CATEGORIES

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Abstract. Everything about Categories, Category Theory, with applications to (relational) databases and other applications.

Contents

From the section on "Terminology" of the Preface of Barr and Wells (1998) [2]:

> "In most scientific disciplines, notation and terminology are standardized, of- ten by an international nomenclature committee. (Would you recognize Ein- steins equation if it said $p = HU^2$?) We must warn the nonmathematician reader that such is not the case in mathematics. There is no standardization body and terminology and notation are individual and often idiosyncratic."

To try to bridge the difference choice of notation and through comparison, suggest the "best" notation that's easy to remember and easy to use, I'll present all the different types of notation that I come across as much as I can.

0.1. **Classes.** From Adámek, Herrlich, and Strecker (2004) [5]:

(1) members of each class are sets
(2) $\forall$ "property" P can form class of all sets with property P
  e.g. **universe** - class of all sets $\mathcal{U}$
(3) if $X_1, X_2, \dots X_n$ classes, $(X_1, X_2 \dots X_n)$ is a class
(4) $\forall$ set is a class (equivalently, every member of a set is a set)
  proper classes - classes that aren't sets
  $\Longrightarrow$ proper classes cannot be members of any class
  proper classes examples:
  - universe $\mathcal{U}$
  - class of all vector spaces
  - class of all topological spaces
  - class of all automata are proper classes

  (4) $\Longrightarrow$ *Axiom of Replacement*
(5) $\nexists$ surjection from set to proper class

1. Categories

Definition 1 (Category)**.** Using the notation of Adámek, Herrlich, and Strecker (2004) [5]:
category C is quadruple $\mathbf{C} = (\mathrm{Ob}, \hom, 1, \circ)$ consisting of

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class Ob, Ob collection, whose members are objects, $A, B, C \in \mathrm{Ob}$,
$\forall (A, B)$, $A, B \in \mathrm{Ob}$, $\hom(A, B)$ collection of morphisms/arrows
$\forall f \in \hom(A, B)$, $f : A \to B$
$\forall A \in \mathrm{Ob}$, $\exists$ identity morphism/arrow, $1_A : A \to A$,
composition law s.t.

(a) *composition* : $\forall A, B, C \in \mathrm{Ob}$, $\begin{array}{l} f : A \to B \\ g : B \to C \end{array}$, then $g \circ f : A \to C$

(b) associativity $\begin{array}{l} f : A \to B \\ g : B \to C \\ h : C \to D \end{array}$ then $h \circ (g \circ f) = (h \circ g) \circ f$

(c) if $f : A \to B$, $1_B \circ f = f = f \circ 1_A$

In my notation,
category $\mathbf{A}$ is quadruple $\mathbf{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}\mathbf{A}, 1, \circ)$

$$\mathbf{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}\mathbf{A}, 1, \circ)$$

s.t.

(1) $A \in \mathrm{Obj}(\mathbf{A})$ is called an *object*
(2) $\mathrm{Mor}\mathbf{A} = \bigcup_{\mathrm{Hom}(A,B) \in \mathbf{A}} \mathrm{Hom}(A, B)$, $f : A \to B \in \mathrm{Hom}(A, B)$ is a *morphism*, i.e.
$A, B \in \mathrm{Obj}\mathbf{A}$, $f \in \mathrm{Hom}_{\mathbf{A}}(A, B)$

$$A \xrightarrow{f} B$$

(3) $\forall A \in \mathrm{Obj}(\mathbf{A})$, $\exists 1_A : A \to A$

$$A \xrightarrow{1_A} A \quad \text{or} \quad \overset{f}{\circlearrowleft} A$$

(4) $\forall A, B, C \in \mathrm{Obj}\mathbf{A}$,

$\forall \begin{array}{l} f : A \to B \in \mathrm{Hom}(A, B), \text{ i.e. } f, g \in \mathrm{Mor}\mathbf{A}, \\ g : B \to C \in \mathrm{Hom}(B, C) \end{array}$ then $g \circ f : A \to C \in \mathrm{Hom}(A, C)$, $g \circ f \in \mathrm{Mor}\mathbf{A}$ i.e.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

s.t.

(a) *associativity* $\forall \begin{array}{l} f : A \to B \\ g : B \to C \\ h : C \to D \end{array}$, $h \circ (g \circ f) = (h \circ g) \circ f$ i.e.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rrr, bend left=40, "(h \circ g) \circ f"] \arrow[rr, bend right, "g \circ f"'] \arrow[rrr, bend right=40, "h \circ (g \circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h \circ g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

(b) $\forall f : A \to B \in \mathrm{Hom}(A, B)$, $1_B \circ f = f$ and $f \circ 1_A = f$ i.e.
$\forall f \in \mathrm{Hom}_{\mathbf{A}}(A, B)$,

$$1_A \circlearrowleft A \xrightarrow{f} B \circlearrowright 1_B$$

(c) $\mathrm{Hom}(A, B) \in \mathrm{Mor}\mathbf{A}$ pairwise disjoint (i.e. $\mathrm{Hom}(A, B) \bigcap \mathrm{Hom}(C, D) \neq \emptyset$ if $C \neq A$ or $D \neq B$)

1.1. **Examples.**

- $\mathrm{Set} = (\mathrm{Ob}_{\mathrm{Set}}, \hom_{\mathrm{Set}}, 1, \circ)$ where
$\mathrm{Ob}_{\mathrm{Set}}$ is the class of all sets
$\hom_{\mathrm{Set}}$ is the class of all functions on a set to another set
- Vec

$$\begin{aligned} \mathrm{ObjVec} &\equiv \text{all real vector spaces} \\ \mathrm{MorVec} &\equiv \text{all linear transformations between them (between real vector spaces)} \end{aligned}$$

- **Monoid**. Consider a monoid as a triple $(M, \cdot, e)$.
Every semigroup $(M, \cdot)$ (recall that a *semigroup* is a set S with binary operation $\cdot$, i.e. s.t.

$S \times S \xrightarrow{\cdot} S$
$\forall a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)
(but no inverse, necessarily!)) that also has a unit e can be made into a category $\mathbf{C}$
$\Longrightarrow \mathbf{C}(M, \cdot, e) = (\mathrm{Ob}, \hom, 1, \circ)$, a category $\mathbf{C}$ with only 1 object, i.e. $\mathrm{Ob} = \{M\}$, so that
$\mathrm{Ob} = \{M\}$
$\hom(M, M) = M$
$1_M = e$
$y \circ x = y \cdot x$

2. Duality

Given a category $\mathbf{A} = (\mathrm{Ob}, \hom_{\mathbf{A}}, 1, \circ)$,

Definition 2 (dual opposite category)**. dual** or **opposite** category of $\mathbf{A}$, denoted $\mathbf{A}^{\mathrm{op}}$, is

$$\mathbf{A}^{\mathrm{op}} = (\mathrm{Ob}, \hom_{\mathbf{A}^{\mathrm{op}}}, 1, \circ^{\mathrm{op}}) \tag{1}$$

s.t.

$$\begin{aligned} \hom_{\mathbf{A}^{\mathrm{op}}}(A, B) &= \hom_{\mathbf{A}}(B, A) \\ f \circ^{\mathrm{op}} g &= g \circ f \end{aligned}$$

i.e.
$\forall$ category $\mathbb{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}\mathbf{A}, 1, \circ)$,
dual (or opposite) category of A is $\mathbf{A}^{\mathrm{op}} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}\mathbf{A}^{\mathrm{op}}, 1, \circ^{\mathrm{op}})$ where $\forall \mathrm{Hom}_{\mathbf{A}^{\mathrm{op}}}(A, B) \in \mathrm{Mor}\mathbf{A}^{\mathrm{op}}$, $\mathrm{Hom}_{\mathbf{A}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathbf{A}}(B, A)$ and

$$f \circ^{\mathrm{op}} g = g \circ f$$

e.g. if $\mathbf{A} = (M, \cdot, e)$ monoid, then $\mathbf{A}^{\text{op}} = (M, \hat{\cdot}, e)$ where $a \hat{\cdot} b = b \cdot a$

2.0.1. *Example.*

- Vec^{op}

$$\text{Vec}^{\text{op}} = (\text{Obj}(\text{Vec}), \text{Hom}_{\text{Vec}^{\text{op}}}, 1, \circ^{\text{op}})$$

s.t.

$$\text{Hom}_{\text{Vec}^{\text{op}}}(W, V) = \text{Hom}_{\text{Vec}}(V, W)$$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & V \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} U & V \arrow[l, "f"'] & W \arrow[l, "g"'] \arrow[ll, bend left, "f \circ^{\text{op}} g"] \end{tikzcd}$$

2.1. Initial and Terminal Objects.

cf. Chapter 2 Objects and Morphisms in Abstract Categories, Sec. 7 Objects and morphisms of Adámek, Herrlich, and Strecker (2004).

Definition 3 (initial object)**. initial object** A is object A if $\forall$ object B, $\exists$ exactly 1 morphism from A to B.

2.1.1. *Examples (of initial object).*

(1) $\emptyset \to X$; $\emptyset$ unique initial object for Set, (Pos, Top)
(2) $\forall G \in \text{ObjGrp}$ s.t. $|G| = 1$, i.e. $G = \{1\}$, $\{1\} \to G \in \text{ObjGrp}$ is initial object for Grp; likewise for Vec.

cf. Prop. 7.3 of Adámek, Herrlich, and Strecker (2004):

Proposition 1. *initial objects essentially unique, i.e.*

(1) *if A, B initial objects, then A, B isomorphic*
(2) *if A initial object, then $\forall$ object B s.t. $B \cong A$ (i.e. isomorphic), then B also initial object*

Proof. (1) By def. of initial object,

$$A \xrightarrow{k} B$$

$$B \xrightarrow{h} A$$

$h \circ k$ unique (since h unique, and k unique), so $h \circ k = 1_A$, and likewise $k \circ h = 1_B \implies k$ isomorphism.

(2) Let $k : A' \to A$ isomorphism. $\forall$ object B, $\exists !\, f : A \to B$ (def.)

$$\begin{tikzcd} A' \arrow[d, "k"] \arrow[dr, "f \circ k"] & \\ A \arrow[r, dashed, "f"] & B \end{tikzcd}$$

$f \circ k : A' \to B$ morphism. If $g : A' \to B$, $g \circ k' = f$ (f unique), so $f \circ k$ unique, and $g = f \circ k$, A' initial. □

Definition 4 (terminal object)**. terminal object** A is object A if $\forall$ object B, $\exists$ exactly 1 morphism from B to A.

Terminal objects dual to initial objects, i.e. A terminal in $\mathbf{A}$ iff A initial in $\mathbf{A}^{\text{op}}$.
Every singleton set is a terminal object for set.

2.1.2. *Examples (of terminal objects).* e.g. singleton set $\{0\}$ is terminal object for $\text{Vec}, \text{Pos}, \text{Grp}, \text{Top}$.

Definition 5. zero object if object both initial and terminal object. zero object self dual.

2.1.3. *Examples (of zero objects).*

(1)
(2) $\text{Vec}, \text{Ban}, \text{Ban}_b, \text{TopVec}, \text{Mon}$ have zero objects, but Sgr doesn't.
(3) Ab and Grp have 0 objects, Ring.

3. Functors

Definition 6 (Functors)**. (covariant) functor**

$$F : \mathbf{C} \to \mathbf{D}$$

if $\forall C \in \text{Ob}_{\mathbf{C}}$, then $F(C) \in \text{Ob}_{\mathbf{D}}$
s.t. $\forall f \in \text{hom}_{\mathbf{C}}$, say $f \in \text{hom}_{\mathbf{C}}(B, C)$
$F(f) \in \text{hom}_{\mathbf{D}}(F(B), F(C))$
and s.t.
$F(1_{\mathbf{C}}) = 1_{F(C)}$

$A, B, C \in \text{Ob}_{\mathbf{C}}$, $f : A \to C$, so $g \circ f : A \to C$
$g : B \to C$
then $F(g \circ f) = F(g) \circ F(f)$

i.e.

$$\mathbf{C} \xrightarrow{F} \mathbf{D}$$

if

$$C \overset{F}{\longmapsto} F(C)$$

$$\mathbf{C} \xrightarrow{F} \mathbf{D}$$

s.t.

$$B \xrightarrow{f} C \quad \xrightarrow{F} \quad F(B) \xrightarrow{F(f)} F(C)$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd} \quad \xrightarrow{F} \quad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g \circ f)"'] & F(B) \arrow[r, "F(g)"] & F(C) \end{tikzcd}$$

i.e.

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "F"] & C \arrow[d, "F"] \\ F(B) \arrow[r, "F(f)"] & F(C) \end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "F"] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, "g"] \arrow[d, "F"] & C \arrow[d, "F"] \\
F(A) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g \circ f) = F(g) \circ F(f)"'] & F(B) \arrow[r, "F(g)"] & F(C)
\end{tikzcd}$$

Definition 7. *(contravariant)* functor F is s.t.

(2) $$\mathbf{C}^{\text{op}} \xrightarrow{F} \mathbf{D}$$

so that

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "F"] & C \arrow[d, "F"] \\
F(B) & F(C) \arrow[l, "F(f)"']
\end{tikzcd}$$

Definition 8 (covariant hom-functor)**.** $\forall$ *locally small category* $\mathbf{C}$ (i.e. $\text{hom}_{\mathbf{C}}$ is actually a set and not a proper class), $\forall A \in \text{Ob}_{\mathbf{C}}$, $\exists$ covariant hom-functor $\text{hom}(A, -) : \mathbf{C} \to \text{Set}$ s.t. $\forall B \xrightarrow{f} C$,

$$\text{hom}(A, -)(f) = \text{hom}(A, B) \xrightarrow{\text{hom}(A,f)} \text{hom}(A, C)$$

where $\text{hom}(A, f)(g) = f \circ g$

i.e. $\forall X, Y \in \text{Ob}_{\mathbf{C}}$, $\forall X \xrightarrow{f} Y$,

then

$$\text{hom}(A, -)(f) = \text{hom}(A, f)$$

$$\text{hom}(A, X) \xrightarrow{\text{hom}(A, f)} \text{hom}(A, Y)$$

and $g \longmapsto f \circ g$ with $g \in \text{hom}(A, X)$ i.e. (20160424 EY)

$\forall$ category $\mathbf{A}$, $\forall A \in \text{Obj}\mathbf{A}$,
$\exists$ **covariant hom-functor**

$$\begin{gathered}
\text{hom}(A, -) : \mathbf{A} \to \text{Set defined by },\ \forall f \in \text{Hom}(B, C) \subset \text{Mor}\mathbf{A} \\
\text{hom}(A, -)(B \xrightarrow{f} C) = \text{Hom}(A, B) \xrightarrow{\text{hom}(A,f)} \text{Hom}(A, C) \\
\text{hom}(A, f)(g) = f \circ g
\end{gathered}$$

M-set is a covariant hom-functor on a monoid $\mathbf{C}(M, \cdot, e) \equiv \mathbf{C}(M)$, M a monoid, i.e. the category that is the domain that the covariant hom-functor maps from is a monoid (category).

Definition 9 (contravariant hom-functor)**.** $\forall$ category $\mathbf{A}$, $\forall A \in \text{Obj}\mathbf{A}$,
$\exists$ **contravariant hom-functor**,

$$\begin{gathered}
\text{hom}(-, A) : \mathbf{A}^{\text{op}} \to \text{Set defined by, } \forall f \in \text{Hom}_{\mathbf{A}^{\text{op}}}(B, C) \subset \text{Mor}\mathbf{A}^{\text{op}} \text{ i.e. } B \xrightarrow{f} C \\
\text{hom}(-, A)(B \xrightarrow{f} C) = \text{Hom}_{\mathbf{A}}(B, A) \xrightarrow{\text{hom}(f,A)} \text{Hom}_{\mathbf{A}}(C, A) \\
\text{hom}(f, A)(g) = g \circ f \equiv g \circ_{\mathbf{A}} f
\end{gathered}$$

i.e.

$$\begin{tikzcd}
A & \\
B \arrow[u, "g"] & C \arrow[l, "f"] \arrow[ul, "g \circ f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "g^{\text{op}}"'] \arrow[dr, "f \circ^{\text{op}} g"] & \\
B \arrow[r, "f^{\text{op}}"'] & C
\end{tikzcd}$$

Definition 10 (forgetful functor)**.** $\forall$ constructs (i.e. categories)

- Vec
- Grp
- Top
- Rel

$\exists U : \mathbf{A} \to \text{Set}$ s.t.

$$\begin{aligned}
U(A) & \quad \text{is underlying set} \\
U(f) = f & \quad \text{is underlying function}
\end{aligned}$$

Definition 11. given functor $F : \mathbf{A} \to \mathbf{B}$,
dual functor or **opposite functor** $F^{\text{op}} : \mathbf{A}^{\text{op}} \to \mathbf{B}^{\text{op}}$ is given by
$\forall f : A \to A'$, $f \in \text{Hom}(A, A')$,

$$F^{\text{op}} f = Ff$$

$Ff : FA \to FA'$, $Ff \in \text{Hom}(FA, FA')$

3.0.4. *Examples.*

- **duality functor for vector spaces** $(*) : \text{Vec}^{\text{op}} \to \text{Vec}$
associates $\forall$ vector space V its dual V^* (i.e. vector space $\text{Hom}(V, \mathbb{R})$ with operations defined pointwise),
associates $\forall V \xrightarrow{f} W$, $f \in \text{MorVec}^{\text{op}}$,
i.e. $\forall$ linear map $W \xrightarrow{f} V$,
morphism $f^* : V^* \to W^*$ defined by
$f^*(g) = g \circ f$ i.e.

$$\text{Vec}^{\text{op}} \xrightarrow{(*)} \text{Vec}$$

$$V \overset{(*)}{\longmapsto} V^*$$

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "(*)"] & V \arrow[d, "(*)"] \\
W^* & V^* \arrow[l, "f^*"']
\end{tikzcd}$$

3.1. Functor properties.

Definition 12. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor.

(1) F **embedding** if F is injective on morphisms ($\forall\, f \in \mathrm{Mor}\mathbf{A}$, if $F(f) = F(g)$, then $f = g$)
$g \in \mathrm{Mor}\mathbf{A}$

(2) F **faithful** if $\forall$ hom-set restrictions,

$$F : \mathrm{Hom}_{\mathbf{A}}(A, A') \to \mathrm{Hom}_{\mathbf{B}}(FA, FA')$$

are injective, i.e.
for hom-set restriction $F : \mathrm{Hom}_{\mathbf{A}}(A, A') \to \mathrm{Hom}_{\mathbf{B}}(FA, FA')$,
if $F(f) = F(f')$, then $f = f'$.

(3) F **full** if all hom-set restrictions are surjective

(4) F **amnestic** if $Ff = 1_{\mathbf{B}}$, then $\mathbf{A}$-isomorphism $f = 1_{\mathbf{A}}$

So

(1) F an embedding iff F faithful and injective on objects
(2) F isomorphism iff F full, faithful, and bijective on objects

cf. Def. 3.33 of Adámek, Herrlich, and Strecker (2004) [5] (note that, again, I base these notes heavily on Adámek, Herrlich, and Strecker (2004) and take definitions, propositions, theorems, etc. liberally from there):

Definition 13 (equivalence)**.** functor $F : \mathbf{A} \to \mathbf{B}$ is an **equivalence** if F full, faithful, isomorphism-dense (meaning $\forall\, B \in \mathrm{Obj}\mathbf{B}$, $\exists$ some $A \in \mathrm{Obj}\mathbf{A}$, s.t. $F(A)$ isomorphic to B, i.e.

(1) faithful: $\forall\, F : \mathrm{Hom}_{\mathbf{A}}(A, A') \to \mathrm{Hom}_{\mathbf{B}}(FA, FA')$, if $F(f) = F(f')$, $f = f'$
(2) full: $\forall\, g \in \mathrm{Hom}_{\mathbf{B}}(FA, FA')$, $FA \xrightarrow{g} FA'$, $\exists\, f \in \mathrm{Hom}_{\mathbf{A}}(A, A')$, $A \xrightarrow{f} A'$ s.t. $g = Ff$
(3) isomorphism-dense: $\forall\, B \in \mathrm{Obj}\mathbf{B}$, $\exists\, A \in \mathrm{Obj}\mathbf{A}$ s.t. $F(A) \xrightarrow{\cong} B$

$\mathbf{A}$, $\mathbf{B}$ are **equivalent** if $\exists$ equivalence F, $F : \mathbf{A} \to \mathbf{B}$.

3.2. Natural Transformation.

Definition 14 (Natural transformation)**.** Let functors $F, G : \mathbf{A} \to \mathbf{B}$.
natural transformation τ from F to $G \equiv \tau : F \to G$ or $F \xrightarrow{\tau} G$ is function that assigns $\forall\, A \in \mathrm{Obj}\mathbf{A}$, $\tau_A : FA \to GA$, $\tau_A \in \mathrm{Mor}\mathbf{B}$, s.t. **naturality condition** holds:
$\forall\, A \xrightarrow{f} A'$, $f \in \mathrm{Mor}\mathbf{A}$

$$\begin{tikzcd} FA \arrow[r, "\tau_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FA' \arrow[r, "\tau_{A'}"] & GA' \end{tikzcd}$$

3.2.1. *Examples.*

- Let $(**) : \mathrm{Vec} \to \mathrm{Vec}$ be **second-dual functor for vector spaces** defined by

$$\mathrm{Vec} \xrightarrow{(**)} \mathrm{Vec} = (\mathrm{Vec}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{(*)^{\mathrm{op}}} \mathrm{Vec}^{\mathrm{op}} \xrightarrow{(*)} \mathrm{Vec}$$

where $(*)^{\mathrm{op}}$ is the dual of the duality functor for vector spaces.
Then linear transformations

$$\tau_V : V \to V^{**}$$
$$(\tau_V(x))(f) = f(x)$$

yield a natural transformation $1_{\mathrm{Vec}} \xrightarrow{\tau} (**)$
Indeed, looking at the definition of the natural transformation, for

$$\mathrm{Vec} \xrightarrow{1_{\mathrm{Vec}}} \mathrm{Vec}$$
$$\mathrm{Vec} \xrightarrow{(**)} \mathrm{Vec}$$

$\forall\, V \in \mathrm{Obj}(\mathrm{Vec})$, $\tau_V : 1_{\mathrm{VeC}} V = V \to (**)V \equiv V^{**}$, $\tau_V \in \mathrm{MorVec}$, and
$\forall\, f : V \to W$, $f \in \mathrm{MorVec}$,

$$\begin{tikzcd} V \arrow[r, "\tau_V"] \arrow[d, "f"] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\tau_W"] & W^{**} \end{tikzcd}$$

- assignment of Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ to each topological space X is a natural transformation from nth homotopy functor $\pi_n : \mathrm{Top} \to \mathrm{Grp}$ to nth homology functor $H_n : \mathrm{Top} \to \mathrm{Grp}$

$$\pi_n \xrightarrow{\tau} H_n$$

Indeed, $\forall\, X \in \mathrm{Obj}(\mathrm{Top})$, $\tau_X : \pi_n(X) \to H_n(X)$, $\tau_X \in \mathrm{MorGrp}$,
$\forall\, X \xrightarrow{\varphi} Y$, $\varphi \in \mathrm{MorTop}$,

$$\begin{tikzcd} \pi_n(X) \arrow[r, "\tau_X"] \arrow[d, "\pi_n \circ \varphi"] & H_n(X) \arrow[d, "H_n \circ \varphi"] \\ \pi_n(Y) \arrow[r, "\tau_Y"] & H_n(Y) \end{tikzcd}$$

Definition 15 (Grothendieck construction)**.** Let category $\mathbf{C}$, a category of small categories CAT,
Let functor $F : \mathbf{C} \to CAT$
Then category $\Gamma(C)$ (also denoted $C \int(F)$) is $\Gamma(C) = (\mathrm{Ob}_{\Gamma(F)}, \mathrm{hom}_{\Gamma(F)}, 1, \circ)$ s.t.

$$(C, X) \in \mathrm{Ob}_{\Gamma(F)}, \quad \begin{aligned} &C \in \mathrm{Ob}_{\mathbf{C}} \\ &X \in \mathrm{Ob}_{F(C)} \end{aligned}$$

and
$\mathrm{hom}_{\Gamma(F)}((C_1, X_1), (C_2, X_2)) \ni (f, x)$ s.t.

$$f : C_1 \to C_2 \in \mathrm{mor}_{\mathbf{C}} := \mathrm{hom}_{\mathbf{C}}$$
$$x : F(f)(X_1) \to X_2 \in \mathrm{mor}_{F(C_2)} := \mathrm{hom}_{F(C_2)}$$

EY : 20150714, to clarify, $f \in \mathrm{hom}_{\mathbf{C}}$, and $x \in \mathrm{hom}_{F(C_2)}$,
and

$$(f, x) \circ (f', x') = (ff', x \circ F(f)(x'))$$

i.e.

$$C_1 \xrightarrow{f} C_2 \quad \Longrightarrow \quad F(C_1) \xrightarrow{F(f)} F(C_2)$$

$$\begin{tikzcd}
(C_1, X_1) \arrow[r, "(f', x')"] \arrow[rr, bend right, "(f \circ f', x \circ F(f)(x'))"'] & (C_2, X_2) \arrow[r, "(f, x)"] & (C_3, X_3)
\end{tikzcd}$$

4. Subcategories

Definition 16. category **A** **subcategory** of category **B**, ($\equiv \quad A \subset B$) if
(1) $\text{Obj}\mathbf{A} \subseteq \text{Obj}\mathbf{B}$
(2) $\forall A, A' \in \text{Obj}\mathbf{A}$, $\text{Hom}_{\mathbf{A}}(A, A') \subseteq \text{Hom}_{\mathbf{B}}(A, A')$
(3) $\forall A \in \text{Obj}\mathbf{A}$, $1_{A \in \text{Obj}\mathbf{A}} = 1_{A \in \text{Obj}\mathbf{B}}$

(4) $\forall A, B, C \in \text{Obj}\mathbf{A}$, $\begin{matrix} \forall f \in \text{Hom}_{\mathbf{A}}(A, B), \\ \forall g \in \text{Hom}_{\mathbf{A}}(B, C) \end{matrix}$ $g \circ f : A \to C$, then $g \circ f = g' \circ f'$, $\begin{matrix} \forall f' \in \text{Hom}_{\mathbf{B}}(A, B), \\ \forall g' \in \text{Hom}_{\mathbf{B}}(B, C) \end{matrix}$ i.e.
composition law in **A** is restriction of composition law in **B** to morphisms of **A**.

full subcategory of **B**, **A**, if, in addition, $\forall A, A' \in \text{Obj}\mathbf{A}$, $\text{Hom}_{\mathbf{A}}(A, A') = \text{Hom}_{\mathbf{B}}(A, A')$

Remark 1. $\forall$ subcategory **A** of category **B**, $\exists$ naturally associated inclusion functor $E : \mathbf{A} \hookrightarrow \mathbf{B}$.
Moreover, such inclusion E is s.t.
(1) E an embedding (i.e. E injective on morphisms, i.e. if $E(f) = E(g)$, then $f = g, \forall f, g \in \text{Hom}_{\mathbf{A}}(A, A'), \quad \forall A, A' \in \text{Obj}\mathbf{A}$)
(2) E full functor iff **A** full subcategory of **B**, i.e. full if all hom-set restrictions surjective, i.e. if $g : EA \to EA'$, then $g = E(f)$ for some $f : A \to A' \in \text{Hom}_{\mathbf{A}}(A, A')$, i.e.

$$\begin{tikzcd}
A \arrow[r, hook, "E"] \arrow[d, "f"'] \arrow[dr, phantom, "\xhookrightarrow{E}"] & EA \arrow[d, "g = E(f)"] \\
A' \arrow[r, hook, "E"] & EA'
\end{tikzcd}$$

cf. Prop 4.5 of Adámek, Herrlich, and Strecker (2004) [5]

Proposition 2. (1) *functor* $F : \mathbf{A} \to \mathbf{B}$ *(full) embedding iff* $\exists$ *(full) subcategory* $\mathbf{C} \subset \mathbf{B}$ *with inclusion functor* $E : \mathbf{C} \to \mathbf{B}$ *and isomorphism* $G : \mathbf{A} \to \mathbf{C}$ *with* $F = E \circ G$, *i.e.*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "F"] \arrow[d, "\mathbf{C}"'] & \mathbf{B} \\
\mathbf{C} \arrow[ur, hook, "E"'] &
\end{tikzcd}$$

(2) *functor* $F : \mathbf{A} \to \mathbf{B}$ *faithful iff* $\exists$ *embeddings* $\begin{matrix} E_1 : \mathbf{D} \to \mathbf{B}, \\ E_2 : \mathbf{A} \to \mathbf{C} \end{matrix}$ *equivalence* $G : \mathbf{C} \to \mathbf{D}$ *s.t.*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "F"] \arrow[d, "E_2"] & \mathbf{B} \\
\mathbf{C} \arrow[r, "G"] & \mathbf{D} \arrow[u, "E_1"']
\end{tikzcd} \quad \textit{i.e. } G(C \xrightarrow{g} C') = FC \xrightarrow{g} FC'$$

Proof. (1)

(2) Let $E_1 : \mathbf{D} \to \mathbf{B}$ be inclusion $E_1 : \mathbf{D} \hookleftarrow \mathbf{B}$, and let **D** be full subcategory of **B**.
Let $\text{Obj}\mathbf{D} = F(\text{Obj}\mathbf{A}) = \{B | B = F(A) \quad \forall A \in \text{Obj}\mathbf{A}\} = \{FA | \forall A \in \text{Obj}\mathbf{A}\}$ = all images (under F) of $\text{Obj}\mathbf{A}$.
Let category **C** s.t. $\text{Obj}\mathbf{C} = \text{Obj}\mathbf{A}$, and
$$\text{Hom}_{\mathbf{C}}(A, A') = \text{Hom}_{\mathbf{B}}(FA, FA')$$
□

Definition 17. full subcategory **A** of category **B** is
(1) **isomorphism-closed** if $\forall B \in \text{Obj}\mathbf{B}$ s.t. B isomorphic to some $A \in \text{Obj}\mathbf{A}$, $B \in \text{Obj}\mathbf{A}$
(2) **isomorphism-dense** if $\forall B \in \text{Obj}\mathbf{B}$, B isomorphic to some $A \in \text{Obj}\mathbf{A}$

4.0.2. *Example.* cf. Example 4.11 of Adámek, Herrlich, and Strecker (2004) [5]:
full subcategory of Set, but consisting of (only) single object $\mathbb{N}$
is neither isomorphism-closed nor isomorphism dense in Set.
This category is equivalent to isomorphism closed full subcategory of Set consisting of all countable infinite sets.
"There are instances when one wishes to consider full subcategories in which different objects can't be isomorphic." -Adámek, Herrlich, and Strecker (2004) [5]

Definition 18. **skeleton** of category is full, isomorphism-dense subcategory in which no 2 distinct objects are isomorphic.

4.0.3. *Examples.* cf. Example 4.13 of Adámek, Herrlich, and Strecker (2004) [5].
(1) full subcategory of all cardinal numbers is skeleton for Set
(2) full subcategory determined by the powers $\mathbb{R}^m$, where $m \in$ all cardinal numbers, is skeleton for Vec

Proposition 3. (1) $\forall$ *category has a skeleton*
(2) $\forall$ *2 skeletons of a category, they're isomorphic (the 2 skeletons)*
(3) $\forall$ *skeleton of category* **C** *is equivalent to* **C**

Proof. (1) from Axiom of Choice [cf. 2.3(4) of Adámek, Herrlich, and Strecker (2004) [5]], applied to equivalence relation "is isomorphic to" on class of objects of the category
(2) Let **A**, **B** be skeletons of **C** Then $\forall A \in \text{Obj}\mathbf{A}$ is isomorphic in **C** to unique $B \in \text{Obj}\mathbf{B}$

$$A \xrightarrow{\cong} B = F(A)$$

Choose $\forall A \in \text{Obj}\mathbf{A}$, **C**-isomorphism $f_A : A \to F(A)$.
Then functor $F : \mathbf{A} \to \mathbf{B}$,

$$F(A \xrightarrow{h} A') = FA \xrightarrow{f_A^{-1}} A \xrightarrow{h} A' \xrightarrow{f_{A'}} FA'$$

is an isomorphism.
(3) The inclusion of skeleton of **C** into **C** is an equivalence.
□

Corollary 1. *2 categories equivalent iff they have isomorphic skeletons.*

5. Limits

5.0.4. *Sources.* It appears Adámek, Herrlich, and Strecker (2004) [5] defines *sources* to simply give a name and formalize a tuple.

Definition 19 (source)**.** **source** is a tuple: $(a, (f_i)_{i \in I})$, $f_i : A \to A_i$

5.1. **Products.**

Definition 20 (Products)**.** (in Turi's notation [4])

Given objects C_1, C_2 of category $\mathbb{C}$, **product** (if exists) consists of object $C_1 \times C_2$ of $\mathbb{C}$ and $\begin{aligned} \pi_1 &: C_1 \times C_2 \to C_1 \text{ s.t.} \\ \pi_2 &: C_1 \times C_2 \to C_2 \end{aligned}$

$$\forall \text{ object } A \text{ of } \mathbb{C}, \quad \forall \begin{aligned} f &: A \to C_1 \\ g &: A \to C_2 \end{aligned} \quad \exists! \quad \langle f, g \rangle : A \to C_1 \times C_2 \text{ s.t. } \begin{aligned} f &= \pi_1 \circ \langle f, g \rangle, \text{ i.e.} \\ g &= \pi_2 \circ \langle f, g \rangle \end{aligned}$$

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ C_1 & C_1 \times C_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & C_2 \end{tikzcd}$$

(compare with Leinster (2014) [3])

Let category $\mathcal{A}$, $X, Y \in \mathcal{A}$, **product** of X, Y consists of object P and maps

(compare this definition with Adámek, Herrlich, and Strecker (2004) [5] and their notation)

product consisting of

$$\begin{aligned} C_1 \times C_2 \times \dots \times C_{\mathcal{N}} &\in \text{Obj}\mathbf{C} \\ \pi_1 : C_1 \times C_2 \times \dots \times C_{\mathcal{N}} &\to C_1 \\ \pi_2 : C_1 \times C_2 \times \dots \times C_{\mathcal{N}} &\to C_2 \\ &\vdots \\ \pi_{\mathcal{N}} : C_1 \times C_2 \times \dots \times C_{\mathcal{N}} &\to C_{\mathcal{N}} \end{aligned}$$

is s.t.

$$\forall \begin{aligned} A &\in \text{Obj}\mathbf{C} \\ f_1 &: A \to C_1 \\ f_2 &: A \to C_2, \\ &\vdots \\ f_{\mathcal{N}} &: A \to C_{\mathcal{N}} \end{aligned}$$

$\exists ! \langle f_1, f_2, \dots, f_{\mathcal{N}} \rangle : A \to C_1 \times C_2 \times \dots \times C_{\mathcal{N}}$ s.t.

$$\begin{aligned} f_1 &= \pi_1 \circ \langle f_1, f_2, \dots f_{\mathcal{N}} \rangle \\ f_2 &= \pi_2 \circ \langle f_1, f_2, \dots f_{\mathcal{N}} \rangle \\ &\vdots \\ f_{\mathcal{N}} &= \pi_{\mathcal{N}} \circ \langle f_1, f_2, \dots f_{\mathcal{N}} \rangle \end{aligned}$$

5.1.1. *Example: Set always has products.* $\forall$ sets $X, Y \in \text{Obj(Set)}$, $\exists$ product $X \times Y \in \text{Obj(Set)}$.

$$\text{Let } A \in \text{Obj(Set)}, \ \begin{aligned} f_1 &: A \to X \\ f_2 &: A \to Y \end{aligned} \qquad \text{Define } \begin{aligned} &\langle f_1, f_2 \rangle : A \to X \times Y \\ &\langle f_1, f_2 \rangle(a) = (f_1(a), f_2(a)) \end{aligned}$$

$$\text{Then } \begin{aligned} \pi_1 \circ \langle f_1, f_2 \rangle(a) &= f_1(a) \\ \pi_2 \circ \langle f_1, f_2 \rangle(a) &= f_2(a) \end{aligned} \qquad \Longrightarrow \begin{aligned} \pi_1 \circ \langle f_1, f_2 \rangle &= f_1 \\ \pi_2 \circ \langle f_1, f_2 \rangle &= f_2 \end{aligned}$$

$$\text{Suppose } f' : A \to X \times Y \text{ s.t. } \begin{aligned} \pi_1 \circ f' &= f_1 \\ \pi_2 \circ f' &= f_2 \end{aligned}$$

Write $f'(a) = (x, y)$

$$\begin{aligned} f_1(a) &= \pi_1 \circ f'(a) = \pi_1(x, y) = x \\ f_2(a) &= \pi_2 \circ f'(a) = \pi_2(x, y) = y \end{aligned} \qquad \Longrightarrow f'(a) = (f_1(a), f_2(a)) = \langle f_1, f_2 \rangle(a)$$

$\langle f_1, f_2 \rangle$ unique.

Proposition 4. *If product* $(A_1 \times \dots \times A_{\mathcal{N}} \xrightarrow{\pi_i} A_i)_{i \in I}$, *if* $\exists \, i_0 \in I$ *s.t.* $Hom(A_{i_0}, A_i) \neq \emptyset$, $\forall i \in I$, *then* π_{i_0} retraction

Proof. $\forall i \in I$, choose $f_i \in \text{Hom}(A_{i_0}, A_i)$ with $f_{i_0} = 1_{A_{i_0}}$.
Then $\langle f_i \rangle : A_{i_0} \to A_1 \times \dots \times A_{\mathcal{N}}$ is a morphism s.t.

$$\pi_{i_0} \circ \langle f_i \rangle = f_{i_0} = 1_{A_{i_0}}$$

□

Adámek, Herrlich, and Strecker (2004) [5] and their notation) calls a **sink** what Leinster (2014) [3] calls a **cocone**.

Definition 21. sink $((f_i)_{i \in I}, A) \equiv (f_i, A)_I \equiv (A_i \xrightarrow{f_i} A)_I$, object A, family of morphisms $f_i : A_i \to A$

For the *coproduct*, consider this enlightening comparision:

$$\begin{array}{cc} \text{product } (\prod_{i \in I} A_i, \pi_j)_{j \in I} & \text{coproduct } (\mu_j, \coprod_{i \in I} A_i)_{j \in I} \\ \text{projection } \pi_j : \prod_{i \in I} A_i \to A_j & \text{injection } \mu_j : A_j \to \coprod_{i \in I} A_i \end{array}$$

$$\begin{tikzcd} & A \arrow[dl, "f_j"'] \arrow[d, "\langle f_i \rangle"] \\ A_j & \prod_{i \in I} A_i \arrow[l, "\pi_j"] \end{tikzcd} \qquad \begin{tikzcd} & A \\ A_j \arrow[ur, "f_j"] \arrow[r, "\mu_j"] & \coprod_{i \in I} A_i \arrow[u, "{[f_i]}"'] \end{tikzcd}$$

$$\begin{array}{cc} C \xrightarrow{\langle f, g \rangle} A \times B & C \xleftarrow{[f,g]} A + B \\ \prod_{i \in I} f_i \,, \text{ or if } i = \{1, 2\}, \ f \times g & \coprod_{i \in I} f_i \,, \text{ or if } i = \{1, 2\}, \ f + g \end{array}$$

5.1.2. *Examples (of coproducts).*

- if $(A_i)_I$ pairwise-disjoint family of sets, then $(\mu_j, \bigcup_{i \in I} A_i)_{j \in I}$ is coproduct in Set.
  If $(A_i)_I$ arbitrary set-indexed family of sets, then it can be "made disjoint" by pairing each A_i with index i, i.e. by working with $A_i \times \{i\}$ rather than A_i.
  So $\bigcup_{i \in I} (A_i \times \{i\})$ disjoint. Consider

$$\begin{aligned} \mu_j : A_j &\to \bigcup_{i \in I} A_i \times \{i\} \\ \mu_i(a) &= (a, j) \end{aligned}$$

  $(\mu_j, \bigcup_{i \in I} A_i \times \{i\})_{j \in I}$ is a coproduct in Set.

Indeed, given $f_j : A_j \to A$,

$f_j(a) \in A$

$$[f_i] : \coprod_{i \in I} A_i \times \{i\} \to A$$

$$[f_i] \circ \mu_j = f_j$$

where

$$f_j(a) = [f_i] \circ \mu_j(a) = [f_i](a, j) = f_j(a)$$

- Top coproducts are "topological sums"; they're "concrete" coproducts (Adámek, Herrlich, and Strecker (2004) [5])
- Vec (nonconcrete) coproducts called *direct sums*
  direct sum $\bigoplus_{i \in I} A_i$ of vector spaces A_i is subspace of direct product $\prod_{i \in I} A_i$
  consisting of all elements $(a_i)_{i \in I}$ with finite carrier (i.e. $\{i \in I | a_i \neq 0\}$ is finite),
  injections

$$\mu_j : A_j \to \bigoplus_{i \in I} A_i$$

$$\mu_j(a) = (a_i)_{i \in I} \text{ with } a_i = \begin{cases} a & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- Grp has nonconcrete coproducts, "free products"

Let *diagram* (functor) $D : \mathbf{I} \to \mathbf{A}$. (diagram is, technically, exactly the same as a functor (Adámek, Herrlich, and Strecker (2004) [5])).

Definition 22. **A**-source $(A \xrightarrow{f_i} D_i)_{i \in \text{Obj}\mathbf{I}}$ **natural** for D if $\forall\, i \xrightarrow{d} j$, $d \in \text{Mor}\mathbf{I}$, then

$$\begin{tikzcd} A \arrow[d, "D_i"'] \arrow[dr, "f_j"] & \\ D_i \arrow[r, "Dd"'] & D_j \end{tikzcd}$$

Definition 23. **limit** of D is a natural source $(L \xrightarrow{l_i} D_i)_{i \in \text{Obj}\mathbf{I}}$ for D with
(universal) property that $\forall$ natural source $(A \xrightarrow{f_i} D_i)_{i \in \text{Obj}\mathbf{I}}$ for D uniquely factors through it, i.e.
$\forall$ natural source $(A \xrightarrow{f_i} D_i)_{i \in \text{Obj}\mathbf{I}}$, $\exists!$ morphism $f : A \to L$ s.t. $f_i = l_i \circ f \qquad \forall\, i \in \text{Obj}(\mathbf{I})$.

It may pay to read and compare with other books because I didn't understand limits the first time reading through Adámek, Herrlich, and Strecker (2004) [5]. So compare with Leinster (2014) [3].
cone from Leinster (2014) [3] is the same as *source* in Adámek, Herrlich, and Strecker (2004) [5]:

Definition 24. **cone** on D (or natural source for D), $A \in \text{Obj}\mathbf{A}$ (vertex of the cone) (i.e. **A**-source), $(A \xrightarrow{A_I} D(I))_{I \in \text{Obj}\mathbf{I}}$ s.t. if $\forall\, I \xrightarrow{u} J$, $u \in \text{Mor}\mathbf{I}$, then

$$\begin{tikzcd} A \arrow[d, "f_I"'] \arrow[dr, "f_J"] & \\ D(I) \arrow[r, "Du"'] & D(J) \end{tikzcd}$$

20160502 EY: I still wasn't clear about the meaning, and so from wikipedia, "Limit (category theory)",
diagram of type J in $\mathbf{C}$ is functor F

$$F : J \to C \text{ or } J \xrightarrow{f} C$$

$J, C \in \text{Cat}$.
One's mostly interested in small or even finite.
Cone to F is $N \in \text{Obj}\mathbf{C}$ and family $\psi_X : N \to F(X)$, $\psi_X \in \text{Mor}\mathbf{C}$, $X \in \text{Obj}J$ s.t.
$\forall\, f : X \to Y$, $f \in \text{Mor}J$

$$F(f) \circ \psi_X = \psi_Y$$

i.e.

$$\begin{tikzcd} N \arrow[d, "\psi_X"'] \arrow[dr, "\psi_Y"] & \\ F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

Comparing with the previous definition for *cone*, then in this notation, $N \equiv$ vertex of cone, and

$$(N \xrightarrow{\psi_X} F(X))_{X \in \text{Obj}J}$$

s.t. $\forall\, X \xrightarrow{f} Y$, $f \in \text{Hom}_J(X;Y)$.
In other words,

$$J \xrightarrow{F} C$$

$$\begin{tikzcd} & & & N \arrow[d, "\psi_X"] \arrow[dr, "\psi_Y"] & \\ X \arrow[r, "f"] \arrow[rrr, bend right, "F"'] & Y \arrow[rrr, bend right, "F"'] & & F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

Definition 25. **limit** of D is natural source (or cone) $(L \xrightarrow{\pi_I} D(I))_{I \in \text{Obj}\mathbf{I}}$ s.t. $\forall$ natural source (or cone) on D, $(A \xrightarrow{f_I} D(I))_{I \in \text{Obj}\mathbf{I}}$,
$\exists!$ morphism $f : A \to L$ s.t. $f_I = \pi_I \circ f \qquad \forall\, I \in \text{Obj}\mathbf{I}$. π_I projections of limit.

$$\begin{tikzcd} L \arrow[dr, "\pi_I"'] & A \arrow[l, "f"'] \arrow[d, "f_I"] \\ & D(I) \end{tikzcd}$$

i.e. this commutes:

EY : 20160502 An *important* example of a limit is the *product* - the product $C_1 \times C_2$ or $\prod_i C_i$ is the vertex of this cone and π_1, π_2 or $(\pi_i)_i$ are the family of morphisms.

Definition 26. Let diagram (functor) $D : \mathbf{I} \to \mathbf{A}$.
Consider functor $D^{\text{op}} : \mathbf{I}^{\text{op}} \to \mathbf{A}^{\text{op}}$.
natural sink $(D(I) \xrightarrow{f_I} A)_{I \in \text{Obj}\mathbf{I}}$ for D s.t. $\forall\, I \xrightarrow{d} J$, $d \in \text{Mor}\mathbf{I}$, then

$$
\begin{tikzcd}
A & \\
D(I) \arrow[u, "f_I"] \arrow[r, "Dd"'] & D(J) \arrow[ul, "f_J"']
\end{tikzcd}
$$

Natural sink of Adámek, Herrlich, and Strecker (2004) [5] is the same as the "cocone" of Leinster (2014) [3].

Definition 27. **colimit** of D is natural sink $(D(I) \xrightarrow{c_I} K)_{I \in \text{Obj}\mathbf{I}}$ for D with (universal) property that

$\forall$ natural sink for D, $(D(I) \xrightarrow{f_I} A)_{I \in \text{Obj}\mathbf{I}}$, $\exists !$ morphism $f : K \to A$ s.t. $f \circ c_I = f_I \qquad \forall I \in \text{Obj}\mathbf{I}$, i.e.

$$
\begin{tikzcd}
K \arrow[r, "f"] & A \\
& D(I) \arrow[ul, "c_I"] \arrow[u, "f_I"']
\end{tikzcd}
$$

5.2. Pullback.

Definition 28. For some category $\mathbf{A}$, and for

$$
\begin{tikzcd}
& Y \arrow[d, "t"] \\
X \arrow[r, "s"'] & Z
\end{tikzcd}
$$

$X, Y, Z \in \text{Obj}\mathbf{A}$.

$$
\begin{aligned}
s &: X \to Z \,; \qquad s, t \in \text{Mor}\mathbf{A} \\
t &: Y \to Z
\end{aligned}
$$

Then the **pullback** or "pullback square" consists of $P \in \text{Obj}\mathbf{A}$, $\begin{array}{l} \pi_1 : P \to X \text{ s.t.} \\ \pi_2 : P \to Y \end{array}$

$$
\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}
$$

commutes and s.t. $\forall$ commutative square in $\mathbf{A}$

$$
\begin{tikzcd}
A \arrow[r, "f_2"] \arrow[d, "f_1"] & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}
$$

then $\exists !\, f : A \to P$ s.t.

$$
\begin{tikzcd}
A \arrow[dr, "f"] \arrow[drr, "f_2"] \arrow[ddr, "f_1"'] & & \\
& P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "t"] \\
& X \arrow[r, "s"] & Z
\end{tikzcd}
$$

6. Adjoint

From the section on "Objects and Morphisms with Respect to a Factor" of Adámek, Herrlich, and Strecker (2004) [5],

Definition 29. Let functor $G : \mathbf{A} \to \mathbf{B}$, $B \in \text{Obj}\mathbf{B}$.
G-structured arrow with domain B is pair (f, A), $A \in \text{Obj}\mathbf{A}$, $f : B \to GA$, $f \in \text{Mor}\mathbf{B}$.
G-structured arrow (f, A) with domain B is called

(1) **generating** provided $\forall$ pair of $\mathbf{A}$-morphism $\begin{array}{l} r : A \to A', \\ s : A \to A' \end{array} \qquad Gr \circ f = Gs \circ f$ implies $r = s$
(2) **extremally generating** provided it's generating and
if $A' \xrightarrow{m} A$ is an $\mathbf{A}$-monomorphism, (g, A') G-structured arrow, s.t. $f = G(m) \circ g$,
then m is $\mathbf{A}$-isomorphism
(3) **G-universal for** B if $\forall G$-structured arrow (f', A') with domain B,
$\exists !$ $\mathbf{A}$-morphism $A \xrightarrow{\widehat{f}} A'$, $f' = G(\widehat{f}) \circ f$ i.e. s.t.

$$
\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "f'"'] & GA \arrow[d, "G\widehat{f}"] \\
& GA'
\end{tikzcd}
$$

commutes.

If you're reading Turi [4], then Turi calls G-universal for B, "**universal arrow**" from an object A of $\mathbf{C}$: inspection of his diagram immediately confirms that they're talking about the exact same thing (I know, it seems as different mathematicians have different names and notation for the exact same thing):

$$U : \mathbf{D} \to \mathbf{C}$$

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "h"'] & U(F_A) \arrow[d, "Uh^\sharp"] & F_A \arrow[d, dashed, "h^\sharp"] \\
& GA' & Y
\end{tikzcd}$$

for $F_A \in \mathrm{Obj}\mathbf{D}$

Definition 30. Let functor $G : \mathbf{A} \to \mathbf{B}$; let $B \in \mathrm{Obj}\mathbf{B}$.

(1) G**-costructured arrow** with codomain B is pair (A, f), $A \in \mathrm{Obj}\mathbf{A}$, $GA \xrightarrow{f} B$, $f \in \mathrm{Mor}\mathbf{B}$.
(2) G-costructured arrow (A, f) with codomain B is called G**-couniversal** for B if $\forall\, G$-costructured arrow (A', f') with codomain B,
$\exists !\, A' \xrightarrow{\widehat{f}} A$, $\widehat{f} \in \mathrm{Mor}\mathbf{A}$, s.t. $f' = f \circ G(\widehat{f})$ i.e.

$$\begin{tikzcd}
B & GA \arrow[l, "f"] \\
& GA' \arrow[ul, "f'"] \arrow[u, "G(\widehat{f})"']
\end{tikzcd}$$

Definition 31 (adjoint)**.** functor $G : \mathbf{A} \to \mathbf{B}$ **adjoint** if $\forall\, B \in \mathrm{Obj}\mathbf{B}$, $\exists$ G-universal arrow with domain B, i.e.
$\forall\, B \in \mathrm{Obj}\mathbf{B}$, $\exists\, (f, A)$ with domain B s.t. $\forall\, (f', A')$ with domain B, $\exists !\, \widehat{f}' \in \mathrm{Mor}\mathbf{A}$ s.t.

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "f'"'] & GA \arrow[d, "G\widehat{f}'"] \\
& GA'
\end{tikzcd}$$

Also called *right* adjoint.

Definition 32 (co-adjoint)**.** functor $G : \mathbf{A} \to \mathbf{B}$ **co-adjoint** if $\forall\, B \in \mathrm{Obj}\mathbf{B}$, $\exists$ G-co-universal arrow with codomain B, i.e.
$\forall\, B \in \mathrm{Obj}\mathbf{B}$, $\exists\, (A, f)$ with codomain B s.t. $\forall\, (A', f')$ with codomain B, $\exists !\, \widehat{f}' \in \mathrm{Mor}\mathbf{A}$ s.t.

$$\begin{tikzcd}
B & GA \arrow[l, "f"] \\
& GA' \arrow[ul, "f'"] \arrow[u, "G(\widehat{f}')"']
\end{tikzcd}$$

Also called *left* adjoint.

In section 19 Adjoint situations of Adámek, Herrlich, and Strecker (2004) [5], their Theorem 19.1 is the same as Exercise 3.1 and Theorem 3.1 on pp. 11 of Turi [4], which Turi says is “Important!”

Theorem 1. *Let adjoint functor $G : \mathbf{A} \to \mathbf{B}$, so (by def. of adjoint), $\forall\, B \in \mathrm{Obj}\mathbf{B}$, let $\eta_B : B \to GA_B$ be the universal arrow. Then $\exists !$ functor $F : \mathbf{B} \to \mathbf{A}$ s.t. $F(B) = A_B$. $\forall\, B \in \mathrm{Obj}\mathbf{B}$, and $1_{\mathbf{B}} \xrightarrow{\eta = (\eta_B)} G \circ F$ natural transformation.*

Moreover, $\exists !$ natural transformation $F \circ G \xrightarrow{\epsilon} 1_{\mathbf{A}}$ s.t.

(1) $$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G = G \xrightarrow{1_G} G$$

(2) $$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F = F \xrightarrow{1_F} F$$

Proof. Given an adjoint functor $G : \mathbf{A} \to \mathbf{B}$. By definition, this means that
$\forall\, B \in \mathrm{Obj}\mathbf{B}$, $\exists$ G-universal arrow with domain B, (f, A), s.t. $\forall\, (f', A')$ (i.e. every other G-structured arrow (f', A')),

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "f'"'] & GA \arrow[d, "G\widehat{f}'"] & A \arrow[d, dashed, "\widehat{f}'"] \\
& GA' & A'
\end{tikzcd}$$

We want to define a function F:

$$F : \mathrm{Obj}\mathbf{B} \to \mathrm{Obj}\mathbf{A}$$
$$F(B) := A_B$$

and make a functor out of it. We know it exists from the definition of an adjoint, so that $\exists$ *a* G-universal arrow (f, A_B), $\forall\, B$.
Is it well defined?

Suppose another $F' : \mathrm{Obj}\mathbf{B} \to \mathrm{Obj}\mathbf{A}$.
$F'(B) = A'$
Using universal arrow definition, then again we have

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "GF'"'] & GA \arrow[d, "G\widehat{f}'"] & A \arrow[d, dashed, "\widehat{f}'"] & B \arrow[l, "F"] \arrow[dl, "F'"] \\
& GA' & A' &
\end{tikzcd}$$

$$\implies F'(B) = A' = \widehat{f}'(A) = \widehat{f}' \circ F(B) \implies F' = \widehat{f}' \circ F$$

So F unique up to a unique morphism, due to universal arrow definition (or property).
Consider how F can act on morphisms.
Take $b \in \mathrm{Mor}\mathbf{B}$. The commutative diagram

$$\begin{tikzcd}
B \arrow[r, "F"] \arrow[d, "b"] & F(B) = A_B \arrow[d, "F(b)"] \\
B' \arrow[r, "F"] & F(B') = A_{B'}
\end{tikzcd}$$

tells us immediately what $F(b) \in \mathrm{Mor}\mathbf{A}$ is (composition $F \circ b$).
A functor has to preserve identity and compositions. The following commutative diagrams show this:

$$\begin{tikzcd}
B \arrow[r, "F"] \arrow[d, "1_{\mathbf{B}}"'] & F(B) = A_B \arrow[d, "F \circ 1_{\mathbf{B}} \equiv 1_{F\mathbf{B}}"] \\
B \arrow[r, "F"] & F(B) = A_B
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "F"] \arrow[d, "b"] \arrow[dd, bend right=60, "b' \circ b"'] & F(B) = A_B \arrow[d, "F(b)"] \arrow[dd, bend left=60, "F(b') \circ F(b)"] \\
B' \arrow[r, "F"] \arrow[d, "b'"'] & F(B') = A_{B'} \arrow[d, "F(b')"] \\
B'' \arrow[r, "F"] & F(B'') = A_{B''}
\end{tikzcd}$$

Thus,

$$\boxed{F : \mathbf{B} \to \mathbf{A} \text{ is a unique functor and it exists, and is defined s.t. } F(B) = A_B\text{, any time you have an adjoint functor } G : \mathbf{A} \to \mathbf{B}.}$$

Given G-universal arrow $\eta_B : B \to G(A_B)$, which exists by adjoint functor def. of G, $\forall\, B \in \text{Obj}\mathbf{B}$. Then

$$\begin{tikzcd}
B \arrow[r, "\eta_B"] & GA_B
\end{tikzcd}$$

$$\begin{tikzcd}
B' \arrow[r, "\eta_{B'}"] & GA_{B'}
\end{tikzcd}$$

So $\forall\, f \in \text{Mor}\mathbf{B}$, $f : B \to B'$,

$$\begin{tikzcd}
B \arrow[r, "\eta_B"] \arrow[d, "f"'] & GA_B \\
B' \arrow[r, "\eta_{B'}"] & GA_{B'}
\end{tikzcd}$$

Use unique functor F, $\begin{matrix} F(B) = A_B \\ F(B') = A_{B'} \end{matrix}$,

$$\begin{tikzcd}
B \arrow[r, "\eta_B"] \arrow[d, "f"'] & GA_B = GF(B) \arrow[d, "GF(f)"] \\
B' \arrow[r, "\eta_{B'}"] & GA_{B'} = GF(B')
\end{tikzcd}$$

where $GF(f) : GF(B) \to GF(B')$, by functor property of G, F, so this holds $\forall\, f \in \text{Mor}\mathbf{B}$.
Thus, $\eta : 1_{\mathbf{B}} \to G \circ F$ is a natural transformation for $1_{\mathbf{B}}, G \circ F : \mathbf{B} \to \mathbf{B}$ (endofunctors, functors that map a category to itself), s.t.
$\forall\, B \in \text{Obj}\mathbf{B}$, $\eta_B : 1_{\mathbf{B}} B = B \to GFB, \quad \eta_B \in \text{Mor}\mathbf{B}$.
Consider $B = GA$, and corresponding universal arrow $\eta_B = \eta_{GA}$, through the unique functor F so that $F(GA) = A_{GA}$.

$$GA \xrightarrow{\eta_{GA}} GA_{GA} = GFGA$$

Consider morphism $1_{GA} : GA \to GA$, then

$$\begin{tikzcd}
GA \arrow[r, "\eta_{GA}"] \arrow[dr, "1_{GA}"'] & GFGA \arrow[d, "G\epsilon_A"] & F(GA) = A_{GA} \arrow[d, dashed, "\epsilon_A"] \\
& GA & A
\end{tikzcd}$$

by definition of an adjoint functor.
Now

$$\begin{gathered}
G(f \circ \epsilon_A) \circ \eta_{GA} = Gf \circ G\epsilon_A \circ \eta_{GA} = Gf = G\epsilon_{A'} \circ \eta_{GA'} \circ Gf = G\epsilon_{A'} \circ GFGf \circ \eta_{GA} = G(\epsilon_{A'} \circ FGf) \circ \eta_{GA} \\
\Longrightarrow f \circ \epsilon_A = \epsilon_{A'} \circ FGf
\end{gathered} \tag{3}$$

since for the first equality in Eq. 3, associativity of functor G was used, i.e.

$$G(f \circ \epsilon_A) = Gf \circ G\epsilon_A$$

and for the second equality, universal arrow definition was used, i.e.

$$\begin{tikzcd}
GA \arrow[r, "\eta_{GA}"] \arrow[dr, "1_{GA}"'] & GFGA \arrow[d, "G\epsilon_A"] \\
& GA
\end{tikzcd}$$

or i.e. $G\epsilon_A \circ \eta_{GA} = 1_{GA}$, and for the third equality, universal arrow definition was used again, i.e.

$$\begin{tikzcd}
GA' \arrow[r, "\eta_{GA'}"] \arrow[dr, "1_{GA'}"'] & GFGA' \arrow[d, "G\epsilon_{A'}"] \\
& GA'
\end{tikzcd}$$

or i.e. $G\epsilon_{A'} \circ \eta_{GA'} = 1_{GA'}$, and for the fourth equality, the natural transformation definition for η and its universal arrow definition was used together, i.e.

$$\begin{tikzcd}
A \arrow[r, "G"] \arrow[d, "f"'] & GA \arrow[r, "\eta_{GA}"] \arrow[d, "Gf"] & GFGA \arrow[d, "GFGf"] \\
A' \arrow[r, "G"] & GA' \arrow[r, "\eta_{GA'}"] \arrow[dr, "1_{GA'}"'] & GFGA' \arrow[d, "G\epsilon_{A'}"] \\
& & GA'
\end{tikzcd}$$

and for the fifth equality, associativity of functor G was used again, i.e. $G\epsilon_{A'} \circ GFGf = G(\epsilon_{A'} \circ FGf)$.
Thus, ϵ is a natural transformation, $\epsilon : FG \to 1_{\mathbf{A}}$, for

$$\begin{tikzcd}
FGA \arrow[r, "\epsilon_A"] \arrow[d, "FGA'"] & A \arrow[d, "f"] \\
FGA' \arrow[r, "\epsilon_{A'}"] & A'
\end{tikzcd}$$

commutes.

□

EY: 20160502 Wikipedia "Adjoint functors" says "any limit functor is right adjoint to a corresponding diagonal functor." I found this pdf, https://www.andrew.cmu.edu/course/80-413-713/notes/chap09.pdf, to be useful.

7. Monad

Definition 33 (monad). **monad** on category $\mathbf{X}$ is triple $\mathbf{T} = (T, \eta, \mu)$, consisting of functor $T : \mathbf{X} \to \mathbf{X}$ (an endofunctor, maps a category to itself), and
natural transformations

$$\eta : 1_{\mathbf{X}} \to T \text{ and}$$
$$\mu : T \circ T \equiv T^2 \to T \text{ s.t.}$$

$$\begin{tikzcd}
T \circ T \circ T \equiv T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] & T \circ T \equiv T^2 \arrow[d, "\mu"] \\
T \circ T \equiv T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "1"'] & T \circ T \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "1"] \\
& T &
\end{tikzcd}$$

commute.

8. Applications

8.1. **Databases.** Let category $\text{db} = (\text{Ob}_{\text{db}}, \text{hom}_{\text{db}}, 1, \circ)$ be a **database schema.**
Ob_{db} is *a* collection of tables τ, $\tau \in \text{Ob}_{\text{db}}$
$c \in \text{hom}_{\text{db}}$ where c is a column (i.e. attribute)
primary key column $c!$ is a primary morphism (or arrow)
Declaring constraints is declaring a composition law, i.e. for tables $\rho, \sigma, \tau \in \text{Ob}_{\text{db}}$,

$$\begin{tikzcd}
\rho \arrow[r, "c_1"] \arrow[rr, bend right, "c_2 \circ c_1"'] & \sigma \arrow[r, "c_2"] & \tau
\end{tikzcd}$$

EY: 20150716 I think it should be emphasized that Ob_{db} is *a* collection of tables associated with this particular database db, not *the* collection of *all* possible tables.
Let **data functor** be a functor $F : \text{db} \to \text{Set}$.
So for tables $\rho, \sigma, \tau \in \text{Ob}_{\text{db}}$, columns $c, c_1, c_2 \in \text{hom}_{\text{db}}(\sigma, \tau)$

$$\begin{tikzcd}
\sigma \arrow[r, "c"] \arrow[d, "F"] & \tau \arrow[d, "F"] \\
F(\sigma) \arrow[r, "F(c)"] & F(\tau)
\end{tikzcd}
\qquad
\begin{tikzcd}
\rho \arrow[r, "c_1"] \arrow[rr, bend left, "c_2 \circ c_1"] \arrow[d, "F"] & \sigma \arrow[r, "c_2"] \arrow[d, "F"] & \tau \arrow[d, "F"] \\
F(\rho) \arrow[r, "F(c_1)"] \arrow[rr, bend right, "F(c_2 \circ c_1) = F(c_2) \circ F(c_1)"'] & F(\sigma) \arrow[r, "F(c_2)"] & F(\tau)
\end{tikzcd}$$

Now note that $F(\rho), F(\sigma), F(\tau) \in \text{Ob}_{\text{Set}}$ means that $F(\rho), F(\sigma), F(\tau)$ are sets. They fill the tables with its data set; the data set of rows.

9. Decorators

Lutz (2009) [6]

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[6] Mark Lutz. **Learning Python**, 4th Edition. O'Reilly Media. 2009.
EY: There's a 5th edition, 2013, but I don't have a copy of the 5th edition; I only have the 4th.

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