Solutions (Quiz 4, section B04)

Problem 1 (5 points): Find the Taylor series generated by $f(x) = -x^5+x^3-6x+1$ at $x=-2$.

solution:

$$\begin{aligned}
f(x) &= -x^5 + x^3 - 6x + 1 && \Rightarrow & f(2) &= 37\\
f'(x) &= -5x^4 + 3x^2 - 6 && \Rightarrow & f'(2) &= -74\\
f''(x) &= -20x^3 + 6x && \Rightarrow & f''(2) &= 148\\
f'''(x) &= -60x^2 + 6 && \Rightarrow & f'''(2) &= -234\\
f^{(4)}(x) &= -120x && \Rightarrow & f^{(4)}(2) &= 240\\
f^{(5)}(x) &= -120 && \Rightarrow & f^{(5)}(2) &= -120
\end{aligned}$$

Note that $f^{(n)}(x) = 0$ for $n \geq 6$. Therefore, the Taylor series generated by $f(x)$ at $x=-2$ is

$$37 - 74(x+2) + \frac{148}{2!}(x+2)^2 + \frac{-234}{3!}(x+2)^3 + \frac{240}{4!}(x+2)^4 + \frac{-120}{5!}(x+2)^5 .$$

Problem 2 (5 points): Let $f(x) = \cos x$ and $P_2(x)$ the Taylor polynomial of $f$ of order 2 centered $x=0$. Using Taylor's remainder of order 2, $R_2(x)$, estimate the bound for the error between $f(x)$ and $P_2(x)$ for $|x| < 10^{-2}$.

solution:

$$f(x) = \cos x = P_2(x) + R_2(x),$$

where $R_2(x) = \frac{f'''(c)}{3!}x^3$ for some $c$ between 0 and $x$. Hence, $R_2(x)$ can be used to estimate the error between $f(x)$ and $P_2(x)$. Here, $f'''(x) = \sin x$ from below and $|\sin x| \leq 1$. Thus,

$$|R_2(x)| = \left|\frac{f'''(c)}{3!}x^3\right| < \frac{|\sin c|}{3!}|x|^3 < \frac{1}{3!}10^{-6} .$$

$$\begin{aligned}
f'(x) &= -\sin x\\
f''(x) &= -\cos x\\
f'''(x) &= \sin x
\end{aligned}$$