Frequentist vs. Bayesian

$X \sim Ber(\theta)$

$P(x)$

$X \sim Ber(\theta) \leftarrow P(x)$

$\theta \sim \underbrace{Uni([0,1])}$

prior $P(\theta)$

posterior $P(\theta|X)$

$\downarrow$

Bayes Estimator : mean

Multinomial Distribution

$X_1, \cdots, X_n \sim \text{Multi}(p_1, \cdots, p_k)$

$$\mathbb{P}(X_1 = 1) = P_1 \rightarrow N_1 = \#\{X_i = 1\}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$\mathbb{P}(X_1 = k) = P_k \rightarrow N_k = \#\{X_i = k\}$$
$$\downarrow$$

① $P_1 + P_2 + \cdots + P_k = 1$. ② $0 \leq P_1 \leq 1$
$\vdots$
$0 \leq P_k \leq 1$

$$f(x_1, \cdots, x_n) \propto P_1^{N_1} P_2^{N_2} \cdots P_k^{N_k} \leftarrow g(x)$$

Dirichlet Distribution

$X_1, \cdots, X_k \sim \text{Dir}(\alpha_1, \cdots, \alpha_k)$

$$f(x_1, \cdots, x_k) = \underbrace{\frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)}}_{C} x_1^{\alpha_1 - 1} \cdots x_k^{\alpha_k - 1}$$

$\Rightarrow 3 \cdot x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1}$

$$f(x_1, \ldots, x_k) = \frac{n!}{N_1! \ldots N_k!} p_1^{N_1} p_2^{N_2} \ldots p_k^{N_k} \quad \text{where } N_1 + \cdots + N_k = n.$$

Setup : $X_1, \cdots, X_n \sim \text{Multi}(P_1, \cdots, P_k)$

$\rightarrow$ ① posterior $\pi(p|x)$?

② Bayes Estimator? (risk func = MSE)

$P_1, \cdots, P_k \sim \text{Dir}(\alpha_1, \cdots, \alpha_k) \rightarrow$ mean: $\dfrac{(\alpha_1, \alpha_2, \cdots, \alpha_k)}{\alpha_1 + \alpha_2 + \cdots \alpha_k} = \left(\dfrac{\alpha_1}{\Sigma\alpha_i}, \cdots, \dfrac{\alpha_k}{\Sigma\alpha_i}\right)$

$\pi(X_1, \cdots, X_n) = P_1^{N_1} \cdots P_k^{N_k}$

$\pi(P_1, \cdots, P_k) = C \cdot P_1^{\alpha_1 - 1} \cdot P_2^{\alpha_2 - 1} \cdots P_k^{\alpha_k - 1}$

$\alpha_1^{P_1}$ $\alpha_1^{X_1}$ $\dfrac{1}{P_1 X_1^{N_1}}$

$\pi(p_1, \cdots, P_k | X_1, \cdots, X_n) \propto \pi(p_1, \cdots, P_k) \cdot \pi(X_1, \cdots, X_n)$

$\propto \underline{P_1^{\alpha_1 - 1}} P_2^{\alpha_2 - 1} \cdots P_k^{\alpha_k - 1} \cdot \underline{P_1^{N_1}} \cdots P_k^{N_k}$

$\propto P_1^{\alpha_1 - 1 + N_1} \cdots P_k^{\alpha_k - 1 + N_k} \rightarrow$ ① $\text{Dir}(\alpha_1 + N_1, \alpha_2 + N_2, \cdots, \alpha_k + N_k)$

$\Downarrow$

$\alpha_1 + N_1 - 1$

$\rightarrow$ Bayes est $= \dfrac{(\alpha_1 + N_1, \cdots, \alpha_k + N_k)}{\alpha_1 + \cdots + \alpha_k + \underbrace{N_1 + \cdots N_k}_{= n}}$

Recap

$X_1,\dots,X_n \mid p_1,\dots,p_k \sim \mathrm{Mult}(p_1,\dots,p_k)$

$p_1,\dots,p_k \sim \mathrm{Dirichlet}(\alpha_1,\dots,\alpha_k)$

$p_1,\dots,p_k \mid X_1,\dots,X_n \sim \mathrm{Dirichlet}(\alpha_1+N_1,\dots,\alpha_k+N_k)$

$\hookrightarrow N_1 = \#\{X_i = 1\}$

$\vdots$

$N_k = \#\{X_i = k\}$

$$\hat{p}_i = \frac{\alpha_i+N_i}{\sum_{i=1}^k \alpha_i+N_i} = \frac{\alpha_i+N_i}{\sum_{i=1}^k \alpha_i + n}$$

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1) Find a 95% Bayesian CI for p_1

2) Plug in some values for $N_1,\dots,N_k,\alpha_1,\dots,\alpha_k$

95% CI for p_1

$\rightarrow$ To find this, we want the quantiles of $\pi(p_1 \mid X_1,\dots,X_n)$.

Simple Example: $k=2$.

$$\pi(p_1,p_2 \mid X_1,\dots,X_n) = \frac{\Gamma(\alpha_1+N_1+\alpha_2+N_2)}{\Gamma(\alpha_1+N_1)\Gamma(\alpha_2+N_2)} p_1^{\alpha_1+N_1-1} p_2^{\alpha_2+N_2-1}$$

$$(p_1+p_2 = 1 \rightarrow p_2 = 1-p_1)$$

$$\pi(p_1 \mid X_1,\dots,X_n) = C\, p_1^{\alpha_1+N_1-1}(1-p_1)^{\alpha_2+N_2-1}$$

$$p_1 \mid X_1,\dots,X_n \sim \mathrm{Beta}(\alpha_1+N_1, \alpha_2+N_2)$$

$$p_1 \mid X_1,\dots,X_n \sim \mathrm{Beta}\left(\alpha_1+N_1, \sum_{i=2}^k \alpha_i+N_i\right)$$

Let $q_{1-\frac{0.05}{2}}$ be the $\frac{0.05}{2}$ quantile of $\mathrm{Beta}(\alpha_1+N_1, \sum_{i=2}^k \alpha_i+N_i)$

$q_{\frac{0.05}{2}}$ be the $1-\frac{0.05}{2}$ quantile

$$I = \left[q_{1-\frac{0.05}{2}}, q_{\frac{0.05}{2}}\right]$$

Recap

$X_1, \dots, X_n \mid p_1, \dots, p_k \sim \text{Mult}(p_1, \dots, p_k)$

$p_1, \dots, p_k \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_k)$

$p_1, \dots, p_k \mid X_1, \dots, X_n \sim \text{Dirichlet}(\alpha_1 + N_1, \dots, \alpha_k + N_k)$

$\hookrightarrow N_1 = \#\{X_i = 1\}$

$\vdots$

$N_k = \#\{X_i = k\}$

$$\hat{p}_i = \frac{\alpha_i + N_i}{\sum_{i=1}^k \alpha_i + N_i} = \frac{\alpha_i + N_i}{\sum_{i=1}^k \alpha_i + n}$$

1) Find a 95% Bayesian CI for p_1

2) Plug in some values for $N_1, \dots, N_k, \alpha_1, \dots, \alpha_k$

Example

- $k = 3$, $N_1 = 10$, $N_2 = 20$, $N_3 = 30$

Case 1: $\alpha_1 = \alpha_2 = \alpha_3 = 1$

$$(\hat{p}_1, \hat{p}_2, \hat{p}_3) = \left(\frac{11}{63}, \frac{21}{63}, \frac{31}{63}\right) \approx \left(0.175, \frac{1}{3}, 0.492\right)$$

Case 2: $\alpha_1 = \alpha_2 = \alpha_3 = 10$

$$(\hat{p}_1, \hat{p}_2, \hat{p}_3) = \left(\frac{20}{90}, \frac{30}{90}, \frac{40}{90}\right) \approx \left(0.222\dots, \frac{1}{3}, 0.444\dots\right)$$