CATEGORIES

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ABSTRACT. Everything about Categories, Category Theory, with applications to (relational) databases and other applications.

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From the section on "Terminology" of the Preface of Barr and Wells (1998) [2]:

"In most scientific disciplines, notation and terminology are standardized, of- ten by an international nomenclature committee. (Would you recognize Ein- steins equation if it said $p = HU^2$?) We must warn the nonmathematician reader that such is not the case in mathematics. There is no standardization body and terminology and notation are individual and often idiosyncratic."

To try to bridge the difference choice of notation and through comparison, suggest the "best" notation that's easy to remember and easy to use, I'll present all the different types of notation that I come across as much as I can.

- 0.1. Classes. From Adámek, Herrlich, and Strecker (2004) [5]:
 - (1) members of each class are sets
 - (2) \forall "property" P can form class of all sets with property P e.g. **universe** class of all sets \mathcal{U}
 - (3) if $X_1, X_2, \ldots X_n$ classes, $(X_1, X_2, \ldots X_n)$ is a class
 - (4) \forall set is a class (equivalently, every member of a set is a set)

proper classes - classes that aren't sets

 \Longrightarrow proper classes cannot be members of any class

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proper classes examples:

- \bullet universe \mathcal{U}
- class of all vector spaces
- class of all topological spaces
- class of all automata are proper classes
- $(4) \Longrightarrow Axiom \ of \ Replacement$
- (5) ∄ surjection from set to proper class

1. Categories

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Definition 1 (Category). Using the notation of Adámek, Herrlich, and Strecker (2004) [5]: **category C** is quadruple $\mathbf{C} = (\mathrm{Ob}, \mathrm{hom}, 1, \circ)$ consisting of class Ob, Ob collection, whose members are objects, $A, B, C \in \mathrm{Ob}$, $\forall (A, B), A, B \in \mathrm{Ob}, \mathrm{hom}(A, B)$ collection of morphisms/arrows $\forall f \in \mathrm{hom}(A, B), f: A \to B$ $\forall A \in \mathrm{Ob}, \exists \mathrm{identity} \mathrm{morphism/arrow}, 1_A: A \to A \mathrm{s.t.}$

(a) composition:
$$\forall A, B, C \in \text{Ob}, f: A \to B, \text{ then } g \circ f: A \to C$$

 $g: B \to C$

(b) associativity
$$\begin{array}{c} f:A\to B\\ g:B\to C\\ h:C\to D \end{array} \quad \text{then } h\circ (g\circ f)=(h\circ g)\circ f$$

(c) if
$$f: A \to B$$
, $1_B \circ f = f = f \circ 1_A$

In my notation,

category **A** is quadruple $\mathbf{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}\mathbf{A}, 1, \circ)$ s.t.

- (1) $A \in \text{Obj}(\mathbf{A})$ is called an *object*
- (2) $\operatorname{Mor} \mathbf{A} = \bigcup_{\operatorname{Hom}(A,B) \in \mathbf{A}} \operatorname{Hom}(A,B), f : A \to B \in \operatorname{Hom}(A,B) \text{ is a morphism}$
- (3) $\forall A \in \text{Obj}(\mathbf{A}), \exists 1_A : A \to A$

(4)
$$\forall f: A \to B \in \operatorname{Hom}(A, B), \qquad g \circ f: A \to C \in \operatorname{Hom}(A, C) \text{ s.t.}$$

 $g: B \to C \in \operatorname{Hom}(B, C)$

(a) associativity
$$\forall \begin{cases} f:A \to B \\ g:B \to C \end{cases}$$
, $h \circ (g \circ f) = (h \circ g) \circ f$
 $h:C \to D$

- (b) $\forall f: A \to B \in \text{Hom}(A, B), 1_B \circ f = f \text{ and } f \circ 1_A = f$
- (c) $\operatorname{Hom}(A, B) \in \operatorname{Mor} \mathbf{A}$ pairwise disjoint (i.e. $\operatorname{Hom}(A, B) \cap \operatorname{Hom}(C, D) \neq \emptyset$ if $C \neq A$ or $D \neq B$)

1.1. Examples.

- Set = $(Ob_{Set}, hom_{Set}, 1, \circ)$ where Ob_{Set} is the class of all sets hom_{Set} is the class of all functions on a set to another set
- Vec

 $MorVec \equiv all linear transformations between them (between real vector spaces)$

• Monoid. Consider a monoid as a triple (M, \cdot, e) . Every semigroup (M, \cdot) (recall that a *semigroup* is a set S with binary operation \cdot , i.e. s.t.

$$S\times S\overset{\cdot}{\to} S$$

$$\forall\,a,b,c\in S,\,(a\cdot b)\cdot c=a\cdot (b\cdot c)\quad \text{(associativity)}$$
 (but no inverse, necessarily!)) that also has a unit e can be made into a category \mathbf{C} $\Longrightarrow \mathbf{C}(M,\cdot,e)=\text{(Ob, hom, 1, \circ)}, \text{ a category }\mathbf{C}$ with only 1 object, i.e. $\text{Ob}=\{M\}, \text{ so that } \text{Ob}=\{M\}$ hom $(M,M)=M$ $1_M=e$ $y\circ x=y\cdot x$

2. Duality

Given a category $\mathbf{A} = (\mathrm{Ob}, \mathrm{hom}_{\mathbf{A}}, 1, \circ),$

Definition 2 (dual opposite category). dual or opposite category of A, denoted A^{op} , is

(1)
$$\mathbf{A}^{\mathrm{op}} = (\mathrm{Ob}, \mathrm{hom}_{\mathbf{A}^{\mathrm{op}}}, 1, \circ^{\mathrm{op}})$$

s.t.

$$hom_{\mathbf{A}^{op}}(A, B) = hom_{\mathbf{A}}(B, A)$$
$$f \circ^{op} g = g \circ f$$

i.e.

 \forall category $\mathbb{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}\mathbf{A}, 1, \circ),$ **dual** (or opposite) category of A is $\mathbf{A}^{\mathrm{op}} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}\mathbf{A}^{\mathrm{op}}, 1, \circ^{\mathrm{op}})$ where $\forall \mathrm{Hom}_{\mathbf{A}^{\mathrm{op}}}(A, B) \in \mathrm{Mor}\mathbf{A}^{\mathrm{op}}, \mathrm{Hom}_{\mathbf{A}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathbf{A}}(B, A)$ and

$$f \circ^{\mathrm{op}} g = g \circ f$$

e.g. if $\mathbf{A} = (M, \cdot, e)$ monoid, then $\mathbf{A}^{op} = (M, \hat{\cdot}, e)$ where $a\hat{\cdot}b = b \cdot a$

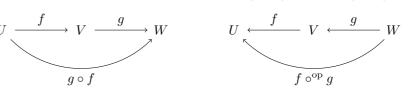
2.0.1. Example.

Vec^{op}

$$\operatorname{Vec}^{\operatorname{op}} = (\operatorname{Obj}(\operatorname{Vec}), \operatorname{Hom}_{\operatorname{Vec}^{\operatorname{op}}}, 1, \circ^{\operatorname{op}})$$

 $\operatorname{Hom}_{\operatorname{Vec}^{\operatorname{op}}}(W,V) = \operatorname{Hom}_{\operatorname{Vec}}(V,W)$

s.t.



3. Functors

Definition 3 (Functors). (covariant) functor

$$F: \mathbf{C} \to \mathbf{D}$$

if $\forall C \in \text{Ob}_{\mathbf{C}}$, then $F(C) \in \text{Ob}_{\mathbf{D}}$ s.t. $\forall f \in \text{hom}_{\mathbf{C}}$, say $f \in \text{hom}_{\mathbf{C}}(B, C)$ $F(f) \in \text{hom}_{\mathbf{D}}(F(B), F(C))$ and s.t. $F(1_{\mathbf{C}}) = 1_{F(C)}$

$$A, B, C \in \mathrm{Ob}_{\mathbf{C}}, \ f : A \to C, \text{ so } g \circ f : A \to C$$

 $g : B \to C$
then $F(g \circ f) = F(g) \circ F(f)$

i.e.

$$\mathbf{C} \xrightarrow{F} \mathbf{D}$$

if

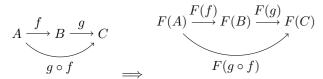
$$C \stackrel{F}{\longmapsto} F(C$$

$$\mathbf{C} \xrightarrow{F} \mathbf{D}$$

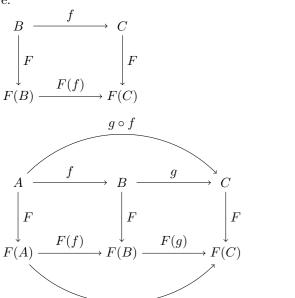
s.t.

$$B \xrightarrow{f} C \implies F(B) \xrightarrow{F(f)} F(C)$$

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i.e.



Definition 4. (contravariant) functor F is s.t.

 $F(g \circ f) = F(g) \circ F(f)$

(2)
$$\mathbf{C}^{\mathrm{op}} \xrightarrow{F} \mathbf{D}$$

so that

$$\begin{array}{ccc}
B & \xrightarrow{f} & C \\
\downarrow F & & \downarrow I \\
F(B) & \longleftarrow & F(C)
\end{array}$$

Definition 5 (covariant hom-functor). \forall locally small category \mathbf{C} (i.e. hom_{\mathbf{C}} is actually a set and not a proper class), \forall $A \in \mathrm{Ob}_{\mathbf{C}}$, \exists covariant hom-functor hom $(A, -) : \mathbf{C} \to \mathrm{Set}$ s.t. \forall $B \xrightarrow{f} C$,

$$hom(A, -)(f) = hom(A, B) \xrightarrow{hom(A, f)} hom(A, C)$$

where $hom(A, f)(g) = f \circ g$

i.e.
$$\forall X, Y \in \text{Ob}_{\mathbf{C}}, \forall X \xrightarrow{f} Y$$
,

then

$$hom(A, -)(f) = hom(A, f)$$

$$hom(A, X) \xrightarrow{hom(A, f)} hom(A, Y)$$

$$g \longmapsto f \circ g$$

with $g \in \text{hom}(A, X)$

M-set is a covariant hom-functor on a monoid $\mathbf{C}(M,\cdot,e) \equiv \mathbf{C}(M)$, M a monoid, i.e. the category that is the domain that the covariant hom-functor maps from is a monoid (category).

Definition 6 (forgetful functor). ∀ constructs (i.e. categories)

• Vec

and

- Grp
- Top
- Rel

 $\exists U : \mathbf{A} \to \text{Set s.t.}$

$$U(A)$$
 is underlying set $U(f) = f$ is underlying function

Definition 7. given functor $F : \mathbf{A} \to \mathbf{B}$, dual functor or opposite functor $F^{\mathrm{op}} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$ is given by $\forall f : A \to A', f \in \mathrm{Hom}(A, A')$,

$$F^{\mathrm{op}}f = Ff$$

$$Ff: FA \to FA', Ff \in \text{Hom}(FA, FA')$$

- 3.0.2. Examples.
 - duality functor for vector spaces $(*): \operatorname{Vec}^{\operatorname{op}} \to \operatorname{Vec}$ associates \forall vector space V its dual V^* (i.e. vector space $\operatorname{Hom}(V,\mathbb{R})$ with operations defined pointwise), associates $\forall V \xrightarrow{f} W, f \in \operatorname{MorVec}^{\operatorname{op}},$

i.e.
$$\forall$$
 linear map $W \xrightarrow{f} V$,
morphism $f^*: V^* \to W^*$ defined by $f^*(g) = g \circ f$ i.e.

$$Vec^{op} \xrightarrow{(*)} Vec$$

$$V \xrightarrow{f} V$$

$$\downarrow (*) \qquad \downarrow (*)$$

3.1. Functor properties.

Definition 8. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor.

- (1) F embedding if F is injective on morphisms $(\forall f \in Mor \mathbf{A}, \text{ if } F(f) = F(g), \text{ then } f = g)$ $g \in Mor \mathbf{B}$
- (2) F faithful if \forall hom-set restrictions,

$$F: \operatorname{Hom}_{\mathbf{A}}(A, A') \to \operatorname{Hom}_{\mathbf{B}}(FA, FA')$$

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are injective

- (3) F full if all hom-set restrictions are surjective
- (4) F amnestic if $Ff = 1_{\mathbf{B}}$, then **A**-ismorphism $f = 1_{\mathbf{A}}$

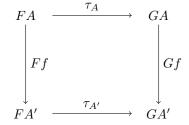
So

- (1) F an embedding iff F faithful and injective on objects
- (2) F isomorphism iff F full, faithful, and bijective on objects

3.2. Natural Transformation.

Definition 9 (Natural transformation). Let functors $F, G : \mathbf{A} \to \mathbf{B}$. **natural transformation** τ from F to $G \equiv \tau : F \to G$ or $F \xrightarrow{\tau} G$ is function that assigns $\forall A \in \text{Obj}\mathbf{A}, \ \tau_A : FA \to GA, \ \tau_A \in \text{Mor}\mathbf{B}$, s.t. **naturality condition** holds:

$$\forall A \xrightarrow{f} A', f \in \text{Mor} \mathbf{A}$$



3.2.1. Examples.

• Let (**): Vec \rightarrow Vec be **second-dual functor for vector spaces** defined by

$$Vec \xrightarrow{(**)} Vec = (Vec^{op})^{op} \xrightarrow{(*)^{op}} Vec^{op} \xrightarrow{(*)} Vec$$

where $(*)^{op}$ is the dual of the duality functor for vector spaces.

Then linear transformations

$$\tau_V: V \to V^{**}$$
$$(\tau_V(x))(f) = f(x)$$

yield a natural transformation $1_{\text{Vec}} \xrightarrow{\tau} (**)$

Indeed, looking at the definition of the natural transformation, for

$$Vec \xrightarrow{1_{Vec}} Vec$$

$$\operatorname{Vec} \xrightarrow{(**)} \operatorname{Vec}$$

 $\forall V \in \text{Obj}(\text{Vec}), \, \tau_V : 1_{\text{VeC}}V = V \to (**)V \equiv V^{**}, \, \tau_V \in \text{MorVec}, \text{ and } \forall f : V \to W, \, f \in \text{MorVec},$

$$\begin{array}{cccc}
V & \xrightarrow{\tau_V} & V^{**} \\
\downarrow f & & \downarrow f^{**} \\
\downarrow W & \xrightarrow{\tau_W} & W^{**}
\end{array}$$

• assignment of Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ to each topological space X is a natural transformation from nth homotopy functor $\pi_n : \text{Top} \to \text{Grp}$ to nth homology functor $H_n : \text{Top} \to \text{Grp}$

$$\pi_n \xrightarrow{\tau} H_n$$

Indeed, $\forall X \in \text{Obj}(\text{Top}), \tau_X : \pi_n(X) \to H_n(X), \tau_X \in \text{MorGrp}$

$$\forall X \xrightarrow{\varphi} Y, \varphi \in \text{MorTop},$$

$$\pi_n(X) \xrightarrow{\tau_X} H_n(X)$$

$$\downarrow \\ \pi_n \circ \varphi \qquad \qquad \downarrow \\ H_n \circ \varphi$$

$$\pi_n(Y) \xrightarrow{\tau_Y} H_n(Y)$$

Definition 10 (Grothendieck construction). Let category C, a category of small categories CAT,

Let functor $F: \mathbf{C} \to CAT$

Then category $\Gamma(C)$ (also denoted $C \cap \Gamma(F)$) is $\Gamma(C) = (\mathrm{Ob}_{\Gamma(F)}, \mathrm{hom}_{\Gamma(F)}, 1, \circ)$ s.t.

$$(C, X) \in \mathrm{Ob}_{\Gamma(F)}, \quad C \in \mathrm{Ob}_{\mathbf{C}}$$

 $X \in \mathrm{Ob}_{F(C)}$

nd

 $\text{hom}_{\Gamma(F)}((C_1, X_1), (C_2, X_2)) \ni (f, x) \text{ s.t.}$

$$f: C_1 \to C_2 \in \operatorname{mor}_{\mathbf{C}} := \operatorname{hom}_{\mathbf{C}}$$

 $x: F(f)(X_1) \to X_2 \in \operatorname{mor}_{F(C_2)} := \operatorname{hom}_{F(C_2)}$

EY : 20150714, to clarify, $f \in \text{hom}_{\mathbf{C}}$, and $x \in \text{hom}_{F(C_2)}$, and

$$(f,x)\circ(f',x')=(ff',x\circ F(f)(x'))$$

i.e.

$$C_{1} \xrightarrow{f} C_{2} \Longrightarrow F(C_{1}) \xrightarrow{F(f)} F(C_{2})$$

$$(C_{1}, X_{1}) \xrightarrow{(f', x')} (C_{2}, X_{2}) \xrightarrow{(f, x)} (C_{3}, X_{3})$$

$$(f \circ f', x \circ F(f)(x'))$$

4. Limits

4.0.2. Sources. It appears Adámek, Herrlich, and Strecker (2004) [5] defines sources to simply give a name and formalize a tuple.

Definition 11 (source). source is a tuple: $(a, (f_i)_{i \in I}), f_i : A \to A_i$

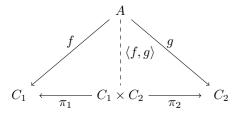
4.1. Products.

Definition 12 (Products). (in Turi's notation [4])

Given objects C_1, C_2 of category \mathbb{C} , **product** (if exists) consists of object $C_1 \times C_2$ of \mathbb{C} and $\pi_1 : C_1 \times C_2 \to C_1$ s.t.

$$\pi_2:C_1\times C_2\to C_2$$

$$\forall$$
 object A of \mathbb{C} , \forall $f: A \to C_1$ \exists ! $\langle f, g \rangle : A \to C_1 \times C_2$ s.t. $f = \pi_1 \circ \langle f, g \rangle$, i.e. $g: A \to C_2$ $g = \pi_2 \circ \langle f, g \rangle$



(compare with Leinster (2014) [3])

Let category A, $X, Y \in A$, **product** of X, Y consists of object P and maps (compare this definition with Adámek, Herrlich, and Strecker (2004) [5] and their notation) **product** consisting of

$$C_{1} \times C_{2} \times \cdots \times C_{N} \in \text{Obj}\mathbf{C}$$

$$\pi_{1} : C_{1} \times C_{2} \times \cdots \times C_{N} \to C_{1}$$

$$\pi_{2} : C_{1} \times C_{2} \times \cdots \times C_{N} \to C_{2}$$

$$\vdots$$

$$\pi_{N} : C_{1} \times C_{2} \times \cdots \times C_{N} \to C_{N}$$

is s.t.

$$A \in \text{Obj}\mathbf{C}$$

$$f_1: A \to C_1$$

$$\forall f_2: A \to C_2,$$

$$\vdots$$

$$f_{\mathcal{N}}: A \to C_{\mathcal{N}}$$

$$\exists ! \langle f_1, f_2, \dots, f_{\mathcal{N}} \rangle : A \to C_1 \times C_2 \times \dots \times C_{\mathcal{N}} \text{ s.t.}$$

$$f_1 = \pi_1 \circ \langle f_1, f_2, \dots f_{\mathcal{N}} \rangle$$

$$f_2 = \pi_2 \circ \langle f_1, f_2, \dots f_{\mathcal{N}} \rangle$$

$$\vdots$$

$$f_{\mathcal{N}} = \pi_{\mathcal{N}} \circ \langle f_1, f_2, \dots f_{\mathcal{N}} \rangle$$

4.1.1. Example: Set always has products. \forall sets $X, Y \in \text{Obj}(\text{Set})$, \exists product $X \times Y \in \text{Obj}(\text{Set})$.

Let
$$A \in \text{Obj(Set)}$$
, $f_1 : A \to X$ Define $\begin{cases} \langle f_1, f_2 \rangle : A \to X \times Y \\ \langle f_1, f_2 \rangle (a) = (f_1(a), f_2(a)) \end{cases}$

Then
$$\pi_1 \circ \langle f_1, f_2 \rangle(a) = f_1(a)$$
 $\Longrightarrow \pi_1 \circ \langle f_1, f_2 \rangle = f_1$
 $\pi_2 \circ \langle f_1, f_2 \rangle(a) = f_2(a)$ $\pi_2 \circ \langle f_1, f_2 \rangle = f_2$

Suppose
$$f': A \to X \times Y$$
 s.t. $\pi_1 \circ f' = f_1$
 $\pi_2 \circ f' = f_2$

Write
$$f'(a) = (x, y)$$

$$f_1(a) = \pi_1 \circ f'(a) = \pi_1(x, y) = x$$

 $f_2(a) = \pi_2 \circ f'(a) = \pi_2(x, y) = y$ $\Longrightarrow f'(a) = (f_1(a), f_2(a)) = \langle f_1, f_2 \rangle (a)$

 $\langle f_1, f_2 \rangle$ unique.

Proposition 1. If product $(A_1 \times \cdots \times A_N \xrightarrow{\pi_i} A_i)_{i \in I}$, if $\exists i_0 \in I$ s.t. $Hom(A_{i_0}, A_i) \neq \emptyset$, $\forall i \in I$, then π_{i_0} retraction

Proof. $\forall i \in I$, choose $f_i \in \text{Hom}(A_{i_0}, A_i)$ with $f_{i_0} = 1_{A_{i_0}}$.

Then $\langle f_i \rangle : A_{i_0} \to A_1 \times \cdots \times A_{\mathcal{N}}$ is a morphism s.t.

$$\pi_{i_0} \circ \langle f_i \rangle = f_{i_0} = 1_{A_{i_0}}$$

Adámek, Herrlich, and Strecker (2004) [5] and their notation) calls a sink what Leinster (2014) [3] calls a cocone.

Definition 13. sink $((f_i)_{i \in I}, A) \equiv (f_i, A)_I \equiv (A_i \xrightarrow{f_i} A)_I$, object A, family of morphisms $f_i : A_i \to A$

For the *coproduct*, consider this enlightening comparision:

4.1.2. Examples (of coproducts).

• if $(A_i)_I$ pairwise-disjoint family of sets, then $(\mu_j, \bigcup_{i \in I} A_i)_{j \in I}$ is coproduct in Set. If $(A_i)_I$ arbitrary set-indexed family of sets, then it can be "made disjoint" by pairing each A_i with index i, i.e. by working with $A_i \times \{i\}$ rather than A_i .

So $\bigcup_{i \in I} (A_i \times \{i\})$ disjoint. Consider

$$\mu_j: A_j \to \bigcup_{i \in I} A_i \times \{i\}$$

$$\mu_i(a) = (a, j)$$

 $(\mu_j, \bigcup_{i \in I} A_i \times \{i\})_{j \in I}$ is a coproduct in Set.

Indeed, given
$$f_j:A_j\to A,$$

$$f_j(a)\in A$$

$$[f_i]:\coprod A_i$$

$$[f_i]: \coprod_{i \in I} A_i \times \{i\} \to A$$

 $[f_i] \circ \mu_i = f_i$

where

$$f_j(a) = [f_i] \circ \mu_j(a) = [f_i](a, j) = f_j(a)$$

• Top coproducts are "topological sums"; they're "concrete" coproducts (Adámek, Herrlich, and Strecker (2004) [5])

• Vec (nonconcrete) coproducts called *direct sums* direct sum $\bigoplus_{i \in I} A_i$ of vector spaces A_i is subspace of direct product $\prod_{i \in I} A_i$ consisting of all elements $(a_i)_{i\in I}$ with finite carrier (i.e. $\{i\in I|a_i\neq 0\}$ is finite), injections

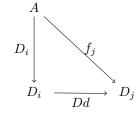
$$\mu_j: A_j \to \bigoplus_{i \in I} A_i$$

$$\mu_j(a) = (a_i)_{i \in I} \text{ with } a_i = \begin{cases} a & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

• Grp has nonconcrete coproducts, "free products"

Let diagram (functor) $D: \mathbf{I} \to \mathbf{A}$. (diagram is, technically, exactly the same as a functor (Adámek, Herrlich, and Strecker

Definition 14. A-source $(A \xrightarrow{f_i} D_i)_{i \in \text{ObjI}}$ natural for D if $\forall i \xrightarrow{d} j$, $d \in \text{MorI}$, then



Definition 15. limit of D is a natural source $(L \xrightarrow{l_i} D_i)_{i \in \text{Obj} \mathbf{I}}$ for D with

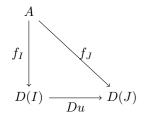
(universal) property that \forall natural source $(A \xrightarrow{f_i} D_i)_{i \in \text{ObiI}}$ for D uniquely factors through it, i.e.

 \forall natural source $(A \xrightarrow{f_i} D_i)_{i \in \text{Obj}}$, $\exists !$ morphism $f : A \to L$ s.t. $f_i = l_i \circ f$ $\forall i \in \text{Obj}(\mathbf{I})$.

It may pay to read and compare with other books because I didn't understand limits the first time reading through Adámek, Definition 20. For some category A, and for Herrlich, and Strecker (2004) [5]. So compare with Leinster (2014) [3].

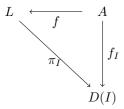
cone from Leinster (2014) [3] is the same as source in Adámek, Herrlich, and Strecker (2004) [5]:

Definition 16. cone on D (or natural source for D), $A \in \text{Obj} \mathbf{A}$ (vertex of the cone) (i.e. \mathbf{A} -source), $(A \xrightarrow{A_I} D(I))_{I \in \text{Obj} \mathbf{I}}$ s.t. if $\forall I \xrightarrow{u} J$, $u \in \text{Mor} \mathbf{I}$, then



Definition 17. limit of D is natural source (or cone) $(L \xrightarrow{\pi_I} D(I))_{I \in \text{Obj} \mathbf{I}}$ s.t. \forall natural source (or cone) on D, Then the **pullback** or "pullback square" consists of $P \in \text{Obj} \mathbf{A}$, $\pi_1 : P \to X$ s.t. $(A \xrightarrow{f_I} D(I))_{I \in \text{Obj}\mathbf{I}},$

 $\exists ! \text{ morphism } f: A \to L \text{ s.t. } f_I = \pi_I \circ f \qquad \forall I \in \text{Obj} \mathbf{I}. \ \pi_I \text{ projections of limit.}$

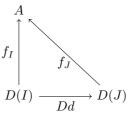


i.e. this commutes:

Definition 18. Let diagram (functor) $D: \mathbf{I} \to \mathbf{A}$.

Consider functor $D^{op}: \mathbf{I}^{op} \to \mathbf{A}^{op}$.

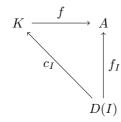
natural sink $(D(I) \xrightarrow{f_I} A)_{I \in \text{Obi} \mathbf{I}}$ for D s.t. $\forall I \xrightarrow{d} J, d \in \text{Mor} \mathbf{I}$, then



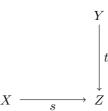
Natural sink of Adámek, Herrlich, and Strecker (2004) [5] is the same as the "cocone" of Leinster (2014) [3].

Definition 19. colimit of D is natural sink $(D(I) \xrightarrow{c_I} K)_{I \in ObiI}$ for D with (universal) property that

 \forall natural sink for D, $(D(I) \xrightarrow{f_I} A)_{I \in \text{Obj}\mathbf{I}}$, $\exists !$ morphism $f : K \to A \text{ s.t. } f \circ c_I = f_I \quad \forall I \in \text{Obj}\mathbf{I}$, i.e.



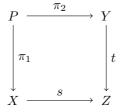
4.2. Pullback.



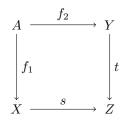
 $X, Y, Z \in \text{Obj} \mathbf{A}$.

 $s: X \to Z$; $s, t \in \text{Mor} \mathbf{A}$

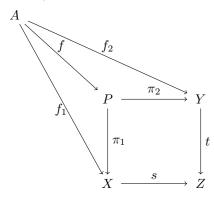
$$\pi_2:P\to Y$$



commutes and s.t. \forall commutative square in **A**



then $\exists ! f : A \to P$ s.t.



5. Adjoint

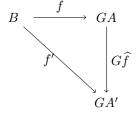
From the section on "Objects and Morphisms with Respect to a Factor" of Adámek, Herrlich, and Strecker (2004) [5],

Definition 21. Let functor $G : \mathbf{A} \to \mathbf{B}$, $B \in \text{Obj}\mathbf{B}$.

G-structured arrow with domain B is pair (f, A), $A \in \text{Obj}\mathbf{A}$, $f : B \to GA$, $f \in \text{Mor}\mathbf{B}$. G-structured arrow (f, A) with domain B is called

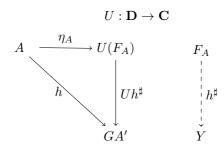
- (1) **generating** provided \forall pair of **A**-morphism $r: A \to A'$, $Gr \circ f = Gs \circ f$ implies r = s $s: A \to A'$
- (2) **extremally generating** provided it's generating and if $A' \xrightarrow{m} A$ is an **A**-monomorphism, (g, A') G-structured arrow, s.t. $f = G(m) \circ g$, then m is **A**-isomorphism
- (3) G-universal for B if $\forall G$ -structured arrow (f', A') with domain B,

$$\exists ! \mathbf{A}\text{-morphism } A \xrightarrow{\widehat{f}} A', f' = G(\widehat{f}) \circ f \text{ i.e. s.t.}$$



commutes.

If you're reading Turi [4], then Turi calls G-universal for B, "universal arrow" from an object A of C: inspection of his diagram immediately confirms that they're talking about the exact same thing (I know, it seems as different mathematicians have different names and notation for the exact same thing):

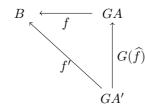


for $F_A \in \text{Obj}\mathbf{D}$

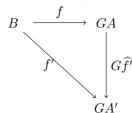
Definition 22. Let functor $G : \mathbf{A} \to \mathbf{B}$; let $B \in \text{Obj}\mathbf{B}$.

- (1) G-costructured arrow with codomain B is pair (A, f), $A \in \text{Obj}\mathbf{A}$, $GA \xrightarrow{f} B$, $f \in \text{Mor}\mathbf{B}$.
- (2) G-costructured arrow (A, f) with codomain B is called G-couniversal for B if \forall G-costructured arrow (A', f') with codomain B,

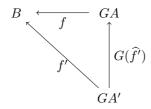
$$\exists ! A' \xrightarrow{\widehat{f}} A, \ \widehat{f} \in \text{Mor} \mathbf{A}, \text{ s.t. } f' = f \circ G(\widehat{f}) \text{ i.e.}$$



Definition 23 (adjoint). functor $G : \mathbf{A} \to \mathbf{B}$ adjoint if $\forall B \in \text{Obj}\mathbf{B}$, $\exists G$ -universal arrow with domain B, i.e. $\forall B \in \text{Obj}\mathbf{B}$, $\exists (f, A)$ with domain B s.t. $\forall (f', A')$ with domain B, $\exists ! \hat{f'} \in \text{Mor}\mathbf{A}$ s.t.



Definition 24 (co-adjoint). functor $G : \mathbf{A} \to \mathbf{B}$ co-adjoint if $\forall B \in \text{Obj}\mathbf{B}$, $\exists G$ -co-universal arrow with codomain B, i.e. $\forall B \in \text{Obj}\mathbf{B}$, $\exists (A, f)$ with codomain B s.t. $\forall (A', f')$ with codomain B, $\exists ! \hat{f'} \in \text{Mor}\mathbf{A}$ s.t.



In section 19 Adjoint situations of Adámek, Herrlich, and Strecker (2004) [5], their Theorem 19.1 is the same as Exercise 3.1 and Theorem 3.1 on pp. 11 of Turi [4], which Turi says is "Important!"

Theorem 1. Let adjoint functor $G: \mathbf{A} \to \mathbf{B}$, so (by def. of adjoint), $\forall B \in Obj\mathbf{B}$, let $\eta_B: B \to GA_B$ be the universal arrow. Then $\exists !$ functor $F: \mathbf{B} \to \mathbf{A}$ s.t. $F(B) = A_B$. $\forall B \in Obj\mathbf{B}$, and $1_{\mathbf{B}} \xrightarrow{\eta = (\eta_B)} G \circ F$ natural transformation. Moreover, $\exists !$ natural transformation $F \circ G \xrightarrow{\epsilon} 1_{\mathbf{A}}$ s.t.

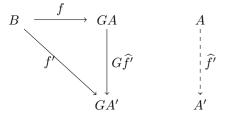
(1)
$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G = G \xrightarrow{1_G} G$$

 $F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F = F \xrightarrow{1_F} F$

$$(2) F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F = F \xrightarrow{1_F} F$$

Proof. Given an adjoint functor $G: \mathbf{A} \to \mathbf{B}$. By definition, this means that

 $\forall B \in \text{Obj}\mathbf{B}, \exists G$ -universal arrow with domain $B, (f, A), \text{ s.t. } \forall (f', A') \text{ (i.e. every other } G$ -structured arrow (f', A')),



We want to define a function F:

$$F : \text{Obj}\mathbf{B} \to \text{Obj}\mathbf{A}$$

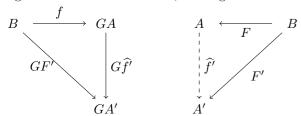
 $F(B) := A_B$

and make a functor out of it. We know it exists from the definition of an adjoint, so that $\exists a G$ -universal arrow $(f, A_B), \forall B$. Is it well defined?

Suppose another $F': \text{Obj}\mathbf{B} \to \text{Obj}\mathbf{A}$.

$$F'(B) = A'$$

Using universal arrow definition, then again we have



$$\Longrightarrow F'(B) = A' = \widehat{f}'(A) = \widehat{f}' \circ F(B) \Longrightarrow F' = \widehat{f}' \circ F$$

So F unique up to a unique morphism, due to universal arrow definition (or property).

Consider how F can act on morphisms.

Take $b \in \text{Mor} \mathbf{B}$. The commutative diagram

$$B \xrightarrow{F} F(B) = A_B$$

$$\downarrow b \qquad \qquad \downarrow F(b)$$

$$B' \xrightarrow{F} F(B') = A_B$$

tells us immediately what $F(b) \in \text{Mor} \mathbf{A}$ is (composition $F \circ b$).

A functor has to preserve identity and compositions. The following commutative diagrams show this:

$$B \xrightarrow{F} F(B) = A_{B}$$

$$\downarrow F \circ 1_{B} \equiv 1_{FB}$$

$$\downarrow B \xrightarrow{F} F(B) = A_{B}$$

$$\downarrow B \xrightarrow{F} F(B) = A_{B}$$

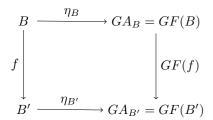
$$\downarrow b \qquad \qquad \downarrow F(b)$$

$$\downarrow b' \circ b \qquad \qquad \downarrow F(b') \circ F(b') \circ$$

 $F: \mathbf{B} \to \mathbf{A}$ is a unique functor and it exists, and is defined s.t. $F(B) = A_B$, any time you have an adjoint functor $G: \mathbf{A} \to \mathbf{B}$.

Given G-universal arrow $\eta_B: B \to G(A_B)$, which exists by adjoint functor def. of $G, \forall B \in \text{Obj} \mathbf{B}$. Then $B \xrightarrow{\eta_B} GA_B$

Use unique functor F, $F(B) = A_B$, $F(B') = A_{B'}$



where $GF(f): GF(B) \to GF(B')$, by functor property of G, F, so this holds $\forall f \in \text{Mor} \mathbf{B}$.

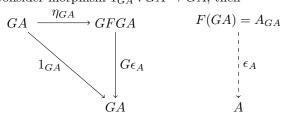
Thus, $\eta: 1_{\mathbf{B}} \to G \circ F$ is a natural transformation for $1_{\mathbf{B}}, G \circ F: \mathbf{B} \to \mathbf{B}$ (endofunctors, functors that map a category to itself), s.t.

 $\forall B \in \text{Obj}\mathbf{B}, \, \eta_B : 1_{\mathbf{B}}B = B \to GFB, \quad \eta_B \in \text{Mor}\mathbf{B}.$

Consider B = GA, and corresponding universal arrow $\eta_B = \eta_{GA}$, through the unique functor F so that $F(GA) = A_{GA}$.

$$GA \xrightarrow{\eta_{GA}} GA_{GA} = GFGA$$

Consider morphism $1_{GA}: GA \to GA$, then



by definition of an adjoint functor.

Now

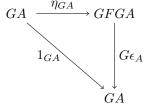
(3)
$$G(f \circ \epsilon_{A}) \circ \eta_{GA} = Gf \circ G\epsilon_{A} \circ \eta_{GA} = Gf = G\epsilon_{A'} \circ \eta_{GA'} \circ Gf = G\epsilon_{A'} \circ GFGf \circ \eta_{GA} = G(\epsilon_{A'} \circ FGf) \circ \eta_{GA}$$

$$\implies f \circ \epsilon_{A} = \epsilon_{A'} \circ FGf$$

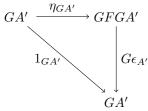
since for the first equality in Eq. 3, associativity of functor G was used, i.e.

$$G(f \circ \epsilon_A) = Gf \circ G\epsilon_A$$

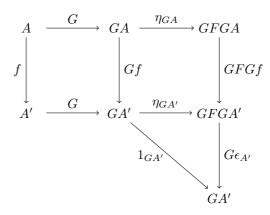
and for the second equality, universal arrow definition was used, i.e.



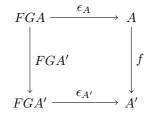
or i.e. $G\epsilon_A \circ \eta_{GA} = 1_{GA}$, and for the third equality, universal arrow definition was used again, i.e.



or i.e. $G\epsilon_{A'} \circ \eta_{GA'} = 1_{GA'}$, and for the fourth equality, the natural transformation definition for η and its universal arrow definition was used together, i.e.



and for the fifth equality, associativity of functor G was used again, i.e. $G\epsilon_{A'}\circ GFGf=G(\epsilon_{A'}\circ FGf)$. Thus, ϵ is a natural transformation, $\epsilon:FG\to 1_{\mathbf{A}}$, for



commutes.

6. Monad

Definition 25 (monad). **monad** on category **X** is triple $\mathbf{T} = (T, \eta, \mu)$, consisting of functor $T : \mathbf{X} \to \mathbf{X}$ (an endofunctor, maps a category to itself), and natural transformations

 $\eta: 1_{\mathbf{X}} \to T \text{ and}$ $\mu: T \circ T \equiv T^2 \to T \text{ s.t.}$

$$T \circ T \circ T \equiv T^{3} \xrightarrow{\qquad T \mu \qquad} T \circ T \equiv T^{2}$$

$$\downarrow \mu T \qquad \qquad \downarrow \mu$$

$$T \circ T = T^{2} \xrightarrow{\qquad \mu \qquad} T$$

 $T \longrightarrow T$

and

commute.

7. Applications

7.1. Databases. Let category $db = (Ob_{db}, hom_{db}, 1, \circ)$ be a database schema.

 Ob_{db} is a collection of tables $\tau, \tau \in Ob_{db}$

 $c \in \text{hom}_{db}$ where c is a column (i.e. attribute)

primary key column c! is a primary morphism (or arrow)

Declaring constraints is declaring a composition law, i.e. for tables $\rho, \sigma, \tau \in Ob_{db}$,

$$\rho \xrightarrow{c_1} \sigma \xrightarrow{c_2} \tau$$

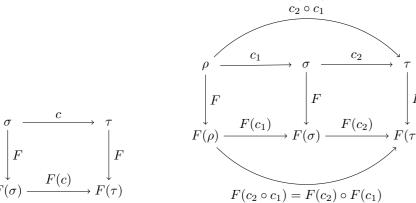
$$c_2 \circ c_1$$

q

EY: 20150716 I think it should be emphasized that Ob_{db} is a collection of tables associated with this particular database db, not the collection of all possible tables.

Let **data functor** be a functor $F : db \rightarrow Set$.

So for tables $\rho, \sigma, \tau \in \text{Ob}_{db}$, columns $c, c_1, c_2 \in \text{hom}_{db}(\sigma, \tau)$



Now note that $F(\rho), F(\sigma), F(\tau) \in \text{Ob}_{\text{Set}}$ means that $F(\rho), F(\sigma), F(\tau)$ are sets. They fill the tables with its data set; the data set of rows.

8. Decorators

Lutz (2009) [6]

10

References

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- [2] Michael Barr, Charles Wells. Category Theory for Computing Science. http://www.tac.mta.ca/tac/reprints/articles/22/tr22.pdf, http://www.math.mcgill.ca/triples/Barr-Wells-ctcs.pdf
- [3] Tom Leinster. Basic Category Theory (Cambridge Studies in Advanced Mathematics) 1st Edition. 2014. ISBN-13: 978-1107044241
- [4] Daniele Turi. Category Theory Lecture Notes. September 1996 December 2001. http://www.dcs.ed.ac.uk/home/dt/CT/categories.pdf
 [5] Jiří Adámek, Horst Herrlich, George E. Strecker. Abstract and Concrete Categories The Joy of Cats. 2004.
- [6] Mark Lutz. Learning Python, 4th Edition. O'Reilly Media. 2009.

EY: There's a 5th edition, 2013, but I don't have a copy of the 5th edition; I only have the 4th.

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