M-estimation

- $X_1, \dots, X_n \overset{iid}{\sim} \mathbb{P}_\mu = f(x-\mu)$
  ↳ pdf "centered" at 0

- Choose a fcn ρ
  e.g. $\rho(x) = |x|$
  $\rho(x) = x^2$

$$\hat{\mu} = \underset{b \in \mathbb{R}}{\operatorname{argmin}} \sum_{i=1}^{n} \rho(X_i - b)$$

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1) Calculate asymp var (avar) of sample med.

2) Compare avar for
   - Mean & Med under Laplace(μ)
   - Mean, Med, Huber est. under Cauchy(μ)

1) • Assume f cts, n is odd, $F(\mu) = \frac{1}{2}$

• Sample med m_n

• $\sqrt{n}(m_n - \mu) \xrightarrow{(d)?} N(0, \sigma^2)$

→ cdf:

$$P(\sqrt{n}(m_n - \mu) \le a) = P\left(m_n \le \frac{a}{\sqrt{n}} + \mu\right)$$

$$= P\left(\#\left(X_i \ge \frac{a}{\sqrt{n}} + \mu\right) \ge \frac{n+1}{2}\right)$$

$$= P\left(\bar{Y}_n \ge \frac{n+1}{2}\right), \quad \left(Y_i = \mathbb{1}\left(X_i \ge \frac{a}{\sqrt{n}} + \mu\right)\right)$$

↳ $\sim \text{Bern}(p_n)$

$$= P\left(\frac{\bar{Y}_n - p_n \cdot n}{\sqrt{n(1-p_n)p_n}} \ge \frac{\frac{n+1}{2} - p_n n}{\sqrt{n(1-p_n)p_n}}\right)$$

$$= P\left(Z \ge \frac{\frac{n+1}{2} - p_n n}{\sqrt{n(1-p_n)p_n}}\right)$$

$Z \sim N(0,1)$

1) ∘ Assume f cts, n is odd, $F(\mu)=\frac{1}{2}$

∘ Sample med m_n

∘ $\underbrace{P(\sqrt{n}(m_n-\mu)\le a)}_{(i)} \to P\left(Z \ge \lim_{n\to\infty} \dfrac{\frac{n+1}{2}-p_n n}{\sqrt{n(1-p_n)p_n}}\right)$

$(Z\sim N(0,1))$

$p_n = P(Y_i=1)$
$= P(X_i \le \frac{a}{\sqrt{n}}+\mu)$
$\to \frac{1}{2}$

∘ $$\frac{\frac{n+1}{2}-p_n n}{\sqrt{n(1-p_n)p_n}} = \frac{\frac{n}{2}-p_n n}{\sqrt{n(1-p_n)p_n}} + \underbrace{\cancel{\frac{\frac{1}{2}}{\sqrt{n(1-p_n)p_n}}}}_{\to 0 \text{ as } n\to\infty}$$

$$= \frac{\frac{1}{2}-p_n}{\frac{\sqrt{(1-p_n)p_n}}{\sqrt{n}}} = \frac{F(\mu)-F(\frac{a}{\sqrt{n}}+\mu)}{a/\sqrt{n}} \cdot \frac{a}{\sqrt{(1-p_n)p_n}}$$

$$= -F'(\mu)\cdot 2a = -2af(\mu)$$

∘ $(i) \to P(Z \ge -2af(\mu)) = P(Z \le 2af(\mu))$
$= P\left(\frac{Z}{2f(\mu)} \le a\right)$, $\frac{Z}{2f(\mu)} \sim N\left(0, \frac{1}{4f(\mu)^2}\right)$

M-estimation

- $X_1, \dots, X_n \overset{iid}{\sim} \mathbb{P}_\mu = f(x-\mu)$
  - $\hookrightarrow$ pdf "centered" at 0
- Choose a fcn ρ
  - e.g. $\rho(x) = |x|$
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$$\hat{\mu} = \underset{b \in \mathbb{R}}{\operatorname{argmin}} \sum_{i=1}^{n} \rho(X_i - b)$$

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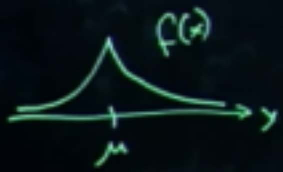
2) $X_1, \dots, X_n \overset{iid}{\sim} \text{Laplace}(\mu)$

$\hookrightarrow f(x) = \frac{1}{2}\exp(-|x-\mu|)$



- $\sqrt{n}(\bar{X} - \mu) \xrightarrow{(d)} N(0, \text{Var}(X_1))$

$$\text{avar}(\bar{X}) = \text{var}(X_1) = \text{var}(X_1') \quad , \quad X_1' \sim \text{Laplace}(0)$$
$$= E(X_1')^2 = 2$$

- $\sqrt{n}(m_n - \mu) \xrightarrow{(d)} N\left(0, \frac{1}{4 f(\mu)^2}\right)$

$$\text{avar}(m_n) = \frac{1}{4 f(\mu)^2} = \frac{1}{4} \cdot \frac{1}{\left[\frac{1}{2}\exp(-|\mu-\mu|)\right]^2}$$
$$= 1$$

- $\text{avar}(m_n) < \text{avar}(\bar{X})$

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2) $X_1, \dots, X_n \sim \text{Cauchy}(\mu)$ if

pdf: $f(x) = \frac{1}{\pi(1+(x-\mu)^2)}$

cdf: $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_{-\infty}^{x} \frac{1}{\pi(1+(t-\mu)^2)}\,dt$$

$$= \frac{\arctan(t-\mu)}{\pi}\Big|_{-\infty}^{x} = \frac{1}{2} + \frac{\arctan(x-\mu)}{\pi}$$

Note: $F(\mu) = \frac{1}{2}$

- Let's use $\bar{X}$ to estimate μ

$$E\bar{X} = EX_1 = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi(1+x^2)}\,dx \quad \text{diverges}$$

$\text{Var}(\bar{X}) = \frac{1}{n}\text{Var}(X_1)$: diverges

$\text{avar}(\bar{X}) = \infty$

- $\text{avar}(m_n) = \frac{1}{4f(\mu)^2} = \frac{1}{4} \cdot \frac{1}{\left[\frac{1}{\pi}\right]^2} = \frac{\pi^2}{4}$

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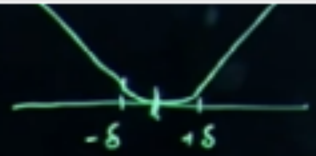
2)
$$\rho(x) = \begin{cases} \frac{x^2}{2} & |x| < \delta \\ \delta|x| - \frac{\delta^2}{2} & |x| > \delta \end{cases}$$



$$\rho'(x) = \begin{cases} x & |x| < \delta \\ \delta \, \operatorname{sign}(x) & |x| \geq \delta \end{cases}$$

$$\rho''(x) = \begin{cases} 1 & |x| < \delta \\ 0 & |x| > \delta \end{cases}$$

∘ Huber est: $\hat{\mu}_{huber} = \underset{b \in \mathbb{R}}{\operatorname{argmin}} \sum_{i=1}^{n} \rho(X_i - b)$

∘ $\sqrt{n}(\hat{\mu}_{huber} - \mu) \xrightarrow{(d)} N\left(0, \frac{Var(\rho'(X))}{[\mathbb{E}\rho''(X)]^2}\right)$

$X \sim f(x)$, $X_i \sim f(x-\mu)$

∘ $$\mathbb{E}\rho''(X) = \int_{-\delta}^{\delta} 1 \cdot f(x)\,dx = \int_{-\delta}^{\delta} \frac{1}{\pi(1+x^2)}\,dx$$
$$= \frac{1}{\pi}(\arctan\delta - \arctan(-\delta))$$
$$= \frac{2}{\pi}\arctan\delta$$

∘ $$Var(\rho'(X)) = \mathbb{E}(\rho'(X))^2 = \int_{-\infty}^{-\delta} \delta^2 \cdot \frac{1}{\pi(1+x^2)}\,dx + \int_{-\delta}^{\delta} x^2 \cdot \frac{1}{\pi(1+x^2)}\,dx + \int_{\delta}^{\infty} \delta^2 \frac{1}{\pi(1+x^2)}\,dx$$

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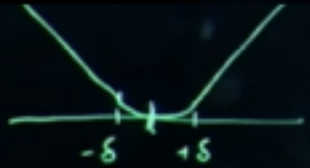

$\mathbb{E}\rho''(x) = \frac{2}{\pi}\arctan(\delta)$, $x \sim f(x)$

$$\mathbb{E}(\rho'(x))^2 = \int_{-\infty}^{-\delta} \delta^2 \frac{1}{\pi(1+x^2)}\,dx + \int_{\delta}^{\infty} \delta^2 \frac{1}{\pi(1+x^2)}\,dx + \int_{-\delta}^{\delta} x^2 \frac{1}{\pi(1+x^2)}\,dx$$

$$= \frac{\delta^2}{\pi}\left(\arctan(-\delta) - \left(-\frac{\pi}{2}\right)\right) + \frac{\delta^2}{\pi}\left(\frac{\pi}{2} - \arctan\delta\right)$$

$$+ \frac{2\delta}{\pi} - \frac{2\arctan\delta}{\pi}$$

$$\int \frac{x^2}{\pi(1+x^2)}\,dx \qquad 1 + \tan^2 x = \sec^2 x$$

$$\tan u = x \longrightarrow \sec^2 u\,du = dx$$

$$= \int \frac{\tan^2 u \cancel{\sec^2 u}\,du}{\pi(\cancel{\sec^2 u})} = \frac{1}{\pi}\int \sec^2 u - 1\,du$$

$$= \frac{1}{\pi}(\tan u - u)$$

$$\int_{-\delta}^{\delta} \frac{x^2}{\pi(1+x^2)}\,dx = \frac{1}{\pi}(x - \arctan x)\Big|_{-\delta}^{\delta}$$

$$= \frac{1}{\pi}\left(\delta - \arctan\delta - (-\delta - \arctan(-\delta))\right)$$

$$= \frac{2\delta}{\pi} - \frac{2\arctan\delta}{\pi}$$

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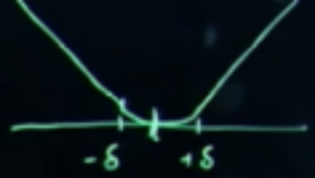

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$$= \frac{\delta^2}{\pi}\left(\arctan(-\delta) - \left(-\frac{\pi}{2}\right)\right) + \frac{\delta^2}{\pi}\left(\frac{\pi}{2} - \arctan\delta\right) + \frac{2\delta}{\pi} - \frac{2\arctan\delta}{\pi}$$

$$\operatorname{avar}(\hat{\mu}_{huber}) = \frac{\mathbb{E}(\rho'(x)^2)}{[\mathbb{E}\rho''(x)]^2}$$

$$= \frac{\frac{2\delta}{\pi} - \frac{2\arctan\delta}{\pi} + \delta^2 - 2\frac{\delta^2}{\pi}\arctan(\delta)}{\left[\frac{2}{\pi}\arctan(\delta)\right]^2}$$

- $\operatorname{avar}(\hat{\mu}_{huber}) \xrightarrow[\delta \to 0]{} \frac{\pi^2}{4} = \operatorname{avar}(m_n)$
- $\delta \to \infty$, $\operatorname{avar}(\hat{\mu}_{huber})$ diverges
- Find which δ minimizes $\operatorname{avar}(\hat{\mu}_{huber})$