$X_1, \dots, X_n \sim N(0, \sigma^2)$ iid.

Goal: Find asymptotic confidence interval for σ^2

① Find estimator $\hat{\sigma^2}$ ($\hat{\sigma}^2$) for σ^2.

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2$$

② Determine the asymptotic distribution of $\hat{\sigma}^2$

③ Construct conf. interval based on $\hat{\sigma}^2$

①

$$\mathrm{Var}(X_1) = \mathbb{E}\left[(X_1 - \mathbb{E}[X_1])^2\right] = \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2$$

By LLN:

$$\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}[X_1^2]$$

$$\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}[X_1]$$

$$\Rightarrow \hat{\sigma}^2 \xrightarrow[n\to\infty]{\mathbb{P}} \mathrm{Var}(X_1)$$

$X_1, \dots, X_n \sim N(0, \sigma^2)$ iid.

Goal: Find asymptotic confidence interval for σ^2

(1) Find estimator $\widehat{\sigma^2}$ ($\hat{\sigma}^2$) for σ^2.

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

(2) Determine the asymptotic distribution of $\hat{\sigma}^2$

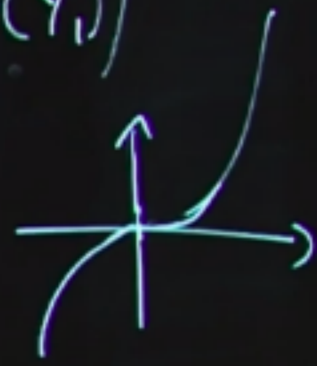
(3) Construct conf. interval based on $\hat{\sigma}^2$

(2) $\begin{pmatrix} Y_1 \\ W_1 \end{pmatrix}, \dots, \begin{pmatrix} Y_n \\ W_n \end{pmatrix}$ iid.

CLT: $$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - E[Y_1]\right) \xrightarrow[n \to \infty]{D} N(0, \mathrm{Var}(Y_1))$$

2D CLT: $$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} Y_i \\ \frac{1}{n}\sum_{i=1}^{n} W_i \end{pmatrix} - \begin{pmatrix} E[Y_1] \\ E[W_1] \end{pmatrix}\right) \xrightarrow[n \to \infty]{D} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathrm{Var}(Y_1) & \mathrm{Cov}(Y_1, W_1) \\ \mathrm{Cov}(Y_1, W_1) & \mathrm{Var}(W_1) \end{pmatrix}\right)$$

$$\boxed{\begin{array}{l} Y_i = X_i^2 \\ W_i = X_i \end{array}}$$



$$\mathrm{Var}(W_i) = \sigma^2$$

$$\mathrm{Var}(Y_i) = E[(X_1^2)^2] - (E[X_1^2])^2 = 3 \cdot \sigma^4 - (\sigma^2)^2 = 2\sigma^4.$$

$$\mathrm{Cov}(Y_1, W_1) = E[X_1^2 \cdot X_1] - E[X_1^2] \cdot E[X_1]$$
$$= 0 - \sigma^2 \cdot 0 = 0$$

$X_1, \dots, X_n \sim N(0, \sigma^2)$ iid.

Goal: Find asymptotic confidence interval for σ^2

① Find estimator $\widehat{\sigma^2}$ ($\hat{\sigma}^2$) for σ^2.

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2$$

② Determine the asymptotic distribution of $\hat{\sigma}^2$

$$\sqrt{n}\cdot\frac{\hat{\sigma}^2 - \sigma^2}{\sqrt{2}\sigma^2} \xrightarrow[n\to\infty]{D} N(0,1)$$

③ Construct conf. interval based on $\hat{\sigma}^2$

② Delta Method: $\sqrt{n}(\underset{\mathbb{R}^d}{T_n} - \Theta) \xrightarrow[n\to\infty]{D} N(0, \underset{\mathbb{R}^{d\times d}}{\Sigma})$

$g: \mathbb{R}^d \to \mathbb{R}$, continuously differentiable at Θ

$$\Rightarrow \sqrt{n}\left(g(T_n) - g(\Theta)\right) \to N(0, \nabla g(\Theta)^T \Sigma \nabla g(\Theta))$$

$Y_i = X_i^2$, $W_i = X_i$, $T_n = \frac{1}{n}\sum_{i=1}^n \begin{pmatrix} X_i^2 \\ X_i \end{pmatrix}$, $\hat{\sigma}^2 = g(T_n)$,

$g(y, w) = y - w^2$, $\nabla g(y,w) = \begin{pmatrix} 1 \\ -2w \end{pmatrix}$, $\Theta = \begin{pmatrix} \mathbb{E}[Y_1] \\ \mathbb{E}[W_1] \end{pmatrix} = \begin{pmatrix} \sigma^2 \\ 0 \end{pmatrix}$

$$\sqrt{n}\cdot(\hat{\sigma}^2 - \sigma^2) \xrightarrow[n\to\infty]{D} N\left(0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 2\sigma^4 & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= N(0, 2\sigma^4)$$

$X_1, \dots, X_n \sim N(0, \sigma^2)$ iid.

<u>Goal</u>: Find asymptotic confidence interval for σ^2.

① Find estimator $\hat{\sigma^2}$ $(\hat{\sigma}^2)$ for σ^2

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

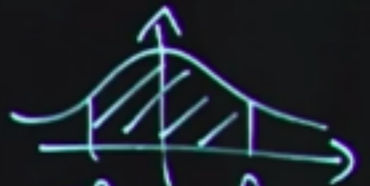
② Determine the asymptotic distribution of $\hat{\sigma}^2$

$$\sqrt{n}\,\frac{\hat{\sigma}^2 - \sigma^2}{\sqrt{2}\,\sigma^2} \xrightarrow[n\to\infty]{D} N(0,1)$$

③ Construct confidence interval based on $\hat{\sigma}^2$.

③ Two-sided confidence interval $I = [\hat{\sigma}^2 - s, \hat{\sigma}^2 + s]$

$$\mathbb{P}(\underbrace{\sigma^2 \in I}) \xrightarrow[n\to\infty]{} \mathbb{P}(Z \in [-q, q]) \overset{!}{=} 1-\alpha, \quad Z \in N(0,1)$$



$$\Leftrightarrow \sigma^2 \in [\hat{\sigma}^2 - s, \hat{\sigma}^2 + s]$$

$$\Leftrightarrow \sigma^2 - \hat{\sigma}^2 \in [-s, s] \qquad \Rightarrow q = q_{\alpha/2}$$

$$\Leftrightarrow \hat{\sigma}^2 - \sigma^2 \in [-s, s] \qquad 1-\frac{\alpha}{2} \text{ quantile of } N(0,1)$$

$$\Leftrightarrow \sqrt{\frac{n}{2}}\,\frac{\hat{\sigma}^2 - \sigma^2}{\hat{\sigma}^2} \in \Big[-\sqrt{\frac{n}{2}}\cdot\frac{s}{\hat{\sigma}^2}, \underbrace{\sqrt{\frac{n}{2}}\cdot\frac{s}{\hat{\sigma}^2}}_{=q}\Big], \quad s = q_{\alpha/2}\cdot\frac{\hat{\sigma}^2\cdot\sqrt{2}}{\sqrt{n}}$$

$$\Rightarrow I = \hat{\sigma}^2 + \left[-\sqrt{\frac{2}{n}}\cdot q_{\alpha/2}\cdot\hat{\sigma}^2, \sqrt{\frac{2}{n}}\, q_{\alpha/2}\cdot\hat{\sigma}^2\right]$$

<u>Slutsky's Thm</u>: $A_n \xrightarrow[n\to\infty]{D} A$, $B_n \xrightarrow[n\to\infty]{D} c \neq 0$, $c \in \mathbb{R}$ $\Rightarrow \dfrac{A_n}{B_n} \xrightarrow[n\to\infty]{D} \dfrac{A}{c}$

$$\Rightarrow \sqrt{n}\,\frac{\hat{\sigma}^2 - \sigma^2}{\sqrt{2}\,\hat{\sigma}^2} \xrightarrow[n\to\infty]{D} N(0,1)$$