Quiz 4 (KEY)

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Problem 1. (5 points)
Find the Taylor series at $x = 0$ for the function

$$\cos\sqrt{x+1}.$$

Answer. Recall the Taylor series at $\theta = 0$ for $\cos\theta$: $\cos\theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}$. This Taylor series converges for all real numbers θ and so in particular the Taylor series converges for $\theta = \sqrt{x+1}$ whenever $x \geq -1$.

Notice that the functions $\cos\sqrt{x+1}$ and $\cos\theta$ agree when $x = -1$ and $\theta = 0$, but disagree when $x = 0$ and $\theta = 0$. This suggests the problem is really to find the Taylor series for $\cos\sqrt{x+1}$ centered at $x = -1$.

Therefore, the the Taylor series for $\cos\sqrt{x+1}$ centered at $x = -1$ is

$$\cos\sqrt{x+1} = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\sqrt{x+1}\right)^{2k}}{(2k)!}.$$

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Problem 2. (5 points)
How close is the approximation $\sin x = x$ when $|x| < 10^{-4}$?

Answer. The formula for computing the bound on the error $R_n(x)$ produced by approximating $f(x)$ by the degree-n polynomial $\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}, \qquad \text{where } |f^{(n+1)}(t)| \leq M, \text{ for all } t \text{ between } a \text{ and } x.$$

The Taylor series of $\sin x$ at $x = 0$ is $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \mp \cdots$. So approximating $\sin x$ by the degree-one polynomial x is actually approximating by the degree-two polynomial

$$x^1 + 0 \cdot x^2 = x \approx \sin x.$$

Therefore, $n = 2$ and the $n+1$ derivative is $f^{(3)}(x) = -\cos x$. For $|t| \leq 10^{-4}$, $|f^{(3)}(t)| \leq 1 = M$.

The formula then gives

$$|R_2(x)| \leq 1 \cdot \frac{|x-0|^{2+1}}{(2+1)!} \leq \frac{10^{-12}}{3!}.$$

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