PROBABILITY REVIEW

Algorithms Analysis

github.com/erngv

1. Suppose you have a fair 6-sided die with the numbers 1 through 6 on the sides and a fair 5-sided die with the numbers 1 through 5 on the sides. What is the probability that a roll of the six-sided die will produce a value larger than the roll of the five-sided die?

   Rolling a fair six-sided die and then a fair five-sided die can only produce the following outcomes on the form (x, y):

$(1,1)$	**(2,1)**	**(3,1)**	**(4,1)**	**(5,1)**	**(6,1)**
$(1,2)$	$(2,2)$	**(3,2)**	**(4,2)**	**(5,2)**	**(6,2)**
$(1,3)$	$(2,3)$	$(3,3)$	**(4,3)**	**(5,3)**	**(6,3)**
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	**(5,4)**	**(6,4)**
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	**(6,5)**

   where x represents the outcome of the fair six-sided die and y represents the outcome of the fair five-sided die. There is a total of 5×6 possible cases; however, only in 15 of those cases $x > y$. Therefore, the probability that a roll of the six-sided die will produce a value larger than the roll of the five-sided die (let us say event E) is:

$$P(E) = \frac{\text{number of outcomes}}{\text{total number of outcomes in sample space}} = \frac{15}{30} = \frac{1}{2} = \mathbf{0.5}$$

2. What is the expected number of rolls until a fair five-sided die rolls a 3? Justify your answer briefly.

In this problem, we are dealing with a **geometric random variable**, since we are counting the number of experiments (in this case the number of rolls) necessary to obtain the first success (in this case the value 3). Therefore,

$$P(X = n) = (1 - p)^{n-1} p$$

p is the probability of getting a 3 when rolling a fair five-sided die, so $p = \dfrac{1}{5}$

Since $E(X) = M'(0)$, let us apply the moment generating function,

$$\begin{aligned}
M(t) &= E\left(e^{tX}\right) = \sum_{k=1}^{\infty} e^{tk} P(X = k) \\
M(t) &= \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} p = pe^t \sum_{k=1}^{\infty} e^{t(k-1)} (1-p)^{k-1} = pe^t \sum_{m=1}^{\infty} \left[(1-p)e^t\right]^m \\
M(t) &= \frac{pe^t}{1 - (1-p)e^t} \\
M'(t) &= \frac{pe^t\left[1 - (1-p)e^t\right] - pe^t(1-p)e^t}{\left[1 - (1-p)e^t\right]^2} = \frac{pe^t}{\left[1 - (1-p)e^t\right]^2} \\
E(X) &= M'(0) = \frac{p}{p^2} = \frac{1}{p} = \frac{1}{\frac{1}{5}} = 5
\end{aligned}$$

Therefore, the expected number of rolls is **5**.