Poisson & Gamma GLMs:

$Y_1, \dots, Y_n$ response

$X_1, \dots, X_n$ covariates

Motivation: count data

1) Derive Poisson GLM

2) Derive a confidence region for β

3) Overdispersion & gamma GLM

Poisson GLM

- $Y_i \sim \text{Poiss}(\lambda_i)$, $\quad g(\lambda_i) = X_i \beta$
- Poisson pmf $\quad f(x|\lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, where $x = 0, 1, 2, \dots$

$$f(x|\lambda) = \exp\left(\log(\lambda^x) - \lambda - \log x! \right)$$

$$= \exp\left(x \log \lambda - \lambda - \log(x!) \right)$$

(Canonical form: $f(x|\theta) = \exp\left(\dfrac{\theta x - b(\theta)}{\phi} - h(x, \phi) \right)$)

$$\theta = \log \lambda$$

For Poiss, $f(x|\theta) = \exp\left(\dfrac{\theta x - e^\theta}{1} - \log x! \right)$

$$EX = b'(\theta) = e^\theta = \lambda, \quad Var(X) = \phi b''(\theta) = e^\theta = \lambda$$

Poisson + Gamma GLMs:

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Poisson GLM

- $Y_i \sim \text{Poiss}(\lambda_i)$, $\quad g(\lambda_i) = X_i \beta$

- Canonical form for Poiss pmf: $f(x|\theta) = \exp\left(\frac{\theta x - e^{\theta}}{1} - \log x!\right)$

$\theta = \log \lambda \quad x = 0, 1, 2, \dots$

- Canonical link

$$g(\lambda) = \theta$$

$$g(\lambda) = \log \lambda$$

$$g(\lambda_i) = \log(\lambda_i) = X_i \beta$$

$$Y_i \sim \text{Poiss}\left(e^{X_i \beta}\right)$$

- How to estimate β: MLE

$$L(Y_1, \dots, Y_n | \beta) = \prod_{i=1}^{n} \frac{\left(e^{X_i \beta}\right)^{Y_i} e^{-\left(e^{X_i \beta}\right)}}{Y_i!}$$

$$= \exp\left(\sum_{i=1}^{n} X_i \beta Y_i - \sum_{i=1}^{n} e^{X_i \beta} - \log\left(\prod_{i=1}^{n} Y_i!\right)\right)$$

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- $L(Y_1, \dots, Y_n | \beta) = \exp\left(\sum_{i=1}^n X_i \beta Y_i - \sum_{i=1}^n e^{X_i\beta} - \log \prod_{i=1}^n Y_i!\right)$
- $\ell(Y_1, \dots, Y_n | \beta) = \sum_{i=1}^n (X_i\beta) Y_i - \sum_{i=1}^n e^{X_i\beta} - \log \prod_{i=1}^n Y_i!$

$$\nabla_\beta \ell = \sum_{i=1}^n Y_i \nabla_\beta (X_i \beta) - \sum_{i=1}^n \nabla_\beta e^{X_i\beta}$$
$$= \sum_{i=1}^n Y_i \cdot X_i^T - \sum_{i=1}^n e^{X_i\beta} \cdot X_i^T$$
$$= \sum_{i=1}^n (Y_i - e^{X_i\beta}) X_i^T \stackrel{?}{=} 0 \quad \text{no closed form}$$

$$\nabla_\beta^2 \ell = \sum_{i=1}^n - \nabla_\beta e^{X_i\beta} X_i^T$$
$$= -\sum_{i=1}^n e^{X_i\beta} X_i^T X_i$$
$$= -\sum_{i=1}^n (e^{X_i\beta/2} X_i^T)(e^{X_i\beta/2} X_i)$$
$$= -AA^T, \text{ where } i^{th} \text{ column } \cancel{\text{column}}\ \cancel{\text{row}} \text{ is } e^{X_i\beta/2} X_i^T$$
$$\preceq 0 \quad (\text{negative eig.}) \rightarrow \text{concave}$$

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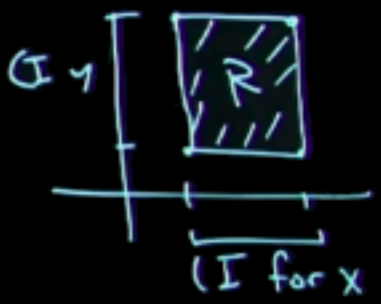
- $\hat{\beta}_{MLE} = \arg\max_{\beta} \ \ell(Y_1, \dots, Y_n \mid \beta)$

$$\sqrt{n}\left(\hat{\beta}_{MLE} - \beta\right) \xrightarrow{(d)} N\left(0, I^{-1}(\beta)\right)$$

$$\sqrt{n}\, I^{1/2}(\beta)\left(\hat{\beta}_{MLE} - \beta\right) \xrightarrow{(d)} N(0, I)$$

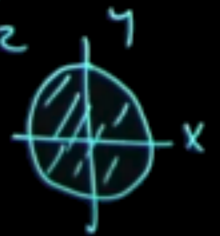
- Simple example: $\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$



$x^2 + y^2 \sim \chi^2_2$

$P(x^2 + y^2 \le q) = 0.95$

0.95 quantile of χ^2_2



I: 95% CI for x

J: 95% CI for y

$P\left(\begin{pmatrix} x \\ y \end{pmatrix} \text{ in } R\right) = (0.95)^2 < 0.95$

I: $\sqrt{0.95}$ region for x

J: $\sqrt{0.95}$ region for y

$P\left(\begin{pmatrix} x \\ y \end{pmatrix} \text{ in } R\right) = 0.95$

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- $\hat{\beta}_{MLE} = \arg\max_{\beta} \; \ell(Y_1, \dots, Y_n \mid \beta)$

$$\sqrt{n}\,(\hat{\beta}_{MLE} - \beta) \xrightarrow{(d)} N(0, I^{-1}(\beta))$$

$$\sqrt{n}\, I^{1/2}(\beta)(\hat{\beta}_{MLE} - \beta) \xrightarrow{(d)} N(0, I)$$

$$\widehat{I(\beta)} = \hat{I}(\beta) = \frac{1}{n}\sum_{i=1}^{n} -\nabla_\beta^2 \ell(Y_i \mid \beta)$$

$$\text{Plug-in } \hat{I}(\hat{\beta}_{MLE}) = \frac{1}{n}\sum_{i=1}^{n} -\nabla_\beta^2 \ell(Y_i \mid \beta)$$

$$\underbrace{n\,(\hat{\beta}_{MLE} - \beta)^T \hat{I}(\hat{\beta}_{MLE})(\hat{\beta}_{MLE} - \beta)} \xrightarrow{(d)} \chi^2_p \quad \text{where } \beta \in \mathbb{R}^p$$

$$P(\qquad \le q) = 0.95$$

0.95 quantile of χ^2_p

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  $\hookrightarrow E Y_i = \lambda_i \quad \text{Var}\, Y_i = \lambda_i$

- $Y_i \sim \text{Gamma}(1, b_i)$

  $E Y_i = b_i, \ \text{Var}(Y_i) = b_i^2$

Gamma(a,b) pdf

$$f(x|a,b) = \frac{1}{\Gamma(a)\, b^a} x^{-a-1} e^{-x/b}$$

$$f(x|b) = \frac{1}{b} e^{-x/b}$$

$$= \exp\left(-\frac{x}{b} + \log \frac{1}{b}\right)$$

$$\left(\theta = -\frac{1}{b}\right)$$

$$f(x|\theta) = \exp(\theta x + \log -\theta)$$

$$b(\theta) = -\log(-\theta)$$

$$\phi = 1$$

Canonical link:

$$g(\mu) = \theta = -\frac{1}{b}$$

$$g(b) = -\frac{1}{b}$$

$$Y_i \sim \text{Gamma}\left(1, -\frac{1}{X_i \beta}\right)$$

$$Y_i \sim \text{Gamma}\left(1, \frac{1}{X_i \beta}\right), \sim \text{Gamma}\left(1, e^{X_i \beta}\right)$$