# Turtle Magic: Royal Turtle Shell Pattern

We claim that there are only 8 configurations that satisfy the condition. The proof is as follows.

Firstly, consider a $2 \times 2$ subgrid that does not lie on any of the grid's corners.

*Claim 1.* Within the $2 \times 2$ subgrid, there must be 2 Os and 2 Xs.

*Proof of Claim 1.* Assume the contrary that there are 3 Os or 4 Os. (The case with 3 Xs or 4 Xs is done too due to symmetry.)

Then, after some logical deduction we will end up with 3 consecutive Xs, as illustrated in the following figure.
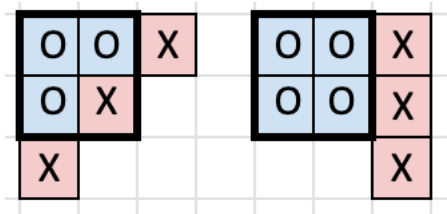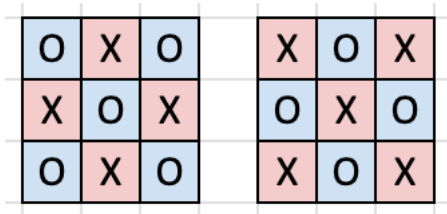


The condition is not satisfied, hence there is a contradiction and there must be 2 Os and 2 Xs within a $2 \times 2$ subgrid that does not lie on any of the grid's corners.

Next, consider a $3 \times 3$ subgrid that does not lie on any of the grid's corners.

*Claim 2.* Within the $3 \times 3$ subgrid, at least one $2 \times 2$ sub-subgrid is one of the four patterns below. We will call the following $2 \times 2$ patterns **good patterns**.
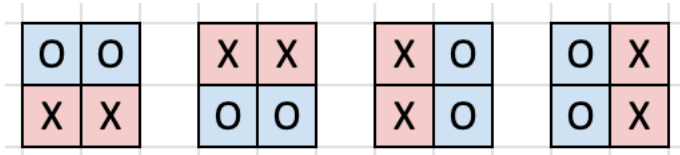


*Proof of Claim 2.* Assume the contrary that none of the 4 sub-subgrids in the $3 \times 3$ subgrid are one of the four given patterns. This naturally means every two cells which share an edge have different shapes. This gives the following two patterns:



The diagonals have the same shape, so the condition is not satisfied. There is a contradiction, hence the claim is true.

Next, consider a **good** $2 \times 2$ subgrid that does not lie on any of the grid's corners. (We have proved its existence in Claim 2.)

Finally, it is possible to uniquely extend a good subgrid to the rest of the grid. For example, you can see in the illustration below, after a few unique logical deductions the grey pattern on the top can tesselate itself a few times, and this can be infinitely repeated.
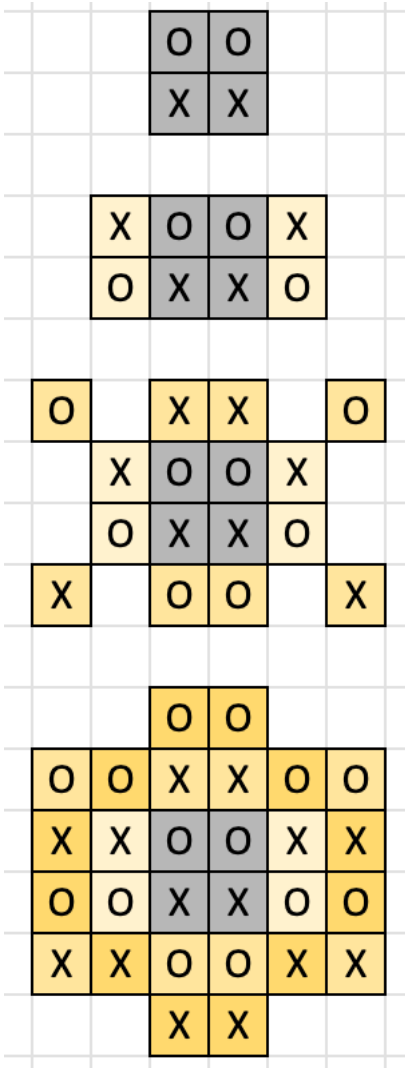
Therefore, the entire grid must be tesselations of one of the four good patterns. The first two patterns may be shifted one column to the right, and the last two patterns may be shifted one column downwards. So there are a total of $4 \times 2 = 8$ ways to satisfy the condition.

Specifically, the 8 ways for $n = 5, m = 5$ are as follows:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| X | X | O | O | X | | X | O | O | X | X |
| O | O | X | X | O | | O | X | X | O | O |
| X | X | O | O | X | | X | O | O | X | X |
| O | O | X | X | O | | O | X | X | O | O |
| X | X | O | O | X | | X | O | O | X | X |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| X | O | X | O | X | | X | O | X | O | X |
| X | O | X | O | X | | O | X | O | X | O |
| O | X | O | X | O | | O | X | O | X | O |
| O | X | O | X | O | | X | O | X | O | X |
| X | O | X | O | X | | X | O | X | O | X |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| O | O | X | X | O | | O | X | X | O | O |
| X | X | O | O | X | | X | O | O | X | X |
| O | O | X | X | O | | O | X | X | O | O |
| X | X | O | O | X | | X | O | O | X | X |
| O | O | X | X | O | | O | X | X | O | O |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| O | X | O | X | O | | O | X | O | X | O |
| O | X | O | X | O | | X | O | X | O | X |
| X | O | X | O | X | | X | O | X | O | X |
| X | O | X | O | X | | O | X | O | X | O |
| O | X | O | X | O | | O | X | O | X | O |

As for some implementation details, the following 8 statements each correspond to one $n \times m$ configuration that satisfies the condition. $a_{i,j}$ represents whether the cell on the $i$-th row and the $j$-th column $(1 \le i \le n, 1 \le j \le m)$ has a circle-shaped fortune cookie.

1. $a_{i,j} = 1$ iff $i + \left\lceil \frac{j}{2} \right\rceil$ is odd.

2. $a_{i,j} = 1$ iff $i + \left\lceil \frac{j}{2} \right\rceil$ is even.

3. $a_{i,j} = 1$ iff $i + \left\lfloor \frac{j}{2} \right\rfloor$ is odd.

4. $a_{i,j} = 1$ iff $i + \left\lfloor \frac{j}{2} \right\rfloor$ is even.

5. $a_{i,j} = 1$ iff $j + \left\lceil \frac{i}{2} \right\rceil$ is odd.

6. $a_{i,j} = 1$ iff $j + \left\lceil \frac{i}{2} \right\rceil$ is even.

7. $a_{i,j} = 1$ iff $j + \left\lfloor \frac{i}{2} \right\rfloor$ is odd.

8. $a_{i,j} = 1$ iff $j + \left\lfloor \frac{i}{2} \right\rfloor$ is even.

**Discussion:** Try to solve the problem for general $n, m$. Specifically that includes the cases when $1 \le \min(n, m) \le 4$.