Algorithms

Heaps and priority queues

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Heap

Data structures allow us to organize data for efficient use.

$$(1, 2, 3, 4, 9, 10, 7, 16, 8, 14) \quad \xRightarrow{\text{build}} \quad \boxed{16 \mid 14 \mid 10 \mid 8 \mid 7 \mid 9 \mid 3 \mid 2 \mid 4 \mid 1}$$

Heap

Data structures allow us to organize data for efficient use.

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

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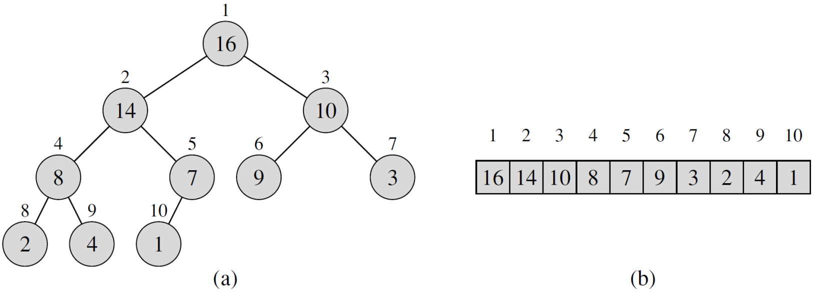

- $\mathrm{left}(i) = 2i$
- $\mathrm{right}(i) = 2i + 1$

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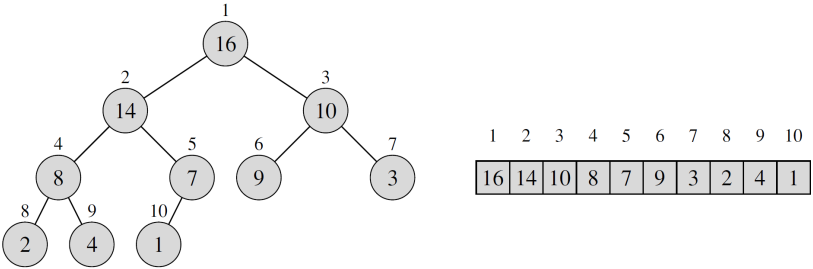

(a) (b)

- $\text{left}(i) = 2i$
- $\text{right}(i) = 2i + 1$
- $\text{parent}(i) = \lfloor i/2 \rfloor$

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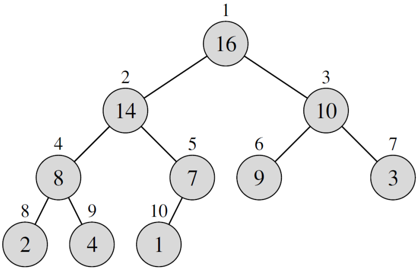

(a)

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

(b)

max-heap property:

$$A[\text{parent}(i)] \geq A[i]$$

which means that root $=$ largest element.

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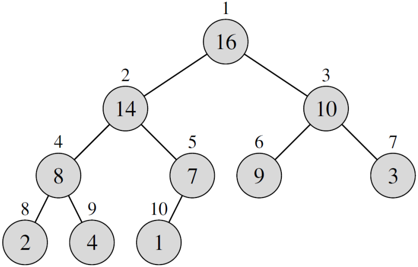

(a)

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

(b)

min-heap property:

$$A[\text{parent}(i)] \leq A[i]$$

which means that root $=$ smallest element.

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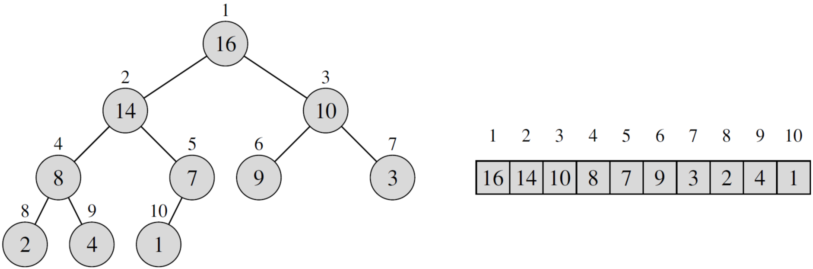

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height of a node: # edges of the longest path to a leaf

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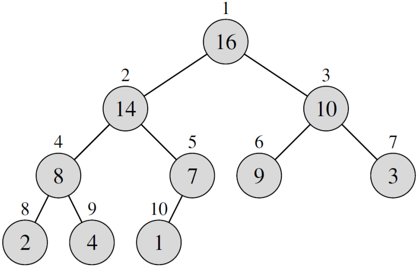

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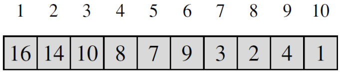

(b)

height of a node: # edges of the longest path to a leaf

height of the heap = height of the root = $\Theta(\lg n)$

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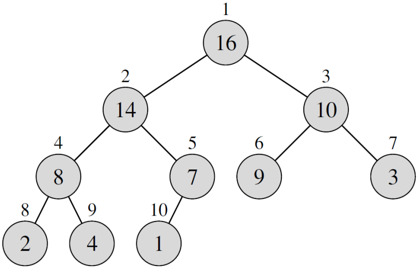

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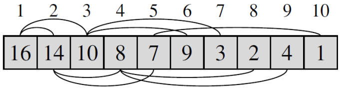

(b)

The heap can be seen as a binary tree or as an array.

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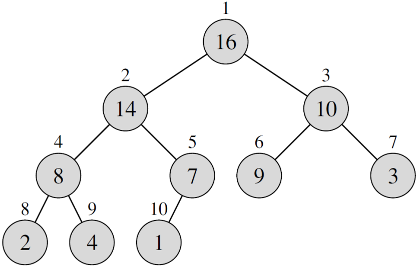

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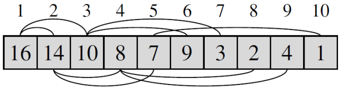

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Each level (except the last one) must be filled completely left-to-right.

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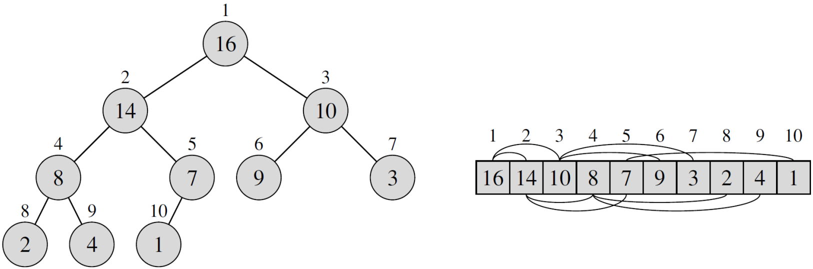

(a) (b)

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Each level (except the last one) must be filled completely left-to-right.

The number of leaves is $n - \lfloor \frac{n}{2} \rfloor$.

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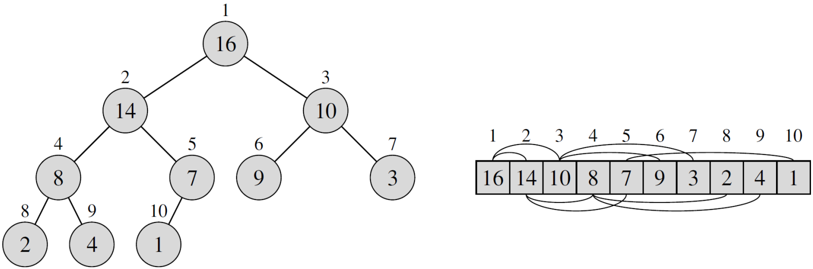


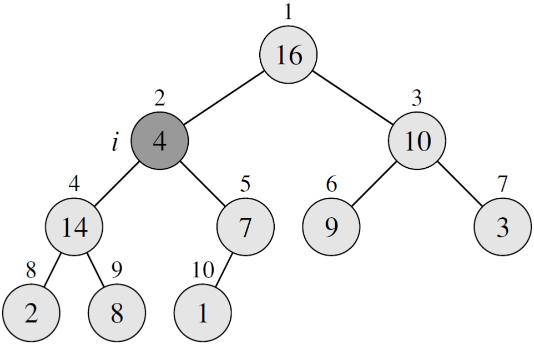
(a) (b)

We will see the following operations:

- Build a (max-)heap from some input array.
- Maintain the (max-)heap property.
- Construct a priority queue on top of a heap.

“Heapify”: Maintaining the heap property

We are given as input the root index i of the sub-tree to check.



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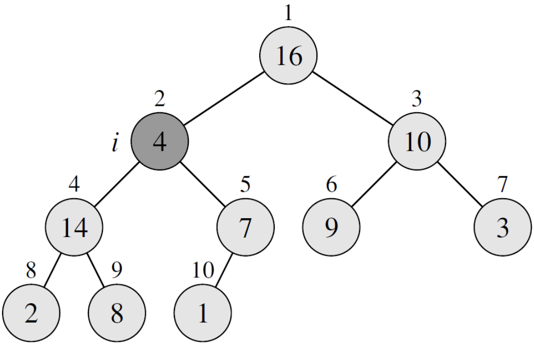

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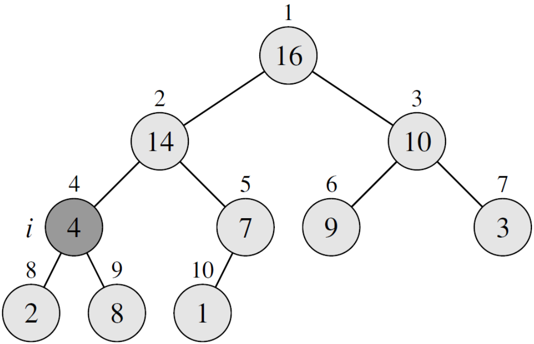

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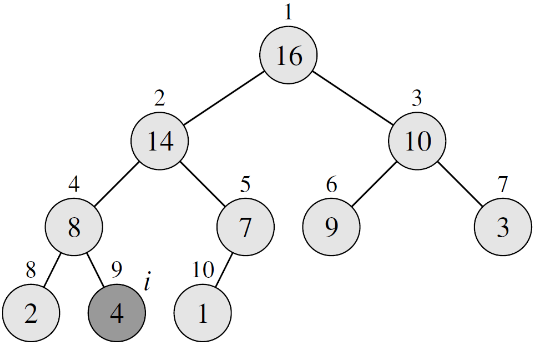

Assumption: $\mathrm{left}(i)$ and $\mathrm{right}(i)$ are roots of valid heaps.

Recursively exchange the violating node with its largest child.

Maintaining the heap property

```
MAX-HEAPIFY(A, i)
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ heap-size[A] and A[l] > A[i]
   then largest ← l
   else largest ← i
if r ≤ heap-size[A] and A[r] > A[largest]
   then largest ← r
if largest ≠ i
   then exchange A[i] ↔ A[largest]
        MAX-HEAPIFY(A, largest)
```

If $A[i]$ is the largest, then the sub-tree is already a max heap.

Maintaining the heap property

MAX-HEAPIFY(A, i)

$\Theta(1)$

$l \leftarrow \text{LEFT}(i)$
$r \leftarrow \text{RIGHT}(i)$
if $l \leq \textit{heap-size}[A]$ and $A[l] > A[i]$
  then $\textit{largest} \leftarrow l$
  else $\textit{largest} \leftarrow i$
if $r \leq \textit{heap-size}[A]$ and $A[r] > A[\textit{largest}]$
  then $\textit{largest} \leftarrow r$
if $\textit{largest} \neq i$
  then exchange $A[i] \leftrightarrow A[\textit{largest}]$
    MAX-HEAPIFY$(A, \textit{largest})$

Maintaining the heap property

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            l ← LEFT(i)
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Θ(1)           else largest ← i
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T(2n/3) →           MAX-HEAPIFY(A, largest)
```

Maintaining the heap property

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                MAX-HEAPIFY(A, i)
                 1  l ← LEFT(i)
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        Θ(1)     5     else largest ← i
                 6  if r ≤ heap-size[A] and A[r] > A[largest]
                 7     then largest ← r
                 8  if largest ≠ i
                 9     then exchange A[i] ↔ A[largest]
T(2n/3) →       10          MAX-HEAPIFY(A, largest)
```

$$T(n) \le T(2n/3) + \Theta(1)$$

Maintaining the heap property

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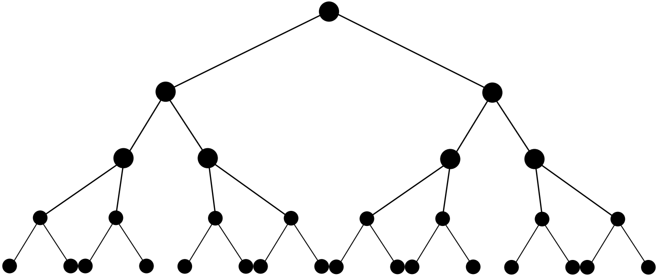
$$T(n) = O(\lg n)$$

by case (2) of the master theorem.

Maximum size of sub-tree in a heap



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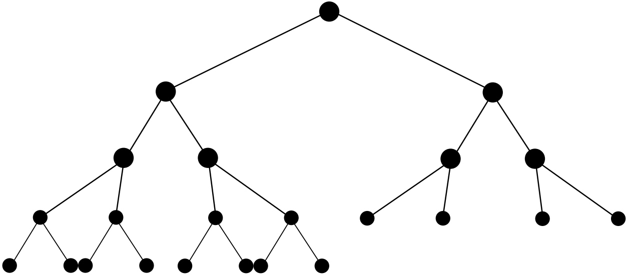

- A valid heap tree **must** be completely filled at each level.

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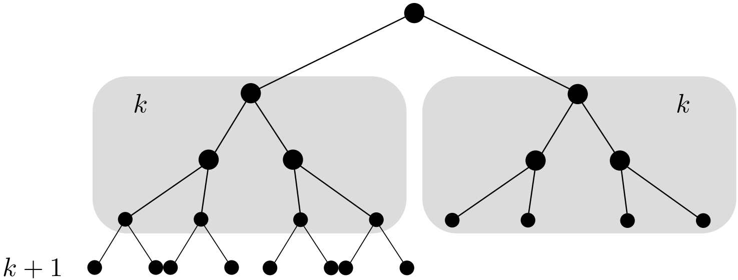

- A valid heap tree must be completely filled at each level.
- The largest sub-tree has $\leq k + (k+1)$ nodes.

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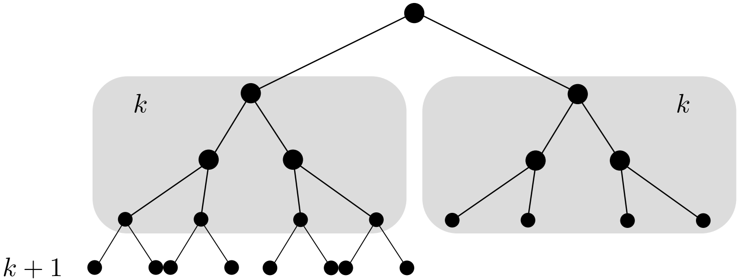

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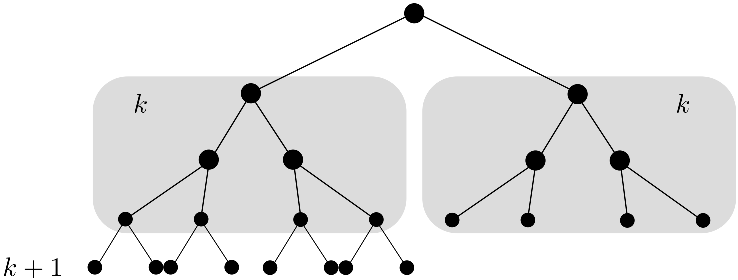

- A valid heap tree must be completely filled at each level.
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- The entire tree has $n = 1 + 2k + (k+1) = 3k+2$ nodes.

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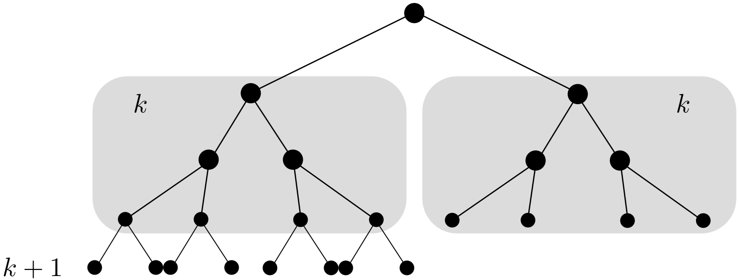

- A valid heap tree must be completely filled at each level.
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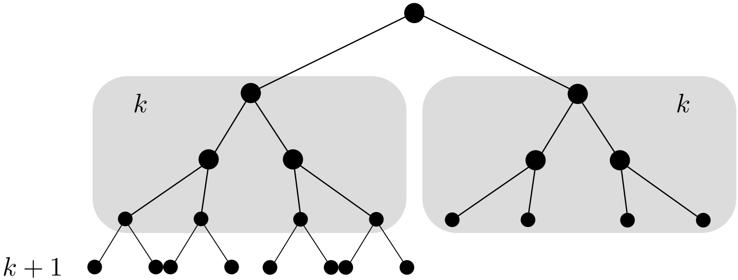
- The entire tree has $n = 1 + 2k + (k+1) = 3k+2$ nodes.
- Thus, the largest sub-tree has $\leq \frac{2k+1}{3k+2} n$ nodes.

Maximum size of sub-tree in a heap



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- The entire tree has $n = 1 + 2k + (k+1) = 3k+2$ nodes.
- Thus, the largest sub-tree has $\leq \frac{2k+1}{3k+2}n$ nodes.
- For arbitrary k, we get $\lim_{k\to\infty} \frac{2k+1}{3k+2}n = \frac{2}{3}n$.

Building a heap

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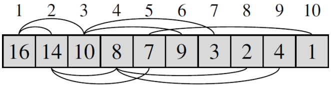
The leaf indices are known a priori: $(\lfloor n/2 \rfloor + 1) \cdots n$



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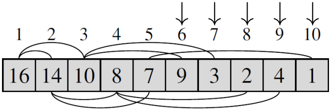

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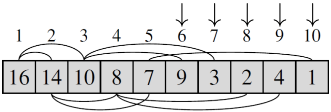

General idea: run max-heapify on the internal nodes.

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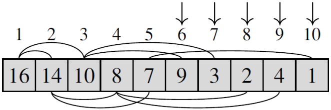

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$\textsc{Build-Max-Heap}(A)$

$\quad heap\text{-}size[A] \leftarrow length[A]$
$\quad$ **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
$\qquad$ **do** $\textsc{Max-Heapify}(A, i)$

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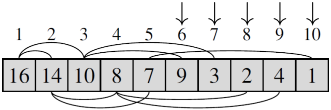

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BUILD-MAX-HEAP(A)

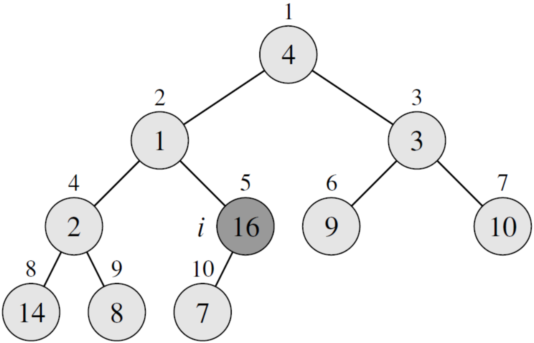
```
heap-size[A] ← length[A]
for i ← ⌊length[A]/2⌋ downto 1
    do MAX-HEAPIFY(A, i)
```

$O(n)$ calls to a $O(\lg n)$ algorithm: $T(n) = O(n \lg n)$.

Building a heap

A	4	1	3	2	16	9	10	14	8	7

Building a heap



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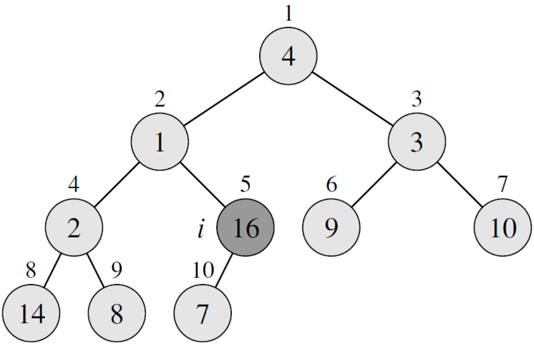
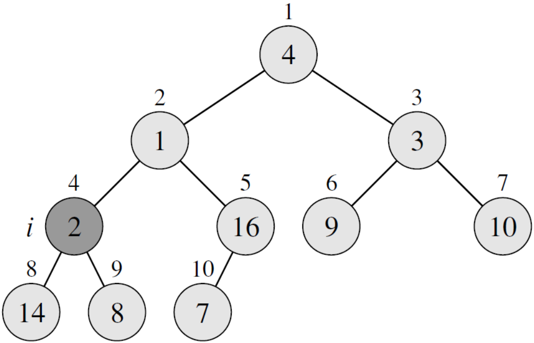


The sub-tree is already a heap.

Building a heap



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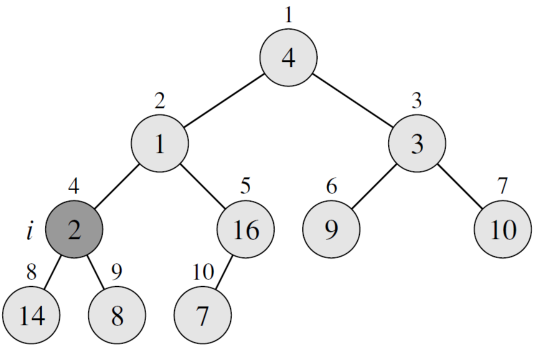
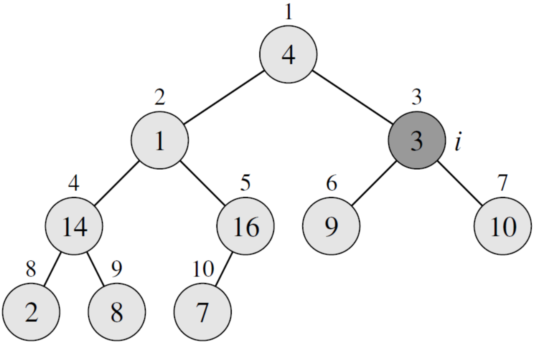


The sub-tree does not satisfy the heap property – run heapify.

Building a heap



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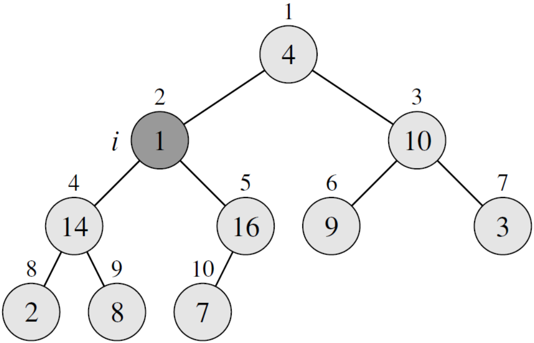
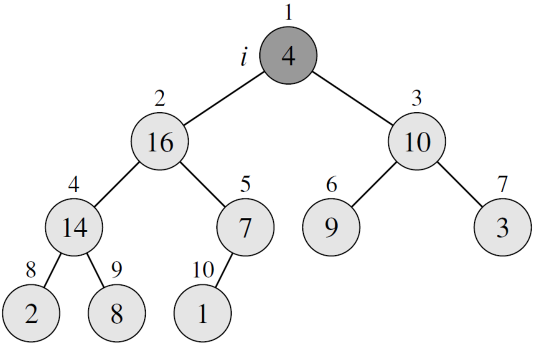


By proceeding bottom-up, we can always run heapify, which assumes that the sub-trees rooted at $\mathrm{left}(i)$ and $\mathrm{right}(i)$ are valid.

This way, after heapify we can guarantee that i is the root of a heap.

Building a heap



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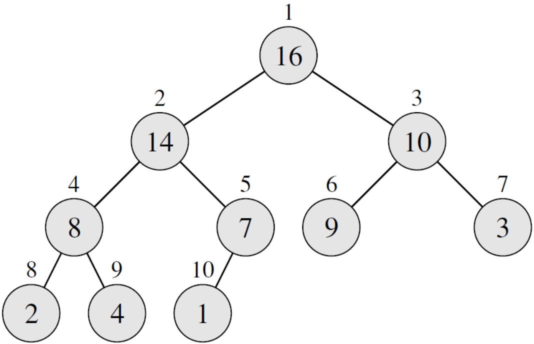

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Priority queues

Example:

Let us be given a set of tasks, each with a priority.

When a task is done, pass to the next with highest priority.

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We need to support the following operations:

- Insert a new element + priority.
- Find the highest-priority element.
- Extract the highest-priority element.
- Increase the priority of a given element.

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- Find the highest-priority element.
- Extract the highest-priority element.
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We will implement the priority queue on top of a heap.

Priority queues

The heap stores the priority values of each element.

In practice the element itself is also attached, but we ignore it here.

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Find the highest priority element in $\Theta(1)$ time:

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HEAP-MAXIMUM(A)
return A[1]
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Extract the highest-priority element in $O(\lg n)$ time:

```
HEAP-EXTRACT-MAX(A)
    if heap-size[A] < 1
        then error "heap underflow"
    max ← A[1]
discard   A[1] ← A[heap-size[A]]
the root  heap-size[A] ← heap-size[A] − 1
    MAX-HEAPIFY(A, 1)
    return max
```

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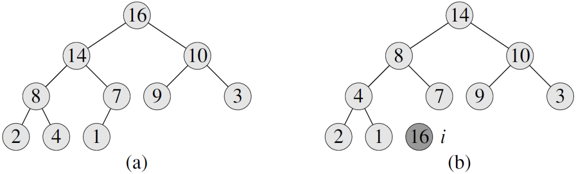
HEAP-MAXIMUM(A)
1 **return** $A[1]$

Extract the highest-priority element in $O(\lg n)$ time:



discarding the root

Priority queues

Increase the priority of a given element in $O(\lg n)$ time:

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
   then error "new key is smaller than current key"
A[i] ← key
while i > 1 and A[PARENT(i)] < A[i]
     do exchange A[i] ↔ A[PARENT(i)]
        i ← PARENT(i)
```

Priority queues

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HEAP-INCREASE-KEY(A, i, key)
1  if key < A[i]
2     then error "new key is smaller than current key"
3  A[i] ← key
4  while i > 1 and A[PARENT(i)] < A[i]
5       do exchange A[i] ↔ A[PARENT(i)]
6          i ← PARENT(i)
```

Line 3 can violate the max-heap property, depending on the value of key.

Priority queues

Increase the priority of a given element in $O(\lg n)$ time:

```
HEAP-INCREASE-KEY(A, i, key)
1  if key < A[i]
2     then error "new key is smaller than current key"
3  A[i] ← key
4  while i > 1 and A[PARENT(i)] < A[i]
5       do exchange A[i] ↔ A[PARENT(i)]
6          i ← PARENT(i)
```

Line 3 can violate the max-heap property, depending on the value of key.

If so, the correct index for key must be up in the tree.

Lines 4-6 find the correct index by moving the value in an upward path.

Priority queues

Insert a new element + priority in $O(\lg n)$ time:

```
MAX-HEAP-INSERT(A, key)
    heap-size[A] ← heap-size[A] + 1
    A[heap-size[A]] ← −∞
    HEAP-INCREASE-KEY(A, heap-size[A], key)
```

Priority queues

Insert a new element + priority in $O(\lg n)$ time:

MAX-HEAP-INSERT(A, key)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
HEAP-INCREASE-KEY$(A, heap\text{-}size[A], key)$

The value of $-\infty$ ensures that line 1 of increase is false.

Priority queues

Insert a new element + priority in $O(\lg n)$ time:

```
MAX-HEAP-INSERT(A, key)
heap-size[A] ← heap-size[A] + 1
A[heap-size[A]] ← −∞
HEAP-INCREASE-KEY(A, heap-size[A], key)
```

The value of $-\infty$ ensures that line 1 of increase is false.

To summarize:

operation	**cost**
find	$\Theta(1)$
extract	$O(\lg n)$
increase	$O(\lg n)$
insert	$O(\lg n)$

Suggested reading

Chapters 6.1, 6.2, 6.3 and 6.5 of:

"Introduction to Algorithms – 2nd Ed.", Cormen et al.