Math 21C - Section B01 - Quiz 3 **SOLUTION**
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Problem 1: Find the interval of convergence of the following power series

$$f(x) = \sum_{n=1}^{\infty} \frac{(3 - \frac{1}{7}x)^n}{n}$$

Solution: We note the *sequence of terms* is given by the formula

$$a_n = \frac{(3 - \frac{1}{7}x)^n}{n}.$$

Therefore,

$$a_{n+1} = \frac{(3 - \frac{1}{7}x)^{n+1}}{n+1}.$$

We'll use the Ratio Test. We want to know for **which** choices of numbers to plug in for x will the series given to us converge.

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(3 - \frac{1}{7}x)^{n+1}}{n+1} \cdot \frac{n}{(3 - \frac{1}{7}x)^n}\right| \\ &= \lim_{n\to\infty} \left|\left(3 - \frac{1}{7}x\right)\right| \cdot \frac{n}{n+1}.\end{aligned}$$

Let's remind ourselves what we're doing. We're asking for **what** choices of x will the series converge. That is, we are asking (since we're using the Ratio Test), for what choices of x will the limit of the sequence $\left|\left(3 - \frac{1}{7}x\right)\right| \cdot \frac{n}{n+1}$ be less than 1 and not equal to 1.

For a fixed value x, the left piece above (in the absolute value) is a fixed number. It's a constant, at least with respect to x. Since the limit is with n running to ∞, we should pull it (this constant coefficient) in front of the limit by the Constant Multiple Rule in Section 11.1. So, we have that the above equation is:

$$\begin{aligned}\lim_{n\to\infty} \cdot \left|\left(3 - \frac{1}{7}x\right)\right| \frac{n}{n+1} &= \left|\left(3 - \frac{1}{7}x\right)\right| \lim_{n\to\infty} \frac{n}{n+1}, \text{ and we can apply L'Hopital's to this to get} \\ &= \left|\left(3 - \frac{1}{7}x\right)\right| \cdot \lim_{n\to\infty} \frac{1}{1} \\ &= \left|\left(3 - \frac{1}{7}x\right)\right| \cdot 1 \\ &= \left|\left(3 - \frac{1}{7}x\right)\right|\end{aligned}$$

Recall, we want this thing (which is the limit, for a fixed x) to be less than 1. So we solve the following[1]:

$$\left|\left(3 - \frac{1}{7}x\right)\right| < 1.$$

Set $y = 3 - \frac{1}{7}x$. Then, we're solving $|y| < 1$. Well, this is the same as saying that $y > -1$ and $y < 1$. Take first $y > -1$. By substituting x back in and solving for x, we see that $28 > x$. Then, by taking $y < 1$, substituting x back in, we'll get $14 < x$. Thus, we know that when $14 < x < 28$, the series will converge.

However, this is not yet the answer to the original question, which was to find the interval of convergence. The answer will be one of the following:

- $14 < x < 28$
- $14 \leq x < 28$
- $14 < x \leq 28$
- $14 \leq x \leq 28$

Which is it? We will plug $x = 14$ and $x = 28$ into the original series and see if the two series that we have converge or diverge.

By plugging in $x = 14$, we have

$$\sum_{n=1}^{\infty} \frac{(3 - \frac{1}{7}x)^n}{n} = \sum_{n=1}^{\infty} \frac{(3 - \frac{1}{7} \cdot 14)^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

and this is the harmonic series[2], which we know diverges[3]. When we plug in $x = 28$, we have the series

$$\sum_{n=1}^{\infty} \frac{(3 - \frac{1}{7}x)^n}{n} = \sum_{n=1}^{\infty} \frac{(3 - \frac{1}{7} \cdot 28)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test. We put our info from $x = 14$ and $x = 28$ together to conclude that the interval of convergence is $14 < x \leq 28$.

[1] Up to this point, you could also get here by using the Root Test. There, you'd end up asking for which x does the sequence $|\sqrt[n]{a_n}|$ have limit < 1.

[2] Careful with your notation. Often, possibly out of a little laziness, some were in the habit of writing all of the above without the summation symbol. While you may have simply been shortcutting and you may have known all along that they are series, the absence of the Σ makes the grader think that you're confusing sequences and series. If the graders on the midterm go too fast, it will typically mean docked points, and I don't want that to happen to you all. Thus, put the series symbol in front every time you are talking about a series. On the flip side, be sure to NOT put the series symbol there when you're talking about a sequence. Make it clear in all your work for this class whether you're dealing with a sequence or series by being intentional with placement of the Σ.

[3] Why does it diverge? You can use the p-test with $p = 1$ since it is a p-series. Alternatively, you can use the Integral Test.

Problem 2: Does the following series converge absolutely or converge conditionally or both?

$$\sum_{n=1}^{\infty} \frac{\tan \frac{(2n-1)\pi}{4}}{n}$$

Solution[4][5]: Note first that the following formula is true:

$$\tan \frac{(2n-1)\pi}{4} = (-1)^{n+1}.$$

This series seemed a bit weird, so I wrote out the first couple terms and I notice that I got 1, −1, 1, −1 by carefully remembering my trig and just computing what that numerator was several times. (So, sometimes, if it seems like you have a crazy monster problem to solve, at least try to feel out what the question is asking by writing out some terms. Without writing out these terms, I felt like I didn't have even a good guess on whether the series converges or diverges. After I wrote some terms out, it was all-of-a-sudden miraculously obvious.)

Thus, the series can simply be rewritten as:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

We note that this is an alternating series. Indeed, every other term is positive and every other term is negative. So, my first guess is to try the Alternating Series Test, and it worked out! Specifically, the terms $a_n = \frac{(-1)^{n+1}}{n}$ alternated in sign, and for the positive version of the sequence terms:

$$u_n = |a_n| = |\frac{(-1)^{n+1}}{n}| = \frac{1}{n},$$

we clearly had the two other necessary[6] properties:

- $u_{n+1} \leq u_n$, and
- $\lim u_n = 0$.

Therefore, our series converges by the Alternating Series Test. Now, does it converge absolutely or converge conditionally? (Note, it can't be both. By definition, if a series does NOT converge absolutely, then that's exactly what it means to converge conditionally. That's a paraphrasing of a box on page 773 in the book.)

We look at the series $\sum u_n$. If this series converges, then the original series converges absolutely. Otherwise, the original series converges conditionally.

Note, $\sum u_n = \sum \frac{1}{n}$ is the Harmonic Series, which diverges. Since $\sum u_n = \sum |a_n|$ diverges, the orginial series $\sum a_n$ converges conditionally.

[6]Both properties are indeed necessary, and you should mention **both** on a test. Most people are in the slightly-lazy habit of listing only the first bullet point.

Remarks

- Your attention to detail is remarkably improved from Quiz 2 to Quiz 3. Keep it up! These are good habits for the midterm!
- Often, you try one test, and it doesn't tell you anything. For example, you might try the ratio test and get a limit of 1 for the sequence $|\frac{a_{n+1}}{a_n}|$. At that point, don't give up! Think of the name of another test! Try another test! Even your teaching staff sometimes have to try several tests on a problem, so don't lose hope!
- Remember that when you add up a whole bunch (infinitely many, or even finitely many) positive numbers, you're going to get either a fixed positive number or $+\infty$. You couldn't possibly get 0 by adding up a whole bunch of positive numbers, right?
- Quite a large handful of people are confusing the alternating series test with the definition of conditional convergence. Recall that the alternating series test is a test for alternating series, which can help you conclude that a series converges. Conditional convergence is not a test, it is rather an additional description on a series. It says that **even when** you treat all of the terms you're adding up as being positive, the series still converges. Making all the terms positive makes the series "harder", in some sense, to converge: at least when there were some negative terms, there was a shot at cancellation. (A third thing that both of these are easy to confuse against is the Absolute Convergence Test.)