COLD NEUTRONS AND TOPOLOGICAL KNOTS

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Contents

Abstract. We investigate exact solutions, via topological gauge theory and knot polynomials, or knot homologies, applied to ultracold neutrons in beta decay, in particular, via the θ-term in the Lagrangian describing neutrons.

Part 1. (Weekly) reports

0.1. **20160422 Things to do.** Clarify Manifold setup $\partial M \hookrightarrow M$; explore various manifold setups; pdf or equation of motion out of $\mathcal{L}$ and Euler-Lagrange equation and compare those equations to instanton equations of Gaiotto and Witten (2011) [3]; understand Virasoro algebra and conformal blocks for the quantum "states" that we can act upon; read more: Gaiotto and Witten (2011) [3]; Gukov (2007) [5], and of course classic Witten (1988) [2]

Part 2. Introduction

1. Geometry; Geometric setup; Manifold setup

Following Eq. 1.28 on pp.5 of Subsection 1.3 The θ-term of Hickerson (2013) [4], for a 3-cylinder ∂M, he has the following setup:

$$\partial M = \mathbb{D}^3(t^+) \bigcup \mathbb{D}^3(t^-) \bigcup \mathbb{S}^2 \times \mathbb{R}$$

Let's count dimensions.

$$\text{dim}\partial M = \text{dim}\mathbb{D}^3(t^+) + \text{dim}\mathbb{S}^2 + 1$$

Now $\text{dim}\mathbb{S}^2 = 2$. So do we have more dimensions than alloted?

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Let's account for the various setups for a 4-dimensional topological gauge theory that involves knot polynomials (i.e. knot homologies). It appears that Gaiotto and Witten (2011) [3] likes to include Riemann surfaces in ∂M, so there setup could be

$$\partial M = \mathbb{C} \times \mathbb{R}(\text{or } \mathbb{C} \times S^1(???))$$

where $\mathbb{C} \equiv$ Riemann surface (i.e. $f : \mathbb{C} \to \mathbb{C}$, f holomorphic, i.e. $\frac{\partial f}{\partial \overline{z}} = 0$). $\frac{\partial \overline{f}}{\partial z} = 0$

knot K is an embedding $f : S^1 \to S^3$

$$S^1 \xrightarrow{f} S^3 \xrightarrow{\cong} \partial M \xhookrightarrow{i} M$$

Let us follow Hickerson (2013) [4] to understand the number of approaches and models for ultracold neutrons [4].

Consider the term

$$\mathcal{L}_\theta = \frac{\theta g^2}{8\pi^2}\text{tr}(F \wedge F) = \frac{\theta g^2}{8\pi^2} d\text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

We've seen how

$$\text{tr}(F_A \wedge F_A) \equiv \text{tr}(F \wedge F) = d\text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{1}$$

is true, without regards to a *metric* g (or i.e. *metric bundle* g over M). In pp. 5 Subsection 1.3 The θ-term of Hickerson (2013) [4], the dual to F was utilized. Let's avoid utilizing "electric-magnetic" dual, or g, or Hodge dual terms in order to think purely "topologically."

Part 3. Preliminaries; (review of) Elementary concepts

2. Curvature

Consider a principal-G bundle with Lie group G, $P \xrightarrow{\pi} M$. Note that an associated bundle, a vector bundle, can be constructed from principal G-bundle P, through representation $\rho : G \to Gl(n; \mathbb{K})$ (cf. 10.9 Associated vector bundles of Taubes (2011) [1]), in that

$$\begin{array}{l} P \\ \downarrow \pi \\ M \xrightarrow{\rho : G \to \text{Gl}(n;\mathbb{K})} P \times_\rho \mathbb{K}^n \equiv P \times \mathbb{K}^n / (p, v) \sim (pg^{-1}, \rho(g)v) \quad \forall \, g \in G \end{array}$$

for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathbb{K}^n$ being a vector space of dimension n over field $\mathbb{K}$.

Recall the exterior covariant derivative D s.t.

$$\begin{aligned} &D : \Omega^p(M; E) \to \Omega^{p+1}(M; E) \\ &D(\theta \otimes s) = d\theta \otimes s + (-1)^p \theta \wedge \nabla s = d\theta \otimes s + \nabla s \wedge \theta \end{aligned}$$

with $E \xrightarrow{\pi} M$ being a vector bundle (from which one can construct the principal G bundle, if so desired).

Proposition 1. *For exterior covariant derivative $D : \Omega^p(M;E) \to \Omega^{p+1}(M;E)$, $\forall\, \eta \in \Omega^p(M;E)$,*

$$D^2\eta \equiv D \circ D\eta = F \wedge \eta$$

where $F \in \Omega^p(M; End(E))$, and F unique

Proof. $\forall\, \eta \in \Omega^p(M;E)$, of the form $\eta = \theta \otimes s$, where $\theta \in \Omega^p(M)$, $s \in \Gamma(E)$,

$$D\eta = d\theta \otimes s + (-1)^p \theta \wedge \nabla s = d\theta \otimes s + (-1)^p\theta \wedge (ds + \omega^k{}_i s^i e_k) = d\theta \otimes s + ds \wedge \theta + \omega^k{}_i s^i \wedge \theta \otimes e_k =$$
$$= (s^k d\theta + ds^k \wedge \theta + \omega^k{}_i s^i \wedge \theta) \otimes e_k$$

$$\begin{aligned} D \circ D\eta \equiv DD\eta = (ds^k \wedge d\theta + (-1) ds^k \wedge d\theta + ds^i \wedge \omega^k{}_i \wedge \theta + s^i d\omega^k{}_i \wedge \theta + (-1)\omega^k{}_i s^i \wedge d\theta) \otimes e_k + \\ + (-1)^{p+1}(s^k d\theta + ds^k \wedge \theta + \omega^k{}_i s^i \wedge \theta) \otimes \wedge \omega^l{}_k e_l = \\ = (ds^i \wedge \omega^l{}_i \wedge \theta + s^i d\omega^l{}_i \wedge \theta + (-1)\omega^l{}_i s^i \wedge d\theta) \otimes e_l + \\ + (s^k \omega^l{}_k \wedge d\theta + \omega^l{}_k \wedge ds^k \wedge \theta + \omega^l{}_k \wedge \omega^k{}_i s^i \wedge \theta) e_l = \\ = (d\omega^l{}_i + \omega^l{}_k \wedge \omega^k{}_i) s^i \wedge \theta e_l \end{aligned}$$

If you're following at home (i.e. independent study), one only needs to be careful with factors of (-1) when "commuting through" the wedge product $\wedge$.

I (still) find it a near miracle that terms cancel such that F takes this form (with, simply a change of notation, $\omega \equiv A$):

$$F = dA + A \wedge A \in \Omega^p(M; \mathrm{End}E)$$

By Lemma 11.1 of Sec. 11.2 the space of covariant derivatives of Taubes (2011) [1], this F is *unique*.

Thus

$$D^2\eta = F \wedge \eta$$

for, notice that for, locally (in components)

$$A = A^k{}_{ij} dx^j \otimes (e_k \otimes e^i)$$

$$\begin{aligned} F = dA + A \wedge A = \frac{\partial A^k{}_{lj}}{\partial x^i} dx^i \wedge dx^j \otimes (e_k \otimes e^l) + A^k{}_{mi} A^m{}_{lj} dx^i \wedge dx^j \otimes (e_k \otimes e^l) = \\ = \left(\frac{\partial A^k{}_{lj}}{\partial x^i} + A^k{}_{mi} A^m{}_{lj} \right) dx^i \wedge dx^j \otimes (e_k \otimes e^l) \end{aligned}$$

and so

$$F \wedge \eta = \left(\frac{\partial A^k{}_{lj}}{\partial x^i} + A^k{}_{mi} A^m{}_{lj} \right) dx^i \wedge dx^j \wedge \theta \otimes e_k s^l$$

□

2.0.1. *Alternative form of curvature F in terms of commutators.* cf. Subsection 12.6 Curvature and commutators of Taubes (2011) [1].

Consider $\forall\, U, V \in \mathfrak{X}(M)$, $X \in \Gamma(E)$,

$$\nabla_U X = U^j \left(\frac{\partial X}{\partial x^j} + A^k{}_{ij} X^i \right) = U^j \left(\frac{\partial}{\partial x^j} + A_j \right) X \in \Gamma(E)$$

and so clearly

$$\nabla_U \in \Gamma(\mathrm{End}(E))$$

Also recall the commutator for vector fields, in component form (locally):

$$[U,V] = \left(U^i \frac{\partial}{\partial x^i} V^j - V^i \frac{\partial}{\partial x^i} U^j \right) \frac{\partial}{\partial x^j} \in \mathfrak{X}(M)$$

and so

$$\nabla_{[U,V]} = \left(U^i \frac{\partial}{\partial x^i} V^j - V^i \frac{\partial}{\partial x^i} U^j \right) \left(\frac{\partial}{\partial x^j} + A_j \right)$$

Consider that

$$\begin{gathered} \nabla_U \nabla_V = \\ = U^i \left[\left(\frac{\partial}{\partial x^i} + A_i \right) V^j \left(\frac{\partial}{\partial x^j} + A_j \right) \right] = \\ = U^i \left[\frac{\partial V^j}{\partial x^i} \left(\frac{\partial}{\partial x^j} + A_j \right) + V^j \left(\frac{\partial^2}{\partial x^i \partial x^j} + \frac{\partial A_j}{\partial x^i} + A_j \frac{\partial}{\partial x^i} \right) + A_i V^j \frac{\partial}{\partial x^j} + A_i V^j A_j \right] \end{gathered}$$

Then by canceling out matching terms,

$$\begin{aligned} [\nabla_U, \nabla_V] - \nabla_{[U,V]} = U^i V^j \frac{\partial A_j}{\partial x^i} - V^i U^j \frac{\partial A_j}{\partial x^i} + U^i V^j A_i A_j - V^i U^j A_i A_j = \\ = \left(\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) + [A_i, A_j] \right) U^i V^j = F(U,V) \end{aligned}$$

and so we have this form for the curvature $F(U,V) \in \Gamma(\text{End}(E))$, $\forall U, V \in \mathfrak{X}(M)$,

$$F(U,V) = \left(\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) + [A_i, A_j] \right) U^i V^j$$

but I think that one should keep in mind that this is just one form that F could take, if it is applied to U, V beforehand.

2.0.2. *deRham cohomology.* I'm going to now follow Section 12.2 Closed forms, exact forms, diffeomorphisms and De Rham cohomology of Taubes (2011) [1].

Recall the definition of *deRham cohomology*:

$$H^p_{\text{deRham}}(M) := \text{ker}d / \text{im}d \qquad \left(= \{\omega | d\omega = 0\} / \{\theta | \theta = d\alpha \text{ for } \alpha \in \Omega^{p-1}(M)\} \right) \tag{2}$$

If M, N smooth manifolds, smooth map $f : M \to N$, $\forall\, \alpha \in \Omega^{p-1}(N)$, then

$$f^*(d\alpha) = d(f^*\alpha) \text{ or } f^* d = df^* \tag{3}$$

i.e.

$$\begin{tikzcd} \Omega^p(M) & \Omega^p(N) \arrow[l, "f^*"'] \\ \Omega^{p-1}(M) \arrow[u, "d"] & \Omega^{p-1}(N) \arrow[l, "f^*"'] \arrow[u, "d"] \\ M \arrow[u] \arrow[r, "f"] & N \arrow[u] \end{tikzcd}$$

Proof. Indeed, this can be shown, by considering local expressions: locally, $\alpha_I dy^I \in \Omega^{p-1}_y(N)$ where $I \equiv (i_1, i_2 \dots i_{p-1})$ s.t. $i_1 < i_2 < \dots < i_{p-1}$, and consider, with $f(x) = y$:

$$d\alpha = \frac{\partial \alpha_I}{\partial y^i} dy^i \wedge dy^I = \frac{\partial \alpha_I}{\partial y^i} \epsilon^{iI}_J dy^J \text{ since there's only 1 way to permute } iI \text{ into } J = (j_1 \dots j_p) \text{ s.t. } j_1 < \dots < j_p$$

$$f^* d\alpha = \frac{\partial \alpha_I}{\partial y^i} \epsilon^{iI}_J \frac{\partial y^J}{\partial x^k} dx^k$$

$$f^* \alpha = \alpha_I \frac{\partial y^I}{\partial x^J} dx^J = \frac{\partial \alpha_I}{\partial y^i} \frac{\partial y^i}{\partial x^j} \frac{\partial y^I}{\partial x^J} \epsilon^{jJ}_K dx^K$$

Now

$$df^*\alpha = \left(\frac{\partial \alpha_I}{\partial x^i} \frac{\partial y^I}{\partial x^j} + \alpha_I \frac{\partial^2 y^I}{\partial x^i \partial x^J} \right) dx^i \wedge dx^J = \left(\frac{\partial \alpha_I}{\partial x^i} \frac{\partial y^I}{\partial x^J} + \alpha_I \frac{\partial^2 y^I}{\partial x^i \partial x^J} \right) \epsilon^{iJ}_K dx^K =$$

$$= \frac{\partial \alpha_I}{\partial y^i} \frac{\partial y^i}{\partial x^j} \frac{\partial y^I}{\partial x^J} \epsilon^{jJ}_K dx^K + \alpha_I \frac{\partial^2 y^J}{\partial x^i \partial x^J} \epsilon^{iJ}_K dx^K = \frac{\partial \alpha_I}{\partial y^i} \frac{\partial y^i}{\partial x^j} \frac{\partial y^I}{\partial x^J} \epsilon^{jJ}_K dx^K + 0 = f^* d\alpha$$

□

Consider this homotopy: for smooth maps $\begin{matrix} f_0 : M \to N \\ f_1 : M \to N \end{matrix}$, $\exists$ smooth map $\begin{matrix} \psi : [0,1] \times M \to N \\ \psi(0, \cdot) = f_0 \\ \psi(1, \cdot) = f_1 \end{matrix}$

Let closed form $\omega \in \Omega^p(N)$; $d\omega = 0$. Then $f_0^*\omega$, $f_1^*\omega$ closed form.

Now consider $\mathbb{R} \times M$, and that

$$\begin{array}{ccc} T^*(\mathbb{R} \times M) = \mathbb{R} \oplus T^* M & & \alpha = \alpha_0 dt + \alpha_M = \alpha_0 dt + (\alpha_M)_i dx^i \\ \uparrow & & \uparrow \\ \mathbb{R} \times M & \text{in that} & (t, x) \end{array}$$

in that i runs through the indices for (some) local chart of M *only*, i.e. $i = 1, 2, \dots \dim M = d$.

Likewise, $\Omega^p(\mathbb{R} \times M) = \Omega^{p-1}(M) \oplus \Omega^p(M)$, in that

$$\forall\, \alpha \in \Omega^p(\mathbb{R} \times M) \text{ then for } \mu = 0, 1, 2, \dots d, \ 0 \text{ standing in for } t \in \mathbb{R} \text{ of } \mathbb{R} \times M,$$

$$\alpha = \alpha_M dx^M = dt \wedge \alpha_I dx^I + \alpha_J dx^J \text{ where } \begin{array}{lll} M = (\mu_1 \dots \mu_p) & \mu_\mu = 0, 1 \dots d & \mu_1 < \dots < \mu_p \\ I = (i_1 \dots i_{p-1}) & i_i = 1 \dots d & i_1 < \dots < i_{p-1} \\ J = (j_1 \dots j_p) & j_j = 1 \dots d & j_1 < \dots < j_p \end{array}$$

and so, naming these components of α as

$$\alpha^{p-1} \equiv \alpha_I dx^I \in \Omega^{p-1}(M)$$
$$\alpha^p \equiv \alpha_J dx^J \in \Omega^p(M)$$

Then $\forall\, \alpha \in \Omega^p(\mathbb{R} \times M)$,

$$\alpha = dt \wedge \alpha^{p-1} + \alpha^p \tag{4}$$

. Taking d on both sides to obtain $d\alpha \in \Omega^{p+1}(\mathbb{R} \times M)$, and $d\alpha$, being a $p+1$-form, taking the form of Eq. 4, then

$$d\alpha = dt \wedge (d\alpha)^p + (d\alpha)^{p+1} = -dt \wedge d^\perp \alpha^{p-1} + \frac{\partial \alpha^p_J}{\partial t} dt \wedge dx^J + d^\perp \alpha^p$$

where $d\alpha^p = \frac{\partial \alpha^p_J}{\partial x^\mu} dx^\mu \wedge dx^J = \frac{\partial \alpha^p_J}{\partial t} dt \wedge dx^J + \frac{\partial \alpha^p_J}{\partial x^i} dx^i \wedge dx^J = \frac{\partial \alpha^p_J}{\partial t} dt \wedge dx^J + d^\perp \alpha^p$, and so $d^\perp$ signifies that this exterior derivative only "acts" on the (local) coordinates of M.

Thus

$$\begin{aligned} (d\alpha)^p &= -d^\perp \alpha^{p-1} + \frac{\partial \alpha^p}{\partial t} \\ (d\alpha)^{p+1} &= d^\perp \alpha^p \end{aligned}$$

Suppose $\alpha = \psi^* \omega$; $\omega \in \Omega^p(N)$; $\psi : [0,1] \times M \to N$.
If ω closed ($d\omega = 0$), then $\psi^*\omega$ closed ($d\psi^*\omega = \psi^* d\omega = 0$).

So using the above facts shown for $\alpha = \psi^*\omega$,

$$\begin{aligned} \alpha = dt \wedge \alpha^{p-1} + \alpha^p &\xrightarrow{\alpha = \psi^*\omega} \psi^*\omega = dt \wedge (\psi^*\omega)^{p-1} + (\psi^*\omega)^p \\ d\psi^*\omega = \psi^* d\omega = 0 &= dt \wedge (d\psi^*\omega)^p + (d\psi^*\omega)^{p+1} \end{aligned}$$

$$(d\psi^*\omega)^{p+1} = 0 = d^\perp (\psi^*\omega)^p$$

$$(d\psi^*\omega)^p = 0 = -d^\perp (\psi^*\omega)^{p-1} + \frac{\partial (\psi^*\omega)^p}{\partial t} \xrightarrow{\int_0^1 dt} (\psi^*\omega)^p|_{t=1} - (\psi^*\omega)^p|_{t=0} = d^\perp \int (\psi^*\omega)^{p-1} \text{ or}$$

$$f_1^*\omega - f_0^*\omega = d^\perp \int (\psi^*\omega)^{p-1}$$

So $f_1^*\omega$ differ from $f_0^*\omega$ by an exact form, $d^\perp \int (\psi^*\omega)^{p-1}$.

$$\implies [f_1^*\omega] = [f_0^*\omega]$$

Thus deRham cohomology classes are invariant under homotopy (homotopy invariant!).

Consider 1-form connection on principal G-bundle $A = A(x) \in \Omega^1(M; \mathbf{g})$, $\forall x \in M$, $\mathfrak{g}$ Lie algebra of G (Recall $\mathbf{g} = T_1 G$).
Consider 1-form connection over $[0,1] \times U$, open $U \subset M$, $A' = A'(t,x)$ in that

$$A' = \mathbf{g}^{-1} d\mathbf{g} + t \mathbf{g}^{-1} A \mathbf{g} = A'(t,x)$$

Note that $\mathbf{g} \in \mathfrak{g}$.

A' interpolates between a flat connection $A'(0,x) = A'|_{t=0} = \mathbf{g}^{-1} d\mathbf{g}$, the connection 1-form for product principal bundle $P = M \times G$ and $A'(1,x) = A'|_{t=1} = \mathbf{g}^{-1} d\mathbf{g} + \mathbf{g}^{-1} A \mathbf{g}$. I think that this could be interpreted as turning off and turning on the gauge field, respectively.

Now, doing the calculation out explicitly,

$$\begin{aligned} F_{A'} = (d + A')^2 = dA' + A' \wedge A' = d\mathbf{g}^{-1} \wedge d\mathbf{g} + dt \wedge \mathbf{g}^{-1} A \mathbf{g} + t(d\mathbf{g}^{-1} \wedge A\mathbf{g} + \mathbf{g}^{-1} dA \mathbf{g} + \mathbf{g}^{-1} A \wedge d\mathbf{g} + \\ + t(\mathbf{g}^{-1} d\mathbf{g} \wedge \mathbf{g}^{-1} A \mathbf{g} + \mathbf{g}^{-1} A \mathbf{g} \wedge \mathbf{g}^{-1} d\mathbf{g}) + t^2 \mathbf{g}^{-1} A \wedge A \mathbf{g} \end{aligned}$$

Using this identity:

$$\begin{aligned} \mathbf{g}^{-1}\mathbf{g} &= 1 \\ \implies d(\mathbf{g}^{-1}\mathbf{g}) &= d\mathbf{g}^{-1}\mathbf{g} + \mathbf{g}^{-1} d\mathbf{g} = 0 \end{aligned}$$

and "commuting" or "moving through" differential forms "through the wedge product", then

$$F_{A'} = tdA + dt \wedge A + t^2 A \wedge A$$

Now consider $\mathrm{tr}(F_{A'} \wedge F_{A'}) \in \Omega^4([0,1] \times U)$.

$\mathrm{tr}(F_{A'} \wedge F_{A'})$ is closed, since $\dim M = 4$.

Now recall that $\forall\, p$-form on $[0,1] \times U$, $\alpha \in \Omega^p([0,1] \times U)$, $\alpha = dt \wedge \alpha^{p-1} + \alpha^p$;

with $\alpha^{p-1} \in \Omega^{p-1}(U)$

$\alpha^p \in \Omega^p(U)$

Thus, in our case currently,

$$\mathrm{tr}(F_{A'} \wedge F_{A'}) = dt \wedge \alpha^3 + \alpha^4$$

Calculating out $F_{A'} \wedge F_{A'}$ explicitly,

$$\begin{gathered} F_{A'} \wedge F_{A'} = dt \wedge A \wedge tdA + t^3 A \wedge A \wedge dA + tdA \wedge dt \wedge A + t^2 dt \wedge A \wedge A \wedge A + \\ + tdA \wedge A \wedge A + t^2 dt \wedge A \wedge A \wedge A = \\ = 2dt \wedge tA \wedge dA + 2t^2 dt \wedge A \wedge A \wedge A + (t^3 + t) A \wedge A \wedge dA \end{gathered}$$

and so

$$\alpha^3 = 2\mathrm{tr}(tA \wedge dA + t^2 A \wedge A \wedge A)$$

Since $\mathrm{tr}(F_{A'} \wedge F_{A'})$ closed, $0 = 0 - dt \wedge d\alpha^3 + d\alpha^4$, and "applying" $\frac{\partial}{\partial t}$ to this expression (i.e. this 4-form "acts" on $\frac{\partial}{\partial t}$, then

$$\frac{\partial \alpha^4}{\partial t} = d\alpha^3$$

$$\xrightarrow{\int dt} \int \frac{\partial \alpha^4}{\partial t} = \int d\alpha^3 = \mathrm{tr}(F_{A'(1)} \wedge F_{A'(1)}) - \mathrm{tr}(F_{A'(0)} \wedge F_{A'(0)}) = d\mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

Explicitly,

$$d(\mathrm{tr}(F_{A'} \wedge F_{A'})) = -dt \wedge d\alpha^3 + d\alpha^4$$

$$\xrightarrow{(\frac{\partial}{\partial t}, \cdot, \cdot, \cdot)} \frac{\partial(\mathrm{tr}(F_{A'} \wedge F_{A'}))}{\partial t} = -d\alpha^3 \xrightarrow{\int_0^1 dt} \int_0^1 dt \frac{\partial \mathrm{tr}(F_{A'} \wedge F_{A'})}{\partial t} = \int_0^1 dt(-d\alpha^3)$$

and so for $\mathrm{tr}(F \wedge F) \in \Omega^4(M)$, $\mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \in \Omega^3(M)$

$$\Longrightarrow \mathrm{tr}(F_A \wedge F_A) \equiv \mathrm{tr}(F \wedge F) = d\mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

For oriented smooth M; $\dim M = 4$

$$\int_M \mathrm{tr}(F \wedge F) = \int_M d\mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = \int_{\partial M} \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

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