Multiple Hypothesis Testing & Bonferroni Correction

$$Y = \mathbb{X}\beta + \varepsilon$$

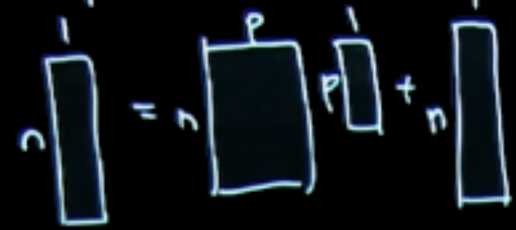
β, σ^2 <u>unknown</u>.

$\mathbb{X} \in \mathbb{R}^{n\times p}$
$\beta \in \mathbb{R}^p$
$\varepsilon \sim \mathcal{N}_n(0, \sigma^2 I)$.

$H_0: 0 \le \beta_1 \le \beta_2$
$H_1:$ NOT $0 \le \beta_1 \le \beta_2$

<u>Q</u>: Design test against H_0 at level $\alpha = 5\%$.

$$Y = \mathbb{X}\ \beta + \varepsilon$$



$H_0: 0 \le \beta_1$ $\underline{\underline{\text{AND}}}$ $\beta_1 \le \beta_2$.

$H_1: 0 > \beta_1$ $\underline{\underline{\text{OR}}}$ $\beta_1 > \beta_2$.

$H_0^{(1)}: 0 \le \beta_1 \rightsquigarrow \psi_{\alpha_1}^{(1)}$
$H_0^{(2)}: \beta_1 \le \beta_2. \rightsquigarrow \psi_{\alpha_2}^{(2)}$

$\downarrow$

$$\psi_\alpha = \psi_{\alpha_1}^{(1)} \ \underline{\text{OR}}\ \psi_{\alpha_2}^{(2)}$$
$$= \max\left(\psi_{\alpha_1}^{(1)}, \psi_{\alpha_2}^{(2)}\right).$$

$\mathbb{P}(\psi$ outputs "reject")
$= \mathbb{P}(\psi^{(1)} = \text{"reject"}$ $\underline{\underline{\text{OR}}}$ $\psi^{(2)} = \text{"reject"})$.
$\ge \mathbb{P}(\psi^{(1)} = \text{"reject"}) = 5\%$

<u>Union Bound</u>:

$A_1: \psi^{(1)} = \text{"reject"}$
$A_2: \psi^{(2)} = \text{"reject"}$

A_1, A_2 events

$$\mathbb{P}(A_1 \cup A_2) \le \underbrace{\mathbb{P}(A_1)}_{\le 2.5\%} + \underbrace{\mathbb{P}(A_2)}_{\le 2.5\%}.$$

$\mathbb{P}(A_1 \cup A_2) \rightsquigarrow \le 5\%$.

$A_1, A_2, \dots, A_K$

$$\mathbb{P}(A_1 \cup A_2 \cup \dots \cup A_K) \le \sum_{i=1}^{K} \underbrace{\mathbb{P}(A_i)}_{\le \alpha/K}$$
$$\le \alpha.$$

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Strategy: $H_0^{(1)}: \beta_1 \geq 0$ $\quad H_0^{(2)}: \beta_2 \geq \beta_1$.

$\psi^{(1)}_{2.5\%} :=$

$\psi^{(2)}_{2.5\%} :=$

$\psi := \max(\psi^{(1)}, \psi^{(2)})$

$$\underset{n \times 1}{Y} = \underset{n \times p}{\mathbb{X}}\ \underset{p \times 1}{\beta} + \underset{n \times 1}{\varepsilon}$$

$$\hat\beta^{LSE} = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T Y.$$

$$\hat\beta^{LSE} \sim \mathcal{N}_p\left(\beta, \sigma^2(\mathbb{X}^T\mathbb{X})^{-1}\right)$$

$$\hat\beta_1 = e_1^T \cdot \hat\beta^{LSE}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$$\hat\beta_1 \sim \mathcal{N}\Big(\underbrace{e_1^T\beta}_{=\beta_1}, \underbrace{e_1^T[\sigma^2(\mathbb{X}^T\mathbb{X})^{-1}]e_1}_{=\sigma^2 M_{11}}\Big)$$

$$(\mathbb{X}^T\mathbb{X})^{-1} = (M_{ij})_{i,j=1}^p$$

$$\hat\sigma^2 = \frac{1}{n-p}\|Y - \mathbb{X}\hat\beta\|_2^2$$

$H_0^{(1)}: \beta_1 \geq 0 \quad H_1^{(1)}: \beta_1 < 0$

$$T^{(1)} = \frac{e_1^T \cdot \hat\beta}{\sqrt{\hat\sigma^2 M_{11}}}$$

$$\psi^{(1)}: \mathbb{1}\{T^{(1)} < q_{2.5\%}(t_{n-p})\}$$

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<u>Strategy</u>: $H_0^{(1)}: \beta_1 \ge 0$ $\quad H_0^{(2)}: \beta_2 \ge \beta_1$

$$\psi^{(1)}_{2.5\%} := \mathbb{1}\left\{\frac{e_1^T\hat\beta}{\sqrt{\hat\sigma^2 M_{11}}} < q_{2.5\%}(t_{n-p})\right\}$$

$$\psi^{(2)}_{2.5\%} := \mathbb{1}\left\{\frac{\hat\beta_2 - \hat\beta_1}{\sqrt{\hat\sigma^2(M_{11}+M_{22}-2M_{12})}} < q_{2.5\%}(t_{n-p})\right\}$$

$$\psi := \max(\psi^{(1)}, \psi^{(2)})$$

$$\underset{n\times 1}{Y} = \underset{n\times p}{\mathbb{X}}\ \underset{p\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

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$$\hat\beta^{LSE} \sim \mathcal{N}_p(\beta, \sigma^2(\mathbb{X}^T\mathbb{X})^{-1})$$

$$\hat\beta_2 - \hat\beta_1 = u^T\hat\beta^{LSE} \ ; \ u = \begin{pmatrix}-1\\+1\\0\\\vdots\\0\end{pmatrix}$$

$$\hookrightarrow \sim \mathcal{N}\Big(\underbrace{u^T\beta}_{\beta_2-\beta_1},\ \underbrace{u^T[\sigma^2(\mathbb{X}^T\mathbb{X})^{-1}]u}_{\sigma^2(M_{11}+M_{22}-2M_{12})}\Big)$$

$$T^{(2)} := \frac{\hat\beta_2 - \hat\beta_1}{\sqrt{\hat\sigma^2(M_{11}+M_{22}-2M_{12})}}$$

$$\psi^{(2)} = \mathbb{1}\left\{T^{(2)} < q_{2.5\%}(t_{n-p})\right\}$$

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$$\psi := \max(\psi^{(1)}, \psi^{(2)})$$

(1) $\psi^{(1)} \rightsquigarrow H_0^{(1)}: 0 \le \beta_1$

"Reject if $\hat{\beta}_1$ is too small"

(2) $\psi^{(2)} \rightsquigarrow H_0^{(2)}: \beta_1 \le \beta_2 \iff 0 \le \beta_2 - \beta_1$

"Reject if $\hat{\beta}_2 - \hat{\beta}_1$ is too small"

$$\mathbb{P}(\psi = \text{"reject"}) \le 2.5\% + 2.5\% = 5\%$$

What if $H_0 = \beta_1, \beta_2, \dots, \beta_p \ge 0$

(Test at level $\alpha = 5\%$)