1 Parallel Reduction

$$\text{v}\ \frac{}{x \rightrightarrows x} \qquad \text{a}\ \frac{M \rightrightarrows M' \qquad N \rightrightarrows N'}{MN \rightrightarrows NN'}$$

$$\lambda\ \frac{M[x/z] \rightrightarrows M'[y/z]}{\lambda x M \rightrightarrows \lambda y M'}\ z\#\lambda xM, \lambda yM' \qquad \beta\ \frac{M[x/y] \rightrightarrows M' \qquad N \rightrightarrows N'}{(\lambda xM)N \rightrightarrows M'[y/N']}\ y\#\lambda xM$$

$$\alpha\ \frac{M \rightrightarrows N \qquad N \sim P}{M \rightrightarrows P}$$

Principal ideas:

1. We have to leave α rule because the substitution operation in the conclusion of the β rule normilises the right term.

Lemma 1. $\lambda xM \sim \lambda yN \Rightarrow M[x/z] \equiv N[y/z]$

Proof. $\lambda\ \dfrac{M[x/w] \sim N[y/w]}{\lambda xM \sim \lambda yN}\ w\#\lambda xM, \lambda yN$

$$\begin{array}{rcl} M[x/z] & \equiv & \{\text{as } w\#\lambda xM\} \\ (M[x/w])[w/z] & \equiv & \{\text{substitution lemma of} \sim \text{relation applied to } M[x/w] \sim N[y/w]\} \\ (N[y/w])[w/z] & \equiv & \{\text{as } w\#\lambda yN\} \\ N[y/z] & & \end{array}$$

$\square$

Lemma 2. $M \sim N, N \rightrightarrows P \Rightarrow M \rightrightarrows P$

Proof. Induction on $\rightrightarrows$ relation.

λ case: <u>Hypotheses:</u> $\lambda wM \sim \lambda xN$ $\qquad \lambda\ \dfrac{N[x/z] \rightrightarrows P[y/z]}{\lambda xN \rightrightarrows \lambda yP}\ z\#\lambda xN, \lambda yP$

<u>Tesis:</u> $\lambda wM \rightrightarrows \lambda yP$

<u>Proof</u>:

$$\dfrac{\text{congruence}\ \dfrac{\text{lemma 1}\ \dfrac{\lambda xN \sim \lambda xN}{M[w/z] \equiv N[x/z]} \qquad N[x/z] \rightrightarrows P[y/z]}{M[w/z] \rightrightarrows P[y/z]}}{\lambda wM \rightrightarrows \lambda yP}\ z\#\lambda wM(*), \lambda yP$$

$$(*) \quad \left.\begin{array}{c} z\#\lambda xN \\ \lambda wM \sim \lambda xN \end{array}\right\} \Rightarrow z\#\lambda wM.$$

β case: <u>Hypotheses:</u> $\quad \text{a}\ \dfrac{\lambda yM \sim \lambda xN \qquad M' \sim N'}{(\lambda yM)M' \sim (\lambda xN)N'}$

. $\quad \beta\ \dfrac{N[x/w] \rightrightarrows P \qquad N' \rightrightarrows P'}{(\lambda xN)N' \rightrightarrows P[w/P']}\ w\#\lambda xN$

Tesis: $(\lambda y M)M' \rightrightarrows P[w/P']$

Proof:

$$\beta\,\dfrac{\text{congruence}\,\dfrac{\text{lemma 1}\,\dfrac{\lambda y M \sim \lambda x N}{M[y/w] \equiv N[x/w]} \qquad N[x/w] \rightrightarrows P}{M[y/w] \rightrightarrows P} \qquad \text{ind. hyp.}\,\dfrac{M' \sim N' \qquad N' \rightrightarrows P'}{M' \rightrightarrows P'}}{(\lambda y M)M' \rightrightarrows P[w/P']}\; w\#\lambda y M(*)$$

□

(*) $\left.\begin{array}{c} w\#\lambda x N \\ \lambda y M \sim \lambda x N \end{array}\right\} \Rightarrow w\#\lambda y M.$

α case: Hypotheses: $M \sim N$ $\qquad \alpha\,\dfrac{N \rightrightarrows P' \qquad P' \sim P}{N \rightrightarrows P}$

Tesis: $M \rightrightarrows P$

Proof:

$$\alpha\,\dfrac{\text{ind. hypothesis}\,\dfrac{M \sim N \qquad N \rightrightarrows P'}{M \rightrightarrows P'} \qquad P' \sim P}{M \rightrightarrows P}$$

□

Lemma 3. $\left.\begin{array}{l} y\#\lambda x M \\ z \,\#\!\!\downarrow (\sigma, \lambda x M) \end{array}\right\} \Rightarrow (M\sigma \prec\!\!+ (x,z))[z/N] \sim (M[x/y])\sigma \prec\!\!+ (y,N)$

Proof.

$$\begin{array}{rcl} M\sigma \prec\!\!+ (x,z))[z/N] & \sim & \{ \text{ as } z \,\#\!\!\downarrow (\sigma, \lambda x M) \text{ by corollary 1 substitution lemma } \} \\ M\sigma \prec\!\!+ (x,N) & \equiv & \{ \text{ as } y\#\lambda x M \text{ by lemma prec } \} \\ (M[x/y])\sigma \prec\!\!+ (y,N) & & \end{array}$$

□

Lemma 4. $M \rightrightarrows M', \sigma \rightrightarrows \sigma' \Rightarrow M\sigma \rightrightarrows M'\sigma'$

Proof. Induction on $\rightrightarrows$ relation.

λ case: Hypotheses: $\lambda\,\dfrac{M[x/z] \rightrightarrows M'[y/z]}{\lambda x M \rightrightarrows \lambda y M'}\, z\#\lambda x M, \lambda y M' \qquad , \quad \sigma \rightrightarrows \sigma'$

Tesis: $(\lambda x M)\sigma \rightrightarrows (\lambda y M')\sigma'$

Proof: Be $t = \chi((\lambda x M)\sigma(\lambda y M')\sigma')$, then $t\#(\lambda x M)\sigma, (\lambda y M')\sigma'$. And $u = \chi(\sigma, \lambda x M), w = \chi(\sigma', \lambda y M')$.

$$\text{ind. hyp.}\,\dfrac{M[x/z] \rightrightarrows M'[y/z] \qquad \dfrac{\sigma \rightrightarrows \sigma'}{\sigma \prec\!\!+ (z,t) \rightrightarrows \sigma' \prec\!\!+ (z,t)}}{(M[x/z])\sigma \prec\!\!+ (z,t) \rightrightarrows (M'[y/z])\sigma \prec\!\!+ (z,t)}$$

$$
\text{lem. 2}\;\dfrac{\text{lem. 3}\;\dfrac{z\#\lambda xM \qquad u\,\#\!\downarrow(\sigma,\lambda xM)}{(M\sigma \prec\!\!+ (x,u))[u/t] \sim (M[x/z])\sigma \prec\!\!+ (z,t)} \qquad (M[x/z])\sigma \prec\!\!+ (z,t) \rightrightarrows (M'[y/z])\sigma \prec\!\!+ (z,t)}{(M\sigma \prec\!\!+ (x,u))[u/t] \rightrightarrows (M'[y/z])\sigma \prec\!\!+ (z,t)}
$$

$$
\lambda\;\dfrac{\alpha\;\dfrac{(M\sigma \prec\!\!+ (x,u))[u/t] \rightrightarrows (M'[y/z])\sigma \prec\!\!+ (z,t) \qquad \text{lem. 3}\;\dfrac{z\#\lambda xM \qquad w\,\#\!\downarrow(\sigma',\lambda yM')}{(M'[y/z])\sigma \prec\!\!+ (z,t) \sim (M'\sigma \prec\!\!+ (y,w))[w/t]}}{(M\sigma \prec\!\!+ (x,u))[u/t] \rightrightarrows (M'\sigma \prec\!\!+ (y,w))[w/t]}}{\underbrace{\lambda u(M\sigma \prec\!\!+ (x,u))}_{=(\lambda xM)\sigma} \rightrightarrows \underbrace{\lambda w(M'\sigma \prec\!\!+ (y,w))}_{=(\lambda yM')\sigma'}}\; t\#(\lambda xM)\sigma,(\lambda yM')\sigma'
$$

$\square$

β case: Hypotheses: $\beta\;\dfrac{M[x/z] \rightrightarrows M' \qquad N \rightrightarrows N'}{(\lambda xM)N \rightrightarrows M'[z/N']}\; z\#\lambda xM \quad , \quad \sigma \rightrightarrows \sigma'$

Tesis: $((\lambda xM)N)\sigma \rightrightarrows (M'[z/N'])\sigma'$

Proof: Be $w = \chi((\lambda xM)\sigma(\lambda zM')\sigma')$ then $w\#(\lambda xM)\sigma,(\lambda zM')\sigma'$. And $y = \chi(\sigma,\lambda xM)$.

$$
\text{ind. hyp.}\;\dfrac{M[x/z] \rightrightarrows M' \qquad \dfrac{\sigma \rightrightarrows \sigma'}{\sigma \prec\!\!+ (z,w) \rightrightarrows \sigma' \prec\!\!+ (z,w)}}{(M[x/z])\sigma \prec\!\!+ (z,w) \rightrightarrows M'\sigma \prec\!\!+ (z,w)}
$$

$$
\text{lem. 2}\;\dfrac{\text{lem. 3}\;\dfrac{z\#\lambda xM \qquad y\,\#\!\downarrow(\sigma,\lambda xM)}{(M\sigma \prec\!\!+ (x,y))[y/w] \sim (M[x/z])\sigma \prec\!\!+ (z,w)} \qquad (M[x/z])\sigma \prec\!\!+ (z,w) \rightrightarrows M'\sigma \prec\!\!+ (z,w)}{(M\sigma \prec\!\!+ (x,y))[y/w] \rightrightarrows M'\sigma \prec\!\!+ (z,w)}
$$

$$
\beta\;\dfrac{(M\sigma \prec\!\!+ (x,y))[y/w] \rightrightarrows M'\sigma \prec\!\!+ (z,w) \qquad \text{ind. hyp.}\;\dfrac{N \rightrightarrows N' \qquad \sigma \rightrightarrows \sigma'}{N\sigma \rightrightarrows N'\sigma'}}{(\lambda y(M\sigma \prec\!\!+ (x,y)))(N\sigma) \rightrightarrows (M'\sigma \prec\!\!+ (z,w))[w/N'\sigma']}
$$

$$
\begin{array}{rcl}
(M'\sigma' \prec\!\!+ (z,w))[w/N'\sigma'] & \sim & \text{corollary subst. lem} \\
M'\sigma' \prec\!\!+ (z,N'\sigma') & \equiv & \text{corollary prop. 7} \\
(M'[z/N'])\sigma' & &
\end{array}
$$

$$
\alpha\;\dfrac{(\lambda y(M\sigma \prec\!\!+ (x,y)))(N\sigma) \rightrightarrows (M'\sigma \prec\!\!+ (z,w))[w/N'\sigma'] \qquad (M'\sigma' \prec\!\!+ (z,w))[w/N'\sigma'] \sim (M'[z/N'])\sigma'}{\underbrace{(\lambda y(M\sigma \prec\!\!+ (x,y)))(N\sigma)}_{=((\lambda xM)N)\sigma} \rightrightarrows (M'[z/N'])\sigma'}
$$

$\square$

α case: Hypotheses: $\alpha\;\dfrac{M \rightrightarrows N \qquad N \sim P}{M \rightrightarrows P} \qquad \sigma \rightrightarrows \sigma'$

Tesis: $M\sigma \rightrightarrows P\sigma'$

Proof:

$$
\alpha\;\dfrac{\text{ind. hypotesis}\;\dfrac{M \rightrightarrows N}{M\sigma \rightrightarrows N\sigma'} \qquad \text{subst. lemma}\;\dfrac{N \sim P}{N\sigma' \sim P\sigma'}}{M\sigma \rightrightarrows P\sigma'}
$$

$\square$

1.1 Reduce all redex function (*)

$$\begin{array}{llcl} x^* & & = & x \\ (\lambda x M)^* & & = & \lambda x M^* \\ (x & M)^* & = & x M^* \\ ((\lambda x M) & N)^* & = & M^*[x/N^*] \\ ((MN) & P)^* & = & (MN)^* P^* \end{array}$$

Lemma 5. *σ renaming* $\Rightarrow (M\sigma)^* \sim M^*\sigma$

Proof. Structural induction on terms.

variable case: Hypotheses: σ renaming Tesis: $(x\sigma)^* \sim x^*\sigma$

Proof:

$$(x\sigma)^* = (\sigma x)^* \stackrel{\sigma \text{ renaming}}{=} \sigma x = x\sigma = x^*\sigma \sim x^*\sigma$$

λ case: Hypotheses: σ renaming Tesis: $((\lambda x M)\sigma)^* \sim (\lambda x M)^*\sigma$

Proof:

$$\begin{array}{lcl} ((\lambda x M)\sigma)^* & = & \text{substitution definition, where } y = \chi(\sigma, \lambda x M) \\ (\lambda y M\sigma[x/y])^* & = & * \text{ definition} \\ \lambda y M\sigma[x/y]^* & \sim & \text{inductive hypothesis with renaming } \sigma[x/y] \\ \lambda y M^*\sigma[x/y] & \sim & \text{stoughton corollary 4.2 with } y \#\!\downarrow (\sigma, \lambda x M^*)(*) \\ \lambda x M^*\sigma & = & * \text{ definition} \\ (\lambda x M)^*\sigma & & \end{array}$$

(*) $y = \chi(\sigma, \lambda x M) \Rightarrow y \#\!\downarrow (\sigma, \lambda x M)$ then by definition: $\forall z, z * \lambda x M \Rightarrow y \# \sigma z$. But if $z * \lambda x M \Rightarrow z * (\lambda x M)^*$ so we can conclude $y \#\!\downarrow (\sigma, \lambda x M^*)$

□

application case: Variable and application case ($M = x \vee M = MM'$).

Hypotheses: σ renaming Tesis: $((MN)\sigma)^* \sim (MN)^*\sigma$

Proof:

$$((MN)\sigma)^* = (M\sigma N\sigma)^* = M\sigma^* N\sigma^* \stackrel{\text{ind. hyp.}}{=} M^*\sigma N^*\sigma = (M^*N^*)\sigma = (MN)^*\sigma$$

application case: Abstraction case ($M = \lambda x M$).

Hypotheses: σ renaming Tesis: $(((\lambda x M)N)\sigma)^* \sim ((\lambda x M)N)^*\sigma$

Proof:

$$\begin{array}{llcl} (((\lambda x M) & N)\sigma)^* & = & \text{subst. definition, with } y = \chi(\sigma, \lambda x M) \\ ((\lambda y M\sigma[x/y]) & N\sigma)^* & = & * \text{ definition} \\ (M\sigma[x/y]^*)[y/N\sigma^*] & & \sim & \text{ind. hypotheses and subtitution lemma} \\ (M^*\sigma[x/y])[y/N^*\sigma] & & \sim & \text{subst. composition} \\ M^*\sigma[x/N^*\sigma] & & \sim & \text{auxiliar lemma} \\ (M^*[x/N^*])\sigma & & = & * \text{ definition} \\ ((\lambda x M) & N)^*\sigma & & \end{array}$$

$\square$

Lemma 6. $M \rightrightarrows N \Rightarrow N \rightrightarrows M^*$

Proof. Induction on $\rightrightarrows$ relation.

application case: Hypotheses: $\text{a}\,\dfrac{M \rightrightarrows M' \qquad N \rightrightarrows N'}{MN \rightrightarrows M'N'}$

Tesis: $M'N' \rightrightarrows (MN)^*$

Proof by cases in M:

- $M = \lambda x M$
- $M = x$
- $M = MP$

λ case: Hypotheses: $\lambda\,\dfrac{M[x/z] \rightrightarrows N[y/z]}{\lambda x M \rightrightarrows \lambda y N}\; z\#\lambda x M$

Tesis: $\lambda y N \rightrightarrows (\lambda x M)^* \equiv \lambda x M^*$

Proof:

$$\left.\begin{array}{l} z\#\lambda x M \\ \lambda x M \rightrightarrows \lambda y N \end{array}\right\} \Rightarrow z\#\lambda y N$$

$$\alpha\,\dfrac{\text{ind. hyp.}\,\dfrac{M[x/z] \rightrightarrows N[y/z]}{N[y/z] \rightrightarrows (M[x/z])^*} \qquad (M[x/z])^* \sim M^*[x/z]}{\lambda\,\dfrac{N[y/z] \rightrightarrows M^*[x/z]}{\lambda y N \rightrightarrows \lambda x M^*}\; z\#\lambda y N}$$

$\square$

β case: Hypotheses: $\beta\,\dfrac{M[x/y] \rightrightarrows M' \qquad N \rightrightarrows N'}{(\lambda x M)N \rightrightarrows M'[y/N']}\; y\#\lambda x M$

Tesis: $M'[y/N'] \rightrightarrows ((\lambda x M)N)^* \equiv M^*[x/N^*]$

Proof:

$y\#\lambda x M \Rightarrow y\#(\lambda x M)^* \equiv y\#\lambda x M^*$

$$\text{subst. lemma}\,\dfrac{\lambda\,\dfrac{\text{ind. hyp.}\,\dfrac{M[x/y] \rightrightarrows M'}{M' \rightrightarrows M[x/y]^*} \qquad M[x/y]^* \sim M^*[x/y]}{M' \rightrightarrows M^*[x/y]} \qquad \dfrac{\text{ind. hyp.}\,\dfrac{N \rightrightarrows N'}{N' \rightrightarrows N^*}}{[y/N'] \rightrightarrows [y/N^*]}}{\text{subst. composition}\,\dfrac{M'[y/N'] \rightrightarrows (M^*[x/y])[y/N^*]}{M'[y/N'] \rightrightarrows M^*[x/N^*]}\; y\#\lambda x M^*}$$

$\square$

α case: Hypotheses: $\alpha \dfrac{M \rightrightarrows N \qquad N \sim P}{M \rightrightarrows P}$

Tesis: $P \rightrightarrows M^*$

Proof:

$$\text{lemma 2}\ \dfrac{P \sim N \qquad \text{ind. hyp.}\ \dfrac{M \rightrightarrows N}{N \rightrightarrows M^*}}{P \rightrightarrows M^*}$$

$\square$