Problem 1

(5 points): Given $a_1 = 4$ and the recursion formula $a_{n+1} = a_n + \frac{1}{4^n}$ for the remaining terms of the sequence. Determine if the sequence converges or diverges. If it converges, determine its limit. If it diverges, give reason why. Hint: Write out the first few terms without simplifying.

Solution:

The first few terms are
$a_1 = 4$
$a_2 = a_1 + 1/4$
$a_3 = a_2 + 1/4^2 = a_1 + 1/4 + 1/4^2$
$a_4 = a_3 + 1/4^3 = a_1 + 1/4 + 1/4^2 + 1/4^3$.
We see that in gerneral

$$a_{n+1} = 4 + \sum_{i=1}^{n} 1/4^i$$

so

$$\lim_{n\to\infty} a_n = 4 + \sum_{i=1}^{\infty} 1/4^i.$$

Notice that $\sum_{i=1}^{\infty} 1/4^i$ is a geometric series with $r = 1/4$ and so it converges to

$$\frac{1}{1 - 1/4} = 4/3.$$

Thus we have $\{a_n\}$ converges to $4 + 4/3$.

Problem 2

(5 points): Determine if the series

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{4n}\right)^n$$

converges or diverges. Give reasons for your answer.

Solution:

We can show this series diverges with the nth term test for divergence. First we recongize that by theorem 5 part 5

$$\begin{aligned}\lim_{n\to\infty} \left(1 - \frac{1}{4n}\right)^n &= \lim_{n\to\infty} \left(1 + \frac{-1/4}{n}\right)^n \\ &= e^{-1/4}.\end{aligned}$$

and since $e^{-1/4} \neq 0$ by the nth term test for divergence the series diverges.