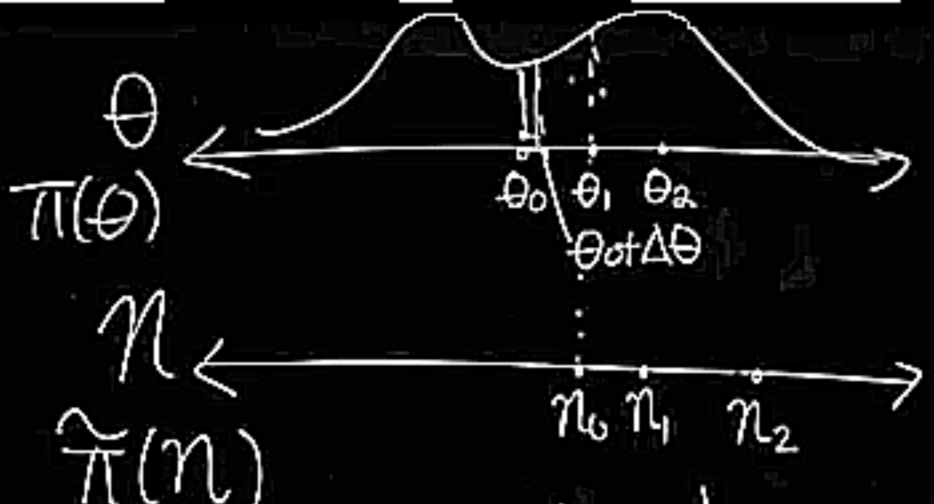
Reparameterizing a Distribution

- Distribution $\pi(\theta)$, $\theta \in \mathbb{R}$.
- Reparameterize with $\eta = \phi(\theta)$
- ϕ strictly monotone
- Corresponding $\boxed{\tilde{\pi}(\eta)?}$



$\eta = \phi(\theta)$.

$\phi(x) = 2x$

$\phi' \quad \dfrac{d\eta}{d\theta}$

$$\int_{\theta_0}^{\theta_1} \pi(\theta)\,d\theta = \int_{\eta_0}^{\eta_1} \tilde{\pi}(\eta)\,d\eta.$$

$$\left|\int_{\theta_0}^{\theta_0+\Delta\theta} \pi(\theta)\,d\theta\right| = |\Delta\theta|\,\pi(\theta_0)$$

$$\left|\int_{\eta_0}^{\eta_0+\Delta\eta} \tilde{\pi}(\eta)\,d\eta\right| = |\Delta\eta|\,\tilde{\pi}(\eta_0).$$

$$\underline{\pi(\theta_0)\,|\Delta\theta| = \tilde{\pi}(\eta_0)\,|\Delta\eta|}.$$

$$\underline{\tilde{\pi}(\eta_0)} = \pi(\theta_0)\left|\frac{\Delta\theta}{\Delta\eta}\right| = \pi(\theta_0)\left|\frac{d\eta}{d\theta}\right|_{\theta_0}$$

$$= \pi(\theta_0)\,\phi'(\theta_0)^{-1} = \boxed{\frac{\pi(\phi^{-1}(\eta_0))}{|\phi'(\phi^{-1}(\eta_0))|}}$$

$$\boxed{\tilde{\pi}(\eta) = \frac{\pi(\phi^{-1}(\eta))}{|\phi'(\phi^{-1}(\eta))|}}$$

Reparametrization Example

$\pi(x)=e^{-x},\ x\geq 0$

$y=\phi(x)=x^2$

$$\tilde{\pi}(y)=\frac{\pi(\phi^{-1}(y))}{|\phi'(\phi^{-1}(y))|}=\frac{\pi(\sqrt{y})}{|\phi'(\sqrt{y})|}=\boxed{\frac{e^{-\sqrt{y}}}{2\sqrt{y}}}$$

$\phi'(x)=2x$

$\phi^{-1}(x)=\sqrt{x}$

CDF in x $\quad 1-e^{-x}$

CDF in y $\quad 1-e^{-\sqrt{y}}$.

Jeffreys Prior Intuition

$\pi_J(\theta) \propto \sqrt{I(\theta)}$

$I(\theta) = E\left[\left(\frac{d}{d\theta}\ln L(X_i|\theta)\right)^2\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\ln L(X_i|\theta)\right]$

$I(\theta)(\Delta\theta)^2 = E[(\Delta \ln L(X_i|\theta))^2]$

- Gives high weight to θ with a high Fisher info., which is where

1) Marginal shifts have a relativitely large effect to X.
2) MLE of θ is more certain (has asymp. var of $I(\theta)^{-1}$)

Why $\sqrt{I(\theta)}$, not $I(\theta)$?

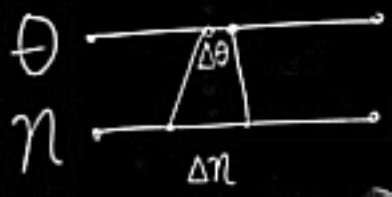
- In MLE terms, $\sqrt{I(\theta)}^{-1}$ is in the same units as θ and thus represents the radius of uncertainty.

Main Idea

- In a model $\theta \to \mathbb{P}_\theta$, parametrizations may have different scales

- Jeffreys prior converts a parametrized distribution into a universal form by taking sensitivity into account.

Jeffreys Prior Reparametrization Invariance Proof



$I_0(\theta)$ $\quad$ $\pi_J(\theta)$

$I_1(\eta)$ $\quad$ $\tilde{\pi}_J(\eta)$

$\theta_0 \quad \eta_0.$

$\phi(\theta_0) = \eta_0.$

$$I_0(\theta) = E\left[\left(\frac{\partial}{\partial\theta} \ln L_0(X_i|\theta)\right)^2\right]$$

$$I_1(\eta) = E\left[\left(\frac{\partial}{\partial\eta} \ln L_1(X_i|\eta)\right)^2\right].$$

$$\Delta \ln L(x_i|\theta) = \left(\frac{d}{d\theta} \ln L_0(X_i|\theta)\right)\Delta\theta = \left(\frac{d}{d\eta} \ln L_1(X_i|\eta)\right)\Delta\eta.$$

$$I_1(\eta) = E\left[\left(\frac{d}{d\theta} \ln L_0(X_i|\theta)\right)^2\right]\left(\frac{\Delta\theta}{\Delta\eta}\right)^2.$$

$$= I_0(\theta)\left|\frac{\Delta\theta}{\Delta\eta}\right|^2$$

$$\tilde{\pi}_J(\eta) = \pi_J(\theta)\left|\frac{\Delta\theta}{\Delta\eta}\right| \Rightarrow \tilde{\pi}_J(\eta)|\Delta\eta| = \pi_J(\theta)|\Delta\theta|$$

Reparametrization Invariance for the Ber(p) Model

Compute the Jeffreys prior for the model $\mathrm{Ber}(q^{10})$ in two ways:

1) Direct from definition
2) Using the reparam. invariance property.

Directly

$$p^x(1-p)^{1-x}$$

$$\tilde{\pi}_J(q) \propto \sqrt{I(q)}, \quad I(q) = -E\left[\frac{\partial^2}{\partial q^2} \ln L(x_i|q)\right]$$

$$L(x_i|q) = (q^{10})^x (1-q^{10})^{1-x}$$

$$\ln L(x_i|q) = 10x\ln(q) + (1-x)\ln(1-q^{10})$$

$$\frac{\partial^2}{\partial q^2} \ln L(x_i|q) = \frac{-10(-11xq^{10}+x+q^{20}+9q^{10})}{q^2(1-q^{10})^2}$$

$$E[x] = q^{10}$$

$$I(q) = -E\left[\frac{\partial^2}{\partial q^2} \ln L(x_i|q)\right] = \frac{10(-11q^{20}+q^{10}+q^{20}+9q^{10})}{q^2(1-q^{10})^2} = \frac{100q^8}{1-q^{10}}$$

$$\boxed{\tilde{\pi}_J(q) \propto \frac{q^4}{\sqrt{1-q^{10}}}}$$

Reparametrization Invariance for the Ber(p) Model

$p = q^{10}$

Compute the Jeffreys prior for the model Ber(q^{10}) in two ways:

1) Direct from definition
2) Using the reparam invariance property

$$\pi_J(p) \propto \frac{1}{\sqrt{p(1-p)}} \qquad I(p) = \frac{1}{p(1-p)}.$$

$$\tilde{\pi}(q) \propto \frac{\pi(\phi^{-1}(q))}{|\phi'(\phi^{-1}(q))|}$$

$$= \frac{\pi(q^{10})}{|\phi'(q^{10})|} = \frac{\frac{1}{\sqrt{q^{10}(1-q^{10})}}}{\frac{1}{10}q^{-9}}$$

$$= \sqrt{\frac{q^{8}}{1-q^{10}}}$$

$q = p^{1/10}$

$\phi(x) = x^{1/10}$

$\phi'(x) = \frac{1}{10}x^{\frac{-9}{10}}$

$\phi^{-1}(x) = x^{10}$