Intr.: Ernest Woei *September 15, 2006*

Last name: Solutions *First name:*

PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!

1. Make sure that your exam contains 8 pages, including this one.
2. **NO** calculators, books, notes or other written material allowed.
3. Express all numbers in exact arithmetic, i.e., no decimal approximations.
4. Read the statement below and sign your name.

I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.

Signature: ______________________________

Page #	Score
2 (*10*)	
3 (*20*)	
4 (*20*)	
5 (*20*)	
6 (*10*)	
7 (*E.C.*)	
Total (*80*)	

1. (*10 pts*) Let x be a continuous random variable with probability density function

$$f(x) = 3x^2, \quad 0 \le x \le 1.$$

The mean of this random variable is $\frac{3}{4}$.

(a) (*4 pts*) Find the variance of x. You do not need to simplify your answer.

$$V(x) = \int_0^1 x^2 f(x)\,dx - \mu^2 = \int_0^1 x^2\, 3x^2\,dx - \left(\frac{3}{4}\right)^2$$

$$= \frac{3x^5}{5}\Big]_0^1 - \left(\frac{3}{4}\right)^2$$

$$= \frac{3}{5} - \frac{9}{16} \quad //$$

(b) (*4 pts*) Find the median of x.

Find m s.t. $\int_0^m f(x)\,dx = .5$

$$m^3 = x^3\Big|_0^m = \int_0^m 3x^2\,dx = .5$$

$$\Longrightarrow \quad m = \frac{1}{\sqrt[3]{2}} \,/$$

(c) (*2 pts*) If the variance of x is $\frac{1}{4}$, what is the standard deviation of x?

$$\sigma = \sqrt{V(x)} = \sqrt{\frac{1}{4}} = 1/2 \quad //$$

2. (*20 pts*) Find the indefinite integrals.

(a) (*10 pts*)

$$\int \arctan x \quad dx$$

$u = \arctan x$ $\quad dv = dx$

$du = \frac{1}{1+x^2} dx$ $\quad V = x$

$= x \arctan x - \int \frac{1}{1+x^2} x \, dx$

$\int \frac{x}{1+x^2} dx$ $\quad u = 1+x^2$, $du = 2x dx$, $\frac{1}{2} du = x dx$

$= \int \frac{\frac{1}{2}}{u} du$

$= \frac{1}{2} \ln |u| = \frac{1}{2} \ln |1+x^2|$

$= x \arctan x - \frac{1}{2} \ln(1+x^2) + C$ //

(b) (*10 pts*)

$$\int \frac{3x^2+3x+1}{x(x^2+2x+1)} \quad dx$$

$x^2 + 2x + 1 = (x+1)^2$

$\frac{3x^2+3x+1}{x(x^2+2x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$3x^2 + 3x + 1 = A(x+1)^2 + Bx(x+1) + Cx$

let $x = 0$

$1 = A$

let $x = -1$

$1 = C(-1) \Rightarrow C = -1$

let $x = 1$

$7 = 4 + B2 - 1$

$\Rightarrow B = 2$

$\int \frac{3x^2+3x+1}{x(x^2+2x+1)} dx = \int \frac{1}{x} + \frac{2}{x+1} + \frac{-1}{(x+1)^2} dx$

$= \ln|x| + 2\ln|x+1| + \frac{1}{x+1} + C$ //

3. (*20 pts*) Evaluate the improper integrals.

   (a) (*10 pts*)

$$\int_{\frac{1}{3}}^{\infty} \frac{1}{\sqrt{3x-1}} \, dx = \underbrace{\int_{1/3}^{1} \frac{1}{\sqrt{3x-1}} \, dx}_{(1)} + \underbrace{\int_{1}^{\infty} \frac{1}{\sqrt{3x-1}} \, dx}_{(2)}$$

$$(1) = \lim_{a \to 1/3^+} \int_a^1 (3x-1)^{-1/3} \, dx = \lim_{a \to 1/3^+} \frac{3 (3x-1)^{2/3}}{2 \cdot \cancel{3}} \Big]_a^1$$

$$= \lim_{a \to 1/3^+} \left[\frac{2^{2/3}}{2} - \cancel{\frac{\overset{\to 0}{(3a-1)^{2/3}}}{2}} \right] = 2^{-1/3}$$

$$(2) = \lim_{b \to \infty} \int_1^b (3x-1)^{-1/3} \, dx = \lim_{b \to \infty} \frac{(3x-1)^{2/3}}{2} \Big]_1^b = \lim_{b \to \infty} \left[\cancel{\frac{\overset{\to \infty}{(3b-1)^{2/3}}}{2}} - 2^{-1/3} \right]$$

$$\therefore \int_{1/3}^{\infty} \frac{1}{\sqrt{3x-1}} \, dx = \infty \text{ diverges. } //$$

   (b) (*10 pts*)

$$\int_0^{\infty} x e^{-x^2} \, dx = \lim_{b \to \infty} \int_0^b x e^{-x^2} \, dx$$

let $u = x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, d$

$$= \lim_{b \to \infty} \int_0^{b^2} e^{-u} \, \frac{1}{2} \, du$$

$$= \lim_{b \to \infty} -\frac{1}{2} e^{-u} \Big]_0^{b^2}$$

$$= \lim_{b \to \infty} \cancel{-\frac{1}{2} e^{\overset{\to 0}{-b^2}}} + \frac{1}{2} = 1/2 \; //$$

4. (*10 pts*) The scores on a Math 16B exam are normally distributed with a mean of 60 and a standard deviation of 12. You scored 70 on the exam.

   (a) (*5 pts*) What percent of those who took the exam had scored lower than yours?

$$P(x< 70) = \int_{-\infty}^{70} \frac{1}{12\sqrt{\pi 2}}\, e^{\frac{-(x-60)^2}{2\cdot 12^2}}\, dx$$

x normal r.v. with $\mu = 60$, $\sigma = 12$

let $u = \frac{x-60}{12}$

$du = \frac{1}{12}\,dx$

$12\,du = dx$

$x \to -\infty \Rightarrow u \to -\infty$

$x = 70 \Rightarrow u = \frac{5}{6} \cong .8\overline{3}$

$$= \int_{-\infty}^{.8\overline{3}} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du$$

$$\cong .7967 \quad //$$

79%

   (b) (*5 pts*) What percent scored above 64?

$$P(x > 64) = 1 - P(x \le 64) = 1 - \int_{-\infty}^{64} \frac{1}{12\sqrt{2\pi}}\, e^{-\frac{(x-60)^2}{2\cdot 12^2}}\, dx$$

$$= 1 - \int_{-\infty}^{1/3} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du$$

$$\cong 1 - .6293 = .3707 \quad //$$

5. (*10 pts*) A disintegrating radioactive substance decreases from 12 to 11 grams in 1 day. Find it's half-life. Hint: $\frac{\ln 2}{\ln(11/12)} \approx -7.9662$.

$y(t) = Ce^{rt}$

$y(0) = 12 = Ce^{0} \Rightarrow C = 12$

$y(1) = 11 = Ce^{r} \Rightarrow \frac{11}{12} = e^{r} \Rightarrow$

$\ln\left(\frac{11}{12}\right) = r$

Find t s.t. $y(t) = 6$

$$6 = 12\, e^{\ln\left(\frac{11}{12}\right)t} \Rightarrow \frac{1}{2} = e^{\ln\frac{11}{12} t} \Rightarrow \frac{\ln\left(\frac{1}{2}\right)}{\ln\frac{11}{12}} = t$$

7.96 days $= t$ //

6. (*10 pts*) Let $f(x) = x + \sin x$ and $g(x) = x$ for $0 \leq x \leq 2\pi$. Find the area of the region bounded by the graphs of $f(x)$ and $g(x)$.

Find where f & g intersect

$f(x) = g(x)$

$x + \sin x = x$

$\sin x = 0 \implies x = 0, \pi, 2\pi$

$f(x) \geq g(x)$ in $[0, \pi]$

$f(x) \leq g(x)$ in $[\pi, 2\pi]$

Area of the region bounded by f & g

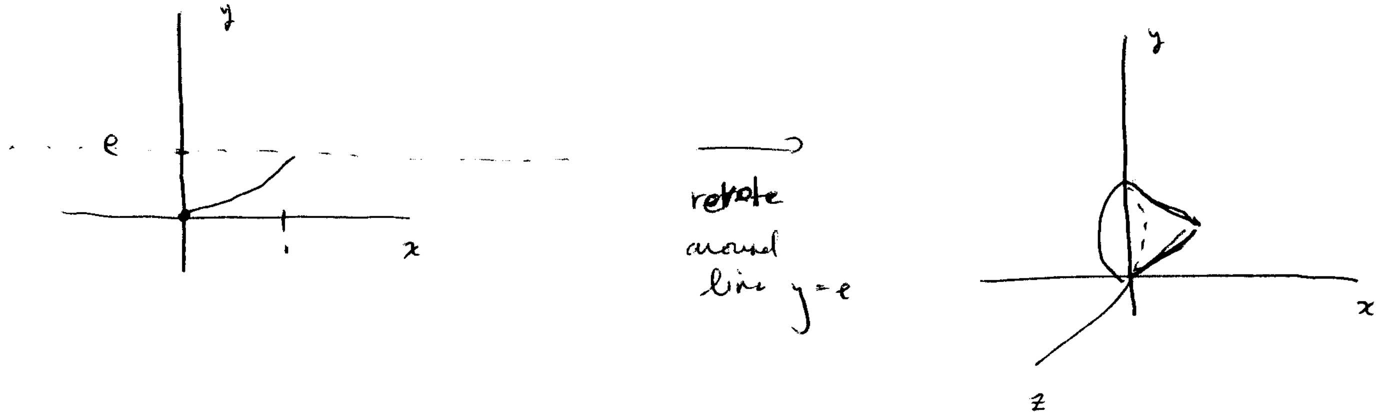
$$= \int_0^{\pi} x + \sin x - x \, dx + \int_{\pi}^{2\pi} x - (x + \sin x) \, dx$$

$$= -\cos x \Big]_0^{\pi} + \cos x \Big]_{\pi}^{2\pi}$$

$$= -\cos \pi + \cos 0 + \cos 2\pi - \cos \pi$$

$$= -(-1) + 1 + 1 - (-1) = 4 //$$

7. (*Extra Credit*) Consider the graph of xe^x for $0 \leq x \leq 1$ in the xy-plane. A solid is formed by revolving this curve about the line $y = e$. Find the volume of this solid. Hint: $\int xe^x dx = xe^x - e^x + C$.



$$\pi \int_0^1 (e - xe^x)^2 \, dx = \pi \int_0^1 e^2 - 2xe^x + x^2 e^{2x} \, dx$$

$$= \pi \left[e^2 - 2\left[xe^x - e^x\right]_0^1 + \int_0^1 x^2 e^{2x} \, dx \right]$$

$$= \pi \left[e^2 - 2\left(e - e - (0 - 1)\right) + \frac{e^2 - 1}{4} \right]$$

$$= \boxed{\pi \left[e^2 - 2 + \frac{e^2 - 1}{4} \right]}$$

$$\int_0^1 x^2 e^{2x} \, dx$$

$u = x^2$ $\quad dv = e^{2x} dx$

$du = 2x \, dx$ $\quad V = \frac{e^{2x}}{2}$

$$= \frac{x^2 e^{2x}}{2} \Big|_0^1 - \int_0^1 \frac{e^{2x}}{\cancel{2}} \cancel{2}x \, dx$$

$u = x$ $\quad dv = e^{2x} dx$

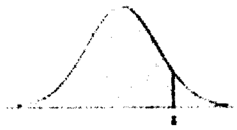
$du = dx$ $\quad v = \frac{e^{2x}}{2}$

$$= \frac{e^2}{2} - \left[\frac{xe^{2x}}{2} \Big|_0^1 - \int_0^1 \frac{e^{2x}}{2} \, dx \right]$$

$$= \frac{e^2}{2} - \left[\frac{e^2}{2} - \left[\frac{e^{2x}}{4}\right]_0^1 \right] = \frac{e^2}{2} - \frac{e^2}{2} + \left(\frac{e^2}{4} - \frac{1}{4} \right)$$

$$= \frac{e^2 - 1}{4}$$

Tables of the Normal Distribution



Probability Content from -oo to Z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990