Find the posterior & Bayes est. (posterior mean) for the following models

1) $X_1, \dots, X_n \sim \text{Geom}(p)$

$p \sim \text{Beta}(a, b)$

2) $X_1, \dots, X_n \sim \text{Pareto}(\theta)$

$\theta \sim \text{Gamma}(a, b)$

3) $X_1, \dots, X_n \sim N(0, \tau)$

$\tau \sim \text{Inv. Gamma}(a, b)$

A, B events

Bayes Thm: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Find the posterior + Bayes est. (posterior mean) for the following models

1) $X_1, \dots, X_n \sim \text{Geom}(p)$

$p \sim \text{Beta}(a, b)$

2) $X_1, \dots, X_n \sim \text{Pareto}(\theta)$

$\theta \sim \text{Gamma}(a, b)$

3) $X_1, \dots, X_n \sim N(0, v)$

$v \sim \text{Inv. Gamma}(a, b)$

A, B events

Bayes Thm: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

1) $X \sim \text{Geom}(p) \quad f(x) = (1-p)^{x-1} p \quad , x = 1, 2, 3, \dots$

$p \sim \text{Beta}(a, b) \quad g(p) = \dfrac{p^{a-1}(1-p)^{b-1}}{B(a,b)} \quad p \in [0,1]$

$a > 0, b > 0$

$$\pi(p|X_1, \dots, X_n) = \frac{\pi(X_1, \dots, X_n|p)\,\pi(p)}{\pi(X_1, \dots, X_n)}$$

$$\propto \pi(X_1, \dots, X_n|p)\,\pi(p)$$

$$= \left(\prod_{i=1}^n (1-p)^{x_i - 1} p\right) \cdot \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)}$$

$$\propto (1-p)^{\sum_{i=1}^n (x_i - 1) + b - 1} \, p^{n+a-1}$$

$$p|X_1, \dots, X_n \sim \text{Beta}\left(n+a, \sum_{i=1}^n (x_i - 1) + b\right)$$

$$\hat{p}_{\text{Bayes}} = \frac{n+a}{n+a+\sum_{i=1}^n (X_i - 1) + b}$$

$$= \frac{n+a}{a + \sum_{i=1}^n X_i + b}$$

Find the posterior + Bayes est. (posterior mean) for the following models

1) $X_1,\dots,X_n \sim Geom(p)$

$p \sim Beta(a,b)$

2) $X_1,\dots,X_n \sim Pareto(\theta)$

$\theta \sim Gamma(a,b)$

3) $X_1,\dots,X_n \sim N(0,v)$

$v \sim Inv.\ Gamma(a,b)$

A, B events

Bayes Thm: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

2) $X \sim Pareto(\theta)$ $\quad f(x) = \theta x^{-\theta-1} \quad x \geq 1$

$\Theta \sim Gamma(a,b) \quad g(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \quad \theta > 0$

$$\pi(\theta|X_1,\dots,X_n) \propto \pi(X_1,\dots,X_n|\theta)\cdot\pi(\theta)$$

$$= \left(\prod_{i=1}^n \theta x_i^{-\theta-1}\right)\cdot\frac{\cancel{b^a}}{\cancel{\Gamma(a)}}\theta^{a-1}e^{-b\theta}$$

$$\propto \theta^{n+a-1}\left(\prod_{i=1}^n X_i\right)^{-\theta} e^{-b\theta}$$

$$= \theta^{n+a-1}\left(e^{\log(\prod_{i=1}^n X_i)}\right)^{-\theta} e^{-b\theta}$$

$$= \theta^{n+a-1} e^{-(\log(\prod_{i=1}^n X_i)+b)\theta}$$

$$\theta|X_1,\dots,X_n \sim Gamma\left(n+a, \log(\prod_{i=1}^n X_i)+b\right)$$

$$\hat{\theta}_{Bayes} = \frac{n+a}{\log(\prod_{i=1}^n X_i)+b}$$

Find the posterior + Bayes est. (posterior mean) for the following models

1) $X_1, \dots, X_n \sim Geom(p)$
$p \sim Beta(a, b)$

2) $X_1, \dots, X_n \sim Pareto(\theta)$
$\theta \sim Gamma(a, b)$

3) $X_1, \dots, X_n \sim N(0, v)$
$v \sim Inv.\ Gamma(a, b)$

A, B events

Bayes Thm: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

3) $X \sim N(0, v)$ $\quad f(x) = \dfrac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}x^2}$ $\quad x \in \mathbb{R}$

$v \sim Inv.\ Gamma(a, b)$ $\quad g(v) = \dfrac{b^a}{\Gamma(a)} v^{-a-1} e^{-b/v}$ $\quad v > 0$

$$\pi(v | X_1, \dots, X_n) \propto \pi(X_1, \dots, X_n | v)\, \pi(v)$$
$$= \left(\prod_{i=1}^n \frac{1}{\sqrt{2\cancel{\pi} v}} e^{-\frac{1}{2v}X_i^2}\right) \cancel{\frac{b^a}{\Gamma(a)}} v^{-a-1} e^{-b/v}$$
$$\propto v^{-n/2} e^{-\frac{1}{2v}\sum_{i=1}^n X_i^2} v^{-a-1} e^{-b/v}$$
$$= v^{-n/2-a-1} e^{-\frac{1}{2v}\sum_{i=1}^n X_i^2 - \frac{b}{v}}$$
$$= v^{-n/2-a-1} e^{-\frac{1}{v}\left(\frac{1}{2}\sum_{i=1}^n X_i^2 + b\right)}$$

$$v | X_1, \dots, X_n \sim Inv.\ Gamma\left(n/2 + a, \tfrac{1}{2}\sum_{i=1}^n X_i^2 + b\right)$$

$$\hat{v}_{Bayes} = \frac{\frac{1}{2}\sum_{i=1}^n X_i^2 + b}{\frac{n}{2} + a - 1}$$