VIBRATION ANALYSIS
AND VIBROACOUSTICS

# VIBRATION ANALYSIS

## Assignment 1 - A.Y. 2023/24


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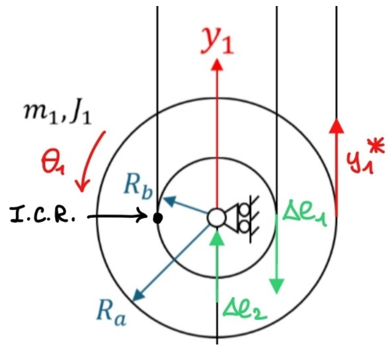
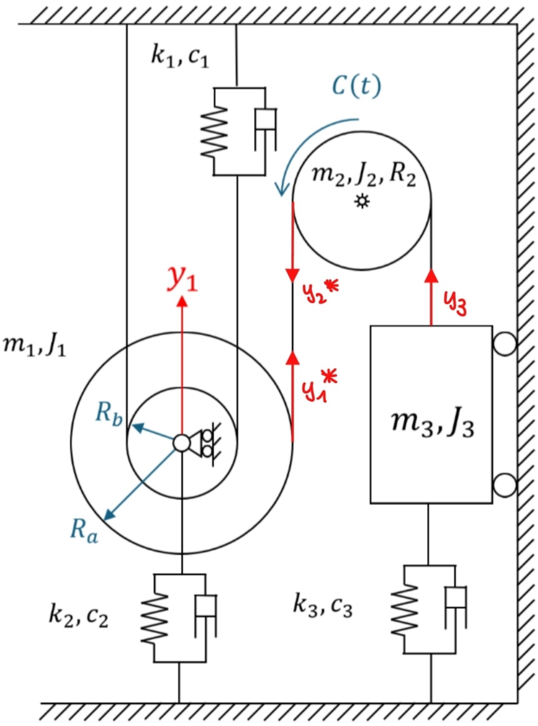
# 1 Question 1

## 1.1 Equation of Motion

The mechanical system in figure presents different physical variables and one independent variable. The motion of the mass $m_1$ is both translational and rotational, so both displacement $y_1$ and angle $\theta_1$ are used to describe its energy forms. The presence of an inextensible string at the left of the disk of radius $R_b$ determines an instantaneous center of rotation (I.C.R.) at the conjunction between string and disk.




The constraints on the disk of mass $m_2$ force it to rotate only, so the angle $\theta_2$ is sufficient to describe its motion.
The mass $m_3$, on the other hand, can only translate vertically, so the displacement $y_3$ describes completely its motion.




By considering the displacements $y_i$ positive upwards and the rotations $\theta_i$ positive counter-clockwise, the physical variables are defined in function of the independent variable $y_1$ as:

$$y_1 = R_b\theta_1 \implies \theta_1 = \frac{y_1}{R_b}, \quad \omega_1 = \dot{\theta}_1 = \frac{v_1}{R_b} = \frac{\dot{y_1}}{R_b}$$

$$y_1^* = -y_2^* \implies (R_a + R_b)\theta_1 = -R_2\theta_2$$

$$\theta_2 = \frac{R_a + R_b}{R_2}\frac{y_1}{R_b} = -\frac{R_a + R_b}{R_2 R_b}y_1, \quad \omega_2 = \dot{\theta}_2 = -\frac{R_a + R_b}{R_2 R_b}\dot{y_1}$$

$$y_3 = -y_2^* = -R_2\theta_2 = \frac{R_a + R_b}{R_b} y_1, \quad v_3 = \dot{y_3} = \frac{R_a + R_b}{R_b} \dot{y_1}$$

Regarding the elongations $\Delta l_i$ of the springs and the altitudes $h_i$ of the masses:

$$\Delta l_1 = -2y_1, \quad \Delta l_2 = y_1, \quad \Delta l_3 = y_3 = \frac{R_a + R_b}{R_b} y_1$$

$$h_1 = y_1, \quad h_3 = y_3 = \frac{R_a + R_b}{R_b} y_1$$

Let's consider the energy forms of the Lagrange equation one by one. The kinetic energy is given by two translational components and two rotational components:

$$E_K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_3 v_3^2 + \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} \omega_2^2 =$$

$$= \frac{1}{2} m_1 \dot{y_1}^2 + \frac{1}{2} m_3 \left( \frac{R_a + R_b}{R_b} \right)^2 \dot{y_1}^2 + \frac{1}{2} J_1 \left( \frac{1}{R_b} \right)^2 \dot{y_1}^2 + \frac{1}{2} J_2 \left( -\frac{R_a + R_b}{R_2 R_b} \right)^2 \dot{y_1}^2 =$$

$$= \frac{1}{2} \left( m_1 + \left( \frac{R_a + R_b}{R_b} \right)^2 m_3 + \frac{J_1}{R_b^2} + J_2 \left( \frac{R_a + R_b}{R_2 R_b} \right)^2 \right) \dot{y_1}^2 = \frac{1}{2} M_{eq} \dot{y_1}^2$$

The potential energy has three elastic components and two gravitational components:

$$V = \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2 + \frac{1}{2} k_3 \Delta l_3^2 + m_1 g h_1 + m_3 g h_3 =$$

$$= \frac{1}{2} k_1 (-2y_1)^2 + \frac{1}{2} k_2 y_1^2 + \frac{1}{2} k_3 \left( \frac{R_a + R_b}{R_b} y_1 \right)^2 + m_1 g y_1 + m_3 g \left( \frac{R_a + R_b}{R_b} \right) y_1 =$$

$$= \frac{1}{2} \left( 4k_1 + k_2 + \left( \frac{R_a + R_b}{R_b} \right)^2 k_3 \right) y_1^2 + \left( m_1 + \frac{R_a + R_b}{R_b} m_3 \right) g y_1 = \frac{1}{2} k_{eq} y_1^2 + m_{eq} g y_1$$

The dissipative function has three terms:

$$D = \frac{1}{2} c_1 \dot{\Delta l_1}^2 + \frac{1}{2} c_2 \dot{\Delta l_2}^2 + \frac{1}{2} c_3 \dot{\Delta l_3}^2 = \frac{1}{2} c_1 (-2\dot{y_1})^2 + \frac{1}{2} c_2 \dot{y_1}^2 + \frac{1}{2} c_3 \left( \frac{R_a + R_b}{R_b} \dot{y_1} \right)^2 =$$

$$= \frac{1}{2} \left( 4c_1 + c_2 + \left( \frac{R_a + R_b}{R_b} \right)^2 c_3 \right) \dot{y_1}^2 = \frac{1}{2} c_{eq} \dot{y_1}^2$$

The only work contribution is given by the torque $C(t)$ applied on the disk $m_2$:

$$\delta W = C(t) \delta\theta_2 = C(t) \left( -\frac{R_a + R_b}{R_2 R_b} \right) \delta y_1 = Q_{y_1} \delta y_1$$

It is now possible to apply the energies into the Lagrange equation:

$$\frac{d}{dt} \left( \frac{\partial E_K}{\partial \dot{y_1}} \right) = \frac{d}{dt} \left( \frac{\partial}{\partial \dot{y_1}} \left( \frac{1}{2} M_{eq} \dot{y_1}^2 \right) \right) = \frac{d}{dt} (M_{eq} \dot{y_1}) = M_{eq} \ddot{y_1}$$

$$\frac{\partial E_K}{\partial y_1} = 0$$

$$\frac{\partial D}{\partial \dot{y_1}} = \frac{\partial}{\partial \dot{y_1}} \left( \frac{1}{2} c_{eq} \dot{y_1}^2 \right) = c_{eq} \dot{y_1}$$

$$\frac{\partial V}{\partial y_1} = \frac{\partial}{\partial y_1} \left( \frac{1}{2} k_{eq} y_1^2 + m_{eq} g y_1 \right) = k_{eq} y_1 + m_{eq} g$$

Giving the result:

$$\frac{d}{dt} \left( \frac{\partial E_K}{\partial \dot{y_1}} \right) + \frac{\partial E_K}{\partial y_1} + \frac{\partial D}{\partial \dot{y_1}} + \frac{\partial V}{\partial y_1} = Q_{y_1} \Longrightarrow M_{eq} \ddot{y_1} + c_{eq} \dot{y_1} + k_{eq} y_1 = \left( -\frac{R_a + R_b}{R_2 R_b} \right) C(t) - m_{eq} g$$

## 1.2 Natural frequency

The natural frequency is a parameter of the system (in free motion) in the undamped case:

$$M_{eq}\ddot{y}_1 + k_{eq}y_1 = 0 \longrightarrow M_{eq}\lambda^2 + k_{eq} = 0 \Longrightarrow \lambda_{1,2} = \pm j\sqrt{\frac{k_{eq}}{M_{eq}}} = \pm j\omega_0 \Longrightarrow \omega_0 = \sqrt{\frac{k_{eq}}{M_{eq}}} = 9.8554 \text{ rad/s}$$

## 1.3 Adimensional damping factor and damped frequency

By considering also the effect of the dampers, the adimensional damping factor is defined as:

$$h = \frac{c_{eq}}{2M_{eq}\omega_0} = \frac{\alpha}{\omega_0} = 0.1057$$

together with the damped frequency:

$$\omega_d = \omega_0\sqrt{1-h^2} = 9.8002 \text{ rad/s } \approx \omega_0$$

# 2 Question 2

## 2.1 Free motion response

A damped system in free motion is described by the homogeneous equation of motion:

$$M_{eq}\ddot{y} + c_{eq}\dot{y} + k_{eq}y = 0$$

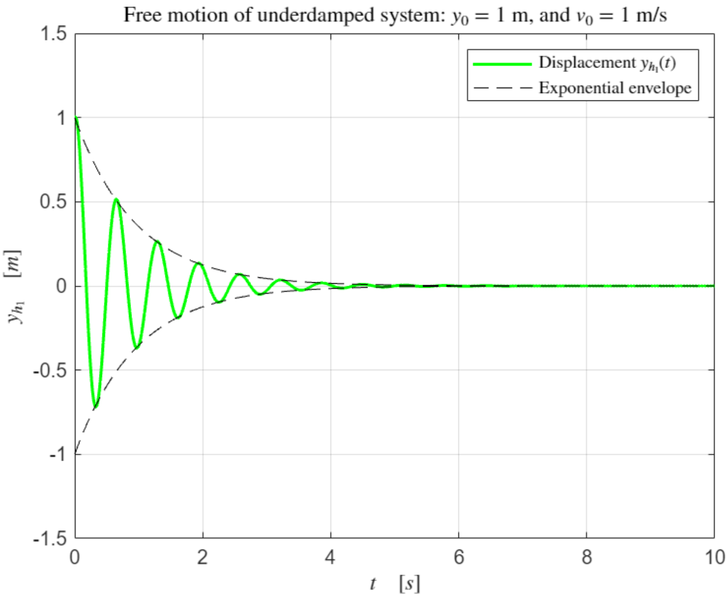
Its solution $y_{1_h}(t)$ is obtained by solving the characteristic equation:

$$M_{eq}\lambda^2 + c_{eq}\lambda + k_{eq} = 0$$

For the underdamped case (given by the small value of $h \approx 10\%$), the homogeneous solution $y_{1_h}(t)$ (in second form) is given by decaying sinusoids:

$$M_{eq}\lambda^2 + c_{eq}\lambda + k_{eq} = 0 \implies \lambda_{1,2} = -\alpha \pm j\omega_0\sqrt{1-h^2} = -\alpha \pm j\omega_d$$

$$\implies y_{h_1}(t) = e^{-\alpha t}\left[y_0 \cos(\omega_d t) + \frac{v_0 + \alpha y_0}{\omega_d}\sin(\omega_d t)\right]$$

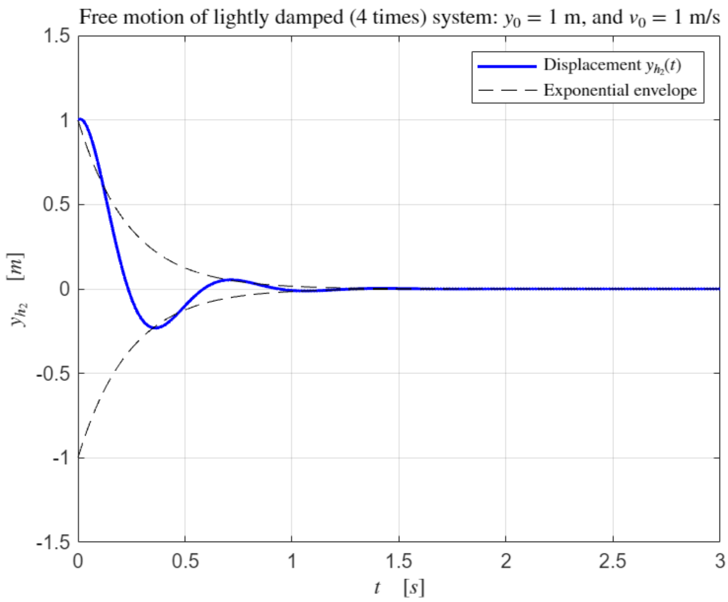



The initial conditions $y_0$ and $v_0$ represent the displacement and velocity values at $t = 0$ and they determine the coefficients of the two sinusoidal components. Considering the small value of $h \approx 10\%$, the system is underdamped, so the damped frequency $\omega_d \approx \omega_0$ as said in section 1.3. The decaying time is given by $\frac{5}{\alpha} \approx 4.80$ seconds, which gives the "slowly" decaying trend to the plot. The steady-state of the system is zero because of the free motion scenario.

## 2.2 Free motion response ($h \to 4h$)

The new value of $h_2 = 8h = 0.4228$, which means the system is lightly damped. The form of the homogeneous solution is the same, only the coefficients change. With respect to the previous point, $\alpha_2 = 4\alpha$ and $\omega_{d_2} = \omega_0\sqrt{1-h_2^2} = 8.9311$ rad/s:

$$y_{h_2}(t) = e^{-\alpha_2 t}\left[y_0 \cos(\omega_{d_2} t) + \frac{v_0 + \alpha_2 y_0}{\omega_{d_2}}\sin(\omega_{d_2} t)\right]$$




The decaying time is given by $\frac{5}{\alpha_2} \approx 1.20$ seconds, which gives a faster (with respect to the previous case) decaying trend to the plot. The steady-state of the system is zero because of the free motion scenario.

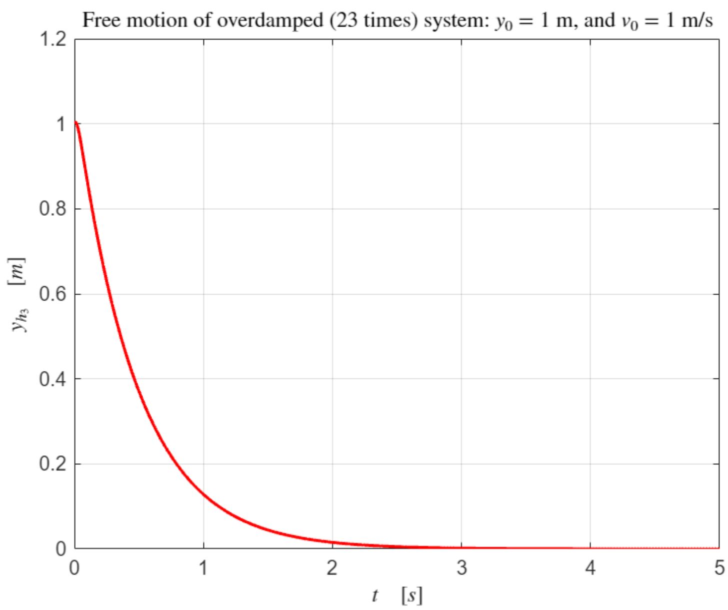
## 2.3 Free motion response ($h \to 23h$)

The new value of $h_3 = 23h = 2.4312$, which means the system is overdamped. In this case, the solutions $\lambda_1$ and $\lambda_2$ of the characteristic equation are real:

$$\lambda_1 = -\alpha_3 - \sqrt{\alpha_3^2 - \omega_0^2} = -a_1, \quad \lambda_2 = -\alpha_3 + \sqrt{\alpha_3^2 - \omega_0^2} = -a_2$$

The homogeneous solution $y_{1_h}(t)$ does not oscillate anymore:

$$y_{h_3}(t) = -\left(\frac{v_0 + a_2 y_0}{a_1 - a_2}\right) e^{-a_1 t} + \left(\frac{v_0 + a_1 y_0}{a_1 - a_2}\right) e^{-a_2 t}$$




The decaying time is mainly given by $\frac{5}{a_2} \approx 2.36$ seconds. The steady-state of the system is zero because of the free motion scenario.

# 3 Question 3

## 3.1 Forced motion: frequency response function

The forced motion adds the external forces and torques contribution to the equation of the free vibrations:

$$M_{eq}\ddot{y} + c_{eq}\dot{y} + k_{eq}y = F(t) = F_0 \cos(\Omega t + \varphi)$$

The complete time solution is the sum of the homogeneous solution $y_{1_h}(t)$ (from section 2.1) and a particular solution given by:

$$y_{1_p}(t) = Y_0 \cos(\Omega t + \varphi + \beta)$$

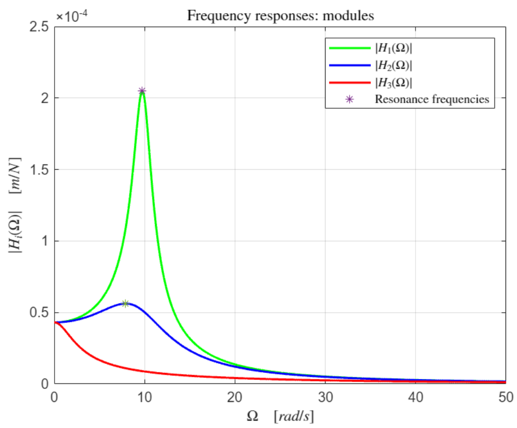
Or equivalently in a complex form:

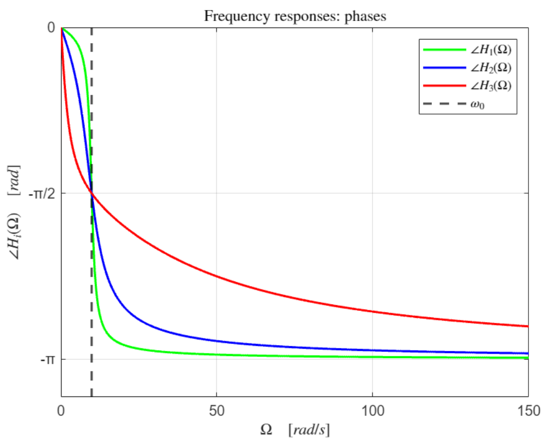
$$\tilde{y}_{1_p}(t) = \tilde{Y}_0 e^{j(\Omega t + \varphi)}, \quad \tilde{Y}_0 = |\tilde{Y}_0| e^{j\beta} \in \mathbb{C}$$

The frequency response function $H(\Omega)$ compares the system result $\tilde{Y}_0$ and the external force contribution $F_0$: in general, an amplification or attenuation $|\tilde{Y}_0|$ and a phase shift $\beta$ will be introduced by the mechanical system.

$$H(\Omega) = \frac{\tilde{Y}_0}{F_0} = \frac{\frac{1}{M_{eq}}}{\omega_0^2 - \Omega^2 + j2\alpha\Omega}$$

Where $\omega_0$ is the natural frequency and $\alpha = \frac{c_{eq}}{2M_{eq}}$ represents the decaying contribution of the dampers.




In the figures above, the three different scenarios described in question 2 are plotted. The system normally presents (first case in green) an underdamped frequency response: an high quality factor describes the narrow bell centered in $\omega_d \approx \omega_0$, where the module response is almost 8 times higher than the static response ($H(\Omega = 0) = 0.43 \cdot 10^{-4}$ m/N), though the system works like an attenuator for all frequencies (module $< 1$). The phase response is given by a steep curve which tends to the one of the undamped case, in which an abrupt counterphase occurs in $\omega_0$ (dotted black line).
In the second case (in blue) $h_2 = 4h$, so the damping effect is more present. The module response is more flat, the quality factor and the damped frequency are lower, and the phase response curve is less steep.
In the third case ($h_3 = 23h$, in red) the damping effect is dominant: there is no natural frequency (the system does not oscillate) and phase shift tends to reach $-\frac{\pi}{2}$ more steeply and $-\pi$ way slower.

## 3.2 Forced motion: complete time response

The complete time response $y_1(t)$ of a forced system is the sum of the homogeneous solution $y_{h_2}(t)$ (defined in section 2.2) and the particular solution $y_{p_2}(t)$ (using the FRF defined in section 3.1):

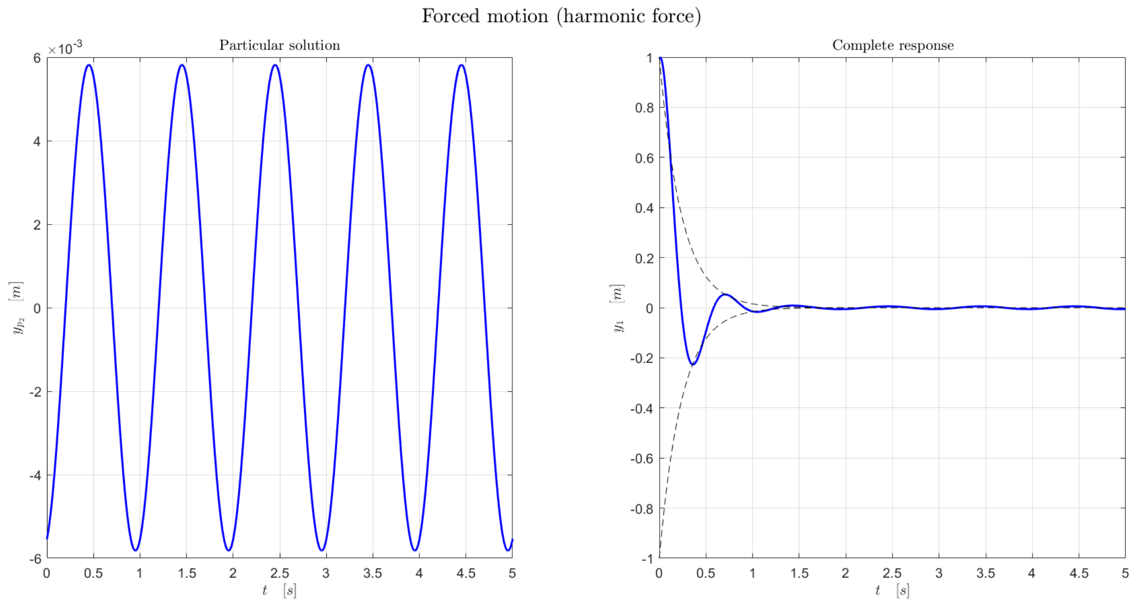
$$y_{h_2}(t) = e^{-\alpha_2 t}\left[y_0\cos(\omega_{d_2}t) + \frac{v_0+\alpha_2 y_0}{\omega_{d_2}}\sin(\omega_{d_2}t)\right]$$

$$y_{p_2}(t) = Y_0\cos(\Omega t+\varphi+\beta) = |H_2(\Omega)|F_0\cos\left(\Omega t+\varphi+\angle H_2(\Omega)\right)$$

By considering a simple harmonic torque ($A = 25$ N·m, $f = 1$ Hz, $\varphi = \pi/3$ rad):

$$C(t) = A\cos(2\pi ft+\varphi) \Longrightarrow F(t) = F_0\cos(2\pi ft+\varphi) = -\frac{R_a+R_b}{R_2R_b}\cdot A\cos(2\pi ft+\varphi)$$

$$\Longrightarrow y_{p_2}(t) = |H_2(2\pi f)|F_0\cos\left(2\pi ft+\varphi+\angle H_2(2\pi f)\right)$$




## 3.3 Steady-state response for two frequencies

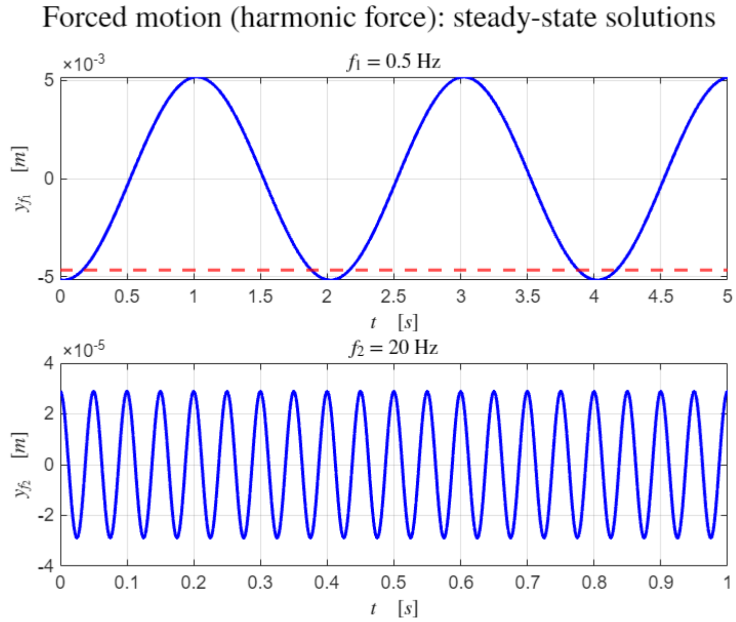
The steady-state response of the system consists in the form that $y_1(t)$ has for $t\to\infty$. Considering that the homogeneous solution $y_{h_1}(t)\to 0$, the steady-state response is given only by the particular solution $y_{p_1}(t)$.
For two different harmonic torques (by considering the first scenario with $h = 0.1057$):

$$C(t) = 25\cos(2\pi 0.5t+0) \Longrightarrow y_{p_1}(t) = |H_1(2\pi 0.5)|F_0\cos\left(2\pi 0.5t+\angle H_1(2\pi 0.5)\right)$$

$$C(t) = 25\cos(2\pi 20t+0) \Longrightarrow y_{p_1}(t) = |H_1(2\pi 20)|F_0\cos\left(2\pi 20t+\angle H_1(2\pi 20)\right)$$

For a torque statically applied, instead:

$$C(t) = 25\cos(2\pi 0t+0) = 25 \Longrightarrow y_{p_1}(t) = |H_1(0)|F_0\cos\left(2\pi 0t+\angle H_1(2\pi 0)\right) = |H_1(0)|F_0$$

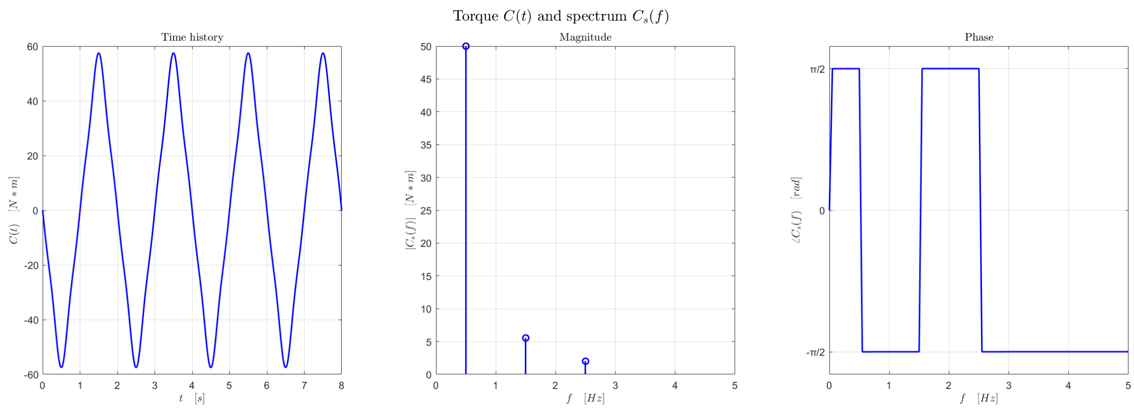



From the frequency response module $|H_1(\Omega)|$ graph, obtained in section 3.1, it is possible to compare the amplitudes of the different steady-state solutions with the static response. For the first frequency $f_1 = 0.5$ Hz (equivalent to $\pi$ rad/s), the amplitude of the steady-state solution (in blue) is close to the static response value $y_{1_{st}} = -4.7 \cdot 10^{-3}$ m (in red): this frequency is located in the quasi-static zone of the frequency response function, where the magnitude is almost constant. For the second frequency $f_2 = 20$ Hz (equivalent to 125.66 rad/s), the amplitude of the steady-state solution is way smaller than the static response: this frequency is located in the seismographic zone of the frequency response function, where the magnitude is way smaller compared to the quasi-static zone and tends to zero.

## 3.4 Superposition of three harmonic torques

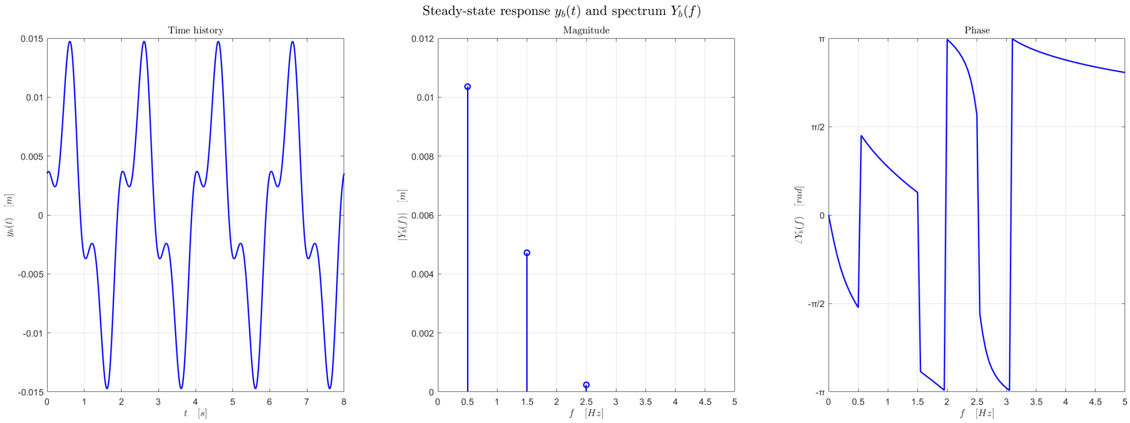
The linearity of the system allows to obtain the output displacement as sum of different input contributions. In case of a torque given by the superposition of three harmonics:

$$C(t) = \sum_{k=1}^{3} B_k \cos\left(2\pi f_k t + \varphi_k\right) = 50\cos\left(2\pi 0.5t + \frac{\pi}{2}\right) + 5.55\cos\left(2\pi 1.5t - \frac{\pi}{2}\right) + 2\cos\left(2\pi 2.5t + \frac{\pi}{2}\right)$$



The overall steady-state response is the superposition of the three particular solutions:

$$y_{p_1}(t) = 50\left(-\frac{R_a+R_b}{R_2R_b}\right)|H_1(2\pi 0.5)|\cos\left(2\pi 0.5t+\frac{\pi}{2}+\angle H_1(2\pi 0.5)\right)+$$
$$+5.55\left(-\frac{R_a+R_b}{R_2R_b}\right)|H_1(2\pi 1.5)|\cos\left(2\pi 1.5t-\frac{\pi}{2}+\angle H_1(2\pi 1.5)\right)+$$
$$+2\left(-\frac{R_a+R_b}{R_2R_b}\right)|H_1(2\pi 2.5)|\cos\left(2\pi 2.5t+\frac{\pi}{2}+\angle H_1(2\pi 2.5)\right)$$




The torque waveform is mainly given by the first harmonic (frequency at 0.5 Hz), considering the amplitude coefficients (coherently represented in the magnitude plot).
Regarding the steady-state solution, the frequency response module $|H_1(\Omega)|$ modifies each harmonic contribution of the torque, giving a dominant first harmonic and a more present second harmonic. The relative amplitude difference between the harmonics $f_1$ and $f_2$ is higher in the input torque spectrum ($f_1$ almost 10 times $f_2$) than in the steady-state spectrum ($f_1$ at least 2 times $f_2$). The third harmonic is negligible in the particular solution (amplitude approximately 19 times lower than the second harmonic amplitude and 43 times lower than the first harmonic amplitude).
The frequency response phase graph is coherent with the model, considering that each harmonic contribution results in sudden phase shift from the frequency response phase graph. In particular, two phase wraps occur around 2 Hz and 3 Hz.