Analysis of Algorithms: Methods and Examples

CS 350 – Algorithms and Complexity
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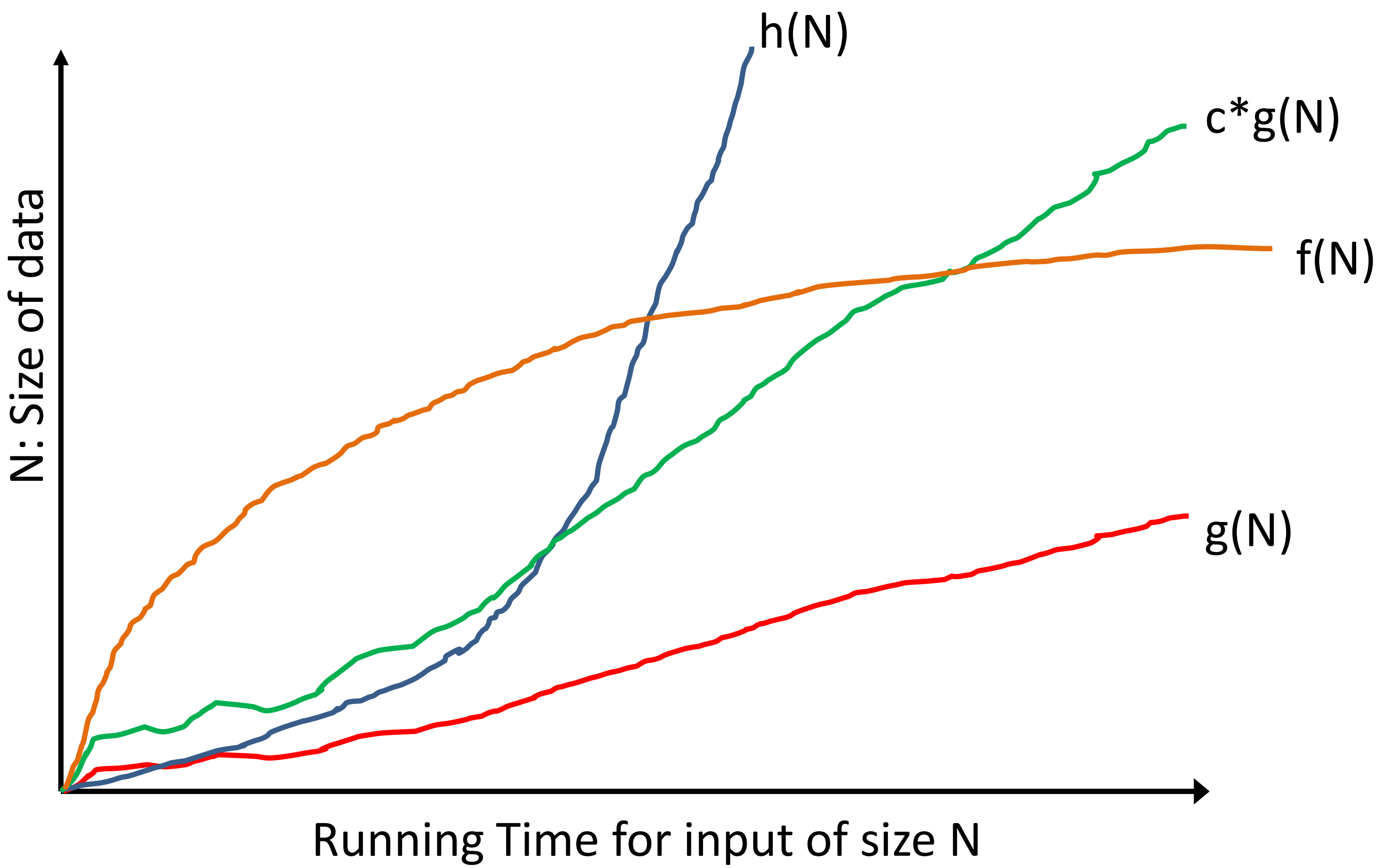
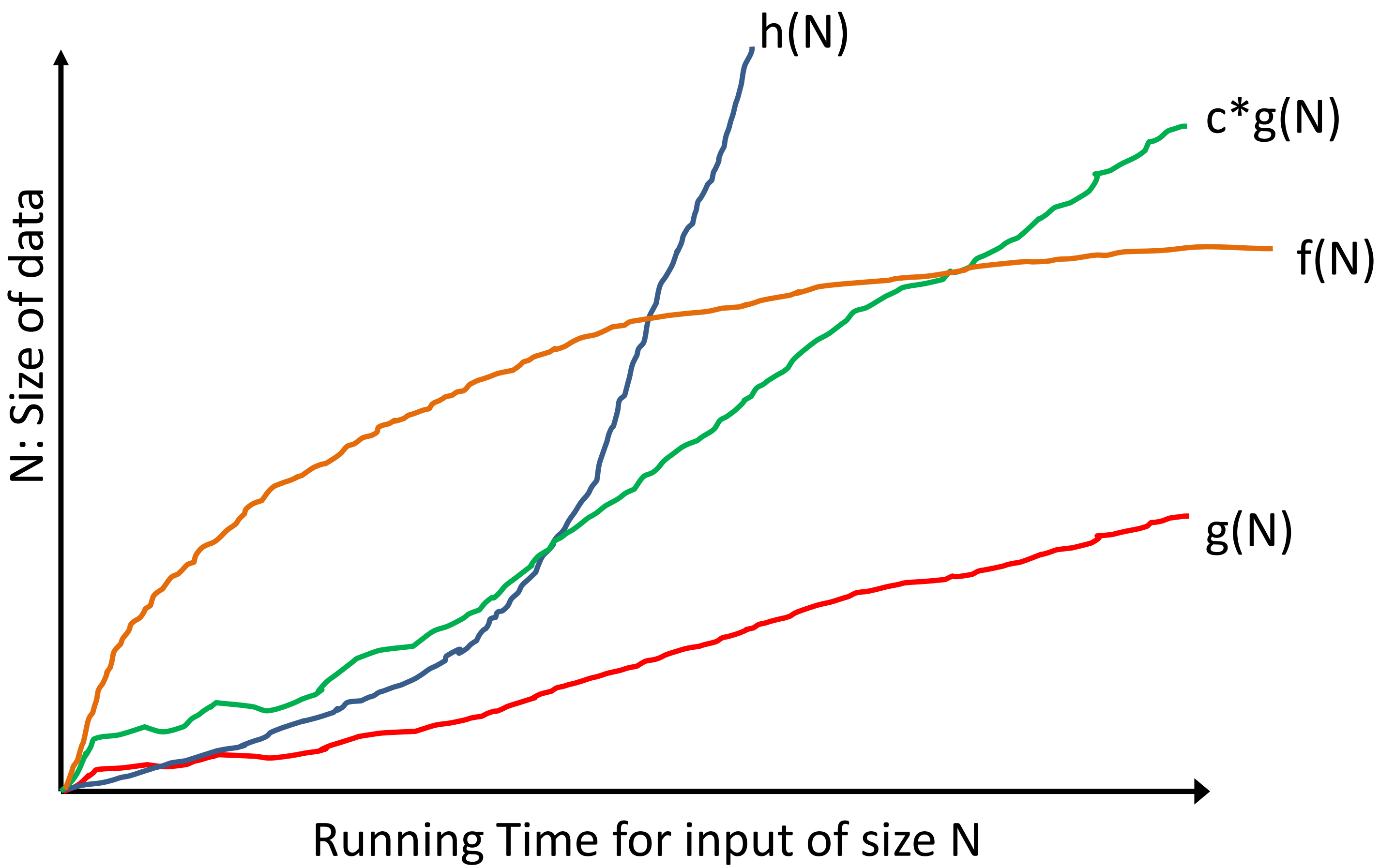
Why Asymptotic Behavior Matters

- Asymptotic behavior: The behavior of a function as the input approaches infinity.



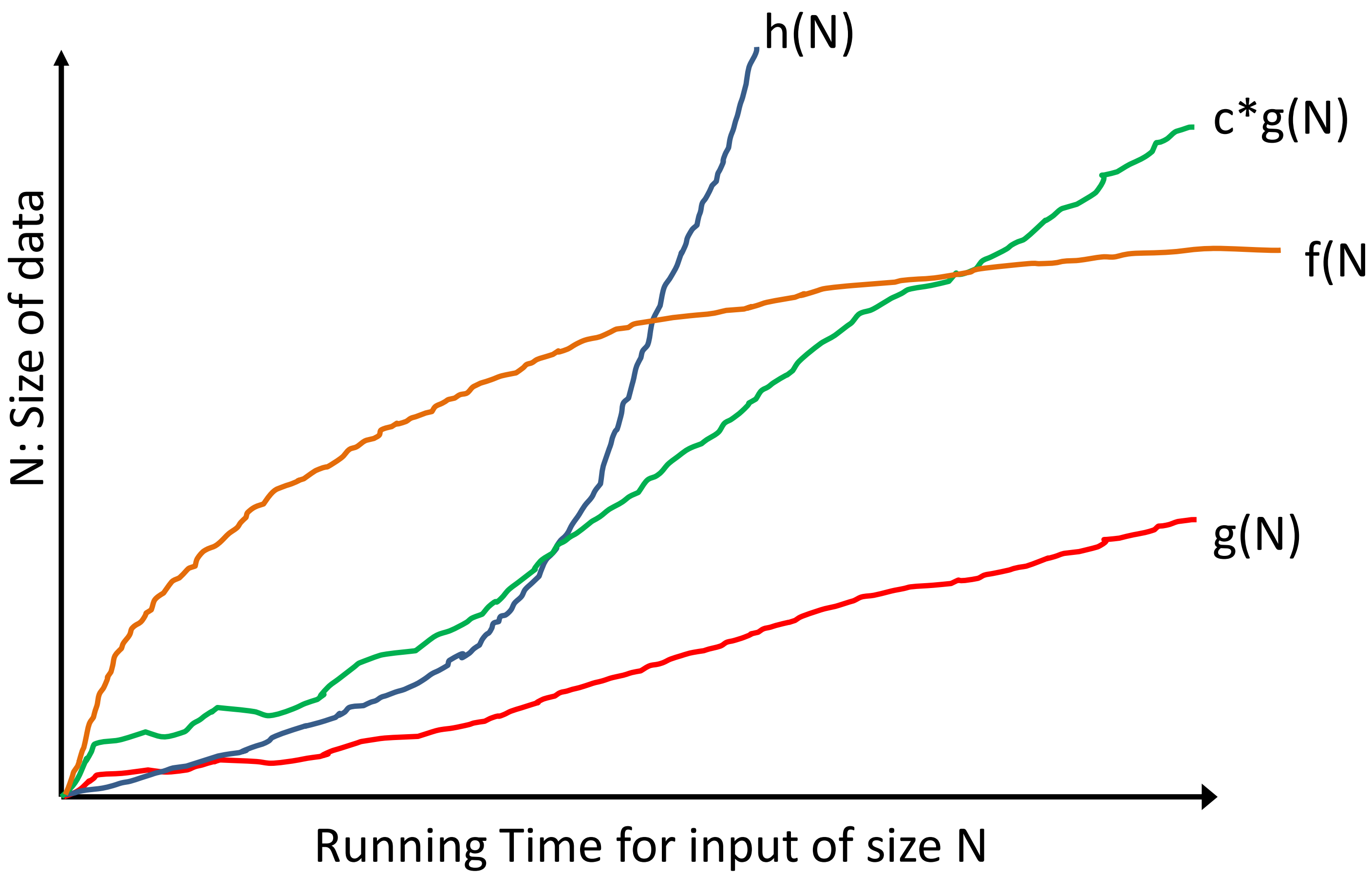
Why Asymptotic Behavior Matters

- Which of these functions is smallest asymptotically?



Why Asymptotic Behavior Matters

- Which of these functions is smallest asymptotically?
  - f(N) seems to grow very slowly after a while.



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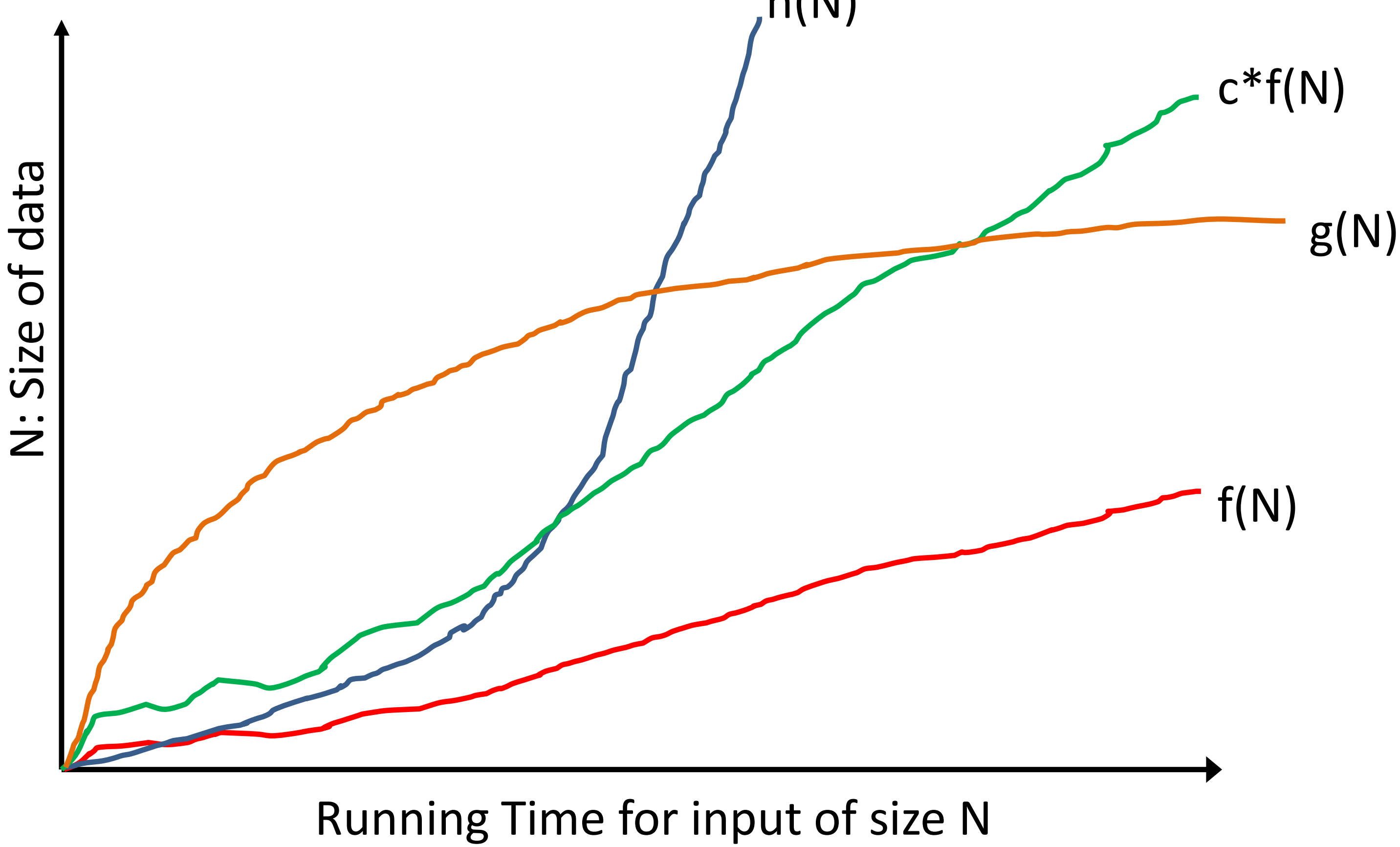
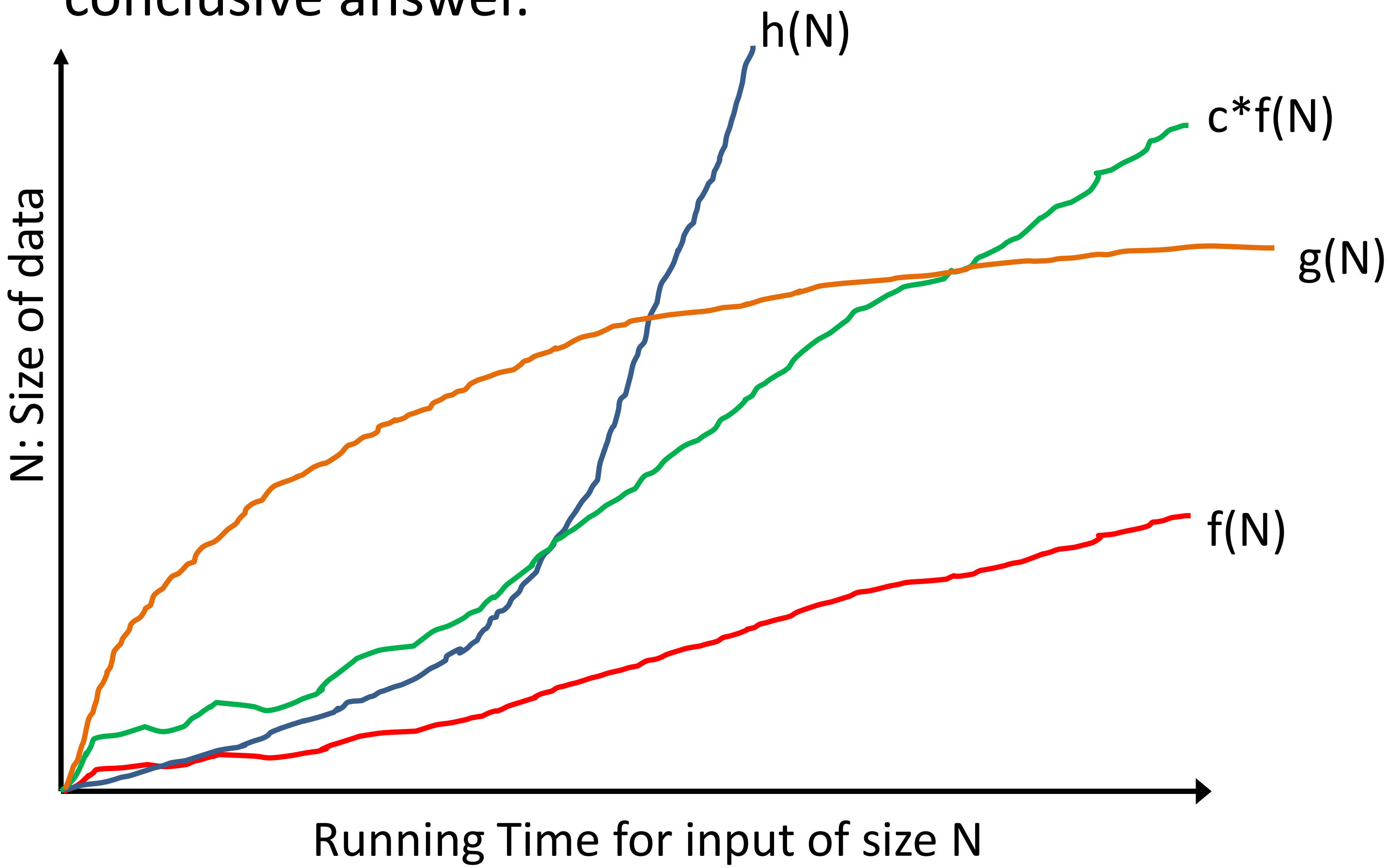
- Which of these functions is smallest asymptotically?
  - However, the picture is not conclusive (need to see what happens for larger N).



Why Asymptotic Behavior Matters

- Which of these functions is smallest asymptotically?
  - Proving that $f(N) = O(g(N))$ would provide a conclusive answer.



Using Limits for Comparing Order of Growth

- $\lim_{n\to\infty} \frac{t(n)}{g(n)} = 0$, **implies that *t(n)* has a smaller order of growth than *g(n)***
- $\lim_{n\to\infty} \frac{t(n)}{g(n)} = c, c > 0$, **implies that *t(n)* has the same order of growth as** *g(n)*
- $\lim_{n\to\infty} \frac{t(n)}{g(n)} = \infty$, **implies that *t(n)* has a larger order of growth than *g(n)***
- Note that the first two cases mean that: $t(n) \in O(g(n))$
- Note that the last two cases mean that: $t(n) \in \Omega(g(n))$
- Note that the second case means that: $t(n) \in \Theta(g(n))$

Using Limits - Comments

- The previous formulas relating limits to big-Oh notation show once again that big-Oh notation ignores:
  - constants
  - behavior for small values of N.
- How do we see that?

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- The previous formulas relating limits to big-Oh notation show once again that big-Oh notation ignores:
  - constants
  - behavior for small values of N.
- How do we see that?
  - In the previous formulas, it is sufficient that the limit is equal to a constant. **The value of the constant does not matter.**
  - In the previous formulas, only **the limit at infinity** matters. This means that we can ignore behavior up to any finite value, if we need to.

Using Limits: An Example

- Show that $\dfrac{n^5+3n^4+2n^3+n^2+n+12}{5n^3+n+3} = \Theta(???)$.

Using Limits: An Example

- Show that $\frac{n^5+3n^4+2n^3+n^2+n+12}{5n^3+n+3} = \Theta(n^2)$.

- Let $\mathrm{f}(n) = \frac{n^5+3n^4+2n^3+n^2+n+12}{5n^3+n+3}$
- Let $g(n) = n^2$.

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\left(\frac{n^5+3n^4+2n^3+n^2+n+12}{5n^3+n+3}\,\frac{1}{n^2}\right)$$
$$= \lim_{n\to\infty}\left(\frac{n^5+3n^4+2n^3+n^2+n+12}{5n^5+n^3+3n^2}\right) = \frac{1}{5}$$

Using Limits: An Example

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- Let $\mathrm{f}(n) = \frac{n^5+3n^4+2n^3+n^2+n+12}{5n^3+n+3}$
- Let $g(n) \;=\; n^2$.

- In the previous slide, we showed that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1}{5}$
- Therefore, $f(n) \;=\; \Theta(g(n))$.

Big-Oh Hierarchy

- $1 = O(\log(N))$
- $\log(N) = O(N)$
- $N = O(N^2)$
- If $c \geq d \geq 0$, then $N^d = O(N^c)$.

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- For any d, if $c > 1$, $N^d = O(c^N)$.

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- For any d, if $c > 1$, $N^d = O(c^N)$.
  - Exponential functions always get larger than polynomial functions, eventually.
- You can use these facts in your assignments.
- You can apply transitivity to derive other facts, e.g., that $\log(N) = O(N^2)$.

Using Substitutions

- If $\lim_{x \to \infty} h(x) = \infty$, then:

$$f(x) = O\big(g(x)\big) \Rightarrow f(h(x)) = O(g(h(x))).$$

- How do we use that?
- For example, prove that $\log\big(\sqrt{N}\big) = O(\sqrt{N})$.

Using Substitutions

- If $\lim_{x \to \infty} h(x) = \infty$, then:

$$g(x) = O\big(f(x)\big) \Rightarrow g(h(x)) = O(f(h(x))).$$

- How do we use that?
- For example, prove that $\log\left(\sqrt{N}\right) = O(\sqrt{N})$.
- Use $\mathrm{h(x)} = \sqrt{N}$. We get:

$$\log(\mathrm{N}) = \mathrm{O}(\mathrm{N}) \Rightarrow \log\left(\sqrt{N}\right) = O\left(\sqrt{N}\right)$$

Big-Oh Notation: Example Problem

- Is $N = O(\sin(N)\, N^2)$?
- Answer:

Big-Oh Notation: Example Problem

- Is $N = O(\sin(N)\, N^2)$?
- Answer: no!
- Why? $\sin(N)$ fluctuates forever between -1 and 1.
- As a result, $\sin(N)\, N^2$ fluctuates forever between negative and positive values.
- Therefore, for every possible $c_0 > 0$ and N_0, we can always find an $N > N_0$ such that:

$$N > c_0 \sin(N) N^2$$