1 χ Lemmas

1.

2.

$$\forall \sigma \in \Sigma, M \in \Lambda, \chi(\sigma, M) \#\downharpoonleft (\sigma, M)$$

2 Corollary 4

1.

2.

$$\forall M \in \Lambda, \sigma \in \Sigma, x, y \in \mathsf{V}, y \#\downharpoonleft (\sigma, \lambda x M) \Rightarrow (\lambda x M)\sigma \sim_\alpha \lambda y M(\sigma, x := y)$$

3 Lemma 7

$$\forall M, N \in \Lambda, x \in \mathsf{V}, x * N \wedge M \rightrightarrows N \Rightarrow x * M$$

3.1 Corollary Lemma 7

$$\forall M, N \in \Lambda, x \in \mathsf{V}, x \# M \wedge M \rightrightarrows N \Rightarrow x \# N$$

4 Substitution Lemma for Parallel Reduction

Induction in $\rightrightarrows$ relation.

- Case var: Immediate by definition of $\sigma \rightrightarrows \sigma'$.
- Case abs: Take $M \rightrightarrows M'$. The induction hypothesis is that for all σ_1, σ_1' s.t. $\sigma_1 \rightrightarrows \sigma_1'$, $M\sigma_1 \rightrightarrows M'\sigma_1'$. Take now σ, σ' s.t. $\sigma \rightrightarrows \sigma'$. We have to show $(\lambda x M)\sigma \rightrightarrows (\lambda x M')\sigma'$. The left hand side is $\lambda y(M(\sigma, x := y))$ where $y = \chi(\sigma, \lambda x M)$. Whatever this y is, we know $(\sigma, x := y) \rightrightarrows (\sigma', x := y)$. Then, by virtue of the inductive hypothesis, we get $M(\sigma, x := y) \rightrightarrows M'(\sigma', x := y)$ and, by rule abs of $\rightrightarrows$, $\lambda y M(\sigma, x := y) \rightrightarrows \lambda y M'(\sigma', x := y)$. Now we show that this right hand side is α-convertible with $(\lambda x M')\sigma'$, which gives the desired result by using the rule α of $\rightrightarrows$. But this is just Corollary 4 part 2, which we could apply if $y \#\downharpoonleft (\sigma', \lambda x M')$, that is, for any variable z such that $z * \lambda x M'$ we must prove that $y \# \sigma' z$. By lemma 7 we know $z * \lambda x M$ because $\lambda x M \rightrightarrows \lambda x M'$ by abs rule of $\rightrightarrows$ and $M \rightrightarrows M'$ hypothesis. We know by the second χ lemma that $y \#\downharpoonleft (\sigma, \lambda x M)$, we can apply this result to variable z and get that $y \# \sigma z$. Also, because $\sigma \rightrightarrows \sigma'$, we know $\sigma z \rightrightarrows \sigma' z$. Then, as we already know that $y \# \sigma z$, lemma 7 corollary allows us to conclude that $y \# \sigma' z$, fullfilling the premises of corollary 4 part 2.
- Case app: Immediate using induction hypothesis and rule app of $\rightrightarrows$ relation.
- Case β:
- Case α: Suppose $M \rightrightarrows N$ and $N \sim_\alpha N'$. The induction hypothesis is that for all σ, σ' s.t. $\sigma \rightrightarrows \sigma'$, $M\sigma \rightrightarrows N\sigma'$. Now take σ, σ' s.t. $\sigma \rightrightarrows \sigma'$. We must show $M\sigma \rightrightarrows N'\sigma'$. But, since $N \sim_\alpha N'$, $N'\sigma' = N\sigma'$ by Lemma 4, and by induction hypothesis $M\sigma \rightrightarrows N\sigma'$, thus getting what was required.