

QUANTUM SUPER-A-POLYNOMIALS

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ABSTRACT. Everything about quantum super-A-polynomials.

I also prepared here, first, for myself, and second, hopefully to help others that begin at an elementary level (advanced undergraduate, early graduate studies) or if one needs to look up or be refreshed upon a previous concept, notes and readings of background material necessary to understand quantum super-A-polynomials and everything behind it fully. This includes notes that I would take as I read through books or articles. I want to note that sometimes I copy complete passages out of books or articles because that’s how I learn as a novice, imitating and copying (if you have another way of learning, please let me know). I

~~also work out some solutions to examples, exercises, and problems out of said books and articles.~~

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Part 1. Introduction

1. LIE GROUPS

- : Lie Groups
- : Groups

- : Ring
- : group algebra
- : Group Ring
- : Representation Theory
- : Modules
- :  $kG$ -modules

From Sec. 8.1 “Noncommutative Rings” of Rotman (2010) [2]:

**Definition 1.** *ring  $R$  - additive abelian group equipped with multiplication  $R \times R \rightarrow R$  s.t.  $\forall a, b \in R$*   
 $(a, b) \mapsto ab$

- (i)  $a(bc) = (ab)c$
- (ii)  $a(b + c) = ab + ac$ ,  $(b + c)a = ba + ca$
- (iii)  $\exists 1 \in R$  s.t.  $\forall a \in R$ ,  $1a = a = a1$

Example 8.1[2]

- (ii) group algebra  $kG$ ,  $k$  commutative ring,  $G$  group, “its additive abelian group is free  $k$ -module having basis labeled by elements of  $G$ ,  
i.e.  $\forall a \in kG$ ,  $a = \sum_{g \in G} a_g g$ ,  $a_g \in k$ ,  $\forall g \in G$ ,  $a_g \neq 0$  for only finitely many  $g \in G$ .

define (ring) multiplication  $kG \times kG \rightarrow kG$   $\forall a, b \in kG$ ,  $a = \sum_{g \in G} a_g g$  to be  
 $ab = ab$   $b = \sum_{h \in G} b_h g$

$$\left( \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} b_h h \right) = \sum_{z \in G} \left( \sum_{gh=z} a_g b_h \right) z$$

**Definition 2.** *Given  $R$  ring, left  $R$ -module is (additive) abelian group  $M$  equipped with*

*scalar multiplication  $R \times M \rightarrow M$  s.t.  $\forall m, m' \in M$ ,  $\forall r, r', 1 \in R$*   
 $(r, m) \mapsto rm$

- (i)  $r(m + m') = rm + rm'$
- (ii)  $(r + r')m = rm + r'm$
- (iii)  $(rr')m = r(r'm)$
- (iv)  $1m = m$

EY : 20150922 Example : for  $kG$ -module  $V^\sigma$ , for  $r \in kG$ , so  $r = \sum_{g \in G} a_g g$

$$\begin{array}{ccc} R \times M \rightarrow M & & kG \times V \rightarrow V \\ (r, m) \mapsto rm & \implies & (r, v) \mapsto tv \end{array}$$

For some representation  $\sigma : G \rightarrow GL(V)$ ,

$$rv = \sum_{g \in G} a_g g \cdot v = \sum_{g \in G} a_g \sigma_g(v)$$

So a  $kG$ -module needs to be associated with some chosen representation.

Note for  $V$  as an additive abelian group,  $\forall u, v, w \in V$ ,

$$v + w = w + v, (u + v) + w = u + (v + w)$$

$$v + 0 = v \quad \forall v \in V \text{ for } 0 \in V$$

$$v + (-v) = 0 \quad \forall v \in V$$

So a vector space can be an additive abelian group.

Note that

$$r(v + w) = \left( \sum_{g \in G} a_g g \right) (v + w) = \left( \sum_{g \in G} a_g \sigma_g \right) (v + w) = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} a_g \sigma_g(w) = rv + rw$$

$$(r + r')v = \left( \sum_{g \in G} a_g g + b_g g \right) v = \sum_{g \in G} (a_g \sigma_g + b_g \sigma_g) v = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} b_g \sigma_g(v) = rv + r'v$$

$$(rr')v = \left( \sum_{g \in G} a_g g \sum_{h \in G} b_h h \right) v = \left( \sum_{z \in G} \sum_{gh=z} a_g b_h z \right) v = \sum_{z \in G} \sum_{gh=z} a_g b_h \sigma_z(v) = \sum_{g \in G} \sum_{h \in G} a_g b_h \sigma_g \sigma_h(v)$$

since  $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g \sigma_h = \sigma_{gh}$  ( $\sigma$  homomorphism)

$$1v = \sigma(1)v = 1v = v$$

From Sec. 8.3 “Semisimple Ring” of Rotman (2010) [2]:

**Definition 3.**  *$k$ -representation of group  $G$  is homomorphism*

$$\sigma : G \rightarrow GL(V)$$

where  $V$  is vector field over field  $k$

**Proposition 1** (8.37 Rotman (2010)[2]).  $\forall k$ -representation  $\sigma : G \rightarrow GL(V)$  equips  $V$  with structure of left  $kG$ -module, denote module by  $V^\sigma$ .

Conversely,  $\forall$  left  $kG$ -module  $V$  determines  $k$ -representation  $\sigma : G \rightarrow GL(V)$

*Proof.* Given  $\sigma : G \rightarrow GL(V)$ ,

$$\sigma_g =: \sigma(g) : V \rightarrow V$$

define

$$kG \times V \rightarrow V$$

$$\left( \sum_{g \in G} a_g g \right) v = \sum_{g \in G} a_g \sigma_g(v)$$

$$v, w \in V$$

Let  $r, r', 1 \in kG$

$$r = \sum_{g \in G} a_g g$$

$$r(v + w) = \left( \sum_{g \in G} a_g g \right) (v + w) = \left( \sum_{g \in G} a_g \sigma_g \right) (v + w) = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} a_g \sigma_g(w) = rv + rw$$

$$(r + r')v = \left( \sum_{g \in G} a_g g + b_g g \right) v = \sum_{g \in G} (a_g \sigma_g + b_g \sigma_g) v = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} b_g \sigma_g(v) = rv + r'v$$

$$(rr')v = \left( \sum_{g \in G} a_g g \sum_{h \in G} b_h h \right) v = \left( \sum_{z \in G} \sum_{gh=z} a_g b_h z \right) v = \sum_{z \in G} \sum_{gh \in z} a_g b_h \sigma_z(v) = \sum_{g \in G} \sum_{h \in G} a_g b_h \sigma_g \sigma_h(v)$$

since  $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g \sigma_h = \sigma_{gh}$  ( $\sigma$  homomorphism)

$$1v = \sigma(1)v = 1v = v$$

Conversely, assume  $V$  left  $kG$ -module.

If  $g \in G$ , then  $v \mapsto gv$  defines  $T_g : V \rightarrow V$ .  $T_g$  nonsingular since  $\exists T_g^{-1} = T_{g^{-1}}$

Define  $\sigma : G \rightarrow GL(V)$

$$\sigma : g \mapsto T_g$$

$\sigma$   $k$ -representation

$$\sigma(gh) = T_{gh} = T_g T_h = \sigma(g)\sigma(h)$$

$$\sigma(gh)(v) = T_{gh}v = ghv = T_g T_h v = \sigma(g)\sigma(h)v \quad \forall v \in V$$

**Proposition 2.** Let group  $G$ , let  $\sigma, \tau : G \rightarrow GL(V)$  be  $k$ -representations, field  $k$ .

If  $V^\sigma, V^\tau$  corresponding  $kG$ -modules in Prop. 1 (Prop. 8.37 in Rotman (2010) [2]), then

$V^\sigma \simeq V^\tau$  as  $kG$ -modules iff  $\exists$  nonsingular  $\varphi : V \rightarrow V$  s.t.

$$\varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$$

*Proof.* If  $\varphi : V^\tau \rightarrow V^\sigma$   $kG$ -isomorphism, then  $\varphi : V \rightarrow V$  isomorphism s.t.

$$\varphi(\sum a_g gv) = (\sum a_g g)\varphi(v) \quad \forall v \in V, \forall g \in G$$

in  $V^\tau$ ,  $kG \times V \rightarrow V$

in  $V^\sigma$ ,  $kG \times V \rightarrow V$  scalar multiplication

$$gv = \tau(g)(v)$$

$$gv = \sigma(g)(v)$$

$$\implies \forall g \in G, v \in V, \quad \varphi(\tau(g)(v)) = \sigma(g)(\varphi(v))$$

I think

$$\varphi(gv) = \varphi(\tau(g)(v)) = g\varphi(v) = \sigma(g)\varphi(v)$$

$$\implies \varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$$

Conversely, if  $\exists$  nonsingular  $\varphi : V \rightarrow V$  s.t.  $\varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$

$$\varphi\tau(g)v = \varphi(\tau(g)v) = \sigma(g)\varphi(v) \quad \forall g \in G, \forall v \in V$$

Consider scalar multiplication

$$kG \times V \rightarrow V$$

$$\sum_{g \in G} a_g g(v) = \sum_{g \in G} a_g \tau_g(v)$$

$$\varphi \left( \sum_{g \in G} a_g \tau_g(v) \right) = \varphi \left( \sum_{g \in G} a_g \tau(g)v \right) = \sum_{g \in G} a_g \sigma(g)\varphi(v) = \left( \sum_{g \in G} a_g g \right) \varphi(v)$$

□

Admittedly, after this exposition from Rotman (2010) [2], I still didn't understand how  $kG$ -modules relate to representation theory and group rings. I turned to Baker (2011) [4], which we'll do right now. Note that I found a lot of links to online resources on representation theory from Khovanov's webpage <http://www.math.columbia.edu/~khovanov/resource/>.

Note,

**Definition 4.** vector subspace  $W \subseteq V$  is called a

$G$ -submodule,  $G$ -subspace, EY : 20150922 “invariant” subspace?

if  $\forall g \in G$ , for representation  $\rho : G \rightarrow GL_k(V)$ ,  $\rho_g(w) \in W$ ,  $\forall w \in W$ ,  $\forall g \in G$  i.e. closed under “action of elements of  $G$ ” with  $\rho_g =: \rho(g) : V \rightarrow V$

Given basis  $\mathbf{v} = \{v_1 \dots v_n\}$  for  $V$ ,  $\dim_k V = n$ ,  $\forall g \in G$ ,

$$\rho_g v_j = \rho(g)v_j = r_{kj}(g)v_k$$

for, indeed,

$$\rho_g x^j v_j = \rho(g)x^j v_j = x^j \rho(g)v_j = x^j r_{kj}(g)v_k = r_{kj} x^j v_k$$

so that

$$\rho : G \rightarrow GL_k(V)$$

$$\rho(g) = [r_{ij}(g)]$$

□

Example 2.1 (Baker (2011) [4]): Let  $\rho : G \rightarrow GL_k(V)$  where  $\dim_k V = 1$

$$\forall v \in V, v \neq 0, \forall g \in G, \lambda_g \in k \text{ s.t. } g \cdot v = \rho_g(v) = \lambda_g v$$

$$\rho(hg)v = \rho_h \rho_g v = \lambda_{hg} v = \lambda_h \lambda_g v \implies \lambda_{hg} = \lambda_h \lambda_g$$

$\implies \exists$  homomorphism  $\Lambda : G \rightarrow k^\times$

$$\Lambda(g) = \lambda_g$$

From Sec. 2.2 “ $G$ -homomorphisms and irreducible representations” of Baker (2011) [4], suppose  $\rho : G \rightarrow GL_k(V)$  and  $\sigma : G \rightarrow GL_k(W)$  are 2

representations

Many names for the same thing:  $G$ -equivalent,  $G$ -linear,  $G$ -homomorphism, EY : 20150922  $kG$ -isomorphic?

If  $\forall g \in G$ ,

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V \\ \tau_g \downarrow & & \downarrow \sigma_g \\ V & \xrightarrow{\varphi} & V \end{array} \iff V^\tau \stackrel{\varphi}{\simeq} V^\sigma$$

Indeed, define

$$\varphi : V^\tau \rightarrow V^\sigma$$

$$\varphi(v+w) = \varphi(v) + \varphi(w)$$

$$\varphi(rv) = \varphi\left(\sum_{g \in G} a_g g \cdot v\right) = \varphi\left(\sum_{g \in G} a_g \tau_g(v)\right) = \sum_{g \in G} a_g \varphi(\tau_g(v)) = \sum_{g \in G} a_g \sigma_g \cdot \varphi(v) = r\varphi(v)$$

EY : 20150922 So  $\varphi$  is a  $kG$ -isomorphism between left  $kG$  modules  $V^\tau$  and  $V^\sigma$  if it's bijective and is “linear” in “scalars”  $r \in kG$ , i.e.  $\varphi(rv) = r\varphi(v)$ .

Define action of  $G$  on  $\text{Hom}_k(V, W)$  ( $\text{Hom}_k(V, W)$  is the vector space of  $k$ -linear transformations  $V \rightarrow W$ )

□

$\forall f \in \text{Hom}_k(V, W)$ ,  $f : V \rightarrow W$

$$f(v) \in W$$

Consider

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ v & \xmapsto{f} & f(v) \end{array}$$

$$G \times \text{Hom}_k(V, W) \rightarrow \text{Hom}_k(V, W)$$

$$(g \cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}} \text{ i.e. } (g \cdot f)(v) = \sigma_g f(\rho_{g^{-1}} v) \quad (f \in \text{Hom}_k(V, W))$$

Let  $g, h \in G$ ,

$$(gh \cdot f)(v) = g \cdot \sigma_h f(\rho_{h^{-1}} v) = \sigma_g \sigma_h f \rho_{h^{-1}} \rho_{g^{-1}}(v) = (\sigma_{gh} f \rho_{(gh)^{-1}})(v)$$

Thus,  $G \times \text{Hom}_k(V, W) \rightarrow \text{Hom}_k(V, W)$  is thus another  $G$ -representation of  $G$ .

$$(g \cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}}$$

For  $k$ -representation  $\rho$ , if the only  $G$ -subspaces of  $V$  are  $\{0\}$ ,  $V$ ,  $\rho$  **irreducible** or **simple**.

$$\rho_g(\{0\}) = \{0\}$$

$$\rho_g(V) = V$$

given subrepresentation  $W \subseteq V$ ,  $V/W$  admits linear action of  $G$ ,  $\bar{\rho}_W : G \rightarrow GL_k(V/W)$  quotient representation

$$\bar{\rho}_W(g)(v + W) = \rho(g)(v) + W$$

if  $v' - v \in W$

$$\rho(g)(v') + W = \rho(g)(v + (v' - v)) + W = (\rho(g)(v) + \rho(g)(v' - v)) + W = \rho(g)(v) + W$$

**Proposition 3** (2.7 Baker (2011)[4]). *if  $f : V \rightarrow W$   $G$ -homomorphism, then*

(a)  *$\ker f$  is  $G$ -subspace of  $V$*

(b)  *$\text{im} f$  is  $G$ -subspace of  $W$*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \rho_g \downarrow & & \downarrow \sigma_g \\ V & \xrightarrow{f} & W \end{array}$$

*Proof.* Recall

(a) Let  $v \in \ker f$ . Then  $\forall g \in G$ ,

$$f(\rho_g v) = \sigma_g f(v) = 0$$

so  $\rho_g v \in \ker f$ ,  $\forall g \in G$ . So  $\ker f$  is  $G$ -subspace of  $V$

(b) Let  $w \in \text{im} f$ . So  $w = f(u)$  for some  $u \in V$

$$\sigma_g w = \sigma_g f(u) = f(\rho_g u) \in \text{im} f$$

So  $\text{im} f$  is  $G$ -subspace of  $W$

**Theorem 1** (Schur's Lemma). *Let  $\rho : G \rightarrow GL_{\mathbb{C}}(V)$  be irreducible representations of  $G$  over field  $k = \mathbb{C}$ ; let  $f : V \rightarrow W$  be*

$$\sigma : G \rightarrow GL_{\mathbb{C}}(W)$$

*$G$ -linear map.*

(a) *if  $f \neq 0$ ,  $f$  isomorphism. True  $\forall k$  field, not just  $\mathbb{C}$*

(b) *if  $V = W$ ,  $\rho = \sigma$ , then for some  $\lambda \in \mathbb{C}$ ,  $f$  given by  $f(v) = \lambda v$  ( $v \in V$ ) (true for algebraically closed fields)*

*Proof.* (a) By Prop. 3,  $\ker f \subseteq V$ ,  $\text{im} f \subseteq W$  are  $G$ -subspaces.

For  $\rho$ , only  $G$ -subspaces are 0 or  $V$ , so if  $\ker f = V$ ,  $f = 0$ . If  $\ker f = 0$ ,  $f$  injective.

For  $\sigma$ , only  $G$ -subspaces are 0 or  $V$ , so  $\text{im} f = 0$ ,  $f = 0$ . If  $\text{im} f = V$ ,  $f$  surjective.

$\implies f$  isomorphism.

(b) Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $f$ ,  $f(v_0) = \lambda v_0$  eigenvector,  $v_0 \neq 0$ .

Let linear  $f_{\lambda} : V \rightarrow V$  s.t.

$$f_{\lambda}(v) = f(v) - \lambda v \quad (v \in V)$$

$\forall g \in G$

$$\rho_g f_{\lambda}(v) = \rho_g f(v) - \rho_g \lambda v = f(\rho_g v) - \lambda \rho_g v = f_{\lambda}(\rho_g v)$$

So  $f_{\lambda}$  is  $G$ -linear, for

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \rho_g \downarrow & & \downarrow \rho_g \\ V & \xrightarrow{f} & V \end{array}$$

Since  $f_{\lambda}(v_0) = 0$ , by Prop. 3,  $\ker f_{\lambda} = V$ , (for  $\ker f_{\lambda} \neq 0$  and so  $\ker f_{\lambda} = V$ )

By rank-nullity theorem,  $\dim V = \dim \ker f_{\lambda} + \dim \text{im} f_{\lambda}$ .

So  $\text{im} f_{\lambda} = 0$ , and so  $f_{\lambda}(v) = 0$  ( $\forall v \in V$ )  $\implies f(v) = \lambda v$

□

Schur's lemma, at least the first part, implies that the left  $kG$ -modules associated with representations  $\rho, \sigma$  are  $kG$ -isomorphic,

i.e.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \rho_g \downarrow & & \downarrow \sigma_g \\ V & \xrightarrow{f} & W \end{array} \iff V^{\rho} \stackrel{f}{\cong} V^{\sigma}$$

with  $f$  being an isomorphism between  $V^{\rho}$  and  $V^{\sigma}$  s.t.

$$f(v + w) = f(v) + f(w) \quad \forall v, w \in (V^{\sigma}, +)$$

$$f(rv) = rf(v) \quad \forall r = \sum_{g \in G} a_g g \in kG$$

Kosmann-Schwarzbach's **Groups and Symmetries**[5] is a very lucid text that's mathematically rigorous enough and practical for physicists. It's really good and very clear. Let's follow its development for  $SU(2), SO(3), SL(2, \mathbb{C})$  and corresponding

□ Lie algebras  $\mathfrak{su}(2)$ ,  $\mathfrak{so}(3)$ ,  $\mathfrak{sl}(2, \mathbb{C})$ .

From Chapter 2 "Representations of Finite Groups" of Kosmann-Schwarzbach (2010) [5]

**Definition 5** (2.1 Kosmann-Schwarzbach (2010)[5]). *On  $L^2(G)$ , scalar product defined by*

$$\langle f_1 | f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

$f_1, f_2 \in \mathcal{F}(G) \equiv \mathbb{C}[G]$  vector space of functions on  $G$  taking values on  $\mathbb{C}$

**Definition 6** (2.3 Kosmann-Schwarzbach (2010)[5]). *Let  $(E, \rho)$  be representation of  $G$*

character of  $\rho \equiv \chi_{\rho} : G \rightarrow \mathbb{C}$

$$\chi_{\rho}(g) = \text{tr}(\rho(g)) = \sum_{i=1}^n (\rho(g))_{ii}$$

Note: equivalent representations have same character  
each conjugacy class of  $G$ , function  $\chi_p$  is constant

Looking at Def. 5

$$\langle \chi_{\rho_1} | \chi_{\rho_2} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2}(g)$$

since  $\overline{\chi_{\rho_1(g)}} = \chi_{\rho_1}(g^{-1})$  by unitarity of representation with respect to scalar product  $\langle , \rangle$

**Proposition 4** (2.7 Kosmann-Schwarzbach (2010)[5]). Let  $(E_1, \rho_1)$  be representations of  $G$ , let linear  $u : E_1 \rightarrow E_2$ .  
 $(E_2, \rho_2)$

Then  $\exists$  linear  $T_u$  s.t.

$$(1) \quad \begin{aligned} T_u &: E_1 \rightarrow E_2 \\ T_u &= \frac{1}{|G|} \sum_{g \in G} \rho_2(g) u \rho_1(g)^{-1} \end{aligned}$$

so that  $\rho_2(g) T_u = T_u \rho_1(g) \quad \forall g \in G$

Proof.

$$\rho_2(g) T_u = \frac{1}{|G|} \sum_{h \in G} \rho_2(gh) u \rho_1(h^{-1}) = \frac{1}{|G|} \sum_{k \in G} \rho_2(k) u \rho_1(k^{-1}g) = T_u \rho_1(g)$$

Thus, diagrammatically, we have that

$$\begin{array}{ccc} E_1 & \xrightarrow{T_u} & E_2 \\ \downarrow \rho_1(g) & & \downarrow \rho_2(g) \\ E_1 & \xrightarrow{T_u} & E_2 \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} E_1 & \xrightarrow{u} & E_2 \end{array}$$

From Definition 1.12 of Kosmann-Schwarzbach [5], “representations  $\rho_1$  and  $\rho_2$  are called **equivalent** if there is a bijective intertwining operator for  $\rho_1$  and  $\rho_2$ .” So I will interpret this as if an intertwining operator is not bijective, then the representations  $\rho_1, \rho_2$  are not equivalent.

**Proposition 5** (2.8 Kosmann-Schwarzbach (2010)[5]). Let  $(E_1, \rho_1)$  be irreducible representations of  $G$ , let linear  $u : E_1 \rightarrow E_2$ ,  
 $(E_2, \rho_2)$

define  $T_u$  by  $T_u = \frac{1}{|G|} \sum_{g \in G} \rho_2(g) u \rho_1(g)^{-1}$  by Eq. 1.

- (i) If  $\rho_1, \rho_2$  inequivalent, then  $T_u = 0$
- (ii) If  $E_1 = E_2 = E$  and  $\rho_1 = \rho_2 = \rho$ , then

$$T_u = \frac{\text{tr}(u)}{\dim E} 1_E$$

Proof. (i) if  $\rho_1, \rho_2$  are inequivalent, by definition,  $T_u$  is not isomorphic. Then by Schur’s lemma (first part),  $T_u = 0$

<sup>1</sup>Mathematics stackexchange Isomorphism between Hom and tensor product [duplicate] <http://math.stackexchange.com/questions/428185/isomorphism-between-hom-and-tensor-product>  
<http://math.stackexchange.com/questions/57189/understanding-isomorphic-equivalences-of-tensor-product>

(ii) By Schur’s lemma,  $T_u(v) = \lambda v \quad \forall v \in E = E_1 = E_2$ . So  $T_u = \lambda 1_E$ .  $\text{tr} T_u = \lambda \dim E$  or  $\lambda = \frac{\text{tr} T_u}{\dim E}$ . Thus,  $T_u = \frac{\text{tr} T_u}{\dim E} 1_E$

□

Let  $(e_1 \dots e_n)$  basis of  $E$

$(f_1 \dots f_p)$  basis of  $F$

$$\begin{aligned} \forall u \in \mathcal{L}(E, F), \quad & u : E \rightarrow F \\ & u(x) = u(x^j e_j) = x^j u(e_j) = x^j u^i_j f_i \quad \text{for } x = x^j e_j \in E \\ & y = y^i f_i \in F \\ & u = u^i_j e^j \otimes f_i \end{aligned}$$

For

$$\begin{aligned} T &: E^* \otimes F \rightarrow \mathcal{L}(E, F) \\ T(\xi \otimes y) &= u^i_j e^j \otimes f_i \text{ i.e. set } T(\xi \otimes y) \text{ to this } u \\ T(\xi \otimes y) &= T(\xi_l e^l \otimes y^k f_k) = \xi_l y^k T(e^l \otimes f_k) = (\xi_l y^k T^{li}_{kj}) e^j \otimes f_i \implies \xi_l y^k T^{li}_{kj} = u^i_j \end{aligned}$$

Exercises. Exercises of Ch. 2 Representations of Finite Groups [5]

**Exercise 2.6.** [5] The dual representation.

Let  $(E, \pi)$  representation of group  $G$ .

□  $\forall g \in G, \xi \in E^*, x \in E$ , set  $\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle$

(a) dual (or contragredient) of  $\pi, \pi^* : G \rightarrow \text{End}(E^*)$ ,  $\pi^*$  is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x \rangle &= \langle \xi, \pi((gh)^{-1})(x) \rangle = \langle \xi, \pi(h^{-1}g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x)) \rangle = \\ &= \langle \pi^*(h)(\xi), \pi(g^{-1})(x) \rangle = \langle \pi^*(g)\pi^*(h)(\xi), x \rangle \end{aligned}$$

since this is true,  $\forall x \in E, \forall \xi \in E^*, \pi^*(gh) = \pi^*(g)\pi^*(h)$ .  
dual  $\pi^*$  of  $\pi$  is a representation.

(b) Consider  $G \times \mathcal{L}(E, F) \rightarrow \mathcal{L}(E, F)$ .

$$g \cdot u = \rho(g) \circ u \circ \pi(g^{-1})$$

Define

$$\begin{aligned} \sigma &: G \rightarrow \text{End}(\mathcal{L}(E, F)) \\ \sigma(g) &: \mathcal{L}(E, F) \rightarrow \mathcal{L}(E, F) \\ \sigma(g)(u) &= \rho(g) \circ u \circ \pi(g^{-1}) \end{aligned}$$

Let  $(e_1 \dots e_n)$  be a basis of  $E$ . Let  $\xi = \xi_i e^i \in E^*, x = x^j e_j \in E$ .

Consider the isomorphism  $T : E^* \otimes F \rightarrow \mathcal{L}(E, F)$  defined as<sup>1</sup>

$$\begin{aligned} T &: E^* \otimes F \rightarrow \mathcal{L}(E, F) = \text{Hom}(E, F) \\ \xi \otimes y &\mapsto (x \mapsto \xi(x)y) \end{aligned}$$

Choose bases  $(e_1 \dots e_n)$  of  $E$   
 $(e^1 \dots e^n)$  of  $E^*$ . Then  
 $(f_1 \dots f_p)$  of  $F$

$$T(e^j \otimes f_i)(x) = T(e^j \otimes f_i)(x^k e_k) = \delta_k^j x^k f_i = x^j f_i$$

$$T(e^j \otimes f_i)(e_k) = \delta_k^j f_i$$

Consider

$$u \in \mathcal{L}(E, F)$$

$$u : E \rightarrow F$$

$$u(x) = u(x^j e_j) = x^j u(e_j) = x^j u^i_j f_i$$

$$u(e_j) = u^i_j f_i \text{ i.e. } u : e_j \rightarrow u^i_j f_i$$

Then  $\forall u \in \mathcal{L}(E, F)$ ,

$$T(u^i_j e^j \otimes f_i)(e_k) = u^i_j \delta_k^j f_i = u^i_k f_i = u(e_k) \implies u = T(u^i_j e^j \otimes f_i)$$

so  $T$  is surjective.

With  $T(\xi \otimes y) = T(\xi' \otimes y')$ ,

$$T(\xi \otimes y)(x) = T(\xi' \otimes y')(x)$$

$$\xi(x)y = \xi'(x)y' \implies \xi(x)y - \xi'(x)y' = 0$$

which implies that  $\xi \otimes y = \xi' \otimes y'$ . So  $T$  is injective. Or, one could consider that  $T^{-1} : \mathcal{L}(E, F) \rightarrow E^* \otimes F$ ,  $T^{-1} : u \mapsto u^i_j e^j \otimes f_i$ , which is the inverse of  $T$ .

**Remark 1.**

$$E^* \otimes F \xrightarrow{T} \mathcal{L}(E, F) = \text{Hom}(E, F)$$

$$(\xi, y) \mapsto (x \mapsto \xi(x)y)$$

and so  $(e^j \otimes f_i) \mapsto (x \mapsto e^j(x)f_i = x^j f_i)$

So  $E^* \otimes F$  is isomorphic to  $\mathcal{L}(E, F) = \text{Hom}(E, F)$

For representation  $\pi$ ,

$$\pi : G \rightarrow \text{End}(E)$$

$$\pi(g) : E \rightarrow E$$

$$\pi(g)(x) = \pi(g)(x^j e_j) = x^j \pi(g)(e_j) = x^j \pi(g)^i_j e_i = (\pi(g)^i_j x^j e_i$$

Consider this matrix formulation:

$$\pi^*(g)(\xi) = \pi^*(g)(\xi_i e^i) = \xi_i \pi^*(g)(e^i) = \xi_i (\pi^*(g))^i_j e^j$$

$$\implies \langle \pi^*(g)(\xi), x \rangle = \xi_i (\pi^*(g))^i_j x^j$$

and

$$\langle \xi, \pi(g^{-1})(x) \rangle = \xi_i \pi(g^{-1})^i_j x^j$$

so that

$$\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle \implies \pi(g^{-1})^i_j = (\pi^*(g))^i_j$$

Thus, given a choice of basis for  $E$ , the *dual* of  $\pi$ ,  $\pi^*(g)^i_j$ , and  $\pi(g^{-1})^i_j$  are formally equal.

So for a choice of basis of  $E$  and of  $F$ ,

$$(\pi^* \otimes \rho)(g)(\xi, y) = (\pi^*(g) \otimes \rho(g))(\xi, y) = \pi^*(g)\xi \otimes \rho(g)y = \xi_i \pi(g^{-1})^l_j e^j \otimes \rho(g)^i_k y^k f_i = \rho(g)^i_k y^k \xi_l \pi(g^{-1})^l_j e^j \otimes f_i$$

Applying  $T$ ,

$$T(\pi^* \otimes \rho)(g)(\xi, \rho) = \rho(g)^i_k y^k \xi_l \pi(g^{-1})^l_j = \rho(g)T(\xi, y)\pi(g^{-1})$$

$$\begin{array}{ccc} E^* \otimes F & \xrightarrow{T} & \mathcal{L}(E, F) \\ \downarrow (\pi^* \otimes \rho)(g) & & \downarrow \sigma(g) \\ E^* \otimes F & \xrightarrow{T} & \mathcal{L}(E, F) \end{array} \quad \begin{array}{ccc} (\xi, y) & \xrightarrow{T} & (x \mapsto \xi(x)y) = y^i \xi_j \\ \downarrow (\pi^* \otimes \rho)(g) & & \downarrow \sigma(g) \\ \pi^*(g)(\xi) \otimes \rho(g)y & \xrightarrow{T} & \rho(g)y^i \xi_j \pi(g^{-1}) = \rho(g)T(\xi, y)\pi(g^{-1}) \end{array}$$

Thus

Thus, representation  $\sigma(g)$  is equivalent to representation  $(\pi^* \otimes \rho)$ , a tensor product of representations.

**Exercise 2.15.** Representation of  $GL(2, \mathbb{C})$  on the polynomials of degree 2

Let group  $G$ , let representation  $\rho$  of  $G$  on  $V = \mathbb{C}^n$ , i.e.  $\rho : G \rightarrow \text{End}(V)$

Let  $P^{(k)}(V)$  vector space of complex polynomials on  $V$  that are homogeneous of degree  $k$ .

For  $f \in P^{(k)}(V)$ , the general form is

$$f = \sum_{\substack{i_1 + i_2 + \dots + i_n = k \\ 0 \leq i_j \leq k}} a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

Given

$$\binom{n+k}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \dots + \binom{n+k-1}{k-1} = \sum_{i=0}^n \binom{k-1+i}{k-1}$$

$\binom{k+n-1}{n-1}$  is number of monomials of degree  $k$ .

So  $\dim P^{(k)}(V) = \binom{k+n-1}{n-1}$ . This is a very lucid and elementary exposition on the basics of polynomials which I found was

useful for the basic facts I forgot<sup>2</sup>.

So we have the graded algebra

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V)$$

$$\rho^{(k)} : G \rightarrow \text{End}(P^{(k)}(V))$$

$$\rho^{(k)}(g) : P^{(k)}(V) \rightarrow P^{(k)}(V)$$

$$\rho^{(k)}(g)(f) = f \circ \rho(g^{-1})$$

This is a representation of  $G$  since

(a)

$$\rho^{(k)}(gh)(f) = f \circ \rho((gh)^{-1}) = f \circ \rho(h^{-1}g^{-1}) = f \circ \rho(h^{-1}\rho(g^{-1})) \implies \rho^{(k)}(gh) = \rho^{(k)}(g)\rho^{(k)}(h)$$

$$\rho^{(k)}(g)\rho^{(k)}(h)(f) = \rho^{(k)}(g)(f \circ \rho(h^{-1})) = f \circ \rho(h^{-1}) \circ \rho(g^{-1})$$

(b) Choose basis  $(e_1 \dots e_n)$  of  $V$ ,  $x = x^j e_j \in V$ ,  $\rho : G \rightarrow \text{End}(V)$ , and so  $\rho(g)(x) = \rho(g)(x^j e_j) = x^j \rho(g)(e_j) = x^j (\rho(g))^i_j e_i$ .

With  $\xi(e_i) = \xi_i \implies \langle \xi, \rho(g^{-1})x \rangle = \xi_i x^j (\rho(g^{-1}))^i_j$

$\forall \xi \in V^*$ ,  $\xi = \xi_i e^i$ ,

$$\rho^*(g)(\xi) = \rho^*(g)(\xi_i e^i) = \xi_i \rho^*(g)^i_j e^j$$

$$\implies \langle \rho^*(g)(\xi), x \rangle = \xi_i x^j (\rho^*(g))^i_j \implies (\rho^*(g))^i_j = (\rho(g^{-1}))^i_j$$

So  $\forall f \in P^{(1)}(V)$ ,  $x \in V$ ,  $\rho(g^{-1})x = x^j (\rho(g^{-1}))^i_j e_i$ . So  $f \circ \rho(g^{-1})(x) = \sum_{i=1}^n a_i (\rho(g^{-1}))^i_j x^j = \sum_{i=1}^n a_i (\rho^*(g))^i_j x^j$

$$\implies \rho^{(1)}(g)(f) = f \circ \rho^*(g)$$

<sup>2</sup>Polynomials. Math 4800/6080 Project Course <http://www.math.utah.edu/~bertram/4800/PolyIntroduction.pdf>



(c) Suppose  $G = GL(2, \mathbb{C})$ ,  $V = \mathbb{C}^2$ ,  $\rho$  fundamental representation  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $g^{-1} = \frac{1}{\det g} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$  for  $\det g = ad - bc$ .

Let  $k = 2$ ,  $\dim P^{(2)}(\mathbb{C}^2) = \binom{2+2-1}{2-1} = \binom{3}{1} = 3$   
 $\forall f \in P^{(2)}(\mathbb{C}^2)$ ,  $f(x, y) = Ax^2 + 2Bxy + Cy^2$

Let

$$P^{(2)}(\mathbb{C}^2) \rightarrow \mathbb{C}^3$$

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 \mapsto \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$$

Call this transformation  $T$ ,  $T : P^{(2)}(\mathbb{C}^2) \rightarrow \mathbb{C}^3$ .

$\forall \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$ ,  $f(x, y) = Ax^2 + 2Bxy + Cy^2$  and  $Tf(x, y) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ .  $T$  surjective.

Suppose  $Tf(x, y) = Tf'(x, y)$ ,

$$\begin{aligned} \implies Ax^2 + 2Bxy + Cy^2 &= A'x^2 + 2B'xy + C'y^2 \\ \implies (A - A')x^2 + 2(B - B')xy + (C - C')y^2 &= 0 \end{aligned}$$

Then since the monomials form a basis, and its basis elements are independent (by definition), then  $A = A'$ ,  $B = B'$ ,  $C = C'$ .  $T$  injective. So  $T$  is bijective, an isomorphism.

(This is all in `groups.sage`)

```
sage: P2CC.<x,y> = PolynomialRing(CC,2) # this declares a PolynomialRing of field of complex numbers,
# of order 2 (i.e. only 2 variables for a polynomial, such as x, y)
sage: A = var('A')
sage: assume(A, 'complex')
sage: B = var('B')
sage: assume(B, 'complex')
sage: C = var('C')
sage: assume(C, 'complex')
sage: f(x,y) = A*x**2 + 2*B*x*y + C*y**2
```

```
sage: a = var('a')
sage: assume(a, 'complex')
sage: b = var('b')
sage: assume(b, 'complex')
sage: c = var('c')
sage: assume(c, 'complex')
sage: d = var('d')
sage: assume(d, 'complex')
sage: g = Matrix([[a,b],[c,d]])
sage: X = Matrix([[x],[y]])
sage: f((g.inverse()*X)[0,0], (g.inverse()*X)[1,0]).expand()
sage: f((g.inverse()*X)[0,0], (g.inverse()*X)[1,0]).expand().coefficient(x^2).full_simplify()
(C*c^2 - 2*B*c*d + A*d^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f((g.inverse()*X)[0,0], (g.inverse()*X)[1,0]).expand().coefficient(x*y).full_simplify()
-2*(C*a*c + A*b*d - (b*c + a*d)*B)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f((g.inverse()*X)[0,0], (g.inverse()*X)[1,0]).expand().coefficient(y^2).full_simplify()
(C*a^2 - 2*B*a*b + A*b^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
```

So

$$\begin{aligned} \rho^{(2)}(g)(f)(x, y) &= f \circ \rho(g^{-1})(x, y) = \\ &= \frac{Cc^2 - 2Bcd + Ad^2}{(ad - bc)^2}x^2 + -2\frac{(Cac + Abd - (bc + ad)B)}{(ad - bc)^2}xy + \frac{Ca^2 - 2Bab + Ab^2}{(ad - bc)^2}y^2 \end{aligned}$$

So define  $\tilde{\rho} : G \rightarrow \text{End}(\mathbb{C}^3)$ .  $\tilde{\rho}$  is a representation, for

$$\forall v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3, \quad \tilde{\rho}(gh)(v) = T \circ f \circ \rho((gh)^{-1}) = T \circ f \circ \rho(h^{-1}g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1})$$

Now  $\tilde{\rho}(h)(v) = T \circ f \circ \rho(h^{-1})$

$$\implies \tilde{\rho}(g)\tilde{\rho}(h)(v) = T \circ (f \circ \rho(h^{-1})) \circ \rho(g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1}) \text{ and so}$$

$$\tilde{\rho}(gh) = \tilde{\rho}(g)\tilde{\rho}(h)$$

And so

$$\tilde{\rho}^*(g)(v) = Tf\rho(g^{-1})$$

and consider this commutation diagram, that (helped me at least and) clarifies the relationships:

$$\begin{array}{ccc} P^{(2)}(\mathbb{C}^2) & \xrightarrow{T} & \mathbb{C}^3 \\ \rho^{(2)}(g) \downarrow & & \downarrow \tilde{\rho}(g) \\ P^{(2)}(\mathbb{C}^2) & \xrightarrow{T} & \mathbb{C}^3 \end{array} \quad \begin{array}{ccc} f & \xrightarrow{T} & \begin{pmatrix} A \\ B \\ C \end{pmatrix} \\ \rho^{(2)}(g) \downarrow & & \downarrow \tilde{\rho}(g) \\ f \circ \rho(g^{-1}) & \xrightarrow{T} & \begin{pmatrix} D \\ E \\ F \end{pmatrix} \end{array}$$

with

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{Cc^2 - 2Bcd + Ad^2}{(ad - bc)^2} \\ -2\frac{(Cac + Abd - (bc + ad)B)}{(ad - bc)^2} \\ \frac{Ca^2 - 2Bab + Ab^2}{(ad - bc)^2} \end{pmatrix}$$

Now define the dual  $\tilde{\rho}^*$  as such:

$$\tilde{\rho}^*(g) : (\mathbb{C}^3)^* \rightarrow (\mathbb{C}^3)^*$$

$$\tilde{\rho}^*(g) = \tilde{\rho}(g^{-1})$$

$$\forall \xi \in (\mathbb{C}^3)^*$$

$$\tilde{\rho}^*(g)\xi = \xi_i(\tilde{\rho}^*(g))^i_j e^j = \xi_i(\tilde{\rho}(g^{-1}))^i_j e^j$$

So for  $v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$ ,  $f = T^{-1}v = Ax^2 + 2Bxy + Cy^2 \in P^2(\mathbb{C}^2)$ ,

$$\tilde{\rho}(g^{-1})(v) = T \circ (f\rho(g)) = \begin{bmatrix} Aa^2 + 2Bac + Cc^2 \\ Aab + Bbc + Bad + Ccd \\ Ab^2 + 2Bbd + Cd^2 \end{bmatrix}$$

which was found using Sage Math:

```
sage: f((g*X)[0,0],(g*X)[1,0])
(a*x + b*y)^2*A + 2*(a*x + b*y)*(c*x + d*y)*B + (c*x + d*y)^2*C
sage: f((g*X)[0,0],(g*X)[1,0]).expand()
A*a^2*x^2 + 2*B*a*c*x^2 + C*c^2*x^2 + 2*A*a*b*x*y + 2*B*b*c*x*y + 2*B*a*d*x*y + 2*C*c*d*x*y + A*b^2*y^2 + 2*B*b*d*y^2 + C*d^2*y^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x^2)
A*a^2 + 2*B*a*c + C*c^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x*y)
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(y^2)
```

$$A*b^2 + 2*B*b*d + C*d^2$$

or

```
sage: T( f((g*X)[0,0] , (g*X)[1,0]) . expand() )
[A*a^2 + 2*B*a*c + C*c^2,
 2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d,
 A*b^2 + 2*B*b*d + C*d^2]
```

So then

$$\tilde{\rho}(g^{-1}) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

So then

$$\tilde{\rho}^*(g) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

and operate on row vectors  $\xi \in (\mathbb{C}^3)^*$  with  $\tilde{\rho}^*(g)$  from the row vector’s right.

More: Let  $G = SU(2)$ . Then  $U = e^{i\phi} \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$

$$\tilde{\rho}: SU(2) \rightarrow \text{End}(\mathbb{C}^3)$$

$$\tilde{\rho}(U): \mathbb{C}^3 \rightarrow \mathbb{C}^3$$

$$\tilde{\rho}(U)(v) = e^{-2i\varphi} \begin{bmatrix} A\bar{a}^2 + 2B\bar{a}\bar{b} + C\bar{b}^2 \\ -A\bar{a}b + B + Ca\bar{b} \\ Ab^2 - 2Bab + Ca^2 \end{bmatrix}$$

$$\implies \tilde{\rho}(U) = e^{-2i\varphi} \begin{bmatrix} -\bar{a}^2 & 2\bar{a}\bar{b} & \bar{b}^2 \\ -\bar{a}b & 1 & a\bar{b} \\ b^2 & -2ab & a^2 \end{bmatrix}$$

From Chapter 4 “Lie Groups and Lie Algebras” of Kosmann-Schwarzbach (2010) [5]

While Proposition 2.6 of Kosmann-Schwarzbach (2010) [5] states that

$$\det(\exp(X)) = \exp(\text{tr}X)$$

here are some other resources online that gave further discussion on the characteristic polynomial,  $\det(A - \lambda 1)$  and the different terms of it, called Newton identities:

- [http://scipp.ucsc.edu/~haber/ph116A/charpoly\\_11.pdf](http://scipp.ucsc.edu/~haber/ph116A/charpoly_11.pdf)
- <http://math.stackexchange.com/questions/1126114/how-to-find-this-lie-algebra-proof-that-mathfraksl-is-trace-zero-matrice>
- <http://mathoverflow.net/questions/131746/derivative-of-a-determinant-of-a-matrix-field>

**Theorem 2** (5.1 [5]). *Consider  $\mathfrak{g} = \{X = \gamma'(0)|\gamma: 1 \rightarrow G \text{ of class } C^1, \gamma(0) = 1\}$*

*Let Lie group  $G$*

- (i)  $\mathfrak{g}$  *vector subspace of  $\mathfrak{gl}(n, \mathbb{R})$*
- (ii)  $X \in \mathfrak{g}$  *iff  $\forall t \in \mathbb{R}, \exp(tX) \in G$*
- (iii) *if  $X \in \mathfrak{g}$ , if  $g \in G$ , then  $gXg^{-1} \in \mathfrak{g}$*
- (iv)  $\mathfrak{g}$  *closed under matrix commutator, i.e. if  $X, Y \in \mathfrak{g}, [X, Y] \in \mathfrak{g}$*

*Proof.*

- (i)
- (ii) If  $\exp(tX) \in G$ , then  $X \frac{d}{dt} \exp(tX) \Big|_{t=0} \in \mathfrak{g}$  (by def.)  
If  $X \in \mathfrak{g}$ , then by def.,  $X = \frac{d}{dt} \gamma(t) \Big|_{t=0}$  with  $\gamma(t) \in G$ .  
Now Taylor expand;  $\forall k \in \mathbb{Z}^+$

$$\gamma\left(\frac{t}{k}\right) = 1 + \frac{t}{k}X + O\left(\frac{1}{k^2}\right) = \exp\left(\frac{t}{k}X + O\left(\frac{1}{k^2}\right)\right)$$

$$\implies \left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp(tX)$$

$$\gamma\left(\frac{t}{k}\right) \in G \quad \forall k \in \mathbb{Z}^+$$

$G$  closed subgroup, so  $\lim_{k \rightarrow \infty} (\gamma(\frac{t}{k}))^k = \exp(tX) \in G$

(iii)

(iv)

□

**Definition 7.** *Lie algebra  $\mathfrak{g}$ , tangent space to  $G$  at 1, i.e.  $\mathfrak{g} := T_1G$  is called Lie algebra of Lie group  $G$ .*

$$\mathfrak{g} := \{X = \gamma'(0)|\gamma: 1 \rightarrow G \text{ of class } C^1, \gamma(0) = 1\} = T_1G$$

This is based on Proposition 5.3 of Kosmann-Schwarzbach (2010) [5].

For Lie group

$$U(n) = \{U \in GL(n, \mathbb{C})|UU^\dagger = 1\}$$

If  $X \in \mathfrak{u}(n)$ , then  $\exp(tX) \in U(n)$ . Then

$$\exp(tX) \exp(tX)^\dagger = (1 + tX + O(t^2))(1 + tX^\dagger + O(t^2)) = 1 + t(X + X^\dagger) + O(t^2) = 1 \forall t \in \mathbb{R} \implies X + X^\dagger = 0$$

i.e.  $X \in \mathfrak{u}(n)$  is an anti-Hermitian complex  $n \times n$  matrix.

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C})|X + X^\dagger = 0\}$$

*Physicists:*  $X = iA$  and so  $A - A^\dagger$ .  $A \in \mathfrak{u}(n)$  is a Hermitian complex  $n \times n$  matrix.

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C})|A - A^\dagger = 0\}$$

Regardless,  $\dim_{\mathbb{R}}\mathfrak{u}(n) = n^2 = 2n^2 - n^2$

For Lie group

$$SU(n) = \{U \in GL(n, \mathbb{C})|UU^\dagger = 1, \det U = 1\}$$

Then

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C})|X + X^\dagger = 1, \text{tr}X = 0\}$$

is the Lie algebra of traceless anti-Hermitian complex  $n \times n$  matrices, and that

$$\dim_{\mathbb{R}}\mathfrak{su}(n) = n^2 - 1$$

In summary,

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C})|X + X^\dagger = 0\}$$

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C})|X + X^\dagger = 0, \text{tr}X = 0\}$$

$$\exp(tX) \Big\downarrow$$

$$\exp(tX) \Big\downarrow$$

$$U(n) = \{U \in GL(n, \mathbb{C})|UU^\dagger = 1\}$$

$$SU(n) = \{U \in GL(n, \mathbb{C})|UU^\dagger = 1, \det U = 1\}$$

$$\dim_{\mathbb{R}}\mathfrak{u}(n) = n^2$$

$$\dim_{\mathbb{R}}\mathfrak{su}(n) = n^2 - 1$$

From Chapter 5 “Lie Groups  $SU(2)$  and  $SO(3)$ ” of Kosmann-Schwarzbach (2010) [5],



1.0.1. *Bases of  $\mathfrak{su}(2)$ , Subsection 1.1 of Chapter 5 of Kosmann-Schwarzbach (2010) [5].* Recall that

$$\begin{array}{c} \mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X + X^\dagger = 0, \operatorname{tr} X = 0\} \\ \exp(tX) \downarrow \\ SU(n) = \{U \in GL(n, \mathbb{C}) \mid UU^\dagger = 1, \det U = 1\} \end{array}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

and so

$$\begin{array}{c} \mathfrak{su}(2) = \{X \in \mathfrak{gl}(2, \mathbb{C}) \mid X + X^\dagger = 0, \operatorname{tr} X = 0\} \\ \exp(tX) \downarrow \\ SU(2) = \{U \in GL(2, \mathbb{C}) \mid UU^\dagger = 1, \det U = 1\} \end{array}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(2) = 3$$

Also, recall that  $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{C})$  is a vector subspace (2) and that  $X \in \mathfrak{g}$  iff  $\forall t \in \mathbb{R}, \exp(tX) \in G$ .  
if  $X \in \mathfrak{g}$ , if  $g \in G$ , then  $gXg^{-1} \in \mathfrak{g}$

$\mathfrak{g}$  closed under  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$

$$(X, Y) \mapsto [X, Y]$$

and so with  $\mathfrak{g}$  as a vector space, we can have a choice of bases.

$$\begin{array}{l} \xi_1 = \frac{i}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \text{(a) } \xi_2 = \frac{1}{2} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\ \xi_3 = \frac{i}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ \text{satisfying} \end{array}$$

$$[\xi_k, \xi_l] = \epsilon_{klm} \xi_m$$

$$\begin{array}{l} \sigma_1 = -2i\xi_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \sigma_2 = 2i\xi_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \\ \sigma_3 = -2i\xi_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{array}$$

satisfying

$$[\sigma_k, \sigma_l] = 2i\epsilon_{klm} \sigma_m$$

EY : 20151001 Sage Math 6.8 doesn't run on Mac OSX El Capitan: I suspect that it's because in Mac OSX El Capitan, `/usr` cannot be modified anymore, even in an Administrator account. The TUG group for MacTeX had a clear, thorough, and useful (i.e. copy UNIX commands, paste, and run examples) explanation of what was going on:

<http://tug.org/mactex/elcapitan.html>

So keep in mind that my code for Sage Math is for Sage Math 6.8 that doesn't run on Mac OSX El Capitan. I'll also use `sympy` in Python as an alternative and in parallel.

One can check in `sympy` the traceless anti-Hermitian (or Hermitian) property of the bases and Pauli matrices, and the commutation relations (see `groups.py`):

```
import itertools
from itertools import product, permutations

import sympy
from sympy import I, LeviCivita
from sympy import Rational as Rat

from sympy.physics.matrices import msigma # <class 'sympy.matrices.dense.MutableDenseMatrix'>

def commute(A,B):
    """
    """
    """commute==commute(A,B)
    """commute_takes_the_commutator_of_A_and_B
    """
    """
    return (A*B - B*A)

def xi(i):
    """
    """
    """xi==xi(i)
    """xi is a function that returns the independent basis for
    """Lie algebra su(2)\equiv su(2,\mathbb{C}) of Lie group SU(2) of
    """traceless anti-Hermitian matrices, based on msigma of sympy
    """cf. http://docs.sympy.org/dev/modules/sympy/physics/matrices.html#msigma
    """
    if i not in [1,2,3]:
        raise IndexError("Invalid Pauli index")
    elif i==1:
        return I/Rat(2)*msigma(1)
    elif i==2:
        return -I/Rat(2)*msigma(2)
    elif i==3:
        return I/Rat(2)*msigma(3)

## check anti-Hermitian property and commutation relations with xi
# xi is indeed anti-Hermitian
xi(1) == -xi(1).adjoint() # True
xi(2) == -xi(2).adjoint() # True
xi(3) == -xi(3).adjoint() # True

# xi obeys the commutation relations
```

(b) *Physics*

```
for i,j in product([1,2,3],repeat=2): print i,j

for i,j in product([1,2,3],repeat=2): print i,j, "\tCommutator: ", commute(xi(i),xi(j))

## check traceless Hermitian property and commutation relations with Pauli matrices
# Pauli matrices i.e. msigam is indeed traceless Hermitian

msigma(1) == msigma(1).adjoint() # True
msigma(2) == msigma(2).adjoint() # True
msigma(3) == msigma(3).adjoint() # True

msigma(1).trace() == 0 # True
msigma(2).trace() == 0 # True
msigma(3).trace() == 0 # True

# Pauli matrices obey commutation relation
print "For Pauli matrices, the commutation relations are:\n"
for i,j in product([1,2,3],repeat=2): print i,j, "\tCommutator: ", commute(msigma(i),msigma(j))

for i,j,k in permutations([1,2,3],3): print "Commutate: ", i,j,k, msigma(i), msigma(j), \
": and is 2*i_of ", msigma(k), commute(msigma(i),msigma(j)) == 2*I*msigma(k)*LeviCivita(i,j,k)
```

And finally the traceless property of the Pauli matrices:

```
>>> msigma(1).trace()
0
>>> msigma(2).trace()
0
>>> msigma(3).trace()
0
```

1.1. **Spin.** Let’s follow the development by Baez and Muniain (1994) on pp. 175 of the Section II.1 “Lie Groups”, the second (II) chapter on “Symmetry” [1].  
Let  $V = \mathbb{C}^2$ ,  $G = SU(2)$ . Then consider the graded algebra of polynomials on  $V = \mathbb{C}^2 \ni (x, y)$

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V) = \bigoplus_{\substack{j=0 \\ 2j \in \mathbb{Z}}}^{\infty} P^{(2j)}(V) = \bigoplus_{\substack{j=0 \\ j \in \mathbb{Z}}}^{\infty} P^{(2j)}(V) \oplus \bigoplus_{\substack{j=1/2 \\ 2j \text{ odd}}}^{\infty} P^{(2j)}(V)$$

$P^{(2j)}(V) \equiv$  vector space of complex polynomials of degree  $2j$

and recall this representation on  $P^{(2j)}(V)$

$$\begin{aligned} \rho^{(2j)} : G &\rightarrow \text{End}(P^{(2j)}(V)) \\ \rho^{(2j)} : P^{(2j)}(V) &\rightarrow P^{(2j)}(V) \\ \rho^{(2j)}(g)(f) &= f \circ \rho(g^{-1}) \text{ where } \rho \text{ is the fundamental representation of } G = SU(2) \\ \rho^{(2j)}(g)(f)(v) &= f \circ \rho(g^{-1})(v) \quad \forall f \in P^{(2j)}(V), \forall v \in V = \mathbb{C}^2 \end{aligned}$$

Note,  
 $\dim P^{(2j)} = \binom{2j+2-1}{2-1} = 2j+1$

**Exercise 21.** [1] *spin-0* Consider the trivial representation  $\tau$ :

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{T} & P^{(0)}(V) \\ \tau(g) \downarrow & & \downarrow \rho^{(0)}(g) \\ \mathbb{C} & \xrightarrow{T} & P^{(0)}(V) \end{array}$$
$$\begin{aligned} \tau : G &\rightarrow \text{End}(\mathbb{C}) \\ \tau(g) : \mathbb{C} &\rightarrow \mathbb{C} \\ \tau(g) &= 1_{\mathbb{C}} \end{aligned}$$

Clearly,  $P^{(0)}(V) = \mathbb{C}$ , since  $P^{(0)}(V)$  consists of polynomials of constants in  $\mathbb{C}$ .  
Consider  $c_0 \in \mathbb{C}$ ,  $f = k_0 \in P^{(0)}(V)$   
 $\rho^{(0)}(g)(f) = f \circ \rho(g^{-1}) = k_0$   
 $\implies \rho^{(0)}(g)T(c_0) = T \circ \tau(g)c_0 = T(c_0)$ . Let  $T = 1_{\mathbb{C}} = 1_{P^{(0)}(V)}$   
So  $\rho^{(0)}(g) = \tau(g) = 1$ .  $T = 1$ . So representations  $\rho^{(0)}$  and trivial representation  $\tau$  on  $G$  are equivalent.

**Exercise 22.** [1] *spin- $\frac{1}{2}$*  For spin- $\frac{1}{2}$ ,  $j = \frac{1}{2}$ ,  $2j = 1$ .

$\forall f \in P^{(1)}(V)$ ,  $V = \mathbb{C}^2$ . So in general form,  $f(x, y) = ax + by \in P^{(1)}(V)$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} \in V = \mathbb{C}^2$

Recall the fundamental representation

$$\begin{aligned} \rho : G &\rightarrow GL(2, \mathbb{C}) \equiv GL(\mathbb{C}^2) \\ \rho(g) : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ \rho(g) &= g \end{aligned}$$

So consider  $T$  such that

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{T} & P^{(1)}(V) \\ \rho(g) \downarrow & & \downarrow \rho^{(1)}(g) \\ \mathbb{C}^2 & \xrightarrow{T} & P^{(1)}(V) \end{array}$$

Consider  $\forall v \in \mathbb{C}^2$ ,  $v = \begin{pmatrix} x \\ y \end{pmatrix}$ , then

$$\rho(g)v = gv = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

```
sage: g*X
[a*x + b*y]
[c*x + d*y]
```

For notation, let  $U \in G = SU(2)$  s.t.  $UU^\dagger = 1$ .  
Consider  $(\rho^{(2j)}(U)(f))(x) = f(U^{-1}x)$ ,  $\forall x \in \mathbb{C}^2$ .

Choose  $f(x, y) = x$ . So for  $f(x, y) = Ax + By$ ,  $A = 1, B = 0$ . Choose  $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$  so  $U^{-1} = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$ . Then

$$U^{-1}x = \begin{pmatrix} \bar{a}x - by \\ \bar{b}x + ay \end{pmatrix}$$

So

$$\begin{aligned} (\rho^{(1)}(U)(f))(x) &= f(U^{-1}x) = \bar{a}x - by \\ (\rho^{(1)}(U)(f))(x) &= f(U^{-1}x) = \bar{b}x + ay \text{ for } f(x, y) = y \end{aligned}$$

Let  $f(x, y) = Ax + By$

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = (A\bar{a} + B\bar{b})x + (Ba - Ab)y = (\bar{a}x - by)A + (\bar{b}x + ay)B = (A\bar{a} + B\bar{b})x + (Ba - Ab)y$$

which was calculated with the assistance of Sage Math:

```
sage: U_try1 = Matrix( [[a.conjugate(), -b], [b.conjugate(), a] ] )
sage: f1( U_try1*X).coefficient(x)
A*conjugate(a) + B*conjugate(b)
sage: f1( U_try1*X).coefficient(y)
B*a - A*b
```

Treating  $P^{(1)}(\mathbb{C}^2)$  as a vector space, in its matrix formulation, then  $f(x, y) = Ax + By \in P^{(1)}(\mathbb{C}^2)$  is treated as  $\begin{bmatrix} A \\ B \end{bmatrix}$ , then  $(\rho^{(1)}(U)f)$  is

$$\implies \begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A\bar{a} + B\bar{b} \\ -Ab + Ba \end{bmatrix}$$

so conclude in general that  $\rho^{(1)}(U) = (U^\dagger)^T$ .

Now, as Kosmann-Schwarzbach (2010) [5] says, on pp. 13, Chapter 2 Representations of Finite Groups, “Two representations  $(E_1, \rho_1)$  and  $(E_2, \rho_2)$  are equivalent if and only if there is a basis  $B_1$  of  $E_1$  and a basis  $B_2$  of  $E_2$  such that for every  $g \in G$ , the matrix of  $\rho_1(g)$  in the basis  $B_1$  is equal to the matrix of  $\rho_2(g)$  in the basis  $B_2$ . In particular, if the representations  $(E_1, \rho_1)$  and  $(E_2, \rho_2)$  are equivalent, then  $E_1$  is isomorphic to  $E_2$ .” So we need a change of basis between  $\rho(U) = U$  and  $\rho^{(1)}(U)$ . What’s the linear transformation  $T$  s.t.

$$T^{-1}\rho^{(1)}(U)T = U?$$

By intuition,

$$T = \sigma_x \sigma_z \equiv \sigma_1 \sigma_3$$

where  $\sigma_i$ ’s are Pauli matrices.

Indeed,

```
sage: Paulimat [3] * Paulimat [1]*U_try*Paulimat [1] * Paulimat [3]
[conjugate(a) conjugate(b)]
[ -b          a          ]
```

Then  $\rho^{(1)}(U) \circ T = TU$ , so this  $T = \sigma_1 \sigma_3$  is an “intertwining operator” between  $\rho^{(1)}(U)$  and fundamental representation  $\rho(U) = U$ , with  $T = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ , and  $T^{-1} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ .

$T$  is an isomorphism between  $\mathbb{C}^2$  and  $P^{(1)}(\mathbb{C}^2)$ . So fundamental representation  $\rho$  of  $G = SU(2)$  is equivalent to  $\rho^{(1)}(U)$  on  $P^{(1)}(\mathbb{C}^2)$ .

**Exercise 23.** [1] (Also from Exercise 2.6 of Kosmann-Schwarzbach (201) [5])

Let  $(E, \pi)$  representation of group  $G$ .

$\forall g \in G, \xi \in E^*, x \in E$ , set  $\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle$

*dual* (or *contragredient*) of  $\pi$ ,  $\pi^* : G \rightarrow \text{End}(E^*)$ ,  $\pi^*$  is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x \rangle &= \langle \xi, \pi((gh)^{-1})(x) \rangle = \langle \xi, \pi(h^{-1}g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x)) \rangle = \\ &= \langle \pi^*(h)(\xi), \pi(g^{-1})(x) \rangle = \langle \pi^*(g)\pi^*(h)(\xi), x \rangle \end{aligned}$$

since this is true,  $\forall x \in E, \forall \xi \in E^*, \pi^*(gh) = \pi^*(g)\pi^*(h)$ .

dual  $\pi^*$  of  $\pi$  is a representation.

## Part 2. Bundles

[6]

<sup>3</sup>Werner Ballmann. “Vector bundles and connections”, <http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/conncurv1999.pdf>

## 2. VECTOR BUNDLES

Ballmann has a lucid and straightforward and useful exposition on vector bundles and connections [?] <sup>3</sup>.

$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

**Definition 8.**  $\mathbb{K}$ -vector bundle over  $M$  of rank  $k$  is bundle  $\pi : E \rightarrow M$ , fibers  $E_p := \pi^{-1}(p)$  are  $\mathbb{K}$ -vector spaces, s.t.  $\forall p \in M, \exists$  open  $U \subseteq M, U \ni p, \exists$  diffeomorphism (called local trivialization)  $\Phi : E|_{\pi^{-1}(U)} \rightarrow U \times \mathbb{K}^k$  s.t.

$$\pi \circ \Phi^{-1} = \pi_1 \quad \pi_1 : U \times \mathbb{R}^k \rightarrow U \text{ is canonical projection}$$

$$\forall q \in U, \Phi_q^{-1} : \mathbb{K}^k \rightarrow E_q \quad \text{is a } \mathbb{K}\text{-linear isomorphism.}$$

$$\Phi_q^{-1}(v) := \Phi^{-1}(q, v)$$

**Definition 9.** *frame* of  $E$  over  $U$  is  $k$ -triple  $(s_1 \dots s_k)$ ,  $s_i \in \Gamma(E|_{\pi^{-1}(U)})$ ,  $\sigma_i$  smooth section of  $E$  over  $U$  s.t.

$$\sigma_1(p) \dots \sigma_k(p)$$

basis of  $E_p \quad \forall p \in U$

By Prop. 9, frames and local trivializations are “equivalent”.

**Proposition 6.** conversely, if  $\Phi : E|_{\pi^{-1}(U)} \rightarrow U \times \mathbb{K}^k$  trivialization of  $E|_{\pi^{-1}(U)}$  and  $e_1 \dots e_k$  standard basis of  $\mathbb{K}^k$ , then  $k$ -tuple  $\sigma_i = \Phi^{-1}(\cdot, e_i)$ ,  $1 \leq i \leq k$  is a frame of  $E$  over  $U$ .

**Definition 10.** Let  $\Phi^{-1} = (s_1, \dots, s^k)$  local trivialization/frame.

Consider arbitrary  $s \in \Gamma(E|_{\pi^{-1}(U)})$

Then  $\exists \sigma = \sigma_{\Phi^{-1}} : U \rightarrow \mathbb{K}^k$  s.t.

$$s(p) = \Phi^{-1}(p, \sigma(p)) \quad \forall p \in U \text{ i.e. } s = \sigma^i s_i, \sigma = (\sigma^1 \dots \sigma^k)$$

$\sigma$  principal part of  $s$  with respect to  $\Phi$ .

Let  $\Psi^{-1} = (t_1 \dots t_k)$  another local trivialization of  $E$  over open  $V \subseteq M$ .

$$\begin{aligned} \forall p \in U \cap V, \text{ isomorphisms } \Phi_p^{-1} : \mathbb{K}^k \rightarrow E_k, \quad \Phi^{-1} = (s_1 \dots s_k), \text{ then} \\ \Psi_p^{-1} : \mathbb{K}^k \rightarrow E_k \quad \Psi^{-1} = (t_1 \dots t_k) \end{aligned}$$

$$s_j = g^i_j t_i$$

with smooth  $g^i_j : U \cap V \rightarrow \mathbb{K}, \forall p \in U \cap V, g^i_j(p) = a^i_j(p) \in \text{Mat}(k, k)$ , ( $k \times k$  matrix).

$g^i_j$  invertible so smooth  $g : U \cap V \rightarrow Gl(k, \mathbb{K})$ .

Let  $s \in \Gamma(E)$  over  $U \cap V$ ,  $\sigma_\Phi, \sigma_\Psi$  principal part of  $s$  with respect to  $\Phi, \Psi$ . Then

$$\sigma_\Psi^i = g^i_j \sigma_\Phi^j \text{ i.e. } \sigma_\Psi = g \cdot \sigma_\Phi$$

Indeed,  $\forall s \in \Gamma(E)$ ,

$$s = \sigma_\Phi^j s_j = \sigma_\Phi^j g^i_j t_i = g^i_j \sigma_\Phi^j t_i = \sigma_\Psi^i t_i$$

$$\begin{aligned} \nabla_X s &= (X(\sigma_\Phi^i) + \sigma_\Phi^j (\omega_\Phi)^i_j(X)) s_i = (X(\sigma_\Psi^i) + \sigma_\Psi^j (\omega_\Psi)^i_j(X)) t_i = (X(g^i_k \sigma_\Psi^k) + g^j_k \sigma_\Phi^k (\omega_\Psi)^i_j(X)) t_i = \\ &= X(\sigma_\Phi^k) s_k + t_i X(g^i_k) \sigma_\Phi^k + t_i (\omega_\Psi)^i_j(X) g^j_k \sigma_\Phi^k \end{aligned}$$

$X(\sigma_{\Phi}^j)s_i$  cancels from both sides and so

$$\begin{aligned} \implies s_i \sigma_{\Phi}^j (\omega_{\Phi})^i_j (X) &= s_i (\omega_{\Phi})^i_j (X) (g^{-1})^j_k (\sigma_{\Psi})^k = t_l g^l_i (\omega_{\Phi})^i_j (X) (g^{-1})^j_k \sigma_{\Psi}^k = \\ &= t_l X(g^l_k) (g^{-1})^k_i \sigma_{\Psi}^i + t_l (\omega_{\Psi})^l_j (X) \sigma_{\Psi}^j \\ \implies \omega_{\Psi}(X) &= g \omega_{\Phi}(X) g^{-1} - X(g)(g^{-1}) \end{aligned}$$

In summary,

$$\begin{aligned} \text{for connection } \nabla : \mathfrak{X}(M) \times \Gamma(E) &\rightarrow \Gamma(E) \\ \nabla(X, s) &= \nabla_X s = (\nabla s)(X) \in \Gamma(E) \end{aligned}$$

For frames  $\Phi^{-1} = (s_1 \dots s_k)$  of  $E$  over open  $U$ ,  $U \cap V \neq \emptyset$ ,  $\exists$  smooth  $g = (g^i_j) : U \cap V \rightarrow \text{Gl}(k; \mathbb{K})$  s.t.

$$\Psi^{-1} = (t_1 \dots t_k) \text{ of } E \text{ over open } V$$

$$s_j = t_i g^i_j \text{ or } \Phi^{-1} = \Psi^{-1} g$$

Then  $\forall s \in \Gamma(E_{\pi^{-1}(U \cap V)})$ ,  $s = \sigma_{\Phi}^j s_j = \sigma_{\Psi}^j t_j$ , so

$$\sigma_{\Psi}^i = g^i_j \sigma_{\Phi}^j \text{ or } \sigma_{\Psi} = g \sigma_{\Phi}$$

then

$$\nabla_X s = s_i (X(\sigma^i) + \omega^i_j(X) \sigma^j)$$

so that

$$(2) \quad \boxed{\omega_{\Psi}(X) = g \omega_{\Phi}(X) g^{-1} - X(g)(g^{-1}) \text{ or } (\omega_{\Psi})^i_j (X) = g^i_k (\omega_{\Phi})^k_l (X) (g^{-1})^l_j - X(g^i_k) (g^{-1})^k_j}$$

define covariant derivative  $d\mathbf{v} = (x, d\mathbf{v}|_x)$

$\forall \sigma \in \Gamma(E)$ , section  $\sigma$  of  $E$ . define

$$\nabla \sigma = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d(\varphi_U \circ \sigma|_U))$$

$$\begin{aligned} \nabla(f\sigma) &= \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d(\varphi_U \circ f \sigma|_U)) = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* ((d\varphi_U)(d(f \sigma|_U))) = \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* ((d\varphi_U)(\sigma|_U df + f d\sigma|_U)) = \\ &= \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d\varphi_U) \sigma|_U \otimes df + \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d\varphi_U) (f d\sigma|_U) = \sum_{U \in \mathcal{U}} \chi_U f \varphi_U^* d(\varphi_U \circ d\sigma|_U) + \sum_{U \in \mathcal{U}} \chi_U \varphi_U^* (d\varphi_U) \sigma|_U \otimes df = \\ &= f \nabla \sigma + \sigma \otimes df \end{aligned}$$

### 3. PRINCIPAL BUNDLES

Let's follow Taubes (2011) from Chapter 10 "Principal bundles", on [6].

**Definition 11.** Let smooth manifold  $M$ , Lie group  $G$ .

principal  $G$ -bundle  $\equiv$  smooth manifold  $P$  s.t.

- smooth action of  $G$  by diffeomorphisms  $\equiv$  map  $m : G \times P \rightarrow P$  s.t.
  - $m(1, p) = p$
  - $m(h, m(g, p)) = m(hg, p)$

Notation:  $(g, p) \mapsto pg^{-1}$

EY : 20151007 note that

$$(g, p) \mapsto pg^{-1} \xrightarrow{m(h)} pg^{-1}h^{-1} = p(hg)^{-1} = m(hg, p)$$

$$\begin{array}{ccc} G \times P & \longrightarrow & G \\ m \downarrow & & \\ P & & \end{array} \quad \begin{array}{ccc} (g, p) & \longmapsto & g \\ m \downarrow & & \\ m(p, g) = pg^{-1} & & \end{array}$$
  

$$\begin{array}{ccc} P|_{\pi^{-1}(U)} & \xrightarrow{\varphi} & U \times G \\ \pi \downarrow & \swarrow & \\ U \subset M & & \end{array} \quad \begin{array}{ccc} p & \xrightarrow{\varphi} & \varphi(p) = (\pi(p), h(p)) \\ \pi \downarrow & \swarrow & \\ \pi(p) & & \end{array}$$
  

$$\begin{array}{ccc} m(p, g) = pg^{-1} & \xrightarrow{\varphi} & \varphi(pg^{-1}) = (\pi(pg^{-1}), h(pg^{-1})) = (\pi(p), h(p)g^{-1}) \\ \pi \downarrow & \swarrow & \\ \pi(pg^{-1}) = \pi(p) & & \end{array}$$

cf. Section 10.1 "The definition" of Taubes (2011) [6]

3.0.1. *Cocycle definition* (Sec. 10.2 "A cocycle definition") [6]. Let locally finite, open cover  $\mathcal{U} = \{U_{\alpha}\}$  of  $M$  (recall *locally finite*:  $\forall p \in M$ ,  $\exists$  open  $\mathcal{O} \ni p$  s.t.  $\mathcal{O} \cap U_{\alpha}$  for only finite many  $\alpha$ 's).

**Definition 12.** principal bundle transition functions  $\equiv \{g_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow G\}_{U_{\alpha}, U_{\beta} \in \mathcal{U}}$  i.e. collection of smooth maps from intersections of  $U_{\alpha}, U_{\beta} \in \mathcal{U}$  s.t. cocycle constraints hold:

- $g_{\alpha\alpha} = 1$
- $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$
- if  $U_{\alpha}, U_{\beta}, U_{\gamma} \in \mathcal{U}$ ,  $U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \neq \emptyset$ ,  $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$

Claim: principle  $G$ -bundle  $P = \coprod_{U \in \mathcal{U}} U_{\alpha} \times G / \sim$

with equivalence relation  $\sim$  s.t.  $\forall (x, g) \in U_{\alpha} \times G$ ,  $(x, g) \sim (x', g')$  iff  $x = x'$  and  $g = g_{\alpha\beta} g'$   
 $(x', g') \in U_{\beta} \times G$

*Proof.* Let  $[(x, g)] \in \coprod_{U_{\alpha} \in \mathcal{U}} U_{\alpha} \times G / \sim$

Now for  $m : G \times P \rightarrow P$ ,  $m(h) : P \rightarrow P$  for  $h \in G$ ,

$$[(x, g)] = (x, g) \xrightarrow{m(h)} (x, gh^{-1})$$

$$[(x, g)] = (x, g') \xrightarrow{m(h)} (x, g'h^{-1}) = (x, g_{\alpha\beta}^{-1} gh^{-1}) = (x, g_{\beta\alpha} gh^{-1})$$

Now  $(x, gh^{-1}) \sim (x, g_{\beta\alpha} gh^{-1})$ . So we have a well-defined smooth action of  $G$  by diffeomorphisms  $m : G \times P \rightarrow P$ , but now on the equivalence classes of  $\coprod_{U_{\alpha} \in \mathcal{U}} U_{\alpha} \times G / \sim$ , i.e.

$$[(x, g)] \xrightarrow{m(h)} [(x, gh^{-1})]$$

Second,

$$[(x, g)] \xrightarrow{\pi} x \in M$$

unequivocally. So we have a surjective projection from  $P$  to  $M$ ,  $\pi$ , but now a surjective projection from  $\coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim$  to  $M$ .

Third, define a  $G$ -equivariant diffeomorphism as such: Let  $[(x, g)] \in \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim$ . In fact, let  $[(x, g)] \in U_\alpha \times G / \sim$ . So  $\forall x \in M, \exists U_\alpha \ni x, U_\alpha \in \mathcal{U}$ , and we *define*  $G$ -equivariant diffeomorphism  $\varphi : \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim \big|_{U_\alpha \times G} \rightarrow U_\alpha \times G$  s.t.

$$(3) \quad \begin{aligned} \varphi : \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G / \sim \bigg|_{U_\alpha \times G} &\rightarrow U_\alpha \times G \\ \varphi : [(x, g)] &\mapsto (x, g) \end{aligned}$$

Indeed, this  $\varphi$  does what we want, for

$$m(h)[(x, g)] = (x, gh^{-1}) \xrightarrow{\varphi} (x, gh^{-1})$$

3.0.2. *Frame bundles.* cf. Sec. 10.3 “Principal bundles constructed from vector bundles”, Subsection 10.3.1 “Frame bundles” of Taubes (2011) [6]

**Definition 13.** Let rank  $n$  vector bundle  $\pi : E \rightarrow M$ .

Let submanifold  $P_{GL(E)} \rightarrow M \subseteq \oplus^n E$  s.t.

$$(e_1 \dots e_n) \in P_{GL(E)}$$

frame bundle of  $E \equiv$  principal  $GL(n; \mathbb{R})$ -bundle over  $M \equiv$  manifold  $P_{GL(E)}$ .

This submanifold  $P_{GL(E)}$  is indeed a principal  $GL(n; \mathbb{R})$ -bundle, as shown below.

Let  $g \in GL(n; \mathbb{R})$ .

$$\begin{aligned} m : GL(E) \times P_{GL(E)} &\rightarrow GL(E) \\ (g, (e_1 \dots e_n)) &\mapsto (e_k g_{k1}^{-1}, e_k g_{k2}^{-1} \dots e_k g_{kn}^{-1}) \\ e_j &\mapsto e_k g_{kj}^{-1} \\ x = x^j e_j &\mapsto x^j e_k g_{kj}^{-1} = g_{kj}^{-1} x^j e_k \end{aligned}$$

Consider open  $U \subset M$  s.t.  $\exists$  vector bundle isomorphism  $\varphi_U : E|_{\pi^{-1}(U)} \rightarrow U \times \mathbb{R}^n$

$$\varphi_U : (\oplus^n E)|_U \rightarrow U \times (\oplus^n \mathbb{R}^n) \implies \varphi_U : P_{GL(E)}|_U \rightarrow U \times GL(n; \mathbb{R})$$

**Definition 14.** We can define a homomorphism of principal  $G$ -bundles from  $(P_G, \pi)$  to  $(Q_H, \theta)$  as a pair  $(\eta, \rho)$ <sup>4</sup>

$$(P_G, \pi) \xrightarrow{(\eta, \rho)} (Q_H, \theta)$$

s.t.

$$(1) \quad \rho \text{ homomorphism } \rho : G \rightarrow H$$

<sup>4</sup>pp. 32 <http://www2.math.umd.edu/~jmr/Toronto/BaumNotes.pdf>

$$(2) \quad P \xrightarrow{\eta} Q \text{ cont. map s.t.}$$

$$\begin{array}{ccc} P & \xrightarrow{\eta} & Q \\ \downarrow \pi & & \downarrow \theta \\ M & \xrightarrow{1} & M \end{array} \quad \begin{array}{ccc} P \times G & \xrightarrow{\eta \times \rho} & Q \times H \\ \downarrow & & \downarrow \\ P & \xrightarrow{\eta} & Q \end{array}$$

$$\pi(p) = \theta(\eta p) \quad \eta(pg) = (\eta p)(\rho g)$$

**Definition 15.** Lee defines the **principal  $G$ -bundle morphism** on pp. 298 of Lee (2009)[10] as a pair  $(\tilde{f}, f)$

$$(P_G, \pi, M_1) \xrightarrow{(\tilde{f}, f)} (Q_G, \theta, M_2)$$

s.t.

$$(1) \quad \text{smooth } f : M_1 \rightarrow M_2$$

$$(2) \quad P \xrightarrow{\tilde{f}} Q \text{ morphism s.t.}$$

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & Q \\ \downarrow \pi & & \downarrow \theta \\ M_1 & \xrightarrow{f} & M_2 \end{array} \quad \begin{array}{ccc} P \times G & \xrightarrow{\tilde{f} \times 1} & Q \times G \\ \downarrow & & \downarrow \\ P & \xrightarrow{\tilde{f}} & Q \end{array}$$

$$f(\pi(p)) = \theta(\tilde{f}(p)) \quad \tilde{f}(pg) = \tilde{f}(p)g$$

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & Q \\ \downarrow \pi & \searrow \theta & \\ M & & \end{array}$$

Now so that  $\pi(p) = \theta(\tilde{f}(p))$ , is a *bundle isomorphism* (EY : 20151008 how is  $\tilde{f} : P \rightarrow Q$  an isomorphism? And how is it a diffeomorphism when restricted to fibers?)

From Taubes (2011) Sec. 10.9 “Associated vector bundles” [6],

Suppose Lie group  $G$ , principal  $G$ -bundle  $\pi : P \rightarrow M$ .

Let representation  $\rho, \rho : G \rightarrow GL(V)$

$\exists$  corresponding vector bundle denoted  $P \times_\rho V := P \times V / \sim$  where  $(p, v) \sim (pg^{-1}, \rho(g)v) \quad \forall g \in G$ .

□

$$\begin{array}{ccc} P \times_\rho V & \ni & (p, v) \\ \downarrow \pi & & \downarrow \pi \\ M & \ni & \pi(p) \end{array}$$

Define *zero section* 0, to be

$$0 \in \Gamma(P \times_\rho V)$$

$$0 = [(p, 0)] \quad (p, 0) \in P \times V$$

Define the usual scalar multiplication by  $z \in \mathbb{R}$  or  $\mathbb{C}$ ,  $(\mathbb{R} \text{ or } \mathbb{C}) \times P \times_{\rho} V \rightarrow P \times_{\rho} V$  by  $(p, v) \mapsto (p, zv)$ .

Recall from the definition of a principal  $G$ -bundle the existence of a  $G$ -equivariant diffeomorphism  $\varphi$ :

$$\begin{array}{ccc} P|_{\pi^{-1}(U)} & \xrightarrow{\varphi} & U \times G \\ \downarrow \pi & & \\ U \subset M & & \end{array} \quad \text{i.e. } \varphi \text{ is } G\text{-equivariant diffeomorphism } \varphi : P|_{\pi^{-1}(U)} \rightarrow U \times G \quad \forall p \in M \text{ (that } \exists \text{ open } U \ni p)$$

s.t.

if  $\varphi(p) = (\pi(p), h(p))$ ,  $h(p) \in G$ , then  $\varphi(pg^{-1}) = (\pi(p), h(p)g^{-1})$ , i.e.

$$\begin{array}{ccc} p & \longmapsto & (\pi(p), h(p)) \\ \downarrow \pi & & \\ \pi(p) & & \end{array} \quad \text{if} \quad \begin{array}{ccc} pg^{-1} & \longmapsto & (\pi(p), h(p)g^{-1}) \\ \downarrow \pi & & \\ \pi(p) & & \end{array} \quad \text{then}$$

i.e. (EY : 20151015 I was wondering how best to diagram this smooth right action of  $G$  by diffeomorphisms,  $pg^{-1}$ )

$$\begin{array}{ccc} G & & g \\ \uparrow & & \uparrow \\ P|_{\pi^{-1}(U)} \times G & & (p, g) \\ \downarrow m & & \downarrow m \\ P|_{\pi^{-1}(U)} & \xrightarrow{\varphi} & U \times G \\ \downarrow \pi & & \downarrow \pi \\ p \in U \subset M & & pg^{-1} \equiv m(p, g) \longmapsto (\pi(p), h(p)g^{-1}) \\ & & \downarrow \pi \\ & & \pi(p) \end{array}$$

or

$$\begin{array}{ccc} m(g) & & m(g) \\ \cap & & \cap \\ P|_{\pi^{-1}(U)} & \xrightarrow{\varphi} & U \times G \\ \downarrow \pi & & \downarrow \pi \\ p \in U \subset M & & pg^{-1} \longmapsto (\pi(p), h(p)g^{-1}) \\ & & \downarrow \pi \\ & & \pi(p) \end{array}$$

Recall  $(\varphi^V)^{-1} : U \times V \rightarrow P \times_{\rho} V$

$$(\varphi^V)^{-1}(x, v) = [(\varphi^{-1}(x, 1), v)]$$

Then

$$\varphi^V \circ (\varphi^V)^{-1}(x, v) = \varphi^V([( \varphi^{-1}(x, 1), v)]) = \varphi^V((\varphi^{-1}(x, 1), v)) = (x, \rho(1)v) = (x, 1v) = (x, v)$$

Checking well-definedness (of the  $\varphi^V$  operation), so given that

$$(\varphi^{-1}(x, 1), v) \sim (\varphi^{-1}(x, 1)g^{-1}, \rho(g)v) \quad \forall g \in G$$

and so

$$\varphi^V(\varphi^{-1}(x, 1)g^{-1}, \rho(g)v) = (\pi(\varphi^{-1}(x, 1)g^{-1}), \rho(\psi(\varphi^{-1}(x, 1)g^{-1}))\rho(g)v) = (x, \rho(\psi(\varphi^{-1}(x, 1)))\rho(g^{-1})\rho(g)v) = (x, v)$$

Thus,  $\varphi^V \circ (\varphi^V)^{-1} = 1_{U \times V}$ .

Let (or recall) principal bundle isomorphisms  $\varphi_{\alpha} : P|_{\pi^{-1}(U_{\alpha})} \rightarrow U_{\alpha} \times G$ . Then  $\varphi_{\beta}\varphi_{\alpha}^{-1} : U_{\alpha} \bigcap U_{\beta} \times G \rightarrow U_{\alpha} \bigcap U_{\beta} \times G$  (i.e.

$$\varphi_{\beta} : P|_{\pi^{-1}(U_{\beta})} \rightarrow U_{\beta} \times G \quad \varphi_{\beta}\varphi_{\alpha}^{-1} : (x, g) \mapsto (x, g_{\beta\alpha}g)$$

principle  $G$ -bundles have principal bundle transition functions  $g_{\beta\alpha}$ 's).

Then for  $\varphi^V : P \times_{\rho} V \rightarrow U \times V$ ,  
 $\varphi^V : [(p, v)] = (p, v) \mapsto (\pi(p), \rho(\psi(p))v)$

$$\begin{array}{l} \varphi_{\alpha}^V : (P \times_{\rho} V)|_{\pi^{-1}(U_{\alpha})} \rightarrow U_{\alpha} \times V \\ \varphi_{\beta}^V : (P \times_{\rho} V)|_{\pi^{-1}(U_{\beta})} \rightarrow U_{\beta} \times V \end{array} \Rightarrow \begin{array}{l} \varphi_{\beta}^V(\varphi_{\alpha}^V)^{-1} : U_{\alpha} \bigcap U_{\beta} \times V \rightarrow U_{\alpha} \bigcap U_{\beta} \times V \\ \varphi_{\beta}^V(\varphi_{\alpha}^V)^{-1} : (x, v) \mapsto (x, \rho(g_{\beta\alpha})v) \end{array}$$

So for our vector bundle  $P \times_{\rho} V$ , we have smooth, invertible “transition” maps  $\varphi_{\beta}^V(\varphi_{\alpha}^V)^{-1}$  defined as immediately above.

Thus

**Proposition 7.** *Given principle  $G$ -bundle  $\pi : P \rightarrow M$ , Lie group  $G$ , representation  $\rho$ ,  $\rho : G \rightarrow GL(V)$ , then  $\exists$  (an associated) vector bundle  $P \times_{\rho} V := P \times V / \sim$  where  $(p, v) \sim (pg^{-1}, \rho(g)v) \quad \forall g \in G$  s.t. for this vector bundle  $P \times_{\rho} V$ , we have local trivialization  $\varphi^V$ ,  $P \times_{\rho} V \xrightarrow{\varphi^V} U \times V$ , defined as such:*

*For principal  $G$ -bundle  $\pi : P \rightarrow M$ ,*

$$\begin{array}{ccccc} & & \psi & & \\ & & \curvearrowright & & \\ P & \supset P|_{\pi^{-1}(U)} & \xrightarrow{\varphi} & U \times G & \longrightarrow G \\ \downarrow \pi & \uparrow \pi^{-1} & \swarrow & & \\ M & \supset U \ni x & & & \end{array}$$

define

$$\begin{array}{l} \varphi^V : P \times_{\rho} V \rightarrow U \times V \\ [(p, v)] = (p, v) \mapsto (\pi(p), \rho(\psi(p))v) \end{array}$$

and

$$\begin{array}{l} (\varphi^V)^{-1} : U \times V \rightarrow P \times_{\rho} V \\ (x, v) \mapsto [(\varphi^{-1}(x, 1), v)] \end{array}$$

and define the transition functions for our (associated) vector bundle  $P \times_{\rho} V$  as such: for principal

bundle isomorphisms  $\varphi_{\alpha} : P|_{\pi^{-1}(U_{\alpha})} \rightarrow U_{\alpha} \times G$ , then resulting principal bundle transition functions are such:

$$\varphi_{\beta} : P|_{\pi^{-1}(U_{\beta})} \rightarrow U_{\beta} \times G$$

$\varphi_{\beta}\varphi_{\alpha}^{-1} : U_{\alpha} \bigcap U_{\beta} \times G \rightarrow U_{\alpha} \bigcap U_{\beta} \times G$ , and so we have transition functions for our (associated) vector bundle

$$\varphi_{\beta}\varphi_{\alpha}^{-1} : (x, g) \mapsto (x, g_{\beta\alpha}g)$$



$$\begin{array}{lcl} \varphi_\alpha^V : (P \times_\rho V)|_{\pi^{-1}(U_\alpha)} \rightarrow U_\alpha \times V & & \varphi_\beta^V(\varphi_\alpha^V)^{-1} : U_\alpha \bigcap U_\beta \times V \rightarrow U_\alpha \bigcap U_\beta \times V \\ \varphi_\beta^V : (P \times_\rho V)|_{\pi^{-1}(U_\beta)} \rightarrow U_\beta \times V & \Longrightarrow & \varphi_\beta^V(\varphi_\alpha^V)^{-1} : (x, v) \mapsto (x, \rho(g_{\beta\alpha})v) \end{array}$$

Claim: If smooth map  $\sigma^P : P \rightarrow V$  s.t.  $\sigma^P(pg^{-1}) = \rho(g)\sigma^P(p)$ ,  $\exists$  section  $\sigma \in \Gamma(P \times_\rho V)$ , defined as

$$\sigma(x) = [(p, \sigma^P(p))] \text{ s.t. } \pi(p) = x \ x \in M$$

If given section  $\sigma \in \Gamma(P \times_\rho V)$ ,  $\exists$  smooth map  $\sigma^P : P \rightarrow V$  s.t.  $\sigma^P(pg^{-1}) = \rho(g)\sigma^P(p)$  cf. Taubes (2001), Subsec. 11.4.1 Part 1

*Proof.* If given  $\sigma^P : P \rightarrow V$  s.t.  $\sigma^P(pg^{-1}) = \rho(g)\sigma^P(p)$ , let

$$\sigma(x) = [(p, \sigma^P(p))] \text{ s.t. } \pi(p) = x$$

Indeed, checking well-definedness for  $(p, \sigma^P(p)) \sim (pg^{-1}, \rho(g)\sigma^P(p))$ ,

$$(pg^{-1}, \rho(g)\sigma^P(p)) = (pg^{-1}, \sigma^P(pg^{-1})) = \sigma(x)$$

since  $\pi(pg^{-1}) = x$ . Since  $\pi(pg^{-1}) = x \ \forall g \in G$ , then indeed this section  $\sigma \in \Gamma(P \times_\rho V)$  is well-defined.

□

**Proposition 8** (cf. Taubes (2001) Subsection 11.4.3 [6]).  $\forall$  principal  $G$ -bundle  $\pi : P \rightarrow M$ ,  $\exists$  short exact sequence of vector bundle homomorphisms

$$(4) \quad 0 \longrightarrow \ker(\pi_*) \longrightarrow TP \longrightarrow \pi^*TM \longrightarrow 0$$

s.t.

$\pi_* : TP \rightarrow TM$  and so

$\ker\pi_* \subset TP$

and so with arrows meaning, as follows:

$\ker\pi_* \xrightarrow{\mathbf{i}} TP$ , inclusion  $\mathbf{i}$ ,  
 $TP \rightarrow \pi^*TM$

$v \in TP \mapsto (\pi(p), \pi_*v) \in \pi^*TM \subset P \times TM$

Recall (smooth, right) action of  $G$  on  $P$ ,  $m(g) \equiv m_g : P \rightarrow P \quad \forall g \in G$ . Then  $(m_g)_* : TP \rightarrow TP$ .

$$m_g : p \mapsto pg^{-1} \qquad (m_g)_* : v_p \mapsto v_{pg^{-1}}$$

If  $v \in TP$ ,  $v \in \ker\pi_*$ ,  $(m_g)_*v \in \ker\pi_*$  since  $\pi_*((m_g)_*v) = (\pi \circ m_g)_*v = (\pi_*)v = 0$   
(for  $\pi \circ m_g(p) = \pi(pg^{-1}) = \pi(p)$ , so  $\pi \circ m_g = \pi$ )

There are many parts of this sequence to parse out and understand.

Begin with

$$\pi_* : TP \rightarrow TM$$

$$\begin{array}{ccccc} P & \ni p & \longrightarrow & T_pP & \ni v_p \equiv v(p) \\ \downarrow \pi & \downarrow \pi & & \downarrow \pi_* & \downarrow \pi_* \\ M & \ni \pi(p) & \longrightarrow & T_{\pi(p)}P & \ni \pi_*v_p \end{array}$$

$\ker\pi_*$  is the so-called vertical bundle of  $\pi_* : TP \rightarrow TM$  <sup>5</sup>.

$$0 \longrightarrow \ker(\pi_*) \xrightarrow{i} TP \xrightarrow{p} \pi^*TM \longrightarrow 0$$

Claim: This sequence is a short exact sequence.

<sup>5</sup>“Vertical bundle”, *Wikipedia* [https://en.wikipedia.org/wiki/Vertical\\_bundle](https://en.wikipedia.org/wiki/Vertical_bundle)

*Proof.* Consider  $i : \ker(\pi_*) \rightarrow TP$  (inclusion)

$i(a) = i(a')$ ,  $a, a' \in \ker\pi_*$ . Then  $a = a'$ . Inclusion  $i$  is injective. So  $0 \rightarrow \ker(\pi_*) \rightarrow TP$  exact. (this is shown mathematically in Rotman, or my notes on Rotman, or in your favorite abstract algebra book or notes)

EY : 20151017 weblinks roundup:

This showed that vector fields are isomorphic to derivations, and explicitly gives the construction of the isomorphism: 1300Y Geometry and Topology <http://www.math.toronto.edu/mgualt/MAT1300/week9.pdf>

This clarifies Taubes, 11.4.4 Part 3-4 <http://xwww.uni-math.gwdg.de/upmeier/notes/connections.pdf>

Since the Snake Lemma and 5-lemma are needed to show, in a civilized manner, the splitting lemma, could we use the Snake Lemma and 5-lemma for connections on a principal-G bundle? <http://www.mathematik.uni-kl.de/~gathmann/class/commalg-2013/chapter-4.pdf>

□

#### 4. CONNECTIONS TO CLASSES

Tsui [8] uses Baez and Muniain (1994) [1] as a reference for Chern-Simons Theory, but in my opinion, Jost (2011) [7], is a better (from a mathematically clear viewpoint) reference to begin with, and to set notation with. I follow his development and warn that I'll liberally copy from Jost (2011) [7] as necessary.

$c : \mathbb{R} \rightarrow \text{open } U \subset M$   
Let smooth curve  $\dot{c} = X$ .

$$D_X s = e_k(X(s^k) + \Gamma^k_{ij}s^iX^j) = e_k(\dot{s}^k + \Gamma^k_{ij}s^i\dot{c}^j)$$

$D_X s = 0$  represents a linear system of  $n$  1st-order ODEs, with initial values  $s(0) \in E_{c(0)}$ , so  $\exists!$  solution  $s$ .

**Definition 16.** solution  $s = s(t)$  of  $D_{\dot{c}}s = 0$  is the **parallel transport** of  $s(0) \in E$  along curve  $c : I \rightarrow M$ .

$$(5) \quad \begin{aligned} A &\equiv \Sigma^k_i \equiv \omega^k_i \in \Omega^1(M; \mathfrak{gl}(n, \mathbb{R})) = \Gamma(\mathfrak{gl}(n, \mathbb{R}) \otimes T^*M|_U) \\ D &: \Gamma(E) \rightarrow \Gamma(E) \otimes \Gamma(T^*M) \\ D &\equiv d + A \\ D\mu &= D(\mu^i e_i) = e_k(d\mu^k + \mu^i \Gamma^k_i) \equiv (d + A)\mu \end{aligned}$$

There can appear to be some confusion (at least I become confused many times) with the matrix multiplication on the *values* of a vector vs. matrix multiplication on the “abstract” choice of basis for a vector space  $V$ ,  $\dim V = n$ . I want to clarify this here.

For a given  $x \in V$ , and given bases  $\{e_j\}_j$ ,  $\{f_i\}_i$ ,  $x$  can be expressed differently (taking different “values” that you’d fill in a so-called “column” vector) in each of these bases, but  $x$  is equal to  $x$ :

$$x = x^j e_j = y^i f_i \qquad i, j = 1 \dots n$$

For

$$e_j = f_i g^i{}_j$$

Then

$$x^j e_j = x^j f_i g^i{}_j = (g^i{}_j x^j) f_i = y^i f_i$$

Thus

$$gx = y \qquad (\text{matrix multiplication on “values” on a specified basis; i.e. matrix} \cdot \text{column vector} = \text{column vector})$$

is the “usual” “linear algebra” matrix multiplication on column vectors containing the “values” of  $x$  expressed in terms of a particular choice of basis, into the vector  $x$ ’s values into another choice of bases.

$$e_j = f_i g^i{}_j$$

is how this automorphism  $g$  transforms these abstract basis vectors into the other abstract basis vectors. Notice how  $g$  is acting from the right of  $f_i$ 's.

Following pp. 137 Eq. (4.1.17) of Jost (2011) [7] and pp. 15 Eq. (2.1.5) Ballmann [9]:

So for open cover  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ , s.t.  $\exists$  local trivializations of  $E$  on each  $U_\alpha$ ,  $\Phi_\alpha : E|_{\pi^{-1}(U_\alpha)} = U_\alpha \times V$  (local trivialization is an isomorphism, hence the = sign).

Considering any 2 overlapping  $U_\alpha, U_\beta \subset M$ ,  $U_\alpha \cap U_\beta \neq \emptyset$ , then for  $\Phi_\alpha : E|_{\pi^{-1}(U_\alpha)} = U_\alpha \times V$  and so the choice of basis for  $\Phi_\beta : E|_{\pi^{-1}(U_\beta)} = U_\beta \times V$

$V$ , for each of these trivializations  $\Phi_\alpha, \Phi_\beta$  is different, in general.

Supposing local trivialization/frame  $\Phi_\alpha^{-1} = (e_1 \dots e_n)$  of  $E$  over open  $U_\alpha \subset M$  s.t.  $U_\alpha \cap U_\beta \neq \emptyset$

$\Phi_\beta^{-1} = (f_1 \dots f_n)$  of  $E$  over open  $U_\beta \subset M$

$\exists$  smooth  $g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow Gl(n; \mathbb{R})$ , in that  $g_{\beta\alpha}(p) : V \rightarrow V$ , s.t.

$$e_j = f_i g^i{}_j \text{ or } \Phi_\alpha^{-1} = \Psi^{-1} g$$

Let  $s \in \Gamma(E)$

$$s = \sigma_\alpha^j e_j = \sigma_\alpha^j f_k g^k{}_j = g^k{}_j \sigma_\alpha^j f_k = \sigma_\beta^k f_k \implies \sigma_\beta^k = g^k{}_j \sigma_\alpha^j$$

$s(p) = \Phi^{-1}(p, \sigma(p))$  (keep in mind this form of the section  $s \in \Gamma(E)$ ).

For

$$Ds = e_k(d\sigma^k + \sigma^i \Gamma^k{}_i) \equiv e_k(d\sigma^k + \sigma^i A^k{}_i) \equiv e_k(d\sigma^k + \sigma^i \omega^k{}_i)$$

$$\begin{aligned} Ds &= f_l g^l{}_k (d((g^{-1})^k{}_m \sigma_\beta^m) + (g^{-1})^i{}_j \sigma_\beta^j (A_\alpha)^k{}_i) = f_l (d\sigma_\beta^l + g^l{}_k d(g^{-1})^k{}_m \sigma_\beta^m + g^l{}_k (A_\alpha)^k{}_i (g^{-1})^i{}_j \sigma_\beta^j) = \\ &= f_l (d\sigma_\beta^l + \sigma_\beta^j (A_\beta)^l{}_m) \\ \implies (A_\beta)^l{}_m &= g^l{}_k d(g^{-1})^k{}_m + g^l{}_k (A_\alpha)^k{}_i (g^{-1})^i{}_m \text{ or } A_\beta = g d(g^{-1}) + g A_\alpha g^{-1} \end{aligned}$$

with  $g \equiv g_{\beta\alpha} : V_\alpha \rightarrow V_\beta$  (so to speak).

Consider dual (vector) bundle  $E^* \rightarrow M$ .

cf. Jost (2011) [7], Def. 4.1.3,

**Definition 17.** Let connection  $D^*$  dual to  $D$  on dual bundle  $E^*$  defined by

$$d(s, t^*) = (Ds, t^*) + (s, D^* t^*) \quad \forall s \in \Gamma(E), t^* \in \Gamma(E^*)$$

For  $\omega \in \Gamma(E^*)$ ,

$$\begin{aligned} D_X^* \omega &= X^j D_{\frac{\partial}{\partial x^j}}^* \omega = X^j D_{\frac{\partial}{\partial x^j}}^* \omega_i e^i = X^j \left( \frac{\partial \omega_i}{\partial x^j} e^i - \omega_k \Gamma^k{}_{ij} e^i \right) = X^j \left( \frac{\partial \omega_i}{\partial x^j} - \omega_k \Gamma^k{}_{ij} \right) e^i \\ D^* \omega &= e^i (d\omega_i + -\omega_k \Gamma^k{}_i) \end{aligned}$$

Thus, for the orthonormal frame and dual frame, so that  $(e_i, e^j) = \delta_i{}^j$  (and remember, it equals a number; derivatives of numbers are always 0),

$$\begin{aligned} d(e_i, e^j) &= d(\delta_i{}^j) = 0 = (De_i, e^j) + (e_i, D^* e^j) = (A^k{}_i e_k, e^j) + (e_i, (A^*)^j{}_k e^k) = A^j{}_i + (A^*)^j{}_i \\ (A^*)^j{}_i &= -A^j{}_i \implies A^* = -A^T \end{aligned}$$

cf. Def. 4.1.4 of Jost (2011) [7], pp. 138,

**Definition 18.** Let  $E_1, E_2$  vector bundles over  $M$  with respective connections  $D_1, D_2$ .

Then **induced connection**  $D$  on  $E := E_1 \otimes E_2$ , defined by

$$D(s_1 \otimes s_2) = D_1 s_1 \otimes s_2 + s_1 \otimes D_2 s_2 \quad \forall s_i \in \Gamma(E_i), \quad i = 1, 2$$

Considering the case of  $\text{End}(E) = E \otimes E^*$ ,

$$D : \Gamma(\text{End}(E)) \rightarrow \Gamma(\text{End}(E)) \otimes \Gamma(T^*M)$$

Let  $\sigma = \sigma^i{}_j e_i \otimes e^j \in \Gamma(\text{End}(E))$

$$\begin{aligned} D\sigma &= D(\sigma^i{}_j e_i) \otimes e^j + \sigma^i{}_j e_i \otimes D^* e^j = (d\sigma^k{}_j + \sigma^i{}_j A^k{}_i) e_k \otimes e^j + \sigma^i{}_j e_i \otimes (A^*)^j{}_k e^k = \\ &= (d\sigma^k{}_j + \sigma^i{}_j A^k{}_i) e_k \otimes e^j + \sigma^k{}_i e_k \otimes -A^i{}_j e^k = (d\sigma^k{}_j + [A, \sigma]^k{}_j) e_k \otimes e^j \end{aligned}$$

Let vector bundle  $E \rightarrow M$ , with bundle metric  $\langle \cdot, \cdot \rangle$ .

**Definition 19.** Metric connection  $D$  on  $E$  if

$$d\langle \mu, \nu \rangle = \langle D\mu, \nu \rangle + \langle \mu, D\nu \rangle \quad \forall \mu, \nu \in \Gamma(E)$$

cf. Lemma 4.2.1. of Jost (2011) [7]

**Lemma 1.** For parallel transport on vector bundle induced by metric connection, preserves bundle metric, in that parallel transport constitutes an isometry of corresponding fibers, i.e. for  $s = s(\tau), t = t(\tau) \in \Gamma(E)$  s.t. they get parallely transported, i.e.  $D_{\dot{c}} s = D_{\dot{c}} t = 0$  for curve  $c : I \rightarrow M$  in  $M$ ,  $\frac{d}{d\tau} \langle s(\tau), t(\tau) \rangle = 0$ .

*Proof.* Let smooth curve  $c : I \rightarrow M$

$$\dot{c} = X$$

in general, let  $X \in T_x M$ , then  $X\langle \mu, \nu \rangle = d\langle \mu, \nu \rangle(X) = \langle D_X \mu, \nu \rangle + \langle \mu, D_X \nu \rangle$

$$\xrightarrow{X=\dot{c}} \langle D_{\dot{c}} \mu, \nu \rangle + \langle \mu, D_{\dot{c}} \nu \rangle$$

$D_{\dot{c}} \mu = D_{\dot{c}} \nu = 0$ , i.e.  $\mu, \nu$  parallel along  $c$ .

$$\implies \frac{d}{dt} \langle \mu(t), \nu(t) \rangle = 0$$

□

cf. Lemma 4.2.2. of Jost (2011) [7]

**Lemma 2.** Let metric connection  $D$  on vector bundle  $E$  with bundle metric  $\langle \cdot, \cdot \rangle$ .

Assume that with respect to metric bundle chart (cf. Def. 2.1.12, Thm. 2.1.3 of Jost (2011) [7].

$$D = d + A$$

$$\begin{aligned} \text{Then, } \forall X \in TM, \quad A(X) &\in \mathfrak{o}(n) & (= \text{Lie algebra of } O(n)) & \quad (n = \text{rank of } E) \\ A(X) &\in \mathfrak{u}(n) & (= \text{Lie algebra of } U(n)) & \quad (n = \text{rank of } E) \end{aligned}$$

*Proof.* Thm. 2.1.3: metric bundle chart  $(f, U)$  generates sections  $e_1 \dots e_n$  on  $U$  s.t.  $e_1 \dots e_n$  form orthonormal basis on fiber  $E_x$ ,  $\forall x \in U$  i.e.  $\langle e_i(x), e_j(x) \rangle = \delta_{ij}$ .

Moreover, since  $e_i$  constant in bundle chart  $(f, U)$ , define exterior derivative  $d$ .

$$d\mu_i \equiv 0 \quad (i = 1 \dots n)$$

Let  $X \in T_x M$ ,  $x \in U$ .

$$0 = X\langle e_k, e_l \rangle = \langle D_X e_k, e_l \rangle + \langle e_k, D_X e_l \rangle = \langle X^j \Gamma^m{}_{kj} e_m, e_l \rangle + \langle e_k, x^j \Gamma^m{}_{lj} e_m \rangle$$

For  $\mathbb{K} = \mathbb{R}$ ,

$$\implies A^l{}_k + A^k{}_l = 0 \text{ so } A = -A^T \text{ skew symmetric}$$

□

For the case/issue of field  $\mathbb{K} = \mathbb{C}$  (complex numbers) and dealing with the so-called *Hermitian* metric on a vector bundle with complex numbers involved, then I took a look at the articles on [Hermitian Inner Product](#), and [Hermitian metric](#):

Hermitian Inner Product:

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

$$\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$$

$$\langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle$$

$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

$$\langle u, u \rangle \geq 0 \quad \langle u, u \rangle = 0 \iff u = 0$$

Then, using the Hermitian inner product rule,

$$\begin{aligned} d\langle e_i, e_j \rangle &= \langle De_i, e_j \rangle + \langle e_i, De_j \rangle = \langle (d + A)e_i, e_j \rangle + \langle e_i, (d + A)e_j \rangle = \\ &= \langle 0 + A^k{}_i e_k, e_j \rangle + \langle e_i, 0 + A^k{}_j e_k \rangle = \langle A^k{}_i e_k, e_j \rangle + \langle e_i, A^k{}_j e_k \rangle = \\ &= A^j{}_i + (A^i{}_j)^* = 0 \implies A^j{}_i = -(A^i{}_j)^* \text{ so } A = -A^\dagger \end{aligned}$$

### Part 3. Gauge Theory

The story of Gauge Theory in Physics is a beautiful story - it is the triumph of geometry in physical theories of matter.

#### 5. ELECTROMAGNETISM

- : Electromagnetism
- : Maxwell's Equations

From “Rewriting Maxwell's Equations” of Baez and Muniain (1994) [1]:

$$*F_+ = \frac{1}{2} [iF_+ - iF_- + i(F_+ + F_-)] = iF_+ \text{ i.e. self dual} \quad *F_- = \frac{1}{2} [iF_+ - iF_- - iF_+ - iF_-] = -iF_- \text{ i.e. anti-self dual}$$

So if  $F \in \Omega^p(M)$  ( $F$  self dual or anti self dual), if  $dF = 0$ ,  $d * F = 0$ , since  $d * F = \begin{cases} d(\pm F) = \pm dF = 0 \\ d(\pm iF) = \pm idF = 0 \end{cases}$

Consider  $\Delta : \Omega^p(M) \rightarrow \Omega^p(M)$

$$\Delta = (\delta + d)^2 = \delta d + d\delta$$

$$\begin{aligned} \Delta F &= \delta dF + d\delta F = 0 + d(- * d *) F = -dJ \\ J &= j - \rho dt \end{aligned}$$

$$dJ = \frac{\partial j_j}{\partial x^\mu} dx^\mu \wedge dx^j - \frac{\partial \rho}{\partial x^i} dx^i \wedge dt$$

Note that for  $d^2 = 0$ ,  $\delta^2 = 0$  (as seen with this example of Maxwell's equations, for  $F$ ),  $\delta = (-1)^{n(p+1)+1} * d * = - * d *$  (especially if  $n$  even),

$$\begin{array}{ccc} \Omega^2(M) & \xrightarrow{*} & \Omega^{n-2}(M) \\ \downarrow \delta & & \downarrow d \\ \Omega^1(M) & \xrightarrow{-*(s)} & \Omega^{n-1}(M) \end{array}$$

$$\begin{array}{ccc} F & \xrightarrow{*} & *F \\ \downarrow \delta & & \downarrow d \\ \delta F & \xrightarrow{-(s)} & d * F = (s) * J \end{array}$$

$$\begin{aligned} F \in \Omega^2(M) & \quad *F \in \Omega^{n-2}(M) \\ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu & \quad *F = \frac{\sqrt{g}}{(n-2)!} \frac{1}{2} F_{\mu\nu} g^{\mu\mu'} g^{\nu\nu'} \epsilon_{\mu'\nu'\rho\sigma} dx^\rho \wedge dx^\nu \end{aligned}$$

$$\begin{aligned} F &= B + E \wedge dt = \frac{B_{ij}}{2} dx^i \wedge dx^j + E_i dx^i \wedge dt \\ *F &= \frac{\sqrt{g}}{(n-2)!} \frac{B_{ij}}{2} g^{i\mu} g^{j\nu} \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma + \frac{\sqrt{g}}{(n-2)!} E_i g^{i\mu} g^{0\nu} \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma \end{aligned}$$

If  $M = \mathbb{R} \times N$ ,

$$\begin{aligned} B_{ij} g^{il} g^{jm} \epsilon_{lm\rho\sigma} dx^\rho \wedge dx^\sigma &= B_{ij} g^{il} g^{jn} (\epsilon_{lmn\sigma} dx^n \wedge dx^\sigma + \epsilon_{lm0\sigma} dx^0 \wedge dx^\sigma) = \\ &= B_{ij} g^{il} g^{jm} (\epsilon_{lmn0} dx^n \wedge dt + \epsilon_{lmn} dx^n \wedge dt) = 2B_{ij} g^{il} g^{jm} \epsilon_{lmn} dx^n \wedge dt \end{aligned}$$

Noting that

$$\epsilon_{lmn} B_{ij} g^{il} g^{jm} = \epsilon_{lmn} \epsilon_{ijk} B^k g^{il} g^{jm} = 2B^n$$

$$\text{If } g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ and } M = \mathbb{R} \times N,$$

$$\begin{aligned} *F &= \frac{1}{(n-2)!} \left[ \frac{B_{ij}}{2} g^{il} g^{jm} \epsilon_{lmn0} dx^n \wedge dt + -E_i g^{il} \epsilon_{l0mn} dx^m \wedge dx^n \right] = \frac{1}{(n-2)!} [-\epsilon_{lmn} g^{li} E_i dx^m \wedge dx^n + (-\epsilon_{lmn}) B_{ij} g^{il} g^{jm} dx^n \wedge dt] = \\ &= \frac{1}{(n-2)!} [-E_i g^{il} \epsilon_{lmn} dx^m \wedge dx^n + (-B_{ij}) g^{il} g^{jm} \epsilon_{lmn} dx^n \wedge dt] = \vec{*}E - \vec{*}B \wedge dt \end{aligned}$$

Indeed, in general, for  $M = \mathbb{R} \times N$ ,

$$\begin{aligned} *F &= \frac{\sqrt{g}}{(n-2)!} B_{ij} g^{il} g^{jm} \epsilon_{lmn0} dx^n \wedge dt + \frac{\sqrt{g}}{(n-2)!} E_i g^{il} (-1) \epsilon_{l0mn} dx^m \wedge dx^n = \\ &= \frac{\sqrt{g}}{(n-2)!} (-1) E_i g^{il} (-\epsilon_{l0mn}) dx^m \wedge dx^n + \frac{\sqrt{g}}{(n-2)!} \epsilon_{ijk} B^k g^{il} g^{jm} (-\epsilon_{lmn}) dx^n \wedge dt \end{aligned}$$

If  $g^{il} = \delta^{il}$ , then applying  $\epsilon_{jmn} \epsilon^{imn} = 2\delta_j^i$ , and  $n = 4$

$$*F = \vec{*}E + \vec{*}B \wedge dt$$

$*F = iF$  (self-dual) if

$$\vec{*}E = iB$$

$$\vec{*}B = -iE$$

And so

$$E = E_i dx^i$$

$$\vec{*}E = \frac{\sqrt{g}}{(d-1)!} E_i g^{il} \epsilon_{lmn} dx^m \wedge dx^n \implies -i\vec{*}E = \frac{-i\sqrt{g}}{(d-1)!} E_i g^{il} \epsilon_{lmn} dx^m \wedge dx^n \implies \frac{1}{2} B^l = \frac{-i\sqrt{g}}{(d-1)!} E_i g^{il} \text{ or } B^k = -i\sqrt{g} E_i g^{ik}$$

$$B = \frac{1}{2} B_{mn} dx^m \wedge dx^n = \frac{1}{2} \epsilon_{mnl} B^l dx^m \wedge dx^n$$

Assume  $F$  self-dual ( $*F = iF$ ) and  $E(x) = E_i e^{ikx} dx^i$ ,  $k \in (\mathbb{R}^4)^*$ ,  $x \in \mathbb{R}^4 = M$ ,  $k$  fixed covector called **energy-momentum**, s.t.

$$k_\mu x^\mu = kx$$

$$B = -i\vec{*}E$$

## 6. CHERN-SIMONS THEORY

From Baez and Muniain (1994) [1]:

**Part 4. Polynomials**

## 7. (SOFTWARE) PACKAGES

This link [Combinatorial Software and Databases](#) has a list of (possibly) useful implementations of the Gosper-Zeilberger Algorithm. There is

- **EKHAD**: written in Maple by Doron Zeilberger, is an implementation of the Gosper-Zeilberger algorithm.
- **:** written in Maple by Doron Zeilberger, is an implementation of the q-Zeilberger algorithm. Also at [Rutgers, Packages Accompanying or Related to A=B](#)
- **Zeilberger**:written in Maxima by Fabrizio Caruso, is an implementation of the Gosper-Zeilberger algorithm.

Tom Koornwinder had a paper [On Zeilberger’s algorithm and its q-analogue: a rigorous description](#) that explicitly and very lucidly reviews Zeilberger’s algorithm, but I cannot find his accompanying Maple software (links don’t work).

I will focus on looking at Caruso’s package because the source code is available and I had emailed Paule and the source code for Mathematica implementations are not available, understandably from their particular policies on distribution.

Wilf and Zeilberger (1992) has a good introduction to hypergeometric polynomials and its significance [11]. In the immediate sections, I will recap, summarize, and copy (or state) some definitions and results from Wilf and Zeilberger (1992), and show (some of its) direct implementations in Sage Math and sympy.

Starting from pp. 586 of Wilf and Zeilberger (1992) [11],

Consider

$$(c)_n := (1 - c)(1 - cq) \dots (1 - cq^{n-1})$$

Let  $c = q$ . Then

$$(q)_n = (1 - q)(1 - q^2) \dots (1 - q^n)$$

Define  $f(n) := (c)_n$ . Notice that

$$\frac{f(n+1)}{f(n)} = 1 - cq^n$$

This is the simplest nonzero rational function in  $q^n$ .

Let  $f(x) := (x)_\infty$ .  $\frac{f(qx)}{f(x)} = \frac{(qx)_\infty}{(x)_\infty} = \frac{1}{1-x}$ .

Note that

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$$

$$(x)_\infty = (1-x)(1-xq) \dots (1-xq^{n-1}) \dots$$

$$(qx)_\infty = (1-qx)(1-q^2x) \dots (1-xq^n) \dots$$

Also note that

$$(q)_k = \frac{(q)_\infty}{(q^{k+1})_\infty} = \frac{(1-q)(1-q^2) \dots (1-q^n)}{(1-q^{k+1})(1-q^{k+2}) \dots}$$

Introduce dilation operator  $\forall x \in \mathbb{F}$ ,  $Q_x f(x, \mathbf{y}) := f(qx, \mathbf{y})$

$q$ -derivative  $D_x^{(q)}$

$$D_x^{(q)} f(x, \mathbf{y}) = \frac{f(qx, \mathbf{y}) - f(x, \mathbf{y})}{(q-1)x} = \frac{(Q_x - 1)f(x, \mathbf{y})}{(x(q-1))}$$

Notice that

$$\lim_{q \rightarrow 1} D_x^{(q)} f(x, \mathbf{y}) = D_x f$$

**Definition 20.**  $q$ -hypergeometric,

$$F(k_1 \dots k_r, y_1 \dots y_s) \equiv F(\mathbf{k}, \mathbf{y})$$

$k_1 \dots k_r \in \mathbb{Z}$  (discrete),  $y_1 \dots y_s$  cont. ( $y_1 \dots y_s \in \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ )  $q$ -hypergeometric

if  $\forall$  (discrete)  $k_i$ ,  $(E_{k_i} F)/F$  rational functions of  $(q^{k_1} \dots q^{k_r}, y_1 \dots y_s)$  and possible other constnat parameters, including  $q$ .

$\forall$  (cont.)  $y_j$ ,  $(Q_{y_j} F)/F$

7.0.3. Examples.

(1) Polynomials  $P(q^{k_1} \dots q^{k_r}, y_1 \dots y_n)$ . Indeed,

$$(E_{k_i} P)/P = \frac{P(q^{k_1} \dots q^{k_i+1} \dots q^{k_r}, \mathbf{y})}{P(q^{k_1} \dots q^{k_r}, \mathbf{y})}$$

$$(Q_{y_j} P)/P = \frac{P(q^{k_1} \dots q^{k_r}, y_1 \dots q y_j \dots y_n)}{P(q^{k_1} \dots q^{k_r}, y_1 \dots y_n)}$$

(2)

$$(cy_1^{\alpha_1} \dots y_s^{\alpha_s}, q^{\beta_1 k_1} \dots q^{\beta_r k_r})_\infty^\gamma$$

cf. (qPH-II) of Wilf and Zeilberger (1992) [11] EY : 20160105 I don’t understand the upper and lower exponent notation of  $()_\infty^\gamma$ , i.e. what is  $\gamma$  and  $\infty$  in this case?

(3)  $q^{\sum_{i,j} a_{i,j} k_i k_j + \sum_i b_i k_i}$   $a_{i,j}, b_i \in \mathbb{Z}$  or  $\{\pm \frac{1}{2}, \pm \frac{3}{2}, \dots\}$ . Indeed

$$\xrightarrow{E_{k_i}} q^{\sum_{i,j} q_{i,j} k_i k_j + \sum_i b_i k_i} q^{\sum_{i,j} a_{i,j} k_j + b_i}$$

(4)  $z_1^{k_1} \dots z_r^{k_r}$ . Indeed,

$$\xrightarrow{E_{k_i}} z_1^{k_1} \dots z_r^{k_r} \cdot z_i$$

**Definition 21.**  $F(k_1 \dots k_r, y_1 \dots y_s)$ , (discrete)  $k_1 \dots k_r \in \mathbb{Z}$ , (cont.)  $y_1 \dots y_r \in \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $q$ -proper-hypergeometric

if  $F$  has form  $F = P(q^{k_1} \dots q^{k_r}, y_1 \dots y_s) \cdot q^{\sum_{i,j} a_{-i,j} k_i k_j + \sum_i b_i k_i} \cdot z_1^{k_1} \dots z_r^{k_r}$  . finite number of  $(cy_1^{\alpha_1} \dots y_s^{\alpha_s} q^{\beta_1 k_1} \dots q^{\beta_r k_r})_\infty^\gamma$

**Lemma 3** ( $q$ -fundamental lemma).  $\forall$   $q$ -proper-hypergeometric  $F(x_1 \dots x_n, a_1 \dots a_m)$  and  $\forall$  cont.  $x_i$ ,  $\forall$  (discrete)  $a_j$ ,  $\exists$  nonzero linear recurrence- $q$ -differential operators

$$P_i(x_i; Q_{x_1} \dots Q_{x_n}; E_{a_1} \dots E_{a_m}) \text{ and respectively } C_j(q^{a_j}; Q_{x_1} \dots Q_{x_n}; E_{a_1} \dots E_{a_m})$$

annihilating  $F$ , i.e.  $P_i F = 0$

$$C_j F = 0$$

Let  $x = q^k$

$$(cq^k)_\infty = \frac{(c)_\infty}{(c)_k} = \frac{(1-c)(1-cq) \dots}{(1-c)(1-cq) \dots (1-cq^{k-1})} = (1-cq^k)(1-cq^{k+1}) \dots$$

$$(cx)_\infty = \frac{(c)_\infty}{(c)_k} = (1-cq)(1-cxq) \dots$$

Consider these transformations or mappings:

$$F(x_1 \dots x_n, a_1 \dots a_m) \xrightarrow{x=q^k} F(q^{k_1} \dots q^{k_n}, a_1 \dots a_m)$$

$$Q_{x_i} F = F(x_1 \dots qx_i \dots x_n, \mathbf{a}) = F(q^{k_1} \dots q^{(1+k_i)} \dots q^{k_n}, \mathbf{a}) = E_{k_i} F$$

$E_k q^k = q q^k E_k$  is “isomorphic” to  $Q_x x = qx Q_x$

Now for  $\frac{(c)_\infty}{(cx)_\infty}$  substitute  $(c)_k$ .

$$\frac{(c)_\infty}{(cx)_\infty} = \frac{(1-c)(1-cq)\dots}{(1-cx)(1-cxq)\dots} \xrightarrow{x=q^k} (1-c)(1-cq)\dots(1-cq^{k-1})$$

**7.1. The fundamental theorem of hypergeometric summation/integration.** cf. Sec. 2. “The fundamental theorem of hypergeometric summation/integration” of Wilf and Zeilberger (1992) [11].

**Definition 22.**  $F(k_1 \dots k_r, y_1 \dots y_s) \equiv F(\mathbf{k}, \mathbf{y})$  vanishes at infinity if  $\forall k_i, y_j$ ,

$$\lim_{|k_i| \rightarrow \infty} F(\mathbf{k}, \mathbf{y}) = 0$$

$$\lim_{|y_j| \rightarrow \infty} F(\mathbf{k}, \mathbf{y}) = 0$$

**Definition 23.** *integral-sum*

$$g(n, x) := \sum_{\mathbf{k}} \int_{\mathbf{y}} F(n, \mathbf{k}, x, \mathbf{y}) d\mathbf{y} \quad (\text{general-integral-sum})$$

is pointwise trivially evaluable, if  $\forall$  given  $\mathbf{n}, \mathbf{x}$ ,  $\exists$  algorithm to evaluate it.

e.g. for pure sums that are terminating,

$$g(\mathbf{n}) := \sum_{\mathbf{k}} F(n, \mathbf{k}) \quad (\text{general-sum})$$

sum is finite  $\forall \mathbf{n}$  since  $F(\mathbf{n}, \cdot)$  has finite support.

7.1.1. *The fundamental Theorem of hypergeometric summation-integration.*

**Theorem 3** (The fundamental theorem). Let  $\Delta_{k_i} \equiv$  forward difference operator in  $k_i$  ;  $\Delta_{k_i} := (K_i - 1)$ .

Let  $F(n, k_1 \dots k_r, y_1 \dots y_s)$  be hypergeometric (holonomic) (both hold if proper-hypergeometric) in  $(\mathbf{k}, \mathbf{y})$  and  $n$   
 $F(x, k_1 \dots k_r, y_1 \dots y_s)$  ( $\mathbf{k}, \mathbf{y}$ ) and  $x$

where discrete  $n, \mathbf{k}$   
cont.  $x, \mathbf{y}$

Then  $\exists$  linear ordinary recurrence operator with polynomial coefficients  $P(N, n)$   
 $\exists$  linear differential operator  $P(D_x, x)$   
and rational functions  $R_1 \dots R_r, S_1 \dots S_s$  s.t.

$$(6) \quad P(N, n)F = \sum_{i=1}^r \Delta_{k_i}(R_i, F) + \sum_{j=1}^s D_{y_j}(S_j F)$$

$$P(D_x, x)F = \sum_{i=1}^r \Delta_{k_i}(R_i, F) + \sum_{j=1}^s D_{y_j}(S_j F)$$

If  $F$  proper-hypergeometric, it's possible to find bounds for order of  $P(N, n), P(D_x, x)$ , and denominators  $R_i, S_j$

Recall “the fundamental lemma”, due to Bernstein, reproduced on pp. 585 of Wilf and Zeilberger (1992) [11],

**Lemma 4** (The fundamental lemma).  $\forall$  holonomic function  $F(x_1 \dots x_n, a_1 \dots a_m)$ ,  $\forall$  cont.  $x_i$ ,  $\forall$  discrete  $a_j$ ,  $\exists$  non-zero linear recurrence differential operators

$$P_i(x_i; D_{x_1} \dots D_{x_n}; E_{a_1} \dots E_{a_m})$$

$$C_j(x_i; D_{x_1} \dots D_{x_n}; E_{a_1} \dots E_{a_m}) \quad \text{respectively}$$

that annihilates  $F$ , i.e.  $PF = 0$

e.g.  $(D_x^2 + I) \cos x = 0$

*Proof.* By fundamental lemma 4,  $\exists$  operator(s)

$$A(n, E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n})$$

$$A(x; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n}) \quad \text{respectively}$$

s.t.  $AF = 0$

It's possible to rewrite  $A$  as

$$A(n; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n}) = P(n, E_n) - \sum_{i=1}^r (E_{k_i} - 1) B_i(n; E_n, E_{k_1} \dots E_{k_r}, D_y \dots D_{y_n}) -$$

$$- \sum_{j=1}^s D_{y_j} \bar{B}_j(n; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n})$$

$$A(x; E_{k_1} \dots E_{k_r}; D_x, D_{y_1} \dots D_{y_n}) = P(x, D_x) - \sum_{i=1}^r (E_{k_i} - 1) B_i(x; E_{k_1} \dots E_{k_r}, D_x, D_y \dots D_{y_n}) -$$

$$- \sum_{j=1}^s D_{y_j} \bar{B}_j(x; E_{k_1} \dots E_{k_r}; D_x, D_{y_1} \dots D_{y_n})$$

$$0 = P(n, E_n)F - \sum_{i=1}^r (E_{k_i} - 1) B_i(n, E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n})F - \sum_{j=1}^s D_{y_j} \bar{B}_j(n, E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_n})F$$

$$0 = P(x, D_x)F - \sum_{i=1}^r (E_{k_i} - 1) B_i F - \sum_{j=1}^s D_{y_j} \bar{B}_j F$$

□

$F$  hypergeometric, s.t.  $\frac{E_k F}{F}, \frac{D_{y_j} F}{F}$  rational functions. By induction,  $\forall$  “operator monomial”

$$\text{Mon} := E_n^{\alpha_0} \prod_{i=1}^r E_{k_i}^{\alpha_i} \prod_{j=1}^s D_{y_j}^{\beta_j} \quad (\text{Op-Mon})$$

$$\text{Mon} := D_x^{\beta_0} \prod_{i=1}^r E_{k_i}^{\alpha_i} \prod_{j=1}^s D_{y_j}^{\beta_j}$$

$\frac{\text{Mon} F}{F}$  rational function.

$\forall$  operator  $T(n, k_1 \dots k_r; y_1 \dots y_s; E_n, E_{k_1} \dots E_{k_r}; D_{y_1} \dots D_{y_s})$   
 $T(k_1 \dots k_r; y_1 \dots y_s; E_{k_1} \dots E_{k_r}; D_x, D_{y_1} \dots D_{y_s})$

$\frac{TF}{F}$  rational function, since  $T$  linear combination with coefficients that are polynomials in all variables, of operator monomials.  
In a sense, one could say  $T$  belongs to

$$k[n, k_1 \dots k_r; y_1 \dots y_s] \{ \text{Mon} \} \rightarrow \{ \text{Mon} \}, \quad k[n, k_1 \dots k_r; y_1 \dots y_s] - \text{module}$$

$$k[k_1 \dots k_r; x, y_1 \dots y_s] \{ \text{Mon} \} \rightarrow \{ \text{Mon} \}, \quad k[k_1 \dots k_r; x, y_1 \dots y_s] - \text{module}$$

In particular,

$$B_i(F) = R_i F$$

$$\bar{B}_j(F) = S_j F$$



$$\begin{aligned}
0 &= P(n, E_n)F - \sum_{i=1}^r (E_{k_i} - 1)R_i F - \sum_{j=1}^s D_{y_j} S_j F \\
\implies \\
0 &= P(x, D_x)F - \sum_{i=1}^r (E_{k_i} - 1)R_i F - \sum_{j=1}^s D_{y_j} S_j F
\end{aligned}$$

$R_i, S_j$  are certificates. Note Gert Almquist observed  $B_i, B_j$  not necessary to be independent of  $(k_1 \dots k_i; y_1 \dots y_s)$

**Corollary 1** (Fundamental corollary). *If  $F(n, \mathbf{k}, \mathbf{y})$  hypergeometric functions, and vanishes at infinity,  $\forall$  fixed  $n$ , then  $F(x, \mathbf{k}, \mathbf{y})$  holonomic functions  $\forall$  fixed  $x$*

$$f(n) := \sum_{\mathbf{k}} \int F(n, \mathbf{k}, \mathbf{y}) d\mathbf{y}$$

$$f(x) := \sum_{\mathbf{k}} \int F(x, \mathbf{k}, \mathbf{y}) d\mathbf{y}$$

satisfies linear recurrence equation with polynomial coefficients  $P(N, n)f(n) = 0$   
differential equation  $P(D_x, x)f(x) = 0$

*Proof.*

$$\begin{aligned}
\sum_{\mathbf{k}} \int d\mathbf{y} (E_{k_i} - 1)R_i F &= 0 \quad \text{since } \lim_{|k_i| \rightarrow \infty} F(\mathbf{k}, \mathbf{y}) = 0 \\
\sum_{\mathbf{k}} \int d\mathbf{y} D_{y_j} S_j F &= 0 \quad \text{since } \lim_{|y_j| \rightarrow \infty} F(\mathbf{k}, \mathbf{y}) = 0
\end{aligned}$$

Again,  $R_i, S_j$  are called *certificates*.  $\square$

7.1.2. *How to find  $P(N, n)$  and the certificates.* cf. 2.2. How to find  $P(N, n)$  and the certificates of Wilf and Zeilberger (1992) [11].

Guess the order of  $P(N, n)$ , say  $L$   
in practice, try  $L = 0$ . Then  $L = 1, \dots$

$$P(N, n) = \sum_{i=0}^L b_i(n) N^i$$

guess  $R_i, S_j$

$$P(N, n) = \sum_{i=1}^r \Delta_{k_i}(R_i, F) + \sum_{j=1}^s D_{y_j}(S_j F)$$

Reading Riese (2003) [12],

Let

$\mathbf{K} = \mathbb{C}(q, \tau_1 \dots \tau_m)$  denote transcendental extension of complex numbers  $\mathbb{C}$  by indeterminates  $q, \tau_1 \dots \tau_m$  (transcendental extension, roughly speaking, includes elements that aren't roots of polynomials over  $\mathbb{C}$ ),

$\mathbf{k} \equiv (k_1 \dots k_r) \in \otimes_{i=1}^r \mathbb{Z}$

polynomial in  $q^n, q^{\mathbf{k}}$  over  $\mathbf{K}$ ,  $P(n, \mathbf{k}) \in \mathbf{K}[q^n, q^{\mathbf{k}}]$  if  $\exists P^* \in \mathbf{K}[x_0, x_1 \dots x_r]$  s.t.  $P(n, \mathbf{K}) = P^*(q^n, q^{k_1} \dots q^{k_r})$

$R(n, \mathbf{k})$  is a rational function in  $q^n, q^{\mathbf{k}}$  over  $\mathbf{K}$ ,  $R(n, \mathbf{k}) \in \mathbf{K}(q^n, q^{\mathbf{k}})$ , if  $\exists$  rational function  $R^* \in \mathbf{K}(x_0, x_1 \dots x_r)$  s.t.

$$R(n, \mathbf{k}) = R^*(q^n, q^{k_1} \dots q^{k_r})$$

**Definition 24.**  $q$ -hypergeometric function  $F(n, \mathbf{k})$  in  $n, \mathbf{k}$  over  $\mathbf{K}$  if

$$\frac{F(n+1, \mathbf{k})}{F(n, \mathbf{k})}, \frac{F(n, k_1 \dots k_i + 1 \dots k_r)}{F(n, \mathbf{k})} \in \mathbf{K}(q^n, q^{\mathbf{k}})$$

**Definition 25.**  $\mathbf{k}$ -free recurrence, if  $\exists$  finite set  $S$ ,  $S = \{s | s \in \otimes i = 1^{r+1} \mathbb{Z}\}$ , if  $\exists \sigma_{i, \mathbf{j}}(n) \in \mathbf{K}[q^n]$  s.t.

$$\sum_{(i, \mathbf{j}) \in S} \sigma_{i, \mathbf{j}}(n) F(n-i, \mathbf{k}-\mathbf{j}) = 0 \quad \forall (n, \mathbf{k})$$

s.t.  $F(n-i, \mathbf{k}-\mathbf{j})$ ,  $\forall (i, \mathbf{j}) \in S$  well-defined.

$S$  is called a structure set.

Thus

$$(7) \quad \sum_{(i, \mathbf{j}) \in S} \sigma_{i, \mathbf{j}}(n) F(n-i, \mathbf{k}-\mathbf{j}) = 0 =$$

$$(8) \quad \left( \sum_{(i, \mathbf{j}) \in S} \sigma_{i, \mathbf{j}}(n) N^{-i} \mathbf{K}^{-\mathbf{j}} \right) F(n, \mathbf{k}) = 0 \xrightarrow{1/F}$$

$$(9) \quad \sum_{(i, \mathbf{j}) \in S} \sigma_{i, \mathbf{j}}(n) R_{F, i, \mathbf{j}}(n, \mathbf{k}) = 0$$

Multiplying by denominators on both sides,

$$(10) \quad \sum_{(i, \mathbf{j}) \in S} \sigma_{i, \mathbf{j}}(n) P_{F, i, \mathbf{j}}(n, \mathbf{k}) = 0$$

**Definition 26.** Let  $F(n, \mathbf{k})$  be  $q$ -proper hypergeometric,  $S$  structure set.

for fixed  $s$ ,  $(I, \mathbf{J}) \in S$  is numerator boundary pt.  $\equiv (I_s^{num}, \mathbf{J}_s^{num})$  if  $Ia_s + \mathbf{Jb}_s \geq ia_s + \mathbf{j}b_s \quad \forall (i, \mathbf{j}) \in S$

cf. pp. 3 Eq. (1.2.15) of Gasper and Rahman (2004) [13] and pp. 4 of Riese (2003) [12]:

**Definition 27** ( $q$ -shifted factorial or  $q$ -Pochhammer symbol).

$$(11) \quad (a; q)_k = \begin{cases} 1 & \text{if } k = 0 \\ (1-a)(1-aq) \dots (1-aq^{k-1}) & \text{if } k > 0 \\ \frac{1}{(1-aq^{-1})(1-aq^{-2}) \dots (1-aq^{-k})} = \frac{1}{(aq^{-k}; q)_k} = \frac{(-q/a)^k q^{\binom{k}{2}}}{(q/a; q)_k} & \text{if } k < 0 \end{cases}$$

and define

$$(12) \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1-aq^k)$$

Notice that

$$(13) \quad \frac{(a; q)_\infty}{(q^k a; q)_\infty} = (a; q)_k$$

From Exercise 1.2 on pp. 24 of Gasper and Rahman (2004) [13] and pp. 4 of Riese (2003) [12],

**Definition 28** ( $q$ -binomial coefficient or Gaussian polynomials).

$$(14) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$



Notice that this definition is exactly the same as the one given by [Wikipedia](#), which is the definition used by [Sage Math](#), and there, a different notation is used,  $\binom{n}{k}_q$  i.e.  $\binom{n}{k}_q \equiv [n]_q$ :

$$\binom{n}{k}_q = \frac{(1 - q^n)(1 - q^{n-1}) \dots (1 - q^{n-k+1})}{(1 - q)(1 - q^2) \dots (1 - q^k)} = \frac{(q; q)_n}{(q; q)_{n-k}} \frac{1}{(q; q)_k}$$

7.2. Reverse Engineering Caruso’s Zeilberger[14].

Part 5. Jones Polynomials and Khovanov Homology

8. JONES POLYNOMIALS FROM SNAPPY

One should have installed successfully SnapPy and its implementation in Sage Math (cf. [10.1](#)). Then, one can obtain the Jones polynomial for various torus knots in such a manner - in Sage Math:

```
sage: import snappy
sage: T0203snap = snappy.Link('T(2,3)')
sage: T0203snap # obtain the number of crossings as a print out!
<Link: 1 comp; 3 cross>
sage: T0203snap.crossings # obtain the crossings as a list!
[0, 1, 2]
sage: len(T0203snap.crossings) # this is another way to obtain the number of crossings
3
```

If one does `dir(T0203)`, then as one can see, there are a number of modules that one can play with. For instance, the signature and morse number can be obtained quickly:

```
sage: T0203snap.signature()
2
sage: T0203snap.morse_number()
2
```

and also the Jones polynomial can be obtained quickly, after declaring the torus knot with `Link`:

```
sage: T0203snap.jones_poly()
-q^4 + q^3 + q
```

Let’s obtain all the Jones polynomials for torus knots with less than 14 crossings (arbitrary choice of 14) with SnapPy, and keep in mind that you can use `latex()` command in Sage Math to immediate get the output in a LaTeX friendly print out:

K	J(K;q)
$T(2,3)$	$-q^4 + q^3 + q$
$T(2,5)$	$-q^7 + q^6 - q^5 + q^4 + q^2$
$T(2,7)$	$-q^{10} + q^9 - q^8 + q^7 - q^6 + q^5 + q^3$
$T(2,9)$	$-q^{13} + q^{12} - q^{11} + q^{10} - q^9 + q^8 - q^7 + q^6 + q^4$
$T(2,11)$	$-q^{16} + q^{15} - q^{14} + q^{13} - q^{12} + q^{11} - q^{10} + q^9 - q^8 + q^7 + q^5$
$T(2,13)$	$-q^{19} + q^{18} - q^{17} + q^{16} - q^{15} + q^{14} - q^{13} + q^{12} - q^{11} + q^{10} - q^9 + q^8 + q^6$
$T(3,4)$	$-q^8 + q^5 + q^3$
$T(3,5)$	$-q^{10} + q^6 + q^4$

Look at `Jonespoly_and_kh_sage.py` for the Python dictionary `TKNOTS14SNAP` for all the Torus knots with less than 14 crossings, and then one can do this to obtain the Jones polynomial:

```
sage: TKNOTS14SNAP[(2,3)].jones_poly()
-q^4 + q^3 + q
```

9. KHOVANOV HOMOLOGY IMPLEMENTATIONS: KHOHO AND KNOTKIT

9.1. **KhoHo**. First, install Pari/GP. I recommend using *Homebrew*: `brew install pari`  
KhoHo was unavailable on the website <http://www.geometrie.ch/KhoHo/> and doesn’t appear to be on [github](#). Dunfield provided KhoHo-0.9.3.5 by email which I’ll try to put on github in the `qsApoly` repository. However, I tried to follow the instructions for KhoHo in the `00README` file, skipping the ‘make’ command since library files `nicematr.so`, `print_ranks.so`, `sparreduce.so` were already there (I tried removing these 3 library files and running make, but 20 errors were generated: see my [wordpress blog](#) for the full printout of the errors), but in Pari/GP, by typing `gp` to start up PARI’s programmable calculator, and then in `gp`, typing in the command to read KhoHo, this occurred:

```
? read(KH)
*** at top-level: read(KH)
***
*** in function read: read(KhoHo)
***
*** in function read: read(KhoHo_basic)
***
*** in function read: kill(print_ranks)
***
*** kill: can't kill that.
****Break loop: type 'break' to go back to GP prompt
break> break
```

I also tried `gp > read(KH)` and obtained an error for the break loop.  
Please contact me or let people know what’s wrong in this case with KhoHo, or how to get it to run.

9.2. **knotkit**.

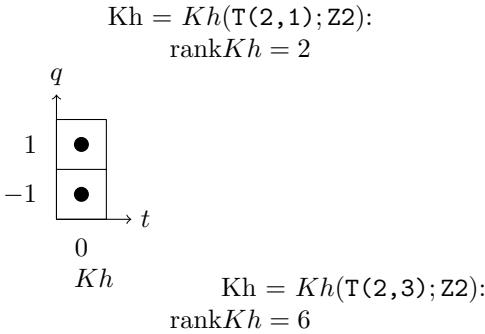
9.2.1. *Installation of knotkit*. I suggest doing a `git clone` of `knotkit` from its github repository:

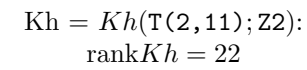
```
git clone https://github.com/cseed/knotkit.git
```

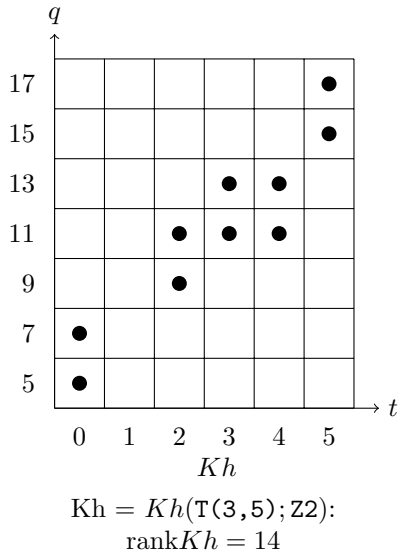
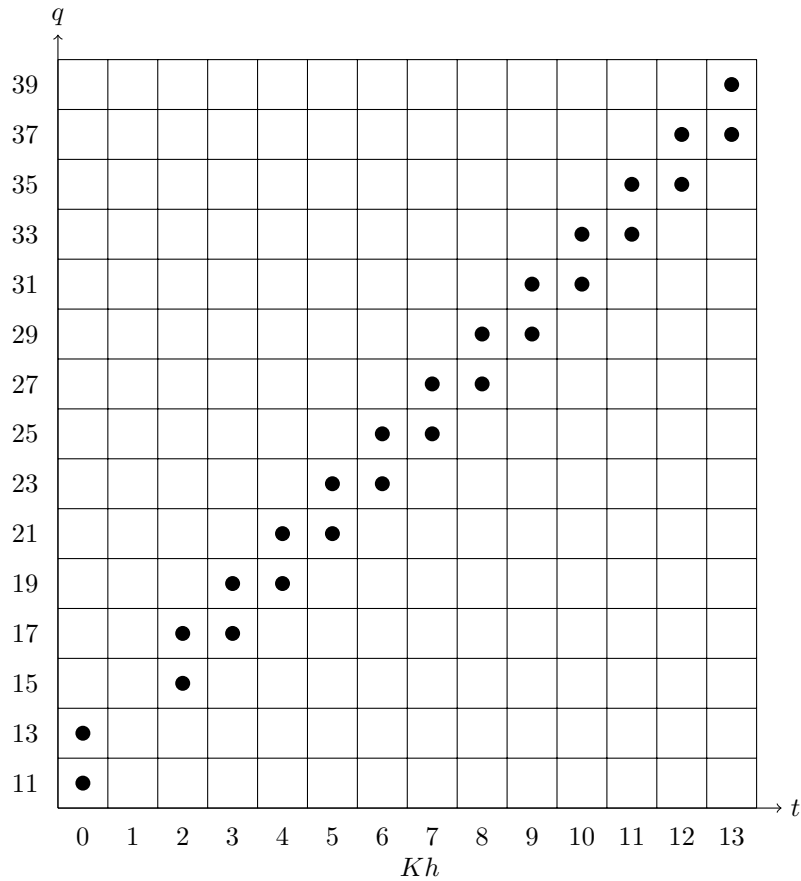
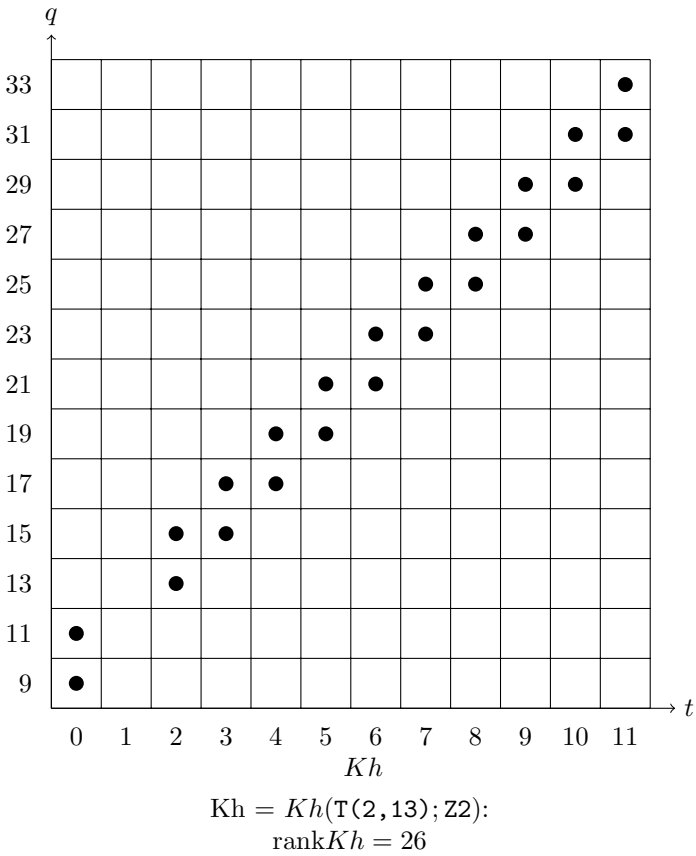
and typing the command ‘make’ (no need to type ‘./’ before, and it should be available to run everywhere on the system) to install and make the executable `kk`.

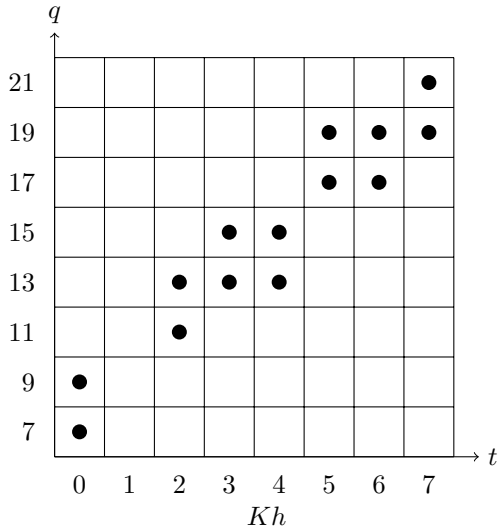
Use `./kk -h` for the help string.  
However, if one opens up the code for `kk.cpp` and also run the executable, `kk`, for Khovanov homology (kh) calculations, then the output is in LaTeX; the display is nice, but we want a nice string format so to do manipulations on it. Dunfield provided an alternative version of `kk.cpp` with the option of “`khplainout`” which outputs a string with the categorified Jones polynomial.

9.2.2. *Khovanov Homology computations from knotkit*. Knotkit or `kk`, with the “kh” flag outputs the following, for various torus knots of less than 14 crossings, the graph of the exponents for variables ( $t, q$ ) as dots:









If you wanted to *save the output* from option flag “khplainout”, then one simple and direct manner is to “pipe” the screen output with the symbol “>” at the Command prompt and specify the target file you’d want to save the output. For example, for the Torus knot  $T(2,5)$ , saving into the file `kh_T0205` in the directory above the current working directory,

```
./kk khplainout -v ‘T(2,5)’ > ../kh_T0205
```

I’ve done this for all torus knots of less than 14 crossings and saved the output and placed them in the github repository `qsApoly` so others can have access to the Khovanov homology computation outputs readily; they are named files `kh_T0203`, `kh_T0205`, ... in `kk_kh_output`.

If you don’t want to do this one-by-one, but a batch of these, one at a time, see the Python script `kk_kh_batch.py`, which you can run directly by typing at the command prompt `python kk_kh_batch.py` or manually, by opening up an interactive shell (i.e. `python -i kk_kh_batch.py`) and then running manually the Python function `direct_output`.

To “clean” or “scrape” the text (strings) of “categorified” Jones polynomial, to get it ready to use for **Sage Math**, I provide the file `kh_scrape.py`. It contains the function `scrape_file` that will take a file containing a *single* categorified Jones polynomial and output a “cleaned up” text (string) for the polynomial, in a Python dictionary, that can be used in Sage Math (just then apply the Sage Math function `sage_eval`). For example:

```
from kh_scrape import scrape_file
Qkh.<t,q> = PolynomialRing(RationalField(),2)
T0203stf = scrape_file('kh_T0203')
T0203poly = sage_eval( T0203stf['poly'], locals={'x1':t,'x2':q} )
T0203poly.substitute(t=-1) # for some reason, subs doesn't work in this case
```

Also, from running `kh_scrape.py` and function `scrape_bat` in *Sage Math* (make sure you’re in the appropriate working directory)

```
sage: from kh_scrape import scrape_bat
sage: Tknots14 = scrape_bat('khTknotsless14')
sage: Tknots14sage = [ sage_eval(line, locals={'x1':t,'x2':q}) for line in Tknots14 ]
sage: for K in Tknots14sage:
....:     print latex(K)
```

Then you’ll have all the torus knots categorified Jones polynomials from Khovanov Homology in a single list “Tknots14sage” (but you’ll have to manually keep track of which entry corresponds to which torus knot).

The result of printing with `latex` command in the last command immediately above is this table of “categorified” Jones polynomials from Khovanov Homology (source:`knotkit`)

K	J(K;t,q)
$T(2,1)$	$\frac{q^2+1}{q}$
$T(2,3)$	$t^3q^9+t^3q^7+t^2q^{\frac{q}{7}}+t^2q^5+q^3+q$
$T(2,5)$	$t^5q^{15}+t^5q^{13}+t^4q^{13}+t^4q^{11}+t^3q^{11}+t^3q^9+t^2q^9+t^2q^7+q^5+q^3$
$T(2,7)$	$t^7q^{21}+t^7q^{19}+t^6q^{19}+t^6q^{17}+t^5q^{17}+t^5q^{15}+t^4q^{15}+t^4q^{13}+t^3q^{13}+t^3q^{11}+t^2q^{11}+t^2q^9+q^7+q^5$
$T(2,9)$	$t^9q^{27}+t^9q^{25}+t^8q^{25}+t^8q^{23}+t^7q^{23}+t^7q^{21}+t^6q^{21}+t^6q^{19}+t^5q^{19}+t^5q^{17}+t^4q^{17}+t^4q^{15}+t^3q^{15}+t^3q^{13}+t^2q^{13}+t^2q^{11}+q^9+q^7$
$T(2,11)$	$t^{11}q^{33}+t^{11}q^{31}+t^{10}q^{31}+t^{10}q^{29}+t^9q^{29}+t^9q^{27}+t^8q^{27}+t^8q^{25}+t^7q^{25}+t^7q^{23}+t^6q^{23}+t^6q^{21}+t^5q^{21}+t^5q^{19}+t^4q^{19}+t^4q^{17}+t^3q^{17}+t^3q^{15}+t^2q^{15}+t^2q^{13}+q^{11}+q^9$
$T(2,13)$	$t^{13}q^{39}+t^{13}q^{37}+t^{12}q^{37}+t^{12}q^{35}+t^{11}q^{35}+t^{11}q^{33}+t^{10}q^{33}+t^{10}q^{31}+t^9q^{31}+t^9q^{29}+t^8q^{29}+t^8q^{27}+t^7q^{27}+t^7q^{25}+t^6q^{25}+t^6q^{23}+t^5q^{23}+t^5q^{21}+t^4q^{21}+t^4q^{19}+t^3q^{19}+t^3q^{17}+t^2q^{17}+t^2q^{15}+q^{13}+q^{11}$
$T(3,4)$	$t^5q^{17}+t^5q^{15}+t^4q^{13}+t^3q^{13}+t^4q^{11}+t^3q^{11}+t^2q^{11}+t^2q^9+q^7+q^5$
$T(3,5)$	$t^7q^{21}+t^7q^{19}+t^6q^{19}+t^5q^{19}+t^6q^{17}+t^5q^{17}+t^4q^{15}+t^3q^{15}+t^4q^{13}+t^3q^{13}+t^2q^{13}+t^2q^{11}+q^9+q^7$

The miracle is that if one sets  $t = -1$  in the above categorified Jones polynomials,  $J(K;t,q)$ , then  $J(K;t=-1,q) = J(K,q)$ , where  $J(K,q)$  are the Jones polynomials (modulo conventions), from Table 8!

To show this, keep in mind that for the categorified Jones polynomials, one must *normalize* to the “unknot” which in this case is the  $T(2,1)$  torus knot, after setting  $t = -1$  (or before? that’s my question; please let me know what “dividing” by the unknot means in both cases):

```
sage: normTknots14sage = [K/Tknots14sage[0] for K in Tknots14sage]

# Compare Jones polynomials from SnaPy to Jones polynomials from decategorified Khovanov Homology:
sage: x = var('x')
sage: for i in range(1,len(TKNOTS14SNAP)):
    SnaPyvskh = TKNOTS14SNAP[TKNOTSLIST14[i]].jones_poly().subs(q=x) ==
               normTknots14sage[i].subs(t=-1).subs(q=sqrt(x))
    print bool( SnaPyvskh )

sage: print "If all True, then J(K;t=-1,q) = J(K;q)!"
```

and indeed it does.

Note that for some reason, the Jones polynomial for  $T(2,1)$  of *SnaPy* produces this error:

```
sage: TKNOTS14SNAP[(2,1)].jones_poly()
# stuff
IndexError: list index out of range
```

EY: 20160224: My immediate question is this: what’s the convention or normalization that results in SnaPy outputting the Jones polynomial for the torus knots, say  $T(2,3)$  trefoil knot, to be

$$-q^4 + q^3 + q$$

vs. the Jones polynomial that results from Khovanov homology, after setting  $t = -1$ , for the trefoil:

$$-q^8 + q^6 + q^2$$

The latter expression is found on pp. 339 of Bar-Natan’s (nicely, pedagogically-friendly) review article [15]

## 10. EXPLORING THE JONES POLYNOMIAL AND KHOVANOV HOMOLOGY: UNANSWERED AVENUES

Armed with our categorified Jones polynomials and Sage Math, there are a number of modules (functions) that can be explored, as seen if one does the `dir()` command on a polynomial (e.g. `dir(Tknots14sage[1])`).

For instance, consider

```
sage: Tknots14sage[1].gradient() # T(2,3)
[3*t^2*q^9 + 3*t^2*q^7 + 2*t*q^7 + 2*t*q^5,
 9*t^3*q^8 + 7*t^3*q^6 + 7*t^2*q^6 + 5*t^2*q^4 + 3*q^2 + 1]
```

which is  $\frac{\partial}{\partial t}J(T(2,3);t,q)$  and  $\frac{\partial}{\partial q}J(T(2,3);t,q)$ . Are there any relationships we can discover between the categorified Jones polynomial and its partial(s) (derivatives)?

One can also use the Sage Math module `newton_polytope` to obtain the Newton Polytope immediately. One does the Newton Polytope tell us about categorified Jones polynomials?

Also, there are many Sage Math modules associated with `newton_polytope` (i.e. do e.g. `dir(Tknots14sage[1].newton_polytope())`) such as `face_lattice`:

```
sage: Tknots14sage[1].newton_polytope().face_lattice().list()
[<>, <0>, <1>, <2>, <3>, <1,2>, <0,1>, <0,3>, <2,3>, <0,1,2,3>]
sage: Tknots14sage[7].newton_polytope().face_lattice().list()
[<>, <0>, <1>, <2>, <3>, <4>, <1,2>, <0,1>, <0,4>, <3,4>, <2,3>, <0,1,2,3,4>]
```

So the face lattice of a  $T(2,2j+1)$  torus knot,  $j = 1 \dots 6$ , is different from the family of torus knots  $T(3,4)$  and  $T(3,5)$ . What does that mean?

This reference page might help with polytopes: [A class to keep information about faces of a polyhedron](#)

Something else one could try to do is to repeat Witten’s celebrated computation of the Jones polynomial from Chern-Simons theory [16] in *Sage Math*. This would entail an understanding, a grasp, of the Weyl group, in this case,  $SU(2)$ . I tried looking up topics on the Weyl group and Weyl character in Sage Math (cf. [Weyl Group](#), [SL versus GL](#))

### 10.1. Installation of SnapPy.

10.1.1. *Mac OS X Installation of SnapPy*. First, one should simply download [SnapPy.dmg](#), and then double-click the .dmg file and then drag-and-drog the SnapPy icon into the Applications Folder <sup>6</sup>.

However, one would like to take advantage of its integration with **Sage Math** and so here’s how to install SnapPy into **Sage**.

- (1) Go to or `cd` into the directory where the main program **sage** is in; for example, the directory that **sage**, the *executable file* is in, is

`/Applications/SageMath-6.10.app/Contents/Resources/sage`

where **sage** here is a *directory*.

- (2) Make sure you have `pip` installed and do this command:

`./sage -pip install --no-use-wheel snappy`

cf. <http://www.math.uic.edu/t3m/SnapPy/installing.html#sage>. It should successfully install.

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<sup>6</sup>cf. [Installing SnapPy](#)

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## 11. CODE LISTINGS

12. AUTOMATIC GENERATION OF DOCUMENTATION

Demonstration using epydoc:

```
epydoc --pdf -o /home/fnielsen/tmp/epydoc/ --name RBBase wikipedia/api.py
```

This example does not use `brede_str_nmf` but another more well-documented module called `api.py` that are used to download material from Wikipedia.