Solutions(Quiz 3, section B04)

1. (2 points): Given

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2.$$

Estimate the magnitude of the error between

$$\sum_{n=1}^{96}(-1)^{n+1}\frac{1}{n}$$

and $\ln 2$.

For an convergent alternating sequence $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, we have the following:

$$|L - s_n| < u_{n+1}$$

where $L = \sum_{n=1}^{\infty}(-1)^{n+1}u_n$ and $s_n = \sum_{j=1}^{n}(-1)^{j+1}u_j$. The given series is convergent because it satisfies all three conditions for the alternating series test. Hence

$$|error| < \frac{1}{97}$$

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2. (8 points): Find the series:

$$\sum_{n=0}^{\infty}\frac{(x-6)^n}{7^n\sqrt{n^2+6}}$$

radius and interval of convergence. For what values of x does the series converge absolutely and conditionally?

Let $u_n = \frac{(x-6)^n}{7^n\sqrt{n^2+6}}$.

First, use the ratio test to find the radius of convergence.

$$\begin{aligned}\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} &= \lim_{n\to\infty}\frac{|x-6|^{n+1}}{7^{n+1}\sqrt{(n+1)^2+6}}\frac{7^n\sqrt{n^2+6}}{|x-6|^n}\\ &= \frac{|x-6|}{7}\underbrace{\lim_{n\to\infty}\frac{\sqrt{n^2+6}}{\sqrt{(n+1)^2+6}}}_{=1}\\ &= \frac{|x-6|}{7}<1\end{aligned}$$

So, $|x-6|<7$. This implies that the radius of convergence is 7. From the previous inequality we obtain $-1<x<13$. Now we need to check the end points (i.e., $x=-1\ ,13$).

$\underline{x=-1}$: In this case, we have the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n^2+6}}$. Then, this is an alternating series and satisfies the following conditions:

(1) $\frac{1}{\sqrt{n^2+6}}\geq 0$

(2) $\frac{1}{\sqrt{n^2+6}}\geq\frac{1}{\sqrt{(n+1)^2+6}}$

(3) $\lim_{n\to\infty}\frac{1}{\sqrt{n^2+6}}=0$

Therefore, the series converges. However, this series does not converge absolutely because $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{\sqrt{n^2+6}}\right| = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+6}}$ and $\frac{1}{\sqrt{n^2+6}}$ behaves like $\frac{1}{n}$ for large n whose infinite series diverges.

$\underline{x=13:}$ For $x=13$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+6}}$ and we just showed that this series diverges. Hence, the interval of convergence is $-1\leq x<13$,the series converges absolutely for $-1<x<13$ and the series converges conditionally for $x=-1$.