Hypothesis Testing

$X_1, \dots, X_n \overset{iid}{\sim} \text{Unif}[0, \theta]$

Test $H_0: \theta = 1$ vs $H_1: \theta < 1$

1) Find a test with sig. level $\alpha = 0$.

2) Calculate power & sig. level of test with rejection region $R = [1-c, 1]$ $c < 1$

3) Same thing for $R = [0, c]$

4) What is the likelihood ratio test?

5) Suppose we observe $X_1, \dots, X_{10} < 0.9$. Would you reject H_0 at sig. level $\alpha = 0.05$?

Basic testing concepts

- Statistic $T(X_1, \dots, X_n)$
- Test $\Psi(T(X_1, \dots, X_n)) = \mathbb{1}(T(X_1, \dots, X_n) \in R)$
- Type I: rejecting H_0 when its true
  Type II: failing to reject H_0 when H_1 is true
- Significance level: $P(\text{Type I error}) = P(\Psi = 1 \mid H_0 \text{ true})$

Power: $1 - P(\text{Type II error}) = 1 - P(\Psi = 0 \mid H_1 \text{ true})$

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1) ∘ Find $T \to$ estimator for θ
   ∘ Find R

Maximum Likelihood

$$L(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}(0 \le X_i \le \theta)$$

$$= \frac{1}{\theta^n} \mathbb{1}(0 \le X_{(1)} \le X_{(n)} \le \theta)$$

> 0 when $\theta < X_{(n)} \Rightarrow \hat{\theta}_{MLE} \ge X$

$>$ L decreasing when $\theta \ge X_{(n)}$

$$\hat{\theta}_{MLE} = X_{(n)} = T(X_1, \dots, X_n)$$

$$\psi(T(X_1, \dots, X_n)) = \psi(\hat{\theta}_{MLE}) = \mathbb{1}(\hat{\theta}_{MLE} \in R)$$

∘ Look at R of the form $[0, c]$

$$\alpha = P(\psi = 1 \mid \theta = 1) = P(X_{(n)} \le c \mid \theta = 1)$$

$$= \left(P(X_1 \le c \mid \theta = 1)\right)^n = c^n = 0$$

Need $c = 0$, $R = \{0\}$

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5) Suppose we observe $X_1, \dots, X_{10} < 0.9$. Would you reject H_0 at sig level $\alpha = 0.05$?

2) $T(X_1, \dots, X_n) = X_{(n)}$, $R = [1-c, \infty)$ $c < 1$

- $\alpha = P(\Psi = 1 \mid \theta = 1)$
  $= P(X_{(n)} \in [1-c, \infty) \mid \theta = 1) = P(X_{(n)} \geq 1-c \mid \theta = 1)$
  $= 1 - P(X_{(n)} \leq 1-c \mid \theta = 1)$
  $= 1 - (P(X_1 \leq 1-c \mid \theta = 1))^n = 1 - (1-c)^n$
- $P(\text{Type II error}) = P(\Psi = 0 \mid H_1 \text{ true})$
  $= P(X_{(n)} \leq 1-c \mid \theta < 1)$
  $= \left(\frac{1-c}{\theta}\right)^n, \quad \theta > 1-c$

$$\text{Power} = 1 - \sup_{1-c < \theta < 1} \left(\frac{1-c}{\theta}\right)^n = 0$$

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3) $T(X_1, \ldots, X_n) = X_{(n)}$, $R = [0, c]$

- $\alpha = P(\Psi = 1 \mid \theta = 1) = P(X_{(n)} \leq c \mid \theta = 1)$
  $= (c)^n$
- $P(\text{Type II error}) = P(\Psi = 0 \mid H_1 \text{ true})$
  $= P(X_{(n)} > c \mid \theta < 1)$ if $\theta \leq c$, $P = 0$
  $= 1 - P(X_{(n)} \leq c \mid \theta < 1)$
  $= 1 - \left(\frac{c}{\theta}\right)^n$ $\theta \geq c$

$$\text{Power} = \inf_{c \leq \theta < 1} 1 - \left(1 - \left(\frac{c}{\theta}\right)^n\right)$$

$$= \inf_{c \leq \theta < 1} \left(\frac{c}{\theta}\right)^n = c^n$$

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4) What is the likelihood ratio test?

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4) Likelihood ratio $\Lambda = \dfrac{L(X_1, \ldots, X_n; \theta = 1)}{\sup_{\theta < 1} L(X_1, \ldots, X_n; \theta)}$

- Reject in LRT when $\Lambda < c < 1$
- $\Lambda = \dfrac{\prod_{i=1}^n 1 \cdot \mathbb{1}(0 \le X_i \le 1)}{\sup_{\theta < 1} \prod_{i=1}^n \frac{1}{\theta} \mathbb{1}(0 \le X_i \le \theta)}$

$$= \frac{1}{\left(\frac{1}{X_{(n)}}\right)^n} = X_{(n)}^n$$

- $R = \{\Lambda < c\} = \{X_{(n)}^n < c\}$

$$= \{X_{(n)} < c^{1/n}\} = \{X_{(n)} < c'\}$$

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5) $R = [0, c]$, Significance level $\alpha = c^n$

- For sig. level 0.05, need $0.05 = c^n$

$$c = 0.05^{1/10} \approx 0.74$$

- $R = [0, 0.74]$
- $\psi(T(X_1, \dots, X_n)) = \psi(0.9)$

$$= \mathbb{1}(0.9 \in [0, 0.74])$$

$$= 0$$

- Do not reject H_0 at sig. level of 0.05