

Solutions(Quiz 3, section B03)

1. (2 points): Given

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2.$$

Estimate the magnitude of the error between

$$\sum_{n=1}^{95} (-1)^{n+1} \frac{1}{n}$$

and $\ln 2$.

For an convergent alternating sequence $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$, we have the following:

$$|L - s_n| < u_{n+1}$$

where $L = \sum_{n=1}^{\infty} (-1)^{n+1} u_n$ and $s_n = \sum_{j=1}^n (-1)^{j+1} u_j$. The given series is convergent because it satisfies all three conditions for the alternating series test. Hence

$$|error| < \frac{1}{96}$$

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2. (8 points): Find the series:

$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{4^n \sqrt{n^2+5}}$$

radius and interval of convergence. For what values of x does the series converge absolutely and conditionally?

$$\text{Let } u_n = \frac{(x-5)^n}{4^n \sqrt{n^2+5}}.$$

First, use the ratio test to find the radius of convergence.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} &= \lim_{n \rightarrow \infty} \frac{|x-5|^{n+1}}{4^{n+1} \sqrt{(n+1)^2 + 5}} \frac{4^n \sqrt{n^2 + 5}}{|x-5|^n} \\ &= \frac{|x-5|}{4} \underbrace{\lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 5}}{\sqrt{(n+1)^2 + 5}}}_{=1} \\ &= \frac{|x-5|}{4} < 1\end{aligned}$$

So, $|x-5| < 4$. This implies that the radius of convergence is 4. From the previous inequality we obtain $1 < x < 9$. Now we need to check the end points (i.e., $x = 1, 9$).

$x = 1$: In this case, we have the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 5}}$. Then, this is an alternating series and satisfies the following conditions:

- (1) $\frac{1}{\sqrt{n^2 + 5}} \geq 0$
- (2) $\frac{1}{\sqrt{n^2 + 5}} \geq \frac{1}{\sqrt{(n+1)^2 + 5}}$
- (3) $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2 + 5}} = 0$

Therefore, the series converges. However, this series does not converge absolutely because $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^2 + 5}} \right| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 5}}$ and $\frac{1}{\sqrt{n^2 + 5}}$ behaves like $\frac{1}{n}$ for large n whose infinite series diverges.

$x = 9$: For $x = 9$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 5}}$ and we just showed that this series diverges. Hence, the interval of convergence is $1 \leq x < 9$, the series converges absolutely for $1 < x < 9$ and the series converges conditionally for $x = 1$.