Last Name: ____________________, First Name: ____________________

Mat 21C-A03 (5:10 - 6:00pm) Quiz #6 Solutions

You have 15 minutes to do the following problems. Justify all solutions. You may not use any electronic devices for the duration of the quiz. Answers without support will receive no credit.

1. (5 points) Let $\mathbf{u} = \langle 10, 11, -2\rangle$ and $\mathbf{v} = \langle 0, 3, 4\rangle$. Find the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$. Find the angle between $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution The formula for a projection is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}.$$

The dot product of the two vectors is $33 - 8 = 25$, while the square of the magnitude of $\mathbf{v}$ is $|\mathbf{v}|^2 = 9 + 16 = 25$

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{25}{25}\langle 0, 3, 4\rangle = \langle 0, 3, 4\rangle.$$

Without computation, the angle between $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is simply $90°$ because they are orthogonal by construction. But a proof could go like. Notice $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$ so that

$$\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)\cdot\left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right) = \frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^2} - \frac{(\mathbf{u}\cdot\mathbf{v})^2\mathbf{v}\cdot\mathbf{v}}{|\mathbf{v}|^4} = \frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^2}\left(1 - \frac{|\mathbf{v}|^2}{|\mathbf{v}|^2}\right) = 0.$$

2. (5 points) Find the distance from the point $(2, 1, -1)$ to the line

$$\begin{aligned} x &= 2t, \\ y &= 1 + 2t, \\ z &= 2t. \end{aligned}$$

Solution Rewrite the parametric equations into vector form

$$\vec{r}(t) = \langle 0, 1, 0\rangle + t\langle 2, 2, 2\rangle = \vec{r}_0 + t\vec{v}.$$

Calling $P(0, 1, 0)$ the point on the line, the distance formula from $Q(2, 1, -1)$ to the line $\vec{r}(t)$ is

$$d = \frac{|\overrightarrow{PQ}\times\vec{v}|}{|\vec{v}|}$$

First computing the cross product,

$$\overrightarrow{PQ}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = \hat{i}[(2)(0) - (-1)(2)] - \hat{j}[(2)(2) - (2)(-1)] + \hat{k}[(2)(2) - (0)(2)] = 2\langle 1, -3, 2\rangle,$$

which has magnitude $2\sqrt{1 + 9 + 4}$. Dividing this by the length of $\vec{v}$, which is $|\vec{v}| = \sqrt{4 + 4 + 4} = \sqrt{12}$, we finally get the distance is $d = 2\sqrt{7/6} \approx 2.1602469$.