Distances between probability distributions

P, Q, R probability distributions over E, with densities p, q, r [pmf p, q, r]

$A \subseteq E$: $P(A) = \int_A p(x)\,dx \left[= \sum_{x \in A} p(x)\right]$ $\quad (P_\theta)_{\theta \in \Theta}$. $\underline{Q}$: $d(P_\theta, P_{\theta'}) = ???$

Total Variation Distance (TV)

$$TV(P,Q) = \sup_{A \subseteq E} |P(A) - Q(A)|$$

$$= \frac{1}{2}\int_E |p(x) - q(x)|\,dx$$

$$\left[= \frac{1}{2}\sum_{x \in E} |p(x) - q(x)|\right]$$

Kullback-Leibler Divergence (KL)

$$KL(P,Q) = KL(P \| Q)$$

$$= \begin{cases} \int_E p(x) \log \frac{p(x)}{q(x)}\,dx, & q(x) = 0 \Rightarrow p(x) = 0 \\ \left[\sum_{x \in E} p(x) \log \frac{p(x)}{q(x)}\right] & \text{— " —} \\ +\infty, & \exists x: q(x) = 0, p(x) \neq 0 \end{cases}$$

Distances between probability distributions

	TV	KL
• Non-negative: $d(P,Q) \geq 0$	✓	✓
• Definite: $d(P,Q) = 0 \Rightarrow P = Q$	✓	✓
• Symmetric: $d(P,Q) = d(Q,P)$	✓	✗
• Triangle inequality: $d(P,Q) \leq d(P,R) + d(R,Q)$	✓	✗
• Amenable to estimation (replace $\mathbb{E}[\cdot]$ by $\frac{1}{n}\sum_{i=1}^{n}[\cdot]$)	✗	✓

Distances between probability distributions

(1) $P_n = \mathrm{Poi}\left(\frac{1}{n}\right)$, $Q = \delta_0$. <u>Show:</u> $TV(P_n, Q) \xrightarrow[n\to\infty]{} 0$

[$\mathrm{Poi}(\lambda)$ has pmf $p_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, 2, \dots$; $q(k) = \overbrace{\begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases}}^{=\delta_0(k)}$]

<u>Proof:</u> $$TV(P_n, Q) = \frac{1}{2} \sum_{k=0}^{\infty} |p_{1/n}(k) - q(k)| = \frac{1}{2} \sum_{k=0}^{\infty} \left| \frac{\left(\frac{1}{n}\right)^k}{k!} e^{-1/n} - \delta_0(k) \right|$$

$$= \underbrace{\frac{1}{2} \left| \frac{\left(\frac{1}{n}\right)^{0}}{\underbrace{0!}_{=1}} e^{-1/n} - 1 \right|}_{\xrightarrow[n\to\infty]{} 0} + \frac{1}{2} \underbrace{\sum_{k\geq 1} \frac{\left(\frac{1}{n}\right)^k}{k!} e^{-1/n}}_{} \xrightarrow[n\to\infty]{} 0$$

($\left(\frac{1}{n}\right)^0 = 1$)

$$= P_n(\{1, 2, \dots\})$$
$$= 1 - P_n(\{0\})$$
$$= 1 - \frac{(1/n)^0}{0!} e^{-1/n} \xrightarrow[n\to\infty]{} 0$$

What about $P_n = \mathrm{Geom}\left(1 - \frac{1}{n}\right)$

Distances between probability distributions

② $P = \mathrm{Bin}(n,p)$, $Q = \mathrm{Bin}(n,q)$, $p,q \in (0,1)$, $f(p,k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$KL(P\|Q) = \sum_{k=0}^{n} f(p,k) \cdot \log \frac{f(p,k)}{f(q,k)} = \sum_{k=0}^{n} f(p,k) \log\left[\frac{\cancel{\binom{n}{k}} p^k (1-p)^{n-k}}{\cancel{\binom{n}{k}} q^k (1-q)^{n-k}}\right]$$

$$= \sum_{k=0}^{n} f(p,k)\left[\log\left(\left(\frac{p}{q}\right)^k\right) + \log\left(\left(\frac{1-p}{1-q}\right)^{n-k}\right)\right]$$

$$= \sum_{k=0}^{n} f(p,k)\left[k \log\left(\frac{p}{q}\right) + (n-k)\log\left(\frac{1-p}{1-q}\right)\right]$$

$$= \log\left(\frac{p}{q}\right)\cdot np + \log\left(\frac{1-p}{1-q}\right)\cdot(n-np)$$

$$\boxed{\begin{array}{l} X \sim \mathrm{Bin}(n,p) \\ E[X] = n \cdot p \end{array}}$$

$q \to 0 : KL(P,Q) \to \infty$

$q = 0, p \in (0,1) : KL(P,Q) = \infty$

$q \to 1; p \to 0; p \to 1$???

Distances between probability distributions

③ $P = \mathcal{N}(a, \underset{\sigma^2}{1})$, $Q = \mathcal{N}(b, \underset{\sigma^2}{1})$, $a, b \in \mathbb{R}$; $f_{a,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\cdot(x-a)^2}$

$$\boxed{KL(P\|Q)} = \int_{\mathbb{R}} f_{a,1}(x) \log\left[\frac{f_{a,1}(x)}{f_{b,1}(x)}\right] dx = \int_{\mathbb{R}} f_{a,1}(x) \log\left[\frac{\frac{1}{\cancel{\sqrt{2\pi}}} \cdot e^{-\frac{1}{2}\cdot(x-a)^2}}{\frac{1}{\cancel{\sqrt{2\pi}}} e^{-\frac{1}{2}\cdot(x-b)^2}}\right] dx$$

$$= \int_{\mathbb{R}} f_{a,1}(x) \cdot \cancel{\log e}^{\underbrace{-\frac{1}{2}(x-a)^2 + \frac{1}{2}(x-b)^2}_{= -\frac{1}{2}(x^2 - 2xa + a^2) + \frac{1}{2}(x^2 - 2xb + b^2)}} dx = \int_{\mathbb{R}} f_{a,1}(x) \cdot \left[x(a-b) - \frac{1}{2}a^2 + \frac{1}{2}b^2\right] dx$$

$$= (a-b) \underbrace{\int_{\mathbb{R}} x f_{a,1}(x)\, dx}_{=a} + \left(-\frac{1}{2}a^2 + \frac{1}{2}b^2\right) \underbrace{\int_{\mathbb{R}} f_{a,1}(x)\, dx}_{=1} = \underbrace{(a-b)a}_{=a^2 - ba} - \frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}(a^2 - 2ab + b^2)$$

$$\boxed{= \frac{1}{2}(a-b)^2}$$