1 Types

$$\alpha, \beta ::= \tau \mid \alpha \to \beta$$

1.1 Typing judgement

$$\frac{x \in \Gamma}{\Gamma \vdash x : \Gamma x} \qquad \frac{\Gamma \vdash M : \alpha \to \beta \qquad \Gamma \vdash N : \alpha}{\Gamma \vdash MN : \beta} \qquad \frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \to \beta}$$

We define the inclusion between contexts $\Gamma \subseteq \Delta$ as $\forall x \in \Gamma, x \in \Delta \wedge \Gamma x = \Delta x$. Next, we present a weakening lemma of the typing judgement. Both lemmas are proved by structural induction on the typing relation, the second lemma uses for the first one for the abstraction case.

Proposition 1 (Weakening Lemma).

1. *$\Gamma \subseteq \Delta, \Gamma \vdash M : \alpha \Rightarrow \Delta \vdash M : \alpha$*
2. *$x \# M, \Gamma \vdash M : \alpha \Rightarrow \Gamma, x : \beta \vdash M : \alpha$*

We define a substitution σ goes from Γ to Δ contexts, denoted as $\sigma : \Gamma \rightharpoonup \Delta$, iff $\forall z \in \Gamma, \Delta \vdash \sigma z : \Gamma z$. We also define the restriction of this concept to some term, denoted as $\sigma : \Gamma \rightharpoonup \Delta \downharpoonleft M$ iff $\forall z * M, z \in \Gamma, \Delta \vdash \sigma z : \Gamma z$. Some usefull properties are:

Proposition 2.

1. *$\iota : \Gamma \rightharpoonup \Gamma$*
2. *$\Gamma \vdash M : \alpha \Rightarrow (\iota, x{:=}M) : \Gamma, x : \alpha \rightharpoonup \Gamma$*
3. *$\sigma : \Gamma \rightharpoonup \Delta \Rightarrow \sigma : \Gamma \rightharpoonup \Delta \downharpoonleft M$*
4. *$(\iota, y{:=}x) : \Gamma, y : \alpha \rightharpoonup \Gamma, x : \alpha \downharpoonleft M(\iota, x{:=}y)$*
5. *$x \# \sigma \downharpoonleft \lambda y M, \sigma : \Gamma \rightharpoonup \Delta \downharpoonleft \lambda y M \Rightarrow (\sigma, y{:=}x) : \Gamma, y : \alpha \rightharpoonup \Delta, x : \alpha \downharpoonleft M$*

Proof. All results are direct except from last one that we discuss next. For all z free in M and decared in context $\Gamma, y : \alpha$, we have to prove that $\Delta, x : \alpha \vdash (\sigma, y{:=}x)z : (\Gamma, y : \alpha)z$. When $y = z$ the proof is inmediate, when not the goal becomes $\Delta, x : \alpha \vdash \sigma z : \Gamma z$. As $y \neq x$ and $z * M$ then $z * \lambda y M$, and we can use the $\sigma : \Gamma \rightharpoonup \Delta \downharpoonleft \lambda y M$ hypotheses with z variable to get $\Delta \vdash \sigma z : \Gamma z$. Using that $z * \lambda y M$ and the $x \# \sigma \downharpoonleft \lambda y M$ premise, we know that $x \# \sigma z$, then we can apply the second weakening lemma to $\Delta \vdash \sigma z : \Gamma z$ and obtain our goal. $\square$

We can now prove the following substitution lemma for the typing judge.

Proposition 3 (Substitution Lemma for Type System). *$\Gamma \vdash M : \alpha, \sigma : \Gamma \rightharpoonup \Delta \downharpoonleft M \Rightarrow \Delta \vdash M\sigma : \alpha$*

Proof. The proof is by induction in the typing judge. The variable and application cases are direct. For the abstraction case we have to prove $\Delta \vdash (\lambda y.M)\sigma : \alpha \to \beta$, where $(\lambda y.M)\sigma$ is equal to $\lambda x.(M(\sigma, y{:=}x))$, and $x = \chi(\sigma, \lambda y.M)$. By the syntax directed definition of the typing judge, using abstraction case, we derive that $\Gamma, y : \alpha \vdash M : \beta$. We can use the last part of previous proposition to derive that $(\sigma, y{:=}x) : \Gamma, y : \alpha \rightharpoonup \Delta, x : \alpha \downharpoonleft M$. Then, applying the inductive hypothesis to previous result we have that $\Delta, x : \alpha \vdash M(\sigma, y{:=}x) : \beta$. Next by abstraction rule of type system we have that $\Delta \vdash \lambda x.(M(\sigma, y{:=}x)) : \alpha \to \beta$, which by substitution definition is desired result. $\square$

Corollary 1 (Typing judge is preserved by β-contraction). *$\Gamma \vdash M : \alpha, M \triangleright N \Rightarrow \Gamma \vdash N : \alpha$*

Proof. We have to prove that $\Gamma \vdash M(\iota, x{:=}N) : \alpha$. By hypotheses $\Gamma \vdash (\lambda x M) : \alpha$, then by typing judge $\Gamma, x : \alpha \vdash M : \beta$ and $\Gamma \vdash N : \alpha$. Using second part and third parts of proposition 2 to last typing judge we have that $(\iota, x{:=}M) : \Gamma, x : \alpha \rightharpoonup \Gamma \downharpoonleft M$, then applying previous subsitution lemma to $(\iota, x{:=}M)$ substitution and $\Gamma, x : \alpha \vdash M : \beta$ we finish the proof.

$\square$

Corollary 2. *$\Gamma \vdash M : \alpha, M \to_\beta N \Rightarrow \Gamma \vdash N : \alpha$*

Proof. Direct induction on the contextual clousure relation. For the β-contraction we directly use last result. $\square$

Proposition 4. *$\Gamma \vdash M\iota : \alpha, \Rightarrow \Gamma \vdash M : \alpha$*

Proof. By induction on the typing relation. Variable and application cases are direct. Next we detail the abstraction case we have that $\Gamma, y : \alpha \vdash M(\iota, x{:=}y) : \beta$, with $y = \chi(\iota, \lambda x M)$. Using the subtitution lemma to previous judge, and with $(\iota, y{:=}x)$ substitution, by part fourth of prop. 2 we get that $\Gamma, x : \alpha \vdash (M(\iota, x{:=}y))(\iota, y{:=}x) : \beta$, but $(M(\iota, x{:=}y))(\iota, y{:=}x) \equiv M\iota$ because $y \# \iota \downharpoonleft \lambda x M$. Then by inductive hypothesis and abstraction rule of typing system we end the proof. $\square$

Proposition 5. *$\Gamma \vdash M : \alpha, M \sim_\alpha N \Rightarrow \Gamma \vdash N : \alpha$*

Proof. As $M \sim_\alpha N$ then $M\iota \equiv N\iota$. Using previous prposition we just need to proof that $\Gamma \vdash M\iota : \alpha$, which is direct by the substitution lemma and first part of proposition 3 . $\square$

Subject redution is proved by induction on the reflexive-transitive closure of $\to_\alpha(\to_\beta \cup \sim_\alpha)$ relation, using directly the two last results for the $\to_\alpha$ and $\sim_\alpha$ cases respectively.

Theorem 1 (Subject Reduction). *$\Gamma \vdash M : \alpha, M \to_\beta^* N \Rightarrow \Gamma \vdash N : \alpha$*