# CS 350 Algorithms and Complexity

*Fall 2015*

## Lecture 9: Divide & Conquer
## The Master Theorem, Mergesort & Quicksort

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True or False?

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*False!*

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$$T(n) \; = \; aT(\frac{n}{b}) + T_{split+combine}(n)$$

# MergeSort

**ALGORITHM** *Mergesort*$(A[0..n-1])$

//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
**if** $n > 1$
    copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
    copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
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    *Mergesort*$(C[0..\lceil n/2 \rceil - 1])$
    *Merge*$(B, C, A)$

# Merge

**ALGORITHM** *Merge*($B[0..p-1]$, $C[0..q-1]$, $A[0..p+q-1]$)

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of $B$ and $C$
$i \leftarrow 0$; $j \leftarrow 0$; $k \leftarrow 0$
**while** $i < p$ **and** $j < q$ **do**
    **if** $B[i] \leq C[j]$
        $A[k] \leftarrow B[i]$; $i \leftarrow i + 1$
    **else** $A[k] \leftarrow C[j]$; $j \leftarrow j + 1$
    $k \leftarrow k + 1$
**if** $i = p$
    copy $C[j..q-1]$ to $A[k..p+q-1]$
**else** copy $B[i..p-1]$ to $A[k..p+q-1]$

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$\cong \lg n$ (neglecting small terms)

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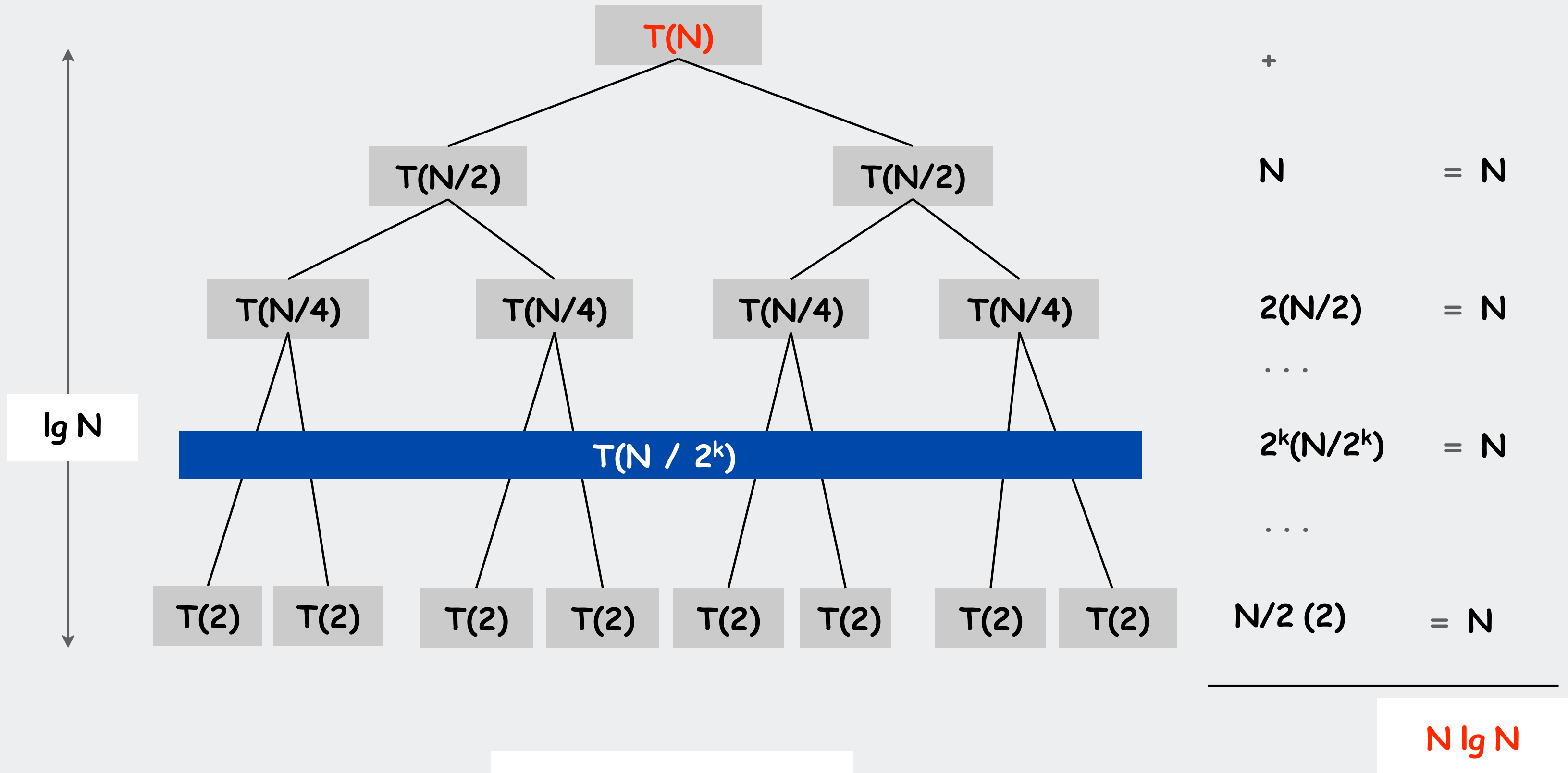
# Look at the recursion tree:

$T(N) = 2\,T(N/2) + N$

for $N > 1$, with $T(1) = 0$

(assume that N is a power of 2)




$T(N) = N \lg N$



from Sedgewick & Wayne http://www.cs.princeton.edu/courses/archive/spr07/cos226/lectures/04MergeQuick.pdf

# Divide & Conquer

- When designing a divide and conquer algorithm, the time for the “split” and “combine” phases contributes to the cost:

  A. not at all
  B. in a small way
  C. in a major way
  D. in a way that dominates the total cost
  E. in a way that depends on the algorithm

# Divide & Conquer

- In mergesort, the time for the “split” and “combine” phases contributes to the cost

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# Divide & Conquer

- In Quicksort, the time for the "split" and "combine" phases contributes to the cost

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- Suppose that $T_{combine}(n) \in \Theta(n^d)$
  Then:

$$T(n) \in \begin{cases} \Theta(n^d) & \text{when } a < b^d \\ \Theta(n^d \log n) & \text{when } a = b^d \\ \Theta(n^{\log_b a}) & \text{when } a > b^d \end{cases}$$

# Application to Mergesort:

- We divide the problem into 2 (roughly equal) parts, so $b$ = 2
- We have to solve *both* of them, so $a$ = 2
- Combining the parts is linear, i.e., in $\Theta(n^1)$, so $d$ = 1
- Hence, in the Master Theorem, $a$ = 2 = $2^1$ = $b^d$

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So $T(n) = \Theta(n \log n)$

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Does the base of this log matter? A: yes B: no

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# Problem:

- Why are there three cases in the Master Theorem?
  - Look at the recursion tree:

At level $i$: $a^i$ problems of size $\frac{n}{b^i}$, *each* requiring combining work in $\Theta\left((\frac{n}{b^i})^d\right)$

So total work is at level $i$ is: $\Theta\left(a^i\left(\frac{n}{b^i}\right)^d\right) = \Theta\left(n^d\left(\frac{a}{b^d}\right)^i\right)$

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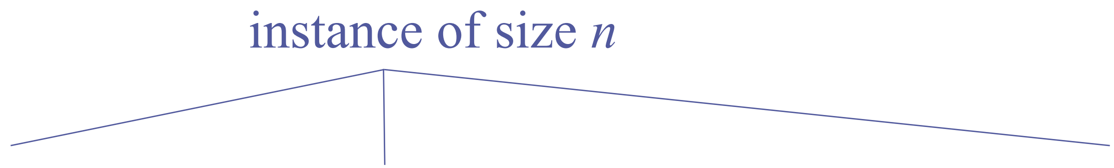


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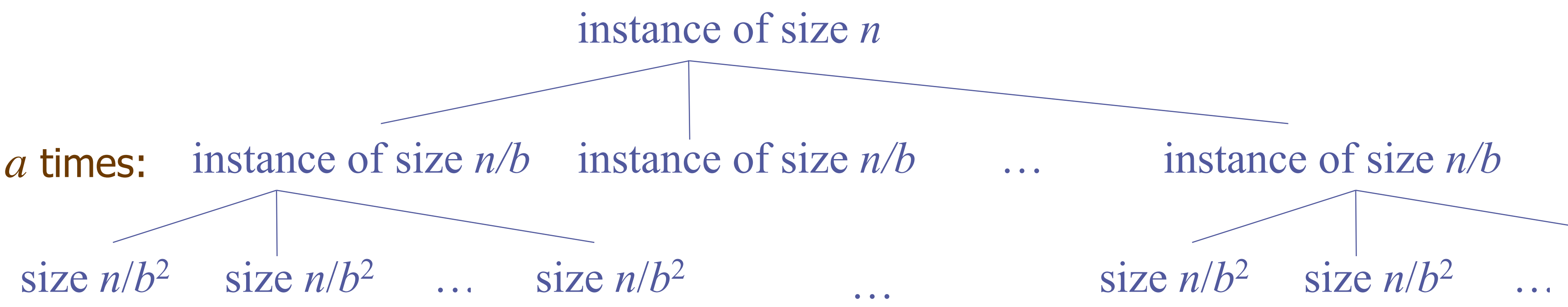


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- Why are there three cases in the Master Theorem?
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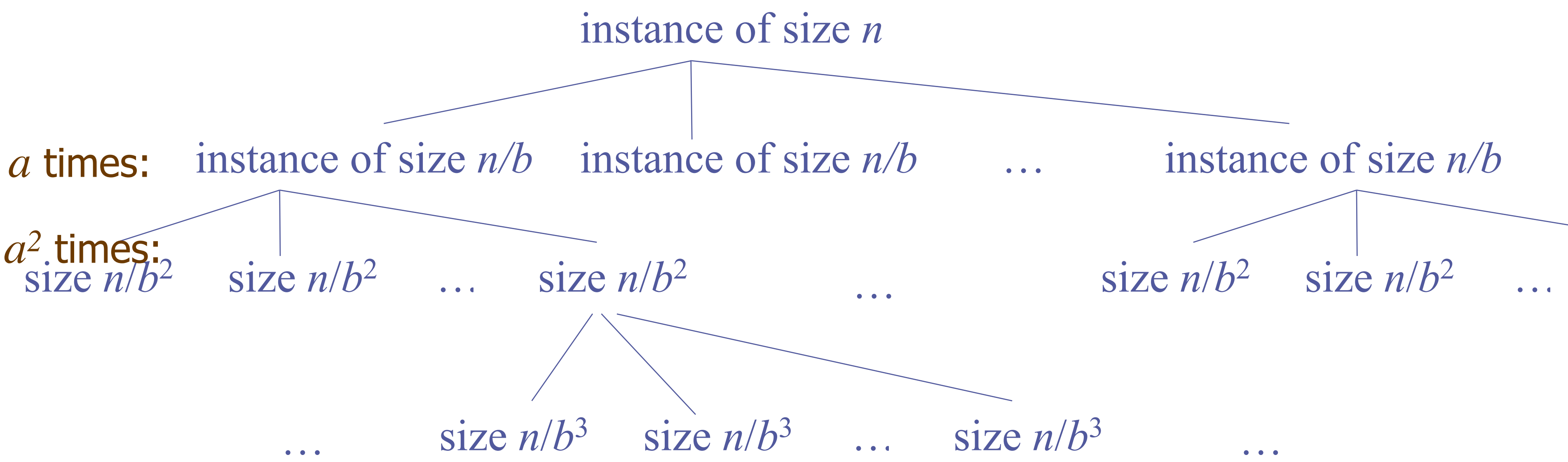


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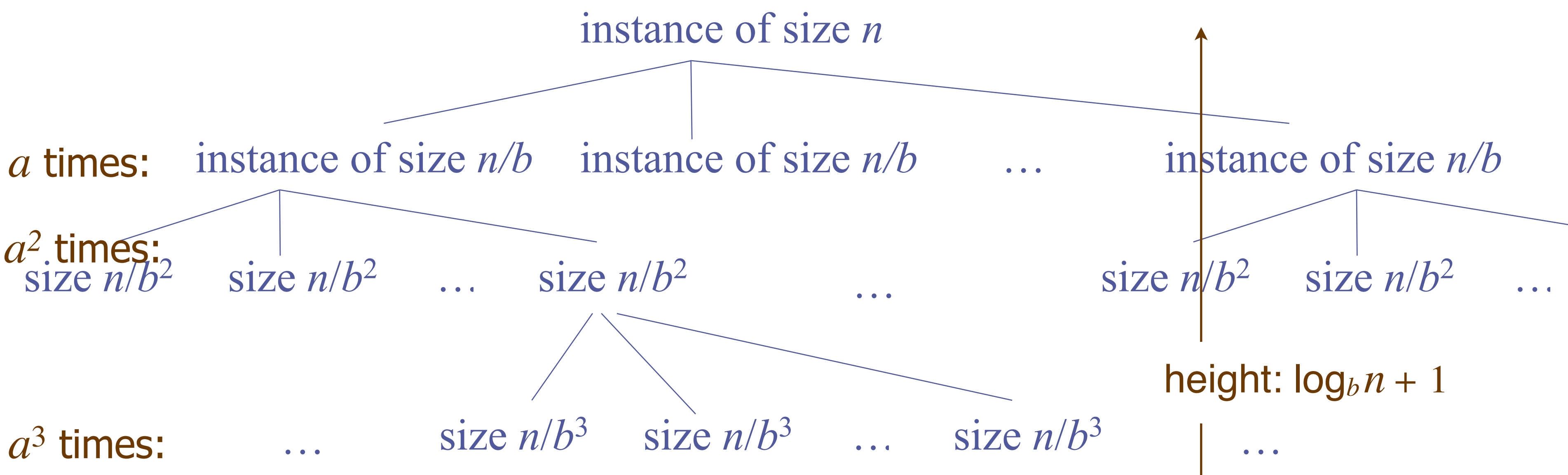


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1. increases
2. decreases
3. constant

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- But there are $\log_b n + 1$ levels
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$$\Theta\left(\sum_{i=0}^{\log_b n} n^d\left(\frac{a}{b^d}\right)^i\right)$$

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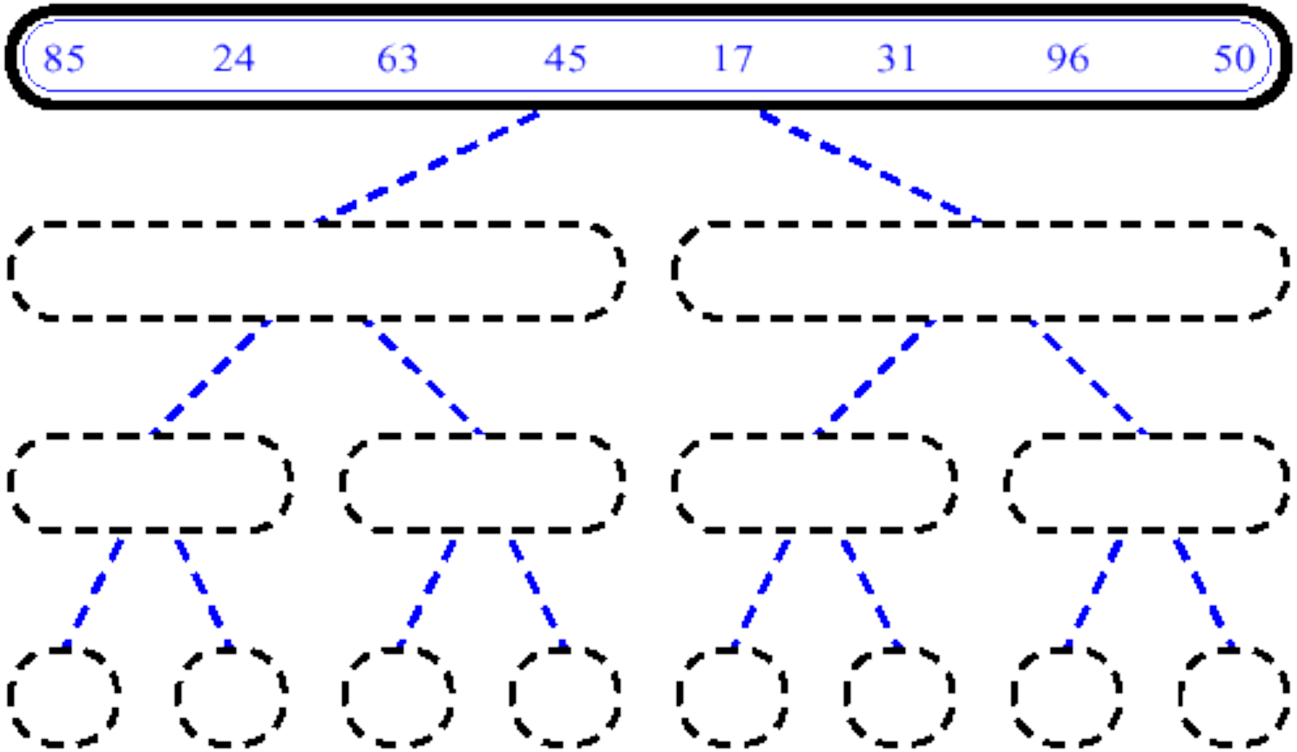
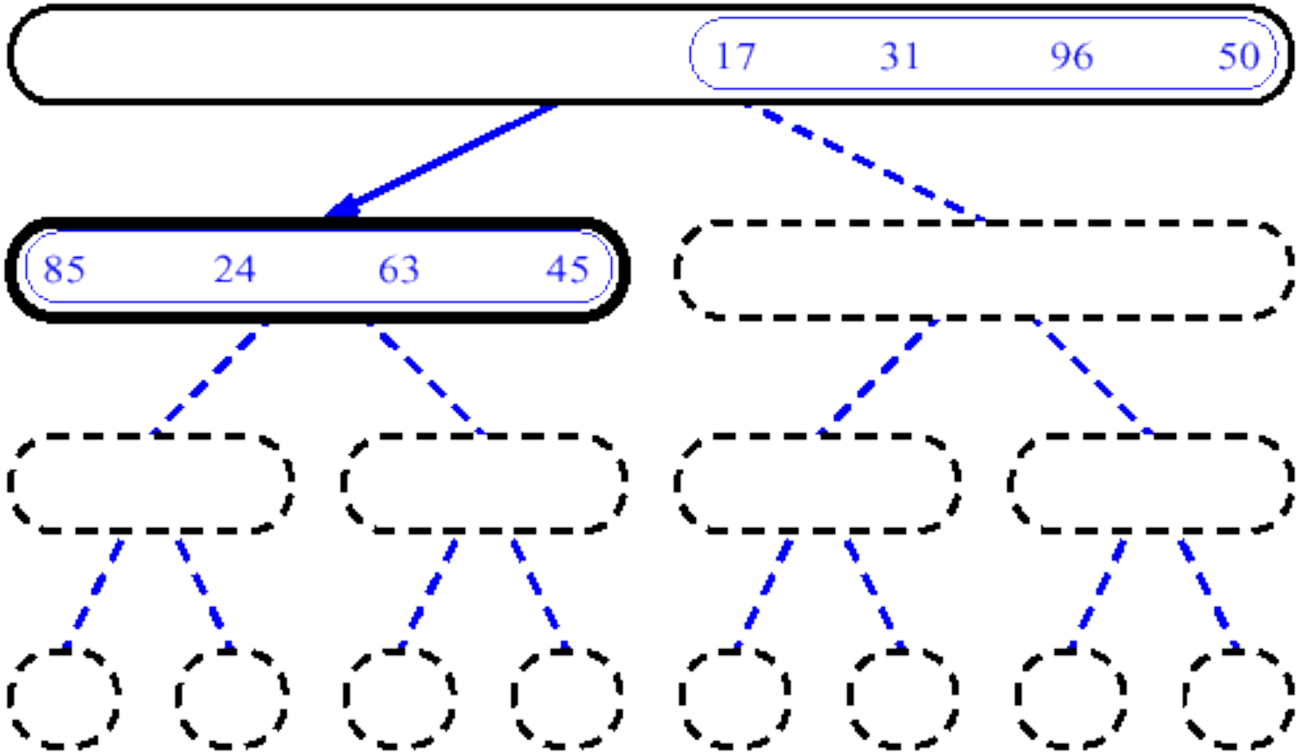
$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r} \approx \frac{1}{1 - r} \quad \text{as } n \to \infty$$

# MergeSort (Example) — 1

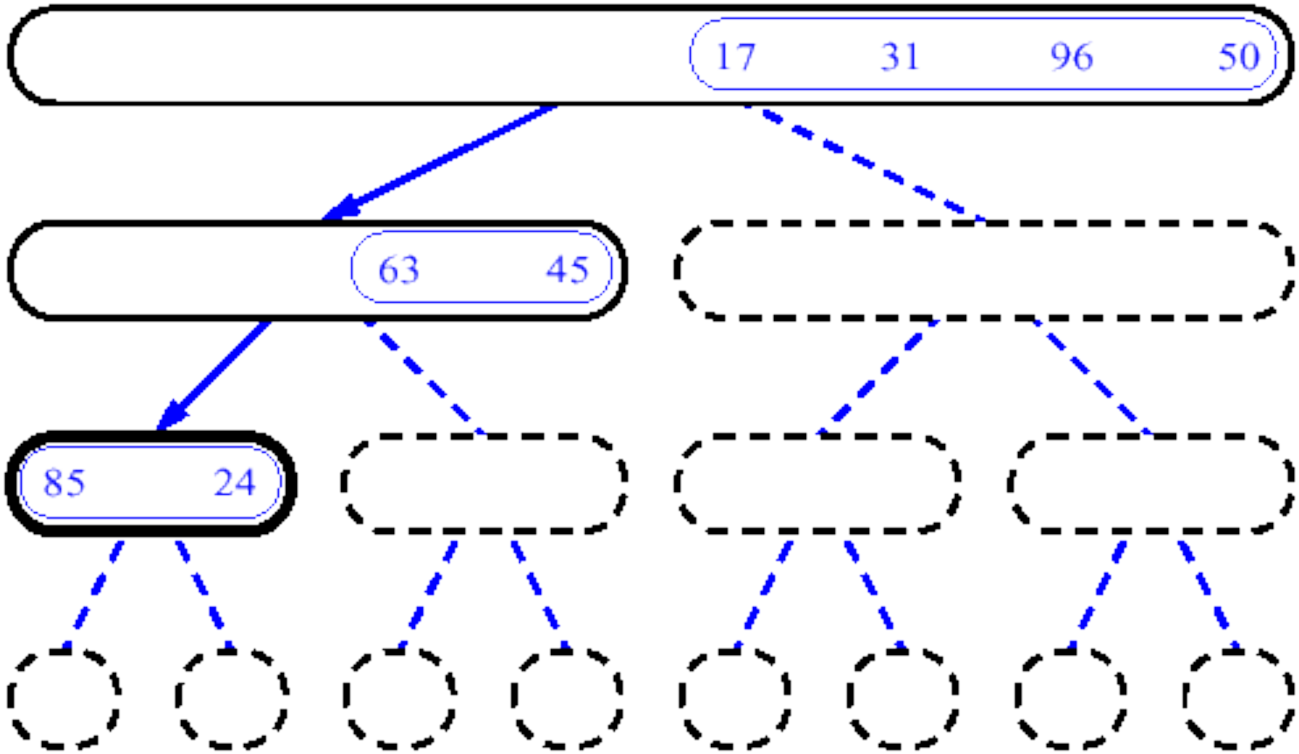



Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 2

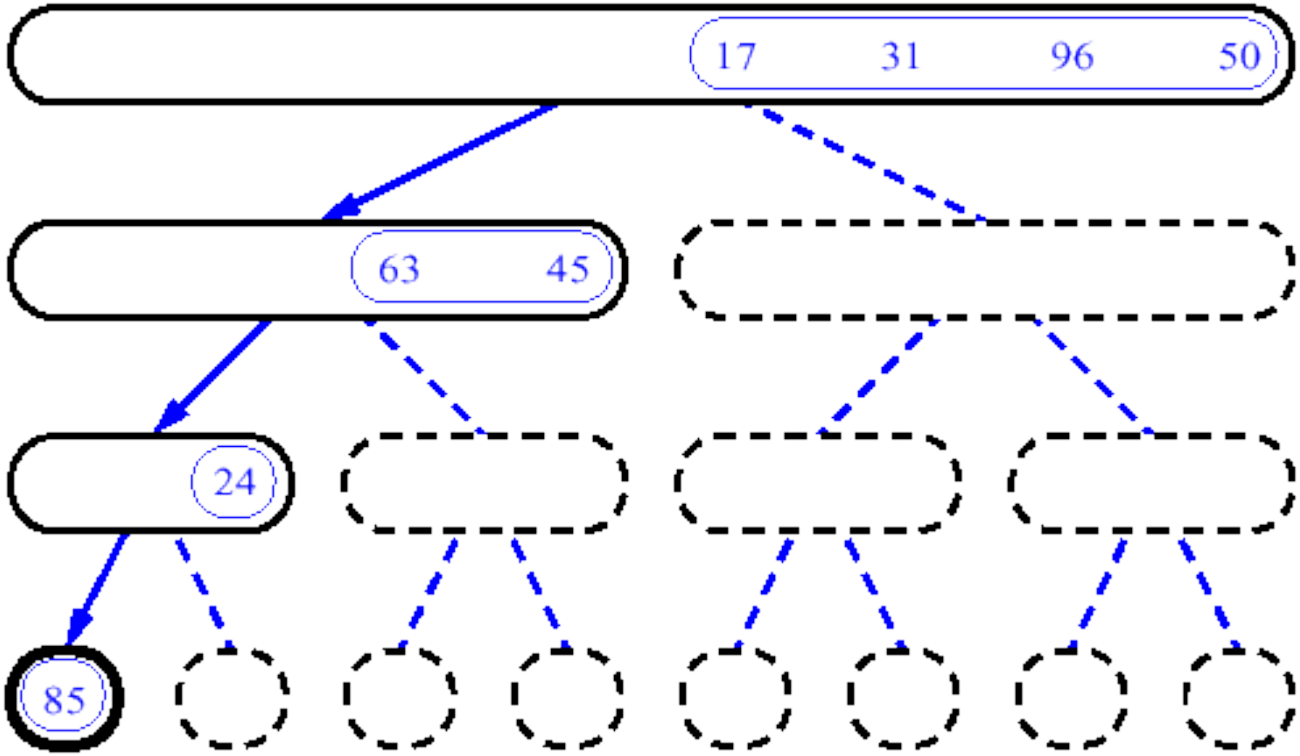


# MergeSort (Example) — 3



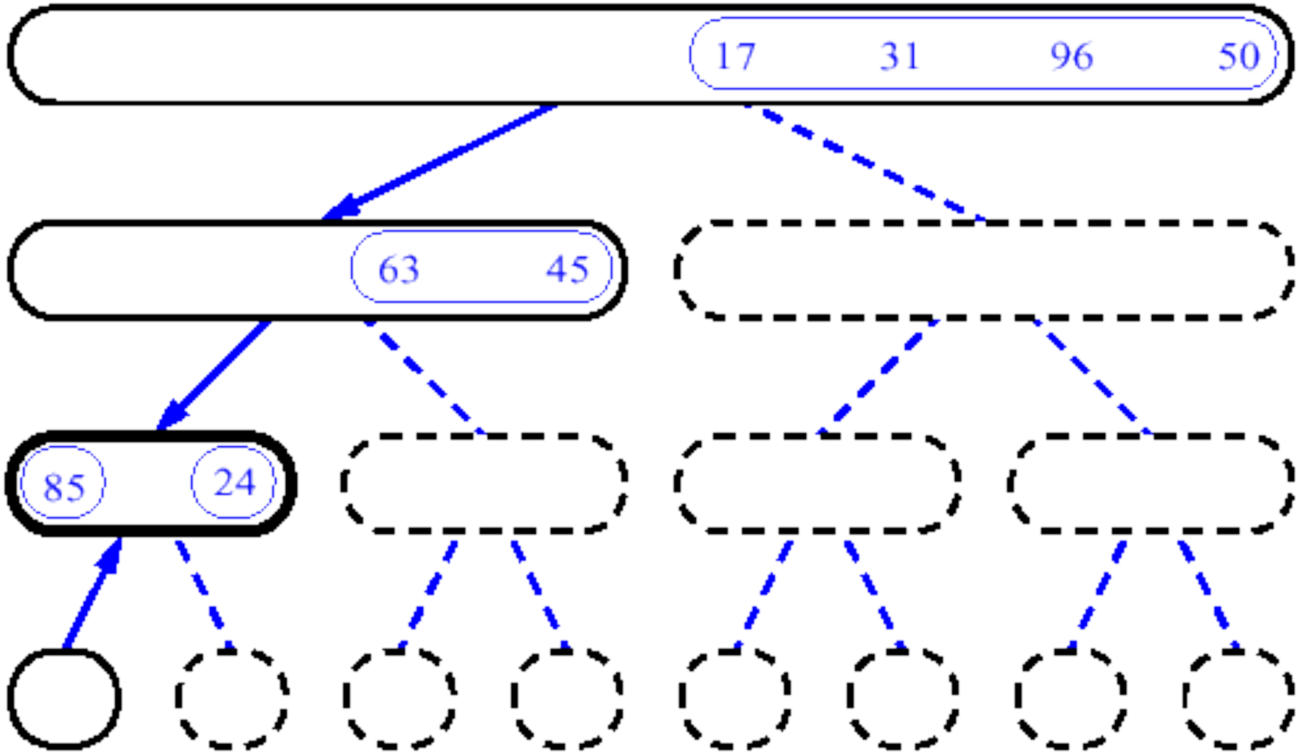

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 4



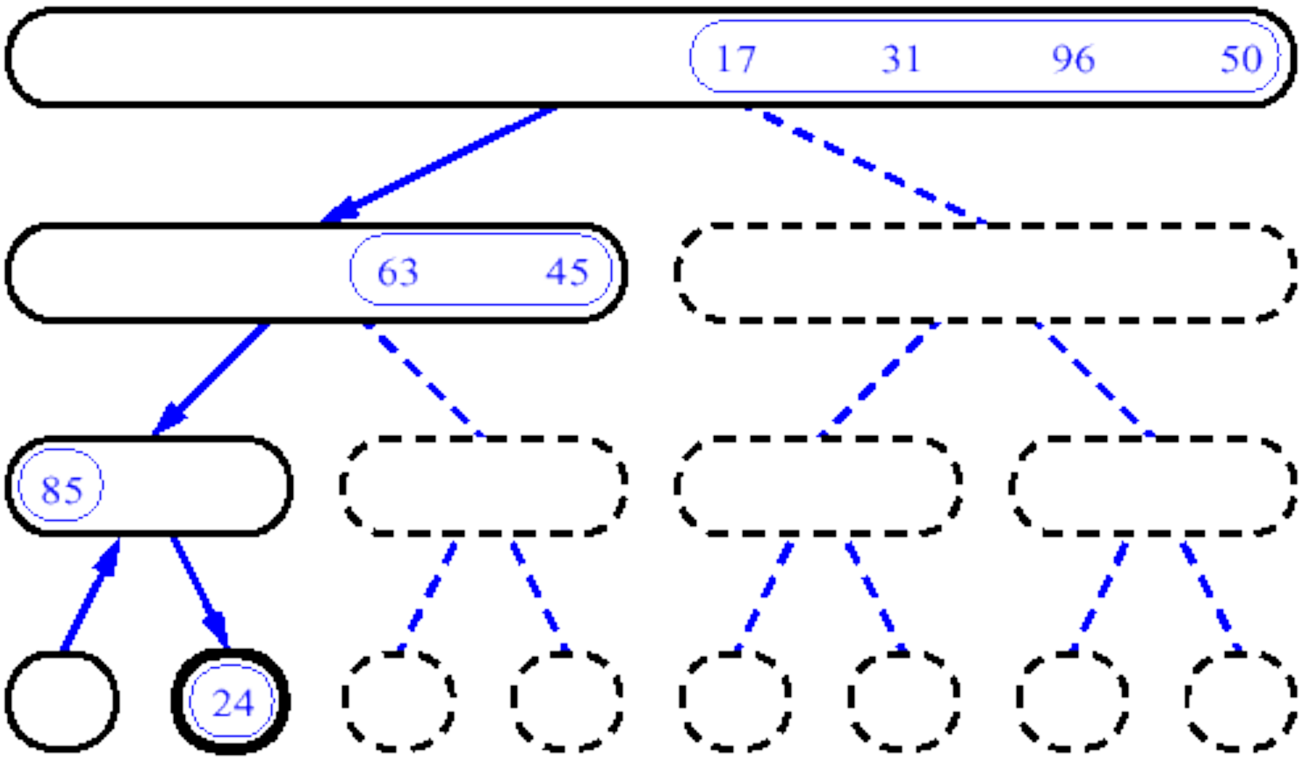

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 5



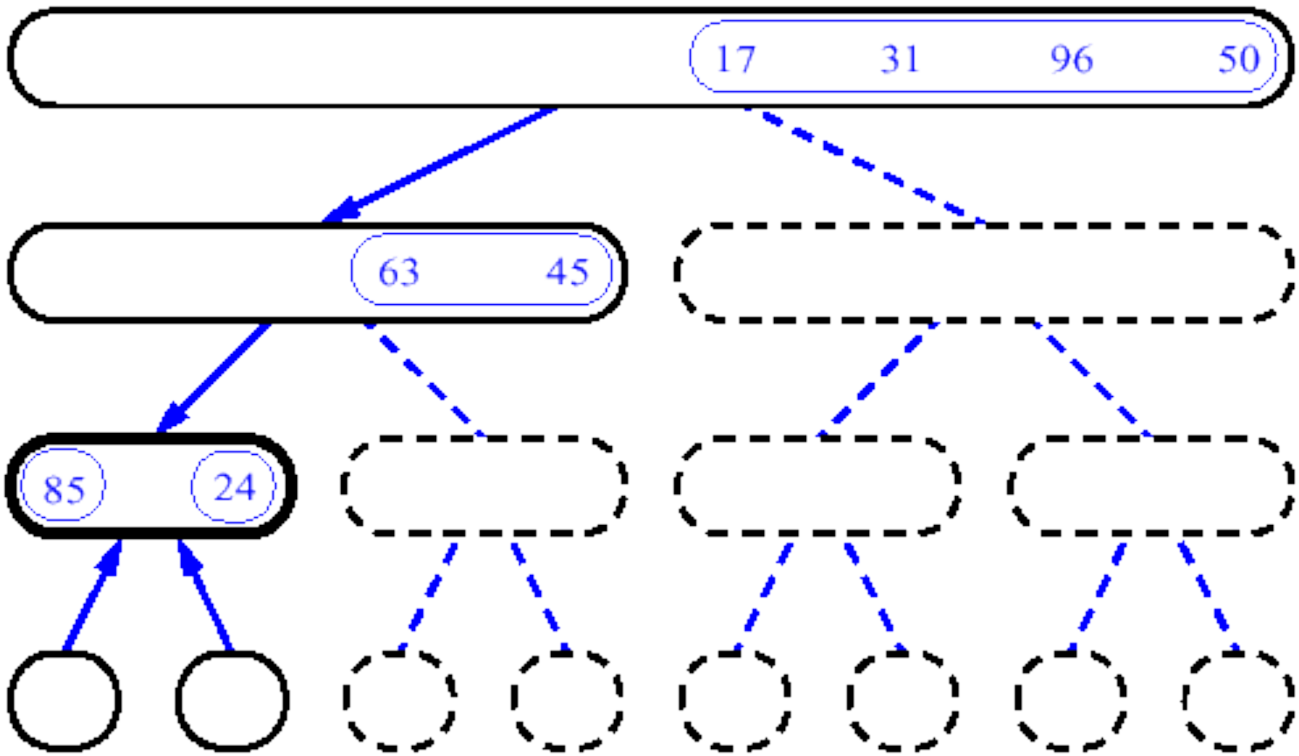

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 6



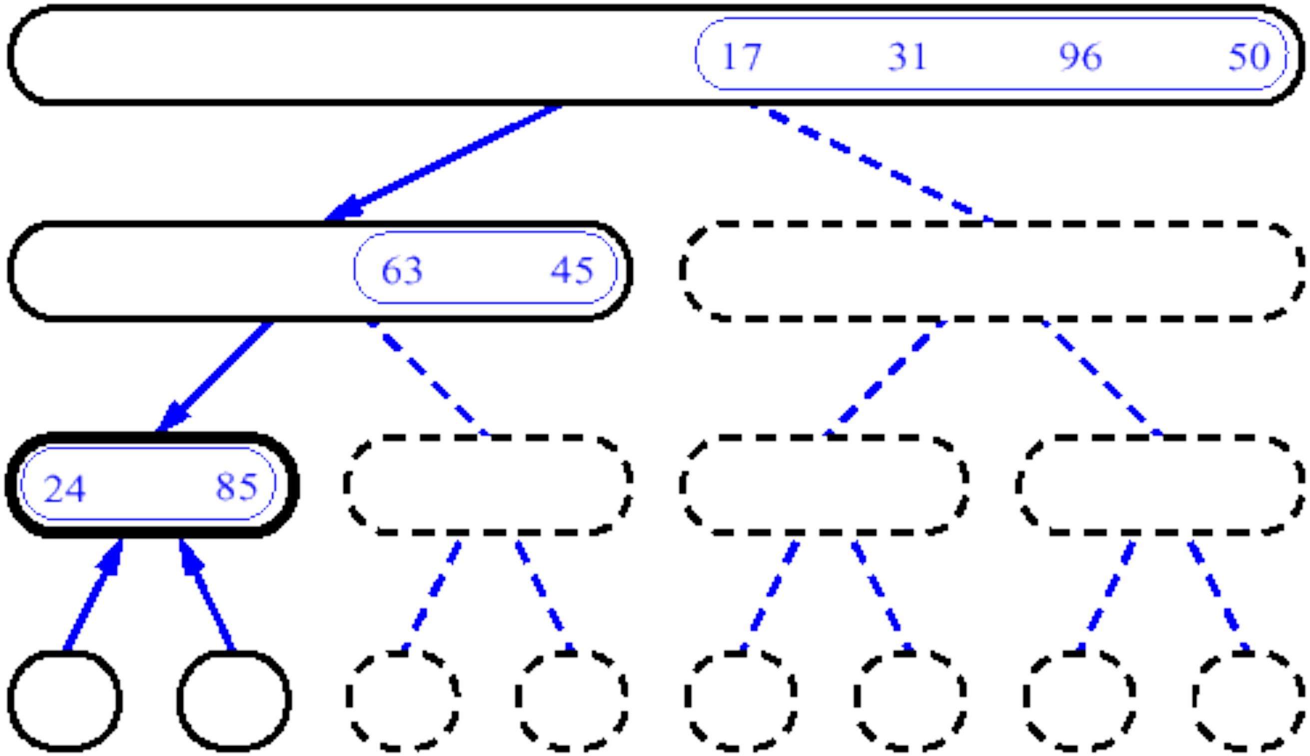

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 7



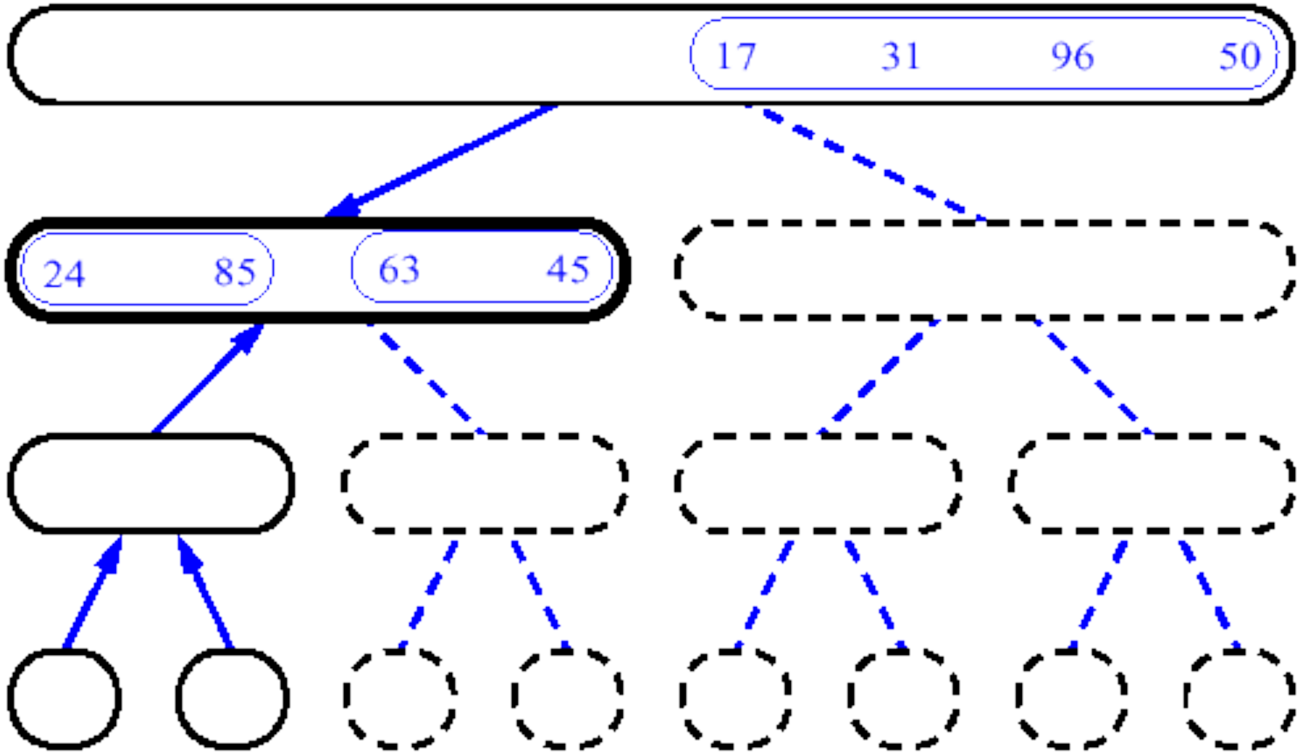

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 8



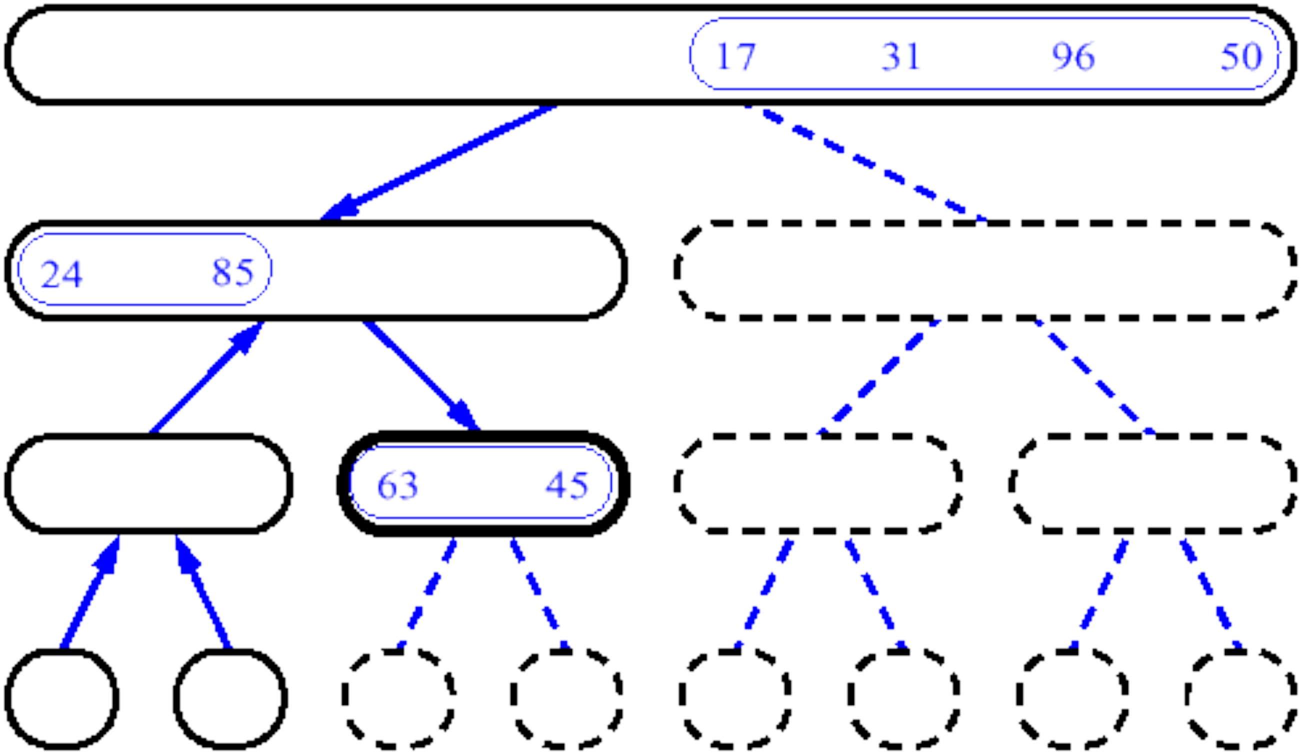

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 9



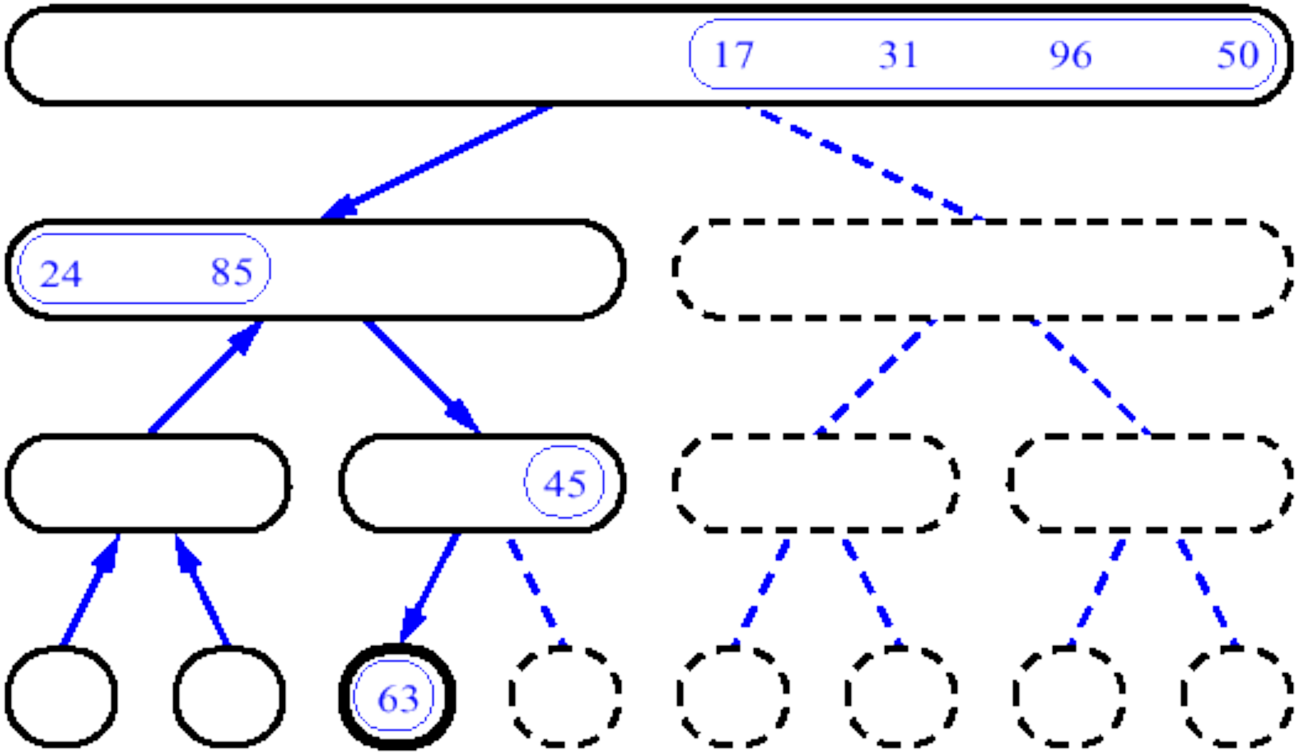

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 10




Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 11




Example from Ján Maňuch, Simon Fraser University

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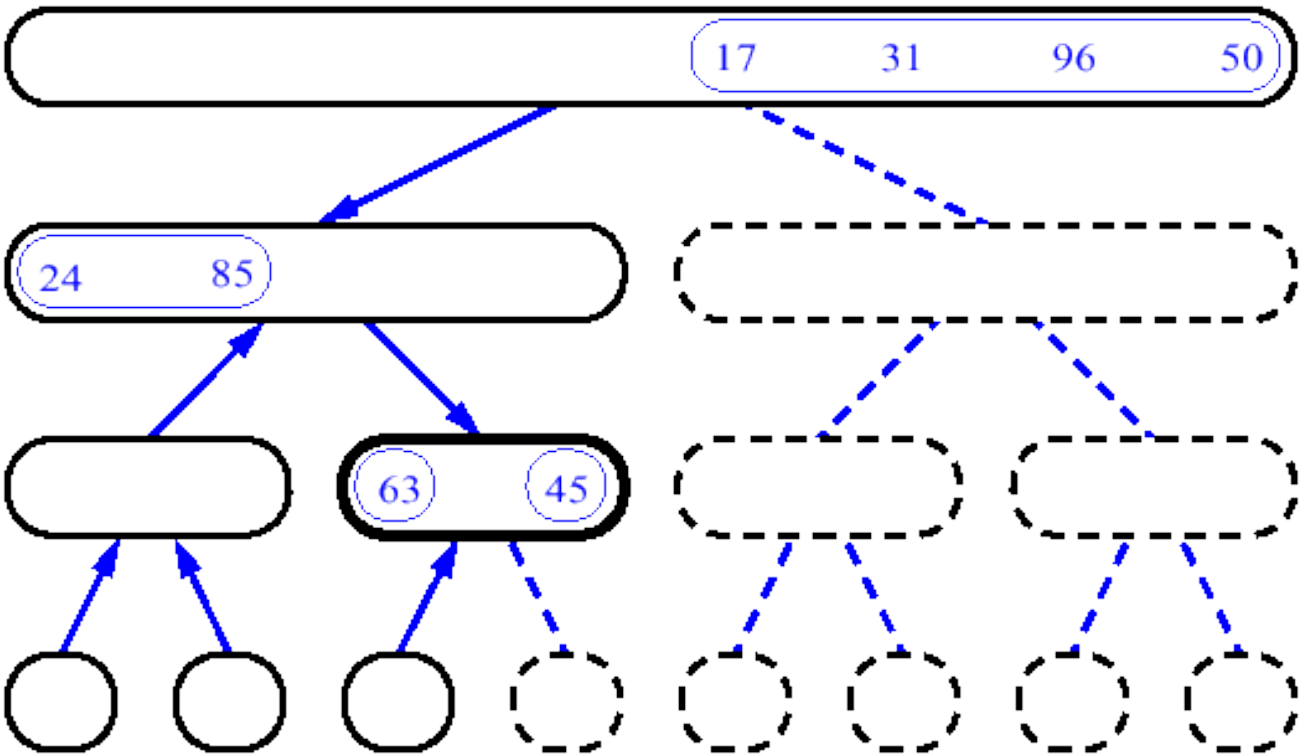


Example from Ján Maňuch, Simon Fraser University

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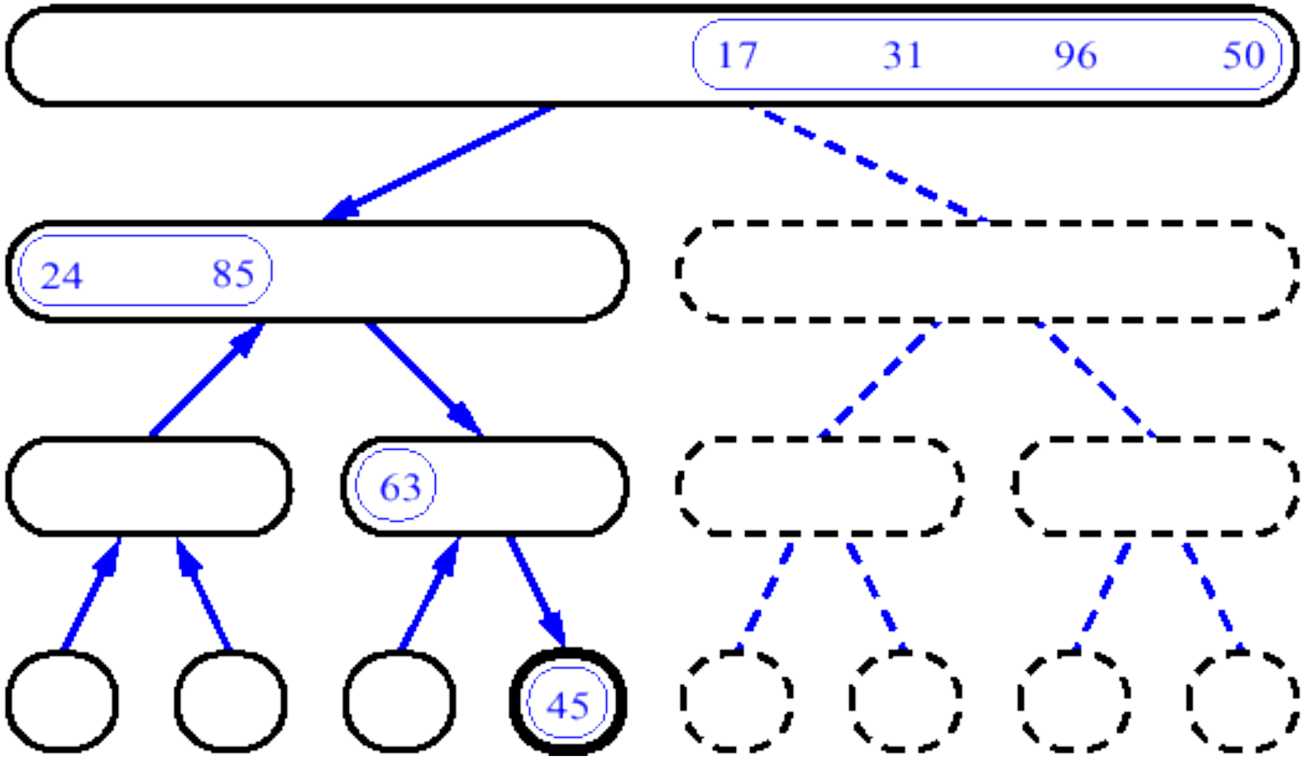
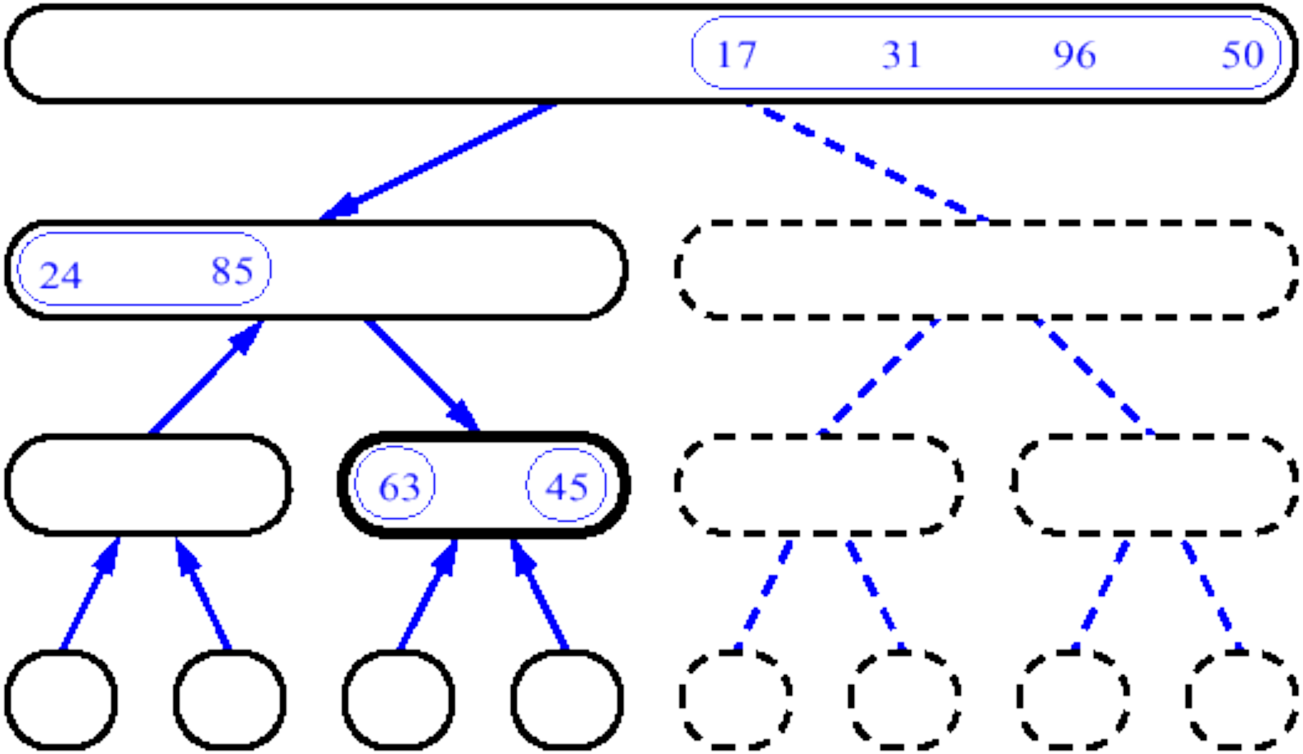


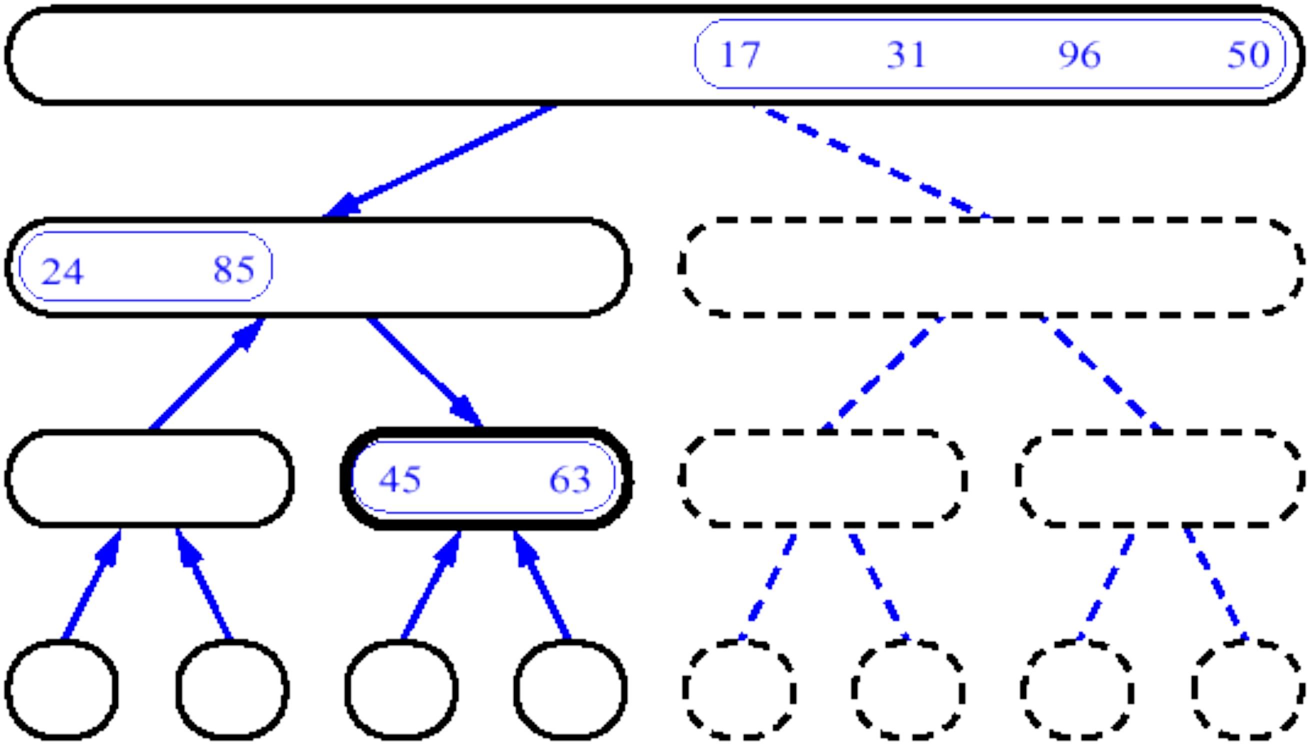

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 14



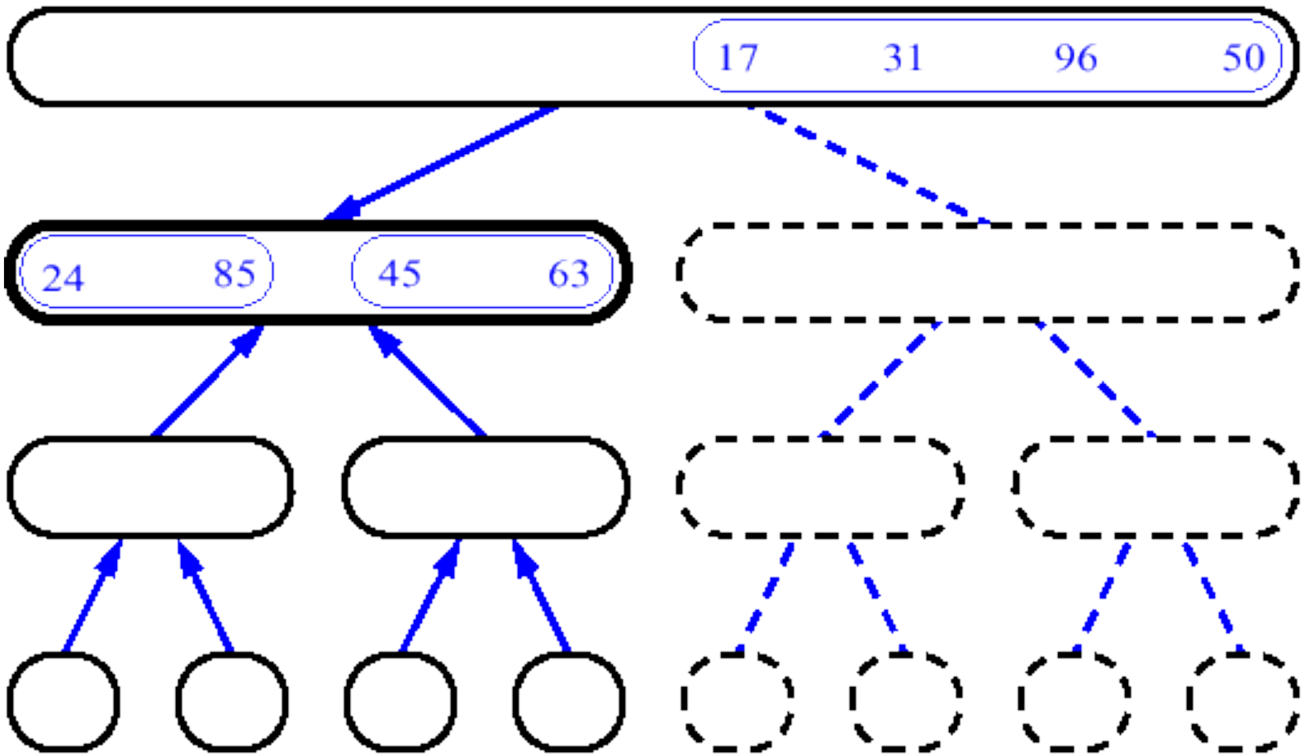

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 15



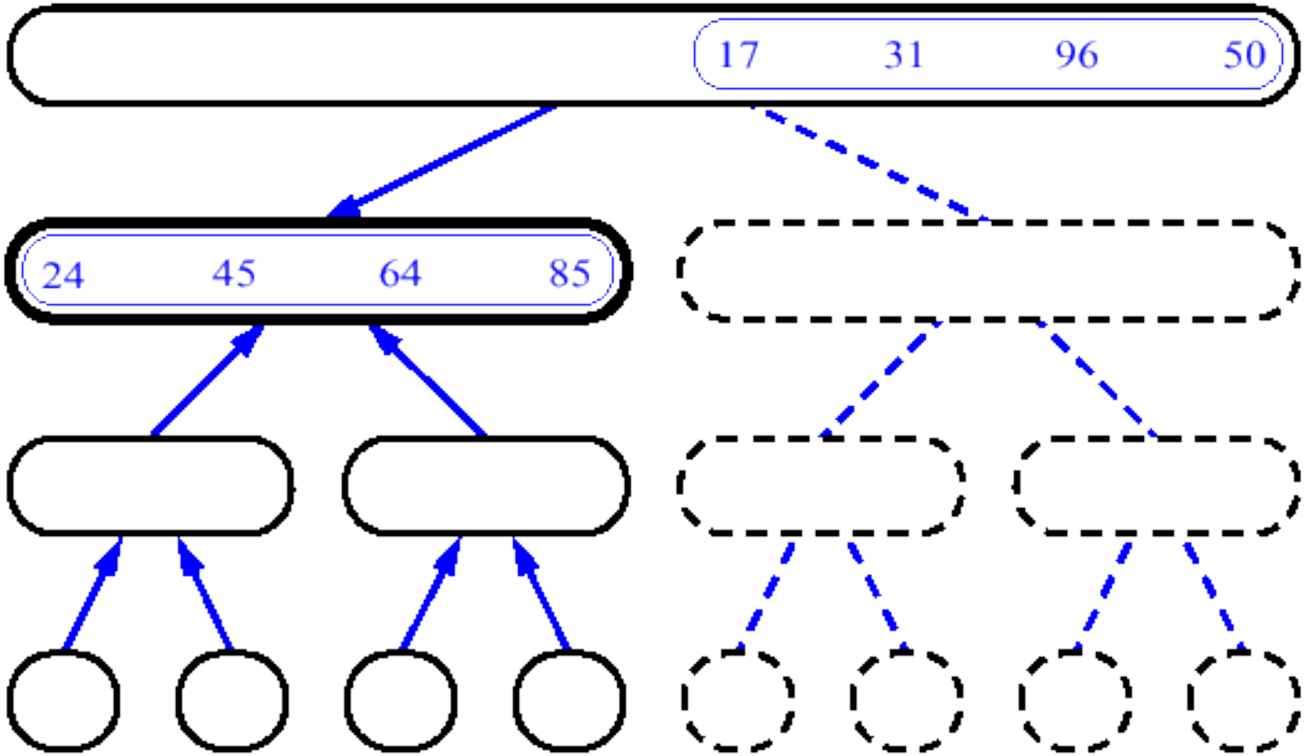

Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 16




Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 17




Example from Ján Maňuch, Simon Fraser University

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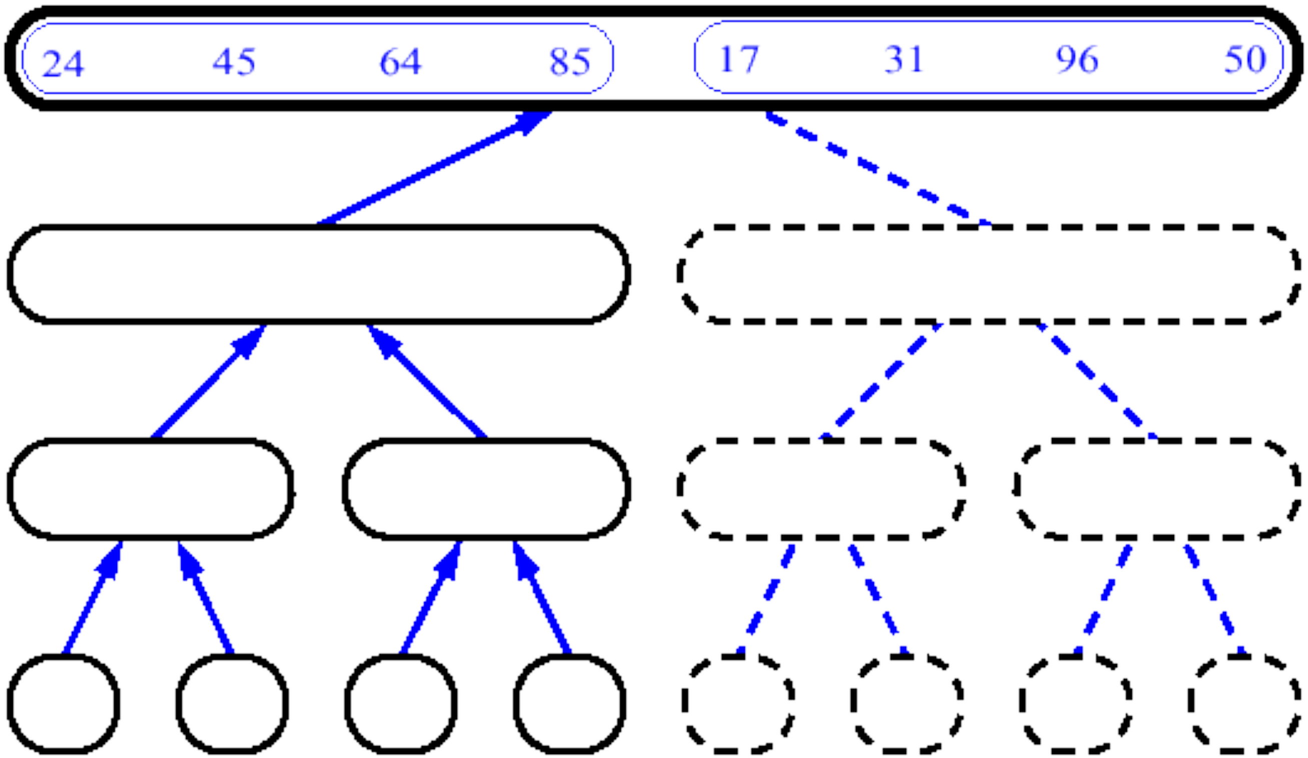
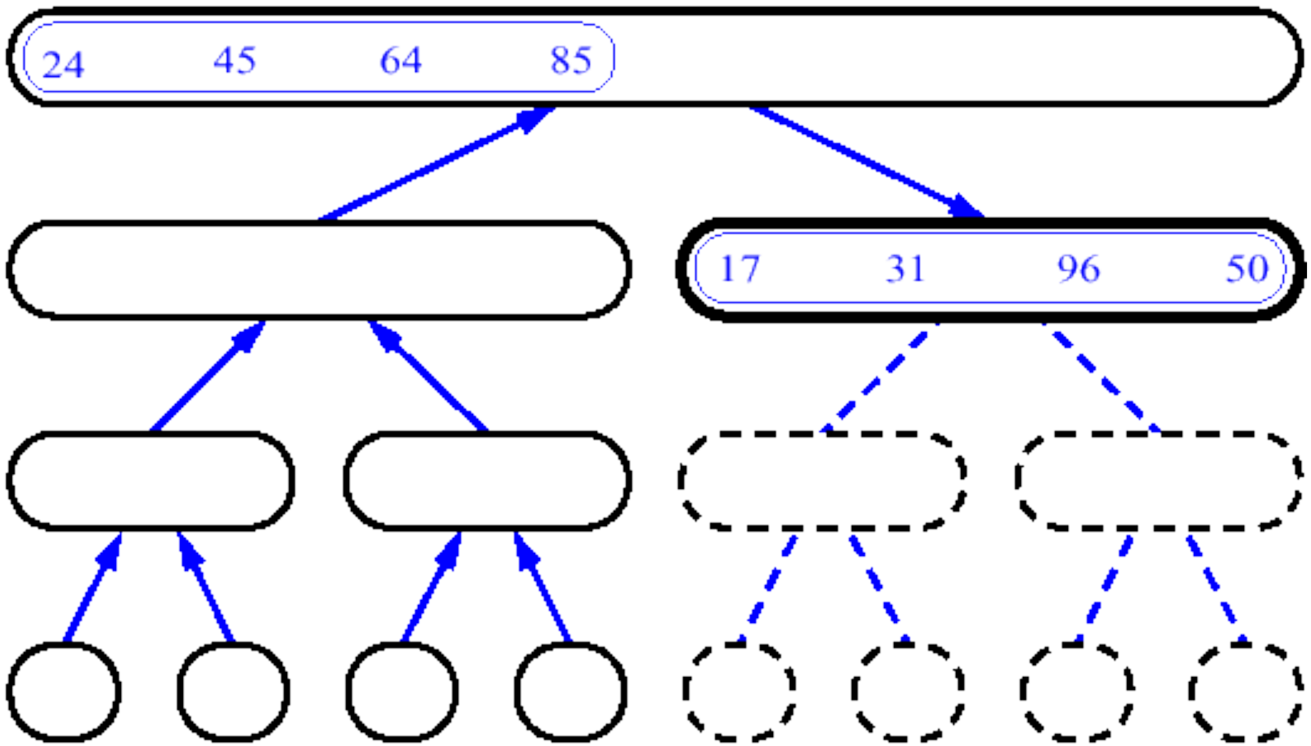



Example from Ján Maňuch, Simon Fraser University

# MergeSort (Example) — 19




Example from Ján Maňuch, Simon Fraser University

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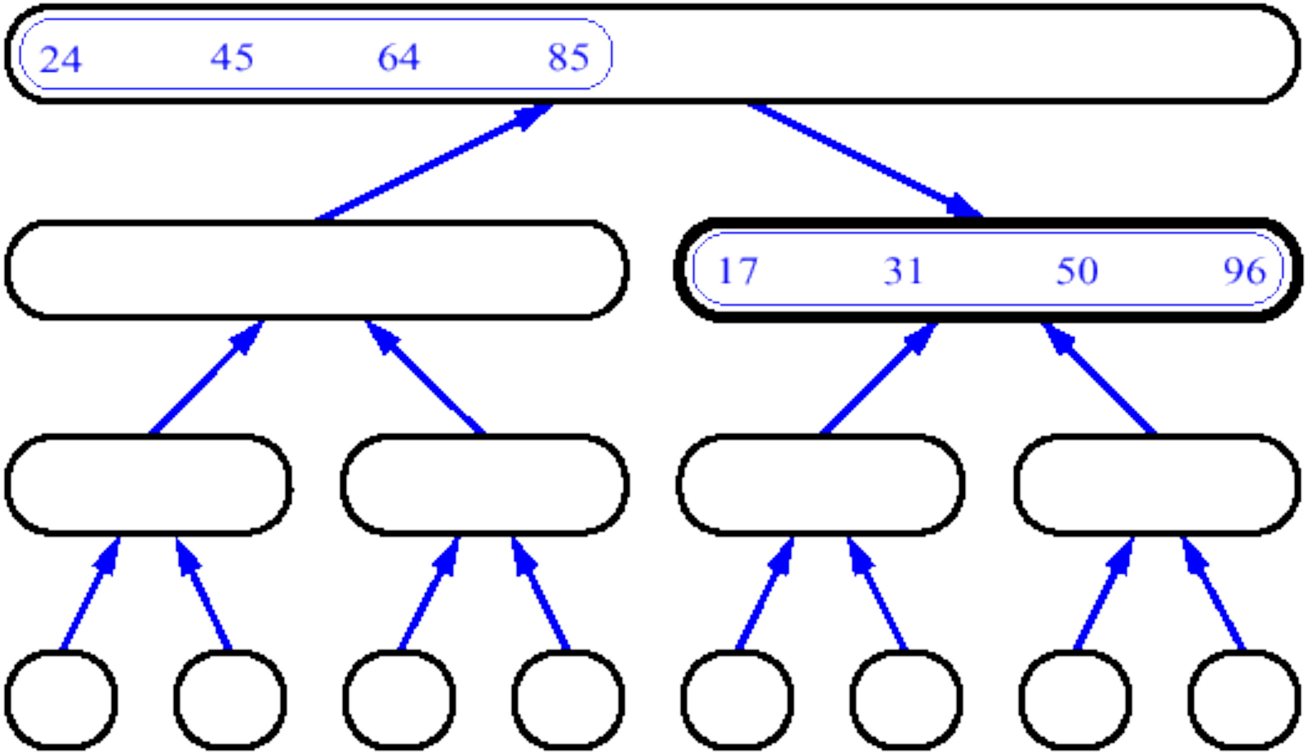


Example from Ján Maňuch, Simon Fraser University

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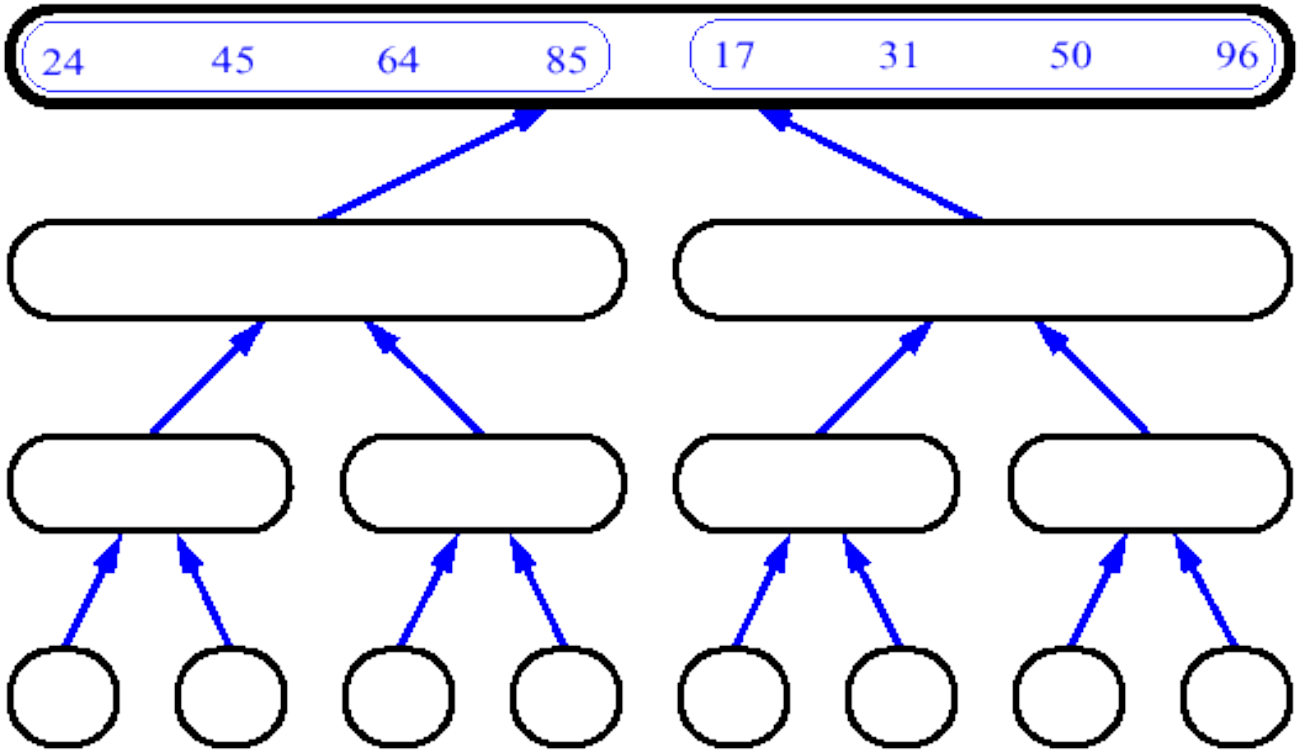


Example from Ján Maňuch, Simon Fraser University

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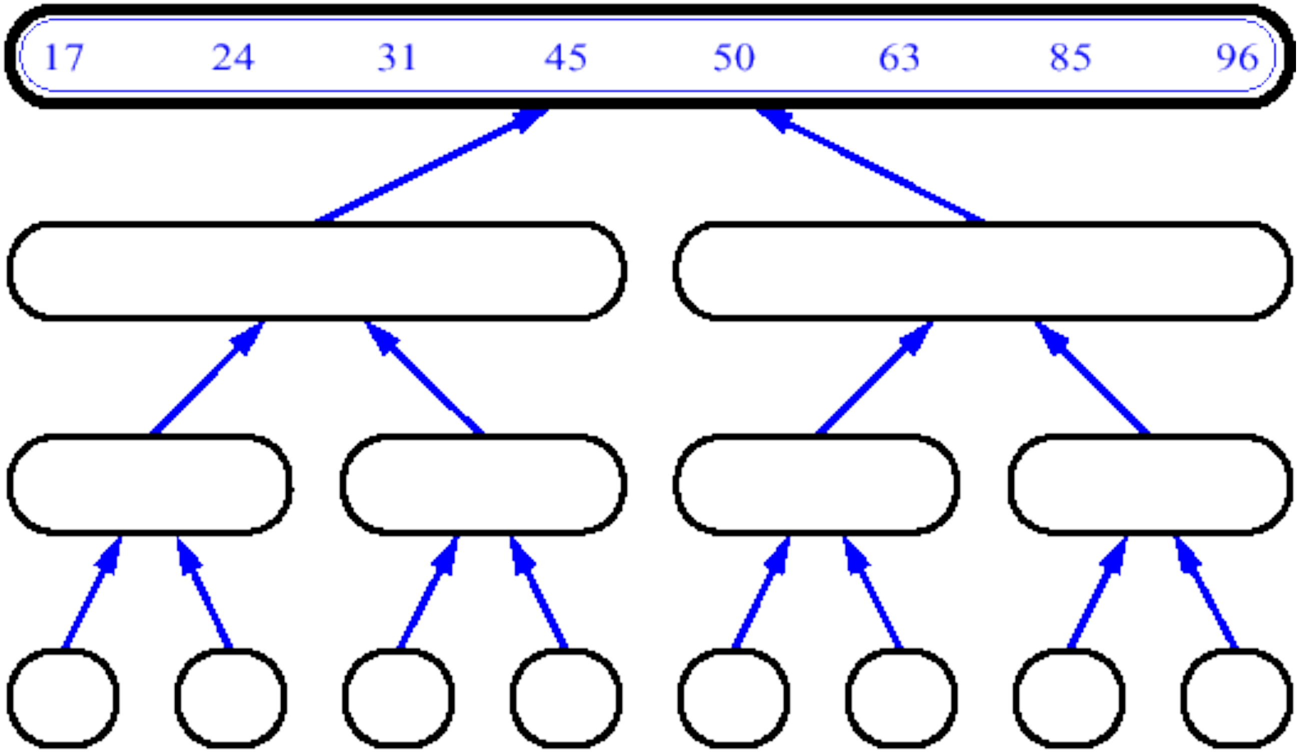


Example from Ján Maňuch, Simon Fraser University

# Merge Phase (Example)

| | | | |
|---|---|---|---|
| 14 | 23 | 45 | 98 |

| | | | |
|---|---|---|---|
| 6 | 33 | 42 | 67 |

Example from Ján Maňuch, Simon Fraser University

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**Merge**

Example from Ján Maňuch, Simon Fraser University

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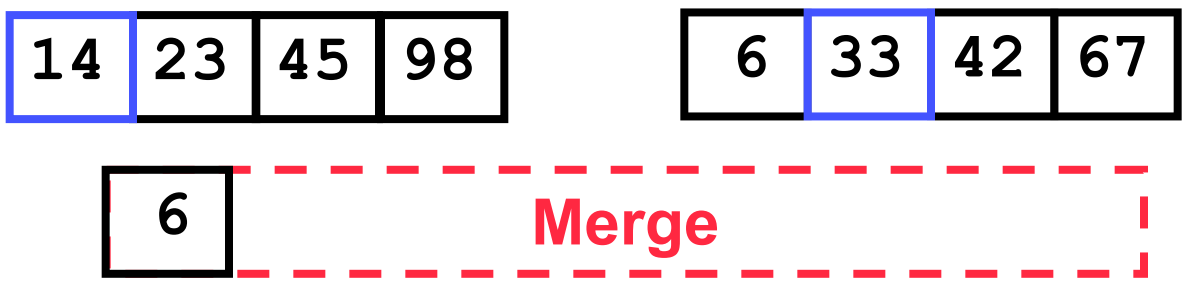


Example from Ján Maňuch, Simon Fraser University

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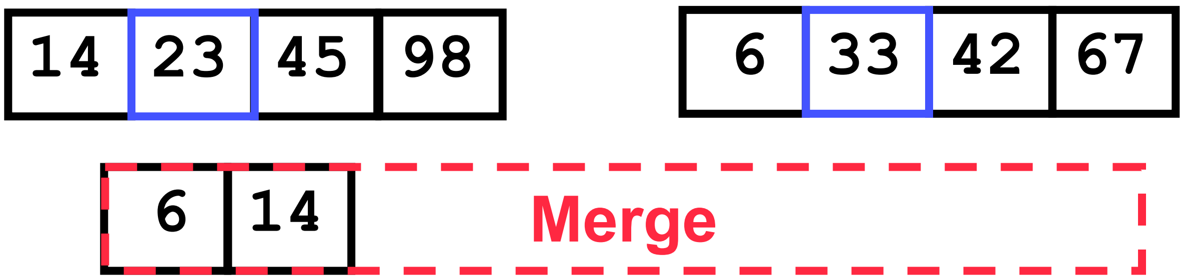


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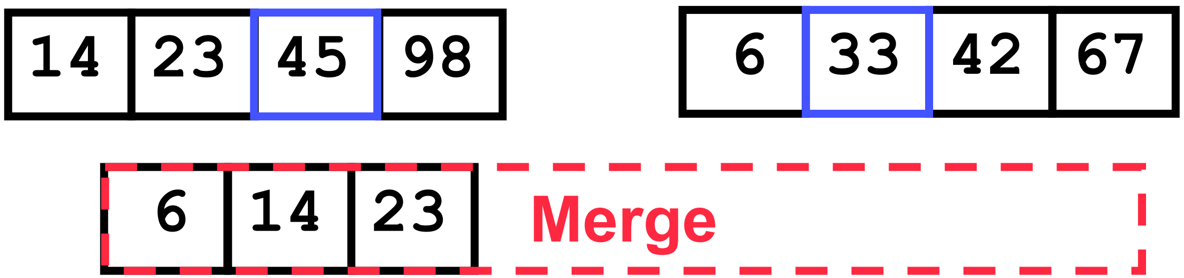


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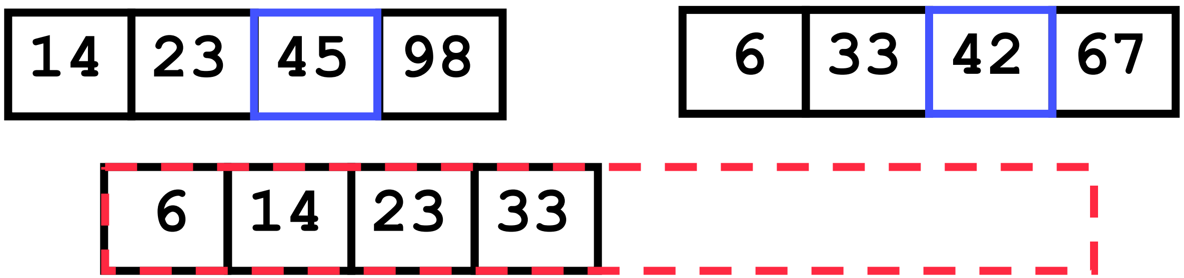


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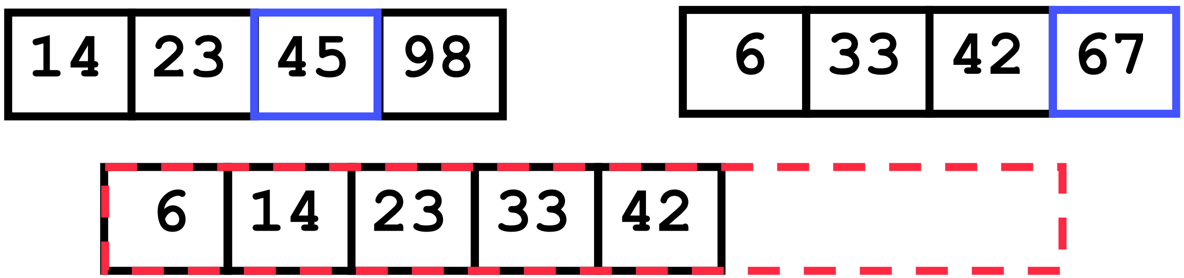


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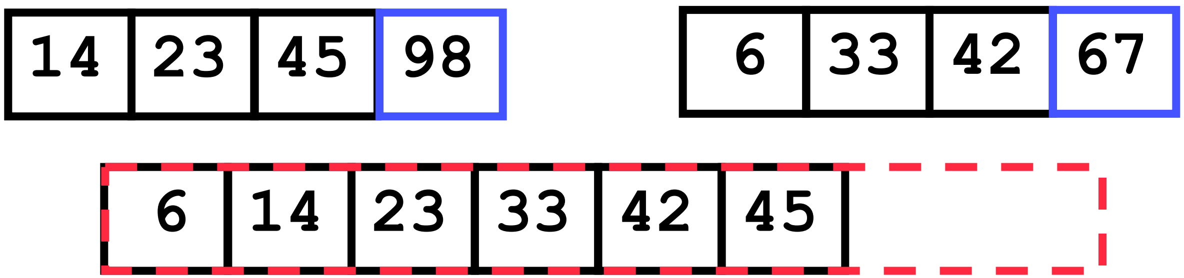


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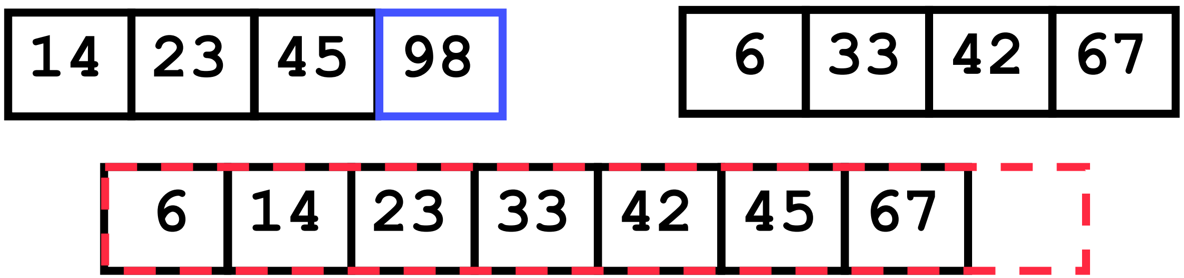


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Example from Ján Maňuch, Simon Fraser University

# What's wrong with this Algorithm?

**ALGORITHM** *Mergesort*$(A[0..n-1])$

//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
**if** $n > 1$
  copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
  copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
  *Mergesort*$(B[0..\lfloor n/2 \rfloor - 1])$
  *Mergesort*$(C[0..\lceil n/2 \rceil - 1])$
  *Merge*$(B, C, A)$

**ALGORITHM** *Merge*$(B[0..p-1], C[0..q-1], A[0..p+q-1])$

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of $B$ and $C$
$i \leftarrow 0$; $j \leftarrow 0$; $k \leftarrow 0$
**while** $i < p$ **and** $j < q$ **do**
  **if** $B[i] \leq C[j]$
    $A[k] \leftarrow B[i]$; $i \leftarrow i + 1$
  **else** $A[k] \leftarrow C[j]$; $j \leftarrow j + 1$
  $k \leftarrow k + 1$
**if** $i = p$
  copy $C[j..q-1]$ to $A[k..p+q-1]$
**else** copy $B[i..p-1]$ to $A[k..p+q-1]$

## Running time estimates:

- Home pc executes $10^8$ comparisons/second.
- Supercomputer executes $10^{12}$ comparisons/second.

| | Insertion Sort ($N^2$) | | | Mergesort (N log N) | | |
|---|---|---|---|---|---|---|
| computer | thousand | million | billion | thousand | million | billion |
| home | instant | 2.8 hours | 317 years | instant | 1 sec | 18 min |
| super | instant | 1 second | 1.6 weeks | instant | instant | instant |

**Lesson 1.** Good algorithms are better than supercomputers.

from Sedgewick & Wayne http://www.cs.princeton.edu/courses/archive/spr07/cos226/lectures/04MergeQuick.pdf

# Problem

- Is Mergesort a stable sorting algorithm?

    - A yes
    - B no

# Problem

- Design an algorithm to rearrange elements of a given array of $n$ real numbers so that all its negative elements precede all its non-negative elements. Your algorithm should be both time-efficient and space-efficient.

$l$ $r$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
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$l$ $i$ $r$

| | |
|---|---|
| $< 0$ | $\geq 0$ |

# Partition based on Lomuto's idea

| $l$ | $s$ | | $i$ | $r$ |
|---|---|---|---|---|
| $< 0$ | 6 | $\geq 0$ | 3 | unknown |

- Examine $A[i]$:

  $\geq 0$: just increment $i$

  $< 0$: make room in the segment $A[i..s]$ by

  increment $s$;

  swap $A[i]$ and $A[s]$;

  increment $i$

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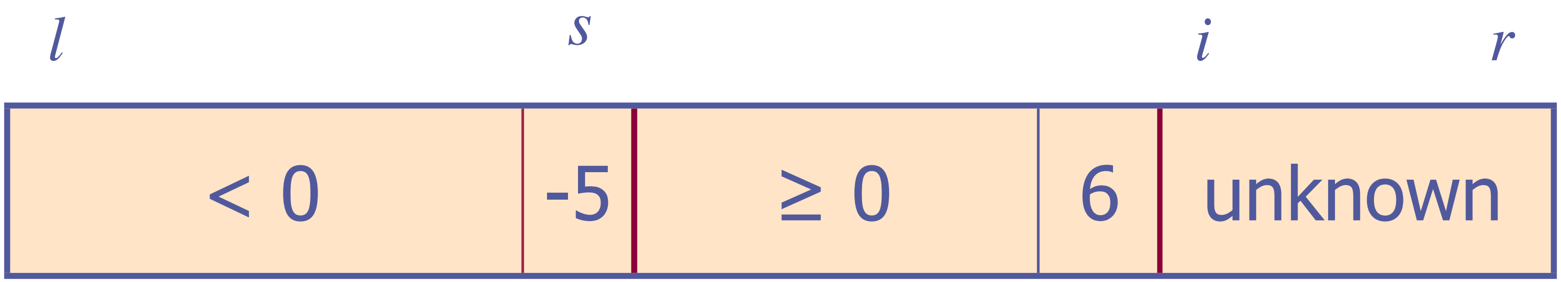


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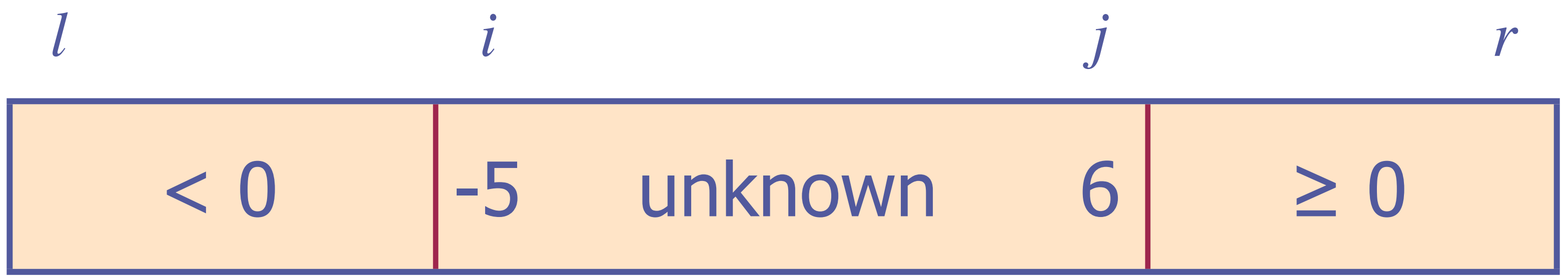
    increment $i$

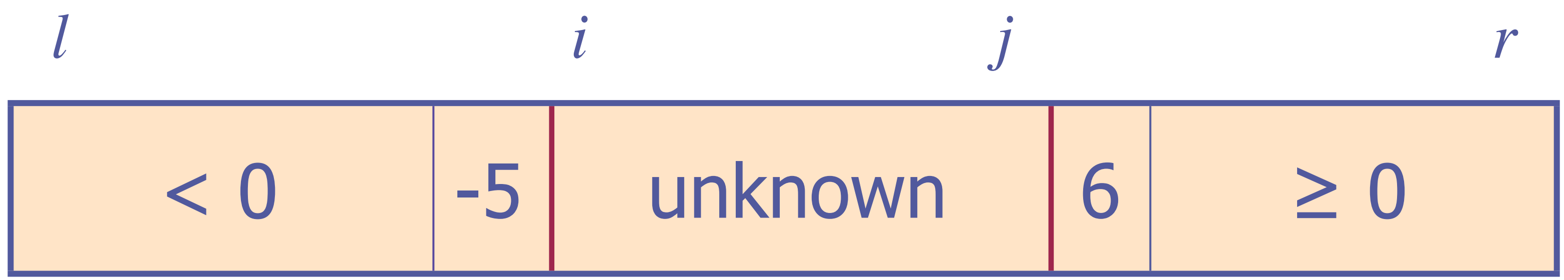
# Partition based on Hoare's idea

$l$ $i$ $j$ $r$

| | | | | |
|---|---|---|---|---|
| < 0 | -5 | unknown | 6 | ≥ 0 |

# Partition based on Hoare's idea



# Partition based on Hoare's idea

| $l$ | $i$ | | $j$ | $r$ |
|---|---|---|---|---|
| < 0 | -5 | unknown | 6 | ≥ 0 |

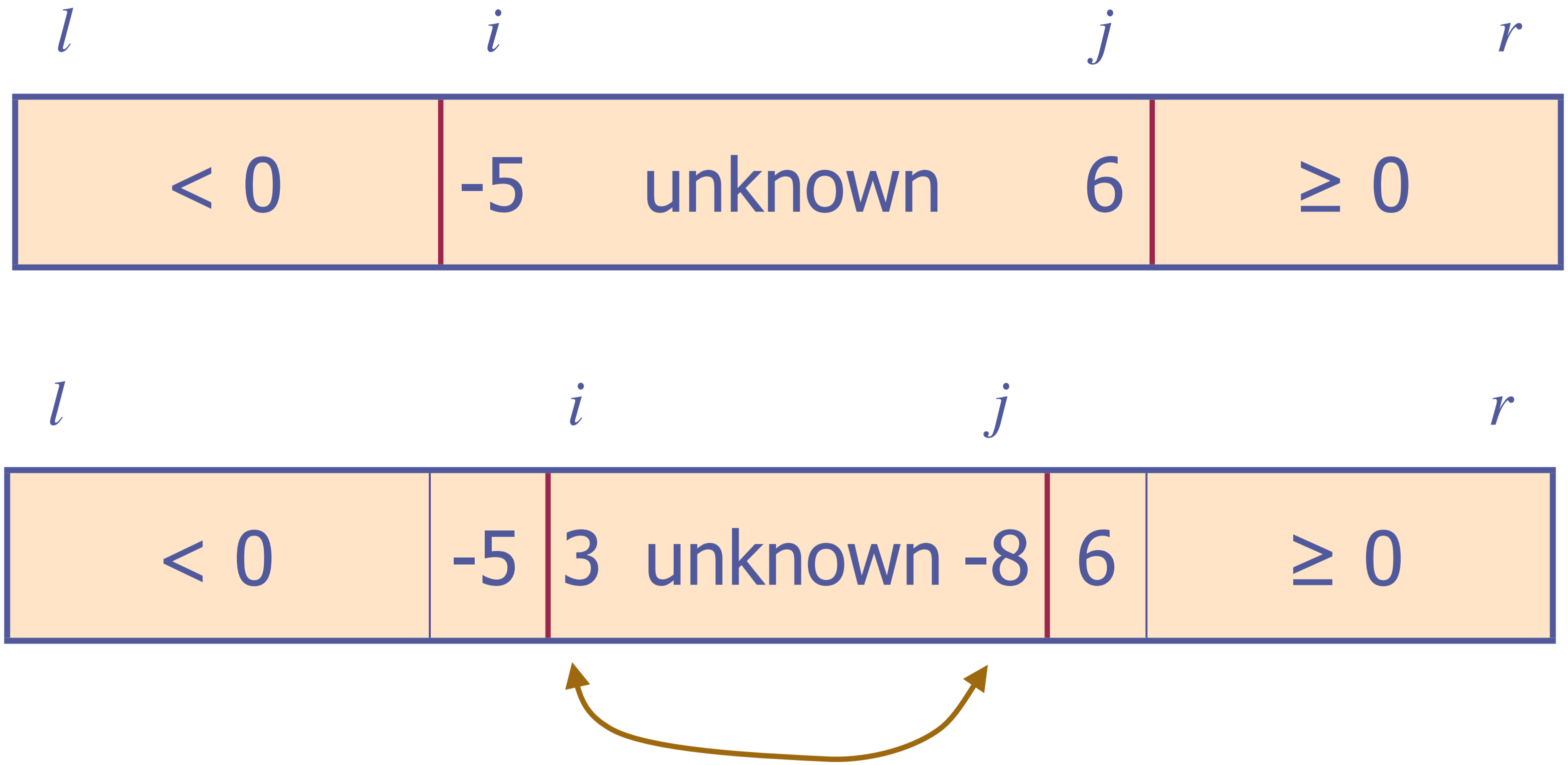
| $l$ | | $i$ | $j$ | | $r$ |
|---|---|---|---|---|---|
| < 0 | -5 | 3 unknown | | 6 | ≥ 0 |

# Partition based on Hoare’s idea

| $l$ | $i$ | $j$ | $r$ |
|---|---|---|---|
| < 0 | -5 unknown 6 | | ≥ 0 |

| $l$ | | $i$ | $j$ | | $r$ |
|---|---|---|---|---|---|
| < 0 | -5 | 3 unknown -8 | 6 | ≥ 0 | |

# Partition based on Hoare's idea



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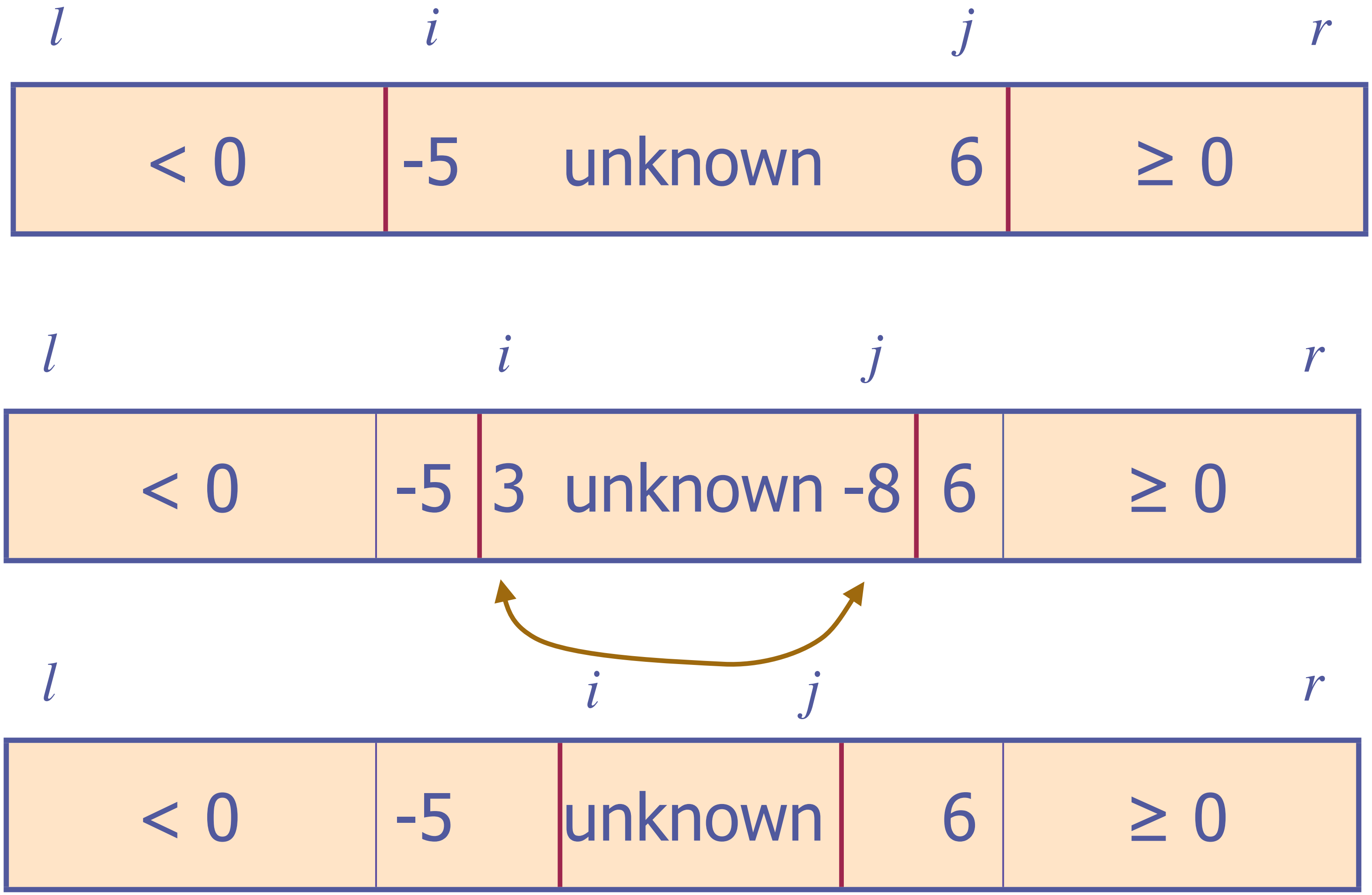

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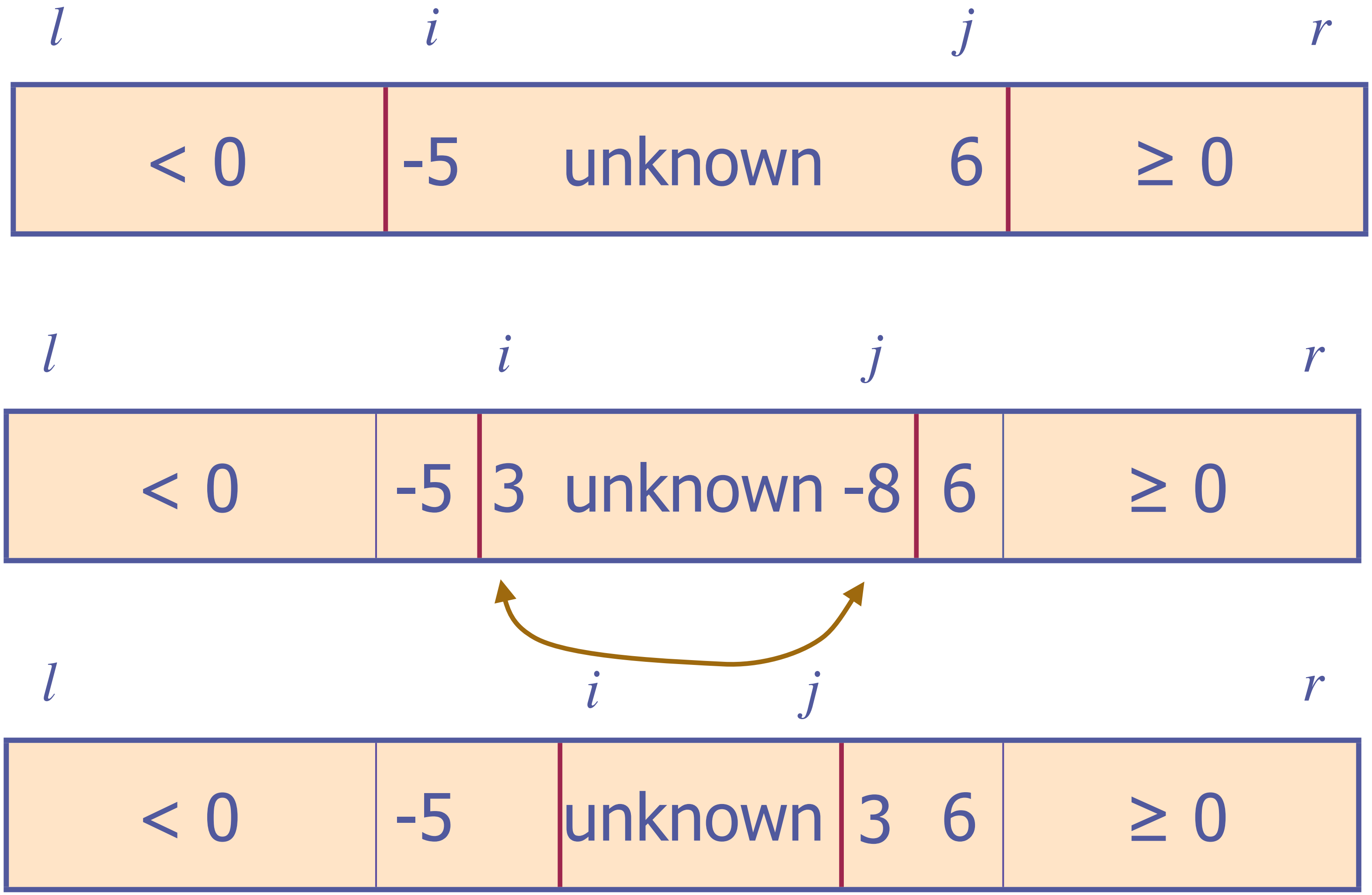

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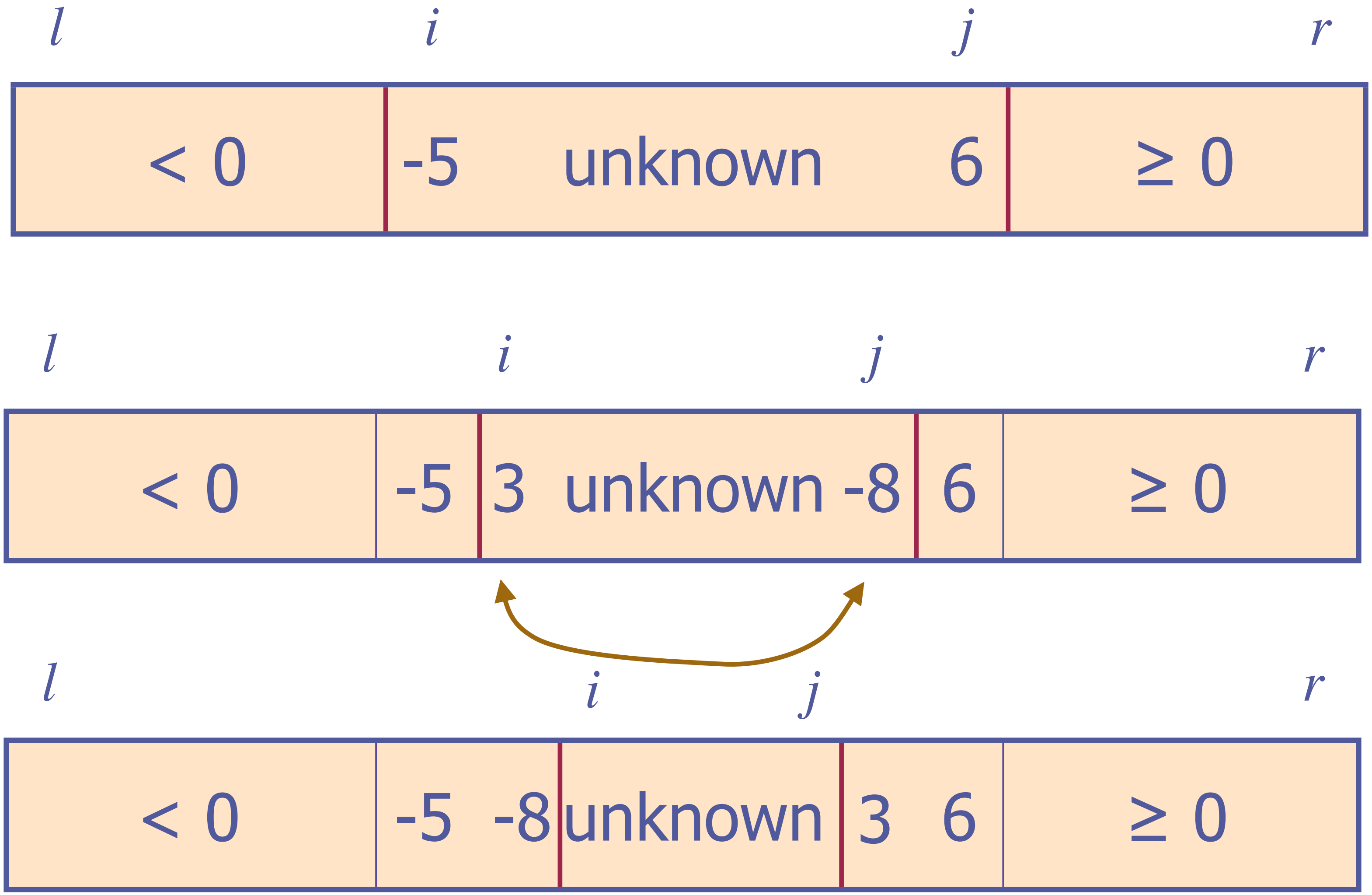
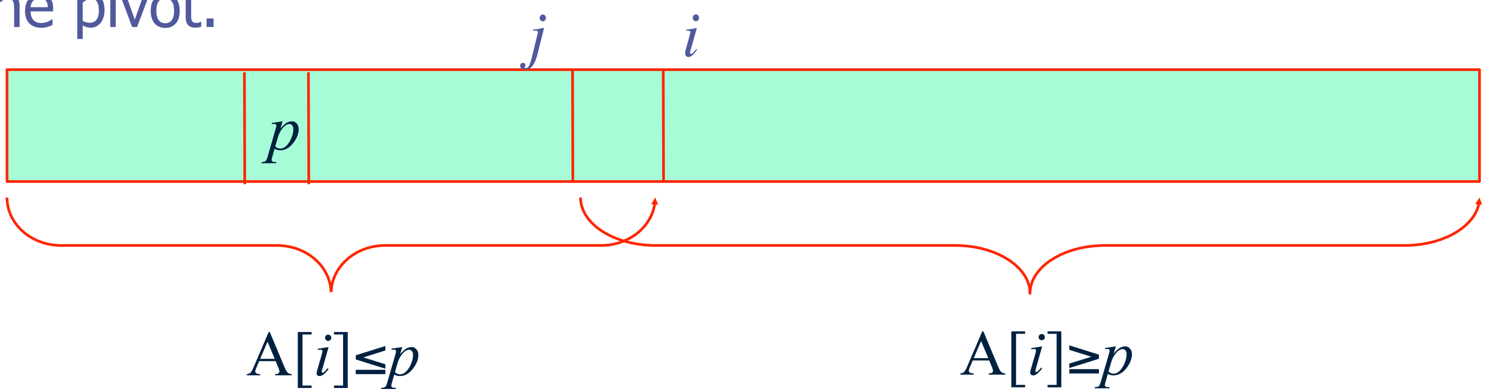


# Quicksort

- Select a pivot (partitioning element) – a random element
- Rearrange the list so that all the elements in the first $j$ positions are smaller than or equal to the pivot and all the elements in the last $n$-$i$ positions are larger than or equal to the pivot.




- Exchange the pivot with the element in its partition closest to the center — the pivot is now in its final position
- Sort the two subarrays recursively

# Hoare Partition Algorithm

```
method partition(A, lo, hi) {
    def pivotIndex = randomBetween(lo)and(hi)
    def pivot = A[pivotIndex]
    var i := lo−1
    var j := hi+1
    while {
        do { i := i + 1 } while { (i ≤ hi).andAlso {A[i] ≤ pivot} }
        do { j := j − 1 } while { (j ≥ lo).andAlso {A[j] ≥ pivot} }
        i < j
    } do { exchange(A, i, j) }
    if (i < pivotIndex) then {
            exchange(A, i, pivotIndex) ;  i := i + 1
    } elseif {j > pivotIndex} then {
            exchange(A, pivotIndex, j) ; j := j − 1
    }
    list.with(i, j)
}
```

# Which Algorithmic Paradigm?

- **Partition**:

A. Brute force
B. Decrease by a constant
C. Decrease by a Variable Amount
D. Divide and Conquer

# Which Algorithmic Paradigm?

- Median Finding using Partition:

  A. Brute force
  B. Decrease by a constant
  C. Decrease by a Variable Amount
  D. Divide and Conquer

# Which Algorithmic Paradigm?

- Sorting using Partition (Quicksort):

  A. Brute force
  B. Decrease by a constant
  C. Decrease by a Variable Amount
  D. Divide and Conquer

# Analysis of Quicksort

- Best case: split in the middle — $\Theta(n \log n)$
- Worst case: choose 1st element from sorted array! — $\Theta(n^2)$
- Average case: random arrays — $\Theta(n \log n)$

- Improvements:
  - better pivot selection: median-of-three partitioning
  - separate partition for keys equal to pivot
  - switch to insertion sort on small sub-problems
  - elimination of recursion
  - These combine to give 20–25% improvement

- Considered the method of choice for internal sorting of large files ($n \geq 10000$)

# Dual-pivot Quicksort

**Vladimir Yaroslavskiy | 11 Sep 12:35 2009**

**Replacement of Quicksort in java.util.Arrays with new Dual-Pivot Quicksort**

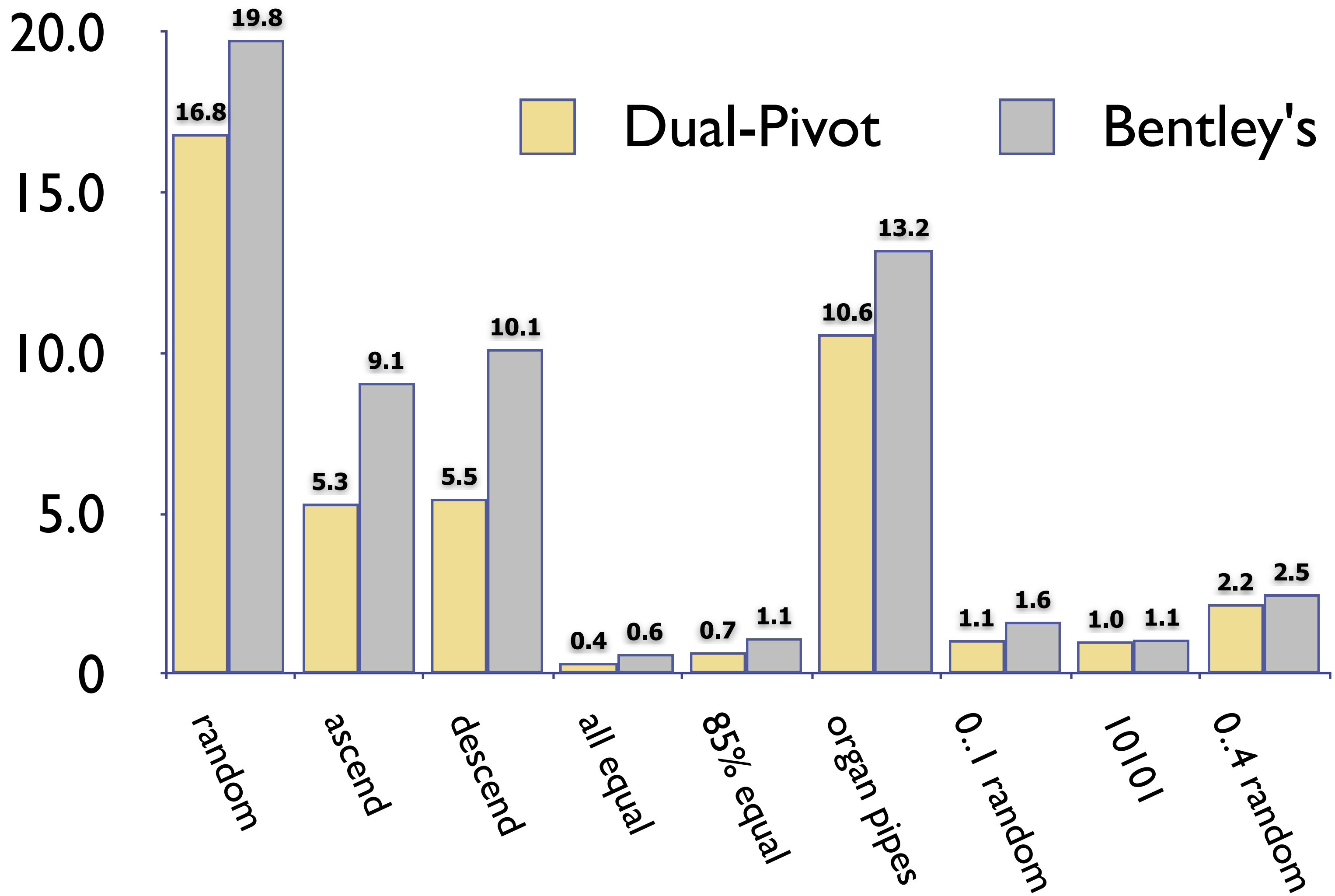
```
Hello All,

I'd like to share with you new Dual-Pivot Quicksort which is
faster than the known implementations (theoretically and
experimental). I'd like to propose to replace the JDK's
Quicksort implementation by new one.
```

...

```
It is proved that for the Dual-Pivot Quicksort the average number of
comparisons is 2*n*ln(n), the average number of swaps is 0.8*n*ln(n),
whereas classical Quicksort algorithm has 2*n*ln(n) and 1*n*ln(n)
respectively.
```

- Read for yourself: http://permalink.gmane.org/gmane.comp.java.openjdk.core-libs.devel/2628

# Performance Comparison



# Jon Bentley (9th Sept 2009):

Vladimir, Josh,

I *finally* feel like I understand what is going on. Now that I (think that) I see it, it seems straightforward and obvious.

Tony Hoare developed Quicksort in the early 1960s. I was very proud to make minor contributions to a particularly clean (binary) quicksort in the mid 1980s, to a relatively straightforward, industrial-strength Quicksort with McIlroy in the early 1990s, and then to algorithms and data structures for strings with Sedgewick in the mid 1990s.

I think that Vladimir's contributions to Quicksort go way beyond anything that I've ever done, and rank up there with Hoare's original design and Sedgewick's analysis. I feel so privileged to play a very, very minor role in helping Vladimir with the most excellent work!

# Problem:

- Is Quicksort stable?

  A. Yes
  B. No

# Problem:

Should we stop scanning when we find an element = pivot?

- Levitin: **repeat** $i \leftarrow i + 1$ **until** $A[i] \geq pivot$
- Hoare: **if** $pivot < A[I]$ **then goto** $down$

# Problem:

Should we stop scanning when we find an element = pivot?

Levitin: **repeat** $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

Hoare: **if** $pivot < A[I]$ **then goto** *down*

Levitin claims: *"Why is it worth stopping the scans after encountering an element equal to the pivot? Because doing this tends to yield more-even splits for arrays with a lot of duplicates, which makes the algorithm run faster."*

QUICKSORT IS OPTIMAL

Robert Sedgewick
Jon Bentley

- "Knuthfest", Stanford University, January, 2002.
  - http://www.sorting-algorithms.com/static/QuicksortIsOptimal.pdf
- Examines an interesting detail: How to handle keys equal to the pivot

# Partitioning with equal keys

How to handle keys equal to the partitioning element?

METHOD A: Put equal keys all on one side?

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

NO: quadratic for n=1 (all keys equal)

METHOD B: Scan over equal keys? (linear for n=1)

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 1 | 1 | 4 | 4 | 4 | 1 | 4 | 1 | 1 | 4 | 4 |
| 1 | 1 | 1 | 1 | 4 | 4 | 4 | 1 | 4 | 1 | 4 | 4 | 4 |

NO: quadratic for n=2

METHOD C: Stop both pointers on equal keys?

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 9 | 4 | 4 | 1 | 4 | 4 | 4 | 9 | 4 | 4 | 1 | 4 |
| 1 | 4 | 4 | 4 | 1 | 4 | 4 | 4 | 9 | 4 | 9 | 4 | 4 |

YES: NlgN guarantee for small n, no overhead if no equal keys

# Partitioning with equal keys

How to handle keys equal to the partitioning element?

METHOD C: Stop both pointers on equal keys?

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 9 | 4 | 4 | 1 | 4 | 4 | 4 | 9 | 4 | 4 | 1 | 4 |
| 1 | 4 | 4 | 4 | 1 | 4 | 4 | 4 | 9 | 4 | 9 | 4 | 4 |

YES: NlgN guarantee for small n, no overhead if no equal keys

METHOD D (3-way partitioning): Put all equal keys into position?

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 9 | 4 | 4 | 1 | 4 | 4 | 4 | 9 | 4 | 4 | 1 | 4 |
| 1 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 9 | 9 |

yes, BUT: early implementations cumbersome and/or expensive

# Quicksort common wisdom (last millennium}

1. Method of choice in practice
   - tiny inner loop, with locality of reference
   - NlogN worst-case "guarantee" (randomized)
   - but use a radix sort for small number of key values
2. Equal keys can be handled (with care)
   - NlogN worst-case guarantee, using proper implementation
3. Three-way partitioning adds too much overhead
   - "Dutch National Flag" problem
4. Average case analysis with equal keys is intractable
   - keys equal to partitioning element end up in both subfiles

# Changes in Quicksort common wisdom

1. Equal keys abound in practice.
   - never can anticipate how clients will use library
   - linear time required for huge files with few key values
2. 3-way partitioning is the method of choice.
   - greatly expands applicability, with little overhead
   - easy to adapt to multikey sort
   - no need for separate radix sort
3. Average case analysis already done!
   - Burge, 1975
   - Sedgewick, 1978
   - Allen, Munro, Melhorn, 1978

# Bentley-McIlroy 3-way partitioning

Partitioning invariant

| equal | less | | greater | equal |
|---|---|---|---|---|

- move from left to find an element that is not less
- move from right to find an element that is not greater
- stop if pointers have crossed
- exchange
- if left element equal, exchange to left end
- if right element equal, exchange to right end

Swap equals to center after partition

| less | equal | greater |
|---|---|---|

KEY FEATURES

- always uses N-1 (three-way) compares
- no extra overhead if no equal keys
- only one “extra” exchange per equal key

# Quicksort with 3-way partitioning

```
void quicksort(Item a[], int l, int r)
{ int i = l-1, j = r, p = l-1, q = r; Item v = a[r];
  if (r <= l) return;
  for (;;)
    {
      while (a[++i] < v) ;
      while (v < a[--j]) if (j == l) break;
      if (i >= j) break;
      exch(a[i], a[j]);
      if (a[i] == v) { p++; exch(a[p], a[i]); }
      if (v == a[j]) { q--; exch(a[j], a[q]); }
    }
  exch(a[i], a[r]); j = i-1; i = i+1;
  for (k = l; k < p; k++, j--) exch(a[k], a[j]);
  for (k = r-1; k > q; k--, i++) exch(a[i], a[k]);
  quicksort(a, l, j);
  quicksort(a, i, r);
}
```

## Running time estimates:

- Home pc executes $10^8$ comparisons/second.
- Supercomputer executes $10^{12}$ comparisons/second.

Insertion Sort ($N^2$)

| computer | thousand | million | billion |
|---|---|---|---|
| home | instant | 2.8 hours | 317 years |
| super | instant | 1 second | 1.6 weeks |

Mergesort (N log N)

| thousand | million | billion |
|---|---|---|
| instant | 1 sec | 18 min |
| instant | instant | instant |

Quicksort (N log N)

| thousand | million | billion |
|---|---|---|
| instant | 0.3 sec | 6 min |
| instant | instant | instant |

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

# The Problem of the Dutch National Flag

- Levitin §5.2, Q9

The ***Dutch flag problem*** is to rearrange an array of characters $R$, $W$, and $B$ (red, white, and blue are the colors of the Dutch national flag) so that all the $R$'s come first, the $W$'s come next, and the $B$'s come last. Design a linear in-place algorithm for this problem.

# Quickhull

- Divide-and-conquer algorithm for Convex Hull of a set of points P

  - Find two points A,B∈P that are both on the convex hull

  divide the set P into two subsets, $P_l$ left and $P_r$ right of chord AB.

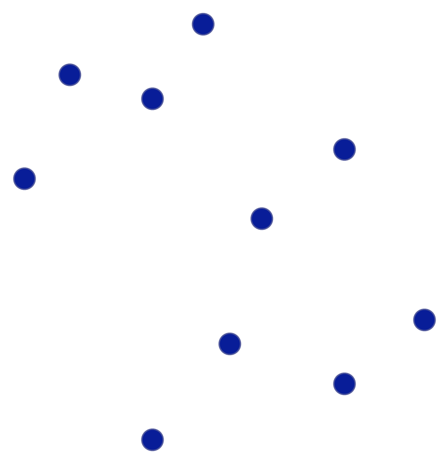
  Find point C in $P_l$ furthest from AB.

  - discard the points inside △ABC

  - recurse with chords AC and CB

    …

  repeat with chord AB and $P_r$



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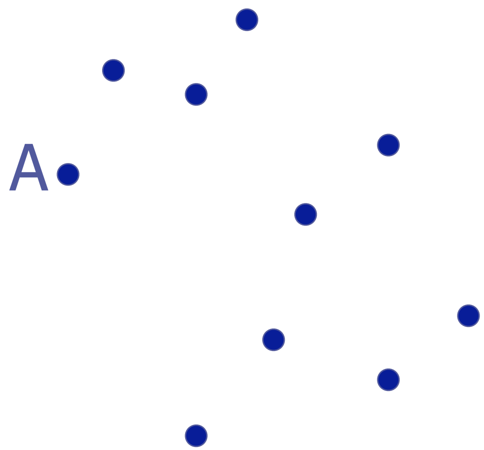

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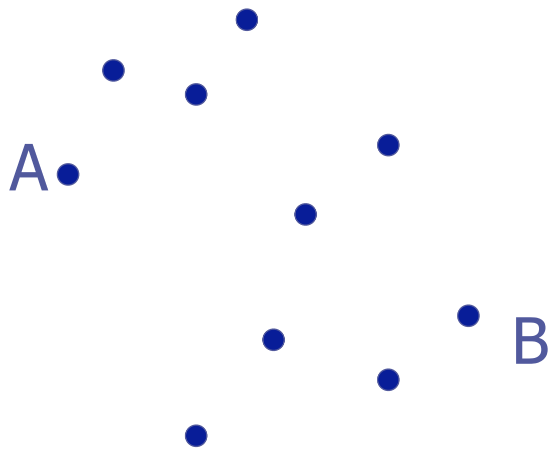

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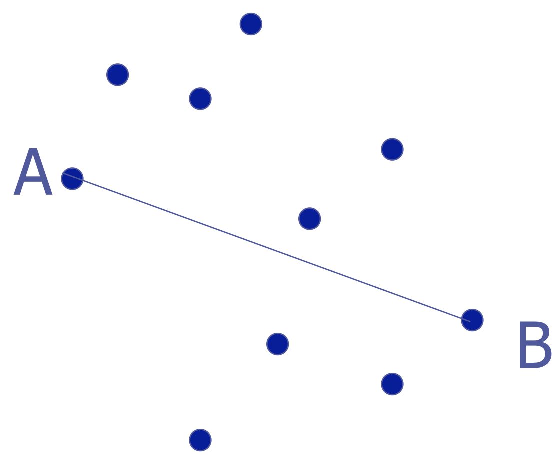

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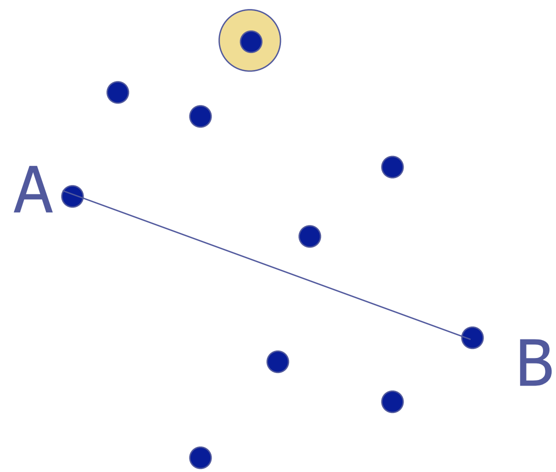

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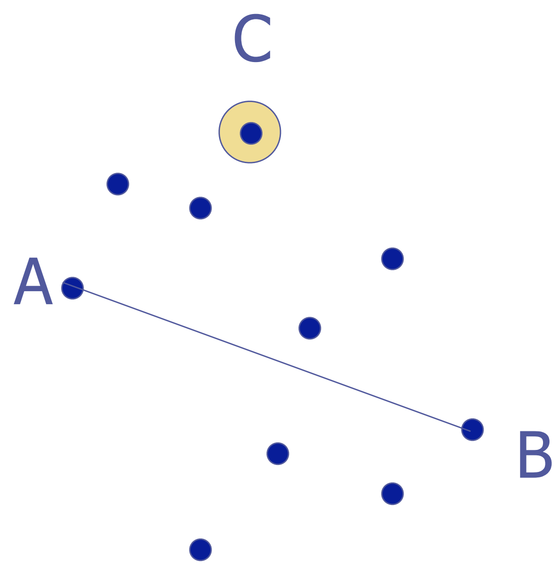

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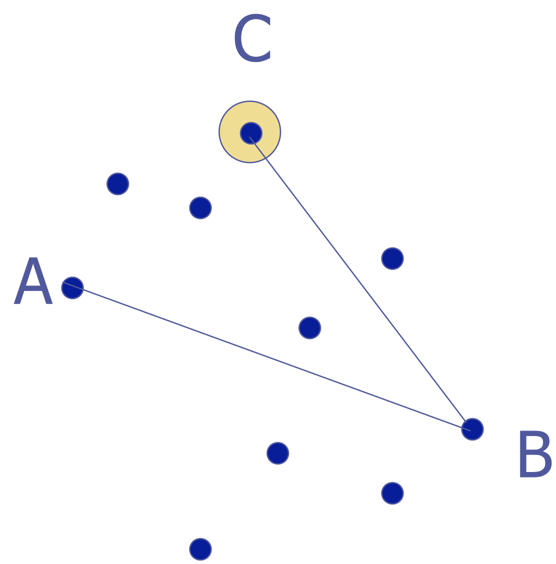

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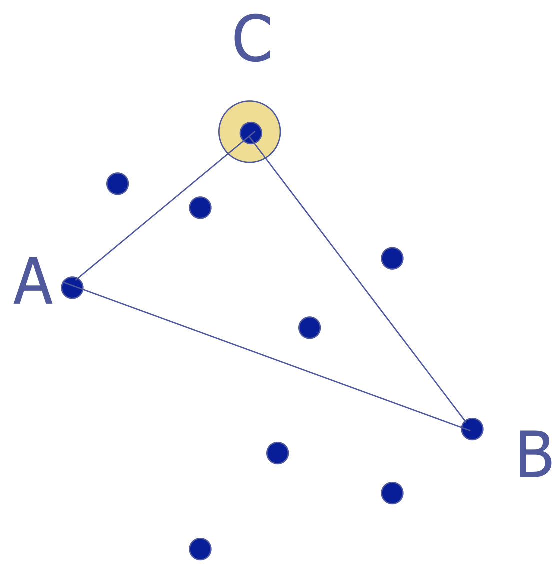

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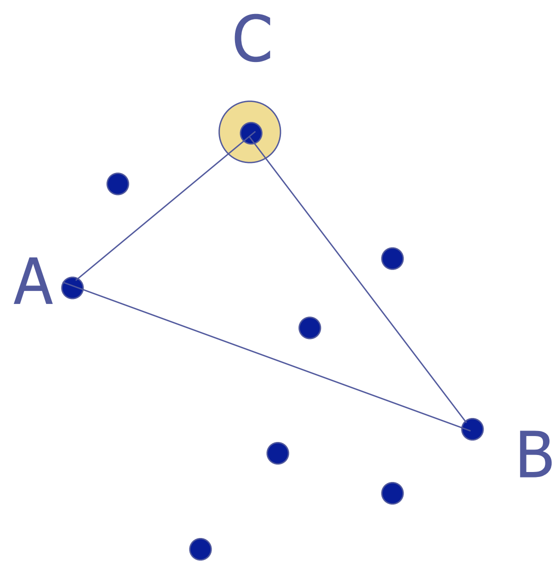

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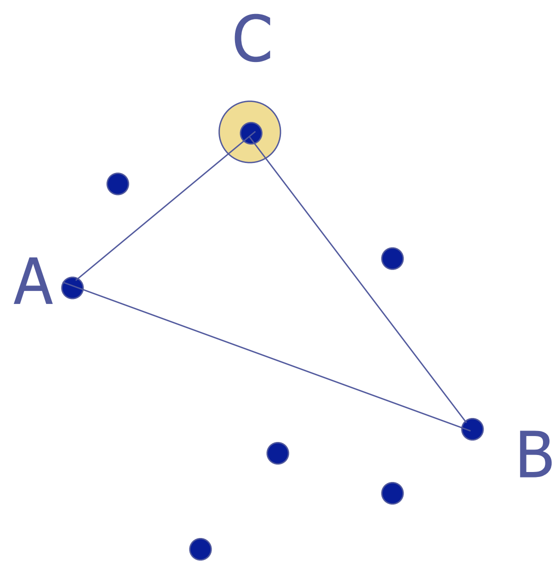

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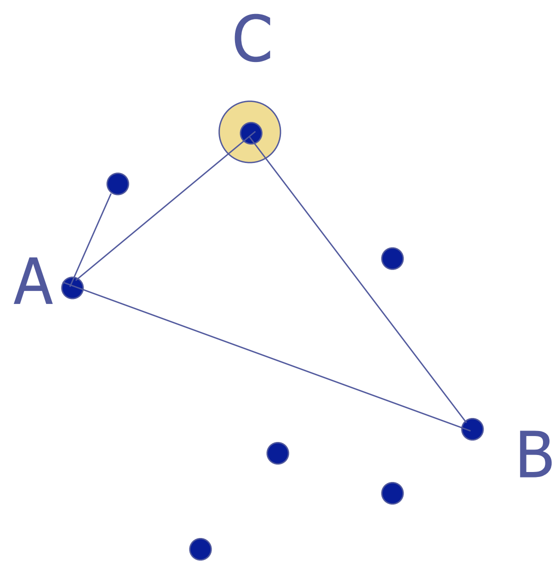

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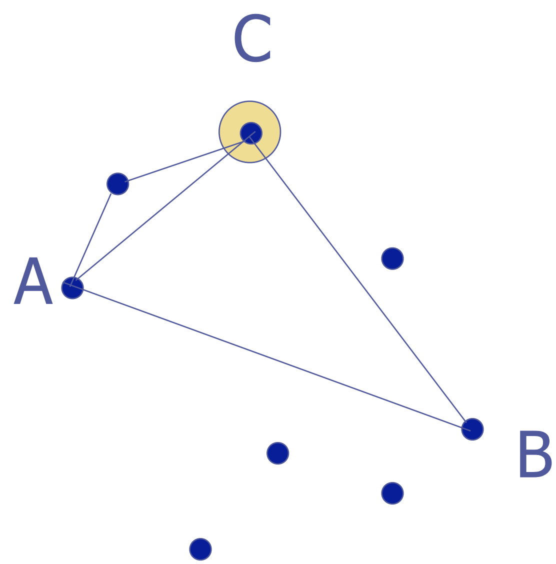

# Animation:

- Quickhull at Princeton