Quiz 4 Solutions

Problem 1 (5 points): Find the Taylor series generated by
$f(x) = x^5 - x^3 + 7x - 2$ at $x = -2$.

$f^{(1)}(x) = 5x^4 - 3x^2 + 7$
$f^{(2)}(x) = 20x^3 - 6x$
$f^{(3)}(x) = 60x^2 - 6$
$f^{(4)}(x) = 120x$
$f^{(5)}(x) = 120$

$f(-2) + f^{(1)}(-2)(x+2) + \frac{f^{(2)}(-2)}{2!}(x+2)^2 + \frac{f^{(3)}(-2)}{3!}(x+2)^3 + \frac{f^{(4)}(-2)}{4!}(x+2)^4 + \frac{f^{(5)}(-2)}{5!}(x+2)^5 =$
$\left((-2)^5 - (-2)^3 + 7(-2) - 2\right) + \left(5(-2)^4 - 3(-2)^2 + 7\right)(x+2) + \frac{20(-2)^3 - 6(-2)}{2!}(x+2)^2 + \frac{60(-2)^2 - 6}{3!}(x+2)^3 + \frac{120(-2)}{4!}(x+2)^4 + (x+2)^5 =$
$-40 + 75(x+2) - 74(x+2)^2 + 39(x+2)^3 - 10(x+2)^4 + (x+2)^5$

Problem 2 (5 points): Let $f(x) = \cos 2x$ and $P_2(x)$ the Taylor polynomial of f of order 2 centered $x = 0$. Using Taylor's remainder of order 2, $R_2(x)$, estimate the bound for the error between $f(x)$ and $P_2(x)$ for $|x| < 10^{-2}$.

Note that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n}$, so that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} (2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{2n!} x^{2n}$.
It follows that $\cos 2x = 1 + \frac{\cos(2c)2^3}{3!} x^3$ for some $c > x$. Therefore:

$|P_2(x) - \cos 2x| = |1 - \cos 2x| = \left|1 - \left(1 + \frac{\cos(2c)2^3}{3!} x^3\right)\right| = \left|\frac{\cos(2c)2^3}{3!} x^3\right| \leq \left|\frac{8}{6} x^3\right|$

Finally, since $|x| < \frac{1}{100}$, $\left|\frac{8}{6} x^3\right| < \frac{8}{6} \frac{1}{100}^3 = \frac{8}{6} \frac{1}{1000000}$.