Solutions (Quiz 4, section B03)

Problem 1 (5 points): Find the Taylor series generated by $f(x) = x^5 - x^3 + 5x - 8$ at $x = 2$.

solution:

$$\begin{array}{lcl} f(x) = x^5 - x^3 + 5x - 8 & \Rightarrow & f(2) = 26 \\ f'(x) = 5x^4 - 3x^2 + 5 & \Rightarrow & f'(2) = 73 \\ f''(x) = 20x^3 - 6x & \Rightarrow & f''(2) = 148 \\ f'''(x) = 60x^2 - 6 & \Rightarrow & f'''(2) = 234 \\ f^{(4)}(x) = 120x & \Rightarrow & f^{(4)}(2) = 240 \\ f^{(5)}(x) = 120 & \Rightarrow & f^{(5)}(2) = 120 \end{array}$$

Note that $f^{(n)}(x) = 0$ for $n \geq 6$. Therefore, the Taylor series generated by $f(x)$ at $x = 2$ is

$$26 + 73(x-2) + \frac{148}{2!}(x-2)^2 + \frac{234}{3!}(x-2)^3 + \frac{240}{4!}(x-2)^4 + \frac{120}{5!}(x-2)^5 .$$

Problem 2 (5 points): Let $f(x) = \mathrm{e}^x$ and $P_2(x)$ the Taylor polynomial of f of order 2 centered $x = 0$. Using Taylor's remainder of order 2, $R_2(x)$, estimate the bound for the error between $f(x)$ and $P_2(x)$ for $|x| < 10^{-2}$.

solution:

$$f(x) = e^x = P_2(x) + R_2(x),$$

where $R_2(x) = \frac{f'''(c)}{3!}x^3$ for some c between 0 and x. Hence, $R_2(x)$ can be used to estimate the error between $f(x)$ and $P_2(x)$. Here, $f'''(x) = e^x$ and in the interval $(-10^{-2}, 10^{-2})$, $f'''(c) = e^c < e^{10^{-2}}$. Thus,

$$|R_2(x)| = \left| \frac{f'''(c)}{3!} x^3 \right| < \frac{e^{10^{-2}}}{3!}|x|^3 < \frac{e^{10^{-2}}}{3!} 10^{-6} .$$