Math 21C - Section B01 - Quiz 2 **SOLUTION**
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<u>Problem 1:</u> Does the series below converge or diverge? Give reasons for your answer: State test(s) you use **by name**.

$$\sum_{n=1}^{\infty} \frac{n}{(\ln n)^2}$$

<u>Solution:</u> We look at the terms associated to the sequence. The n^{th} term of the sequence is:

$$a_n = \frac{n}{(\ln n)^2}.$$

We examine the limit of a_n as $n \to \infty$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{(\ln n)^2} \overset{\text{L'Hôpital}\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{1}{2\ln n \cdot \frac{1}{n}} \overset{\text{alg.}}{=} \lim_{n\to\infty} \frac{n}{2\ln n} \overset{\text{L'Hôpital}\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{1}{2 \cdot \frac{1}{n}}.$$

By algebra, this is the same[1] as

$$\lim_{n\to\infty} \frac{n}{2},$$

and this sequence diverges to ∞.

Since the sequence a_n does not converge to zero, the series $\sum a_n$ diverges by the Nth Term Test.

<u>Summary of Alternate Solution:</u> Let a_n be defined as in the previous solution, and let

$$b_n = \frac{1}{\ln n}.$$

We first demonstrate that the series

$$\sum b_n$$

diverges. Note that

$$n \geq \ln n$$

for all $n > 0$. Thus,

$$\frac{1}{n} \leq \frac{1}{\ln n}$$

Let $c_n = \frac{1}{n}$. We already know the harmonic series $\sum c_n$ diverges[2]. Since $\sum c_n$ diverges and $c_n \leq b_n$ for each n, the series $\sum b_n$ diverges as well.

[1]Note, that I put "limit" in front: The previous expression and what follows would **NOT** be equal otherwise! Write in limit where it you need to. That is, make sure that two things are equal before you put an equals sign between them!

[2]You can also say $\sum c_n$ diverges by the p-test, since $\sum c_n$ is a p-series.

Now, we'll use the fact that the series $\sum b_n$ diverges in the Limit Comparison Test. We consider the limit of the sequence $\frac{a_n}{b_n}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n}{(\ln n)^2} \cdot \ln n \\
&= \lim_{n\to\infty} \frac{n}{\ln n} \\
&\overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} \\
&= \lim_{n\to\infty} n \\
&= \infty.
\end{aligned}$$

We have shown:

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and
- the series $\sum b_n$ diverges

That is, we have met both[3] requirements to use part 3 of the Limit Comparison Test (Theorem 11). By the Limit Comparison Test, we conclude that the series $\sum a_n$ diverges.

Problem 2: Does the following series converge or diverge? Give reasons for your answer: State test(s) you use **by name**.

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$$

Solution: Consider the sequence

$$d_n = \frac{(\ln n)^2}{n}.$$

We note that it converges to zero (this requires two uses of L'Hôpital). In fact, note that the d_n sequence is, so to speak, the "reciprocal" of the sequence a_n from the first problem. So, all of the L'Hôpital work from above can just be

[3]It is not enough to just meet one condition.

copied here (just swap the roles of numerator and denominator). Specifically,

$$\begin{aligned}
\lim_{n\to\infty} d_n &= \lim_{n\to\infty} \frac{(\ln n)^2}{n} \\
&= \lim_{n\to\infty} \frac{2(\ln n)\cdot\frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{2(\ln n)}{n} \\
&= \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{2}{n} \\
&= 0.
\end{aligned}$$

Since the sequence d_n converges to 0, it settles down on 0. To settle down, at some point, it must stay below 1. Thus, there is some[4] N such that for all $n \geq N$, it is the case that $d_n \leq 1$. We use this fact with some algebra now. For $n \geq N$,

$$\frac{(\ln n)^2}{n} \leq 1$$

Then, divide by n^2 on both sides to get

$$\frac{(\ln n)^2}{n^3} \leq \frac{1}{n^2}.$$

The series

$$\sum_{n=N}^{\infty} \frac{1}{n^2}$$

converges since it is a p-series with $p = 2 > 1$. By the Direct Comparison Test, the series

$$\sum \frac{(\ln n)^2}{n^3}$$

converges.

Summary of an Alternate Solution: Let b_n be the sequence

$$b_n = \frac{1}{n^2}.$$

We note that the series $\sum b_n$ converges by the p-test. Let a_n be the sequence

$$a_n = \frac{(\ln n)^2}{n^3}.$$

[4] By saying there is some N, on paper, it might be N is a million...

Then, we study the limit of the sequence $\frac{a_n}{b_n}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} \cdot n^2 \\
&= \lim_{n\to\infty} \frac{(\ln n)^2}{n} \\
&\overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{2(\ln n) \cdot \frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{2\ln n}{n} \\
&\overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{2 \cdot \frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{2}{n} \\
&= 0.
\end{aligned}$$

Both requirements of part 2 of the Limit Comparison Test are met, so the series $\sum a_n$ converges.

Remarks

- Simply saying that a sequence or a series converges or diverges is not enough, even if you are right in your answer. That will zero points: You can't earn anything for guessing a 50/50 question. In general, partial credit is awarded only when the work you do form the initial steps of a successful solution! On a midterm, one can not earn partial credit for writing down things that are true in general about series, but irrelevant for the series at-hand.

- Be careful to not confuse sequences with series. A series has the capital Greek Σ in front. Σ is the Greek for "S", and so you should think "sum" whenever you see Σ. That is, with Σ, you're adding things up.

  - On that note, you need to note that *some* tests for series will make you study a sequence, and *other* tests for series will ask you to study a still different series from your original one. There is a new object to study. If the new object is a sequence, it must be studied using sequence rules. If the new object is a series, it must be studied using series rules. The complaint you should have at this point is "When does it stop?" It's just like for integrals: by using u-sub or integration by parts, you don't do away with integrating altogether: you just hope that the new integral that you get is easier than the old one. Just like with integration, having acumen to say which series is "harder" than which takes practice!

- Be careful with your calculus and algebra. While it is true that $\ln(n^2) = 2\ln n$, it is **not** true that $(\ln n)^2 = 2\ln n$.

- If you have a sequence that is a fraction, there is a tendency to say things like "the numerator grows faster than the denominator" or "the denominator grows faster than the numerator." That's not a very mathematically precise statement, and it's easy to made an honest mistake. When you think this, it's probably because both numerator and denominator do grow. So, it must mean that it meets the L'Hôpital's form $\frac{\infty}{\infty}$. Thus, you should use L'Hôpital's rule.

- For the Comparison tests, you have to have a second series, labelled in the book as $\sum b_n$, to compare **to**!

- Be careful: the converse of the Nth term test is not true. It is NOT true in general that the sequence $a_n \to 0$ means the series $\sum a_n$ converges. For example, $\frac{1}{n}$ and $\frac{1}{n^2}$ are both sequences that converge to zero, yet $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges. Therefore, if you know that $a_n \to 0$, it is **not enough information** to conclude that $\sum a_n$ converges or diverges (sorta like that "D" multiple choice selection on those annoying SAT quantity-comparison problems).