1. (a) (*8 pts*) State the Mean Value Theorem (MVT).

Assume f is continuous on $[a, b]$ and $f'(x)$ exist on (a, b) then there is a c in (a, b) s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$

(b) (*10 pts*) Determine if the following function satisfies the assumption of the MVT. If so, find all values c guaranteed by the conclusion of the MVT: $f(x) = 2x + 5x^{1/5}$ on the interval $[-1, 0]$

+2

$2x + 5x^{1/5}$ are continuous on $[-1, 0]$ and $f'(x) = 2 + \frac{1}{x^{4/5}}$ exist on $(-1, 0)$ then by MVT there is a c in $(-1, 0)$ s.t.

+2

$f'(c) = \frac{f(0) - f(-1)}{0 - (-1)} = \frac{-(-2-5)}{1} = 7$

+2

$f'(c) = 2 + \frac{1}{c^{4/5}} = 7 \Rightarrow \frac{1}{5} = c^{4/5} \Rightarrow$

$c^4 = \frac{1}{5^5} \Rightarrow c = \pm\sqrt[4]{\frac{1}{5^5}}$ since c must be in $(0, 1)$ then the only c that works is

$c = -\sqrt[4]{\frac{1}{5^5}}$ +2

2. (*8 pts*) Is there a differentiable function f satisfying $f(1) = 5$, $f(4) = 14$ and $f'(x) \geq 4$ for all real x? Justify your answer.

No, if there were a differentiable function satisfying the above condition then by MVT there is a c in $(1, 4)$ s.t. $f'(c) = \frac{f(4) - f(1)}{4 - 1} = 3$ but $f'(x) \geq 4$ for all real x. So there is no differentiable function.

3. (*14 pts*) Show that the equation $3x^7 + 2x^5 + 7x + 10 = 0$ has exactly one real solution.

First show there is a solution to this equation. Let $f(x) = 3x^7 + 2x^5 + 7x + 10$ then f is continuous on $[-1, 1]$ and $-2 = f(-1) \leq 0 \leq f(1) = 22$ so by IVT there is a c in $[-1, 1]$ s.t. $f(c) = 0$.

Now since $f'(x) = 21x^6 + 10x^4 + 7 \geq 7 > 0$ for all x then f is increasing on the real line. In fact on $[-1, 1]$ there is a solution and that is the only solution for the whole real line.

4. (*10 pts each*) Find $\frac{dy}{dx}$ for each of the following functions. *You do not need to simplify your answer.*

(a) $y = \frac{2+\cos x}{(x+\sin x)(3-x^4)}$.

Use quotient rule.

$$\frac{dy}{dx} = \frac{([redacted]\sin x)(x + \sin x)(3 - x^4) - \left((1 + \cos x)(3 - x^4) + (-4x^3)(x + \sin x)\right)(2 + \cos x)}{(x + \sin x)^2 (3 - x^4)^2}$$

(b) $y = \sin^5(\sec(x^3 + 3x + 1))$

Let $u = x^3 + 3x + 1$ then

$v = \sec u$

$w = \sin v$

$y = w^5$

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dv}\frac{dv}{du}\frac{du}{dx}$$

$$= 4w^5 \cos v \sec u \tan u (3x^2 + 3)$$

$$= 4\sin^5(\sec(x^3 + 3x + 1)) \cos(\sec(x^3 + 3x + 1)) \sec(x^3 + 3x + 1) \tan(x^3 + 3x + 1)(3x^2 + 3)$$

$$(\sec u)' = \left(\frac{1}{\cos u}\right)' = \frac{\sin u}{\cos^2 u} = \tan u \sec u$$

5. (*10 pts*) Use implicit differentiation to find the relative extrema(s) of $2y^2 + xy + x^2 = 7$.

To find relative extrema we must implicitly differentiate and solve for $\frac{dy}{dx} \Leftrightarrow y' = 0$. So

$$4yy' + y + xy' + 2x = 0$$

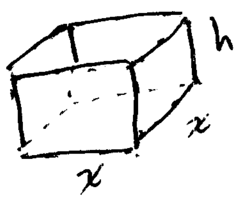
$\Rightarrow y' = \frac{-2x-y}{x+4y} = 0 \Rightarrow y = -2x$ substituting into $2y^2 + xy + x^2 = 7$ we get

$2(-2x)^2 - 2x^2 + x^2 = 7 \Rightarrow 8x^2 - 2x^2 + x^2 = 7 \Rightarrow$

$x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow y = \mp 2$ so at

$(1, -2)$ and $(-1, 2)$ we have relative extrema's.

6. (*12 pts*) Of all topless rectangular boxes with square bases that have a volume of 3 cubic feet, which uses the least material?



Minimize Surface Area

$S = x^2 + 4xh$

with constraint $x^2 h = 3$.

$$\Rightarrow S = x^2 + 4x\frac{3}{x^2} = x^2 + \frac{12}{x}$$

Now to minimize take derivative and set to 0, so

$S' = 2x - \frac{12}{x^2} = 0 \Rightarrow x^3 = 6 \Rightarrow x = \sqrt[3]{6}$

Note Domain $0 < x < \infty$ we look at

$\lim_{x \to 0^+} S(x) = +\infty$ and $\lim_{x \to \infty} S(x) = +\infty$ so minimum occurs in $0 < x < \infty$. Now verify $\sqrt[3]{6} = x$ is a minimum

	$(0, \sqrt[3]{6})$	$(\sqrt[3]{6}, \infty)$
S'	$-$	$+$

1 is in $(0, \sqrt[3]{6})$
2 is in $(\sqrt[3]{6}, \infty)$ $\Rightarrow$ S has minimum at $x = \sqrt[3]{6}$

Minimum Surface Area is $6^{2/3} + \frac{12}{\sqrt[3]{6}}$ $\left(h = \frac{3}{6^{2/3}}\right)$

7. *(20 pts)* Let $f(x) = \frac{x+1}{\sqrt{x^2+1}}$. Sketch the graph of $f(x)$. Clearly indicate any intercepts, asymptotes, critical points, inflection points, or local or global extrema.

$f(0) = 1$ $(0,1)$ y-intercept $f(x) = 0 \Rightarrow x = -1$

so $(-1, 0)$ x-intercept. No V. A. since

$x^2+1 \geq 1 \Rightarrow \sqrt{x^2+1} \geq 1$. Assume $x \neq 0$ then

$f(x) = \dfrac{1+\frac{1}{x}}{\frac{|x|}{x}\sqrt{1+\frac{1}{x^2}}}$ so $\lim\limits_{x\to\infty} f(x) = 1$ and

$\lim\limits_{x\to-\infty} f(x) = -1$. So H. A. at $y = 1$ & $y = -1$. Now find critical points. So

$f'(x) = \dfrac{\sqrt{x^2+1} - \frac{1(2x)}{2\sqrt{x^2+1}}(x+1)}{x^2+1} = \dfrac{x^2+1-x^2-x}{(x^2+1)^{3/2}}$

$= \dfrac{1-x}{(x^2+1)^{3/2}} = 0 \Rightarrow x = 1$ so critical point at

$(1, \sqrt{2})$. For possible inflection points. $f''(x) = \dfrac{-(x^2+1)^{3/2} - \frac{3}{2}(x^2+1)^{1/2}(2x)(1-x)}{(x^2+1)^3}$

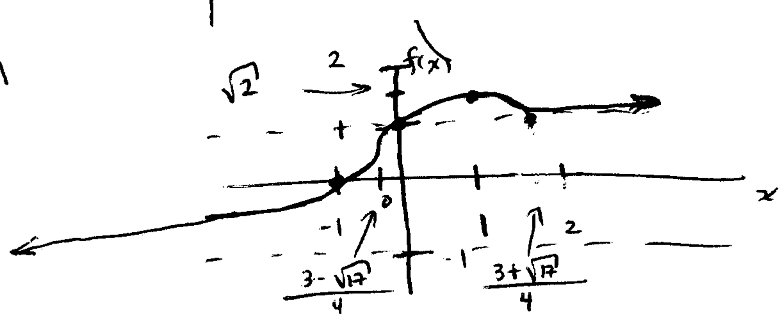
$= \dfrac{(x^2+1)^{1/2}\left(-(x^2+1) - 3x(1-x)\right)}{(x^2+1)^3} = \dfrac{-x^2-1-3x+3x^2}{(x^2+1)^{5/2}} = \dfrac{2x^2-3x-1}{(x^2+1)^{5/2}} = 0$

$\Rightarrow x = \dfrac{3 \pm \sqrt{9-4(2)(-1)}}{4} = \dfrac{3\pm\sqrt{17}}{4} \approx \dfrac{3\pm 4}{4} = \dfrac{7}{4}$ or $\dfrac{-1}{4}$.

x	-1	$\frac{3-\sqrt{17}}{4}$	0	1	$\frac{3}{2}$	$\frac{3+\sqrt{17}}{4}$	2
f'	$+$		$+$		$-$		$-$
f''	$+$		$-$		$-$		$+$

Inflection points at $\left(x = \dfrac{3\pm\sqrt{17}}{4}, f(x)\right)$

Global max at $(1, \sqrt{2})$



8. Standing on a 33 foot high cliff, you throw an apple straight up at a speed of 16 $feet/sec$. Assume the acceleration due to gravity is -32 $feet/sec^2$. Let $f(t)$ be the height of the apple above ground at time t.

(a) (*3 pts*) Write the equation for $f(t)$.

$f(t) = -16t^2 + 16t + 33$

(b) (*5 pts*) At what time is the apple's height above the ground the greatest?

$f'(t) = -32t + 16 = 0 \Rightarrow t = \frac{1}{2}$ sec //

$(0, 1/2)$ $(1/2, \infty)$

f' $-$ $+$

or $f'' = -32 < 0 \Rightarrow$ max.

(c) (*2 pts*) What is the maximum height the apple reaches?

$f(\frac{1}{2}) = \frac{-16}{4} + \frac{16}{2} + 33$

$= -4 + 8 + 33 = 37$ feet //

9. (*10 pts*) Consider the composition of functions $h(x) = f(g^2(x))$. In addition assume that $g(1) = 2$, $g'(1) = -3$, $f'(2) = -1$, and $f'(4) = 5$. Determine the value of $h'(1)$.

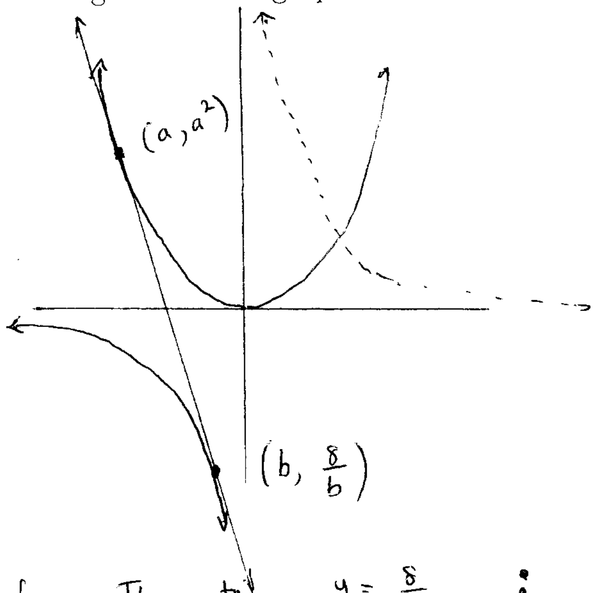
$h'(x) = f'(g^2(x))\, 2g(x)\, g'(x)$

$h'(1) = f'(g^2(1))\, 2g(1)\, g'(1)$

$= f'(4)\, 2(2)(-3)$

$= 5 \cdot 4 \cdot (-3) = -60$ //

10. (*18 pts*) Consider the graphs of $y = x^2$ and $y = \frac{8}{x}$. Find equations of all lines simultanenously tangent to both graphs.



If there was a line tangent to both curves. The general tangent line equation for the curve $y=x^2$ should be the same as the general tangent line curve for $y = \frac{8}{x}$ at some point

(1) Eqn for TL to $y = \frac{8}{x}$: $-\frac{8}{b^2}(x-b) = y - \frac{8}{b}$

(2) " $y = x^2$: $2a(x-a) = y - a^2$

so the slopes of both must match i.e. $2a = \frac{-8}{b^2}$

$\Longrightarrow$ $b^2 = \frac{-8}{2a}$ or $a = \frac{-8}{2b^2} = \frac{-4}{b^2}$ so

$$\overset{(1)}{-\frac{8}{b^2}(x-b) + \frac{8}{b}} = \overset{(2)}{-\frac{8}{b^2}\left(x + \frac{4}{b^2}\right) + \frac{16}{b^4}}$$

$$-\frac{8}{b^2}x + \frac{8}{b} + \frac{8}{b} = -\frac{8}{b^2}x - \frac{32}{b^4} + \frac{16}{b^4}$$

$$\frac{16}{b} = \frac{-16}{b^4} \Longrightarrow b^3 = -1 \Longrightarrow b = -1$$

$\Longrightarrow$ $a = -4$ $\Longrightarrow$ Tangent Line to both graphs : $\underline{-8(x+1) - 8 = y}$

Page	2 (*26*)	3 (*34*)	4 (*22*)	5 (*20*)	6 (*20*)	7 (*18*)	Total (*140*)
Scores							