$X_1, \dots, X_n \sim \mathcal{N}(\underbrace{\mu, \sigma^2}_{\text{unknown}})$ iid.

<u>Goal</u>: Test with level $\alpha$

$H_0: \mu \geq 0 \quad H_1: \mu < 0$

① Estimate $\mu, \sigma^2$

$\hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i, \ \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2$

② Test statistic / pivot

$\psi = \mathbb{1}\{T_n > s\}$

③ Adjust $s$

<u>Non-asymptotic hypothesis test for mean of Gaussians (t-test)</u>

① MLE. Likelihood: $f(\mu, \sigma^2, X_1, \dots, X_n) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}\cdot(X_i - \mu)^2}$

$$\ell(\mu, \sigma^2) = \log f = \sum_{i=1}^n \left[ -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(X_i - \mu)^2 \right]$$

$$= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu)^2$$

$$\partial_\mu \ell(\mu, \sigma^2) = -\frac{1}{2\sigma^2}\sum_{i=1}^n 2\cdot(X_i - \mu)(-1) \overset{!}{=} 0$$

$$\Rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$$

$$\partial_{\sigma^2} \ell(\mu, \sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (X_i - \mu)^2 \overset{!}{=} 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2$$

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③ Adjust $s$

② $\tilde{T}_n^{(1)} = \sqrt{n}\frac{(\mu - \hat{\mu})}{\sqrt{\sigma^2}} \sim \mathcal{N}(0,1)$. Problem: $\sigma^2$ unknown, replace by $\hat{\sigma}^2$!

What is distribution of $\hat{\sigma}^2$?

Review: Orthogonal matrices: $[\underset{\in \mathbb{R}^n}{v_1} | \dots | v_n] = V \in \mathbb{R}^{n \times n}$

$v_i^T v_j = 0, \ i \neq j, \ v_i^T v_i = \|v\|_2^2 = 1 \Rightarrow \{v_i\}_{i=1,\dots,n}$ orthonormal

$V$ orthogonal

$$V^T V = \begin{bmatrix} - v_1^T - \\ \vdots \\ - v_n^T - \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n = V V^T$$

$$x \in \mathbb{R}^n, \ \|Vx\|_2^2 = (Vx)^T(Vx) = x^T \underbrace{V^T V}_{= I_n} x = x^T x = \|x\|_2^2$$

$W = [\underset{\in \mathbb{R}^n}{w_1} | \dots | w_k] \in \mathbb{R}^{n \times k}$, $\{w_i\}_{i=1,\dots,k}$ orthonormal

$$\|W W^T x\|_2^2 = (WW^Tx)^T(WW^Tx) = x^T W \underbrace{W^T W}_{I_k} W^T x = \|W^T x\|_2^2$$

$X_1, \dots, X_n \sim \mathcal{N}(\underbrace{\mu, \sigma^2}_{\text{unknown}})$ iid.

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③ Adjust $s$

② $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2$

$$n\hat{\sigma}^2 = \Big\| \underbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}}_{=X} - \begin{pmatrix} \hat{\mu} \\ \vdots \\ \hat{\mu} \end{pmatrix} \Big\|_2^2$$

$$= \|X - v_1 v_1^T X\|_2^2$$

$$= \|X - \mu \cdot \mathbb{1} - \underbrace{(v_1 \underset{\frac{1}{\sqrt{n}}\mathbb{1}}{\underset{\parallel}{v_1^T}} X - \mu \mathbb{1})}_{= v_1 v_1^T (X - \mu \mathbb{1})}\|_2^2$$

$$\Rightarrow \frac{n\hat{\sigma}^2}{\sigma^2} = \|Y - v_1 v_1^T Y\|_2^2$$

$$= \Big\|\sum_{i=1}^n v_i v_i^T Y - v_1 v_1^T Y\Big\|_2^2$$

$$= \Big\|\sum_{i=2}^n v_i v_i^T Y\Big\|_2^2$$

$$\hat{\mu} = \frac{1}{n} \underset{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}{\underset{\parallel}{\mathbb{1}^T}} X, \quad \begin{pmatrix} \hat{\mu} \\ \vdots \\ \hat{\mu} \end{pmatrix} = \frac{1}{n} \underbrace{\mathbb{1}\mathbb{1}^T}_{=\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}} X$$

$$= \underbrace{\frac{1}{\sqrt{n}}\mathbb{1}}_{=v_1} \underbrace{\left(\frac{1}{\sqrt{n}}\mathbb{1}\right)^T}_{v_1^T} X$$

$$Y = \frac{X - \mu\mathbb{1}}{\sigma} = \begin{pmatrix} \frac{X_1 - \mu}{\sigma} \\ \vdots \\ \frac{X_n - \mu}{\sigma} \end{pmatrix} \sim \mathcal{N}(0, I_n)$$

$V = [v_1 | \dots | v_n] \in \mathbb{R}^{n \times n}$ orthogonal

$$I_n = VV^T = [v_1 | \dots | v_n] \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$= \Big(\sum_{i=1}^n v_i e_i^T\Big)\Big(\sum_{j=1}^n v_j e_j^T\Big)^T = \sum_{i,j=1}^n v_i \underbrace{e_i^T e_j}_{=\begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}} v_j^T$$

$$i \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad = \sum_{i=1}^n v_i v_i^T$$

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② Test statistic / pivot

$\tilde{T}_n^{(1)} = \frac{(\mu - \hat{\mu})\sqrt{n}}{\sqrt{\sigma^2}} \sim N(0,1)$

$T_n = \frac{-\sqrt{n-1} \cdot \hat{\mu}}{\sqrt{\hat{\sigma}^2}}$

$\psi = \mathbb{1}\{T_n > s\}$

③ Adjust $s$

② $\frac{n\hat{\sigma}^2}{\sigma^2} = \left\| \sum_{i=2}^n v_i v_i^T Y \right\|_2^2,$

$= \| W W^T Y \|_2^2 = \| \underbrace{W^T Y}_{\in \mathbb{R}^{n-1}} \|_2^2$

$\operatorname{Cov}(W^T Y) = W^T \cdot \operatorname{Cov}(Y) \cdot (W^T)^T$

$= W^T \cdot I_n \cdot W = I_{n-1}$

$\Rightarrow \frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$

$\operatorname{Cov}(\underbrace{W^T Y}_{\in \mathbb{R}^{n-1}}, \underbrace{v_1^T Y}_{\in \mathbb{R}}) = \mathbb{E}[W^T Y Y^T v_1]$

$= W^T \underbrace{\mathbb{E}[Y Y^T]}_{I_n} v_1 = W^T v_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{n-1}$

$\rightarrow W^T Y \perp\!\!\!\perp v_1^T Y \Rightarrow \hat{\sigma}^2 \perp\!\!\!\perp \hat{\mu}$

$Y = \begin{pmatrix} \frac{X_1 - \mu}{\sigma} \\ \vdots \\ \frac{X_n - \mu}{\sigma} \end{pmatrix} \sim \mathcal{N}(0, I_n)$

$V = [v_1 | \underbrace{v_2 | \dots | v_n}_{= W \in \mathbb{R}^{n \times (n-1)}}]$ orthogonal

$\sum_{i=2}^n v_i v_i^T = W W^T$

$\hat{\mu} = \frac{1}{\sqrt{n}} \cdot v_1^T \cdot X = \mu + \sigma \cdot \frac{1}{\sqrt{n}} v_1^T \cdot Y$

$\tilde{T}_n^{(2)} = \frac{\dfrac{(\mu - \hat{\mu})\cancel{\sqrt{n}}}{\cancel{\sqrt{\sigma^2}}}}{\sqrt{\dfrac{\cancel{n}}{n-1}} \sqrt{\dfrac{\hat{\sigma}^2}{\cancel{\sigma^2}}}}$

$= \frac{(\mu - \hat{\mu})\sqrt{n-1}}{\sqrt{\hat{\sigma}^2}} \sim t_{n-1}$

$X_1, \dots, X_n \sim \mathcal{N}(\underbrace{\mu, \sigma^2}_{\text{unknown}})$ iid.

<u>Goal</u>: Test with level $\alpha$

$H_0: \mu \geq 0 \quad H_1: \mu < 0$

① Estimate $\mu, \sigma^2$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

② Test statistic / pivot

$$\tilde{T}_n^{(1)} = \frac{(\mu - \hat{\mu})\sqrt{n}}{\sqrt{\sigma^2}} \sim N(0,1)$$

$$T_n = \frac{-\sqrt{n-1} \cdot \hat{\mu}}{\sqrt{\hat{\sigma}^2}}$$

$$\psi = \mathbb{1}\{T_n > s\}$$

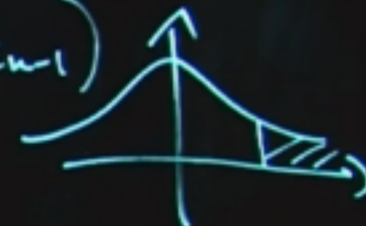
③ Adjust $s$

$$s = q_{1-\alpha}(t_{n-1})$$

③ $$\tilde{T}_n^{(2)} = \frac{(-\hat{\mu} + \mu)\sqrt{n-1}}{\sqrt{\hat{\sigma}^2}} \sim t_{n-1} \quad \Big| \quad T_n = \frac{-\hat{\mu}\sqrt{n-1}}{\sqrt{\hat{\sigma}^2}}$$

For $\mu = 0$: $\mathbb{P}_{(0,\sigma^2)}(\psi = 1) = \mathbb{P}_{(0,\sigma^2)}(T_n > s)$

$$= \mathbb{P}(\underset{\sim t_{n-1}}{Z} > s) \overset{!}{=} \alpha \Rightarrow s = q_{1-\alpha}(t_{n-1})$$



For $\mu > 0$: $\mathbb{P}_{(\mu,\sigma^2)}(\psi = 1) = \mathbb{P}_{(\mu,\sigma^2)}(T_n > s)$

$$= \mathbb{P}_{(\mu,\sigma^2)}\left(\frac{-\hat{\mu}\sqrt{n-1}}{\sqrt{\hat{\sigma}^2}} > s\right) = \mathbb{P}_{(\mu,\sigma^2)}\Bigg(\underbrace{\frac{(\mu - \hat{\mu})\sqrt{n-1}}{\sqrt{\hat{\sigma}^2}}}_{= Z \sim t_{n-1}} > s + \underbrace{\frac{\mu\sqrt{n-1}}{\sqrt{\hat{\sigma}^2}}}_{> 0}\Bigg)$$

$$\leq \mathbb{P}_{(\mu,\sigma^2)}(Z > s) = \alpha$$

type II error: