QUANTUM SUPER-A-POLYNOMIALS

ERNEST YEUNG ERNESTYALUMNI@GMAIL.COM

Contents

Part 1. Introduction 1
1. Lie Groups 1
1.1. Spin 10
1.2. Adjoint Representation 11

Part 2. Bundles 11
2. Vector bundles 11
3. Principal bundles 12
4. Connections to classes 16

Part 3. Gauge Theory 17
5. Electromagnetism 17
6. Chern-Simons Theory 18

Part 4. Supersymmetry 18

Part 5. Polynomials 18
7. (Software) Packages 18
7.1. The fundamental theorem of hypergeometric summation/integration 19
7.2. Reverse Engineering Caruso's Zeilberger[17] 21

Part 6. Hypergeometric summations and recursion 21

Part 7. Jones Polynomials and Khovanov Homology 22
8. Jones Polynomials from SnapPy 22
9. Khovanov Homology implementations: KhoHo and knotkit 22
9.1. KhoHo 22
9.2. knotkit 22
10. Exploring the Jones Polynomial and Khovanov Homology: Unanswered Avenues 25
10.1. Installation of SnapPy 25

Part 8. BPS Spectra 25

Part 9. Cold Neutrons and Topological Knots 26
11. Curvature 26
11.1. Geometry; Geometric setup; Manifold setup 28
11.2. Reviewing Witten's Quantum Field Theory and the Jones Polynomial 28
12. Standard Model 28
13. Spin Structures, Fermions (fermionic field) 30
13.1. Clifford Algebra 30
14. Seiberg-Witten 30
15. End Game; Desires 30
References 30

Abstract. Everything about quantum super-A-polynomials.

I also prepared here, first, for myself, and second, hopefully to help others that begin at an elementary level (advanced undergraduate, early graduate studies) or if one needs to look up or be refreshed upon a previous concept, notes and readings of background material necessary to understand quantum super-A-polynomials and everything behind it fully. This includes notes that I would take as I read through books or articles. I want to note that sometimes I copy complete passages out of books or articles because that's how I learn as a novice, imitating and copying (if you have another way of learning, please let me know). I also work out some solutions to examples, exercises, and problems out of said books and articles.

Part 1. Introduction

1. Lie Groups

- **:** Lie Groups
- **:** Groups
- **:** Ring
- **:** group algebra
- **:** Group Ring
- **:** Representation Theory
- **:** Modules
- **:** kG-modules

From Sec. 8.1 "Noncommutative Rings" of Rotman (2010) [2]:

Definition 1. *ring R - additive abelian group equipped with multiplication* $\begin{aligned} R \times R &\to R \\ (a,b) &\mapsto ab \end{aligned}$ *s.t.* $\forall\, a, b \in R$

(i) $a(bc) = (ab)c$
(ii) $a(b+c) = ab + ac$, $(b+c)a = ba + ca$
(iii) $\exists\, 1 \in R$ *s.t.* $\forall\, a \in R$, $1a = a = a1$

Example 8.1[2]

Date: 15 septembre 2015.
Key words and phrases. representation theory, groups, Lie groups, Lie algebra, quantum super-A-polynomials, super-A-polynomials, gauge theory, Chern-Simons theory, quantum field theory.

(ii) group algebra kG, k commutative ring, G group, "its additive abelian group is free k-module having basis labeled by elements of G,
i.e. $\forall\, a \in kG$, $a = \sum_{g\in G} a_g g$, $a_g \in k$, $\forall\, g \in G$, $a_g \neq 0$ for only finitely many $g \in G$.

define (ring) multiplication $\begin{aligned} kG \times kG &\to kG \\ ab &= ab \end{aligned}$ $\forall\, a, b \in kG$, $\begin{aligned} a &= \sum_{g\in G} a_g g \\ b &= \sum_{h\in G} b_h g \end{aligned}$ to be

$$\left(\sum_{g\in G} a_g g \right) \left(\sum_{h\in G} b_h h \right) = \sum_{z\in G} \left(\sum_{gh=z} a_g b_h \right) z$$

Definition 2. *Given R ring, left R-module is (additive) abelian group M equipped with*

scalar multiplication $\begin{aligned} R \times M &\to M \\ (r,m) &\mapsto rm \end{aligned}$ *s.t.* $\forall\, m, m' \in M$, $\forall\, r, r', 1 \in R$

(i) $r(m+m') = rm + rm'$
(ii) $(r+r')m = rm + r'm$
(iii) $(rr')m = r(r'm)$
(iv) $1m = m$

EY : 20150922 Example : for kG-module V^σ, for $r \in kG$, so $r = \sum_{g\in G} a_g g$

$$\begin{aligned} R \times M &\to M \\ (r,m) &\mapsto rm \end{aligned} \Longrightarrow \begin{aligned} kG \times V &\to V \\ (r,v) &\mapsto tv \end{aligned}$$

For some representation $\sigma : G \to GL(V)$,

$$rv = \sum_{g\in G} a_g g \cdot v = \sum_{g\in G} a_g \sigma_g(v)$$

So a kG-module needs to be associated with some chosen representation.

Note for V as an additive abelian group, $\forall\, u, v, w \in V$,

$$\begin{aligned} & v + w = w + v, \ (u+v)+w = u + (v+w) \\ & v + 0 = v \quad \forall\, v \in V \text{ for } 0 \in V \\ & v + (-v) = 0 \quad \forall\, v \in V \end{aligned}$$

So a vector space can be an additive abelian group.

Note that

$$\begin{aligned} r(v+w) &= \left(\sum_{g\in G} a_g g \right)(v+w) = \left(\sum_{g\in G} a_g \sigma_g \right)(v+w) = \sum_{g\in G} a_g \sigma_g(v) + \sum_{g\in G} a_g\sigma_g(w) = rv + rw \\ (r+r')v &= \left(\sum_{g\in G} a_g g + b_g g \right) v = \sum_{g\in G} (a_g\sigma_g + b_g \sigma_g) v = \sum_{g\in G} a_g \sigma_g(v) + \sum_{g\in G} b_g \sigma_g(v) = rv + r'v \\ (rr')v &= \left(\sum_{g\in G} a_g g \sum_{h\in G} b_h h \right) v = \left(\sum_{z\in G} \sum_{gh=z} a_g b_h z \right) v = \sum_{z\in G} \sum_{gh\in z} a_g b_h \sigma_z(v) = \sum_{g\in G}\sum_{h\in G} a_g b_h \sigma_g \sigma_h(v) \end{aligned}$$

since $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g\sigma_h = \sigma_{gh}$ (σ homomorphism)

$1v = \sigma(1)v = 1v = v$

From Sec. 8.3 "Semisimple Ring" of Rotman (2010) [2]:

Definition 3. *k-representation of group G is homomorphism*

$$\sigma : G \to GL(V)$$

where V is vector field over field k

Proposition 1 (8.37 Rotman (2010)[2])**.** *$\forall$ k-representation $\sigma : G \to GL(V)$ equips V with structure of left kG-module, denote module by V^σ.*
Conversely, $\forall$ left kG-module V determines k-representation $\sigma : G \to GL(V)$

Proof. Given $\begin{aligned} \sigma &: G \to GL(V), \\ \sigma_g &=: \sigma(g) : V \to V \end{aligned}$

define

$$kG \times V \to V$$
$$\left(\sum_{g\in G} a_g g \right) v = \sum_{g\in G} a_g \sigma_g(v)$$

$v, w \in V$

Let $r, r', 1 \in kG$

$r = \sum_{g\in G} a_g g$

$$\begin{aligned} r(v+w) &= \left(\sum_{g\in G} a_g g \right)(v+w) = \left(\sum_{g\in G} a_g \sigma_g \right)(v+w) = \sum_{g\in G} a_g \sigma_g(v) + \sum_{g\in G} a_g\sigma_g(w) = rv + rw \\ (r+r')v &= \left(\sum_{g\in G} a_g g + b_g g \right) v = \sum_{g\in G} (a_g\sigma_g + b_g \sigma_g) v = \sum_{g\in G} a_g \sigma_g(v) + \sum_{g\in G} b_g \sigma_g(v) = rv + r'v \\ (rr')v &= \left(\sum_{g\in G} a_g g \sum_{h\in G} b_h h \right) v = \left(\sum_{z\in G} \sum_{gh=z} a_g b_h z \right) v = \sum_{z\in G} \sum_{gh\in z} a_g b_h \sigma_z(v) = \sum_{g\in G}\sum_{h\in G} a_g b_h \sigma_g \sigma_h(v) \end{aligned}$$

since $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g\sigma_h = \sigma_{gh}$ (σ homomorphism)

$1v = \sigma(1)v = 1v = v$

Conversely, assume V left kG-module.

If $g \in G$, then $v \mapsto gv$ defines $T_g : V \to V$. T_g nonsingular since $\exists\, T_g^{-1} = T_{g^{-1}}$

Define $\begin{aligned} \sigma &: G \to GL(V) \\ \sigma &: g \mapsto T_g \end{aligned}$

σ k-representation

$$\begin{aligned} \sigma(gh) &= T_{gh} = T_g T_h = \sigma(g)\sigma(h) \\ \sigma(gh)(v) &= T_{gh} v = ghv = T_g T_h v = \sigma(g)\sigma(h) v \quad \forall\, v \in V \end{aligned}$$

$\square$

Proposition 2. *Let group G, let $\sigma, \tau : G \to GL(V)$ be k-representations, field k. If V^σ, V^τ corresponding kG-modules in Prop. 1 (Prop. 8.37 in Rotman (2010) [2]), then $V^\sigma \simeq V^\tau$ as kG-modules iff $\exists$ nonsingular $\varphi : V \to V$ s.t.*

$$\varphi\tau(g) = \sigma(g)\varphi \quad \forall\, g \in G$$

Proof. If $\varphi : V^\tau \to V^\sigma$ kG-isomorphism, then $\varphi : V \to V$ isomorphism s.t.

$$\varphi(\sum a_g g v) = (\sum a_g g)\varphi(v) \quad \forall\, v \in V, \forall\, g \in G$$

in V^τ, $kG \times V \to V$ $\quad$ in V^σ, $kG \times V \to V$ scalar multiplication

$gv = \tau(g)(v)$ $\quad$ $gv = \sigma(g)(v)$

$$\Longrightarrow \forall\, g \in G, v \in V, \quad \varphi(\tau(g)(v)) = \sigma(g)(\varphi(v))$$

I think

$$\varphi(gv) = \varphi(\tau(g)(v)) = g\varphi(v) = \sigma(g)\varphi(v)$$
$$\Longrightarrow \varphi\tau(g) = \sigma(g)\varphi \quad \forall\, g \in G$$

Conversely, if $\exists$ nonsingular $\varphi : V \to V$ s.t. $\varphi\tau(g) = \sigma(g)\varphi \quad \forall\, g \in G$

$$\varphi\tau(g)v = \varphi(\tau(g)v) = \sigma(g)\varphi(v) \quad \forall\, g \in G, \forall\, v \in V$$

Consider scalar multiplication

$$kG \times V \to V$$
$$\sum_{g\in G} a_g g(v) = \sum_{g\in G} a_g \tau_g(v)$$

$$\varphi\left(\sum_{g\in G} a_g\tau_g(v)\right) = \varphi\left(\sum_{g\in G} a_g \tau(g)v\right) = \sum_{g\in G} a_g\sigma(g)\sigma(g)\varphi(v) = \left(\sum_{g\in G} a_g g\right)\varphi(v)$$

$\square$

Admittedly, after this exposition from Rotman (2010) [2], I still didn't understand how kG-modules relate to representation theory and group rings. I turned to Baker (2011) [4], which we'll do right now. Note that I found a lot of links to online resources on representation theory from Khovanov's webpage http://www.math.columbia.edu/~khovanov/resource/.

Note,

Definition 4. *vector subspace $W \subseteq V$ is called a G-submodule, G-subspace, EY : 20150922 "invariant" subspace? if $\forall\, g \in G$, for representation $\rho : G \to GL_k(V)$, $\rho_g(w) \in W$, $\forall\, w \in W, \forall\, g \in G$ i.e. closed under "action of elements of G" with $\rho_g =: \rho(g) : V \to V$*

Given basis $\mathbf{v} = \{v_1 \dots v_n\}$ for V, $\dim_k V = n$, $\forall\, g \in G$,

$$\rho_g v_j = \rho(g)v_j = r_{kj}(g)v_k$$

for, indeed,

$$\rho_g x^j v_j = \rho(g)x^j v_j = x^j\rho(g)v_j = x^j r_{kj}(g)v_k = r_{kj}x^j v_k$$

so that

$$\rho : G \to GL_k(V)$$
$$\rho(g) = [r_{ij}(g)]$$

Example 2.1 (Baker (2011) [4]): Let $\rho : G \to GL_k(V)$ where $\dim_k V = 1$

$$\forall\, v \in V, v \neq 0, \forall\, g \in G, \lambda_g \in k \text{ s.t. } g \cdot v = \rho_g(v) = \lambda_g v$$
$$\rho(hg)v = \rho_h\rho_g v = \lambda_{hg}v = \lambda_h\lambda_g v \Longrightarrow \lambda_{hg} = \lambda_h\lambda_g$$

$$\Longrightarrow \exists \text{ homomorphism } \Lambda : G \to k^\times$$
$$\Lambda(g) = \lambda_g$$

From Sec. 2.2 "G-homomorphisms and irreducible representations" of Baker (2011) [4], suppose $\begin{aligned} \rho &: G \to GL_k(V) \\ \sigma &: G \to GL_k(W) \end{aligned}$ are 2 representations

Many names for the same thing: G-equivalent, G-linear, G-homomorphism, EY : 20150922 kG-isomorphic?

If $\forall\, g \in G$,

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V \\ \tau_g \downarrow & & \downarrow \sigma_g \\ V & \xrightarrow{\varphi} & V \end{array} \quad \Longleftrightarrow V^\tau \overset{\varphi}{\simeq} V^\sigma$$

Indeed, define

$$\varphi : V^\tau \to V^\sigma$$
$$\varphi(v + w) = \varphi(v) + \varphi(w)$$
$$\varphi(rv) = \varphi(\sum_{g\in G} a_g g \cdot v) = \varphi(\sum_{g\in G} a_g\tau_g(v)) = \sum_{g\in G} a_g\varphi(\tau_g(v)) = \sum_{g\in G} a_g\sigma_g \cdot \varphi(v) = r\varphi(v)$$

EY : 20150922 So φ is a kG-isomorphism between left kG modules V^τ and V^σ if it's bijective and is "linear" in "scalars" $r \in kG$, i.e. $\varphi(rv) = r\varphi(v)$.

Define action of G on $\mathrm{Hom}_k(V, W)$ ($\mathrm{Hom}_k(V, W)$ is the vector space of k-linear transformations $V \to W$)

$$V \xrightarrow{f} W$$
$$v \overset{f}{\longmapsto} f(v)$$

$\forall\, f \in \mathrm{Hom}_k(V, W)$, $\quad f : V \to W$

$\qquad f(v) \in W$

Consider

$$G \times \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(V, W)$$
$$(g \cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}} \text{ i.e. } (g \cdot f)(v) = \sigma_g f(\rho_{g^{-1}}v) \quad (f \in \mathrm{Hom}_k(V, W))$$

Let $g, h \in G$,

$$(gh \cdot f)(v) = g \cdot \sigma_h f(\rho_{h^{-1}}v) = \sigma_g\sigma_h f\rho_{h^{-1}}\rho_{g^{-1}}(v) = (\sigma_{gh} f\rho_{(gh)^{-1}})(v)$$

Thus, $G \times \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(V, W)$ is thus another G-representation of G.

$(g \cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}}$

For k-representation ρ, if the only G-subspaces of V are $\{0\}$, V, ρ **irreducible** or **simple**.

$$\rho_g(\{0\}) = \{0\}$$
$$\rho_g(V) = V$$

given subrepresentation $W \subseteq V$, V/W admits linear action of G, $\overline{\rho}_W : G \to GL_k(V/W)$ quotient representation

$$\overline{\rho}_W(g)(v + W) = \rho(g)(v) + W$$

if $v' - v \in W$

$$\rho(g)(v') + W = \rho(g)(v + (v' - v)) + W = (\rho(g)(v) + \rho(g)(v' - v)) + W = \rho(g)(v) + W$$

Proposition 3 (2.7 Baker (2011)[4])**.** *if $f : V \to W$ G-homomorphism, then*

(a) *$\ker f$ is G-subspace of V*
(b) *$\mathrm{im} f$ is G-subspace of W*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\sigma_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

Proof. Recall

(a) Let $v \in \ker f$. Then $\forall g \in G$,
$$f(\rho_g v) = \sigma_g f(v) = 0$$
so $\rho_g v \in \ker f$, $\forall g \in G$. So $\ker f$ is G-subspace of V

(b) Let $w \in \mathrm{im} f$. So $w = f(u)$ for some $u \in V$
$$\sigma_g w = \sigma_g f(u) = f(\rho_g u) \in \mathrm{im} f$$
So $\mathrm{im} f$ is G-subspace of W

□

Theorem 1 (Schur's Lemma). *Let $\begin{matrix} \rho : G \to GL_{\mathbb{C}}(V) \\ \sigma : G \to GL_{\mathbb{C}}(W) \end{matrix}$ be irreducible representations of G over field $k = \mathbb{C}$; let $f : V \to W$ be G-linear map.*

(a) *if $f \neq 0$, f isomorphism. True $\forall k$ field, not just $\mathbb{C}$*
(b) *if $V = W$, $\rho = \sigma$, then for some $\lambda \in \mathbb{C}$, f given by $f(v) = \lambda v$ ($v \in V$) (true for algebraically closed fields)*

Proof. (a) By Prop. 3, $\ker f \subseteq V$, $\mathrm{im} f \subseteq W$ are G-subspaces.
For ρ, only G-subspaces are 0 or V, so if $\ker f = V$, $f = 0$. If $\ker f = 0$, f injective.
For σ, only G-subspaces are 0 or V, so $\mathrm{im} f = 0$, $f = 0$. If $\mathrm{im} f = V$, f surjective.
$\implies f$ isomorphism.

(b) Let $\lambda \in \mathbb{C}$ be an eigenvalue of f, $f(v_0) = \lambda v_0$ eigenvector, $v_0 \neq 0$.
Let linear $f_\lambda : V \to V$ s.t.
$$f_\lambda(v) = f(v) - \lambda v \quad (v \in V)$$
$\forall g \in G$
$$\rho_g f_\lambda(v) = \rho_g f(v) - \rho_g \lambda v = f(\rho_g v) - \lambda \rho_g v = f_\lambda(\rho_g v)$$
So f_λ is G-linear, for

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & V \arrow[d, "\rho_g"] \\ V \arrow[r, "f"] & V \end{tikzcd}$$

Since $f_\lambda(v_0) = 0$, by Prop. 3, $\ker f_\lambda = V$, (for $\ker f_\lambda \neq 0$ and so $\ker f_\lambda = V$)
By rank-nullity theorem, $\dim V = \dim\ker f_\lambda + \dim\mathrm{im} f_\lambda$.
So $\mathrm{im} f_\lambda = 0$, and so $f_\lambda(v) = 0$ ($\forall v \in V$) $\implies f(v) = \lambda v$

□

Schur's lemma, at least the first part, implies that the left kG-modules associated with representations ρ, σ are kG-isomorphic, i.e.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\sigma_g"] \\ V \arrow[r, "f"] & W \end{tikzcd} \iff V^\rho \overset{f}{\simeq} V^\sigma$$

with f being an isomorphism between V^ρ and V^σ s.t.
$$f(v + w) = f(v) + f(w) \quad \forall v, w \in (V^\sigma, +)$$
$$f(rv) = rf(v) \quad \forall r = \sum_{g \in G} a_g g \in kG$$

Kosmann-Schwarzbach's **Groups and Symmetries**[5] is a very lucid text that's mathematically rigorous enough and practical for physicists. It's really good and very clear. Let's follow its development for $SU(2), SO(3), SL(2,\mathbb{C})$ and corresponding Lie algebras $\mathfrak{su}(2)$, $\mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{C})$.

From Chapter 2 "Representations of Finite Groups" of Kosmann-Schwarzbach (2010) [5]

Definition 5 (2.1 Kosmann-Schwarzbach (2010)[5]). *On $L^2(G)$, scalar product defined by*
$$\langle f_1 | f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$
$f_1, f_2 \in \mathcal{F}(G) \equiv \mathbb{C}[G]$ vector space of functions on G taking values on $\mathbb{C}$

Definition 6 (2.3 Kosmann-Schwarzbach (2010)[5]). *Let (E, ρ) be representation of G*

character of $\begin{matrix} \rho \equiv \chi_\rho : G \to \mathbb{C} \\ \chi_\rho(g) = tr(\rho(g)) = \sum_{i=1}^n (\rho(g))_{ii} \end{matrix}$

Note: equivalent representations have same character
each conjugacy class of G, function χ_p is constant

Looking at Def. 5
$$\langle \chi_{\rho_1} | \chi_{\rho_2} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2(g)}$$
since $\overline{\chi_{\rho_1(g)}} = \chi_{\rho_1}(g^{-1})$ by unitarity of representation with respect to scalar product $\langle\, ,\, \rangle$

Proposition 4 (2.7 Kosmann-Schwarzbach (2010)[5]). *Let $\begin{matrix} (E_1, \rho_1) \\ (E_2, \rho_2) \end{matrix}$ be representations of G, let linear $u : E_1 \to E_2$.*

Then $\exists$ linear T_u s.t.

$$\begin{gathered} T_u : E_1 \to E_2 \\ T_u = \frac{1}{|G|} \sum_{g \in G} \rho_2(g) u \rho_1(g)^{-1} \end{gathered} \tag{1}$$

so that $\rho_2(g) T_u = T_u \rho_1(g) \quad \forall g \in G$

Proof.
$$\rho_2(g)T_u = \frac{1}{|G|}\sum_{h\in G}\rho_2(gh)u\rho_1(h^{-1}) = \frac{1}{|G|}\sum_{k\in G}\rho_2(k)u\rho_1(k^{-1}g) = T_u\rho_1(g)$$
□

Thus, diagrammatically, we have that

$$E_1 \xrightarrow{u} E_2 \Longrightarrow \begin{tikzcd} E_1 \arrow[r, "T_u"] \arrow[d, "\rho_1(g)"] & E_2 \arrow[d, "\rho_2(g)"] \\ E_1 \arrow[r, "T_u"] & E_2 \end{tikzcd}$$

From Definition 1.12 of Kosmann-Schwarzbach [5], "representations ρ_1 and ρ_2 are called **equivalent** if there is a bijective intertwining operator for ρ_1 and ρ_2." So I will interpret this as if an intertwining operator is not bijective, then the representations ρ_1, ρ_2 are not equivalent.

Proposition 5 (2.8 Kosmann-Schwarzbach (2010)[5])**.** *Let* $\begin{matrix} (E_1,\rho_1) \\ (E_2,\rho_2)\end{matrix}$ *be irreducible representations of* G, *let linear* $u : E_1 \to E_2$, *define* T_u *by* $T_u = \frac{1}{|G|}\sum_{g\in G}\rho_2(g)u\rho_1(g)^{-1}$ *by Eq. 1.*

(i) *If* ρ_1, ρ_2 *inequivalent, then* $T_u = 0$
(ii) *If* $E_1 = E_2 = E$ *and* $\rho_1 = \rho_2 = \rho$, *then*
$$T_u = \frac{tr(u)}{dimE}1_E$$

Proof. (i) if ρ_1,ρ_2 are inequivalent, by definition, T_u is not isomorphic. Then by Schur's lemma (first part), $T_u = 0$
(ii) By Schur's lemma, $T_u(v) = \lambda v \quad \forall\, v \in E = E_1 = E_2$. So $T_u = \lambda 1_E$. $\mathrm{tr}T_u = \lambda \mathrm{dim}E$ or $\lambda = \frac{\mathrm{tr}T_u}{\mathrm{dim}E}$. Thus, $T_u = \frac{\mathrm{tr}T_u}{\mathrm{dim}E}1_E$ □

Let $\begin{matrix}(e_1\dots e_n) \text{ basis of } E \\ (f_1\dots f_p) \text{ basis of } F\end{matrix}$

$$\forall\, u \in \mathcal{L}(E,F), \begin{aligned} & u: E\to F \\ & u(x) = u(x^je_j) = x^ju(e_j) = x^ju^i{}_jf_i \\ & u = u^i{}_je^j\otimes f_i \end{aligned} \text{ for } \begin{aligned} x &= x^je_j \in E \\ y &= y^if_i \in F\end{aligned}$$

For
$$\begin{aligned} & T : E^*\otimes F \to \mathcal{L}(E,F) \\ & T(\xi\otimes y) = u^i{}_je^j\otimes f_i \text{ i.e. set } T(\xi\otimes y) \text{ to this } u \\ & T(\xi\otimes y) = T(\xi_le^l\otimes y^kf_k) = \xi_ly^kT(e^l\otimes f_k) = (\xi_ly^kT^{li}_{kj})e^j\otimes f_i \Longrightarrow \xi_ly^kT^{li}_{kj} = u^i{}_j\end{aligned}$$

Exercises. Exercises of Ch. 2 Representations of Finite Groups [5]

Exercise 2.6. [5] *The dual representation.*

Let (E,π) representation of group G.
$\forall\, g\in G$, $\xi \in E^*$, $x \in E$, set $\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle$

(a) *dual* (or *contragredient*) of π, $\pi^* : G \to \mathrm{End}(E^*)$, π^* is a representation, since
$$\begin{aligned}\langle \pi^*(gh)(\xi),x\rangle &= \langle \xi, \pi((gh)^{-1})(x)\rangle = \langle \xi, \pi(h^{-1}g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x))\rangle = \\ &= \langle \pi^*(h)(\xi), \pi(g^{-1})(x)\rangle = \langle \pi^*(g)\pi^*(h)(\xi), x\rangle\end{aligned}$$
since this is true, $\forall\, x\in E$, $\forall\,\xi \in E^*$, $\pi^*(gh) = \pi^*(g)\pi^*(h)$.
dual π^* of π is a representation.

(b) Consider $\begin{matrix} G\times\mathcal{L}(E,F) \to \mathcal{L}(E,F). \\ g\cdot u = \rho(g)\circ u\circ \pi(g^{-1})\end{matrix}$

Define
$$\begin{aligned} & \sigma : G \to \mathrm{End}(\mathcal{L}(E,F)) \\ & \sigma(g) : \mathcal{L}(E,F) \to \mathcal{L}(E,F) \\ & \sigma(g)(u) = \rho(g)\circ u \circ \pi(g^{-1})\end{aligned}$$

Let $(e_1\dots e_n)$ be a basis of E. Let $\xi = \xi_ie^i \in E^*$, $x = x^je_j \in E$.
Consider the isomorphism $T : E^*\otimes F \to \mathcal{L}(E,F)$ defined as[1]

$$\begin{aligned} & T : E^*\otimes F \to \mathcal{L}(E,F) = \mathrm{Hom}(E,F) \\ & \xi\otimes y \mapsto (x\mapsto \xi(x)y)\end{aligned}$$

Choose bases $\begin{matrix}(e_1\dots e_n) \text{ of } E \\ (e^1\dots e^n) \text{ of } E^* \\ (f_1\dots f_p) \text{ of } F\end{matrix}$. Then

$$\begin{aligned} & T(e^j\otimes f_i)(x) = T(e^j\otimes f_i)(x^ke_k) = \delta^j{}_kx^kf_i = x^jf_i \\ & T(e^j\otimes f_i)(e_k) = \delta^j{}_kf_i\end{aligned}$$

Consider

$$\begin{aligned} & u \in \mathcal{L}(E,F) \\ & u : E\to F \\ & u(x) = u(x^je_j) = x^ju(e_j) = x^ju^i{}_jf_i \\ & u(e_j) = u^i{}_jf_i \text{ i.e. } u : e_j \to u^i{}_jf_i\end{aligned}$$

Then $\forall u \in \mathcal{L}(E,F)$,
$$T(u^i{}_je^j\otimes f_i)(e_k) = u^i{}_j\delta^j{}_kf_i = u^i{}_kf_i = u(e_k) \Longrightarrow u = T(u^i{}_je^j\otimes f_i)$$
so T is surjective.
With $T(\xi\otimes y) = T(\xi'\otimes y')$,
$$\begin{gathered} T(\xi\otimes y)(x) = T(\xi'\otimes y')(x) \\ \xi(x)y = \xi'(x)y' \Longrightarrow \xi(x)y - \xi'(x)y' = 0\end{gathered}$$
which implies that $\xi\otimes y = \xi'\otimes y'$. So T is injective. Or, one could consider that $T^{-1} : \mathcal{L}(E,F) \to E^*\otimes F$, $T^{-1} : u \mapsto u^i{}_je^j\otimes f_i$, which is the inverse of T.

[1] *Mathematics stackexchage* Isomorphism between Hom and tensor product [duplicate] http://math.stackexchange.com/questions/428185/isomorphism-between-hom-and-tensor-product http://math.stackexchange.com/questions/57189/understanding-isomorphic-equivalences-of-tensor-product

Remark 1.

$$E^* \otimes F \overset{T}{\simeq} \mathcal{L}(E,F) = Hom(E,F)$$
$$(\xi, y) \mapsto (x \mapsto \xi(x)y)$$

and so $(e^j \otimes f_i) \mapsto (x \mapsto e^j(x) f_i = x^j f_i)$

So $E^* \otimes F$ *is isomorphic to* $\mathcal{L}(E,F) = Hom(E,F)$

For representation π,

$$\begin{aligned} & \pi : G \to \text{End}(E) \\ & \pi(g) : E \to E \\ & \pi(g)(x) = \pi(g)(x^j e_j) = x^j \pi(g)(e_j) = x^j \pi(g)^i{}_j e_i = (\pi(g)^i{}_j x^j e_i \end{aligned}$$

Consider this matrix formulation:

$$\begin{aligned} \pi^*(g)(\xi) = \pi^*(g)(\xi_i e^i) = \xi_i \pi^*(g)(e^i) = \xi_i (\pi^*(g))^i{}_j e^j \\ \implies \langle \pi^*(g)(\xi), x\rangle = \xi_i (\pi^*(g))^i{}_j x^j \end{aligned}$$

and

$$\langle \xi, \pi(g^{-1})(x)\rangle = \xi_i \pi(g^{-1})^i{}_j x^j$$

so that

$$\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle \implies \pi(g^{-1})^i{}_j = (\pi^*(g))^i{}_j$$

Thus, given a choice of basis for E, the *dual* of π, $\pi^*(g)^i{}_j$, and $\pi(g^{-1})^i{}_j$ are formally equal.

So for a choice of basis of E and of F,

$$(\pi^* \otimes \rho)(g)(\xi, y) = (\pi^*(g) \otimes \rho(g))(\xi, y) = \pi^*(g)\xi \otimes \rho(g) y = \xi_l \pi(g^{-1})^l{}_j e^j \otimes \rho(g)^i{}_k y^k f_i = \rho(g)^i{}_k y^k \xi_l \pi(g^{-1})^l{}_j e^j \otimes f_i$$

Applying T,

$$T(\pi^* \otimes \rho)(g)(\xi, \rho) = \rho(g)^i{}_k y^k \xi_l \pi(g^{-1})^l{}_j = \rho(g) T(\xi, y) \pi(g^{-1})$$

$$\begin{tikzcd} E^* \otimes F \arrow[r, "T"] \arrow[d, "(\pi^*\otimes\rho)(g)"'] & \mathcal{L}(E,F) \arrow[d, "\sigma(g)"] \\ E^* \otimes F \arrow[r, "T"'] & \mathcal{L}(E,F) \end{tikzcd} \qquad \begin{tikzcd} (\xi, y) \arrow[r, mapsto, "T"] \arrow[d, mapsto, "(\pi^*\otimes\rho)(g)"'] & (x \mapsto \xi(x)y) = y^i \xi_j \arrow[d, mapsto, "\sigma(g)"] \\ \pi^*(g)(\xi) \otimes \rho(g) y \arrow[r, mapsto, "T"'] & \rho(g) y^i \xi_j \pi(g^{-1}) = \rho(g) T(\xi, y) \pi(g^{-1}) \end{tikzcd}$$

Thus

Thus, representation $\sigma(g)$ is equivalent to representation $(\pi^* \otimes \rho)$, a tensor product of representations.

Exercise 2.15. *Representation of* $GL(2,\mathbb{C})$ *on the polynomials of degree 2*

Let group G, let representation ρ of G on $V = \mathbb{C}^n$, i.e. $\rho : G \to \text{End}(V)$

Let $P^{(k)}(V)$ vector space of complex polynomials on V that are homogeneous of degree k.

For $f \in P^{(k)}(V)$, the general form is

$$f = \sum_{\substack{i_1 + i_2 + \dots + i_n = k \\ 0 \leq i_j \leq k}} a_{i_1 i_2 \dots i_d} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

Given

$$\binom{n+k}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \dots + \binom{n+k-1}{k-1} = \sum_{i=0}^n \binom{k-1+i}{k-1}$$

$\binom{k+n-1}{n-1}$ is number of monomials of degree k.

So $\dim P^{(k)}(V) = \binom{k+n-1}{n-1}$. This is a very lucid and elementary exposition on the basics of polynomials which I found was useful for the basic facts I forgot[2].

So we have the graded algebra

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V)$$

$$\begin{aligned} & \rho^{(k)} : G \to \text{End}(P^{(k)}(V)) \\ & \rho^{(k)}(g) : P^{(k)}(V) \to P^{(k)}(V) \\ & \rho^{(k)}(g)(f) = f \circ \rho(g^{-1}) \end{aligned}$$

This is a representation of G since

(a)

$$\left. \begin{aligned} \rho^{(k)}(gh)(f) = f \circ \rho((gh)^{-1}) = f \circ \rho(h^{-1} g^{-1}) = f \circ \rho(h^{-1} \rho(g^{-1}) \\ \rho^{(k)}(g) \rho^{(k)}(h)(f) = \rho^{(k)}(g)(f \circ \rho(h^{-1})) = f \circ \rho(h^{-1}) \circ \rho(g^{-1}) \end{aligned} \right. \implies \rho^{(k)}(gh) = \rho^{(k)}(g) \rho^{(k)}(h)$$

(b) Choose basis $(e_1 \dots e_n)$ of V, $x = x^j e_j \in V$, $\rho : G \to \text{End}(V)$, and so $\rho(g)(x) = \rho(g)(x^j e_j) = x^j \rho(g)(e_j) = x^j (\rho(g))^i{}_j e_i$.
With $\xi(e_i) = \xi_i \implies \langle \xi, \rho(g^{-1}) x\rangle = \xi_i x^j (\rho(g^{-1}))^i{}_j$

$\forall\, \xi \in V^*$, $\xi = \xi_i e^i$,

$$\begin{aligned} \rho^*(g)(\xi) = \rho^*(g)(\xi_i e^i) = \xi_i \rho^*(g)^i{}_j e^j \\ \implies \langle \rho^*(g)(\xi), x \rangle = \xi_i x^j (\rho^*(g))^i{}_j \implies (\rho^*(g))^i{}_j = (\rho(g^{-1}))^i{}_j \end{aligned}$$

So $\forall\, f \in P^{(1)}(V)$, $x \in V$, $\rho(g^{-1}) x = x^j (\rho(g^{-1}))^i{}_j e_i$. So $f \circ \rho(g^{-1})(x) = \sum_{i=1}^n a_i (\rho(g^{-1}))^i{}_j x^j = \sum_{i=1}^n a_i (\rho^*(g))^i{}_j x^j$
$\implies \rho^{(1)}(g)(f) = f \circ \rho^*(g)$

(c) Suppose $G = GL(2,\mathbb{C})$, $V = \mathbb{C}^2$, ρ fundamental representation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g^{-1} = \frac{1}{\det g} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for $\det g = ad - bc$.

Let $k = 2$, $\dim P^{(2)}(\mathbb{C}^2) = \binom{2+2-1}{2-1} = \binom{3}{1} = 3$

$\forall\, f \in P^{(2)}(\mathbb{C}^2)$, $f(x,y) = Ax^2 + 2Bxy + Cy^2$

Let

$$P^{(2)}(\mathbb{C}^2) \to \mathbb{C}^3$$

$$f(x,y) = Ax^2 + 2Bxy + Cy^2 \mapsto \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$$

Call this transformation T, $T : P^{(2)}(\mathbb{C}^2) \to \mathbb{C}^3$.

$\forall \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$, $f(x,y) = Ax^2 + 2Bxy + Cy^2$ and $Tf(x,y) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$. T surjective.

Suppose $Tf(x,y) = Tf'(x,y)$,

$$\begin{aligned} \implies Ax^2 + 2Bxy + Cy^2 = A'x^2 + 2B'xy + C'y^2 \\ \implies (A - A')x^2 + 2(B - B')xy + (C - C')y^2 = 0 \end{aligned}$$

Then since the monomials form a basis, and its basis elements are independent (by definition), then $A = A'$, $B = B'$, $C = C'$. T injective. So T is bijective, an isomorphism.

(This is all in `groups.sage`)

[2]Polynomials. Math 4800/6080 Project Course http://www.math.utah.edu/~bertram/4800/PolyIntroduction.pdf

```
sage: P2CC.<x,y> = PolynomialRing(CC,2) # this declares a PolynomialRing of field of complex numbers,
# of order 2 (i.e. only 2 variables for a polynomial, such as x, y)
sage: A = var('A')
sage: assume(A,''complex'')
sage: B = var('B')
sage: assume(B,''complex'')
sage: C = var('C')
sage: assume(C,''complex'')
sage: f(x,y) = A*x**2 +2*B*x*y + C*y**2

sage: a = var('a')
sage: assume(a,''complex'')
sage: b = var('b')
sage: assume(b,''complex'')
sage: c = var('c')
sage: assume(c,''complex'')
sage: d = var('d')
sage: assume(d,''complex'')
sage: g = Matrix([[a,b],[c,d]] )
sage: X = Matrix([[x],[y]])
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand()
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x^2).full_simplify()
(C*c^2 - 2*B*c*d + A*d^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x*y).full_simplify()
-2*(C*a*c + A*b*d - (b*c + a*d)*B)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(y^2).full_simplify()
(C*a^2 - 2*B*a*b + A*b^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
```

So

$$\rho^{(2)}(g)(f)(x,y) = f \circ \rho(g^{-1})(x,y) =$$

$$= \frac{Cc^2 - 2Bcd + Ad^2}{(ad-bc)^2} x^2 + -2\frac{(Cac + Abd - (bc+ad)B)}{(ad-bc)^2} xy + \frac{Ca^2 - 2Bab + Ab^2}{(ad-bc)^2} y^2$$

So define $\widetilde{\rho} : G \to \mathrm{End}(\mathbb{C}^3)$. $\widetilde{\rho}$ is a representation, for

$$\forall v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3, \quad \widetilde{\rho}(gh)(v) = T \circ f \circ \rho((gh)^{-1}) = T \circ f \circ \rho(h^{-1}g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1})$$

$$\text{Now } \widetilde{\rho}(h)(v) = T \circ f \circ \rho(h^{-1})$$

$$\Longrightarrow \widetilde{\rho}(g)\widetilde{\rho}(h)(v) = T \circ (f \circ \rho(h^{-1})) \circ \rho(g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1}) \text{ and so}$$

$$\widetilde{\rho}(gh) = \widetilde{\rho}(g)\widetilde{\rho}(h)$$

And so

$$\widetilde{\rho}^*(g)(v) = Tf\rho(g^{-1})$$

and consider this commutation diagram, that (helped me at least and) clarifies the relationships:

$$\begin{tikzcd} P^{(2)}(\mathbb{C}^2) \arrow[r, "T"] \arrow[d, "\rho^{(2)}(g)"'] & \mathbb{C}^3 \arrow[d, "\widetilde{\rho}(g)"] \\ P^{(2)}(\mathbb{C}^2) \arrow[r, "T"'] & \mathbb{C}^3 \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto, "T"] \arrow[d, mapsto, "\rho^{(2)}(g)"'] & \begin{pmatrix} A \\ B \\ C \end{pmatrix} \arrow[d, mapsto, "\widetilde{\rho}(g)"] \\ f \circ \rho(g^{-1}) \arrow[r, mapsto, "T"'] & \begin{pmatrix} D \\ E \\ F \end{pmatrix} \end{tikzcd}$$

with

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{Cc^2 - 2Bcd + Ad^2}{(ad-bc)^2} \\ -2\frac{(Cac + Abd - (bc+ad)B)}{(ad-bc)^2} \\ \frac{Ca^2 - 2Bab + Ab^2}{(ad-bc)^2} \end{pmatrix}$$

Now define the dual $\widetilde{\rho}^*$ as such:

$$\widetilde{\rho}^*(g) : (\mathbb{C}^3)^* \to (\mathbb{C}^3)^*$$

$$\widetilde{\rho}^*(g) = \widetilde{\rho}(g^{-1})$$

$$\forall \xi \in (\mathbb{C}^3)^*$$

$$\widetilde{\rho}^*(g)\xi = \xi_i(\widetilde{\rho}^*(g))^i{}_j e^j = \xi_i(\widetilde{\rho}(g^{-1}))^i{}_j e^j$$

So for $v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$, $f = T^{-1}v = Ax^2 + 2Bxy + Cy^2 \in P^2(\mathbb{C}^2)$,

$$\widetilde{\rho}(g^{-1})(v) = T \circ (f\rho(g)) = \begin{bmatrix} Aa^2 + 2Bac + Cc^2 \\ Aab + Bbc + Bad + Ccd \\ Ab^2 + 2Bbd + Cd^2 \end{bmatrix}$$

which was found using Sage Math:

```
sage: f((g*X)[0,0],(g*X)[1,0])
(a*x + b*y)^2*A + 2*(a*x + b*y)*(c*x + d*y)*B + (c*x + d*y)^2*C
sage: f((g*X)[0,0],(g*X)[1,0]).expand()
A*a^2*x^2 + 2*B*a*c*x^2 + C*c^2*x^2 + 2*A*a*b*x*y + 2*B*b*c*x*y + 2*B*a*d*x*y + 2*C*c*d*x*y + A*b^2*y^2 + 2*B*b*d*y^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x^2)
A*a^2 + 2*B*a*c + C*c^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x*y)
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(y^2)
A*b^2 + 2*B*b*d + C*d^2
```

or

```
sage: T( f((g*X)[0,0],(g*X)[1,0]).expand() )
[A*a^2 + 2*B*a*c + C*c^2,
 2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d,
 A*b^2 + 2*B*b*d + C*d^2]
```

So then

$$\widetilde{\rho}(g^{-1}) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

So then

$$\widetilde{\rho}^*(g) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

and operate on row vectors $\xi \in (\mathbb{C}^3)^*$ with $\widetilde{\rho}^*(g)$ from the row vector's right.

More: Let $G = SU(2)$. Then $U = e^{i\phi} \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}$

$$\widetilde{\rho} : SU(2) \to \text{End}(\mathbb{C}^3)$$
$$\widetilde{\rho}(U) : \mathbb{C}^3 \to \mathbb{C}^3$$
$$\widetilde{\rho}(U)(v) = e^{-2i\varphi} \begin{bmatrix} A\overline{a}^2 + 2B\overline{a}\overline{b} + C\overline{b}^2 \\ -A\overline{a}b + B + Ca\overline{b} \\ Ab^2 - 2Bab + Ca^2 \end{bmatrix}$$
$$\Longrightarrow \widetilde{\rho}(U) = e^{-2i\varphi} \begin{bmatrix} -\overline{a}^2 & 2\overline{a}\overline{b} & \overline{b}^2 \\ -\overline{a}b & 1 & a\overline{b} \\ b^2 & -2ab & a^2 \end{bmatrix}$$

From Chapter 4 "Lie Groups and Lie Algebras" of Kosmann-Schwarzbach (2010) [5]

While Proposition 2.6 of Kosmann-Schwarzbach (2010) [5] states that

$$\det(\exp(X)) = \exp{(\text{tr}X)}$$

here are some other resources online that gave further discussion on the characteristic polynomial, $\det(A - \lambda 1)$ and the different terms of it, called Newton identities:

- http://scipp.ucsc.edu/~haber/ph116A/charpoly_11.pdf
- http://math.stackexchange.com/questions/1126114/how-to-find-this-lie-algebra-proof-that-mathfraksl-is-trace-zero-matrice
- http://mathoverflow.net/questions/131746/derivative-of-a-determinant-of-a-matrix-field

Theorem 2 (5.1 [5]). *Consider* $\mathfrak{g} = \{X = \gamma'(0) | \gamma : 1 \to G \text{ of class } C^1 \,, \gamma(0) = 1\}$
Let Lie group G

(i) $\mathfrak{g}$ *vector subspace of* $\mathfrak{gl}(n, \mathbb{R})$
(ii) $X \in \mathfrak{g}$ *iff* $\forall \, t \in \mathbb{R}$, $\exp{(tX)} \in G$
(iii) *if* $X \in \mathfrak{g}$, *if* $g \in G$, *then* $gXg^{-1} \in \mathfrak{g}$
(iv) $\mathfrak{g}$ *closed under matrix commutator, i.e. if* $X, Y \in \mathfrak{g}$, $[X, Y] \in \mathfrak{g}$

Proof. (i)
(ii) If $\exp{(tX)} \in G$, then $X \frac{d}{dt} \exp{(tX)}\big|_{t=0} \in \mathfrak{g}$ (by def.)
If $X \in \mathfrak{g}$, then by def., $X = \frac{d}{dt}\gamma(t)\big|_{t=0}$ with $\gamma(t) \in G$.
Now Taylor expand; $\forall \, k \in \mathbb{Z}^+$

$$\gamma\left(\frac{t}{k}\right) = 1 + \frac{t}{k}X + O\left(\frac{1}{k^2}\right) = \exp\left(\frac{t}{k}X + O\left(\frac{1}{k^2}\right)\right)$$
$$\Longrightarrow \left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp{(tX)}$$
$$\gamma\left(\frac{t}{k}\right) \in G \quad \forall \, k \in \mathbb{Z}^+$$

G closed subgroup, so $\lim_{k\to\infty}(\gamma\left(\frac{t}{k}\right))^k = \exp{(tX)} \in G$
(iii)
(iv)

$\square$

Definition 7. *Lie algebra* $\mathfrak{g}$, *tangent space to* G *at* 1, *i.e.* $\mathfrak{g} := T_1G$ *is called* Lie algebra *of Lie group* G.

$$\mathfrak{g} := \{X = \gamma'(0) | \gamma : 1 \to G \text{ of class } C^1 \,, \gamma(0) = 1\} = T_1 G$$

This is based on Proposition 5.3 of Kosmann-Schwarzbach (2010) [5].

For Lie group

$$U(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1\}$$

If $X \in \mathfrak{u}(n)$, then $\exp{(tX)} \in U(n)$. Then

$$\exp{(tX)}\exp{(tX)}^\dagger = (1 + tX + O(t^2))(1 + tX^\dagger + O(t^2)) = 1 + t(X + X^\dagger) + O(t^2) = 1 \forall \, t \in \mathbb{R} \Longrightarrow X + X^\dagger = 0$$

i.e. $X \in \mathfrak{u}(n)$ is an anti-Hermitian complex $n \times n$ matrix.

$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0\}$

Physicists: $X = iA$ and so $A - A^\dagger$. $A \in \mathfrak{u}(n)$ is a Hermitian complex $n \times n$ matrix.

$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C}) | A - A^\dagger = 0\}$

Regardless, $\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 = 2n^2 - n^2$

For Lie group

$$SU(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1\}$$

Then

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 1, \, \text{tr}X = 0\}$$

is the Lie algebra of traceless anti-Hermitian complex $n \times n$ matrices, and that

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

In summary,

$$\begin{tikzcd} \mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0\} \arrow[d, "\exp{(tX)}"'] & \mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0, \text{tr}X = 0\} \arrow[d, "\exp{(tX)}"'] \\ U(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1\} & SU(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1\} \end{tikzcd}$$

$$\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 \qquad \dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

From Chapter 5 "Lie Groups $SU(2)$ and $SO(3)$" of Kosmann-Schwarzbach (2010) [5],

1.0.1. *Bases of* $\mathfrak{su}(2)$, *Subsection 1.1 of Chapter 5of Kosmann-Schwarzbach (2010)* [5]. Recall that

$$\begin{tikzcd} \mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0, \text{tr}X = 0\} \arrow[d, "\exp{(tX)}"'] \\ SU(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1\} \end{tikzcd}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

and so

$$\begin{tikzcd} \mathfrak{su}(2) = \{X \in \mathfrak{gl}(2,\mathbb{C}) | X + X^\dagger = 0, \mathrm{tr}X = 0\} \arrow[d, "\exp{(tX)}"'] \\ SU(2) = \{U \in GL(n,\mathbb{C}) | UU^\dagger = 1, \mathrm{det}U = 1\} \end{tikzcd}$$

$$\dim_{\mathbb{R}}\mathfrak{su}(2) = 3$$

Also, recall that $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{C})$ is a vector subspace (2) and that
$X \in \mathfrak{g}$ iff $\forall t \in \mathbb{R}$, $\exp{(tX)} \in G$.
if $X \in \mathfrak{g}$, if $g \in G$, then $gXg^{-1} \in \mathfrak{g}$

$\mathfrak{g}$ closed under $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$
$(X,Y) \mapsto [X,Y]$

and so with $\mathfrak{g}$ as a vector space, we can have a choice of bases.

(a) $\xi_1 = \frac{i}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$\xi_2 = \frac{1}{2}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

$\xi_3 = \frac{i}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

satisfying

$$[\xi_k,\xi_l] = \epsilon_{klm}\xi_m$$

(b) *Physics*

$\sigma_1 = -2i\xi_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$\sigma_2 = 2i\xi_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}$

$\sigma_3 = -2i\xi_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

satisfying

$$[\sigma_k,\sigma_l] = 2i\epsilon_{klm}\sigma_m$$

EY : 20151001 Sage Math 6.8 doesn't run on Mac OSX El Capitan: I suspect that it's because in Mac OSX El Capitan, `/usr` cannot be modified anymore, even in an Administrator account. The TUG group for MacTeX had a clear, thorough, and useful (i.e. copy UNIX commands, paste, and run examples) explanation of what was going on:

http://tug.org/mactex/elcapitan.html

So keep in mind that my code for Sage Math is for Sage Math 6.8 that doesn't run on Mac OSX El Capitan. I'll also use `sympy` in Python as an alternative and in parallel.

One can check in `sympy` the traceless anti-Hermitian (or Hermitian) property of the bases and Pauli matrices, and the commutation relations (see `groups.py`):

```
import itertools
from itertools import product, permutations
import sympy
from sympy import I, LeviCivita
from sympy import Rational as Rat

from sympy.physics.matrices import msigma # <class 'sympy.matrices.dense.MutableDenseMatrix'>

def commute(A,B):
    """
    commute = commute(A,B)
    commute takes the commutator of A and B
    """
    return (A*B - B*A)

def xi(i):
    """
    xi = xi(i)
    xi is a function that returns the independent basis for
    Lie algebra su(2)\equiv su(2,\mathbb{C}) of Lie group SU(2) of
    traceless anti-Hermitian matrices, based on msigma of sympy
    cf. http://docs.sympy.org/dev/_modules/sympy/physics/matrices.html#msigma
    """
    if i not in [1,2,3]:
        raise IndexError("Invalid Pauli index")
    elif i==1:
        return I/Rat(2)*msigma(1)
    elif i==2:
        return -I/Rat(2)*msigma(2)
    elif i==3:
        return I/Rat(2)*msigma(3)
```

```
## check anti-Hermitian property and commutation relations with xi
# xi is indeed anti-Hermitian
xi(1) == -xi(1).adjoint() # True
xi(2) == -xi(2).adjoint() # True
xi(3) == -xi(3).adjoint() # True

# xi obeys the commutation relations

for i,j in product([1,2,3],repeat=2): print i,j

for i,j in product([1,2,3],repeat=2): print i,j, "\t Commutator: ", commute(xi(i),xi(j))

## check traceless Hermitian property and commutation relations with Pauli matrices
# Pauli matrices i.e. msigam is indeed traceless Hermitian

msigma(1) == msigma(1).adjoint() # True
msigma(2) == msigma(2).adjoint() # True
msigma(3) == msigma(3).adjoint() # True

msigma(1).trace() == 0 # True
msigma(2).trace() == 0 # True
msigma(3).trace() == 0 # True

# Pauli matrices obey commutation relation
print "For Pauli matrices, the commutation relations are:\n"
for i,j in product([1,2,3],repeat=2): print i,j, "\t Commutator: ", commute(msigma(i),msigma(j))

for i,j,k in permutations([1,2,3],3): print "Commute: ", i,j,k, msigma(i), msigma(j), \
": and is 2*i of ", msigma(k), commute(msigma(i),msigma(j)) == 2*I*msigma(k)*LeviCivita(i,j,k)
```

And finally the traceless property of the Pauli matrices:

```
>>> msigma(1).trace()
0
>>> msigma(2).trace()
```

```
0
>>> msigma(3).trace()
0
```

1.1. **Spin.** Let's follow the development by Baez and Muniain (1994) on pp. 175 of the Section II.1 "Lie Groups", the second (II) chapter on "Symmetry" [1].

Let $V = \mathbb{C}^2$, $G = SU(2)$. Then consider the graded algebra of polynomials on $V = \mathbb{C}^2 \ni (x, y)$

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V) = \bigoplus_{\substack{j=0 \\ 2j \in \mathbb{Z}}}^{\infty} P^{(2j)}(V) = \bigoplus_{\substack{j=0 \\ j \in \mathbb{Z}}}^{\infty} P^{(2j)}(V) \oplus \bigoplus_{\substack{j=1/2 \\ 2j \text{ odd}}}^{\infty} P^{(2j)}(V)$$

$$P^{(2j)}(V) \equiv \text{ vector space of complex polynomials of degree } 2j$$

and recall this representation on $P^{(2j)}(V)$

$$\begin{aligned} & \rho^{(2j)} : G \to \text{End}(P^{(2j)}(V)) \\ & \rho^{(2j)} : P^{(2j)}(V) \to P^{(2j)}(V) \\ & \rho^{(2j)}(g)(f) = f \circ \rho(g^{-1}) \text{ where } \rho \text{ is the fundamental representation of } G = SU(2) \\ & \rho^{(2j)}(g)(f)(v) = f \circ \rho(g^{-1})(v) \quad \forall f \in P^{(2j)}(V), \forall v \in V = \mathbb{C}^2 \end{aligned}$$

Note,
$\dim P^{(2j)} = \binom{2j+2-1}{2-1} = 2j + 1$

Exercise 21. [1] *spin-0* Consider the trivial representation τ:

$$\begin{aligned} & \tau : G \to \text{End}(\mathbb{C}) \\ & \tau(g) : \mathbb{C} \to \mathbb{C} \\ & \tau(g) = 1_{\mathbb{C}} \end{aligned}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "T"] \arrow[d, "\tau(g)"'] & P^{(0)}(V) \arrow[d, "\rho^{(0)}(g)"] \\ \mathbb{C} \arrow[r, "T"] & P^{(0)}(V) \end{tikzcd}$$

Clearly, $P^{(0)}(V) = \mathbb{C}$, since $P^{(0)}(V)$ consists of polynomials of constants in $\mathbb{C}$.

Consider $c_0 \in \mathbb{C}$, $f = k_0 \in P^{(0)}(V)$
$\rho^{(0)}(g)(f) = f \circ \rho(g^{-1}) = k_0$
$\Longrightarrow \rho^0(g)T(c_0) = T \circ \tau(g)c_0 = T(c_0)$. Let $T = 1_{\mathbb{C}} = 1_{P^0(V)}$

So $\rho^{(0)}(g) = \tau(g) = 1$. $T = 1$. So representations $\rho^{(0)}$ and trivial representation τ on G are equivalent.

Exercise 22. [1] *spin-$\frac{1}{2}$* For spin-$\frac{1}{2}$, $j = \frac{1}{2}$, $2j = 1$.

$\forall f \in P^{(1)}(V)$, $V = \mathbb{C}^2$. So in general form, $f(x, y) = ax + by \in P^{(1)}(V)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in V = \mathbb{C}^2$

Recall the fundamental representation
$$\begin{aligned} & \rho : G \to GL(2, \mathbb{C}) \equiv GL(\mathbb{C}^2) \\ & \rho(g) : \mathbb{C}^2 \to \mathbb{C}^2 \\ & \rho(g) = g \end{aligned}$$

So consider T such that

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "T"] \arrow[d, "\rho(g)"'] & P^{(1)}(V) \arrow[d, "\rho^{(1)}(g)"] \\ \mathbb{C}^2 \arrow[r, "T"] & P^{(1)}(V) \end{tikzcd}$$

Consider $\forall v \in \mathbb{C}^2$, $v = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\rho(g)v = gv = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

```
sage: g*X
[a*x + b*y]
[c*x + d*y]
```

For notation, let $U \in G = SU(2)$ s.t. $UU^{\dagger} = 1$.

Consider $(\rho^{(2j)}(U)(f))(x) = f(U^{-1}x)$, $\forall x \in \mathbb{C}^2$.

Choose $f(x, y) = x$. So for $f(x, y) = Ax + By$, $A = 1, B = 0$. Choose $U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ so $U^{-1} = \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix}$. Then $U^{-1}x = \begin{pmatrix} \overline{a}x - by \\ \overline{b}x + ay \end{pmatrix}$

So

$$\begin{aligned} & (\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \overline{a}x - by \\ & (\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \overline{b}x + ay \text{ for } f(x, y) = y \end{aligned}$$

Let $f(x, y) = Ax + By$

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = (A\overline{a} + B\overline{b})x + (Ba - Ab)y = (\overline{a}x - by)A + (\overline{b}x + ay)B = (A\overline{a} + B\overline{b})x + (Ba - Ab)y$$

which was calculated with the assistance of Sage Math:

```
sage: U_try1 = Matrix( [[a.conjugate(),-b],[b.conjugate(),a ] ] )
sage: f1( U_try1*X).coefficient(x)
A*conjugate(a) + B*conjugate(b)
sage: f1( U_try1*X).coefficient(y)
B*a - A*b
```

Treating $P^{(1)}(\mathbb{C}^2)$ as a vector space, in its matrix formulation, then $f(x, y) = Ax + By \in P^{(1)}(\mathbb{C}^2)$ is treated as $\begin{bmatrix} A \\ B \end{bmatrix}$, then $(\rho^{(1)}(U)f)$ is

$$\Longrightarrow \begin{bmatrix} \overline{a} & \overline{b} \\ -b & a \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A\overline{a} + B\overline{b} \\ -Ab + Ba \end{bmatrix}$$

so conclude in general that $\rho^{(1)}(U) = (U^{\dagger})^T$.

Now, as Kosmann-Schwarzbach (2010) [5] says, on pp. 13, Chapter 2 Representations of Finite Groups, "Two representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent if and only if there is a basis B_1 of E_1 and a basis B_2 of E_2 such that for every $g \in G$, the matrix of $\rho_1(g)$ in the basis B_1 is equal to the matrix of $\rho_2(g)$ in the basis B_2. In particular, if the representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent, then E_1 is isomorphic to E_2." So we need a change of basis between $\rho(U) = U$ and $\rho^{(1)}(U)$. What's the linear transformation T s.t.

$$T^{-1}\rho^{(1)}(U)T = U?$$

By intuition,

$$T = \sigma_x \sigma_z \equiv \sigma_1 \sigma_3$$

where σ_i's are Pauli matrices.

Indeed,

```
sage: Paulimat[3] * Paulimat[1]*U_try*Paulimat[1] * Paulimat[3]
[conjugate(a) conjugate(b)]
[           -b           a]
```

Then $\rho^{(1)}(U) \circ T = TU$, so this $T = \sigma_1\sigma_3$ is an "intertwining operator" between $\rho^{(1)}(U)$ and fundamental representation $\rho(U) = U$, with $T = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, and $T^{-1} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$.

T is an isomorphism between $\mathbb{C}^2$ and $P^{(1)}(\mathbb{C}^2)$. So fundamental representation ρ of $G = SU(2)$ is equivalent to $\rho^{(1)}(U)$ on $P^{(1)}(\mathbb{C}^2)$.

Exercise 23. [1] (Also from Exercise 2.6 of Kosmann-Schwarzbach (201) [5])

Let (E, π) representation of group G.
$\forall\, g \in G$, $\xi \in E^*$, $x \in E$, set $\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle$

dual (or *contragredient*) of π, $\pi^* : G \to \text{End}(E^*)$, π^* is a representation, since

$$\begin{gathered}\langle \pi^*(gh)(\xi), x\rangle = \langle \xi, \pi((gh)^{-1})(x)\rangle = \langle \xi, \pi(h^{-1}g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x))\rangle = \\ = \langle \pi^*(h)(\xi), \pi(g^{-1})(x)\rangle = \langle \pi^*(g)\pi^*(h)(\xi), x\rangle\end{gathered}$$

since this is true, $\forall\, x \in E$, $\forall\, \xi \in E^*$, $\pi^*(gh) = \pi^*(g)\pi^*(h)$.

dual π^* of π is a representation.

1.2. **Adjoint Representation.** I will first follow Sec. 7.3 The Adjoint Representation of Ch. 4 Lie Groups and Lie Algebras of Kosmann-Schwarzbach (201) [5]).

The *conjugation action* $\mathcal{C}_g : G \to G$ is defined as

$$\begin{aligned}&\mathcal{C}_g : G \to G \\ &\mathcal{C}_g : h \mapsto ghg^{-1}\end{aligned}$$

So

$$\begin{aligned}&\mathcal{C} : G \to \text{Aut}(G) \\ &\mathcal{C}g = \mathcal{C}_g\end{aligned}$$

Now define the *adjoint action* of g as the differential or push forward of $\mathcal{C}_g$:

$$\text{Ad}_g := D_1\mathcal{C}_g \equiv (\mathcal{C}_g)_{*1} \equiv (\mathcal{C}_g)_*|_{g=1} \qquad \text{(adjoint action of } g)$$

Now $\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, so $\begin{aligned}&\text{Ad} : G \to \text{End}(\mathfrak{g}) \\ &\text{Ad}(g) \equiv \text{Ad}_g\end{aligned}$

Note $\mathcal{C}_{gg'} = \mathcal{C}_g\mathcal{C}_{g'} \equiv \mathcal{C}(gg') = \mathcal{C}(g) \circ \mathcal{C}(g')$ and so

$$\xrightarrow{D_1} \text{Ad}_{gg'} = \text{Ad}_g \circ \text{Ad}_{g'}$$

Kosmann-Schwarzbach (201) [5]) claims, because $\text{Ad}_g = 1_{\mathfrak{g}}$ when $g = 1$,

$$\begin{aligned}&\text{Ad} : G \to GL(\mathfrak{g}) \text{ is a representation of } G \text{ on } \mathfrak{g}. \text{ (EY : 20160505 ???)} \\ &\text{Ad} : g \mapsto \text{Ad}_g\end{aligned}$$

Definition 8. *representation Ad of G on $V = \mathfrak{g}$ is called adjoint representation of Lie group G.*

Denote adjoint representation of Lie algebra $\mathfrak{g}$, ad.
By definition, $\text{Ad}_{\exp(tX)} = \exp{(t\text{ad}_X)}$
cf. Prop. 7.8 of Kosmann-Schwarzbach (201) [5])

[3] Werner Ballmann. "Vector bundles and connections", http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/conncurv1999.pdf

Proposition 6. (1) *Let A invertible matrix, $A \in$ Lie group G.*
Let X matrix s.t. $X \in \mathfrak{g}$. Then

$$Ad_A(X) = AXA^{-1}$$

(2) *Let $X, Y \in \mathfrak{g}$. Then*

$$ad_X(Y) = [X, Y]$$

(3) *Let $X, Y \in \mathfrak{g}$. Then*

$$ad_{[X,Y]} = [ad_X, ad_Y]$$

Proof. (1) By def., $\forall\, B \in G$, $\mathcal{C}_A(B) = ABA^{-1}$, and thus

$$\text{Ad}_A(X) = \frac{d}{dt}A\exp{(tX)}A^{-1}\Big|_{t=0} = AXA^{-1}$$

(2)

$$\begin{gathered}\text{ad}_X(Y) = \frac{d}{dt}\text{Ad}_{\exp{(tX)}}(Y)\Big|_{t=0} = \frac{d}{dt}\exp{(tX)}Y\exp{(tX)}\Big|_{t=0} = \\ = XY - YX = [X, Y]\end{gathered}$$

(3) Use Jacobi identity:

$$\begin{gathered}[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \text{ or} \\ [[A, B], C] = [A, [B, C]] - [B, [A, C]] \\ \text{ad}_{[X,Y]}C = [[X, Y], C] = [X, [Y, C]] - [Y, [X, C]] = [X, \text{ad}_Y C] - [Y, \text{ad}_X C] \text{ and that} \\ \text{ad}_X\text{ad}_Y C = [X, [Y, C]] \Longrightarrow \text{ad}_{[X,Y]}C = [\text{ad}_X, \text{ad}_Y]C\end{gathered}$$

□

Part 2. Bundles

[6]

2. Vector bundles

Ballmann has a lucid and straightforward and useful exposition on vector bundles and connections [10] [3].
$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

Definition 9. *$\mathbb{K}$-vector bundle over M of rank k is bundle $\pi : E \to M$, fibers $E_p := \pi^{-1}(p)$ are $\mathbb{K}$-vector spaces, s.t. $\forall\, p \in M$, $\exists$ open $U \subseteq M$, $U \ni p$, $\exists$ diffeomorphism (called local trivialization) $\Phi : E|_{\pi^{-1}(U)} \to U \times \mathbb{K}^k$ s.t.*

$\pi \circ \Phi^{-1} = \pi_1 \qquad \pi_1 : U \times \mathbb{R}^k \to U$ *is canonical projection*

$$\begin{aligned}&\forall\, q \in U,\ \Phi_q^{-1} : \mathbb{K}^k \to E_q \qquad \text{is a } \mathbb{K}\text{-linear isomorphism.} \\ &\Phi_q^{-1}(v) := \Phi^{-1}(q, v)\end{aligned}$$

Definition 10. ***frame*** *of E over U is k-triple $(s_1 \dots s_k)$, $s_i \in \Gamma(E|_{\pi^{-1}(U)})$, σ_i smooth section of E over U s.t.*

$$\sigma_1(p) \dots \sigma_k(p)$$

basis of E_p $\forall\, p \in U$

By Prop. 10, frames and local trivializations are "equivalent".

Proposition 7. *conversely, if $\Phi : E|_{\pi^{-1}(U)} \to U \times \mathbb{K}^k$ trivialization of $E|_{\pi^{-1}(U)}$ and $e_1 \dots e_k$ standard basis of $\mathbb{K}^k$, then k-tuple $\sigma_i = \Phi^{-1}(\cdot, e_i)$, $1 \le i \le k$ is a frame of E over U.*

Definition 11. *Let $\Phi^{-1} = (s_1, \dots, s^k)$ local trivialization/frame.*
Consider arbitrary $s \in \Gamma(E|_{\pi^{-1}(U)})$
Then $\exists\, \sigma = \sigma_{\Phi^{-1}} : U \to \mathbb{K}^k$ s.t.

$$s(p) = \Phi^{-1}(p, \sigma(p)) \quad \forall\, p \in U \text{ i.e. } s = \sigma^i s_i \, , \, \sigma = (\sigma^1 \dots \sigma^k)$$

σ principal part of s with respect to Φ.

Let $\Psi^{-1} = (t_1 \dots t_k)$ another local trivialization of E over open $V \subseteq M$.

$$\forall\, p \in U \bigcap V, \text{ isomorphisms } \begin{array}{ll} \Phi_p^{-1} : \mathbb{K}^k \to E_k \, , & \Phi^{-1} = (s_1 \dots s_k), \text{ then} \\ \Psi_p^{-1} : \mathbb{K}^k \to E_k & \Psi^{-1} = (t_1 \dots t_k) \end{array}$$

$$s_j = g^i_{\ j} t_i$$

with smooth $g^i_{\ j} : U \bigcap V \to \mathbb{K}$, $\forall\, p \in U \bigcap V$, $g^i_{\ j}(p) = a^i_{\ j}(p) \in \text{Mat}(k,k)$, ($k \times k$ matrix).
$g^i_{\ j}$ invertible so smooth $g : U \bigcap V \to Gl(k, \mathbb{K})$.
Let $s \in \Gamma(E)$ over $U \bigcap V$, σ_Φ, σ_Ψ principal part of s with respect to Φ,Ψ. Then

$$\sigma^i_\Psi = g^i_{\ j} \sigma^j_\Phi \text{ i.e. } \sigma_\Psi = g \cdot \sigma_\Phi$$

Indeed, $\forall\, s \in \Gamma(E)$,

$$s = \sigma^j_\Phi s_j = \sigma^j_\Phi g^i_{\ j} t_i = g^i_{\ j} \sigma^j_\Phi t_i = \sigma^i_\Psi t_i$$

$$\begin{aligned} \nabla_X s = (X(\sigma^i_\Phi) + \sigma^j_\Phi (\omega_\Phi)^i_{\ j}(X)) s_i = (X(\sigma^i_\Psi) + \sigma^j_\Psi (\omega_\Psi)^i_{\ j}(X)) t_i = (X(g^i_{\ k}\sigma^k_\Psi) + g^j_{\ k} \sigma^k_\Phi (\omega_\Psi)^i_{\ j}(X)) t_i = \\ = X(\sigma^k_\Phi) s_k + t_i X(g^i_{\ k}) \sigma^k_\Phi + t_i (\omega_\Psi)^i_{\ j}(X) g^j_{\ k} \sigma^k_\Phi \end{aligned}$$

$X(\sigma^j_\Phi) s_i$ cancels from both sides and so

$$\begin{aligned} \Longrightarrow s_i \sigma^j_\Phi (\omega_\Phi)^i_{\ j}(X) = s_i (\omega_\Phi)^i_{\ j}(X) (g^{-1})^j_{\ k} (\sigma_\Psi)^k = t_l g^l_{\ i} (\omega_\Phi)^i_{\ j}(X) (g^{-1})^j_{\ k} \sigma^k_\Psi = \\ = t_l X(g^l_{\ k}) (g^{-1})^k_{\ i} \sigma^i_\Psi + t_l (\omega_\Psi)^l_{\ j}(X) \sigma^j_\Psi \\ \Longrightarrow \omega_\Psi(X) = g \omega_\Phi(X) g^{-1} - X(g)(g^{-1}) \end{aligned}$$

In summary,

$$\text{for connection } \begin{array}{l} \nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \\ \nabla(X, s) = \nabla_X s = (\nabla s)(X) \in \Gamma(E) \end{array}$$

$$\text{For frames } \begin{array}{l} \Phi^{-1} = (s_1 \dots s_k) \text{ of } E \text{ over open } U \\ \Psi^{-1} = (t_1 \dots t_k) \text{ of } E \text{ over open } V \end{array} , \quad U \cap V \neq \emptyset, \exists \text{ smooth } g = (g^i_{\ j}) : U \cap V \to \text{Gl}(k; \mathbb{K}) \text{ s.t.}$$

$$s_j = t_i g^i_{\ j} \text{ or } \Phi^{-1} = \Psi^{-1} g$$

Then $\forall\, s \in \Gamma(E_{\pi^{-1}(U \cap V)})$, $s = \sigma^j_\Phi s_j = \sigma^j_\Psi t_j$, so

$$\sigma^i_\Psi = g^i_{\ j} \sigma^j_\Phi \text{ or } \sigma_\Psi = g \sigma_\Phi$$

then

$$\nabla_X s = s_i (X(\sigma^i) + \omega^i_{\ j}(X) \sigma^j)$$

so that

$$\boxed{\omega_\Psi(X) = g \omega_\Phi(X) g^{-1} - X(g)(g^{-1}) \text{ or } (\omega_\Psi)^i_{\ j}(X) = g^i_{\ k} (\omega_\Phi)^k_{\ l}(X) (g^{-1})^l_{\ j} - X(g^i_{\ k})(g^{-1})^k_{\ j}} \tag{2}$$

define covariant derivative $d\mathbf{v} = (x, d\mathbf{v}|_x)$
$\forall\, \sigma \in \Gamma(E)$, section σ of E. define

$$\nabla \sigma = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d(\varphi_U \circ \sigma|_U))$$

$$\begin{aligned} \nabla(f\sigma) = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d(\varphi_U \circ f\, \sigma|_U)) = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* ((d\varphi_U)(d(f\, \sigma|_U))) = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* ((d\varphi_U)(\sigma|_U\, df + f\, d\sigma|_U) = \\ = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d\varphi_U)\, \sigma|_U \otimes df + \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d\varphi_U)(f\, d\sigma|_U) = \sum_{U \in \mathcal{U}} \chi_U f \varphi_U^* d(\varphi_U \circ d\sigma|_U) + \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d\varphi_U)\, \sigma|_U \otimes df = \\ = f \nabla \sigma + \sigma \otimes df \end{aligned}$$

3. Principal bundles

Let's follow Taubes (2011) from Chapter 10 "Principal bundles", on [6].

Definition 12. *Let smooth manifold M, Lie group G.*
principal *G-bundle $\equiv$ smooth manifold P s.t.*

- *smooth action of G by diffeomorphisms $\equiv$ map $m : G \times P \to P$ s.t.*
  - *$m(1, p) = p$*
  - *$m(h, m(g, p)) = m(hg, p)$*

  Notation: $(g, p) \mapsto pg^{-1}$
  EY : 20151007 note that

$$(g, p) \mapsto pg^{-1} \xrightarrow{m(h)} pg^{-1}h^{-1} = p(hg)^{-1} = m(hg, p)$$

$$\begin{tikzcd} G \times P \arrow[r] \arrow[d, "m"] & G \\ P & \end{tikzcd} \qquad \begin{tikzcd} (g,p) \arrow[r, mapsto] \arrow[d, mapsto, "m"] & g \\ m(p,g) = pg^{-1} & \end{tikzcd}$$

- *projection from P to M, $\pi \equiv$ surjective $\pi : P \to M$ that's G-invariant, i.e. $\pi(pg^{-1}) = p$*
- *$\forall\, p \in M$, $\exists$ open $U \ni p$, with G-equivariant diffeomorphism $\varphi : P|_U \to U \times G$ s.t. if $\varphi(p) = (\pi(p), h(p))$, $h(p) \in G$, then $\varphi(pg^{-1}) = (\pi(p), h(p)g^{-1})$*

$$\begin{tikzcd} P|_{\pi^{-1}(U)} \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times G \arrow[dl] \\ U \subset M & \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\pi"] & \varphi(p) = (\pi(p), h(p)) \arrow[dl, mapsto] \\ \pi(p) & \end{tikzcd}$$

$$\begin{tikzcd} m(p,g) = pg^{-1} \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\pi"] & \varphi(pg^{-1}) = (\pi(pg^{-1}), h(pg^{-1})) = (\pi(p), h(p)g^{-1}) \arrow[dl, mapsto] \\ \pi(pg^{-1}) = \pi(p) & \end{tikzcd}$$

cf. Section 10.1 "The definition" of Taubes (2011) [6]

3.0.1. *Cocycle definition (Sec. 10.2 "A cocycle definition")* [6]. Let locally finite, open cover $\mathcal{U} = \{U_\alpha\}$ of M (recall *locally finite*: $\forall\, p \in M$, $\exists$ open $\mathcal{O} \ni p$ s.t. $\mathcal{O} \bigcap U_\alpha$ for only finite many α's).

Definition 13. principal bundle transition functions $\equiv \{g_{\alpha\beta} : U_\alpha \bigcap U_\beta \to G\}_{U_\alpha, U_\beta \in \mathcal{U}}$ *i.e. collection of smooth maps from intersections of* $U_\alpha, U_\beta \in \mathcal{U}$ *s.t.* cocycle constraints *hold:*

- $g_{\alpha\alpha} = 1$
- $g^{-1}_{\alpha\beta} = g_{\beta\alpha}$
- *if* $U_\alpha, U_\beta, U_\gamma \in \mathcal{U}$, $U_\alpha \bigcap U_\beta \bigcap U_\gamma \neq \emptyset$, $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$

Claim: principle G-bundle $P = \coprod_{U \in \mathcal{U}} U_\alpha \times G / \sim$

with equivalence relation $\sim$ s.t. $\forall \begin{array}{c}(x,g) \in U_\alpha \times G \\ (x', g') \in U_\beta \times G\end{array}$, $(x,g) \sim (x', g')$ iff $x = x'$ and $g = g_{\alpha\beta} g'$

Proof. Let $[(x,g)] \in \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim$
Now for $m : G \times P \to P$, $m(h) : P \to P$ for $h \in G$,

$$[(x,g)] = (x,g) \xrightarrow{m(h)} (x, gh^{-1})$$
$$[(x,g)] = (x,g') \xrightarrow{m(h)} (x, g'h^{-1}) = (x, g^{-1}_{\alpha\beta} g h^{-1}) = (x, g_{\beta\alpha} g h^{-1})$$

Now $(x, gh^{-1}) \sim (x, g_{\beta\alpha} g h^{-1})$. So we have a well-defined smooth action of G by diffeomorphisms $m : G \times P \to P$, but now on the equivalence classes of $\coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim$, i.e.

$$[(x,g)] \xrightarrow{m(h)} [(x, gh^{-1})]$$

Second,

$$[(x,g)] \xrightarrow{\pi} x \in M$$

unequivocally. So we have a surjective projection from P to M, π, but now a surjective projection from $\coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim$ to M.

Third, define a G-equivariant diffeomorphism as such: Let $[(x,g)] \in \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim$. In fact, let $[(x,g)] \in U_\alpha \times G / \sim$. So $\forall\, x \in M$, $\exists\, U_\alpha \ni x$, $U_\alpha \in \mathcal{U}$, and we *define* G-equivariant diffeomorphism $\varphi : \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim\big|_{U_\alpha \times G} \to U_\alpha \times G$ s.t.

$$\begin{aligned} &\varphi : \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim \Big|_{U_\alpha \times G} \to U_\alpha \times G \\ &\varphi : [(x,g)] \mapsto (x,g) \end{aligned} \tag{3}$$

Indeed, this φ does what we want, for

$$m(h)[(x,g)] = (x, gh^{-1}) \xrightarrow{\varphi} (x, gh^{-1})$$

3.0.2. *Frame bundles.* cf. Sec. 10.3 "Principal bundles constructed from vector bundles", Subsection 10.3.1 "Frame bundles" of Taubes (2011) [6]

Definition 14. *Let rank n vector bundle $\pi : E \to M$.*

Let submanifold $\begin{array}{l} P_{GL(E)} \to M \subseteq \oplus^n E \\ (e_1 \dots e_n) \in P_{GL(E)} \end{array}$ *s.t.*
frame bundle of $E \equiv$ *principal $GL(n;\mathbb{R})$-bundle over $M \equiv$ manifold $P_{GL(E)}$.*

[4] pp. 32 http://www2.math.umd.edu/~jmr/Toronto/BaumNotes.pdf

This submanifold $P_{\mathrm{GL}(E)}$ is indeed a principal $\mathrm{GL}(n;\mathbb{R})$-bundle, as shown below.
Let $g \in GL(n;\mathbb{R})$.

$$\begin{aligned} m : \mathrm{GL}(E) \times P_{\mathrm{GL}(E)} &\to \mathrm{GL}(E) \\ (g, (e_1 \dots e_n)) &\mapsto (e_k g^{-1}_{k1}, e_k g^{-1}_{k2} \dots e_k g^{-1}_{kn}) \\ e_j &\mapsto e_k g^{-1}_{kj} \\ x = x^j e_j &\mapsto x^j e_k g^{-1}_{kj} = g^{-1}_{kj} x^j e_k \end{aligned}$$

Consider open $U \subset M$ s.t. $\exists$ vector bundle isomorphism $\varphi_U : E|_{\pi^{-1}(U)} \to U \times \mathbb{R}^n$

$$\varphi_U : (\oplus^n E)\big|_U \to U \times (\oplus^n \mathbb{R}^n) \Longrightarrow \varphi_U : P_{\mathrm{GL}(E)}\big|_U \to U \times \mathrm{GL}(n;\mathbb{R})$$

□

Definition 15. *We can define a homomorphism of principal G-bundles from (P_G, π) to (Q_H, θ) as a pair (η, ρ)*[4]

$$(P_G, \pi) \xrightarrow{(\eta,\rho)} (Q_H, \theta)$$

s.t.

(1) *ρ homomorphism $\rho : G \to H$*
(2) *$P \xrightarrow{\eta} Q$ cont. map s.t.*

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\pi"] & Q \arrow[d, "\theta"] \\ M \arrow[r, "1"] & M \end{tikzcd} \qquad \begin{tikzcd} P \times G \arrow[r, "\eta \times \rho"] \arrow[d] & Q \times H \arrow[d] \\ P \arrow[r, "\eta"] & Q \end{tikzcd}$$

$$\pi(p) = \theta(\eta p) \qquad \eta(pg) = (\eta p)(\rho g)$$

Definition 16. *Lee defines the **principal G-bundle morphism** on pp. 298 of Lee (2009)*[11] *as a pair $(\widetilde{f}, f)$*

$$(P_G, \pi, M_1) \xrightarrow{(\widetilde{f}, f)} (Q_G, \theta, M_2)$$

s.t.

(1) *smooth $f : M_1 \to M_2$*
(2) *$P \xrightarrow{\widetilde{f}} Q$ morphism s.t.*

$$\begin{tikzcd} P \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"] & Q \arrow[d, "\theta"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd} \qquad \begin{tikzcd} P \times G \arrow[r, "\widetilde{f} \times 1"] \arrow[d] & Q \times G \arrow[d] \\ P \arrow[r, "\widetilde{f}"] & Q \end{tikzcd}$$

$$f(\pi(p)) = \theta(\widetilde{f}(p)) \qquad \widetilde{f}(pg) = \widetilde{f}(p)g$$

$$\begin{tikzcd} P \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"] & Q \arrow[dl, "\theta"] \\ M & \end{tikzcd}$$

Now so that $\pi(p) = \theta(\widetilde{f}(p))$, is a *bundle isomorphism* (EY : 20151008 how is $\widetilde{f} : P \to Q$ an isomorphism? And how is it a diffeomorphism when restricted to fibers?)

From Taubes (2011) Sec. 10.9 "Associated vector bundles" [6],

Suppose Lie group G, principal G-bundle $\pi : P \to M$.
Let representation ρ, $\rho : G \to GL(V)$
$\exists$ corresponding vector bundle denoted $P \times_\rho V := P \times V / \sim$ where $(p, v) \sim (pg^{-1}, \rho(g)v) \quad \forall g \in G$.

$$\begin{tikzcd} P \times_\rho V \arrow[d, "\pi"] & \ni (p,v) \arrow[d, mapsto, "\pi"] \\ M & \ni \pi(p) \end{tikzcd}$$

Define *zero section* 0, to be

$$\begin{aligned} & 0 \in \Gamma(P \times_\rho V) \\ & 0 = [(p, 0)] \quad (p, 0) \in P \times V \end{aligned}$$

Define the usual scalar multiplication by $z \in \mathbb{R}$ or $\mathbb{C}$, ($\mathbb{R}$ or $\mathbb{C}$) $\times P \times_\rho V \to P \times_\rho V$ by $(p, v) \mapsto (p, zv)$.

Recall from the definition of a principal G-bundle the existence of a G-equivariant diffeomorphism φ:

$$\begin{tikzcd} P|_{\pi^{-1}(U)} \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times G \\ U \subset M & \end{tikzcd}$$

i.e. φ is G-equivariant diffeomorphism $\varphi : P|_{\pi^{-1}(U)} \to U \times G \ \forall p \in M$ (that $\exists$ open $U \ni p$) s.t.

if $\varphi(p) = (\pi(p), h(p))$, $\quad h(p) \in G$, then $\varphi(pg^{-1}) = (\pi(p), h(p)g^{-1})$, i.e.

if
$$\begin{tikzcd} p \arrow[r, mapsto] \arrow[d, mapsto, "\pi"] & (\pi(p), h(p)) \\ \pi(p) & \end{tikzcd}$$
then
$$\begin{tikzcd} pg^{-1} \arrow[r, mapsto] \arrow[d, mapsto, "\pi"] & (\pi(p), h(p)g^{-1}) \\ \pi(p) & \end{tikzcd}$$

i.e. (EY : 20151015 I was wondering how best to diagram this smooth right action of G by diffeomorphisms, pg^{-1})

$$\begin{tikzcd} G & \\ P|_{\pi^{-1}(U)} \times G \arrow[u] \arrow[d, "m"] & \\ P|_{\pi^{-1}(U)} \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times G \\ p \in U \subset M & \end{tikzcd} \qquad \begin{tikzcd} g & \\ (p, g) \arrow[u, mapsto] \arrow[d, mapsto, "m"] & \\ pg^{-1} \equiv m(p,g) \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\pi"] & (\pi(p), h(p)g^{-1}) \\ \pi(p) & \end{tikzcd}$$

or

$$\begin{tikzcd} P|_{\pi^{-1}(U)} \arrow[loop above, "m(g)"] \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times G \\ p \in U \subset M & \end{tikzcd} \qquad \begin{tikzcd} pg^{-1} \arrow[loop above, "m(g)"] \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\pi"] & (\pi(p), h(p)g^{-1}) \\ \pi(p) & \end{tikzcd}$$

Recall $\begin{aligned} & (\varphi^V)^{-1} : U \times V \to P \times_\rho V \\ & (\varphi^V)^{-1}(x, v) = [(\varphi^{-1}(x, 1), v)] \end{aligned}$

Then

$$\varphi^V \circ (\varphi^V)^{-1}(x, v) = \varphi^V([(\varphi^{-1}(x, 1), v)]) = \varphi^V((\varphi^{-1}(x, 1), v)) = (x, \rho(1)v) = (x, 1v) = (x, v)$$

Checking well-definedness (of the φ^V operation), so given that

$$(\varphi^{-1}(x, 1), v) \sim (\varphi^{-1}(x, 1)g^{-1}, \rho(g)v) \quad \forall g \in G$$

and so

$$\varphi^V(\varphi^{-1}(x, 1)g^{-1}, \rho(g)v) = (\pi(\varphi^{-1}(x, 1)g^{-1}), \rho(\psi(\varphi^{-1}(x, 1)g^{-1}))\rho(g)v) = (x, \rho(\psi(\varphi^{-1}(x, 1)))\rho(g^{-1})\rho(g)v) = (x, v)$$

Thus, $\varphi^V \circ (\varphi^V)^{-1} = 1_{U \times V}$.

Let (or recall) principal bundle isomorphisms $\begin{aligned} & \varphi_\alpha : P|_{\pi^{-1}(U_\alpha)} \to U_\alpha \times G \\ & \varphi_\beta : P|_{\pi^{-1}(U_\beta)} \to U_\beta \times G \end{aligned}$. Then $\begin{aligned} & \varphi_\beta \varphi_\alpha^{-1} : U_\alpha \bigcap U_\beta \times G \to U_\alpha \bigcap U_\beta \times G \\ & \varphi_\beta \varphi_\alpha^{-1} : (x, g) \mapsto (x, g_{\beta\alpha} g) \end{aligned}$ (i.e. principle G-bundles have principal bundle transition functions $g_{\beta\alpha}$'s).

Then for $\begin{aligned} & \varphi^V : P \times_\rho V \to U \times V \\ & \varphi^V : [(p, v)] = (p, v) \mapsto (\pi(p), \rho(\psi(p))v) \end{aligned}$,

$$\begin{aligned} & \varphi_\alpha^V : (P \times_\rho V)\big|_{\pi^{-1}(U_\alpha)} \to U_\alpha \times V \\ & \varphi_\beta^V : (P \times_\rho V)\big|_{\pi^{-1}(U_\beta)} \to U_\beta \times V \end{aligned} \Longrightarrow \begin{aligned} & \varphi_\beta^V(\varphi_\alpha^V)^{-1} : U_\alpha \bigcap U_\beta \times V \to U_\alpha \bigcap U_\beta \times V \\ & \varphi_\beta^V(\varphi_\alpha^V)^{-1} : (x, v) \mapsto (x, \rho(g_{\beta\alpha})v) \end{aligned}$$

So for our vector bundle $P \times_\rho V$, we have smooth, invertible "transition" maps $\varphi_\beta^V(\varphi_\alpha^V)^{-1}$ defined as immediately above.

Thus

Proposition 8. *Given principle G-bundle $\pi : P \to M$, Lie group G, representation ρ, $\rho : G \to GL(V)$, then $\exists$ (an associated) vector bundle $P \times_\rho V := P \times V / \sim$ where $(p,v) \sim (pg^{-1}, \rho(g)v) \quad \forall g \in G$ s.t. for this vector bundle $P \times_\rho V$, we have local trivialization φ^V, $P \times_\rho V \xrightarrow{\varphi^V} U \times V$, defined as such:*

For principal G-bundle $\pi : P \to M$,

$$\begin{tikzcd}
P \arrow[d, "\pi"] & \supset P|_{\pi^{-1}(U)} \arrow[r, "\varphi"] \arrow[rr, bend left, "\psi"] & U \times G \arrow[r] \arrow[dl] & G \\
M & \supset U \ni x \arrow[u, "\pi^{-1}"] & &
\end{tikzcd}$$

define

$$\varphi^V : P \times_\rho V \to U \times V$$
$$[(p,v)] = (p,v) \mapsto (\pi(p), \rho(\psi(p))v)$$

and

$$(\varphi^V)^{-1} : U \times V \to P \times_\rho V$$
$$(x,v) \mapsto [(\varphi^{-1}(x,1), v)]$$

and define the transition functions for our (associated) vector bundle $P \times_\rho V$ as such: for principal bundle isomorphisms $\begin{aligned} \varphi_\alpha &: P|_{\pi^{-1}(U_\alpha)} \to U_\alpha \times G, \\ \varphi_\beta &: P|_{\pi^{-1}(U_\beta)} \to U_\beta \times G \end{aligned}$ *then resulting principal bundle transition functions are such:*

$\varphi_\beta \varphi_\alpha^{-1} : U_\alpha \bigcap U_\beta \times G \to U_\alpha \bigcap U_\beta \times G$, *and so we have transition functions for our (associated) vector bundle*

$\varphi_\beta \varphi_\alpha^{-1} : (x,g) \mapsto (x, g_{\beta\alpha} g)$

$$\begin{aligned} \varphi_\alpha^V &: (P \times_\rho V)\big|_{\pi^{-1}(U_\alpha)} \to U_\alpha \times V \\ \varphi_\beta^V &: (P \times_\rho V)\big|_{\pi^{-1}(U_\beta)} \to U_\beta \times V \end{aligned} \Longrightarrow \begin{aligned} &\varphi_\beta^V (\varphi_\alpha^V)^{-1} : U_\alpha \bigcap U_\beta \times V \to U_\alpha \bigcap U_\beta \times V \\ &\varphi_\beta^V (\varphi_\alpha^V)^{-1} : (x,v) \mapsto (x, \rho(g_{\beta\alpha})v) \end{aligned}$$

<u>Claim:</u> If smooth map $\sigma^P : P \to V$ s.t. $\sigma^P(pg^{-1}) = \rho(g)\sigma^P(p)$, $\exists$ section $\sigma \in \Gamma(P \times_\rho V)$, defined as

$$\sigma(x) = [(p, \sigma^P(p))] \text{ s.t. } \pi(p) = x \, x \in M$$

If given section $\sigma \in \Gamma(P \times_\rho V)$, $\exists$ smooth map $\sigma^P : P \to V$ s.t. $\sigma^P(pg^{-1}) = \rho(g)\sigma^P(p)$ cf. Taubes (2011), Subsec. 11.4.1 Part 1

Proof. If given $\sigma^P : P \to V$ s.t. $\sigma^P(pg^{-1}) = \rho(g)\sigma^P(p)$, let

$$\sigma(x) = [(p, \sigma^P(p))] \text{ s.t. } \pi(p) = x$$

Indeed, checking well-definedness for $(p, \sigma^P(p)) \sim (pg^{-1}, \rho(g)\sigma^P(p))$,

$$(pg^{-1}, \rho(g)\sigma^P(p)) = (pg^{-1}, \sigma^P(pg^{-1})) = \sigma(x)$$

since $\pi(pg^{-1}) = x$. Since $\pi(pg^{-1}) = x \quad \forall g \in G$, then indeed this section $\sigma \in \Gamma(P \times_\rho V)$ is well-defined. □

Proposition 9 (cf. Taubes (2011) Subsection 11.4.3 [6])**.** *$\forall$ principal G-bundle $\pi : P \to M$, $\exists$ short exact sequence of vector bundle homomorphims*

$$0 \longrightarrow ker(\pi_*) \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0 \tag{4}$$

s.t.

$\pi_* : TP \to TM$ and so

$ker\pi_* \subset TP$

and so with arrows meaning, as follows:

$ker\pi_* \xrightarrow{\mathbf{i}} TP$, inclusion $\mathbf{i}$,

$TP \to \pi^* TM$

$v \in TP \mapsto (\pi(p), \pi_* v) \in \pi^* TM \subset P \times TM$

Recall (smooth, right) action of G on P, $\begin{aligned} m(g) \equiv m_g &: P \to P \\ m_g &: p \mapsto pg^{-1} \end{aligned}$ $\forall g \in G$. Then $\begin{aligned} (m_g)_* &: TP \to TP \\ (m_g)_* &: v_p \mapsto v_{pg^{-1}} \end{aligned}$.

If $v \in TP$, $v \in \ker\pi_*$, $(m_g)_* v \in \ker\pi_*$ since $\pi_*((m_g)_* v) = (\pi \circ m_g)_* v = (\pi_*)v = 0$ (for $\pi \circ m_g(p) = \pi(pg^{-1}) = \pi(p)$, so $\pi \circ m_g = \pi$)

There are many parts of this sequence to parse out and understand.

Begin with

$$\pi_* : TP \to TM$$

$$\begin{tikzcd}
P \arrow[d, "\pi"] & \ni p \arrow[r] \arrow[d, mapsto, "\pi"] & T_p P \arrow[d, "\pi_*"] & \ni v_p \equiv v(p) \arrow[d, mapsto, "\pi_*"] \\
M & \ni \pi(p) \arrow[r] & T_{\pi(p)} P & \ni \pi_* v_p
\end{tikzcd}$$

$\ker\pi_*$ *is* the so-called vertical bundle of $\pi_* : TP \to TM$ [5].

<u>Claim:</u> This sequence $0 \longrightarrow \ker(\pi_*) \xrightarrow{i} TP \xrightarrow{p} \pi^* TM \longrightarrow 0$ is a short exact sequence.

Proof. Consider $i : \ker(\pi_*) \to TP$ (inclusion)

$i(a) = i(a')$, $a, a' \in \ker\pi_*$. Then $a = a'$. Inclusion i is injective. So $0 \to \ker(\pi_*) \to TP$ exact. (this is shown mathematically in Rotman, or my notes on Rotman, or in your favorite abstract algebra book or notes)

EY : 20151017 weblinks roundup:

This showed that vector fields are isomorphic to derivations, and explicitly gives the construction of the isomorphism: 1300Y Geometry and Topology `http://www.math.toronto.edu/mgualt/MAT1300/week9.pdf`

This clarifies Taubes, 11.4.4 Part 3-4 `http://xwww.uni-math.gwdg.de/upmeier/notes/connections.pdf`

Since the Snake Lemma and 5-lemma are needed to show, in a civilized manner, the splitting lemma, could we use the Snake Lemma and 5-lemma for connections on a principal-G bundle? `http://www.mathematik.uni-kl.de/~gathmann/class/commalg-2013/chapter-4.pdf` □

[5]"Vertical bundle", *Wikipedia* `https://en.wikipedia.org/wiki/Vertical_bundle`

4. Connections to classes

Tsui [9] uses Baez and Muniain (1994) [1] as a reference for Chern-Simons Theory, but in my opinion, Jost (2011) [7], is a better (from a mathematically clear viewpoint) reference to begin with, and to set notation with. I follow his development and warn that I'll liberally copy from Jost (2011) [7] as necessary.

Let smooth curve $\begin{aligned} & c : \mathbb{R} \to \text{ open } U \subset M \\ & \dot{c} = X \end{aligned}$.

$$D_X s = e_k (X(s^k) + {\Gamma^k}_{ij} s^i X^j) = e_k (\dot{s}^k + {\Gamma^k}_{ij} s^i \dot{c}^j)$$

$D_X s = 0$ represents a linear system of n 1st-order ODEs, with initial values $s(0) \in E_{c(0)}$, so $\exists !$ solution s.

Definition 17. *solution $s = s(t)$ of $D_{\dot{c}} s = 0$ is the* ***parallel transport*** *of $s(0) \in E$ along curve $c : I \to M$.*

$$\begin{aligned} & A \equiv \Sigma^k_{\ i} \equiv \omega^k_{\ i} \in \Omega^1(M; \mathfrak{gl}(n, \mathbb{R})) = \Gamma(\mathfrak{gl}(n,\mathbb{R}) \otimes T^* \, M|_U) \\ & D : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^* M) \\ & D \equiv d + A \\ & D\mu = D(\mu^i e_i) = e_k (d\mu^k + \mu^i {\Gamma^k}_i) \equiv (d+A)\mu \end{aligned} \tag{5}$$

There can appear to be some confusion (at least I become confused many times) with the matrix multiplication on the *values* of a vector vs. matrix multiplication on the "abstract" choice of basis for a vector space V, $\dim V = n$. I want to clarify this here.

For a given $x \in V$, and given bases $\{e_j\}_j$, $\{f_i\}_i$, x can be expressed differently (taking different "values" that you'd fill in a so-called "column" vector) in each of these bases, but x is equal to x:

$$x = x^j e_j = y^i f_i \qquad i, j = 1 \dots n$$

For

$$e_j = f_i {g^i}_j$$

Then

$$x^j e_j = x^j f_i {g^i}_j = ({g^i}_j x^j) f_i = y^i f_i$$

Thus

$$gx = y \qquad \text{(matrix multiplication on "values" on a specified basis; i.e. matrix} \cdot \text{column vector = column vector)}$$

is the "usual" "linear algebra" matrix multiplication on column vectors containing the "values" of x expressed in terms of a particular choice of basis, into the vector x's values into another choice of bases.

$$e_j = f_i {g^i}_j$$

is how this automorphism g transforms these abstract basis vectors into the other abstract basis vectors. Notice how g is acting from the right of f_i's.

Following pp. 137 Eq. (4.1.17) of Jost (2011) [7] and pp. 15 Eq. (2.1.5) Ballmann [10]:

So for open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$, s.t. $\exists$ local trivializations of E on each U_α, $\Phi_\alpha : E|_{\pi^{-1}(U_\alpha)} = U_\alpha \times V$ (local trivialization is an isomorphism, hence the $=$ sign).

Considering any 2 overlapping $U_\alpha, U_\beta \subset M$, $U_\alpha \cap U_\beta \neq \emptyset$, then for $\begin{aligned} & \Phi_\alpha : E|_{\pi^{-1}(U_\alpha)} = U_\alpha \times V \\ & \Phi_\beta : E|_{\pi^{-1}(U_\beta)} = U_\beta \times V \end{aligned}$ and so the choice of basis for V, for each of these trivializations Φ_α, Φ_β is different, in general.

Supposing local trivialization/frame $\begin{aligned} \Phi_\alpha^{-1} & = (e_1 \dots e_n) \text{ of } E \text{ over open } U_\alpha \subset M \\ \Phi_\beta^{-1} & = (f_1 \dots f_n) \text{ of } E \text{ over open } U_\beta \subset M \end{aligned}$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$

$\exists$ smooth $g_{\beta\alpha} : U_\alpha \cap U_\beta \to Gl(n; \mathbb{R})$, in that $g_{\beta\alpha}(p) : V \to V$, s.t.

$$e_j = f_i {g^i}_j \text{ or } \Phi_\alpha^{-1} = \Psi^{-1} g$$

Let $s \in \Gamma(E)$

$$s = \sigma^j_\alpha e_j = \sigma^j_\alpha f_k {g^k}_j = {g^k}_j \sigma^j_\alpha f_k = \sigma^k_\beta f_k \Longrightarrow \sigma^k_\beta = {g^k}_j \sigma^j_\alpha$$

$s(p) = \Phi^{-1}(p, \sigma(p))$ (keep in mind this form of the section $s \in \Gamma(E)$).

For

$$Ds = e_k (d\sigma^k + \sigma^i {\Gamma^k}_i) \equiv e_k (d\sigma^k + \sigma^i {A^k}_i) \equiv e_k (d\sigma^k + \sigma^i {\omega^k}_i)$$

$$\begin{aligned} Ds = f_l {g^l}_k (d(({g^{-1}})^k_{\ m} \sigma^m_\beta) + ({g^{-1}})^i_{\ j} \sigma^j_\beta (A_\alpha)^k_{\ i}) & = f_l (d\sigma^l_\beta + {g^l}_k d({g^{-1}})^k_{\ m} \sigma^m_\beta + {g^l}_k (A_\alpha)^k_{\ i} (g^{-1})^i_{\ j} \sigma^j_\beta) = \\ & = f_l (d\sigma^l_\beta + \sigma^j_\beta (A_\beta)^l_{\ m}) \end{aligned}$$

$$\Longrightarrow (A_\beta)^l_{\ m} = {g^l}_k d(g^{-1})^k_{\ m} + {g^l}_k (A_\alpha)^k_{\ i} (g^{-1})^i_{\ m} \text{ or } A_\beta = g d(g^{-1}) + g A_\alpha g^{-1}$$

with $g \equiv g_{\beta\alpha} : V_\alpha \to V_\beta$ (so to speak).

Consider dual (vector) bundle $E^* \to M$.

cf. Jost (2011) [7], Def. 4.1.3,

Definition 18. *Let connection D^* dual to D on dual bundle E^* defined by*

$$d(s, t^*) = (Ds, t^*) + (s, D^* t^*) \qquad \forall \, s \in \Gamma(E), \, t^* \in \Gamma(E^*)$$

For $\omega \in \Gamma(E^*)$,

$$\begin{gathered} D^*_X \omega = X^j D^*_{\frac{\partial}{\partial x^j}} \omega = X^j D^*_{\frac{\partial}{\partial x^j}} \omega_i e^i = X^j \left(\frac{\partial \omega_i}{\partial x^j} e^i - \omega_k {\Gamma^k}_{ij} e^i \right) = X^j \left(\frac{\partial \omega_i}{\partial x^j} - \omega_k {\Gamma^k}_{ij} \right) e^i \\ D^* \omega = e^i (d\omega_i + - \omega_k {\Gamma^k}_i) \end{gathered}$$

Thus, for the orthonormal frame and dual frame, so that $(e_i, e^j) = {\delta_i}^j$ (and remember, it equals a number; derivatives of numbers are always 0),

$$\begin{gathered} d(e_i, e^j) = d({\delta_i}^j = 0 = (De_i, e^j) + (e_i, D^* e^j) = ({A^k}_i e_k, e^j) + (e_i, {(A^*)_k}^j e^k) = {A^j}_i + {(A^*)_i}^j \\ {(A^*)_i}^j = -{A^j}_i \Longrightarrow A^* = -A^T \end{gathered}$$

cf. Def. 4.1.4 of Jost (2011) [7], pp. 138,

Definition 19. *Let E_1, E_2 vector bundles over M with respective connections D_1, D_2.*
Then ***induced connection*** *D on $E := E_1 \otimes E_2$, defined by*

$$D(s_1 \otimes s_2) = D_1 s_1 \otimes s_2 + s_1 \otimes D_2 s_2 \qquad \forall \, s_i \in \Gamma(E_i), \qquad i = 1, 2$$

Considering the case of $\text{End}(E) = E \otimes E^*$,

$$D : \Gamma(\text{End}(E)) \to \Gamma(\text{End}(E)) \otimes \Gamma(T^* M)$$

Let $\sigma = {\sigma^i}_j e_i \otimes e^j \in \Gamma(\text{End}(E))$

$$\begin{aligned} D\sigma & = D({\sigma^i}_j e_i) \otimes e^j + {\sigma^i}_j e_i \otimes D^* e^j = (d{\sigma^k}_j + {\sigma^i}_j {A^k}_i) e_k \otimes e^j + {\sigma^i}_j e_i \otimes {(A^*)_k}^j e^k = \\ & = (d{\sigma^k}_j + {\sigma^i}_j {A^k}_i) e_k \otimes e^j + {\sigma^k}_i e_k \otimes - {A^i}_j e^k = (d{\sigma^k}_j + {[A, \sigma]^k}_j) e_k \otimes e^j \end{aligned}$$

Let vector bundle $E \to M$, with bundle metric $\langle \cdot, \cdot \rangle$.

Definition 20. Metric connection *D on E if*

$$d\langle \mu, \nu \rangle = \langle D\mu, \nu \rangle + \langle \mu, D\nu \rangle \qquad \forall \, \mu, \nu \in \Gamma(E)$$

cf. Lemma 4.2.1. of Jost (2011) [7]

Lemma 1. *For parallel transport on vector bundle induced by metric connection, preserves bundle metric, in that parallel transport constitutes an isometry of corresponding fibers, i.e. for $s = s(\tau), t = t(\tau) \in \Gamma(E)$ s.t. they get parallely transported, i.e. $D_{\dot{c}} s = D_{\dot{c}} t = 0$ for curve $c : I \to M$* in M, $\frac{d}{d\tau} \langle s(\tau), t(\tau) \rangle = 0$.

Proof. Let smooth curve $c : I \to M$
$\dot{c} = X$
in general, let $X \in T_xM$, then $X\langle \mu, \nu \rangle = d\langle \mu, \nu\rangle (X) = \langle D_X \mu, \nu \rangle + \langle \mu, D_X \nu \rangle$

$$\xrightarrow{X=\dot{c}} \langle D_{\dot{c}} \mu, \nu \rangle + \langle \mu, D_{\dot{c}}\nu \rangle$$

$D_{\dot{c}}\mu = D_{\dot{c}}\nu = 0$, i.e. μ, ν parallel along c.

$$\Longrightarrow \frac{d}{dt} \langle \mu(t), \nu(t) \rangle = 0$$

□

cf. Lemma 4.2.2. of Jost (2011) [7]

Lemma 2. *Let metric connection D on vector bundle E with bundle metric $\langle \cdot, \cdot \rangle$.*
Assume that with respect to metric bundle chart (cf. Def. 2.1.12, Thm. 2.1.3 of Jost (2011) [7].

$$D = d + A$$

Then, $\forall X \in TM$, $A(X) \in \mathfrak{o}(n)$ *(= Lie algebra of $O(n)$)* *($n =$ rank of E)*
$A(X) \in \mathfrak{u}(n)$ *(= Lie algebra of $U(n)$)* *($n =$ rank of E)*

Proof. Thm. 2.1.3: metric bundle chart (f, U) generates sections $e_1 \dots e_n$ on U s.t. $e_1 \dots e_n$ form orthonormal basis on fiber E_x, $\forall x \in U$ i.e. $\langle e_i(x), e_j(x) \rangle = \delta_{ij}$.

Moreover, since e_i constant in bundle chart (f, U), define exterior derivative d.

$$d\mu_i \equiv 0 \qquad (i = 1 \dots n)$$

Let $X \in T_xM$, $x \in U$.

$$0 = X\langle e_k, e_l \rangle = \langle D_X e_k, e_l \rangle + \langle e_k, D_X e_l \rangle = \langle X^j \Gamma^m_{\ \ kj} e_m, e_l \rangle + \langle e_k, x^j \Gamma^m_{\ \ lj} e_m \rangle$$

For $\mathbb{K} = \mathbb{R}$,

$$\Longrightarrow A^l_{\ \ k} + A^k_{\ \ l} = 0 \text{ so } A = -A^T \text{ skew symmetric}$$

□

For the case/issue of field $\mathbb{K} = \mathbb{C}$ (complex numbers) and dealing with the so-called *Hermitian* metric on a vector bundle with complex numbers involved, then I took a look at the articles on Hermitian Inner Product, and Hermitian metric:

Hermitian Inner Product:

$$\begin{aligned} \langle u + v, w \rangle &= \langle u, w \rangle + \langle v, w \rangle \\ \langle u, v + w \rangle &= \langle u, v \rangle + \langle u, w \rangle \\ \langle \alpha u, v \rangle &= \alpha \langle u, v \rangle \\ \langle u, \alpha v \rangle &= \overline{\alpha} \langle u, v \rangle \\ \langle u, v \rangle &= \overline{\langle v, u \rangle} \\ \langle u, u \rangle &\geq 0 \qquad \langle u, u \rangle = 0 \iff u = 0 \end{aligned}$$

vs.

Hermitian manifold or "Hermitian bundle metric on a vector bundle"

$$\langle \cdot, \cdot \rangle \equiv h \in \Gamma(E \otimes E^*)^* \text{ s.t.}$$

$$h_p(\eta, \overline{\xi}) = \overline{h_p(\eta, \overline{\xi})} \qquad \forall \xi, \eta \in E_p$$

$$h_p(\xi, \overline{\xi}) > 0 \qquad \forall \xi \in E_p, \, \xi \neq 0$$

Then, using the Hermitian inner product rule,

$$\begin{aligned} d\langle e_i, e_j \rangle &= \langle De_i, e_j \rangle + \langle e_i, De_j \rangle = \langle (d+A)e_i, e_j \rangle + \langle e_i, (d+A)e_j \rangle = \\ &= \langle 0 + A^k_{\ \ i} e_k, e_j \rangle + \langle e_i, 0 + A^k_{\ \ j} e_k \rangle = \langle A^k_{\ \ i} e_k, e_j \rangle + \langle e_i, A^k_{\ \ j} e_k \rangle = \\ &= A^j_{\ \ i} + (A^i_{\ \ j})^* = 0 \Longrightarrow A^j_{\ \ i} = -(A^i_{\ \ j})^* \text{ so } A = -A^\dagger \end{aligned}$$

Part 3. Gauge Theory

The story of Gauge Theory in Physics is a beautiful story - it is the triumph of geometry in physical theories of matter.

5. Electromagnetism

- : Electromagnetism
- : Maxwell's Equations

From "Rewriting Maxwell's Equations" of Baez and Muniain (1994) [1]:

$$*F_+ = \frac{1}{2}\left[iF_+ - iF_- + i(F_+ + F_-) \right] = iF_+ \text{ i.e. self dual} \qquad *F_- = \frac{1}{2}\left[iF_+ - iF_- - iF_+ - iF_- \right] = -iF_- \text{ i.e. anti-self dual}$$

So if $F \in \Omega^p(M)$ (F self dual or anti self dual), if $dF = 0$, $d * F = 0$, since $d * F = \begin{cases} d(\pm F) = \pm dF = 0 \\ d(\pm iF) = \pm i dF = 0 \end{cases}$

Consider $\Delta : \Omega^p(M) \to \Omega^p(M)$
$\Delta = (\delta + d)^2 = \delta d + d\delta$

$$\Delta F = \delta dF + d\delta F = 0 + d(- * d *)F = -dJ$$

$$J = j - \rho dt$$

$$dJ = \frac{\partial j_j}{\partial x^\mu} dx^\mu \wedge dx^j - \frac{\partial \rho}{\partial x^i} dx^i \wedge dt$$

Note that for $d^2 = 0$, $\delta^2 = 0$ (as seen with this example of Maxwell's equations, for F), $\delta = (-1)^{n(p+1)+1} * d * = - * d *$ (especially if n even),

$$\begin{tikzcd} \Omega^2(M) \arrow[r, "*"] \arrow[d, "\delta"'] & \Omega^{n-2}(M) \arrow[d, "d"] \\ \Omega^1(M) \arrow[r, "-*(s)"] & \Omega^{n-1}(M) \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, mapsto, "*"] \arrow[d, mapsto, "\delta"'] & *F \arrow[d, "d"] \\ \delta F \arrow[r, mapsto, "-*(s)"] & d * F = (s) * J \end{tikzcd}$$

$$F \in \Omega^2(M) \qquad * F \in \Omega^{n-2}(M)$$

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \qquad * F = \frac{\sqrt{g}}{(n-2)!} \frac{1}{2} F_{\mu\nu} g^{\mu\mu'} g^{\nu\nu'} \epsilon_{\mu'\nu'\rho\sigma} dx^\rho \wedge dx^\nu$$

$$F = B + E \wedge dt = \frac{B_{ij}}{2} dx^i \wedge dx^j + E_i dx^i \wedge dt$$

$$*F = \frac{\sqrt{g}}{(n-2)!} \frac{B_{ij}}{2} g^{i\mu} g^{j\nu} \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma + \frac{\sqrt{g}}{(n-2)!} E_i g^{i\mu} g^{0\nu} \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma$$

If $M = \mathbb{R} \times N$,

$$\begin{aligned} & B_{ij} g^{il} g^{jm} \epsilon_{lm\rho\sigma} dx^\rho \wedge dx^\sigma = B_{ij} g^{il} g^{jn} (\epsilon_{lmn\sigma} dx^n \wedge dx^\sigma + \epsilon_{lm0\sigma} dx^0 \wedge dx^\sigma) = \\ & = B_{ij} g^{il} g^{jm} (\epsilon_{lmn0} dx^n \wedge dt + \epsilon_{lmn} dx^n \wedge dt) = 2 B_{ij} g^{il} g^{jm} \epsilon_{lmn} dx^n \wedge dt \end{aligned}$$

Noting that

$$\epsilon_{lmn} B_{ij} g^{il} g^{jm} = \epsilon_{lmn} \epsilon_{ijk} B^k g^{il} g^{jm} = 2B^n$$

If $g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $M = \mathbb{R} \times N$,

$$*F = \frac{1}{(n-2)!}\left[\frac{B_{ij}}{2}g^{il}g^{jm}\epsilon_{lmn0}dx^n \wedge dt + -E_i g^{il}\epsilon_{l0mn}dx^m \wedge dx^n\right] = \frac{1}{(n-2)!}\left[-\epsilon_{lmn}g^{li}E_i dx^m \wedge dx^n + (-\epsilon_{lmn})B_{ij}g^{il}g^{jm}dx^n \wedge dt\right] = \tag{6}$$
$$= \frac{1}{(n-2)!}\left[-E_i g^{il}\epsilon_{lmn}dx^m \wedge dx^n + (-B_{ij})g^{il}g^{jm}\epsilon_{lmn}dx^n \wedge dt\right] = \vec{*}E - \vec{*}B \wedge dt$$

Indeed, in general, for $M = \mathbb{R} \times N$,

$$*F = \frac{\sqrt{g}}{(n-2)!}B_{ij}g^{il}g^{jm}\epsilon_{lmn0}dx^n \wedge dt + \frac{\sqrt{g}}{(n-2)!}E_i g^{il}(-1)\epsilon_{l0mn}dx^m \wedge dx^n =$$
$$= \frac{\sqrt{g}}{(n-2)!}(-1)E_i g^{il}(-\epsilon_{0lmn})dx^m \wedge dx^n + \frac{\sqrt{g}}{(n-2)!}\epsilon_{ijk}B^k g^{il}g^{jm}(-\epsilon_{lmn})dx^n \wedge dt$$

If $g^{il} = \delta^{il}$, then applying $\epsilon_{jmn}\epsilon^{imn} = 2\delta^i_j$, and $n = 4$

$$*F = \vec{*}E + \vec{*}B \wedge dt$$

$*F = iF$ (self-dual) if

$$\vec{*}E = iB$$
$$\vec{*}B = -iE$$

And so

$$E = E_i dx^i$$

$$\vec{*}E = \frac{\sqrt{g}}{(d-1)!}E_i g^{il}\epsilon_{lmn}dx^m \wedge dx^n \Longrightarrow -i\vec{*}E = \frac{-i\sqrt{g}}{(d-1)!}E_i g^{il}\epsilon_{lmn}dx^m \wedge dx^n \Longrightarrow \frac{1}{2}B^l = \frac{-i\sqrt{g}}{(d-1)!}E_i g^{il} \text{ or } B^k = -i\sqrt{g}E_i g^{ik}$$
$$B = \frac{1}{2}B_{mn}dx^m \wedge dx^n = \frac{1}{2}\epsilon_{mnl}B^l dx^m \wedge dx^n$$

Assume F self-dual ($*F = iF$) and $E(x) = E_i e^{ikx}dx^i$, $k \in (\mathbb{R}^4)^*$, $x \in \mathbb{R}^4 = M$, k fixed covector called **energy-momentum**, s.t.

$$k_\mu x^\mu = kx$$

$B = -i\vec{*}E$

6. Chern-Simons Theory

From Baez and Muniain (1994) [1]:

Part 4. Supersymmetry

Readings

- Guillemin and Sternberg (1999) [8]

I will follow Guillemin and Sternberg (1999) [8].

$$\begin{aligned}
&i_\xi i_\eta + i_\eta i_\xi = 0 && \{i_\xi, i_\eta\} = 0 \\
&L_\xi i_\eta - i_\eta L_\xi = i_{[\xi,\eta]} && \\
&L_\xi L_\eta - L_\eta L_\xi = L_{[\xi,\eta]} && \\
&di_\xi + i_\xi d = L_\xi && \{d, i_\xi\} = L_\xi \\
&dL_\xi - L_\xi d = 0 && [d, L_\xi] = 0 \\
&d^2 = 0 && \\
&\rho_a \circ L_\xi \circ \rho_a^{-1} = L_{\mathrm{Ad}_a \xi} && \\
&\rho_a \circ i_\xi \circ \rho_a^{-1} = i_{\mathrm{Ad}_a \xi} &&
\end{aligned}$$

Ad denotes adjoint representation of G on $\mathfrak{g}$.

$L_j \equiv L_{\xi_j}$, $i_j \equiv i_{\xi_j}$ in terms of basis for $\mathfrak{g}$.

supervector space - vector space V with $\mathbb{Z}/2\mathbb{Z}$ gradation: $V = V_0 \oplus V_1$, where $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$.
most of the time, our vector spaces come equipped with $\mathbb{Z}$-gradation $V = \oplus_{i\in\mathbb{Z}}V_i$, $V_0 := \oplus V_{2j}$ even; $V_1 := \oplus V_{2j+1}$ odd.
an element of V_i have degree i.

superalgebra is supervector space A with multiplication satisfying $A_i \cdot A_j \subset A_{i+j}$ if A is $\mathbb{Z}$-graded.

e.g. if V supervector space, then End$V$ superalgebra where

$$(\mathrm{End}V)_i := \{A \in \mathrm{End}V | A : V_j \to V_{j+1}\}$$

in $\mathbb{Z}$-graded case, if only finitely many $V_i \neq \{0\}$.

Part 5. Polynomials

7. (Software) Packages

This link Combinatorial Software and Databases has a list of (possibly) useful implementations of the Gosper-Zeilberger Algorithm. There is

- EKHAD: written in Maple by Doron Zeilberger, is an implementation of the Gosper-Zeilberger algorithm.
- : written in Maple by Doron Zeilberger, is an implementation of the q-Zeilberger algorithm. Also at Rutgers, Packages Accompanying or Related to A=B
- Zeilberger:written in Maxima by Fabrizio Caruso, is an implementation of the Gosper-Zeilberger algorithm.

Tom Koornwinder had a paper On Zeilberger's algorithm and its q-analogue: a rigorous description that explicitly and very lucidly reviews Zeilberger's algorithm, but I cannot find his accompanying Maple software (links don't work).

I will focus on looking at Caruso's package because the source code is available and I had emailed Paule and the source code for Mathematica implementations are not available, understandably from their particular policies on distribution.

Wilf and Zeilberger (1992) has a good introduction to hypergeometric polynomials and its significance [13]. In the immediate sections, I will recap, summarize, and copy (or state) some definitions and results from Wilf and Zeilberger (1992), and show (some of its) direct implementations in Sage Math and sympy.

Starting from pp. 586 of Wilf and Zeilberger (1992) [13],
Consider

$$(c)_n := (1-c)(1-cq)\dots(1-cq^{n-1})$$

Let $c = q$. Then

$$(q)_n = (1-q)(1-q^2)\dots(1-q^n)$$

Define $f(n) := (c)_n$. Notice that

$$\frac{f(n+1)}{f(n)} = 1 - cq^n$$

This is the simplest nonzero rational function in q^n.

Let $f(x) := (x)_\infty$. $\frac{f(qx)}{f(x)} = \frac{(qx)_\infty}{(x)_\infty} = \frac{1}{1-x}$.

Note that

$$\frac{1}{1-x} = \sum_{j=0}^\infty x^j$$

$$(x)_\infty = (1-x)(1-xq)\dots(1-xq^{n-1})\dots$$
$$(qx)_\infty = (1-qx)(1-q^2x)\dots(1-xq^n)\dots$$

Also note that

$$(q)_k = \frac{(q)_\infty}{(q^{k+1})_\infty} = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q^{k+1})(1-q^{k+2})\dots}$$

Introduce dilation operator $\forall\, x\in\mathbb{F}$, $Q_xf(x,\mathbf{y}) := f(qx,\mathbf{y})$

q-derivative $D_x^{(q)}$

$$D_x^{(q)}f(x,\mathbf{y}) = \frac{f(qx,\mathbf{y})-f(x,\mathbf{y})}{(q-1)x} = \frac{(Q_x-1)f(x,\mathbf{y})}{(x(q-1))}$$

Notice that

$$\lim_{q\to 1} D_x^{(q)}f(x,\mathbf{y}) = D_xf$$

Definition 21. *q-hypergeometric,*

$$F(k_1\dots k_r,y_1\dots y_s)\equiv F(\mathbf{k},\mathbf{y})$$

$k_1\dots k_r\in\mathbb{Z}$ *(discrete),* $y_1\dots y_s$ *cont.* $(y_1\dots y_s\in\mathbb{F}\in\{\mathbb{R},\mathbb{C}\})$ q-hypergeometric

if $\forall$ *(discrete)* k_i, $(E_{k_i}F)/F$ *rational functions of* $(q^{k_1}\dots q^{k_r},y_1\dots y_s)$ *and possible other constnat parameters, including q.*

$\forall$ *(cont.)* y_j, $(Q_{y_j}F)/F$

7.0.3. *Examples.*

(1) Polynomials $P(q^{k_1}\dots q^{k_r},y_1\dots y_n)$. Indeed,

$$(E_{k_i}P)/P = \frac{P(q^{k_1}\dots q^{k_i+1}\dots q^{k_r},\mathbf{y})}{P(q^{k_1}\dots q^{k_r},\mathbf{y})}$$

$$(Q_{y_j}P)/P = \frac{P(q^{k_1}\dots q^{k_r},y_1\dots qy_j\dots y_n)}{P(q^{k_1}\dots q^{k_r},y_1\dots y_n)}$$

(2)

$$(cy_1^{\alpha_1}\dots y_s^{\alpha_s},q^{\beta_1k_1}\dots q^{\beta_rk_r})_\infty^\gamma$$

cf. (qPH-II) of Wilf and Zeilberger (1992) [13] EY : 20160105 I don't understand the upper and lower exponent notation of $()^\gamma_\infty$, i.e. what is γ and ∞ in this case?

(3) $q^{\sum_{i,j}a_{i,j}k_ik_j+\sum_i b_ik_i}$ $\quad a_{i,j},b_i\in\mathbb{Z}$ or $\{\pm\frac{1}{2},\pm\frac{3}{2},\dots\}$. Indeed

$$\xrightarrow{E_{k_i}} q^{\sum_{i,j}q_{i,j}k_ik_j+\sum_i b_ik_i}q^{\sum_{i,j}a_{i,j}k_j+b_i}$$

(4) $z_1^{k_1}\dots z_r^{k_r}$. Indeed,

$$\xrightarrow{E_{k_i}} z_1^{k_1}\dots z_r^{k_r}\cdot z_i$$

Definition 22. $F(k_1\dots k_r,y_1\dots y_s)$, *(discrete)* $k_1\dots k_r\in\mathbb{Z}$, *(cont.)* $y_1\dots y_r\in\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$, q-proper-hypergeometric *if F has form* $F = P(q^{k_1}\dots q^{k_r},y_1\dots y_s)\cdot q^{\sum_{i,j}a-i,jk_ik_j+\sum_i b_ik_i}\cdot z_1^{k_1}\dots z_r^{k_r}\cdot$ *finite number of* $(cy_1^{\alpha_1}\dots y_s^{\alpha_s}q^{\beta_1k_1}\dots q^{\beta_rk_r})^\gamma_\infty$

Lemma 3 (q-fundamental lemma)**.** $\forall$ *q-proper-hypergeometric* $F(x_1\dots x_n,a_1\dots a_m)$ *and* $\forall$ *cont.* x_i, $\forall$ *(discrete)* a_j, $\exists$ *nonzero linear recurrence-q-differential operators*

$$P_i(x_i;Q_{x_1}\dots Q_{x_n};E_{a_1}\dots E_{a_m}) \text{ and respectively } C_j(q^{a_j};Q_{x_1}\dots Q_{x_n};E_{a_1}\dots E_{a_m})$$

annihilating F, i.e. $P_iF=0$
$C_jF=0$

Let $x=q^k$

$$(cq^k)_\infty = \frac{(c)_\infty}{(c)_k} = \frac{(1-c)(1-cq)\dots}{(1-c)(1-cq)\dots(1-cq^{k-1})} = (1-cq^k)(1-cq^{k+1})\dots$$

$$(cx)_\infty = \frac{(c)_\infty}{(c)_k} = (1-cq)(1-cxq)\dots$$

Consider these transformations or mappings:

$$F(x_1\dots x_n,a_1\dots a_m)\xrightarrow{x=q^k} F(q^{k_1}\dots q^{k_n},a_1\dots a_m)$$
$$Q_{x_i}F = F(x_1\dots qx_i\dots x_n,\mathbf{a}) = F(q^{k_1}\dots q^{(1+k_i)}\dots q^{k_n},\mathbf{a}) = E_{k_i}F$$

$E_kq^k = qq^kE_k$ is "isomorphic" to $Q_xx = qxQ_x$

Now for $\frac{(c)_\infty}{(cx)_\infty}$ substitute $(c)_k$.

$$\frac{(c)_\infty}{(cx)_\infty} = \frac{(1-c)(1-cq)\dots}{(1-cx)(1-cxq)\dots}\xrightarrow{x=q^k}(1-c)(1-cq)\dots(1-cq^{k-1})$$

7.1. **The fundamental theorem of hypergeometric summation/integration.** cf. Sec. 2. "The fundamental theorem of hypergeometric summation/integration" of Wilf and Zeilberger (1992) [13].

Definition 23. $F(k_1\dots k_r,y_1\dots y_s)\equiv F(\mathbf{k},\mathbf{y})$ *vanishes at infinity if* $\forall\, k_i,y_j$,

$$\lim_{|k_i|\to\infty}F(\mathbf{k},\mathbf{y}) = 0$$
$$\lim_{|y_j|\to\infty}F(\mathbf{k},\mathbf{y}) = 0$$

Definition 24. *integral-sum*

$$g(n,x) := \sum_k\int_y F(n,k,x,y)dy \qquad \textit{(general-integral-sum)}$$

is pointwise trivially evaluable, if $\forall$ *given* $\mathbf{n},\mathbf{x}$, $\exists$ *algorithm to evaluate it.*

e.g. for pure sums that are terminating,

$$g(\mathbf{n}) := \sum_{\mathbf{k}}F(n,\mathbf{k}) \qquad \textit{(general-sum)}$$

sum is finite $\forall\,\mathbf{n}$ *since* $F(\mathbf{n},\cdot)$ *has finite support.*

7.1.1. *The fundamental Theorem of hypergeometric summation-integration.*

Theorem 3 (The fundamental theorem)**.** *Let* $\Delta_{k_i} \equiv$ *forward difference operator in* k_i ; $\Delta_{k_i} := (K_i - 1)$.

Let $F(n, k_1 \dots k_r, y_1 \dots y_s)$ *be hypergeometric (holonomic) (both hold if proper-hypergeometric) in* $(\mathbf{k}, \mathbf{y})$ *and* n
$F(x, k_1 \dots k_r, y_1 \dots y_s)$ $(\mathbf{k}, \mathbf{y})$ *and* x

where *discrete* $n, \mathbf{k}$
cont. $x, \mathbf{y}$

Then $\exists$ *linear ordinary recurrence operator* *with polynomial coefficients* $P(N, n)$
$\exists$ *linear differential operator* $P(D_x, x)$
and rational functions $R_1 \dots R_r, S_1 \dots S_s$ *s.t.*

$$\begin{aligned} P(N,n)F &= \sum_{i=1}^r \Delta_{k_i}(R_i, F) + \sum_{j=1}^s D_{y_j}(S_j F) \\ P(D_x,x)F &= \sum_{i=1}^r \Delta_{k_i}(R_i, F) + \sum_{j=1}^s D_{y_j}(S_j F) \end{aligned} \tag{7}$$

If F *proper-hypergeometric, it's possible to find bounds for order of* $P(N,n), P(D_x, x)$, *and denominators* R_i, S_j

Recall "the fundamental lemma", due to Bernstein, reproduced on pp. 585 of Wilf and Zeilberger (1992) [13],

Lemma 4 (The fundamental lemma)**.** $\forall$ *holonomic function* $F(x_1 \dots x_n, a_1 \dots a_m)$, $\forall$ *cont.* x_i, $\forall$ *discrete* a_j, $\exists$ *non-zero linear recurrence differential operators*

$$\begin{aligned} &P_i(x_i; D_{x_1} \dots D_{x_n}; E_{a_1} \dots E_{a_m}) \\ &C_j(x_i; D_{x_1} \dots D_{x_n}; E_{a_1} \dots E_{a_m}) \qquad \textit{respectively} \end{aligned}$$

that annihilates F, *i.e.* $PF = 0$

e.g. $(D_x^2 + I)\cos x = 0$

Proof. By fundamental lemma 4, $\exists$ operator(s)

$$\begin{aligned} &A(n, E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n}) \\ &A(x; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n}) \qquad \text{respectively} \end{aligned}$$

s.t. $AF = 0$

It's possible to rewrite A as

$$A(n; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n}) = P(n, E_n) - \sum_{i=1}^r (E_{k_i} - 1) B_i(n; E_n, E_{k_1} \dots E_{k_r}, D_y \dots D_{y_n}) - $$
$$- \sum_{j=1}^s D_{y_j} \overline{B}_j(n; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n})$$

$$A(x; E_{k_1} \dots E_{k_r}; D_x, D_{y_1} \dots D_{y_n}) = P(x, D_x) - \sum_{i=1}^r (E_{k_i} - 1) B_i(x; E_{k_1} \dots E_{k_r}, D_x, D_y \dots D_{y_n}) - $$
$$- \sum_{j=1}^s D_{y_j} \overline{B}_j(x; E_{k_1} \dots E_{k_r}; D_x, D_{y_1} \dots D_{y_n})$$

$$0 = P(n, E_n)F - \sum_{i=1}^r (E_{k_i} - 1) B_i(n, E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n})F - \sum_{j=1}^s D_{y_j} \overline{B}_j(n, E_n, E_{k_1} \dots E_{k_i}; D_{y_1} \dots D_{y_n})F$$

$$0 = P(x, D_x)F - \sum_{i=1}^r (E_{k_i} - 1) B_i F - \sum_{j=1}^s D_{y_j} \overline{B}_j F$$

□

F hypergeometric, s.t. $\frac{E_k F}{F}$, $\frac{D_{y_j} F}{F}$ rational functions. By induction, $\forall$ "operator monomial"

$$\text{Mon} := E_n^{\alpha_0} \prod_{i=1}^r E_{k_i}^{\alpha_i} \prod_{j=1}^s D_{y_j}^{\beta_j} \qquad \text{(Op-Mon)}$$

$$\text{Mon} := D_x^{\beta_0} \prod_{i=1}^r E_{k_i}^{\alpha_i} \prod_{j=1}^s D_{y_j}^{\beta_j}$$

$\frac{\text{Mon} F}{F}$ rational function.

$\forall$ operator $T(n, k_1 \dots k_r; y_1 \dots y_s; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_s})$
$T(k_1 \dots k_r; y_1 \dots y_s; E_{k_1} \dots E_{k_r}; D_x, D_{y_1} \dots D_{y_s})$

$\frac{TF}{F}$ rational function, since T linear combination with coefficients that are polynomials in all variables, of operator monomials. In a sense, one could say T belongs to

$$\begin{aligned} &k[n, k_1 \dots k_r; y_1 \dots y_s]\{\text{ Mon }\} \to \{\text{ Mon }\}, \qquad k[n, k_1 \dots k_r; y_1 \dots y_s] - \text{module} \\ &k[k_1 \dots k_r; x, y_1 \dots y_s]\{\text{ Mon }\} \to \{\text{ Mon }\}, \qquad k[k_1 \dots k_r; x, y_1 \dots y_s] - \text{module} \end{aligned}$$

In particular,

$$\begin{aligned} B_i(F) &= R_i F \\ \overline{B}_j(F) &= S_j F \end{aligned}$$

$$\Longrightarrow \begin{aligned} 0 &= P(n, E_n)F - \sum_{i=1}^r (E_{k_i} - 1) R_i F - \sum_{j=1}^s D_{y_j} S_j F \\ 0 &= P(x, D_x)F - \sum_{i=1}^r (E_{k_i} - 1) R_i F - \sum_{j=1}^s D_{y_j} S_j F \end{aligned}$$

R_i, S_j are certificates. Note Gert Almkinst observed B_i, B_j not necessary to be independent of $(k_1 \dots k_i; y_1 \dots y_s)$

Corollary 1 (Fundamental corollary)**.** *If* $F(n, \mathbf{k}, \mathbf{y})$ *hypergeometric functions* , *and vanishes at infinity,* $\forall$ *fixed* n, *then*
$F(x, \mathbf{k}, \mathbf{y})$ *holonomic functions* $\forall$ *fixedx*

$$f(n) := \sum_k \int F(n, \mathbf{k}, \mathbf{y}) d\mathbf{y}$$
$$f(x) := \sum_k \int F(x, \mathbf{k}, \mathbf{y}) d\mathbf{y}$$

satisfies *linear recurrence equation* *with polynomial coefficients* $P(N,n)f(n) = 0$
differential equation $P(D_x, x)f(x) = 0$

Proof.

$$\sum_{\mathbf{k}} \int d\mathbf{y}(E_{k_i}-1)R_iF = 0 \quad \text{since } \lim_{|k_i|\to\infty} F(\mathbf{k},\mathbf{y}) = 0$$

$$\sum_{\mathbf{k}} \int d\mathbf{y} D_{y_j} S_j F = 0 \quad \text{since } \lim_{|y_j|\to\infty} F(\mathbf{k},\mathbf{y}) = 0$$

Again, R_i, S_j are called *certificates*. □

7.1.2. *How to find $P(N,n)$ and the certificates.* cf. 2.2. How to find $P(N,n)$ and the certificates of Wilf and Zeilberger (1992) [13].

Guess the order of $P(N,n)$, say L

in practice, try $L=0$. Then $L=1,\dots$

$$P(N,n) = \sum_{i=0}^{L} b_i(n)N^i$$

guess R_i, S_j

$$P(N,n) = \sum_{i=1}^{r} \Delta_{k_i}(R_i,F) + \sum_{j=1}^{s} D_{y_j}(S_jF)$$

Reading Riese (2003) [14],

Let

$\mathbf{K} = \mathbb{C}(q,\tau_1\dots\tau_m)$ denote transcendental extension of complex numbers $\mathbb{C}$ by indeterminates $q,\tau_1\dots\tau_m$ (transcendental extension, roughly speaking, includes elements that aren't roots of polynomials over $\mathbb{C}$),

$\mathbf{k} \equiv (k_1\dots k_r) \in \otimes_{i=1}^{r}\mathbb{Z}$

polynomial in q^n, $q^{\mathbf{k}}$ over $\mathbf{K}$, $P(n,\mathbf{k}) \in \mathbf{K}[q^n,q^{\mathbf{k}}]$ if $\exists\, P^* \in \mathbf{K}[x_0,x_1\dots x_r]$ s.t. $P(n,\mathbf{K}) = P^*(q^n,q^{k_1}\dots q^{k_r})$

$R(n,\mathbf{k})$ is a rational function in $q^n, q^{\mathbf{k}}$ over $\mathbf{K}$, $R(n,\mathbf{k}) \in \mathbf{K}(q^n,q^{\mathbf{k}})$, if $\exists$ rational function $R^* \in \mathbf{K}(x_0,x_1\dots x_r)$ s.t.

$$R(n,\mathbf{k}) = R^*(q^n,q^{k_1}\dots q^{k_r})$$

Definition 25. *q-hypergeometric function $F(n,\mathbf{k})$ in $n,\mathbf{k}$ over $\mathbf{K}$ if*

$$\frac{F(n+1,\mathbf{k})}{F(n,\mathbf{k})}, \frac{F(n,k_1\dots k_i+1\dots k_r)}{F(n,\mathbf{k})} \in \mathbf{K}(q^n,q^{\mathbf{k}})$$

Definition 26. *$\mathbf{k}$-free recurrence, if $\exists$ finite set S, $S=\{s|s\in\otimes i = 1^{r+1}\mathbb{Z}\}$, if $\exists\,\sigma_{i,\mathbf{j}}(n)\in\mathbf{K}[q^n]$ s.t.*

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)F(n-i,\mathbf{k}-\mathbf{j}) = 0 \quad \forall\,(n,\mathbf{k})$$

s.t. $F(n-i,\mathbf{k}-\mathbf{j})$, $\forall\,(i,\mathbf{j})\in S$ well-defined.

S is called a structure set.

Thus

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)F(n-i,\mathbf{k}-\mathbf{j}) = 0 = \tag{8}$$

$$\left(\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)N^{-i}\mathbf{K}^{-\mathbf{j}}\right)F(n,\mathbf{k}) = 0 \xrightarrow{1/F} \tag{9}$$

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)R_{F,i,\mathbf{j}}(n,\mathbf{k}) = 0 \tag{10}$$

Multiplying by denominators on both sides,

$$\sum_{(i,\mathbf{j})\in S} \sigma_{i,\mathbf{j}}(n)P_{F,i,\mathbf{j}}(n,\mathbf{k}) = 0 \tag{11}$$

Definition 27. *Let $F(n,\mathbf{k})$ be q-proper hypergeometric, S structure set.*

for fixed s, $(I,\mathbf{J})\in S$ is numerator boundary pt. $\equiv (I_s^{num},\mathbf{J}_s^{num})$ if $Ia_s+\mathbf{J}\mathbf{b}_s \geq ia_s+\mathbf{j}\mathbf{b}_s \quad \forall\,(i,\mathbf{j})\in S$

cf. pp. 3 Eq. (1.2.15) of Gasper and Rahman (2004) [15] and pp. 4 of Riese (2003) [14]:

Definition 28 (q-shifted factorial or q-Pochhammer symbol)**.**

$$(a;q)_k = \begin{cases} 1 & \text{if } k=0 \\ (1-a)(1-aq)\dots(1-aq^{k-1}) & \text{if } k>0 \\ \frac{1}{(1-aq^{-1})(1-aq^{-2})\dots(1-aq^{-k})} = \frac{1}{(aq^{-k};q)_k} = \frac{(-q/a)^kq^{\binom{k}{2}}}{(q/a;q)_k} & \text{if } k<0 \end{cases} \tag{12}$$

and define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{13}$$

Notice that

$$\frac{(a;q)_\infty}{(q^ka;q)_\infty} = (a;q)_k \tag{14}$$

From Exercise 1.2 on pp. 24 of Gasper and Rahman (2004) [15] and pp. 4 of Riese (2003) [14],

Definition 29 (q-binomial coefficient or Gaussian polynomials)**.**

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0\leq k\leq n \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Notice that this definition is exactly the same as the one given by Wikipedia, which is the definition used by Sage Math, and there, a different notation is used, $\binom{n}{k}_q$ i.e. $\binom{n}{k}_q \equiv \begin{bmatrix} n \\ k \end{bmatrix}_q$:

$$\binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-k+1})}{(1-q)(1-q^2)\dots(1-q^k)} = \frac{(q;q)_n}{(q;q)_{n-k}}\frac{1}{(q;q)_k}$$

7.2. **Reverse Engineering Caruso's Zeilberger**[17]**.**

Part 6. **Hypergeometric summations and recursion**

Maxima (Lisp) and Pari/GP (C,C++)

[16]

indefinite summation of rational functions using Gosper's algorithm (by Ralf Stephan). `http://pari.math.u-bordeaux.fr/Scripts/gsum1.gp`

And so Gosper's algorithm had been implemented in Pari/GP.

Part 7. Jones Polynomials and Khovanov Homology

8. Jones Polynomials from SnapPy

One should have installed successfully SnapPy and its implementation in Sage Math (cf. 10.1).

Then, one can obtain the Jones polynomial for various torus knots in such a manner - in Sage Math:

```
sage: import snappy
sage: T0203snap = snappy.Link('T(2,3)')
sage: T0203snap # obtain the number of crossings as a print out!
<Link: 1 comp; 3 cross>
sage: T0203snap.crossings # obtain the crossings as a list!
[0, 1, 2]
sage: len(T0203snap.crossings) # this is another way to obtain the number of crossings
3
```

If one does `dir(T0203)`, then as one can see, there are a number of modules that one can play with. For instance, the signature and morse number can be obtained quickly:

```
sage: T0203snap.signature()
2
sage: T0203snap.morse_number()
2
```

and also the Jones polynomial can be obtained quickly, after declaring the torus knot with `Link`:

```
sage: T0203snap.jones_poly()
-q^4 + q^3 + q
```

Let's obtain all the Jones polynomials for torus knots with less than 14 crossings (arbitrary choice of 14) with SnapPy, and keep in mind that you can use `latex()` command in Sage Math to immediate get the output in a LaTeX friendly print out:

K	J(K;q)
$T(2,3)$	$-q^4+q^3+q$
$T(2,5)$	$-q^7+q^6-q^5+q^4+q^2$
$T(2,7)$	$-q^{10}+q^9-q^8+q^7-q^6+q^5+q^3$
$T(2,9)$	$-q^{13}+q^{12}-q^{11}+q^{10}-q^9+q^8-q^7+q^6+q^4$
$T(2,11)$	$-q^{16}+q^{15}-q^{14}+q^{13}-q^{12}+q^{11}-q^{10}+q^9-q^8+q^7+q^5$
$T(2,13)$	$-q^{19}+q^{18}-q^{17}+q^{16}-q^{15}+q^{14}-q^{13}+q^{12}-q^{11}+q^{10}-q^9+q^8+q^6$
$T(3,4)$	$-q^8+q^5+q^3$
$T(3,5)$	$-q^{10}+q^6+q^4$

Look at `Jonespoly_and_kh_sage.py` for the Python dictionary `TKNOTS14SNAP` for all the Torus knots with less than 14 crossings, and then one can do this to obtain the Jones polynomial:

```
sage: TKNOTS14SNAP[(2,3)].jones_poly()
-q^4 + q^3 + q
```

9. Khovanov Homology implementations: KhoHo and knotkit

9.1. KhoHo. First, install Pari/GP. I recommend using *Homebrew*: `brew install pari`

`KhoHo` was unavailable on the website http://www.geometrie.ch/KhoHo/ and doesn't appear to be on `github`. Dunfield provided `KhoHo-0.9.3.5` by email which I'll try to put on github in the `qsApoly` repository. However, I tried to follow the instructions for KhoHo in the `00README` file, skipping the 'make' command since library files `nicematr.so, print_ranks.so, sparreduce.so` were already there (I tried removing these 3 library files and running make, but 20 errors were generated: see my wordpress blog for the full printout of the errors), but in Pari/GP, by typing `gp` to start up PARI's programmable calculator, and then in `gp`, typing in the command to read KhoHo, this occurred:

```
? read(KH)
  ***   at top-level: read(KH)
  ***                 ^--------
  ***   in function read: read(KhoHo)
  ***                     ^-----------
  ***   in function read: read(KhoHo_basic)
  ***                     ^-----------------
  ***   in function read: kill(print_ranks)
  ***                     ^-----------------
  *** kill: can't kill that.
  ***   Break loop: type 'break' to go back to GP prompt
break> break
```

I also tried `gp > read(KH)` and obtained an error for the break loop.

Please contact me or let people know what's wrong in this case with KhoHo, or how to get it to run.

9.2. knotkit.

9.2.1. *Installation of knotkit.* I suggest doing a `git clone` of `knotkit` from its github repository:

`git clone https://github.com/cseed/knotkit.git`

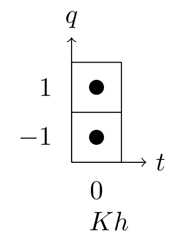
and typing the command 'make' (no need to type './' before, and it should be available to run everywhere on the system) to install and make the executable `kk`.

Use `./kk -h` for the help string.

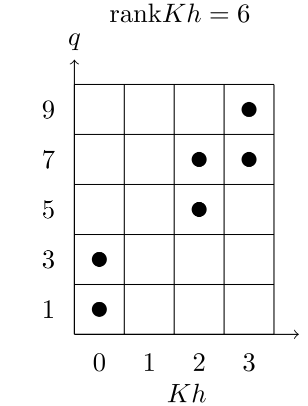
However, if one opens up the code for `kk.cpp` and also run the executable, `kk`, for Khovanov homology (kh) calculations, then the output is in LaTeX; the display is nice, but we want a nice string format so to do manipulations on it. Dunfield provided an alternative version of `kk.cpp` with the option of "`khplainout`" which outputs a string with the categorified Jones polynomial.

9.2.2. *Khovanov Homology computations from knotkit.* Knotkit or `kk`, with the "kh" flag outputs the following, for various torus knots of less than 14 crossings, the graph of the exponents for variables (t,q) as dots:

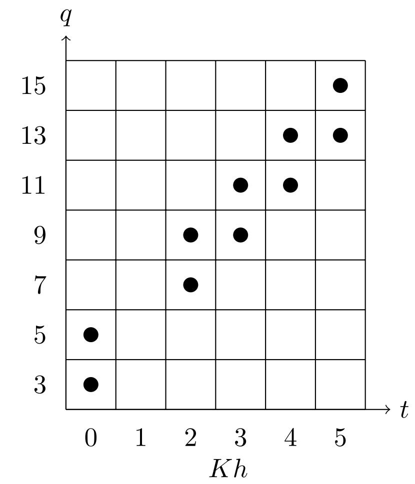
Kh = Kh(`T(2,1)`; `Z2`):
rank$Kh = 2$



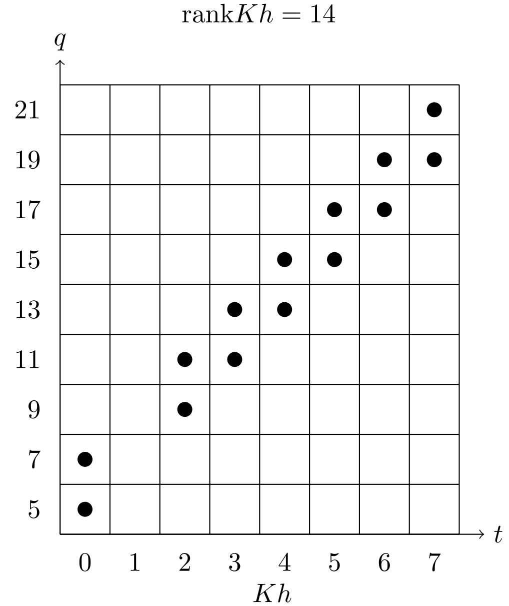
Kh = Kh(`T(2,3)`; `Z2`):
rank$Kh = 6$



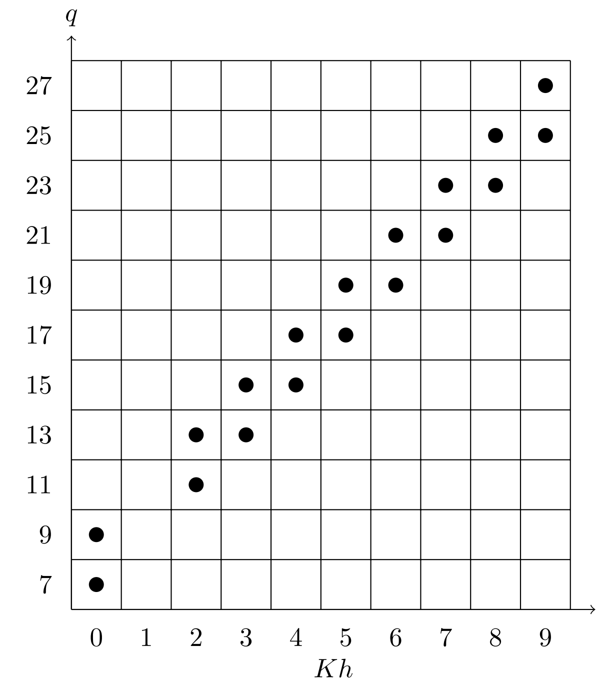
Kh = Kh(`T(2,5)`; `Z2`):
rank$Kh = 10$



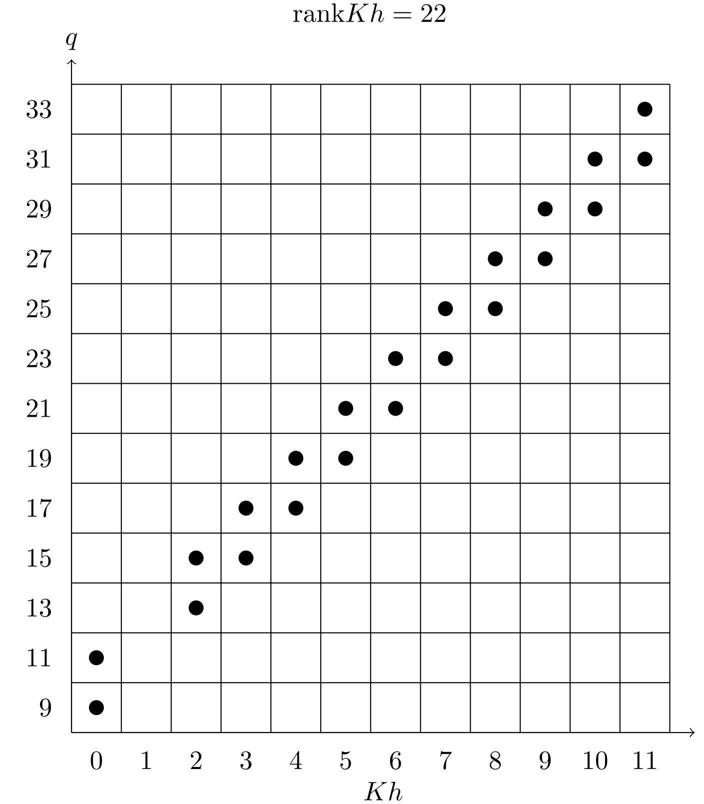
Kh = $Kh(\texttt{T(2,7)};\texttt{Z2})$:
rank$Kh = 14$



Kh = $Kh(\texttt{T(2,9)};\texttt{Z2})$:
rank$Kh = 18$

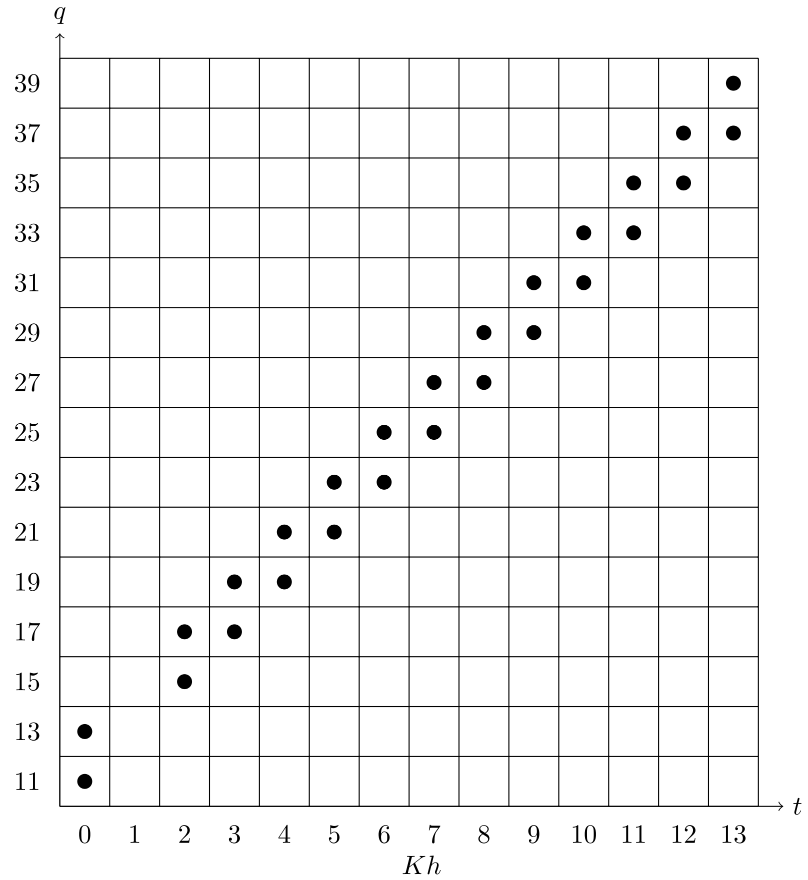


Kh = $Kh(\texttt{T(2,11)};\texttt{Z2})$:
rank$Kh = 22$

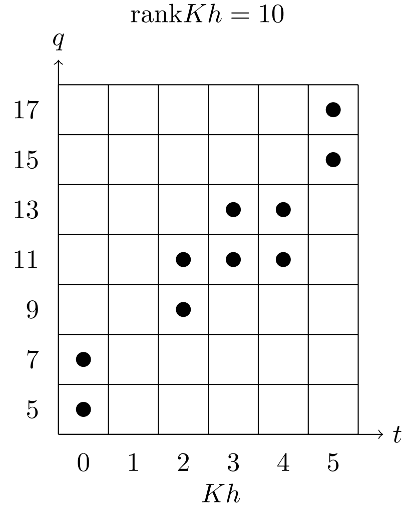


Kh = $Kh(\texttt{T(2,13)};\texttt{Z2})$:

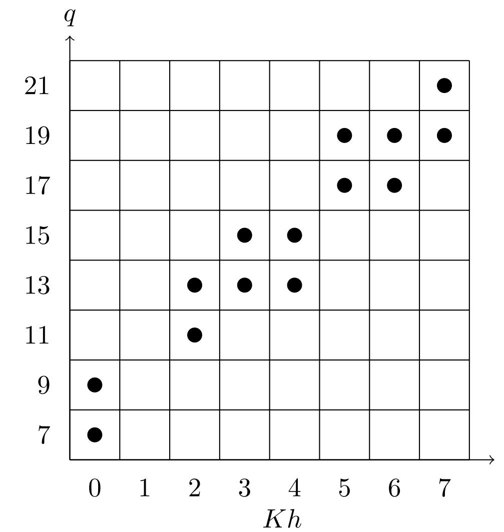
rank$Kh = 26$



Kh $= Kh($`T(3,4);Z2`$)$:
rank$Kh = 10$



Kh $= Kh($`T(3,5);Z2`$)$:
rank$Kh = 14$



If you wanted to *save the output* from option flag "khplainout", then one simple and direct manner is to "pipe" the screen output with the symbol ">" at the Command prompt and specify the target file you'd want to save the output. For example, for the Torus knot $T(2,5)$, saving into the file `kh_T0205` in the directory above the current working directory,

```
./kk khplainout -v ''T(2,5)'' > ../kh_T0205
```

I've done this for all torus knots of less than 14 crossings and saved the output and placed them in the github repository `qsApoly` so others can have access to the Khovanov homology computation outputs readily; they are named files `kh_T0203`, `kh_T0205`, ... in `kk_kh_output`.

If you don't want to do this one-by-one, but a batch of these, one at a time, see the Python script `kk_kh_batch.py`, which you can run directly by typing at the command prompt `python kk_kh_batch.py` or manually, by opening up an interactive shell (i.e. `python -i kk_kh_batch.py`) and then running manually the Python function `direct_output`.

To "clean" or "scrape" the text (strings) of "categorified" Jones polynomial, to get it ready to use for `Sage Math`, I provide the file `kh_scrape.py`. It contains the function `scrape_file` that will take a file containing a *single* categorified Jones polynomial and output a "cleaned up" text (string) for the polynomial, in a Python dictionary, that can be used in Sage Math (just then apply the Sage Math function `sage_eval`). For example:

```
from kh_scrape import scrape_file
Qkh.<t,q> = PolynomialRing(RationalField(),2)
T0203stf = scrape_file('kh_T0203')
T0203poly = sage_eval( T0203stf['poly'], locals={'x1':t,'x2':q} )
T0203poly.substitute(t=-1) # for some reason, subs doesn't work in this case
```

Also, from running `kh_scrape.py` and function `scrape_bat` in *Sage Math* (make sure you're in the appropriate working directory)

```
sage: from kh_scrape import scrape_bat
sage: Tknots14 = scrape_bat('khTknotsless14')
sage: Tknots14sage = [ sage_eval(line, locals={'x1':t,'x2':q}) for line in Tknots14 ]
sage: for K in Tknots14sage:
....:           print latex(K)
```

Then you'll have all the torus knots categorified Jones polynomials from Khovanov Homology in a single list "Tknots14sage" (but you'll have to manually keep track of which entry corresponds to which torus knot).

The result of printing with `latex` command in the last command immediately above is this table of "categorified" Jones polynomials from Khovanov Homology (source:`knotkit`)

K	J(K;t,q)
$T(2,1)$	$\frac{q^2+1}{q}$
$T(2,3)$	$t^3q^9+t^3q^7+t^2q^7+t^2q^5+q^3+q$
$T(2,5)$	$t^5q^{15}+t^5q^{13}+t^4q^{13}+t^4q^{11}+t^3q^{11}+t^3q^9+t^2q^9+t^2q^7+q^5+q^3$
$T(2,7)$	$t^7q^{21}+t^7q^{19}+t^6q^{19}+t^6q^{17}+t^5q^{17}+t^5q^{15}+t^4q^{15}+t^4q^{13}+t^3q^{13}+t^3q^{11}+t^2q^{11}+t^2q^9+q^7+q^5$
$T(2,9)$	$t^9q^{27}+t^9q^{25}+t^8q^{25}+t^8q^{23}+t^7q^{23}+t^7q^{21}+t^6q^{21}+t^6q^{19}+t^5q^{19}+t^5q^{17}+t^4q^{17}+t^4q^{15}+t^3q^{15}+t^3q^{13}+t^2q^{13}+t^2q^{11}+q^9+q^7$
$T(2,11)$	$t^{11}q^{33}+t^{11}q^{31}+t^{10}q^{31}+t^{10}q^{29}+t^9q^{29}+t^9q^{27}+t^8q^{27}+t^8q^{25}+t^7q^{25}+t^7q^{23}+t^6q^{23}+t^6q^{21}+t^5q^{21}+t^5q^{19}+t^4q^{19}+t^4q^{17}+t^3q^{17}+t^3q^{15}+t^2q^{15}+t^2q^{13}+q^{11}+q^9$
$T(2,13)$	$t^{13}q^{39}+t^{13}q^{37}+t^{12}q^{37}+t^{12}q^{35}+t^{11}q^{35}+t^{11}q^{33}+t^{10}q^{33}+t^{10}q^{31}+t^9q^{31}+t^9q^{29}+t^8q^{29}+t^8q^{27}+t^7q^{27}+t^7q^{25}+t^6q^{25}+t^6q^{23}+t^5q^{23}+t^5q^{21}+t^4q^{21}+t^4q^{19}+t^3q^{19}+t^3q^{17}+t^2q^{17}+t^2q^{15}+q^{13}+q^{11}$
$T(3,4)$	$t^5q^{17}+t^5q^{15}+t^4q^{13}+t^3q^{13}+t^4q^{11}+t^3q^{11}+t^2q^{11}+t^2q^9+q^7+q^5$
$T(3,5)$	$t^7q^{21}+t^7q^{19}+t^6q^{19}+t^5q^{19}+t^6q^{17}+t^5q^{17}+t^4q^{15}+t^3q^{15}+t^4q^{13}+t^3q^{13}+t^2q^{13}+t^2q^{11}+q^9+q^7$

The miracle is that if one sets $t=-1$ in the above categorified Jones polynomials, $J(K;t,q)$, then $J(K;t=-1,q)=J(K,q)$, where $J(K,q)$ are the Jones polynomials (modulo conventions), from Table 8!

To show this, keep in mind that for the categorified Jones polynomials, one must *normalize* to the "unknot" which in this case is the $T(2,1)$ torus knot, after setting $t=-1$ (or before? that's my question; please let me know what "dividing" by the unknot means in both cases):

```
sage: normTknots14sage = [K/Tknots14sage[0] for K in Tknots14sage]

# Compare Jones polynomials from SnaPy to Jones polynomials from decategorified Khovanov Homology:
sage: x = var('x')
sage: for i in range(1,len(TKNOTS14SNAP)):
          SnaPyvskh = TKNOTS14SNAP[TKNOTSLIST14[i]].jones_poly().subs(q=x) ==
                      normTknots14sage[i].subs(t=-1).subs(q=sqrt(x))
          print bool( SnaPyvskh )

sage: print "If all True, then J(K;t=-1,q) = J(K;q)!"
```

and indeed it does.

Note that for some reason, the Jones polynomial for $T(2,1)$ of *SnaPy* produces this error:

```
sage: TKNOTS14SNAP[(2,1)].jones_poly()
# stuff
IndexError: list index out of range
```

EY: 20160224: My immediate question is this: what's the convention or normalization that results in SnaPy outputting the Jones polynomial for the torus knots, say $T(2,3)$ trefoil knot, to be

$$-q^4+q^3+q$$

vs. the Jones polynomial that results from Khovanov homology, after setting $t=-1$, for the trefoil:

$$-q^8+q^6+q^2$$

The latter expression is found on pp. 339 of Bar-Natan's (nicely, pedagogically-friendly) review artcle [18]

10. Exploring the Jones Polynomial and Khovanov Homology: Unanswered Avenues

Armed with our categorified Jones polynomials and Sage Math, there are a number of modules (functions) that can be explored, as seen if one does the `dir()` command on a polynomial (e.g. `dir(Tknots14sage[1])`).

For instance, consider

```
sage: Tknots14sage[1].gradient() # T(2,3)
[3*t^2*q^9 + 3*t^2*q^7 + 2*t*q^7 + 2*t*q^5,
  9*t^3*q^8 + 7*t^3*q^6 + 7*t^2*q^6 + 5*t^2*q^4 + 3*q^2 + 1]
```

which is $\frac{\partial}{\partial t}J(T(2,3);t,q)$ and $\frac{\partial}{\partial q}J(T(2,3);t,q)$. Are there any relationships we can discover between the categorified Jones polynomial and its partial(s) (derivatives)?

One can also use the Sage Math module `newton_polytope` to obtain the Newton Polytope immediately. One does the Newton Polytope tell us about categorified Jones polynomials?

Also, there are many Sage Math modules associated with `newton_polytope` (i.e. do e.g. `dir(Tknots14sage[1].newton_polytope())`) such as `face_lattice`:

```
sage: Tknots14sage[1].newton_polytope().face_lattice().list()
[<>, <0>, <1>, <2>, <3>, <1,2>, <0,1>, <0,3>, <2,3>, <0,1,2,3>]
sage: Tknots14sage[7].newton_polytope().face_lattice().list()
[<>, <0>, <1>, <2>, <3>, <4>, <1,2>, <0,1>, <0,4>, <3,4>, <2,3>, <0,1,2,3,4>]
```

So the face lattice of a $T(2,2j+1)$ torus knot, $j=1\dots6$, is different from the family of torus knots $T(3,4)$ and $T(3,5)$. What does that mean?

This reference page might help with polytopes: A class to keep information about faces of a polyhedron

Something else one could try to do is to repeat Witten's celebrated computation of the Jones polynomial from Chern-Simons theory [19] in *Sage Math*. This would entail an understanding, a grasp, of the Weyl group, in this case, $SU(2)$. I tried looking up topics on the Weyl group and Weyl character in Sage Math (cf. Weyl Group, SL versus GL)

10.1. Installation of SnapPy.

10.1.1. *Mac OS X Installation of SnapPy.* First, one should simply download SnapPy.dmg, and then double-click the .dmg file and then drag-and-drog the SnapPy icon into the Applications Folder [6].

However, one would like to take advantage of its integration with `Sage Math` and so here's how to install SnapPy into `Sage`.

(1) Go to or `cd` into the directory where the main program `sage` is in; for example, the directory that `sage`, the *executable file* is in, is

    `/Applications/SageMath-6.10.app/Contents/Resources/sage`

    where `sage` here is a *directory*.
(2) Make sure you have `pip` installed and do this command:

    `./sage -pip install --no-use-wheel snappy`

    cf. `http://www.math.uic.edu/t3m/SnapPy/installing.html#sage`. It should successfully install.

Part 8. BPS Spectra

Gukov, Nawata, Saberi, Stosic, Sułkowski (2015) [12]

[6] cf. Installing SnapPy

Part 9. Cold Neutrons and Topological Knots

11. Curvature

Consider a principal-G bundle with Lie group G, $P \xrightarrow{\pi} M$. Note that an associated bundle, a vector bundle, can be constructed from principal G-bundle P, through representation $\rho : G \to Gl(n;\mathbb{K})$ (cf. 10.9 Associated vector bundles of Taubes (2011) [6]), in that

$$\begin{tikzcd} P \arrow[d, "\pi"'] & \\ M \arrow[r, "\rho:G\to \mathrm{Gl}(n;\mathbb{K})"] & P\times_\rho \mathbb{K}^n \equiv P\times \mathbb{K}^n/(p,v)\sim (pg^{-1},\rho(g)v) \quad \forall g \in G \end{tikzcd}$$

for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathbb{K}^n$ being a vector space of dimension n over field $\mathbb{K}$.

Recall the exterior covariant derivative D s.t.

$$\begin{gathered} D : \Omega^p(M;E) \to \Omega^{p+1}(M;E) \\ D(\theta\otimes s) = d\theta\otimes s + (-1)^p\theta\wedge\nabla s = d\theta\otimes s + \nabla s\wedge\theta \end{gathered}$$

with $E \xrightarrow{\pi} M$ being a vector bundle (from which one can construct the principal G bundle, if so desired).

Proposition 10. *For exterior covariant derivative $D : \Omega^p(M;E) \to \Omega^{p+1}(M;E)$, $\forall\,\eta\in\Omega^p(M;E)$,*

$$D^2\eta \equiv D\circ D\eta = F\wedge\eta$$

where $F\in\Omega^p(M;End(E))$, and F unique

Proof. $\forall\,\eta\in\Omega^p(M;E)$, of the form $\eta = \theta\otimes s$, where $\theta\in\Omega^p(M)$, $s\in\Gamma(E)$,

$$\begin{aligned} D\eta = d\theta\otimes s + (-1)^p\theta\wedge\nabla s = d\theta\otimes s + (-1)^p\theta\wedge(ds + \omega^k{}_i s^i e_k) = d\theta\otimes s + ds\wedge\theta + \omega^k{}_i s^i\wedge\theta\otimes e_k = \\ = (s^k d\theta + ds^k\wedge\theta + \omega^k{}_i s^i\wedge\theta)\otimes e_k \end{aligned}$$

$$\begin{aligned} D\circ D\eta \equiv DD\eta = (ds^k\wedge d\theta + (-1)ds^k\wedge d\theta + ds^i\wedge\omega^k{}_i\wedge\theta + s^i d\omega^k{}_i\wedge\theta + (-1)\omega^k{}_i s^i\wedge d\theta)\otimes e_k + \\ +(-1)^{p+1}(s^k d\theta + ds^k\wedge\theta + \omega^k{}_i s^i\wedge\theta)\otimes\wedge\omega^l{}_k e_l = \\ = (ds^i\wedge\omega^l{}_i\wedge\theta + s^i d\omega^l{}_i\wedge\theta + (-1)\omega^l{}_i s^i\wedge d\theta)\otimes e_l + \\ +(s^k\omega^l{}_k\wedge d\theta + \omega^l{}_k\wedge ds^k\wedge\theta + \omega^l{}_k\wedge\omega^k{}_i s^i\wedge\theta)e_l = \\ = (d\omega^l{}_i + \omega^l{}_k\wedge\omega^k{}_i)s^i\wedge\theta e_l \end{aligned}$$

If you're following at home (i.e. independent study), one only needs to be careful with factors of (-1) when "commuting through" the wedge product $\wedge$.

I (still) find it a near miracle that terms cancel such that F takes this form (with, simply a change of notation, $\omega\equiv A$):

$$F = dA + A\wedge A \in \Omega^p(M;\mathrm{End}E)$$

By Lemma 11.1 of Sec. 11.2 the space of covariant derivatives of Taubes (2011) [6], this F is *unique*.

Thus

$$D^2\eta = F\wedge\eta$$

for, notice that for, locally (in components)

$$A = A^k{}_{ij}dx^j\otimes(e_k\otimes e^i)$$

$$\begin{aligned} F = dA + A\wedge A = \frac{\partial A^k{}_{lj}}{\partial x^i}dx^i\wedge dx^j\otimes(e_k\otimes e^l) + A^k{}_{mi}A^m{}_{lj}dx^i\wedge dx^j\otimes(e_k\otimes e^l) = \\ = \left(\frac{\partial A^k{}_{lj}}{\partial x^i} + A^k{}_{mi}A^m{}_{lj}\right)dx^i\wedge dx^j\otimes(e_k\otimes e^l) \end{aligned}$$

and so

$$F\wedge\eta = \left(\frac{\partial A^k{}_{lj}}{\partial x^i} + A^k{}_{mi}A^m{}_{lj}\right)dx^i\wedge dx^j\wedge\theta\otimes e_k s^l$$

□

11.0.2. *Alternative form of curvature F in terms of commutators.* cf. Subsection 12.6 Curvature and commutators of Taubes (2011) [6].

Consider $\forall\,U,V\in\mathfrak{X}(M)$, $X\in\Gamma(E)$,

$$\nabla_U X = U^j\left(\frac{\partial X}{\partial x^j} + A^k{}_{ij}X^i\right) = U^j\left(\frac{\partial}{\partial x^j} + A_j\right)X \in\Gamma(E)$$

and so clearly

$$\nabla_U\in\Gamma(\mathrm{End}(E))$$

Also recall the commutator for vector fields, in component form (locally):

$$[U,V] = \left(U^i\frac{\partial}{\partial x^i}V^j - V^i\frac{\partial}{\partial x^i}U^j\right)\frac{\partial}{\partial x^j}\in\mathfrak{X}(M)$$

and so

$$\nabla_{[U,V]} = \left(U^i\frac{\partial}{\partial x^i}V^j - V^i\frac{\partial}{\partial x^i}U^j\right)\left(\frac{\partial}{\partial x^j} + A_j\right)$$

Consider that

$$\begin{gathered} \nabla_U\nabla_V = \\ = U^i\left[\left(\frac{\partial}{\partial x^i} + A_i\right)V^j\left(\frac{\partial}{\partial x^j} + A_j\right)\right] = \\ = U^i\left[\frac{\partial V^j}{\partial x^i}\left(\frac{\partial}{\partial x^j} + A_j\right) + V^j\left(\frac{\partial^2}{\partial x^i\partial x^j} + \frac{\partial A_j}{\partial x^i} + A_j\frac{\partial}{\partial x^i}\right) + A_iV^j\frac{\partial}{\partial x^j} + A_iV^jA_j\right] \end{gathered}$$

Then by canceling out matching terms,

$$\begin{gathered} [\nabla_U,\nabla_V] - \nabla_{[U,V]} = U^iV^j\frac{\partial A_j}{\partial x^i} - V^iU^j\frac{\partial A_j}{\partial x^i} + U^iV^jA_iA_j - V^iU^jA_iA_j = \\ = \left(\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) + [A_i,A_j]\right)U^iV^j = F(U,V) \end{gathered}$$

and so we have this form for the curvature $F(U,V)\in\Gamma(\mathrm{End}(E)),\ \forall\,U,V\in\mathfrak{X}(M)$,

$$F(U,V) = \left(\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) + [A_i,A_j]\right)U^iV^j$$

but I think that one should keep in mind that this is just one form that F could take, if it is applied to U,V beforehand.

11.0.3. *deRham cohomology.* I'm going to now follow Section 12.2 Closed forms, exact forms, diffeomorphisms and De Rham cohomology of Taubes (2011) [6].

Recall the definition of *deRham cohomology*:

$$H^p_{\text{deRham}}(M) := \ker d/\text{im} d \qquad \left(= \{\omega | d\omega = 0\}/\{\theta|\theta = d\alpha \text{ for } \alpha \in \Omega^{p-1}(M)\}\right) \tag{16}$$

If M,N smooth manifolds, smooth map $f : M \to N$, $\forall\, \alpha \in \Omega^{p-1}(N)$, then

$$f^*(d\alpha) = d(f^*\alpha) \text{ or } f^*d = df^* \tag{17}$$

$$\begin{tikzcd}
\Omega^p(M) & \Omega^p(N) \arrow[l, "f^*"'] \\
\Omega^{p-1}(M) \arrow[u, "d"] & \Omega^{p-1}(N) \arrow[l, "f^*"'] \arrow[u, "d"'] \\
M \arrow[u] \arrow[r, "f"] & N \arrow[u]
\end{tikzcd}$$

i.e.

Proof. Indeed, this can be shown, by considering local expressions: locally, $\alpha_I dy^I \in \Omega^{p-1}_y(N)$ where $I \equiv (i_1, i_2 \dots i_{p-1})$ s.t. $i_1 < i_2 < \dots < i_{p-1}$, and consider, with $f(x) = y$:

$$d\alpha = \frac{\partial \alpha_I}{\partial y^i} dy^i \wedge dy^I = \frac{\partial \alpha_I}{\partial y^i} \epsilon^{iI}_J dy^J \text{ since there's only 1 way to permute } iI \text{ into } J = (j_1 \dots j_p) \text{ s.t. } j_1 < \dots < j_p$$
$$f^* d\alpha = \frac{\partial \alpha_I}{\partial y^i} \epsilon^{iI}_J \frac{\partial y^J}{\partial x^k} dx^k$$
$$f^*\alpha = \alpha_I \frac{\partial y^I}{\partial x^J} dx^J = \frac{\partial \alpha_I}{\partial y^i}\frac{\partial y^i}{\partial x^j}\frac{\partial y^I}{\partial x^J}\epsilon^{jJ}_K dx^K$$

Now

$$df^*\alpha = \left(\frac{\partial \alpha_I}{\partial x^i}\frac{\partial y^I}{\partial x^j} + \alpha_I \frac{\partial^2 y^I}{\partial x^i \partial x^J}\right) dx^i \wedge dx^J = \left(\frac{\partial \alpha_I}{\partial x^i}\frac{\partial y^I}{\partial x^J} + \alpha_I \frac{\partial^2 y^I}{\partial x^i \partial x^J}\right)\epsilon^{iJ}_K dx^K =$$
$$= \frac{\partial \alpha_I}{\partial y^i}\frac{\partial y^i}{\partial x^j}\frac{\partial y^I}{\partial x^J}\epsilon^{jJ}_K dx^K + \alpha_I \frac{\partial^2 y^J}{\partial x^i \partial x^J}\epsilon^{iJ}_K dx^K = \frac{\partial \alpha_I}{\partial y^i}\frac{\partial y^i}{\partial x^j}\frac{\partial y^I}{\partial x^J}\epsilon^{jJ}_K dx^K + 0 = f^*d\alpha$$

□

Consider this homotopy: for smooth maps $\begin{matrix} f_0 : M \to N \\ f_1 : M \to N \end{matrix}$, $\exists$ smooth map $\begin{matrix} \psi : [0,1]\times M \to N \\ \psi(0,\cdot) = f_0 \\ \psi(1,\cdot) = f_1 \end{matrix}$

Let closed form $\omega \in \Omega^p(N)$; $d\omega = 0$. Then $f_0^*\omega$, $f_1^*\omega$ closed form.
Now consider $\mathbb{R}\times M$, and that

$$\begin{tikzcd}
T^*(\mathbb{R}\times M) = \mathbb{R}\oplus T^*M & \alpha = \alpha_0 dt + \alpha_M = \alpha_0 dt + (\alpha_M)_i dx^i \\
\mathbb{R}\times M \arrow[u] & (t,x) \arrow[u]
\end{tikzcd}$$

in that

in that i runs through the indices for (some) local chart of M *only*, i.e. $i = 1, 2, \dots \dim M = d$.

Likewise, $\Omega^p(\mathbb{R}\times M) = \Omega^{p-1}(M) \oplus \Omega^p(M)$, in that

$$\forall\, \alpha \in \Omega^p(\mathbb{R}\times M) \text{ then for } \mu = 0,1,2,\dots d, \; 0 \text{ standing in for } t \in \mathbb{R} \text{ of } \mathbb{R}\times M,$$

$$\begin{aligned} & & M &= (\mu_1 \dots \mu_p) & \mu_\mu &= 0,1\dots d & \mu_1 &< \dots < \mu_p \\ \alpha = \alpha_M dx^M &= dt\wedge \alpha_I dx^I + \alpha_J dx^J \text{ where} & I &= (i_1 \dots i_{p-1}) & i_i &= 1\dots d & i_1 &< \dots < i_{p-1} \\ & & J &= (j_1 \dots j_p) & j_j &= 1 \dots d & j_1 &< \dots < j_p \end{aligned}$$

and so, naming these components of α as

$$\alpha^{p-1} \equiv \alpha_I dx^I \in \Omega^{p-1}(M)$$
$$\alpha^p \equiv \alpha_J dx^J \in \Omega^p(M)$$

Then $\forall\, \alpha \in \Omega^p(\mathbb{R}\times M)$,

$$\alpha = dt \wedge \alpha^{p-1} + \alpha^p \tag{18}$$

. Taking d on both sides to obtain $d\alpha \in \Omega^{p+1}(\mathbb{R}\times M)$, and $d\alpha$, being a $p+1$-form, taking the form of Eq. 18, then

$$d\alpha = dt \wedge (d\alpha)^p + (d\alpha)^{p+1} = -dt\wedge d^\perp \alpha^{p-1} + \frac{\partial \alpha^p_J}{\partial t} dt \wedge dx^J + d^\perp \alpha^p$$

where $d\alpha^p = \frac{\partial \alpha^p_J}{\partial x^\mu} dx^\mu \wedge dx^J = \frac{\partial \alpha^p_J}{\partial t} dt\wedge dx^J + \frac{\partial \alpha^p_J}{\partial x^i} dx^i \wedge dx^J = \frac{\partial \alpha^p_J}{\partial t} dt \wedge dx^J + d^\perp\alpha^p$, and so $d^\perp$ signifies that this exterior derivative only "acts" on the (local) coordinates of M.

Thus

$$(d\alpha)^p = -d^\perp \alpha^{p-1} + \frac{\partial \alpha^p}{\partial t}$$
$$(d\alpha)^{p+1} = d^\perp \alpha^p$$

Suppose $\alpha = \psi^*\omega$; $\omega \in \Omega^p(N)$; $\psi : [0,1]\times M \to N$.
If ω closed ($d\omega = 0$), then $\psi^*\omega$ closed ($d\psi^*\omega = \psi^* d\omega = 0$).

So using the above facts shown for $\alpha = \psi^*\omega$,

$$\alpha = dt\wedge \alpha^{p-1} + \alpha^p \xrightarrow{\alpha = \psi^*\omega} \psi^*\omega = dt \wedge (\psi^*\omega)^{p-1} + (\psi^*\omega)^p$$
$$d\psi^*\omega = \psi^* d\omega = 0 = dt \wedge (d\psi^*\omega)^p + (d\psi^*\omega)^{p+1}$$
$$(d\psi^*\omega)^{p+1} = 0 = d^\perp (\psi^*\omega)^p$$
$$(d\psi^*\omega)^p = 0 = -d^\perp (\psi^*\omega)^{p-1} + \frac{\partial (\psi^*\omega)^p}{\partial t} \xrightarrow{\int_0^1 dt} (\psi^*\omega)^p|_{t=1} - (\psi^*\omega)^p|_{t=0} = d^\perp \int (\psi^*\omega)^{p-1} \text{ or}$$
$$f_1^*\omega - f_0^*\omega = d^\perp \int (\psi^*\omega)^{p-1}$$

So $f_1^*\omega$ differ from $f_0^*\omega$ by an exact form, $d^\perp \int (\psi^*\omega)^{p-1}$.

$$\Longrightarrow [f_1^*\omega] = [f_0^*\omega]$$

Thus deRham cohomology classes are invariant under homotopy (homotopy invariant!).

Consider 1-form connection on principal G-bundle $A = A(x) \in \Omega^1(M; \mathbf{g})$, $\forall\, x \in M$, $\mathfrak{g}$ Lie algebra of G (Recall $\mathbf{g} = T_1 G$).
Consider 1-form connection over $[0,1]\times U$, open $U \subset M$, $A' = A'(t,x)$ in that

$$A' = \mathbf{g}^{-1} d\mathbf{g} + t\mathbf{g}^{-1} A \mathbf{g} = A'(t,x)$$

Note that $\mathbf{g} \in \mathfrak{g}$.

A' interpolates between a flat connection $A'(0,x) = A'|_{t=0} = \mathbf{g}^{-1}d\mathbf{g}$, the connection 1-form for product principal bundle $P = M\times G$ and $A'(1,x) = A'|_{t=1} = \mathbf{g}^{-1}d\mathbf{g} + \mathbf{g}^{-1}A\mathbf{g}$. I think that this could be interpreted as turning off and turning on the gauge field, respectively.

Now, doing the calculation out explicitly,

$$F_{A'} = (d+A')^2 = dA' + A'\wedge A' = d\mathbf{g}^{-1}\wedge d\mathbf{g} + dt\wedge \mathbf{g}^{-1}A\mathbf{g} + t(d\mathbf{g}^{-1}\wedge A\mathbf{g} + \mathbf{g}^{-1}dA\mathbf{g} + \mathbf{g}^{-1}A\wedge d\mathbf{g} + \\ + t(\mathbf{g}^{-1}d\mathbf{g}\wedge \mathbf{g}^{-1}A\mathbf{g} + \mathbf{g}^{-1}A\mathbf{g}\wedge \mathbf{g}^{-1}d\mathbf{g}) + t^2\mathbf{g}^{-1}A\wedge A\mathbf{g}$$

Using this identity:

$$\mathbf{g}^{-1}\mathbf{g} = 1$$
$$\implies d(\mathbf{g}^{-1}\mathbf{g}) = d\mathbf{g}^{-1}\mathbf{g} + \mathbf{g}^{-1}d\mathbf{g} = 0$$

and "commuting" or "moving through" differential forms "through the wedge product", then

$$F_{A'} = tdA + dt\wedge A + t^2 A\wedge A$$

Now consider $\text{tr}(F_{A'}\wedge F_{A'}) \in \Omega^4([0,1]\times U)$.
$\text{tr}(F_{A'}\wedge F_{A'})$ is closed, since $\dim M = 4$.

Now recall that $\forall\, p$-form on $[0,1]\times U$, $\alpha\in\Omega^p([0,1]\times U)$, $\alpha = dt\wedge\alpha^{p-1} + \alpha^p$; with $\begin{aligned} &\alpha^{p-1}\in\Omega^{p-1}(U) \\ &\alpha^p \in \Omega^p(U)\end{aligned}$

Thus, in our case currently,

$$\text{tr}(F_{A'}\wedge F_{A'}) = dt\wedge\alpha^3 + \alpha^4$$

Calculating out $F_{A'}\wedge F_{A'}$ explicitly,

$$F_{A'}\wedge F_{A'} = dt\wedge A\wedge tdA + t^3 A\wedge A\wedge dA + tdA\wedge dt\wedge A + t^2 dt\wedge A\wedge A\wedge A + tdA\wedge A\wedge A + t^2 dt\wedge A\wedge A\wedge A = \\ = 2dt\wedge tA\wedge dA + 2t^2 dt\wedge A\wedge A\wedge A + (t^3+t)A\wedge A\wedge dA$$

and so

$$\alpha^3 = 2\text{tr}(tA\wedge dA + t^2 A\wedge A\wedge A)$$

Since $\text{tr}(F_{A'}\wedge F_{A'})$ closed, $0 = 0 - dt\wedge d\alpha^3 + d\alpha^4$, and "applying" $\frac{\partial}{\partial t}$ to this expression (i.e. this 4-form "acts" on $\frac{\partial}{\partial t}$, then

$$\frac{\partial\alpha^4}{\partial t} = d\alpha^3$$

$$\xrightarrow{\int dt} \int\frac{\partial\alpha^4}{\partial t} = \int d\alpha^3 = \text{tr}(F_{A'(1)}\wedge F_{A'(1)}) - \text{tr}(F_{A'(0)}\wedge F_{A'(0)}) = d\text{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$$

Explicitly,

$$d(\text{tr}(F_{A'}\wedge F_{A'})) = -dt\wedge d\alpha^3 + d\alpha^4$$

$$\xrightarrow{(\frac{\partial}{\partial t},\cdot,\cdot,\cdot)} \frac{\partial(\text{tr}(F_{A'}\wedge F_{A'}))}{\partial t} = -d\alpha^3 \xrightarrow{\int_0^1 dt} \int_0^1 dt\frac{\partial\text{tr}(F_{A'}\wedge F_{A'})}{\partial t} = \int_0^1 dt(-d\alpha^3)$$

and so for $\text{tr}(F\wedge F)\in\Omega^4(M)$, $\text{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\in\Omega^3(M)$

$$\implies \text{tr}(F_A\wedge F_A) \equiv \text{tr}(F\wedge F) = d\text{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$$

For oriented smooth M; $\dim M = 4$

$$\int_M \text{tr}(F\wedge F) = \int_M d\text{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) = \int_{\partial M}\text{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$$

Let us follow Hickerson (2013) [21] to understand the number of approaches and models for ultracold neutrons [21].
Consider the term

$$\mathcal{L}_\theta = \frac{\theta g^2}{8\pi^2}\text{tr}(F\wedge F) = \frac{\theta g^2}{8\pi^2}d\text{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$$

We've seen how

$$\text{tr}(F_A\wedge F_A)\equiv\text{tr}(F\wedge F) = d\text{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \tag{19}$$

is true, without regards to a *metric* g (or i.e. *metric bundle* g over M). In pp. 5 Subsection 1.3 The θ-term of Hickerson (2013) [21], the dual to F was utilized. Let's avoid utilizing "electric-magnetic" dual, or g, or Hodge dual terms in order to think purely "topologically."

11.1. **Geometry; Geometric setup; Manifold setup.** Following Eq. 1.28 on pp.5 of Subsection 1.3 The θ-term of Hickerson (2013) [21], for a 3-cylinder ∂M, he has the following setup:

$$\partial M = \mathbb{D}^3(t^+)\bigcup\mathbb{D}^3(t^-)\bigcup\mathbb{S}^2\times\mathbb{R}$$

Let's count dimensions.

$$\dim\partial M = \dim\mathbb{D}^3(t^+) + \dim\mathbb{S}^2 + 1$$

Now $\dim\mathbb{S}^2 = 2$. So do we have more dimensions than alloted?

Let's account for the various setups for a 4-dimensional topological gauge theory that involves knot polynomials (i.e. knot homologies). It appears that Gaiotto and Witten (2011) [20] likes to include Riemann surfaces in ∂M, so there setup could be

$$\partial M = \mathbb{C}\times\mathbb{R}(\text{or } \mathbb{C}\times S^1(???))$$

where $\mathbb{C}\equiv$ Riemann surface (i.e. $f:\mathbb{C}\to\mathbb{C}$, f holomorphic, i.e. $\begin{aligned}&\frac{\partial f}{\partial\overline{z}} = 0 \\ &\frac{\partial\overline{f}}{\partial z} = 0\end{aligned}$).

11.1.1. *20160422 Things to do.* Clarify Manifold setup $\partial M\hookrightarrow M$; explore various manifold setups; pdf or equation of motion out of $\mathcal{L}$ and Euler-Lagrange equation and compare those equations to instanton equations of Gaiotto and Witten (2011) [20]; understand Virasoro algebra and conformal blocks for the quantum "states" that we can act upon; read more: Gaiotto and Witten (2011) [20]; Gukov (2007) [22], and of course classic Witten (1988) [19]

knot K is an embedding $f:S^1\to S^3$

$$S^1\xrightarrow{f} S^3 \xrightarrow{\cong}\partial M\xhookrightarrow{i} M$$

11.2. **Reviewing Witten's Quantum Field Theory and the Jones Polynomial.** Let's review Witten's seminal paper (of which he won a Fields medal for, and is, as of 20160503, the only physicist to have won the Fields medal) (cf. Witten (1988) [19]).

Note that this is *not* a perturbative theory, i.e. no *perturbations* were used. This is a non-perturbative theory yielding *exact* solutions, exact topological solutions, i.e. with deal relation to topology. **Conformal Field Theory** was employed in the quantization. Thus, tools from conformal field theory are needed and maybe very unfamiliar to someone that was raised in the usual quantum field theory class with Feynman diagrams. Thus, a generous amount of time will be taken to understand *conformal field theory.*

11.2.1. *Framing.*

12. Standard Model

From *Wikipedia*, on "Standard Model (mathematical formulation)", (cf. https://en.wikipedia.org/wiki/Standard_Model_(mathematical_formulation)), they have a nice and neat chart of the field content of the standard model (SM) and I'll reproduce the gauge fields table:

Field content of the standard model (SM)

Spin 1 - gauge fields

Symbol	Associated charge	Group	Coupling	Representation
B	Weak hypercharge	$U(1)_Y$	g'	$(\mathbf{1},\mathbf{1},0)$
W	Weak isospin	$SU(2)_L$	g_w	$(\mathbf{1},\mathbf{3},0)$
G	Color	$SU(3)_C$	g_s	$(\mathbf{8},\mathbf{1},0)$

12.0.2. *Representation for SM.* As a warmup, consider the group $GL(N, \mathbb{C})$:

$$GL(N;\mathbb{C}) = \{X | X \in \mathrm{Mat}_{\mathbb{C}}(N,N), \det X \neq 0\}$$

where $\mathrm{Mat}_{\mathbb{C}}(N,N)$ denotes the space (not group) of all $N \times N$ matrices with complex numbers in (all of) its entries.

Clearly,

$$\dim_{\mathbb{C}} GL(N;\mathbb{C}) = N^2$$
$$\dim_{\mathbb{R}} GL(N;\mathbb{C}) = 2 * N^2 = 2N^2 \quad \text{(for the 2 real numbers for \textit{each} complex entry)}$$

where I denote whether we're talking about *complex* dimensions (i.e. in using complex numbers) or dimension with real numbers; hence the subscript under dim.

Consider

$$U(N) = \{U \in GL(N, \mathbb{C} | UU^{\dagger} = 1\}$$

Now

Proposition 11.

$$dim_{\mathbb{R}} U(N) = N^2 \quad dim_{\mathbb{R}} SU(N) = N^2 - 1$$

Proof. A great way to calculate this is found in http://www.phys.nthu.edu.tw/~class/Group_theory/Chap%207.pdf, pp. 112, Chapter 7 Classical Lie Groups.

Each row of $\forall U \in U(N)$ has unit norm (analogous to $O(N)$), and different rows are "orthogonal", with respect to some inner product.

First row determined by N complex numbers (parameters), or $2N$ real numbers.

subtract 1 for unit norm condition. $2N - 1$.

2nd row determined by $N - 1$ complex numbers; orthgonality determined by one of the complex numbers (parameters). So $2N - 2$ real numbers.

subtract 1 for unit norm condition: $2N - 3$.

Thus,

$$\sum_{j=1}^{N} 2N - (2j - 1) = 2N^2 - 2\left(\frac{N(N+1)}{2}\right) + N = N^2$$

For $SU(N)$, $\det SU(N) = 1$. This fixes the overall phase of $\det X$, $X \in SU(N)$ (since $\det(UU^{\dagger}) = \det U \det U^{\dagger} = 1$ and so $\det U$ has norm 1, i.e. $|\det U| = 1$, $\forall U \in U(N)$, already). So that costs another *real* number (parameter). So for $SU(N)$, $N^2 - 1$ *real* number parameters are independent.

□

Recall that an automorphism is an isomorphism from a mathematical object to itself. Denote $\mathrm{Aut}(V)$ to be the space of all possible automorphisms on mathematical object V (prominent example is vector space V).

From Löh's notes on representation theory of Lie algebra (cf. Clara Löh "Representation theory of Lie algebra." http://deferentialgeometry.org/papers/Loeh%20-%20Representation%20Theory%20of%20Lie%20Algebras.pdf),

if (given) $\phi : G \to \mathrm{Aut}(V)$ is a representation of G, then $T_1\phi : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of the Lie algebra, i.e. making

$$\begin{tikzcd}
\mathfrak{g} = T_1 G \arrow[r, "T_1\phi"] \arrow[d, "\exp"] & \mathrm{End}(V) = \mathfrak{gl}(V) \arrow[d, "\exp"] \\
G \arrow[r, "\phi"] & \mathrm{Aut}(V)
\end{tikzcd}$$

Since $SU(N)$ is simply connected (recall, $\det A = 1 \, \forall A \in SU(N)$, by definition, and so $\det : SU(N) \to \mathbb{C}$, being a continuous function, and with the preimage of $\{1\}$, $\det^{-1}(\{1\})$, being the entire $SU(N)$, and $\{1\}$, clearly a closed set, then $SU(N)$ simply connected). Note that $U(N)$ is not simply connected. Note that this has a good chart of simply connected vs. not simply connected: 1 Matrix Lie Groups - Springer, we can go the other way, in that given representation of Lie algebra $\mathfrak{g}$, say $\rho : \mathfrak{g} \to Gl(V)$, $\exists \exp(\rho) : G \to \mathrm{Aut}(V)$ group representation.

Ciaran Hughes makes the poignant remark/observation in distinguishing the *representation space* (cf. A brief discussion on representations. While

$$\dim(SU(N)) = N^2 - 1 \dim(\mathfrak{su}(N)) = N^2 - 1$$
$$\dim(SU(2)) = 3 \dim(\mathfrak{su}(2)) = 3$$
$$\dim(SU(3)) = 8 \dim(\mathfrak{su}(3)) = 8$$

so the dimensions of $SU(N)$ and corresponding $\mathfrak{su}(N)$ *look* equal, but they were arrived at from entirely different conditions (they are definitely not equal mathematical objects: by definition $\mathfrak{su}(N)$, as a Lie algebra, is a *vector space*, equipped with Lie bracket, i.e. commutator).

The (3) generators (a basis) for $\mathfrak{su}(2)$ are the celebrated Pauli matrices $\sigma^1, \sigma^2, \sigma^3$. There are 8 generators for $\mathfrak{su}(3)$; we can denote them as T^a, obeying a "structure equation": $[T^a, T^b] = f^{ab}_{\ \ c} T^c$.

Note to take care that mathematicians work with anti-Hermitian generators and physicists work with Hermitian generators (!!!).

The vector space V I've been using is called the *representation space* (a good name) by Hughes in A brief discussion on representations in the context of representations: a representation ρ of group G is a homomorphism, s.t.

$$\rho : G \to GL(V)$$

So $\forall g \in G$, $\rho(g) \in GL(V)$, and so $\rho(g)$ itself is a $\dim V \times \dim V$ *matrix*.

This representation $\rho(g)$ of element $g \in G$ *acts* on this vector space V which Hugehes calls the *representation space* $\mathcal{V}_{\text{rep}}$ (I'll use V as my notation; they mean the same thing):

$$\rho(g) : V \to V$$

A representation ρ of a Lie algebra $\mathfrak{g}$ is s.t.

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

s.t. $\forall A \in \mathfrak{g}$, $\rho(A) \in \mathfrak{gl}(V)$ is a $\dim V \times \dim V$ *matrix*.

Now, as Hughes noted *"Different representations can act on different spaces"* (emphasis mine).

For the example of $SU(2)$,

In $\rho : G \to GL(V)$, label irreducible representation (irrep) ρ by j for "spin" $V \to V_j$.

$T^a = J^a$, J^1, J^2, J^3 are the "usual" angular momentum generators (of quantum mechanics).

Hughes provides this useful table:

j	basis for V_j	$\dim V_j$	$T^a = J^a$
$j = 0$	$\|0,0\rangle$	1	$T^a = 0$ so that it doesn't transform under $SU(2)$
$j = 1$	$\|\frac{1}{2}, m\rangle$ (i.e. $\{\|\frac{1}{2}, \frac{1}{2}\rangle, \|\frac{1}{2}, \frac{-1}{2}\rangle\}$)	2	$T^a = \sigma^a/2$
$j = 2$	$\|1, m\rangle$ (i.e. $\{\|1,1\rangle, \|1,0\rangle, \|1,-1\rangle\}$)	3	$T^a = \sigma^a$ are 3×3 matrices
j	$\|j, m\rangle$, $m = -j, -j+1, \dots, j-1, j$	2j+1	T^a are $2j+1 \times 2j+1$ matrices

Consider this Lagrangian, as does Hughes:

$$\mathcal{L} = \overline{\Psi}(i\gamma^{\mu} D_{\mu} - m)\Psi = \overline{\Psi}(i\not{D} - m)\Psi$$

with

field Ψ is in a representation space $V = V_j$ of gauge group G,

D_{μ} is covariant derivative.

D_{μ} defined to transform under an element of the representation U of gauge group as

$$D_{\mu}\Psi \to UD_{\mu}\Psi = (UD_{\mu}D^{-1})(U\Psi)$$

Keeping in mind that when we write U, we really mean the *representation* of $U \in G$,

i.e. $U = \rho(U)$, $U \in G$.

This transformation property is called by Hughes the *adjoint action* of group on the Lie algebra.

So $D_{\mu} \to UD_{\mu}U^{-1}$, i.e. the covariant derivative transform in the adjoint action, or just adjointly.

$D_{\mu} = \partial_{\mu} - igA_{\mu}(x)$, where $A_{\mu}(x)$ is element of representation of the Lie algebra i.e. $A_{\mu}(x) \in \rho(\mathfrak{g})$.

So $A_{\mu}(x) = A^a_{\mu}(x)T^a$, $T^a \in \rho(\mathfrak{g})$.

Now, globally,

$$A_\mu(x) \to U(x)A_\mu U(x)^\dagger - \frac{1}{e}(\partial_\mu U(x))U^\dagger(x)$$

Consider now infinitesimal transformations.

Let $U = \exp(\lambda_a T^a) = 1 + \lambda_a T^a + O((\lambda_a)^2)$. Denote $\lambda = \lambda_a T^a$.

$$\begin{aligned} A_\mu(x) \to UA_\mu U^{-1} - \frac{1}{e}(\partial_\mu U)U^{-1} \simeq (1+\lambda)A_\mu(1-\lambda) - \frac{1}{e}(\partial_\mu \lambda)(1-\lambda) = \\ = A_\mu - \frac{1}{e}\{\partial_\mu \lambda + e[A_\mu, \lambda]\} = A_\mu - \frac{1}{e}D_\mu \lambda \end{aligned}$$

Call $[A_\mu, \lambda]$ the adjoint action of the Lie algebra itself. At this point, I will change up notation because if T^a's are generators of Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$ is a *vector space*, then go to this notation, T_a, for the generators. Then the components of vectors have superscripts.

$$\begin{aligned} D_\mu \lambda = \partial_\mu \lambda + e[A_\mu, \lambda] = \partial_\mu \lambda^a T_a + eA^a{}_\mu \lambda^b [T_a, T_b] = \\ = \partial_\mu \lambda^a T_a + eA^a{}_\mu \lambda^b f_{ab}^{\ \ c} T_c \qquad \text{using } [T_a, T_b] = f^c_{ab} T_c \end{aligned}$$

Let's try to connect this with what mathematicians, mathematical physicists, and theorists say about vector bundles and principal-G bundles.

Consider a vector bundle $E \xrightarrow{\pi} M$. The covariant derivative $D_j : E \to E$ involves a connection 1-form on E:

$$D_j X = \partial_j X + A^k{}_{ij} X^i e_k \qquad \forall X \in E,\ e_k \text{ a (local) frame over } M$$

Now consider principal-G bundle $P \xrightarrow{\pi} M$ with $\lambda \in P$:

$$\begin{tikzcd} P \arrow[d, "\pi"'] \\ M \end{tikzcd} \qquad \begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi_U"] & U \times \mathfrak{g} \arrow[dl] \\ U \subset M \arrow[u, "\pi^{-1}"] \end{tikzcd}$$

Then for covariant derivative $D_\mu : P \to P$,

$$D_\mu \lambda = \partial_\mu \lambda + A^c{}_{b\mu} \lambda^b e_c$$

with $A^c{}_{b\mu}(e_c \otimes e^b)dx^\mu \in \Omega^1(M; \mathrm{End}(\mathfrak{g}))$, a connection 1-form, in which it (a connection) is a $\mathrm{End}(\mathfrak{g})$-valued 1-form.

Compare this expression with the one from physics:

$$D_\mu \lambda = \partial_\mu \lambda + A^c{}_{b\mu} \lambda^b e_c = \partial_\mu \lambda^a T_a + (eA^a{}_\mu f_{ab}^{\ \ c})\lambda^b T_c$$

Thus, what I'll need to check myself more thoroughly, what theorists call the connection 1-form on the principal-G bundle, A, is related to what physicists call the gauge field, but in its *adjoint representation*; the clue to why this is the adjoint representation is the $f_{ab}^{\ \ c}$ factors in $eA^a{}_\mu f_{ab}^{\ \ c}$. Also, it is unclear to me at least, how to scale or deal with the e numerical factors, which is in the theory.

12.0.3. *Certain Types of Representations (cf. Hughes, Sec. 6).*

- **trivial representation**: $\rho(g) = 0 \quad \forall g \in G$ or $\mathfrak{g}$; $T^a = 0$. $\dim V = 1$.
- **fundamental representation**: $\rho(g) = g \quad \forall g \in G$ or $\mathfrak{g}$. Then $\dim V = N$, for $g \in \mathrm{Mat}(N,N)$ if G is a "matrix Lie group" (similar with $\mathfrak{g}$)
- *adjoint representation*: Hughes mentions that $f_{ab}^{\ \ c}$ are the components of the representation. Recall that $[T_a, T_b] = f_{ab}^{\ \ c} T_c$, where $f_{ab}^{\ \ c}$ are the antisymmetric *structure constants*. Now $T_a, T_b \in \mathfrak{g}$. $[T_a, T_b] \in \mathfrak{g}$ ($\mathfrak{g}$ is a Lie algebra, equipped with Lie bracket, by definition).

  So $f_{ab}^{\ \ c} \in \mathbb{K}$, field $\mathbb{K}$ over which $\mathfrak{g}$ is over with ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ usually).

Redi and Sato (2016) [24] gives a good overall review of the current status of axions in a theory context.

13. Spin Structures, Fermions (fermionic field)

The clearest, most lucid, explanation of Spin Geometry is, in my opinion, from Salamon (1996) [23]: I will interpolate between Salamon (1996) and Jost (2011) [7], in particular, Jost's Chapter 2 Lie Groups, Section 2.4 Spin Structures.

Salamon (1996) [23] uses $C(V)$ to denote the Clifford Algebra generated by V; Jost and Wikipedia uses $Cl(V)$; I'm going to use $Cl(V)$ to stand in for $C(V)$.

13.1. Clifford Algebra.

14. Seiberg-Witten

$\mathcal{A}$

15. End Game; Desires

References

[1] John Baez, Javier P Muniain. **Gauge Fields, Knots and Gravity** (Series on Knots and Everything), World Scientific Publishing Company (October 24, 1994), ISBN-13: 978-9810220341

[2] Joseph J. Rotman, **Advanced Modern Algebra** (Graduate Studies in Mathematics), (Book 114), American Mathematical Society; 2 edition, (August 10, 2010), ISBN-13: 978-0821847411

[3] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina, "Introduction to representation theory", January 10, 2011 http://math.mit.edu/~etingof/replect.pdf

[4] Andrew Baker, "Representations of Finite Groups", 07112011 http://www.maths.gla.ac.uk/~ajb/dvi-ps/groupreps.pdf

[5] Yvette Kosmann-Schwarzbach, **Groups and Symmetries: From Finite Groups to Lie Groups**, Springer, 2010. e-ISBN 978-0-387-78866-1 http://www.caam.rice.edu/~yad1/miscellaneous/References/Math/Groups/Groups%20and%20Symmetries.pdf

[6] Clifford Henry Taubes, **Differential Geometry: Bundles, Connections, Metrics and Curvature** (Oxford Graduate Texts in Mathematics, Vol. 23) Oxford University Press; 1 edition (December 1, 2011). ISBN-13: 978-0199605873

[7] Jürgen Jost. **Riemannian Geometry and Geometric Analysis (Universitext)**. 6th ed. 2011 Edition. Springer; 6th ed. 2011 edition (August 9, 2011). ISBN-13: 978-3642212970

[8] Victor W Guillemin, Shlomo Sternberg. **Supersymmetry and Equivariant de Rham Theory**. Springer; 1999 edition (June 11, 1999). ISBN-13: 978-3540647973

[9] Lokman Tsui. "Applications of Chern-Simons Theory in knot theory." http://ms.mcmaster.ca/~boden/students/Tsui-MSc.pdf

[10] Werner Ballmann. "Vector bundles and connections", http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/conncurv1999.pdf

[11] Jeffrey M. Lee. **Manifolds and Differential Geometry**, *Graduate Studies in Mathematics* Volume: 107, American Mathematical Society, 2009. ISBN-13: 978-0-8218-4815-9

[12] Sergei Gukov, Satoshi Nawata, Ingmar Saberi, Marko Stosic, Piotr Sułkowski. *Sequencing BPS Spectra.* 2015. arXiv:1512.07883v2 [hep-th]

[13] H.S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math., **108** (1992), 575–633. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.443.4980&rep=rep1&type=pdf

[14] Axel Riese. `qMultiSum` — A Package for Proving q-Hypergeometric Multiple Summation Identities. **Journal of Symbolic Computation**. Volume 35, Issue 3, March 2003, Pages 349–376. http://www.risc.jku.at/publications/download/risc_427/qmultisum.pdf

[15] George Gasper, Mizan Rahman. **Basic Hypergeometric Series** (Encyclopedia of Mathematics and its Applications). Cambridge University Press; 2 edition (October 4, 2004). ISBN-13: 978-0521833578

[16] Marko Petkovšek, Herbert S. Wilf, Doron Zeilberger. $A = B$. A K Peters, Ltd. 1997. DOWNLOAD THE BOOK "$A = B$"

[17] F. Caruso, **A Macsyma Implementation of Zeilberger's Fast Algorithm**, RISC-Linz Report Series n. **99-21**, J. Kepler University, Linz, 1999.

[18] Dror Bar-Natan. "On Khovanov's categorification of the Jones polynomial." **Algebraic & Geometric Topology**. Volume 2 (2002) 337–370. arXiv:math/0201043v3 [math.QA]

[19] E. Witten, "Quantum Field Theory and the Jones Polynomial," Commun. Math. Phys. **121**, 351 (1989). doi:10.1007/BF01217730

[20] Davide Gaiotto, Edward Witten. *Knot Invariants from Four-Dimensional Gauge Theory.* 2011. arXiv:1106.4789 [hep-th]

[21] Kevin Peter Hickerson. *The physics of ultracold neutrons and Fierz interference in beta decay.* Dissertation (Ph.D.), California Institute of Technology. 2013. http://resolver.caltech.edu/CaltechTHESIS:11022012-135115177
[22] Sergei Gukov. "Gauge theory and knot homologies." Fortschr. Phys. **55**, No. 5 – 7, 473 – 490 (2007) / **DOI** 10.1002/prop.200610385 http://www.maths.ed.ac.uk/~aar/papers/gukov2.pdf
[23] Dietmar Salamon. "Spin Geometry and Seiberg–Witten Invariants." 1996. https://people.math.ethz.ch/~salamon/PREPRINTS/witsei.pdf
[24] Michele Redi, Ryosuke Sato. *Composite Accidental Axions.* arXiv:1602.05427 [hep-ph]
http://arxiv.org/abs/1602.05427