Math 21C - Section B01 - Quiz 6 **SOLUTION**
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Problem 1: Determine if the limit below exists or not. Give reasons for your answers.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y}$$

Solution: By following different paths, we approach different output values of the function $f(x,y) = \frac{x^2+y}{y}$ and thus the limit does not exist:

- First[1], by restricting our attention to $y = x^2$, then we are examining

$$\lim_{(x,y)\to(0,0)} \frac{x^2+x^2}{x^2} = \lim_{(x,y)\to(0,0)} 2.$$

  Here, we are evaluating the limit of the number 2 (which remains constant) as we travel along the parabola $y = x^2$ (and avoiding $x = 0$, because then we'd be dividing by zero). Since we get the constant number 2, this converges to 2.

- For a second path, we consider (for example), the parabola $y = -x^2$. We'll get a different limit for the output of $f(x,y)$. Specifically,

$$\lim_{(x,y)\to(0,0)} \frac{x^2-x^2}{-x^2} = \lim_{(x,y)\to(0,0)} 0 - x^2 = 0.$$

Since different [restricted] paths[2] produce different limit values, the limit doesn't exist[3].

Problem 2: Find an equation for the level curve of f that passes through the point $(1,5)$

$$f(x,y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$$

Solution: When a specific x and $y \neq 0$ are chosen, this is a geometric series. We want to describe the level curve for the output value $f(1,5)$, so we should first evaluate what output value we have.

$$f(1,5) = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}.$$

[1] You may have chosen different paths, or done a general argument using $y = kx^2$, which is fine.

[2] which avoid the line $y = 0$, since that would be division by zero

[3] Recall, that, for the limit to exist, **all** paths (x,y) to the origin need to produce the same output limit value.

We can use the geometric series formula because the ratio r is $\frac{1}{5}$. (Up to this point, some people did not plug in $x = 1$ and $y = 5$. Just recall how functions work up to this point.) So, we want to describe an equation for the level set $f(x, y) = \sum (x/y)^n = \frac{5}{4}$. That is, which x and y inputs make $f(x, y) = \frac{5}{4}$?

Note that $\frac{x}{y}$ is really the common ratio of the geometric series. Because we start with first term $(\frac{x}{y})^0$, the first term is always 1. Thus, the sum of the series is $\frac{1}{1-\frac{x}{y}}$, the 1 in the numerator being the first term.

Which x and y choices make

$$\frac{1}{1 - \frac{x}{y}} = \frac{5}{4}$$

to be true? Well, the common ratio r will have to be $\frac{1}{5}$. In other words, we require that

$$\frac{x}{y} = \frac{1}{5}.$$

Saying it again: the expression within the parentheses of the summation, which IS the common ratio, needs to be $\frac{1}{5}$ to get the same number at the end. So, an equation[4] is

$$\frac{x}{y} = \frac{1}{5}.$$

If you want, you can simplify by cross multiplying, and describe the equation a different way, and your final answer will be

$$y = 5x, \text{ except } y \neq 0.$$

Note, you technically need to say $y \neq 0$ to not divide by zero.

Remarks

- On problem 1, some people chose paths (that's a good thing) that involved traveling on the line $y = 0$ (that's bad). This doesn't work because $y = 0$ means you're dividing by zero. You needed to consider paths that go to the origin but avoid the line $y = 0$ which is not part of the domain. (Of course, this is not true of the endpoint of the paths: The endpoint is the exception, but that's what limits "are all about": trying to find the "suggested" value of the function f independent of its value there, or whether or not it even has a value there.)

- Several people tried using L'Hopital's rule on Question 1. It doesn't apply here: L'Hopital's rule was meant for 1-variable limits. Besides, with two (or more) variables, how do you differentiate? You have to do some kind of partial differentiation, and we've never discussed how to evaluate limits like that. Note also that the section on partial derivatives in the book comes after the section on multi-input limits.

[4]Note, my equation has an equals sign in it: All equations need an equals sign.

- Several people tried using the product rule on Question 1. Like in Section 11.1, the limits of both factors have to exist to use the product rule.
- Many people got up to
$$\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n = \frac{5}{4},$$
which is an equation, but needed to be simplified just a bit. Don't worry: one point off is negligible!