Math 21C - Section B01 - Quiz 4 **SOLUTION**
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**Problem 1:** Find the Taylor series at $x = 0$ for the function $f(x) = \cos(x^2)$.

**Solution:** Since the Maclaurin series for $\cos(x)$ is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

we can replace (by substitution) $x^2$ for $x$ to get

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}.$$

This is a great final answer. Alternatively, by algebra, you can simplify it just a bit to

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}.$$

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**Problem 2:** How close is the approximation $\sin x = x$ when $|x| < 10^{-5}$?

**Solution:** Let $f(x) = \sin x$. Now $x$ is $P_1(x)$ for $\sin x$. We want to know about $\sin x \approx x$, so we want to know how small $|f(x) - P_1(x)|$ gets. Since

$$f(x) = P_n(x) + R_n(x),$$

we want to know about $|R_1(x)|$. There is the formula

$$|R_n(x)| \leq M \cdot \frac{|x|^{n+1}}{(n+1)!}$$

Now $M$ is supposed to be bigger than the value of the $(n+1)$-st derivative of $f(x)$ for all choices $x$ such that $|x| \leq 10^{-5}$. Since the derivatives of $f$ are sine waves with amplitude 1, we can just take $M = 1$.

So, we have

$$|R_1(x)| \leq 1 \cdot \frac{|x|^{1+1}}{(1+1)!}$$

and our $|x|$ is always less than $10^{-5}$, so

$$\text{error} = |R_1(x)| \leq 1 \cdot \frac{(10^{-5})^2}{2!}$$

**Comment:** Since $P_2(x)$ is also $x$ for $\sin x$, you could also have chosen $n = 2$ instead of $n = 1$. We still choose $M = 1$. But now, we get

$$\text{error} = |R_2(x)| \leq 1 \cdot \frac{(10^{-5})^3}{3!}$$

**A third idea (brief comment):** ... uses the fact that the Maclaurin Series for $\sin(x)$ is alternating (at least on $x > 0$).

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# Remarks

- Another approach to solving the first problem is by using the formula for the Taylor Series. Many of you found out that it got very very messy to do this. This would be a good exercise to try, but not such a fun thing to do in a time-limited forum like a quiz or midterm. Many of you need to remember to be a bit more careful with the 21A stuff. Remember that with the first derivative, you need the chain rule. After that, you need to use the product rule for derivatives (remember, "The first times d of the second, plus the second times d the first"?)

- Study hard for the midterm: The quizzes have been graded with leniency in mind. Points that are taken off have been mainly a feeble attempt to be a "scare tactic" into making sure you have everything in your mind clear. Note that the midterm grading will almost certainly be **a lot more rigorous**.

  - In particular, for the same error on a problem, you might not have had **any** points taken off during a quiz, but it might mean 10-20% (I'm guessing) points off on the midterm!!