Algorithms

Growth of functions

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Θ-notation

Given a function $g(n)$, define the set of functions:

$$\Theta(g(n)) = \{f(n) \ : \ 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}$$

where c_1, c_2, n_0 are positive constants.

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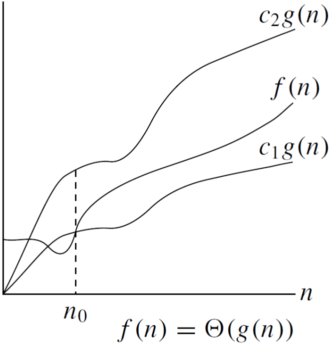

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If this is true, then we can find positive constants c_1, c_2, n_0 such that:

$$c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2 \quad \text{for all } n \geq n_0$$

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This holds, for instance, with $c_1 = \frac{1}{14}, c_2 = \frac{1}{2}, n_0 = 7$.

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- A coefficient that is different from $\frac{1}{2}$ in $\frac{1}{2}n^2$ would just change c_1, c_2.

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Does it meet the intuition?

- A coefficient that is different from $\frac{1}{2}$ in $\frac{1}{2}n^2$ would just change c_1, c_2.
- The mere presence of $\frac{1}{2}$ allows us to find c_1, c_2, so we can always dominate the linear term.

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Example:

$$an^2 + bn + c \in \Theta(n^2) \quad \text{with } a > 0$$

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Which includes constant functions:

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Which includes constant functions:

$$a \in \Theta(1) \quad \text{with } a > 0$$

O-notation

Given a function $g(n)$, define the set of functions:

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where c, n_0 are positive constants.

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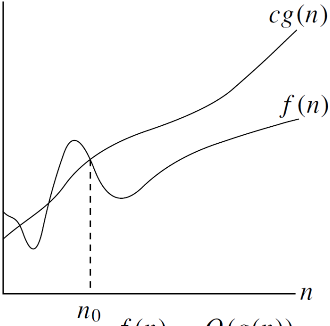

$f(n) = O(g(n))$

O-notation: Example

Intuition: Start from the observation that:

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Thus, lower-order terms are also allowed now.

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- Useful to write (asymptotic) upper bounds.

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- Useful to write (asymptotic) upper bounds.
- We use it to represent the worst case performance.

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Given a function $g(n)$, define the set of functions:

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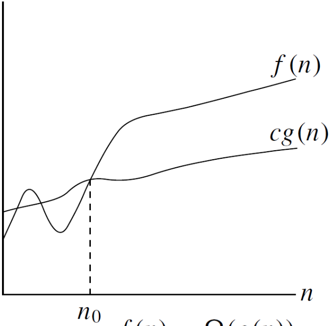

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- Useful to write (asymptotic) lower bounds.
- We use it to bound the best case.

Properties

Not too difficult to prove:

$$f(n) \in \Theta(g(n)) \;\Leftrightarrow\; f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n))$$

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Reflexivity:

$$\begin{aligned}
f(n) &= \Theta(f(n)) \\
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Transpose symmetry: $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$

Intuition and counter

The following analogy helps as a mnemonic:

$$\begin{aligned} f(n) = O(g(n)) &\quad\approx\quad a \le b \\ f(n) = \Omega(g(n)) &\quad\approx\quad a \ge b \\ f(n) = \Theta(g(n)) &\quad\approx\quad a = b \end{aligned}$$

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But two functions are not always asymptotically comparable.

$$f(n) = n \qquad g(n) = n^{1+\sin n}$$

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But two functions are not always asymptotically comparable.

$$f(n) = n \qquad g(n) = n^{1+\sin n}$$

For these f and g, we have:

$$f(n) \notin O(g(n)) \quad \text{and} \quad f(n) \notin \Omega(g(n))$$

Suggested reading

Chapter 3.1 of:

"Introduction to Algorithms – 2nd Ed.", Cormen et al.

Skip the o-**notation** and ω-**notation** paragraphs.