Quiz 1

Name:______________________________

MAT21C-B04, Saito
Spring 2008

Student ID:______________________________

Problem 1. (5 points)
Determine if the sequence $\left(\frac{2008+n}{n}\right)^n$ converges or diverges. If it converges, find the limit.

Answer. Notice

$$\frac{2008+n}{n} = 1 + \frac{2008}{n}.$$

We want the limit of the sequence $a_n = \left(1 + \frac{2008}{n}\right)^n$. By Theorem 5.5 of Section 11.1, $\lim_{n\to\infty} a_n = e^{2008}$.

To show directly:

$$\lim_{n\to\infty} \ln a_n \;=\; \lim_{n\to\infty} \frac{\ln(1+\frac{2008}{n})}{\frac{1}{n}} \;=\; \lim_{n\to\infty} \frac{\frac{-2008/n^2}{1+\frac{2008}{n}}}{-1/n^2} \;=\; \lim_{n\to\infty} \frac{2008}{1+\frac{2008}{n}} \;=\; 2008$$

Use the fact that $e^{\ln x} = x$ for all x. The answer is $\lim\limits_{n\to\infty} e^{\ln a_n} \;=\; e^{\lim\limits_{n\to\infty} \ln a_n} = e^{2008}$. □

Problem 2. (5 points)
Write the binary number $0.\overline{100} = 0.100100100\ldots$ as a rational number.
Hint: Recall the binary number 0.100 *is* $1 * 2^{-1} + 0 * 2^{-2} + 0 * 2^{-3}$ *in decimal.*

Answer. The binary number $0.\overline{100}$ is the sum of all $2^{-(1+3^n)} = \frac{1}{2}\left(\frac{1}{2^3}\right)^n$.

$$\begin{aligned}
0.\overline{100} &= 2^{-1} + 2^{-4} + 2^{-7} + \cdots + 2^{-(1+3^n)} + \cdots \\
&= \tfrac{1}{2}\left(\tfrac{1}{2^3}\right)^0 + \tfrac{1}{2}\left(\tfrac{1}{2^3}\right)^1 + \tfrac{1}{2}\left(\tfrac{1}{2^3}\right)^2 + \cdots + \tfrac{1}{2}\left(\tfrac{1}{2^3}\right)^n + \cdots \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\tfrac{1}{2^3}\right)^n
\end{aligned}$$

$\sum_{n=0}^{\infty}\left(\frac{1}{2^3}\right)^n$ is a geometric series with $r = 2^{-3}$ and so has sum $\frac{1}{1-r} = \frac{8}{7}$.

The answer is $\frac{1}{2}\left(\frac{8}{7}\right) = \frac{4}{7}$. □