MATRIX CHAIN MULTIPLICATION

Algorithms Analysis

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1. Find an optimal parenthesization of a matrix chain product whose sequence of array dimensions is $< 3, 8, 4, 10, 6 >$. How many multiplications are performed using this optimal parenthesization to multiply the matrices?

   To answer this question, let us use the following formula:

$$m[i,j] = \begin{cases} 0, & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_kp_j\}, & \text{if } i < j \end{cases}$$

   Since we have four matrices $A_1, ..., A_4$ we will construct an auxiliary table m, in which $m[i,i] = 0$, for storing the $m[i,j]$ costs and another auxiliary table s to record which index of k achieved the optimal cost in computing $m[i,j]$.

m	1	2	3	4
1	0	?	?	?
2		0	?	?
3			0	?
4				0

s	1	2	3	4
1	0	?	?	?
2		0	?	?
3			0	?
4				0

   Let us fill out the missing values of these tables:

   $m[1,2] = p_0 \times p_1 \times p_2 = 3 \times 8 \times 4 = 96$ for $k = 1$

   $m[2,3] = p_1 \times p_2 \times p_3 = 8 \times 4 \times 10 = 320$ for $k = 2$

   $m[3,4] = p_2 \times p_3 \times p_4 = 4 \times 10 \times 6 = 240$ for $k = 3$

Now, let us consider a window of length three:

$m[1,3]$ for $k=1 \longrightarrow m[1,1]+m[2,3]+(3\times 8\times 10) = 0+320+240 = 560$

$m[1,3]$ for $k=2 \longrightarrow m[1,2]+m[3,3]+(3\times 4\times 10) = 96+0+240 = 216$

Therefore, since the minimum is 216, this is the value recorded in table m and $k=2$ is the value recorded in table s, as shown below.

Applying the same logic for $m[2,3]$ and $m[1,4]$, we get:

$$m[2,4] = \begin{cases} \text{for } k=2 \longrightarrow m[2,2]+m[3,4]+(8\times 4\times 6) = 0+240+192 = \color{red}{432} \\ \text{for } k=3 \longrightarrow m[2,3]+m[4,4]+(8\times 10\times 6) = 320+0+480 = 800 \end{cases}$$

$$m[1,4] = \begin{cases} \text{for } k=1 \longrightarrow m[1,1]+m[2,4]+(3\times 8\times 6) = 0+432+144 = 576 \\ \text{for } k=2 \longrightarrow m[1,2]+m[3,4]+(3\times 4\times 6) = 96+240+72 = 408 \\ \text{for } k=3 \longrightarrow m[1,3]+m[4,4]+(3\times 10\times 6) = 216+0+180 = \color{red}{396} \end{cases}$$

Therefore, once we fill out the tables,

m	1	2	3	4
1	0	96	216	396
2		0	320	432
3			0	240
4				0

s	1	2	3	4
1	0	1	2	3
2		0	2	2
3			0	3
4				0

we can use the values stored in s to find out that the optimal parenthesization of the matrix chain product sequence in question is:

$$\boldsymbol{(((A_1 A_2) A_3) A_4)}$$

Therefore, the number of multiplications performed using this optimal parenthesization is:

$$(3\times 8\times 4)+(3\times 4\times 10)+(3\times 10\times 6) = \mathbf{396}$$

2. Determine an optimal LCS of $< 1, 1, 0, 0, 1, 0, 1, 0 >$ and $< 1, 0, 1, 0, 1, 1, 0, 0 >$

To answer this question, we will use the following LCS table:

–	–	**1**	**1**	**0**	**0**	**1**	**0**	**1**	**0**
–	0	**0**	0	0	0	0	0	0	0
1	0	1	**1**	**1**	1	1	1	1	1
0	0	1	1	2	**2**	2	2	2	2
1	0	1	2	2	2	**3**	3	3	3
0	0	1	2	3	3	3	**4**	4	4
1	0	1	2	3	3	4	**4**	5	5
1	0	1	2	3	3	4	4	**5**	5
0	0	1	2	3	4	4	5	**5**	6
0	0	1	2	3	4	4	5	5	**6**

Therefore, an optimal longest common subsequence is:

$$\boldsymbol{< 1, 0, 1, 0, 1, 0 >}$$