Quiz 3 Solutions

Problem 1 (2 points): Given $\sum_{n=1}^{\infty} -1^{n+1}\frac{1}{n} = \ln 2$, estimate the magnitude of the between $\sum_{n=1}^{97} -1^{n+1}\frac{1}{n}$ and $\ln 2$.

The magnitude of the error is $\left|\sum_{n=1}^{\infty} -1^{n+1}\frac{1}{n} - \sum_{n=1}^{97} -1^{n+1}\frac{1}{n}\right| \leq \left|-1^{99}\frac{1}{98}\right| = \frac{1}{98}$.

Problem 2 (8 points): Find the radius and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-7)^n}{6^n\ \sqrt{n^2+7}}$. For what values of x does the series converge absolutely, and for what values of x does it converge conditionally?

Using the ratio test:

$$\frac{u_{n+1}}{u_n} = \frac{(x-7)^{n+1}}{6^{n+1}\ \sqrt{(n+1)^2+7}}\frac{6^n\ \sqrt{n^2+7}}{(x-7)^n} = \frac{x-7}{6}\frac{\sqrt{n^2+7}}{\sqrt{(n+1)^2+7}} = \frac{x-7}{6}\sqrt{\frac{n^2+7}{n^2+2n+8}}$$

Hence $\lim \frac{u_{n+1}}{u_n} = \frac{x-7}{6}$. It follows that the series converges absolutely if:

$$\left|\frac{x-7}{6}\right| < 1 \Leftrightarrow -1 < \frac{x-7}{6} < 1 \Leftrightarrow -6 < x-7 < 6 \Leftrightarrow 1 < x < 13$$

We must also check absolute convergence at the endpoints of the interval $(1,13)$, so note that the series' $\sum_{n=0}^{\infty}\left|\frac{6^n}{6^n\ \sqrt{n^2+7}}\right| = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+7}}$ and $\sum_{n=0}^{\infty}\left|\frac{-6^n}{6^n\ \sqrt{n^2+7}}\right| = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+7}}$ diverge by the direct comparison test and the integral test as follows.

$$\frac{1}{\sqrt{n^2+7}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$
$$\int_0^{\infty}\frac{1}{x}dx = \lim_{x\to\infty}\ln|x| = \infty$$

Finally, we have to find out if it converges conditionally at the endpoints. That is, we have to check convergence of the series $\sum_{n=0}^{\infty}\frac{6^n}{6^n\ \sqrt{n^2+7}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+7}}$

and $\displaystyle\sum_{n=0}^{\infty} \frac{-6^n}{6^n\ \sqrt{n^2+7}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+7}}$. The first series diverges as before, and the second series converges by the alternating series test.

Therefore, the interval of convergence is $(1, 13]$ with absolute convergence on $(1, 13)$ and conditional convergence at $x = 13$. The radius of convergence is $\frac{13-1}{2} = 6$.