1 Reflexive and Transitve Clousure of a Relation

The reflexive and transitive closure of a relation is more naturally introduced by allowing more than one steps in the transitive clause. Some immediate facts about the transitive closure of an operation is that it is monotone and idempotent (besides these two properties one asks that the closure of R contains R, but this is immediate from the definition). From these two properties we prove that $R \subseteq S^*$ implies $R^* \subseteq S^*$ (alternatively one can proceed by induction on $a\, R^*\, b$); it is straightforward to prove the other

$$
\begin{array}{l}
\textbf{data}\ \mathit{star}\ (\rightsquigarrow\ :\ \mathit{Rel})\ :\ \mathit{Rel}\ \textbf{where} \\
\quad \mathit{refl}\ :\ \forall\ \{\,a\,\} \rightarrow \mathit{star} \rightsquigarrow a\ a \\
\quad \mathit{just}\ :\ \forall\ \{\,a\ b\,\} \rightarrow \rightsquigarrow a\ b \rightarrow \mathit{star} \rightsquigarrow a\ b \\
\quad \mathit{trans}\ :\ \forall\ \{\,a\ b\ c\,\} \rightarrow \mathit{star} \rightsquigarrow a\ b \rightarrow \mathit{star} \rightsquigarrow b\ c \rightarrow \mathit{star} \rightsquigarrow a\ c
\end{array}
$$

$$
\begin{array}{l}
\textbf{data}\ \mathit{equiv}\ (\rightsquigarrow\ :\ \mathit{Rel})\ :\ \mathit{Rel}\ \textbf{where} \\
\quad \equiv -\mathit{refl}\ :\ \forall\ \{\,a\,\} \rightarrow \mathit{equiv} \rightsquigarrow a\ a \\
\quad \equiv -\mathit{just}\ :\ \forall\ \{\,a\ b\,\} \rightarrow \rightsquigarrow a\ b \rightarrow \mathit{equiv} \rightsquigarrow a\ b \\
\quad \equiv -\mathit{sym}\ :\ \forall\ \{\,a\ b\,\} \rightarrow \mathit{equiv} \rightsquigarrow a\ b \rightarrow \mathit{equiv} \rightsquigarrow b\ a \\
\quad \equiv -\mathit{trans}\ :\ \forall\ \{\,a\ b\ c\,\} \rightarrow \mathit{equiv} \rightsquigarrow a\ b \rightarrow \mathit{equiv} \rightsquigarrow b\ c \rightarrow \mathit{equiv} \rightsquigarrow a\ c
\end{array}
$$

$$
\begin{array}{l}
\mathit{mon} - \mathit{star}\ :\ \{\,R\ S\ :\ \mathit{Rel}\,\} \rightarrow R \subseteq S \rightarrow \mathit{star}\ R \subseteq \mathit{star}\ S \\
\mathit{mon} - \mathit{star}\ R \subseteq S\ \mathit{refl}\ =\ \mathit{refl} \\
\mathit{mon} - \mathit{star}\ R \subseteq S\ (\mathit{just}\ aRb)\ =\ \mathit{just}\ (R \subseteq S\ aRb) \\
\mathit{mon} - \mathit{star}\ R \subseteq S\ (\mathit{trans}\ aR^*b\ bR^*c) \\
\quad =\ \mathit{trans}\ (\mathit{mon} - \mathit{star}\ R \subseteq S\ aR^*b)\ (\mathit{mon} - \mathit{star}\ R \subseteq S\ bR^*c)
\end{array}
$$

$$
\begin{array}{l}
\mathit{idem} - \mathit{star}\ :\ \{\,R\ :\ \mathit{Rel}\,\} \rightarrow \mathit{star}\ (\mathit{star}\ R) \subseteq \mathit{star}\ R \\
\mathit{idem} - \mathit{star}\ \mathit{refl}\ =\ \mathit{refl} \\
\mathit{idem} - \mathit{star}\ (\mathit{just}\ aRb)\ =\ aRb \\
\mathit{idem} - \mathit{star}\ (\mathit{trans}\ aR^*b\ bR^*c)\ =\ \mathit{trans}\ (\mathit{idem} - \mathit{star}\ aR^*b)\ (\mathit{idem} - \mathit{star}\ bR^*c)
\end{array}
$$

$$
\begin{array}{l}
\mathit{trans}- \subseteq -\mathit{star}\ :\ \{\,R\ S\ :\ \mathit{Rel}\,\} \rightarrow R \subseteq \mathit{star}\ S \rightarrow \mathit{star}\ R \subseteq \mathit{star}\ S \\
\mathit{trans}- \subseteq -\mathit{star}\ \{\,R\,\}\ \{\,S\,\}\ R \subseteq S^* \\
\quad =\ \mathit{trans}- \subseteq \{\,\mathit{star}\ R\,\}\ \{\,\mathit{star}\ (\mathit{star}\ S)\,\}\ \{\,\mathit{star}\ S\,\} \\
\qquad\qquad\qquad (\mathit{mon} - \mathit{star}\ R \subseteq S^*) \\
\qquad\qquad\qquad \mathit{idem} - \mathit{star}
\end{array}
$$

Our next goal is to prove that the diamond property implies Church-Rosser; for this it turns out that dealing with the usual definition of the reflexive transitive closure is not convenient, because the termination checker is not convinced about the use of the inductive hypothesis in one case. In order to bypass this obstacle we present another formalisation of the reflexive and transitive closure of a relation $R^\omega = \bigcup_{n\in\mathbb{N}} R^n$; although these two notions are not isomorphic (when passing from R^* to R^ω we lane all the single steps to the left), they are equivalent.

$$
\begin{array}{l}
\textbf{data}\ \mathit{steps}\ (\rightsquigarrow\ :\ \mathit{Rel})\ :\ \mathit{Rel}\ \textbf{where} \\
\quad \mathit{zero}\quad :\ \forall\ \{\,a\,\} \rightarrow \mathit{steps} \rightsquigarrow a\ a \\
\quad \mathit{one}\quad\ :\ \forall\ \{\,a\ b\,\}\quad \rightarrow \rightsquigarrow a\ b \rightarrow \mathit{steps} \rightsquigarrow a\ b \\
\quad \mathit{more}\ \ :\ \forall\ \{\,a\ b\ c\,\} \rightarrow \rightsquigarrow a\ b \rightarrow \mathit{steps} \rightsquigarrow b\ c \rightarrow \mathit{steps} \rightsquigarrow a\ c
\end{array}
$$

$$
\begin{array}{l}
\_ + +\_ \;:\; \{\rightsquigarrow \;:\; Rel\}\; \{a\ b\ c\ :\ A\} \\
\qquad\qquad \to steps \rightsquigarrow a\ b \to steps \rightsquigarrow b\ c \to steps \rightsquigarrow a\ c \\
zero\ +\!\!+\ s' \ =\ s' \\
one\ a \rightsquigarrow b\ +\!\!+\ b \rightsquigarrow *c \ =\ more\ a \rightsquigarrow b\ b \rightsquigarrow *c \\
more\ a \rightsquigarrow b\ b \rightsquigarrow *c\ +\!\!+\ c \rightsquigarrow *d \ =\ more\ a \rightsquigarrow b\ (b \rightsquigarrow *c\ +\!\!+\ c \rightsquigarrow *d)
\end{array}
$$

$$
\begin{array}{l}
{}^{*}\!- to -\ : \forall\, \{a\ b \rightsquigarrow\} \to star \rightsquigarrow a\ b \to steps \rightsquigarrow a\ b \\
{}^{*}\!- to -\ refl \ =\ zero \\
{}^{*}\!- to -\ (just\ a \rightsquigarrow b) \ =\ one\ a \rightsquigarrow b \\
{}^{*}\!- to -\ (trans\ a \rightsquigarrow *b\ b \rightsquigarrow *c) \ =\ {}^{*}\!- to -\ a \rightsquigarrow *b\ +\!\!+\ {}^{*}\!- to -\ b \rightsquigarrow *c
\end{array}
$$

$$
\begin{array}{l}
- to -^{*} : \forall\, \{a\ b \rightsquigarrow\} \to steps \rightsquigarrow a\ b \to star \rightsquigarrow a\ b \\
- to -^{*}\, zero \ =\ refl \\
- to -^{*}\, (one\ a \rightsquigarrow b) \ =\ just\ a \rightsquigarrow b \\
- to -^{*}\, (more\ a \rightsquigarrow b\ b \rightsquigarrow *c) \ =\ trans\ (just\ a \rightsquigarrow b)\ (- to -^{*}\, b \rightsquigarrow *c)
\end{array}
$$

A relation R is said to have the *diamond* property if whenever $a\,R\,b$ and $a\,R\,c$, there is a d such that $b\,R\,d$ and $c\,R\,d$; we will say that R is *diamantine* if it satisfies the diamond property.

$$
\begin{array}{l}
diamond\ :\ (\_ \rightsquigarrow \_\ :\ Rel) \to Set\ l \\
diamond\ \_ \rightsquigarrow \_ \ =\ \{a\ b\ c\ :\ A\} \\
\qquad\qquad\qquad \to a \rightsquigarrow b \to a \rightsquigarrow c \to \\
\qquad\qquad\qquad \exists\ (\ d \to b \rightsquigarrow d \times c \rightsquigarrow d)
\end{array}
$$

$$
\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, dashed] & & c \arrow[dl, dashed] \\
 & d &
\end{tikzcd}
$$

A relation R is *Church-Rosser* if its transitive closure is diamantine; as we have already mentioned it is easier to deal with the statement of Church-Rosser for the n-fold version of the transitive closure.

$$
\begin{array}{l}
cr\ :\ (\rightsquigarrow\ :\ Rel) \to Set\ l \\
cr \rightsquigarrow \ =\ diamond\ (star \rightsquigarrow)
\end{array}
$$

$$
\begin{array}{l}
cr - steps\ :\ (\rightsquigarrow\ :\ Rel) \to Set\ l \\
cr - steps \rightsquigarrow \ =\ diamond\ (steps \rightsquigarrow)
\end{array}
$$

$$
\begin{array}{l}
cr - steps - to - cr\ :\ \{\rightsquigarrow\ :\ Rel\} \to cr - steps \rightsquigarrow \to cr \rightsquigarrow \\
cr - steps - to - cr\ cr\ a \rightsquigarrow *b\ a \rightsquigarrow *c \\
\quad \mathbf{with}\ cr\ ({}^{*}\!- to -\ a \rightsquigarrow *b)\ ({}^{*}\!- to -\ a \rightsquigarrow *c) \\
...\ \mid\ d, b \rightsquigarrow *d, c \rightsquigarrow *d \ =\ d,\ - to -^{*}\, b \rightsquigarrow *d,\ - to -^{*}\, c \rightsquigarrow *d
\end{array}
$$

Lemma 1. *Let $\to$ have the diamond property; if there is a reduction $a \to b$ and also a reduction $a \to^{n} c$, then there exists d such that $b \to^{n} d$ and $c \to d$.*

Proof. The proof is by induction in the length of $a \to c$:

$$
\begin{tikzcd}[column sep=small]
 & a \arrow[dl] \arrow[dr, equal] & \\
b \arrow[dr, equal] & & a \arrow[dl] \\
 & b &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr] & \diamond & c \arrow[dl] \\
 & d &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & a \arrow[dl] \arrow[dr] & & & \\
b \arrow[dr] & \diamond & c \arrow[dl] \arrow[ddrr] & & \\
 & d_1 \arrow[ddrr] & \text{i.h.} & & \\
 & & & & c_n \arrow[dl] \\
 & & & d_n &
\end{tikzcd}
$$

□

```
leg : ∀ { a b c ⇝}
        → diamond ⇝
        → ⇝ a b
        → steps ⇝ a c
        → ∃ ( d → steps ⇝ b d × ⇝ c d)
leg { a} { b} ◇ a ⇝ b zero  =  b, zero, a ⇝ b
leg { a} { b} { c} ◇ a ⇝ b (one a ⇝ c)
  with ◇ a ⇝ b a ⇝ c
... | d, b ⇝ d, c ⇝ d          =  d, one b ⇝ d, c ⇝ d
leg ◇ a ⇝ b (more a ⇝ c c ⇝ ∗dn)
  with ◇ a ⇝ b a ⇝ c
... | d1, b ⇝ d1, c ⇝ d1
  with leg ◇ c ⇝ d1 c ⇝ ∗dn
... | dn, e ⇝ ∗dn, d ⇝ dn  =  dn, more b ⇝ d1 e ⇝ ∗dn, d ⇝ dn
```

With this result at hand is easy to prove that the diamond property implies Church-Rosser for the n-fold composition of the relation.

Lemma 2. *Let $\rightarrow$ be diamantine, then it satisfies Church-Rosser for the n-fold closure.*

Proof. The proof is by induction on the second reduction and cases in the first one, using previous result all the non trivial cases. There exists another two cases symmetrical to the first two diagrams, so we omit them.

$$
\begin{tikzcd}[column sep=small]
 & a \arrow[dl, equal] \arrow[ddr] & \\
a \arrow[ddr] & & \\
 & & c \arrow[dl, equal] \\
 & c &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & a \arrow[dl] \arrow[ddr] & \\
b \arrow[ddr] & \text{leg} & \\
 & & c_n \arrow[dl] \\
 & d &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & a \arrow[dl] \arrow[dr] & & \\
 & b_1 \arrow[ddl] & & c_1 \arrow[ddll] \arrow[ddr] & \\
 & & \text{leg} & & \\
b_n \arrow[dr] & & & \text{i.h.} & c_n \arrow[ddll] \\
 & d_1 \arrow[dr] & & & \\
 & & d_n & &
\end{tikzcd}
$$

□

$diamond - cr - steps\ :\ \{\rightsquigarrow\ :\ Rel\} \rightarrow diamond \rightsquigarrow \rightarrow cr - steps \rightsquigarrow$
$diamond - cr - steps\ \diamondsuit\ \{a\}\ \{.a\}\ \{c\}\quad zero\ a \rightsquigarrow *c$
$\quad =\ c, a \rightsquigarrow *c, zero$
$diamond - cr - steps\ \diamondsuit \qquad\qquad\qquad (one\ a \rightsquigarrow b)\ a \rightsquigarrow *c$
$\quad$ **with** $leg\ \diamondsuit\ a \rightsquigarrow b\ a \rightsquigarrow *c$
$...\ |\ d, b \rightsquigarrow *d, c \rightsquigarrow d\ =\ d, b \rightsquigarrow *d, one\ c \rightsquigarrow d$
$diamond - cr - steps\ \diamondsuit\ \{a\}\ \{b_n\}\ \{.a\}\ (more\ a \rightsquigarrow b_1\ b_1 \rightsquigarrow *b_n)\ zero$
$\quad =\ b_n, zero, more\ a \rightsquigarrow b_1\ b_1 \rightsquigarrow *b_n$
$diamond - cr - steps\ \diamondsuit \qquad\qquad\qquad (more\ a \rightsquigarrow b_1\ b_1 \rightsquigarrow *b_n)\ (one\ a \rightsquigarrow c)$
$\quad$ **with** $leg\ \diamondsuit\ a \rightsquigarrow c\ (more\ a \rightsquigarrow b_1\ b_1 \rightsquigarrow *b_n)$
$...\ |\ d, c \rightsquigarrow *d, b_n \rightsquigarrow d\ =\ d, one\ b_n \rightsquigarrow d, c \rightsquigarrow *d$
$diamond - cr - steps\ \diamondsuit \qquad\qquad\qquad (more\ a \rightsquigarrow b_1\ b_1 \rightsquigarrow *b_n)\ (more\ a \rightsquigarrow c_1\ c_1 \rightsquigarrow *c_n)$
$\quad$ **with** $leg\ \diamondsuit\ a \rightsquigarrow c_1\ (more\ a \rightsquigarrow b_1\ b_1 \rightsquigarrow *b_n)$
$...\ |\ d_1, c_1 \rightsquigarrow *d_1, b_n \rightsquigarrow d_1$
$\quad$ **with** $diamond - cr - steps\ \diamondsuit\ c_1 \rightsquigarrow *d_1\ c_1 \rightsquigarrow *c_n$
$...\ |\ d_n, d_1 \rightsquigarrow *d_n, c_n \rightsquigarrow *d_n$
$\quad =\ d_n, more\ b_n \rightsquigarrow d_1\ d_1 \rightsquigarrow *d_n, c_n \rightsquigarrow *d_n$

Since both statements of the transitive closure are equivalent, from cr-steps we can deduce church-rosser for any relation having the diamond property.

$diamond - cr\ :\ \{\rightsquigarrow\ :\ Rel\} \rightarrow diamond \rightsquigarrow \rightarrow cr \rightsquigarrow$
$diamond - cr\ \diamondsuit\ =\ cr - steps - to - cr\ (diamond - cr - steps\ \diamondsuit)$

Lemma 3. *Let R and S be two relations such that $S \subseteq R$ and $R^* \subseteq S$, if R is Church-Rosser, then S is also.*

Proof. The proof is immediate: let $a\, S^*\, b$ and $a\, S^*\, c$, by the first hypothesis and $mon - star$ lemma we know both $a\, R^*\, b$ and $a\, R^*\, c$. By Church-Rosser for R we get an element d with proofs of $b\, R^*\, d$ and $c\, R^*\, d$; by the second hypothesis and $trans\!-\subseteq-star$ lemma we conclude $b\, S^*\, d$ and $c\, S^*\, d$. □

$cr\!-\subseteq\ :\ \{R\ S\ :\ Rel\} \rightarrow S \subseteq R \rightarrow R \subseteq star\ S \rightarrow cr\ R \rightarrow cr\ S$
$cr\!-\subseteq S \subseteq R\ R \subseteq S^*\ cr\ aR^*b\ aR^*c$
$\quad$ **with** $cr\ (mon - star\ S \subseteq R\ aR^*b)\ (mon - star\ S \subseteq R\ aR^*c)$
$...\ |\ d, bR^*d, cR^*d$
$\quad =\ d\qquad\qquad ,$
$\qquad trans\!-\subseteq-star\ R \subseteq S^*\ bR^*d,$
$\qquad trans\!-\subseteq-star\ R \subseteq S^*\ cR^*d$