1. (*16 pts*) Is the function f given below continous at $x = 0$? You must show your work to receive credit.

$$f(x) = \begin{cases} \frac{\frac{1}{x+1}-1}{x} & , \text{ if } x < 0 \\ -1 & , \text{ if } x = 0 \\ \frac{x^2+x}{3x^2-x} & , \text{ if } x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\frac{1}{x+1}-1}{x} = \lim_{x\to 0^-} \frac{-x}{x(x+1)} = \lim_{x\to 0^-} \frac{-1}{x+1} = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+x}{3x^2-x} = \lim_{x\to 0^+} \frac{x(x+1)}{x(3x-1)} = \lim_{x\to 0^+} \frac{x+1}{3x-1} = -1$$

$\Rightarrow \lim_{x\to 0} f(x) = -1 = f(0)$ so $\boxed{f \text{ is continuous at } x = 0}$

2. (*10 pts each*) Find the limit if it exists. If it doesn't, explain why.

(a) $\displaystyle \lim_{x\to 2} \frac{x^2-4}{x^2+x-6} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x+2}{x+3} = \boxed{\frac{4}{5}}$

(b) $\displaystyle \lim_{x\to 0} \frac{1-\sin^2 x}{x} = \lim_{x\to 0} \frac{\cos^2 x}{x}$ "$\frac{1}{0}$"

Since $\displaystyle \lim_{x\to 0^-} \frac{\cos^2 x}{x} = -\infty$ and $\displaystyle \lim_{x\to 0^+} \frac{\cos^2 x}{x} = +\infty$

$\Rightarrow \displaystyle \lim_{x\to 0} \frac{1-\sin^2 x}{x}$ $\boxed{\text{D.N.E.}}$

(c) $\displaystyle \lim_{x\to 0} \frac{\frac{\sin x}{x}-1}{x^2} = \lim_{x\to 0} \frac{\sin x - x}{x^3} \underset{L'H}{\overset{\text{"}\frac{0}{0}\text{"}}{=}} \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$

$\displaystyle \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{-\sin x}{6x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{-\cos x}{6} = \boxed{\frac{-1}{6}}$

(d) $\lim_{x\to+\infty}\left(1+\frac{1}{3x^2}\right)^{4x^2}$ "1^∞" $=$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}$

let $u = 3x^2$ then $x \to +\infty \Rightarrow u \to +\infty$

$$\lim_{u\to+\infty}\left(1+\frac{1}{u}\right)^{4/3\,u} = \lim_{x\to\infty}\left(\left(1+\frac{1}{u}\right)^{u}\right)^{4/3} = \boxed{e^{4/3}}$$

3. (*10 pts each*) Find $\frac{dy}{dx}$ for each of the following functions. *You do not need to simplify your answer.*

(a) $y = \frac{2^x - \log_2 x}{x^2}$

$$\frac{dy}{dx} = \boxed{\frac{\left(\ln 2\; 2^x - \frac{1}{\ln 2}\frac{1}{x}\right)x^2 - 2x\left(2^x - \log_2 x\right)}{x^4}}$$

(b) $y = \frac{(\sec x)(3^x)\sqrt[3]{1-x}}{x\cos(x^2)}$

$$\ln y = \ln \sec x + \ln 3^x + \ln\sqrt[3]{1-x} - \ln x - \ln\cos(x^2)$$
$$= \ln \sec x + x\ln 3 + \frac{1}{3}\ln(1-x) - \ln x - \ln\cos(x^2)$$

$$\frac{y'}{y} = \frac{1}{\sec x}\sec x\tan x + \ln 3 + \frac{1}{3(1-x)}(-1) - \frac{1}{x} - \frac{1}{\cos(x^2)}(-\sin(x^2))\,2x$$

$$\frac{dy}{dx} = \boxed{\left(\tan x + \ln 3 - \frac{1}{3(1-x)} - \frac{1}{x} + \frac{2x\sin(x^2)}{\cos x^2}\right)y}$$

(c) $y = (e^2 + \sin x)^{(1-x^2)}$

$$\ln y = (1-x^2)\ln(e^2+\sin x)$$

$$\frac{y'}{y} = -2x\ln(e^2+\sin x) + \frac{1}{e^2+\sin x}\cos x\,(1-x^2)$$

$$\frac{dy}{dx} = \boxed{\left(-2x\ln(e^2+\sin x) + \frac{\cos x\,(1-x^2)}{e^2+\sin x}\right)y}$$

4. (*10 pts each*) Given that $\ln 2 = A$, $\ln 3 = B$, $\ln 5 = C$. Compute each of the following in term(s) of A, B, C.

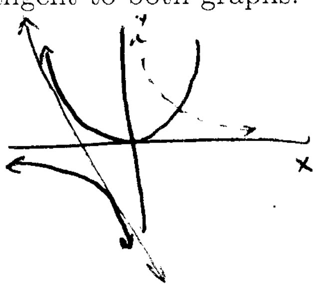
(a) $\ln\left(\frac{9}{10}\right) = \ln 9 - \ln 10 = \ln 3^2 - \ln 2\cdot 5 = 2\ln 3 - \ln 2 - \ln 5$

$= \boxed{2B - A - C}$

(b) $\log_6(10e^2) = \dfrac{\ln 10e^2}{\ln 6} = \dfrac{\ln 10 + \ln e^2}{\ln 2\cdot 3} = \dfrac{\ln 2\cdot 5 + 2\ln e}{\ln 2 + \ln 3}$

$= \dfrac{\ln 2 + \ln 5 + 2}{\ln 2 + \ln 3} = \boxed{\dfrac{A + C + 2}{A + B}}$

5. (*18 pts*) Consider the graphs of $y = x^2$ and $y = \frac{8}{x}$. Find equations of all lines simultanenously tangent to both graphs.



1) (a, a^2)
2) $(b, \frac{8}{b})$ General point on each graph

1) TL at (a, a^2) $\quad 2a(x-a) = y - a^2$

2) TL at $(b, \frac{8}{b})$ $\quad -\frac{8}{b^2}(x-b) = y - \frac{8}{b}$

$\Rightarrow \quad 2a = -\frac{8}{b^2} \quad \Rightarrow \quad a = -\frac{4}{b^2}$

Equate two lines

$-\frac{8}{b^2}\left(x + \frac{4}{b^2}\right) + \frac{16}{b^4} = -\frac{8}{b^2}(x - b) + \frac{8}{b}$

$-\frac{8}{b^2}x - \frac{32}{b^4} + \frac{16}{b^4} = -\frac{8}{b^2}x + \frac{8}{b} + \frac{8}{b}$

$-\frac{16}{b^4} = \frac{16}{b} \quad \Rightarrow \quad b^3 = -1 \quad \Rightarrow b = -1$

Equation of Line Tangent to both graphs

is $\boxed{-8(x+1) - 8 = y}$

6. (*20 pts*) Given that

$$f(x) = \frac{e^x}{x-1}, \quad f'(x) = \frac{e^x(x-2)}{(x-1)^2}, \quad f''(x) = \frac{e^x(x^2-4x+5)}{(x-1)^3}.$$

Sketch the graph of $f(x)$. Include the table that shows signs of f' and f''. Calculate and show the following information. Write "NA" if the information is not applicable.

Domain: $x \neq 1$

x–intercept(s): NA

y–intercept(s): -1

Vertical asymptote: $x = 1$

Horizontal asymptote: $y = 0$

Relative extrema: $(2, e^2)$

Inflection point(s): NA

Range: $(-\infty, 0)$ (e^2, ∞)

$\lim_{x \to +\infty} f(x) = +\infty$

$\lim_{x \to -\infty} f(x) = 0$

$f'(x) = 0 \mid \text{undef} \Rightarrow x = 2$ or $x = 1$

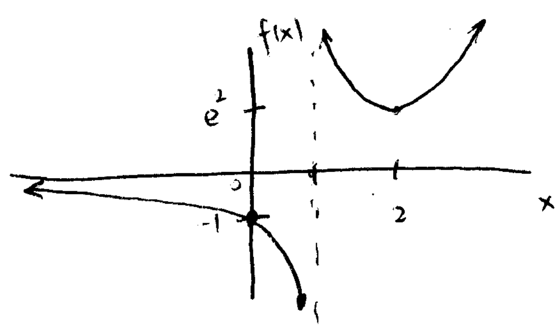
$f''(x) = 0 \mid \text{undef} \Rightarrow x = 1$

$x = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2}$ not defined.

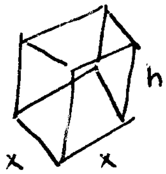
$(-\infty, 1)$ $(1, 2)$ $(2, \infty)$

	0	1	$\frac{3}{2}$	2	3
f'	$-$	|	$-$	|	$+$
f''	$-$	|	$+$	|	$+$

$\left(\frac{3}{2}\right)^2 - 4\left(\frac{3}{2}\right) + 5 = \frac{9}{4} - \frac{24}{4} + \frac{20}{4} = +$



7. (*16 pts*) A rectangular house is to have a square base and total volume of 40,000 cubic feet. In one second, the amount of heat per unit area that leaves each wall is a constant p and 10 times this amount leaves the roof. No heat leaves through the floor. What should the dimensions of the house be to minimize heat loss?



$x^2 h = 40000 \text{ ft}^3 \Rightarrow h = \frac{40000}{x^2}$

$4pxh + 10px^2 = HL$ (Heat Loss)

Minimize HL

$\frac{160000}{x}p + 10px^2 = 4p\left(\frac{40000}{x}\right) + 10px^2 = HL$

Domain $(0, \infty)$ $\lim_{x \to +\infty} HL = +\infty$ $\lim_{x \to 0^+} HL = +\infty$

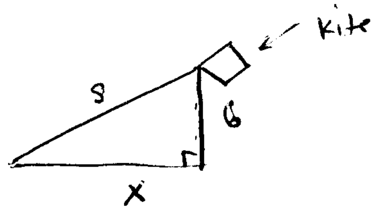
$\Rightarrow$ Minimum lies in $(0, \infty)$

$HL' = \frac{-160000p}{x^2} + 20px = 0 \Rightarrow 20x = \frac{160000p}{x^2}$

$20x^3 = 160000 \Rightarrow x^3 = 8000 \Rightarrow \boxed{x = 20 \text{ ft}}$

$\Rightarrow \boxed{h = 100 \text{ ft}}$

8. (*20 pts*) A kite is flying at a height of 6 feet in a horizontal wind. When 10 feet of string is out, the kite is pulling the string out at a rate of 4 feet per second. What is the kite's velocity?



$\frac{ds}{dt} = 4 \frac{ft}{sec}$

Want $\frac{dx}{dt}$

$s^2 = 6^2 + x^2 \Rightarrow 2s\frac{ds}{dt} = 0 + 2x\frac{dx}{dt}$

$\Rightarrow \frac{ds}{dt}s = x\frac{dx}{dt}$ $s = 10 \Rightarrow x = 8$

$5\frac{ft}{sec} = 4\frac{ft}{sec}\frac{10 ft}{8 ft} = \frac{dx}{dt}$

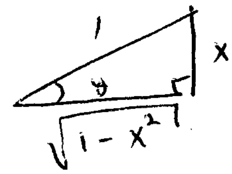
The kite's velocity is $\boxed{5\frac{ft}{sec}}$

9. (a) (*14 pts*) Let $f(x) = \cos^{-1} x$. Prove that $f'(x) = \frac{-1}{\sqrt{1-x^2}}$.

Let $y = \cos^{-1} x \Rightarrow \cos y = x \Rightarrow -\sin y \; y' = 1$

$\Rightarrow y' = \dfrac{-1}{\sin y}$ Find $\sin y$

$\cos y = x$



$\Rightarrow \sin y = \sqrt{1-x^2}$

$\Rightarrow y' = \dfrac{-1}{\sqrt{1-x^2}}$.

(b) (*4 pts*) Compute $\frac{d}{dx}[\cos^{-1}(\tan x)]$.

$= \dfrac{-1}{\sqrt{1-\tan^2 x}} \cdot \sec^2 x$.

10. (*10 pts*) Consider the function $f(x) = \ln(\sin(x) + 1) + x$. Use f' to show that f is a one-to-one function on the interval $(-\pi/2, \pi/2)$.

$f'(x) = \dfrac{1}{\sin x + 1} \cos x + 1 = \dfrac{\cos x}{\sin x + 1} + 1$

when $-\pi/2 < x < \pi/2$ $\quad -1 < \sin x < 1$

$\Downarrow$

and $0 < \cos x < 1$ $\quad 0 < \sin x + 1 < 2$

$\Downarrow$

$0 < \dfrac{\cos x}{\sin x + 1} < \infty \Leftarrow \dfrac{1}{2} < \dfrac{1}{\sin x + 1} < \infty$

$1 < \dfrac{\cos x}{\sin x + 1} + 1 < \infty$ so $f' > 0$ on $(-\pi/2, \pi/2)$

$\Rightarrow f$ is increasing on $(-\pi/2, \pi/2)$

$\Rightarrow f$ is $1-1$

11. (*10 pts*) Find the inverse function f^{-1} for $f(x) = 10^{arctan(x^3+1)}$.

$y = 10^{\arctan(x^3+1)}$

To find inverse swap x & y

$x = 10^{\arctan(y^3+1)} \Rightarrow \log_{10} x = \arctan(y^3+1)$

$\Rightarrow \tan(\log_{10} x) = y^3 + 1 \Rightarrow \tan(\log_{10} x) - 1 = y^3$

$\Rightarrow \boxed{y = \sqrt[3]{\tan(\log_{10} x) - 1}}$

12. (*10 pts*) Determine whether or not the function

$$f(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

is differentiable at $x = 0$. Hint: Do not use the product rule.

check with limit definition of derivative from left & right of 0.

$$\lim_{\Delta x \to 0^-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x^2}{\Delta x} = \lim_{\Delta x \to 0^-} -\Delta x = 0$$

$$\lim_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x^2}{\Delta x} = \lim_{\Delta x \to 0^+} \Delta x = 0$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = 0$$ so f is differentiable at $x = 0$.

13. (*12 pts*) Extra Credit: Using the limit definition of the derivative. Show that the derivative of $\ln x$ is $\frac{1}{x}$.

$$\lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(\frac{x+\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \ln\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} = \lim_{\Delta x \to 0} \ln\left(\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right)^{\frac{1}{x}}$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \ln\left(\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right) = \frac{1}{x} \ln e = \frac{1}{x}$$

Page	2 (*46*)	3 (*40*)	4 (*38*)	5 (*20*)	6 (*36*)	7 (*38*)	8 (*10*)	Total (*228*)
Scores								