{q_{ii}}\omega_{d_i}^2 = \omega_{d_i}^2, \quad c_{q_{ii}} = 2m_{q_{ii}}\omega_{d_i}h_i = 2\omega_{d_i}h_i$$

$$\Rightarrow H_{j,k}(\Omega_f) = \frac{A_j+iB_j}{-\Omega_f^2+i\Omega_f 2\omega_{d_i}h_i+\omega_{d_i}^2}$$

It is of interest to evaluate it at the resonance frequency:

$$H_{j,k}(\omega_{d_i}) = \frac{A_j+iB_j}{-\omega_{d_i}^2+i\omega_{d_i}2\omega_{d_i}h_i+\omega_{d_i}^2} = \frac{A_j+iB_j}{i\omega_{d_i}2\omega_{d_i}h_i} = -i\frac{A_j+iB_j}{2\omega_{d_i}^2h_i} = \frac{B_j}{2\omega_{d_i}^2h_i} - i\frac{A_j}{2\omega_{d_i}^2h_i}$$

The real part of a 1-dof frequency response function is equal to zero at ω_{d_i}:

$$\mathrm{Re}\{H_{j,k}(\Omega)\} = \frac{1}{k_{q_{ii}}}\frac{1-\frac{\Omega^2}{\omega_{d_i}^2}}{\left(1-\frac{\Omega^2}{\omega_{d_i}^2}\right)+i\left(2h_i\frac{\Omega}{\omega_{d_i}}\right)}$$

So $B_j = 0$.

$$\mathrm{Im}\{H_{j,k}(\omega_{d_i})\} = -\frac{A_j}{2\omega_{d_i}^2h_i} \Longrightarrow A_j = -2\omega_{d_i}^2h_i\,\mathrm{Im}\{H_{j,k}(\omega_{d_i})\}$$

The value A_j represents the mode shape $X_j^{(i)}$ of the *i-th* mode for the *j-th* measurement.

2.2 Results

After computation from the Matlab script, we obtain 16 different estimations of the adimensional damping ratios h_i:

	Mode 1	Mode 2	Mode 3	Mode 4
h_{FRF_1}	0.0286	0.0161	0.0113	0.0144
h_{FRF_2}	0.0283	0.0160	0.0113	0.0099
h_{FRF_3}	0.0283	0.0160	0.0115	0.0097
h_{FRF_4}	0.0282	0.0158	0.0111	0.0096

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
h_{mean}	0.0283	0.0160	0.0113	0.0109
h_{std}	0.0002	0.0002	0.0002	0.0023

Considering the very small values of standard deviation, we can keep the averages of each adimensional damping ratio for each mode as the estimated value. This leads us to an underdamped system as we have:

$$h_1 = 0.0283, \quad h_2 = 0.0160, \quad h_3 = 0.0113, \quad h_4 = 0.0109$$

Doing the same for the estimated resonance frequencies (Hz), we obtain the following tables:

	Mode 1	Mode 2	Mode 3	Mode 4
f_{d,FRF_1}	1.0500	1.9666	2.6666	3.0833
f_{d,FRF_2}	1.0500	1.9833	2.6666	3.0666
f_{d,FRF_3}	1.0500	1.9666	2.6666	3.0666
f_{d,FRF_4}	1.0500	1.9833	2.6666	3.0666

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$f_{d,mean}$	1.0500	1.9750	2.6666	3.0708
$f_{d,std}$	0	0.0096	0	0.0083

Considering the very small values of standard deviation, we can keep the averages of each frequency of resonance for each mode as the estimated value. This leads us to values highly coherent with the considered FRFs plots as we have:

$$f_{d,1} = 1.0500 \text{ Hz}, \quad f_{d,2} = 1.9750 \text{ Hz}, \quad f_{d,3} = 2.6666 \text{ Hz}, \quad f_{d,4} = 3.0708 \text{ Hz}$$

Finally, for the mode shapes (m), we obtain:

	Mode 1	Mode 2	Mode 3	Mode 4
$X_1^{(i)}$	0.1891	0.4335	0.3852	0.1209
$X_2^{(i)}$	0.2928	0.2031	-0.3121	-0.1943
$X_3^{(i)}$	0.2704	-0.3393	-0.1325	0.2123
$X_4^{(i)}$	0.1308	-0.3565	0.4094	-0.1736

We remind that each estimated mode shape is the parameter used to express the approximated FRF for each mode (i.e. considering a finite segment of the overall FRF), for each different point of measurement. There is no need in computing the mean or standard deviation here as each mode shape helps to characterize a different section of the FRFs.

3 Residual minimization

3.1 Estimation procedure

The generic FRF $H_{j,k}(\Omega)$ is expressed as:

$$H_{j,k}(\Omega) = \sum_{i=1}^{n=4} \frac{X_j^{(i)} X_k^{(i)}}{-\Omega^2 m_{q_{ii}} + i\Omega c_{q_{ii}} + k_{q_{ii}}}$$

The experimental FRF, evaluated in the frequency range of interest, is given by a resonating contribution, a quasi-static contribution and a seismographic contribution as:

$$H_{j,k}^{\exp}(\Omega_f) = \frac{A_j + iB_j}{-\Omega_f^2 m_{q_{ii}} + i\Omega_f c_{q_{ii}} + k_{q_{ii}}} + (C_j + iD_j) + \frac{E_j + iF_j}{\Omega_f^2}, \quad \Omega_f \in [\Omega_A, \Omega_B]$$

The angular frequencies Ω_A and Ω_B are chosen, as said before, so that the resonating contribution is predominant with respect to the others.
By fixing the modal normalization with $m_{q_{ii}} = 1$, it is possible to obtain the values of $\overline{A}_j$ and $\overline{B}_j$:

$$H_{j,k}^{\exp}(\Omega_f) = \frac{\overline{A}_j + i\overline{B}_j}{-\Omega_f^2 + i\Omega_f c_{q_{ii}} + k_{q_{ii}}} + (C_j + iD_j) + \frac{E_j + iF_j}{\Omega_f^2}$$

The modal parameters identification consists in a problem of $n_f \times n_m$, where n_f is the number of frequency bins (between Ω_A and Ω_B) and n_m is the number of measurements (equal to 4, one for each displacement).
The unknowns of this problem are $c_{q_{ii}}$, $k_{q_{ii}}$, $\overline{A}_j$, $\overline{B}_j$, C_j, D_j, E_j and F_j. So, for each *i-th* mode, the number of unknowns is $2 + 6n_m$ (because $j = 1, 2, 3, 4$).

In order to have an over-determined system, n_f should be sufficiently large. It is now possible to solve the minimization problem with the mean square method.
The minimization problem consists in finding the parameters vector $\underline{x}$ as:

$$\underline{x}_{(2+6n_m)\times 1} = \left(c_{q_{ii}}, k_{q_{ii}}, \bar{A}_1, \bar{B}_1, ..., F_1, \bar{A}_2, \bar{B}_2, ..., F_2 ... F_{n_m}\right)^T$$

So that the energy J of ϵ the error of the approximation is minimum:

$$\epsilon_r = H_{j,k}(\Omega_f, \underline{x}) - H_{j,k}^{(\exp)}(\Omega_f) \Longrightarrow J = \sum_{r=1}^{n_f \times n_m} \left(\mathrm{Re}\{\epsilon_r\}^2 + \mathrm{Im}\{\epsilon_r\}^2\right)$$

So, to goal is to find the parameters vector $\underline{x}^*$ that best satisfies the Mean-Square Error (MSE) problem also known as minimization problem:

$$\underline{x}^* = \arg\left\{\min_{\underline{x}} (J)\right\}$$

The values of $c_{q_{ii}}$ and $k_{q_{ii}}$ are obtained directly, while the other coefficients depend on the chosen resonance zone of the experimental FRF.
In order to check the validity of the results, it is possible to observe that B_j should be small, so that $A_j + iB_j = X_j^{(i)} X_k^{(i)}$. In addition to that, the other four coefficients should also be small with respect to A_j, since the resonating contribution is the predominant one.

The parameters vector search is implemented in MATLAB by means of the `fminsearch` function, which is a built-in optimization function that aims to find the minimum of an unconstrained multi-variable function. More specifically, this function is called as follows in the code:

```
xpar = fminsearch(@(xpar) errHjki_cw(xpar, rfHjki, Hjkiexp(:, jj)), xpar0, options);
```

The instruction `errHjki_cw(xpar, rfHjki, Hjkiexp(:, jj))` calls the function `errHjki_cw`, which computes the energy of the approximation error given the approximation parameters contained in the `xpar` array.
The minimization algorithm is initialised with the `xpar0` array, obtained from the section 2 results. Specifying `@xpar` informs the algorithm that the energy is studied when varying the values of `xpar`.

Morever, the `options` argument is declared as follows:

```
options = optimset('fminsearch');
options = optimset(options, 'TolFun', 1e-8, 'TolX', 1e-8);
```

Those commands initialize the `options` structure with the default settings for the `fminsearch` function and set the termination tolerance on the function value and solution vector: i.e. the minimization algorithm will stop running when the changes in both the function and solution are $\leq 10^{-8}$.
When the minimization problem is solved, it is possible to reconstruct the experimental FRF defined earlier thanks to the `funHjk` function. Note that the function reconstruction occurs in the arbitrary selected frequency range.

3.2 Results

Solving the minimization problem with the Matlab script, we obtain 16 different estimations of the adimensional damping ratios h_i:

	Mode 1	Mode 2	Mode 3	Mode 4
$h_{FRF_1}^{MSE}$	0.0286	0.0283	0.0283	0.0282
$h_{FRF_2}^{MSE}$	0.0127	0.0134	0.0136	0.0137
$h_{FRF_3}^{MSE}$	0.0099	0.0097	0.0102	0.0103
$h_{FRF_4}^{MSE}$	0.0086	0.0081	0.0092	0.0087

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
h_{mean}^{MSE}	0.0283	0.0134	0.0100	0.0087
h_{std}^{MSE}	$0.1581 \cdot 10^{-3}$	$0.4280 \cdot 10^{-3}$	$0.3020 \cdot 10^{-3}$	$0.4246 \cdot 10^{-3}$

Considering the very small values of standard deviation, we can keep the averages of each adimensional damping ratio for each mode as the estimated value. This leads us to an underdamped system as we have:

$$h_1^{MSE} = 0.0283, \quad h_2^{MSE} = 0.0134, \quad h_3^{MSE} = 0.0100, \quad h_4^{MSE} = 0.0087$$

Doing the same for the estimated resonance frequencies (Hz), we obtain the following tables:

	Mode 1	Mode 2	Mode 3	Mode 4
f_{d,FRF_1}^{MSE}	1.0500	1.9757	2.6689	3.0693
f_{d,FRF_2}^{MSE}	1.0500	1.9752	2.6697	3.0673
f_{d,FRF_3}^{MSE}	1.0500	1.9753	2.6675	3.0652
f_{d,FRF_4}^{MSE}	1.0500	1.9751	2.6686	3.0635

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$f_{d,mean}^{MSE}$	1.0500	1.9753	2.6687	3.0663
$f_{d,std}^{MSE}$	0	0.0003	0.0009	0.0025

Considering the very small values of standard deviation, we can keep the averages of each frequency of resonance for each mode as the estimated value. This leads us to values highly coherent with the considered FRFs plots as we have:

$$f_{d,1}^{MSE} = 1.0500 \text{ Hz}, \quad f_{d,2}^{MSE} = 1.9753 \text{ Hz}, \quad f_{d,3}^{MSE} = 2.6687 \text{ Hz}, \quad f_{d,4}^{MSE} = 3.0663 \text{ Hz}$$

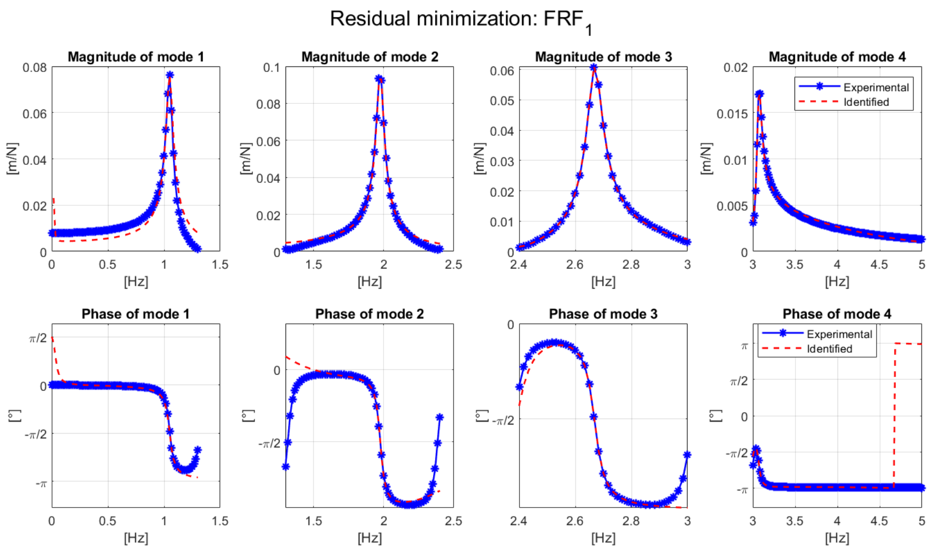
Finally, for the mode shapes (m), we obtain:

	Mode 1	Mode 2	Mode 3	Mode 4
$X_1^{(i)MSE}$	0.1891	0.3795	0.3396	0.0958
$X_2^{(i)MSE}$	0.2928	0.1841	-0.2685	-0.1569
$X_3^{(i)MSE}$	0.2704	-0.3217	-0.1193	0.2033
$X_4^{(i)MSE}$	0.1308	-0.3366	0.3872	-0.1589

Once again, there is no need in computing the mean or standard deviation here as each mode shape helps to characterize a different section of the FRFs.

3.3 Experimental and Approximated FRFs comparison

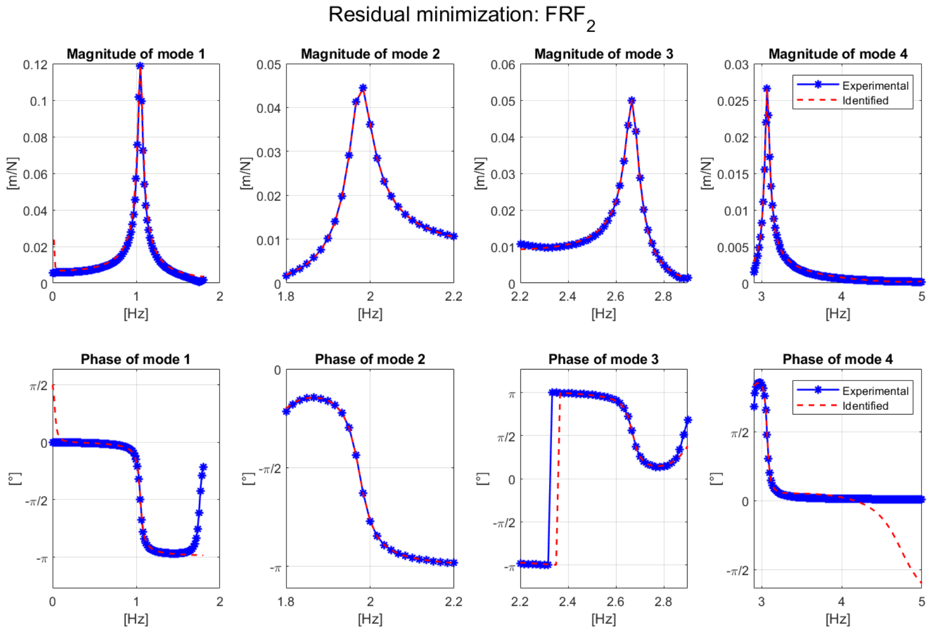
The identification plots for the first FRF is given:



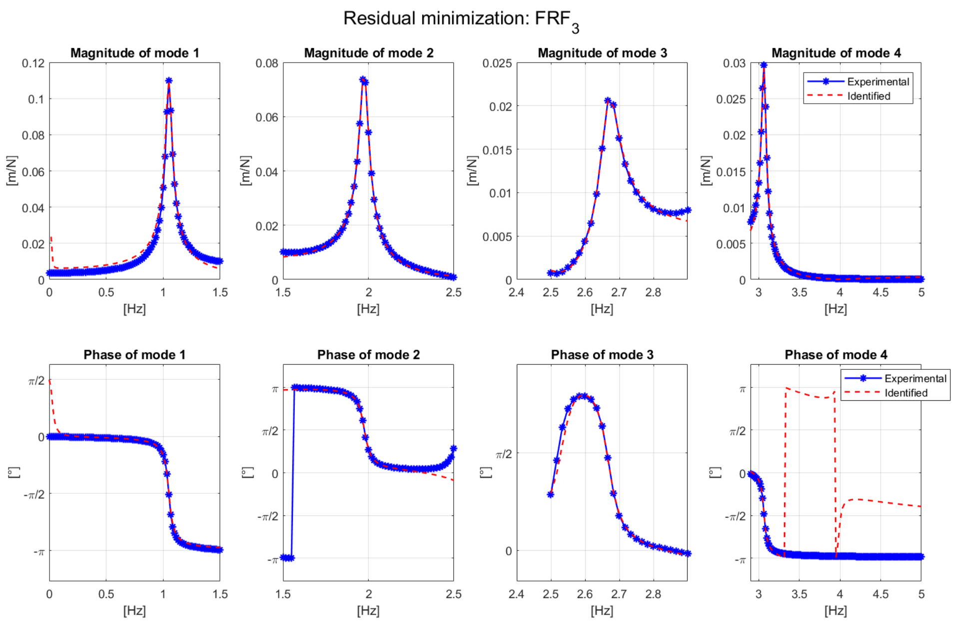
We can see that both the approximated and experimental FRFs are very similar both in amplitude and phase. However, we also see that the FRF approximation becomes less accurate as $f \to 0$. Indeed, the approximated FRF is based on the assumption that we study the response of the system around frequencies for which the resonating mode is the prevalent one, which becomes less and less relevant as $f \to 0$.
The phase response of the mode 4 for the experimental and identified FRFs differ for $f \geq 4$ Hz (roughly). Indeed, this is due to the fact that we are in the seismographic zone (experimental FRF) of the system while still considering that the resonating mode is the prevalent one (identified FRF).
Between the first and second modes, the experimental FRF shows a node of vibration around 1.3 Hz, while the identified one does not show it.
The plot of the phase response for the mode 4 presents a phase wrap around 4.6 Hz.

Comments about the similarity of the experimental and identified FRFs still stand for the next graphs : overall similarity, and difference for $f \to 0$ and $f \geq 4$ Hz.
Differences in phase responses between the 2 plotted FRFs around the phase wrappings should not be accounted.

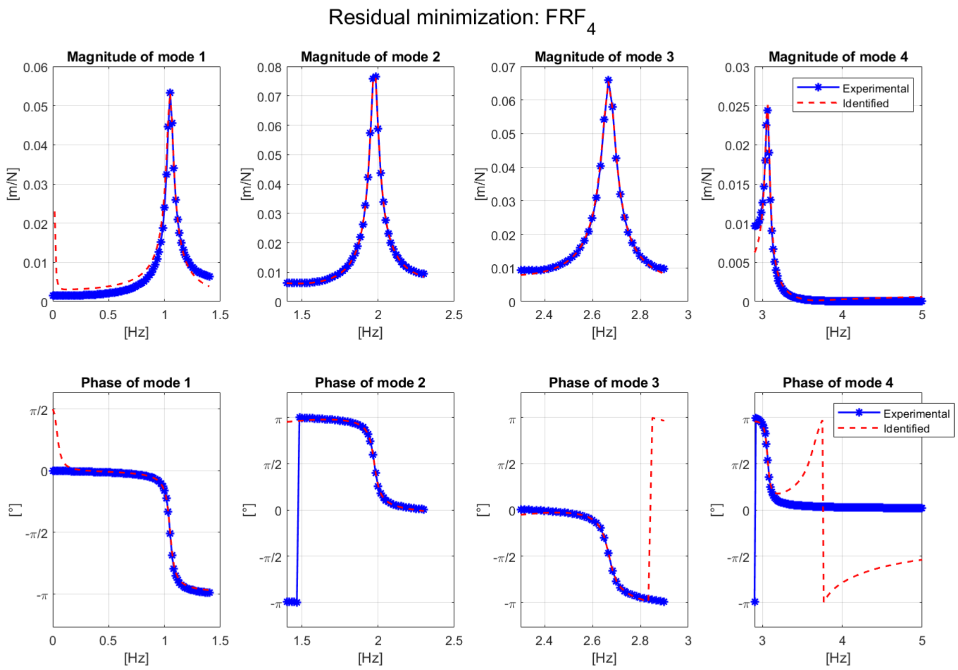
The identification plots for the second FRF is given:



The identification plots for the third FRF is given:



The identification plots for the fourth FRF is given:



4 Modal parameters comparison

4.1 Simplified method

Using a simplified method for estimating the different relevant parameters of our FRFs (section 2), and using the following formulas

$$k_{q_{ii}} = m_{q_{ii}}\omega_{d_i}^2 = \omega_{d_i}^2, \quad c_{q_{ii}} = 2m_{q_{ii}}\omega_{d_i}h_i = 2\omega_{d_i}h_i, \quad \text{such that } m_{q_{ii}} = 1$$

we obtain the following numerical values for the modal stiffness (N/m) from our script :

	Mode 1	Mode 2	Mode 3	Mode 4
$k_{q_{ii}}^{FRF_1}$	43.5235	152.6887	280.7261	375.3066
$k_{q_{ii}}^{FRF_2}$	43.5235	155.2876	280.7261	371.2602
$k_{q_{ii}}^{FRF_3}$	43.5235	152.6887	280.7261	371.2602
$k_{q_{ii}}^{FRF_4}$	43.5235	155.2876	280.7261	371.2602

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$k_{q_{ii},mean}$	43.5235	153.9881	280.7261	372.2718
$k_{q_{ii},std}$	0	1.5005	0	2.0232

Considering the very small values of standard deviation, we can keep the averages of each modal stiffness for each mode as the estimated value:

$$k_{q_{11}} = 43.5235 \text{ N/m}, \quad k_{q_{22}} = 153.9881 \text{ N/m}, \quad k_{q_{33}} = 280.7261 \text{ N/m}, \quad k_{q_{44}} = 372.2718 \text{ N/m}$$

We obtain the following numerical values for the modal damping ($N.s/m$) from our script :

	Mode 1	Mode 2	Mode 3	Mode 4
$c_{q_{ii}}^{FRF_1}$	0.1886	0.1989	0.1894	0.2787
$c_{q_{ii}}^{FRF_2}$	0.1867	0.1997	0.1895	0.1900
$c_{q_{ii}}^{FRF_3}$	0.1864	0.1979	0.1928	0.1861
$c_{q_{ii}}^{FRF_4}$	0.1863	0.1963	0.1862	0.1856

Computing the mean and standard deviation of this set of values, we obtain :

	Mode 1	Mode 2	Mode 3	Mode 4
$c_{q_{ii},mean}$	0.1870	0.1982	0.1895	0.2101
$c_{q_{ii},std}$	0.0010	0.0015	0.0027	0.0458

Considering the very small values of standard deviation, we can keep the averages of each modal stiffness for each mode as the estimated value:

$$c_{q_{11}} = 0.1870 \text{ Ns/m}, \quad c_{q_{22}} = 0.1982 \text{ Ns/m}, \quad c_{q_{33}} = 0.1895 \text{ Ns/m}, \quad c_{q_{44}} = 0.2101 \text{ Ns/m}$$

Due to the normalization we used, we have that $m_{q_{ii}} = 1 \quad \forall i \in [1, 4]$.

4.2 Minimization method

From the MSE algorithm, we simply extract the values for each modal parameters and their mean value and standard deviation, after computation, give us:

	Mode 1	Mode 2	Mode 3	Mode 4
$m_{q_{ii},mean}^{MSE}$	1	1	1	1
$m_{q_{ii},std}^{MSE}$	0	0	0	0
$k_{q_{ii},mean}^{MSE}$	43.5235	154.0408	281.1602	371.1879
$k_{q_{ii},std}^{MSE}$	0	0.0431	0.1931	0.6156
$c_{q_{ii},mean}^{MSE}$	0.3740	0.3314	0.3367	0.3336
$c_{q_{ii},std}^{MSE}$	0.0021	0.0106	0.0100	0.0163

Considering the very small values of standard deviation, we can keep the averages of each modal stiffness for each mode as the estimated value:

$$k_{q_{11}}^{MSE} = 43.5235 \text{ N/m}, \quad k_{q_{22}}^{MSE} = 154.0408 \text{ N/m}, \quad k_{q_{33}}^{MSE} = 281.1602 \text{ N/m}, \quad k_{q_{44}}^{MSE} = 371.1879 \text{ N/m}$$

$$c_{q_{11}}^{MSE} = 0.3740 \text{ Ns/m}, \quad c_{q_{22}}^{MSE} = 0.3314 \text{ Ns/m}, \quad c_{q_{33}}^{MSE} = 0.3367 \text{ Ns/m}, \quad c_{q_{44}}^{MSE} = 0.3336 \text{ Ns/m}$$

4.3 Comparison

We see that both methods provide satisfying (very low standard deviation) yet different results. Indeed, values of the estimated modal damping factors $c_{q_{ii}}$ are significantly different from the simplified method compared to the MSE. This will be clear in the modal FRF plots in next section.
The values of $k_{q_{ii}}$ and $c_{q_{ii}}$, obtained with the two methods, are given:

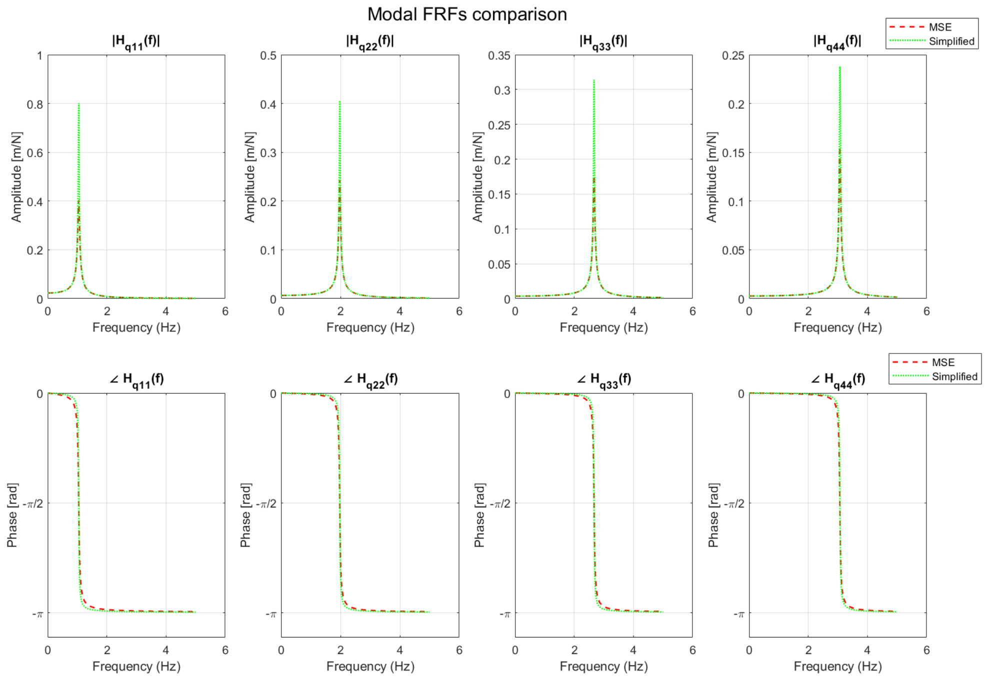
	Mode 1	Mode 2	Mode 3	Mode 4
$c_{q_{ii}}^{MSE}$	0.3740	0.3314	0.3367	0.3336
$c_{q_{ii}}^{simp}$	0.1870	0.1982	0.1895	0.2101
$k_{q_{ii}}^{MSE}$	43.5235	154.0408	281.1602	371.1879
$k_{q_{ii}}^{simp}$	43.5235	153.9881	280.7261	372.2718

Moreover, the MSE algorithm also tries to provide a parametric representation of the different experimental FRFs to best fit the measurements. One might want to compare the reconstructed FRFs of each method to better see which one gives the most satisfying results.

5 FRF reconstruction

In order to reconstruct the experimental FRFs, it is possible to apply the modal approach. Each mode is described by a modal FRF as:

$$H_{q_{ii}}(\Omega) = \frac{1}{-\Omega^2 m_{q_{ii}} + i\Omega c_{q_{ii}} + k_{q_{ii}}}, \quad m_{q_{ii}} = 1$$



It is possible to notice that the different $c_{q_{ii}}$ values, between the two different methods, result in different characteristics: higher dampings result in lower amplitude peaks and smoother phase shift around resonance.

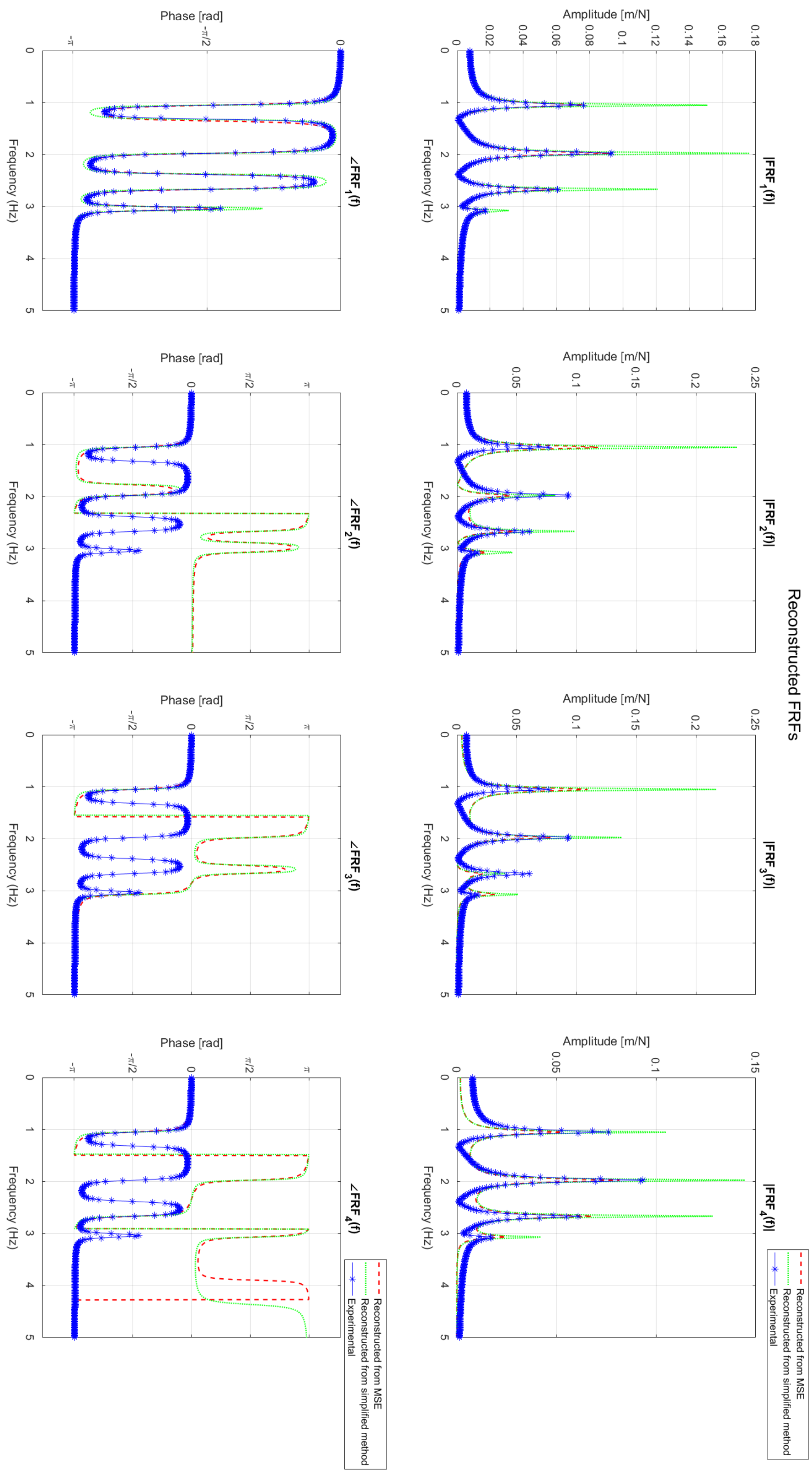
The generic FRF $H_{j,k}$ is given by the product between modal FRF and mode shapes:

$$H_{j,k}(\Omega) = \sum_{i=1}^{4} X_j^{(i)} X_k^{(i)} H_{q_{ii}}(\Omega) = \sum_{i=1}^{4} \frac{A_j + iB_j}{-\Omega^2 m_{q_{ii}} + i\Omega c_{q_{ii}} + k_{q_{ii}}}, \quad B_j << A_j$$

The reconstructed FRF plots (FRF_i in the figure, with $i = 1, 2, 3, 4$) are given in the next page.

Reconstructing the FRFs for each point of measurement allow us to see that the minimization algorithm gives way better results than the simplified method. Indeed, the MSE's amplitude response is way closer to the one obtained from experimental results, as for the phase response. However, it doesn't perfectly fit the experimental FRFs.
The nodes of the experimental FRFs and the ones obtained through reconstruction do not perfectly match. Once again, differences due to phase wraps should not be accounted.

Note that each FRF $H_{j,k}$ describes the system response to a force $F_k(t)$ (applied at the *k-th* point force) at the measurement $x_j(t)$. So, the co-located FRF at the location of the displacement $x_2(t)$ would be equal to $H_{2,2}$.



Reconstructed FRFs
|FRF_1(f)|
|FRF_2(f)|
|FRF_3(f)|
|FRF_4(f)|
∠FRF_1(f)
∠FRF_2(f)
∠FRF_3(f)
∠FRF_4(f)
Amplitude [m/N]
Phase [rad]
Frequency (Hz)
Reconstructed from MSE
Reconstructed from simplified method
Experimental