1 Parallel Reduction

$$\text{v}\ \frac{}{x \rightrightarrows x} \qquad \text{a}\ \frac{M \rightrightarrows M' \qquad N \rightrightarrows N'}{MN \rightrightarrows NN'}$$

$$\lambda\ \frac{M[x/z] \rightrightarrows M'[y/z]}{\lambda xM \rightrightarrows \lambda yM'}\ z\#\lambda xM, \lambda yM' \qquad \beta\ \frac{M \rightrightarrows M' \qquad N \rightrightarrows N'}{(\lambda xM)N \rightrightarrows M'[x/N']}$$

Principal ideas:

1. λ rule subsumes α rule.
2. Syntax directed, except for the superposition between application and β rules, in the case that the left subterm of an application is an abstraction, which is needed to express the choice to reduce or not a redex.

Because we modify the parallel reduction relation we have to prove again the substitution lemma for parallel reduction holds.

Lemma 1. $M \rightrightarrows M', \sigma \rightrightarrows \sigma' \Rightarrow M\sigma \rightrightarrows M'\sigma'$

Proof. Induction on $\rightrightarrows$ relation.

λ case: Hypotheses: $\lambda\ \dfrac{M[x/z] \rightrightarrows M'[y/z]}{\lambda xM \rightrightarrows \lambda yM'}\ z\#\lambda xM, \lambda yM' \quad , \quad \sigma \rightrightarrows \sigma'$

Tesis: $(\lambda xM)\sigma \rightrightarrows (\lambda yM')\sigma'$

Proof:

λxM	$\rightrightarrows$	$\lambda yM'$	hypothesis
$M[x/z]$	$\rightrightarrows$	$M'[y/z]$	$\rightrightarrows$ λ rule $\exists z/z\#\lambda xM, \lambda yM'$
$(M[x/z])\sigma[z/z]$	$\rightrightarrows$	$(M'[y/z])\sigma'[z/z]$	inductive hypothesis with $\sigma = \sigma[z/z], \sigma' = \sigma'[z/z]$
$M(\sigma[z/z] \circ [x/z])$	$\rightrightarrows$	$M'(\sigma'[z/z] \circ [y/z])$	substitution composition
$M\sigma[x/z]$	$\rightrightarrows$	$M'\sigma'[y/z]$	lemmas and $u = \chi(\sigma, \lambda xM), w = \chi(\sigma', \lambda yM')$
$M([u/z] \circ \sigma[x/u])$	$\rightrightarrows$	$M'([w/z] \circ \sigma'[y/w])$	lemmas
$(M\sigma[x/u])[u/z]$	$\rightrightarrows$	$(M'\sigma'[y/w])[w/z]$	substitution composition
$\lambda uM\sigma[x/u]$	$\rightrightarrows$	$\lambda wM'\sigma'[y/w]$	$\rightrightarrows$ λ rule
$(\lambda xM)\sigma$	$\rightrightarrows$	$(\lambda yM')\sigma'$	substitution definition

□

β case: Hypotheses: $\beta\ \dfrac{M \rightrightarrows M' \qquad N \rightrightarrows N'}{(\lambda xM)N \rightrightarrows M'[x/N']} \quad , \quad \sigma \rightrightarrows \sigma'$

Tesis: $((\lambda xM)N)\sigma \rightrightarrows (M'[x/N'])\sigma'$

Proof:

$(\lambda xM)N$	$\rightrightarrows$	$M'[x/N']$	hypothesis
$M \rightrightarrows M'$	$\wedge$	$N \rightrightarrows N'$	β rule
$M\sigma[x/y] \rightrightarrows M'\sigma'[x/y]$	$\wedge$	$N\sigma \rightrightarrows N'\sigma'$	inductive hypothesis, with $y = \chi(\sigma, \lambda xM)$
$(\lambda yM\sigma[x/y])(N\sigma)$	$\rightrightarrows$	$(M'\sigma'[x/y])[y/N'\sigma']$	β rule
$(\lambda yM\sigma[x/y])(N\sigma)$	$\rightrightarrows$	$M'\sigma'[x/N'\sigma']$	auxiliary lemmas
$((\lambda xM)N)\sigma$	$\rightrightarrows$	$(M'[x/N'])\sigma'$	substitution definition and auxiliar lemma

$\square$

$\square$

Now we prove the needed lemma. Now is needed indeed ?

Lemma 2. $M \sim N, N \rightrightarrows P, \Rightarrow M \rightrightarrows P$

Proof. Induction on $\rightrightarrows$ relation.

λ case: Hypotheses: $\lambda\,\dfrac{M[w/u] \sim N[x/u]}{\lambda wM \sim \lambda xN}\; u\#\lambda wM, \lambda xN \qquad \lambda\,\dfrac{N[x/z] \rightrightarrows P[y/z]}{\lambda xN \rightrightarrows \lambda yP}\; z\#\lambda xN, \lambda yP$

Tesis: $\lambda wM \rightrightarrows \lambda yP$

Proof:

$$\begin{array}{rcll} M[w/u] & \sim & N[x/u] & \text{hypothesis} \\ M[w/u][u/z] & \sim & N[x/u][u/z] & \text{substitution lemma of alpha equivalence relation} \\ M[w/z] & \sim & N[x/z] & \text{auxiliar lemma and } u\#\lambda wM, \lambda xN \end{array}$$

So $M[w/z] \sim N[x/z]$ and $N[x/z] \rightrightarrows P[y/z]$ then by inductive hypothesis $M[w/z] \rightrightarrows P[y/z]$ with $z\#\lambda xN, \lambda yP$. As $\lambda wM \sim \lambda xN$ then $z\#\lambda wM, \lambda yP$.

Finally, applying lambda rule of parallel reduction relation we have the desired result.

$$\lambda\,\dfrac{M[w/z] \rightrightarrows P[y/z]}{\lambda wM \rightrightarrows \lambda yP}\; z\#\lambda wM, \lambda yP$$

$\square$

β case: Hypotheses: $$\text{a}\,\dfrac{\lambda\,\dfrac{M[y/z] \sim N[x/z]}{\lambda yM \sim \lambda xN}\; z\#\lambda yM, \lambda xN \qquad M' \sim N'}{(\lambda yM)M' \sim (\lambda xN)N'}$$

. $$\beta\,\dfrac{N \rightrightarrows P \qquad N' \rightrightarrows P'}{(\lambda xN)N' \rightrightarrows P[x/P']}$$

Tesis: $(\lambda yM)M' \rightrightarrows P[x/P']$

Proof:

$$\text{inductive hypothesis}\;\dfrac{M' \sim N' \qquad N' \rightrightarrows P'}{M' \rightrightarrows P'}$$

$$\text{subst. lemma}\;\dfrac{\text{inductive hypothesis}\;\dfrac{\dfrac{\text{subst.lemma}\;\dfrac{M[y/z] \sim N[x/z]}{M[y/z][z/x] \sim N[x/z][z/x]}}{M[y/x] \sim N} \qquad N \rightrightarrows P}{M[y/x] \rightrightarrows P}}{\dfrac{M[y/x][x/y] \rightrightarrows P[x/y]}{M[y/y] \rightrightarrows P[x/y]}}$$

$\square$