$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$ $\leftarrow \theta_1$

2 MLE based tests:

1) Wald's test
2) Likelihood ratio test

$X_1, \ldots, X_n \sim f(x|\theta) = \theta x^{-\theta-1}$

$x \geq 1$

$\theta > 0$

Pareto distribution

$H_0: \theta = 2$ vs. $H_1: \theta \neq 2$

$\hat{\theta}_{MLE} = 2.45, \ n = 100$

Wald's test: $\sqrt{n}\,(\hat{\theta}_{MLE} - \theta) \xrightarrow{(d)} N(0, I^{-1}(\theta))$

$$\sqrt{n}\sqrt{I(\theta)}\,(\hat{\theta}_{MLE} - \theta) \xrightarrow{(d)} N(0,1)$$

$$n\, I(\hat{\theta}_{MLE})(\hat{\theta}_{MLE} - \theta)^2 \xrightarrow{(d)} \chi^2_1$$

Likelihood ratio test

$$\frac{\sup_{\theta \in \Theta_1} \cancel{\max} \, L(X_1, \ldots, X_n | \theta)}{L(X_1, \ldots, X_n | \theta_0)} > c > 1$$

$$\frac{L(X_1, \ldots, X_n | \hat{\theta}_{MLE})}{L(X_1, \ldots, X_n | \theta_0)} > c$$

$$2\left(\ell(X_1, \ldots, X_n | \hat{\theta}_{MLE}) - \ell(X_1, \ldots, X_n | \theta_0)\right) > \log c$$

Wilk's thm: $\xrightarrow{(d)} \chi^2_1$

$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$ ← θ_1

2 MLE based tests:

1) Wald's test
2) Likelihood ratio test

$X_1, \dots, X_n \sim f(x|\theta) = \theta x^{-\theta-1}$

$x \geq 1$

$\theta > 0$

Pareto distribution

$H_0: \theta = 2$ vs. $H_1: \theta \neq 2$

$\hat{\theta}_{MLE} = 2.45,\ n = 100$

MLE:

$$L(x_1, \dots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \prod_{i=1}^{n} \theta x_i^{-\theta-1}$$

$$= \theta^n \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1}$$

$$\ell(x_1, \dots, x_n|\theta) = \log\left(\theta^n \left(\prod_{i=1}^{n} x_i\right)^{-\theta-1}\right)$$

$$= n\log\theta + (-\theta-1)\sum_{i=1}^{n} \log X_i$$

$$\frac{d\ell}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \log X_i = 0$$

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \log X_i}$$

$$\frac{d^2\ell}{d\theta^2} = -\frac{n}{\theta^2} < 0$$

$$I(\theta) = E - \frac{d^2 \ell(x|\theta)}{d\theta^2} = E - \left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

θ_1

2 MLE based tests:

1) Wald's test
2) Likelihood ratio test

$$X_1, \dots, X_n \sim f(x|\theta) = \theta x^{-\theta - 1}$$

$x \geq 1$

$\theta > 0$

Pareto distribution

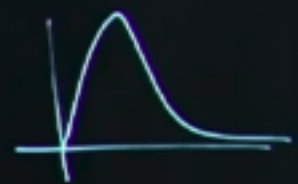
$H_0: \theta = 2$ vs. $H_1: \theta \neq 2$

$\hat{\theta}_{MLE} = 2.45$, $n = 100$

- $\ell(X_1, \dots, X_n | \theta) = n \log \theta + (-\theta - 1) \sum_{i=1}^n \log X_i$
- $\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^n \log X_i}$
- $I(\theta) = \frac{1}{\theta^2}$

- Wald's test

$$\underbrace{n \cdot \frac{1}{\hat{\theta}_{MLE}^2} \left(\hat{\theta}_{MLE} - \theta_0\right)^2}_{T_n} \xrightarrow{(d)} \chi^2_1 \text{ (Under } H_0)$$



$$\psi = \mathbb{1}_{(T_n \geq \chi^2_{1, 0.05})}$$

$\chi^2_{1,0.05} \rightarrow$ $1 - 0.05$ quantile of a χ^2_1

- Likelihood ratio test

$$\underbrace{2\left(\ell(X_1, \dots, X_n | \hat{\theta}_{MLE}) - \ell(X_1, \dots, X_n | \theta_0)\right)}_{T_n'} \xrightarrow{(d)} \chi^2_1$$

(Under H_0)

$$\psi' = \mathbb{1}_{(T_n' \geq \chi^2_{1, 0.05})}$$

$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$ ← θ_1

2 MLE based tests:

1) Wald's test
2) Likelihood ratio test

$X_1, \dots, X_n \sim f(x|\theta) = \theta x^{-\theta-1}$

$x \geq 1$

$\theta > 0$

Pareto distribution

$H_0: \theta = 2$ vs. $H_1: \theta \neq 2$

$\hat{\theta}_{MLE} = 2.45$, $n = 100$

- $\ell(X_1, \dots, X_n|\theta) = n \log\theta + (-\theta - 1)\sum_{i=1}^{n} \log X_i$
- $\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \log X_i}$
- $I(\theta) = \frac{1}{\theta^2}$

- Wald's test : $\psi = \mathbb{1}_{(T_n \geq \chi^2_{1,0.05})} = \mathbb{1}_{(T_n \geq 3.84)}$

$$T_n = n \cdot \frac{1}{\hat{\theta}^2_{MLE}} (\hat{\theta}_{MLE} - \theta_0)^2 = 100 \cdot \frac{1}{2.45^2} (2.45 - 2)^2$$

$$= 3.37$$

Wald's test fails to reject H_0

- Likelihood ratio test: $\psi = \mathbb{1}_{(T_n' \geq 3.84)}$

$$T_n' = 2\left(\ell(X_1, \dots, X_n|\hat{\theta}_{MLE}) - \ell(X_1, \dots, X_n|\theta_0)\right)$$

$$\ell(X_1, \dots, X_n|\theta_0) = n \cdot \log\theta_0 + (-\theta_0 - 1) \cdot \underbrace{\sum_{i=1}^{n} \log X_i}_{= \frac{1}{mle} \cdot n}$$

$$= n \cdot \log\theta_0 + (-\theta_0 - 1) \cdot \frac{n}{\hat{\theta}_{MLE}}$$

$T_n' = 3.85 \rightarrow$ Likelihood ratio test rejects H_0