1. Let $f(x) = \sqrt{1-x^2}$.

   (a) (*4 pts*) Find the domain and range of $f(x)$.

   $1-x^2 \geq 0 \implies 1 \geq x^2 \implies \boxed{1 \geq |x| \therefore \text{Domain of } f}$

   Since $|x| \leq 1 \implies \boxed{\text{Range of } [0,1]}$

   (b) (*4 pts*) Compute $(f \circ f \circ f)(\frac{1}{2})$

   $f(f(f(\frac{1}{2}))) = f(f(\sqrt{\frac{3}{4}})) = f(\sqrt{\frac{1}{4}}) = \boxed{\sqrt{\frac{3}{4}}}$

2. (*6 pts*) Write $f(x) = \sin\left(\sqrt{x^2+2}\right)$ as a composition of 3 functions.

   $$\boxed{\begin{aligned} u(x) &= x^2+2 \\ v(u) &= \sqrt{u} \\ w(v) &= \sin v \end{aligned}}$$

3. (*8 pts each*) Find the limit if it exists. If it doesn't, explain why.

   (a) $\lim\limits_{x\to -1} \dfrac{x^2-x-2}{x^2+4x+3} \overset{\text{"}0\text{"}}{=} \dfrac{}{0}$ $\lim\limits_{x\to -1} \dfrac{(x-2)(x+1)}{(x+1)(x+3)} = \lim\limits_{x\to -1} \dfrac{x-2}{x+3}$

   $= \boxed{\dfrac{-3}{2}}$

(b) $\lim_{x\to\infty} \left(\frac{x^2}{2x-1} - \frac{x}{2}\right) \overset{\text{“}\infty-\infty\text{”}}{=} \lim_{x\to\infty} \frac{2x^2 - 2x^2 + x}{2(2x-1)} = \lim_{x\to\infty} \frac{x}{4x-2}$

$= \lim_{x\to\infty} \frac{1}{4 - \frac{2}{x}} = \boxed{\frac{1}{4}}$

(c) $\lim_{x\to 0} \frac{2x^3 + \sin 3x}{4x} \overset{\text{“}\frac{0}{0}\text{”}}{=} \lim_{x\to 0} \frac{2x^3}{4x} + \frac{\sin 3x}{4x} = \lim_{x\to 0} \frac{1}{2}x^2 + \frac{\sin 3x}{4x}$

$= 0 + \lim_{x\to 0} \frac{3}{4} \frac{\sin 3x}{3x} = \boxed{3/4}$

(d) $\lim_{x\to 0} \frac{\cos x - 1}{x^2} \overset{\text{“}\frac{0}{0}\text{”}}{=} -\lim_{x\to 0} \frac{1-\cos x}{x^2} \frac{(1+\cos x)}{(1+\cos x)} = -\lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$

Hint: $1 - \cos^2(x) = sin^2(x)$.

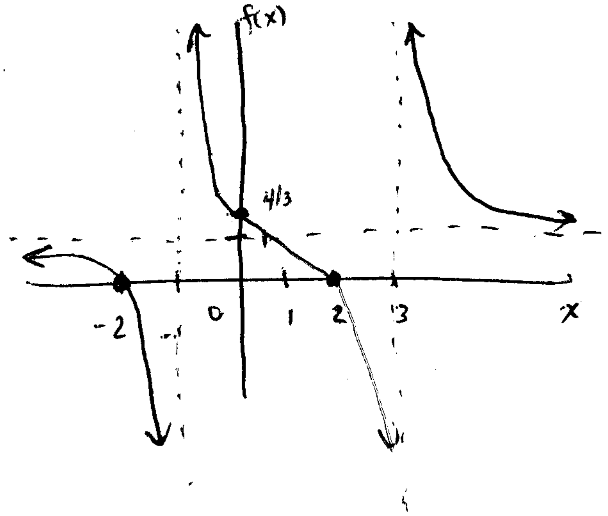
$= -\lim_{x\to 0} \frac{\sin^2 x}{x^2} \frac{1}{1+\cos x}$

$= -\lim_{x\to 0} \frac{\sin x}{x} \frac{\sin x}{x} \frac{1}{1+\cos x}$

$= \boxed{-\frac{1}{2}}$

4. (*22 pts*) Sketch the graph of $f(x) = \frac{x^2-4}{x^2-2x-3}$. Clearly specify the domain, range, intercept(s), asymptotes and symmetry.

$$f(x) = \frac{(x-2)(x+2)}{(x+1)(x-3)}$$



Domain of f : $x \neq -1$ & $x \neq 3$

Intercepts : $f(0) = \frac{4}{3}$ $f(x) = 0$ when $x = 2$ or -2.

Symmetry : Neither even nor odd since $f(4) = \frac{12}{5} \neq f(-4) = \frac{12}{21}$

You can also see this since the asymptotes are not at $x = -1$ & $x = 1$

Asymptotes : H. A. $\lim_{x \to \pm\infty} f(x) = 1$.

V. A. @ $x = -1, 3$ $\lim_{x \to -1^-} f(x) = -\infty$ $\lim_{x \to -1^+} f(x) = \infty$

check both limits from both sides $\lim_{x \to 3^-} f(x) = -\infty$ $\lim_{x \to 3^+} f(x) = +\infty$

Range of f = $\mathbb{R}$

5. (*16 pts*) Is the function f given below continious at $x = 0$? You must show your work to receive credit.

$$f(x) = \begin{cases} \frac{\frac{1}{x+1}-1}{x} & , \text{ if } x < 0 \\ -1 & , \text{ if } x = 0 \\ \frac{x^2+x}{3x^2-x} & , \text{ if } x > 0 \end{cases}$$

check limits from both sides of 0

$$\lim_{x \to 0^-} \frac{\frac{1}{x+1} - 1}{x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0^-} \frac{1 - x - 1}{(x+1)x} = \lim_{x \to 0^-} \frac{-1}{x+1} = -1$$

$$\lim_{x \to 0^+} \frac{x^2 + x}{3x^2 - x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0^+} \frac{x(x+1)}{x(3x-1)} = \lim_{x \to 0^+} \frac{x+1}{3x-1} = \frac{1}{-1} = -1$$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = -1 \implies \lim_{x \to 0} f(x) = -1 = f(0)$

so f is continuous at 0.

6. (*6 pts*) Let $f(x) = \frac{1-\cos x}{x}$ for $x \neq 0$. Is it possible to define $f(0)$ in such a way that f is continuous throughout the x-axis? Explain your answer.

Since $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$ if we define

$f(0) = 0$ then f will be continuous throughout

the x-axis.

7. (a) (*8 pts*) State the intermediate value theorem.

Let f be a continuous function on a closed interval $[a, b]$ if m is in between $f(a)$ and $f(b)$ then there exist a c in $[a, b]$ s.t. $f(c) = m$.

(b) (*8 pts*) Use the intermediate value theorem to show that the equation $x^2 + \sin(\frac{\pi x}{2}) = 3$ has a solution on $[0, 2]$.

Let $f(x) = x^2 + \sin \frac{\pi x}{2}$

Since $f(x)$ is continuous $[0, 2]$

and $f(0) = 0$ and $f(2) = 4$

and 3 is in between 0 and 4

Then by IVT there is a c in $[0, 2]$

s.t. $f(c) = 3$.

8. (a) (*8 pts*) State the definition of $\lim_{x\to a} f(x) = L$.

For any $\varepsilon > 0$ there is a $\delta > 0$ s.t.
if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

(b) (*16 pts*) Use the definition of limit to prove $\lim_{x\to 0} \frac{x-1}{x+1} = -1$.

Scratch

$\left|\frac{x-1}{x+1} + 1\right| < \varepsilon$ want $|x| < \delta$

$\left|\frac{x-1+x+1}{x+1}\right| < \varepsilon$

$\left|\frac{2x}{x+1}\right| < \varepsilon$

$\left|\frac{x}{x+1}\right| < \varepsilon/2$ now bound $\left|\frac{1}{x+1}\right|$

assume $|x| < 1/2$

$-1/2 < x < 1/2$

$1/2 < x+1 < 3/2$

$\Downarrow$

$\frac{2}{3} < \frac{1}{x+1} < 2$

so

$\left|\frac{2}{3}x\right| < \left|\frac{x}{x+1}\right| < |2x| < \varepsilon/2$

$\Rightarrow |x| < \varepsilon/4 \Rightarrow \delta = \min\{\frac{1}{2}, \varepsilon/4\}$

Formal Proof

Let $\varepsilon > 0$ choose $\delta = \min\{\frac{1}{2}, \varepsilon/4\}$

if $|x| < \varepsilon/4$

$-\varepsilon/4 < x < \varepsilon/4$

$1 - \varepsilon/4 < x+1 < 1 + \varepsilon/4$

$\frac{1}{1+\varepsilon/4} < \frac{1}{x+1} < \frac{1}{1-\varepsilon/4} < 2$

So $\left|\frac{x}{x+1}\right| < \varepsilon/2$

$\left|\frac{2x}{x+1}\right| < \varepsilon \Rightarrow \left|\frac{x-1+x+1}{x+1}\right| < \varepsilon$

$\left|\frac{x-1}{x+1} + 1\right| < \varepsilon$

Thus $\lim_{x\to 0} \frac{x-1}{x+1} = -1$

9. (*10 pts*) The position of a rocket at time t is given by the function

$$f(t) = t + \sqrt{t}$$

What is the velocity of the rocket at time $t = t_0 > 0$?
Use the definition of the derivative but the power rule.
Hint: $(a-b)(a+b) = a^2 - b^2$.

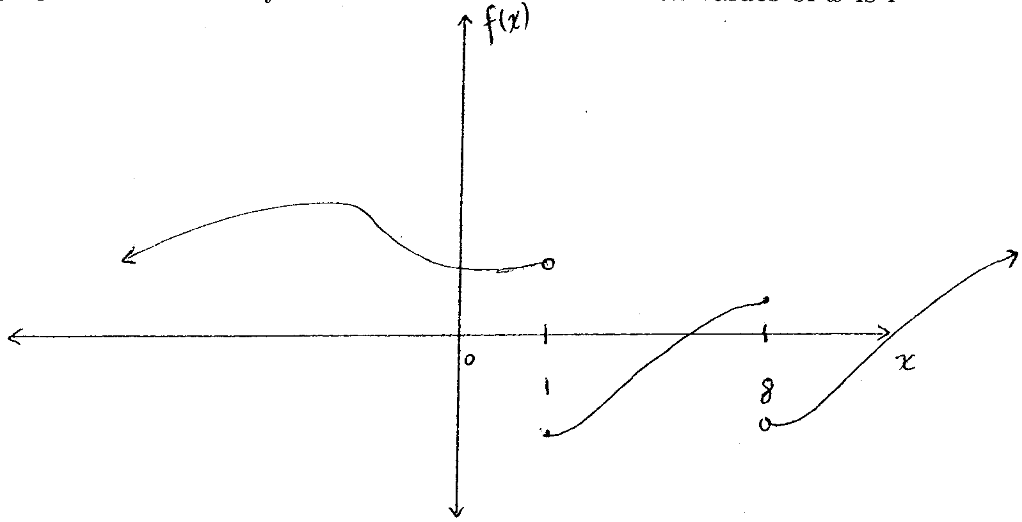
$$\lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h} = \lim_{h\to 0} \frac{t_0 + h + \sqrt{t_0+h} - t_0 - \sqrt{t_0}}{h}$$

$$= \lim_{h\to 0} \frac{h + \sqrt{t_0+h} - \sqrt{t_0}}{h} = \lim_{h\to 0} 1 + \frac{\sqrt{t_0+h} - \sqrt{t_0}}{h} = 1 + \lim_{h\to 0} \frac{\sqrt{t_0+h} - \sqrt{t_0}}{h}$$

$$= 1 + \lim_{h\to 0} \frac{\sqrt{t_0+h} - \sqrt{t_0}}{h} \cdot \frac{\sqrt{t_0+h} + \sqrt{t_0}}{\sqrt{t_0+h} + \sqrt{t_0}} = 1 + \lim_{h\to 0} \frac{t_0 + h - t_0}{h(\sqrt{t_0+h} + \sqrt{t_0})}$$

$$= 1 + \lim_{h\to 0} \frac{1}{\sqrt{t_0+h} + \sqrt{t_0}} = 1 + \frac{1}{2\sqrt{t_0}} \quad //$$

10. The graph of a function f is sketched below. For which values of x is f



(a) (*4 pts*) Left continuous

$(-\infty, 1) \cup (1, 8] \cup (8, \infty) = (-\infty, 1) \cup (1, \infty)$

(b) (*4 pts*) Right continuous

$(-\infty, 1) \cup [1, 8) \cup (8, \infty) = (-\infty, 8) \cup (8, \infty)$

(c) (*2 pts*) Continuous

$(-\infty, 1) \cup (1, 8) \cup (8, \infty)$

Page	2 (*22*)	3 (*24*)	4 (*38*)	5 (*22*)	6 (*34*)	7 (*10*)	Total (*150*)
Scores							