Algorithms

Recursion I

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Recursive algorithms

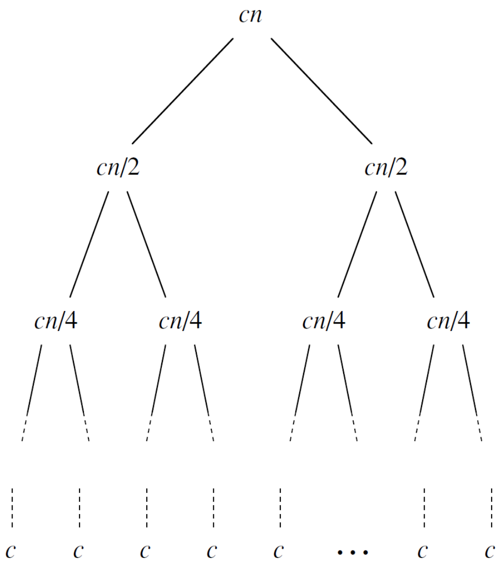
```
MERGE-SORT(A, p, r)
if p < r
   then q ← ⌊(p + r)/2⌋
        MERGE-SORT(A, p, q)
        MERGE-SORT(A, q + 1, r)
        MERGE(A, p, q, r)
```

Recursive algorithms

```python
def binsearch(S, x, i, f):
    if i < f:
        q = int((i + f) / 2)
        if x < S[q]:
            return binsearch(S, x, i, q)
        elif x > S[q]:
            return binsearch(S, x, q + 1, f)
        elif x == S[q]:
            return q + 1
    else:
        return -1


A = [11, 21, 31, 41, 51, 61, 71, 91]
v = 71
index = binsearch(A, v, 0, len(A))
print(index)
```

Recursion tree



The factorial

$$4! = 4 \times 3 \times 2 \times 1$$

The factorial

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots$$

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\begin{aligned}
n! &= n \times (n-1)! \\
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&\vdots \\
1! &= 1
\end{aligned}
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&\vdots \\
1! &= 1 \qquad \textbf{base case}
\end{aligned}
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Q: What is the complexity $T(n)$ of this algorithm?

Each function call generates a context, containing all the data associated with that function call.

The recursion stack

Recursion involves nested function calls.

The contexts are memorized one on top of the other, in a stack.

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        return 1
    else:
               3            2
        return n*factorial(n-1)
    end
```

```
          2
factorial(n):
    if n == 1:
        return 1
    else:
               2            1
        return n*factorial(n-1)
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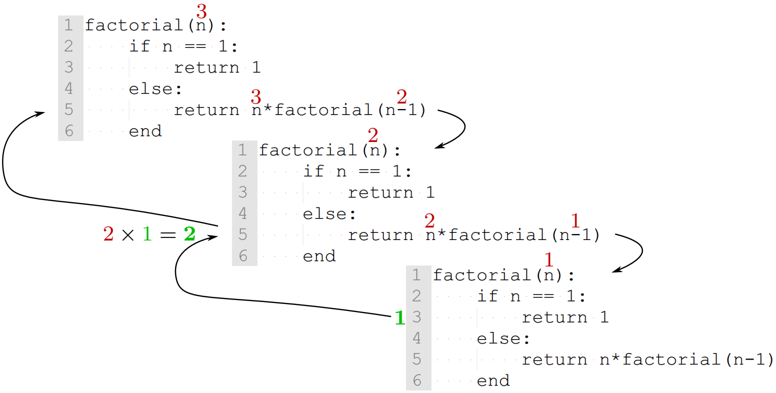

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Tip: Use the debugger to trace recursive calls step by step.

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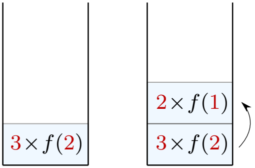

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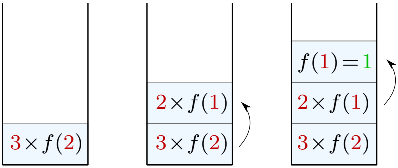

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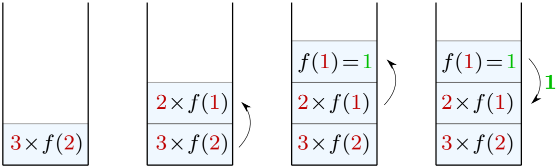

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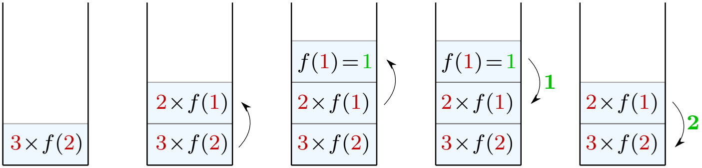

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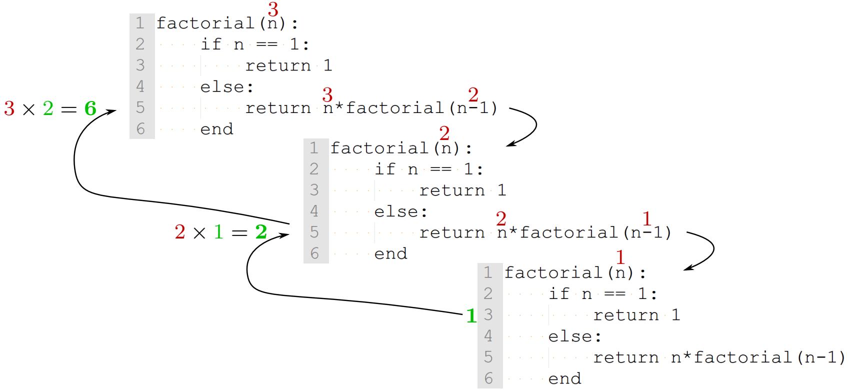

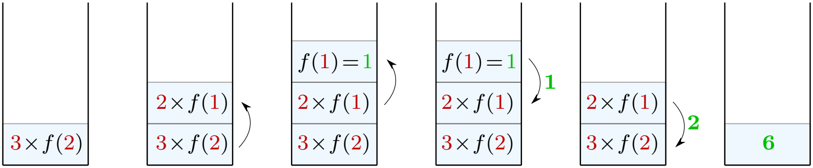

Backtracking

General algorithm for a wide family of computational problems.

- Incrementally build a possible solution.
- Abandon the solution as soon as it does not meet the constraints. (backtrack)

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Each intermediate step is a node of a search tree.

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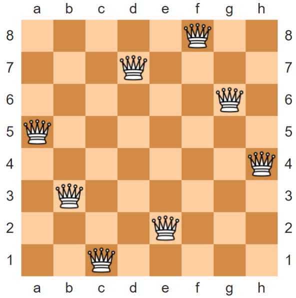
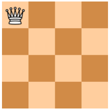
Classical example: 8 queens problem.



Example: 4 queens



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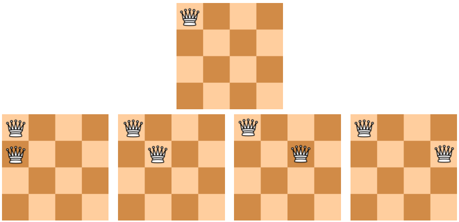

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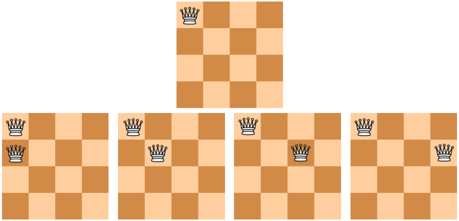

invalid **invalid**

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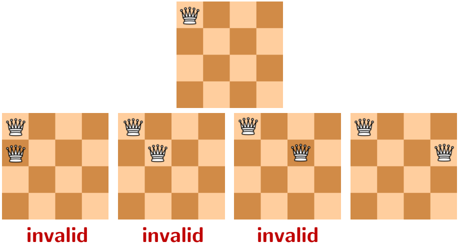

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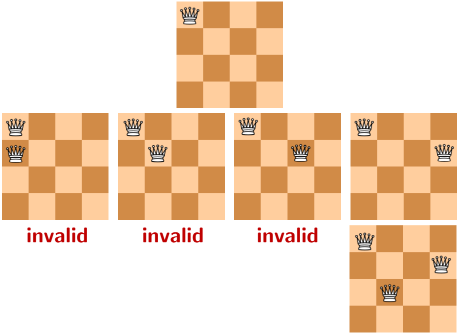

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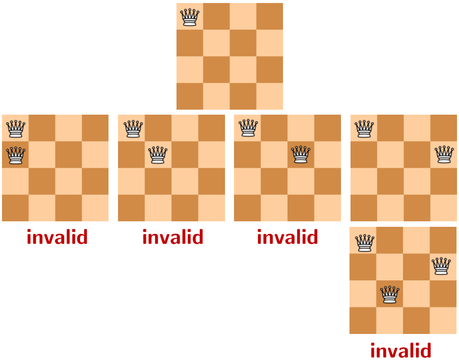

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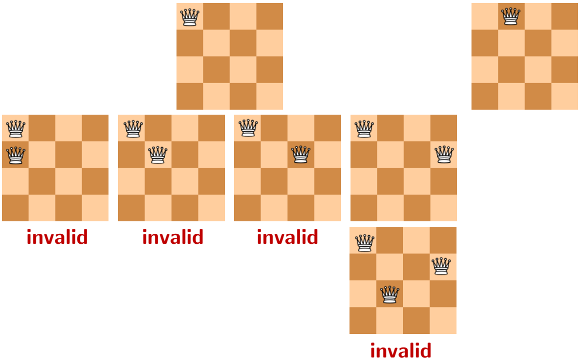

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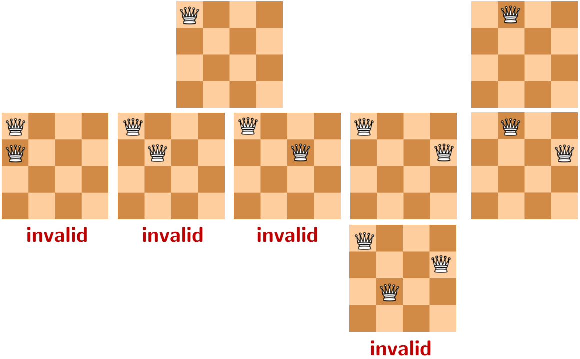

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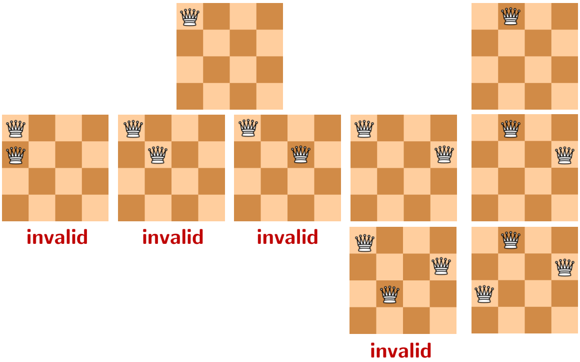

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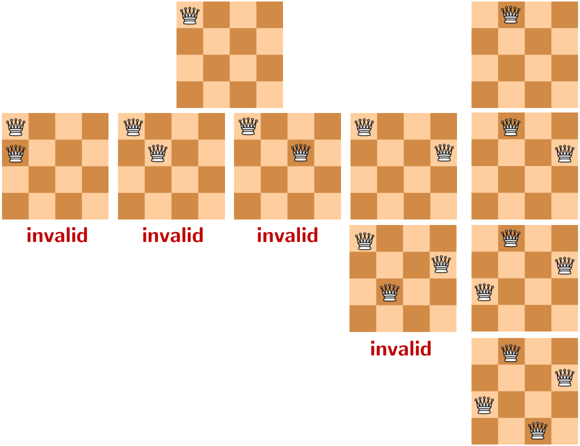

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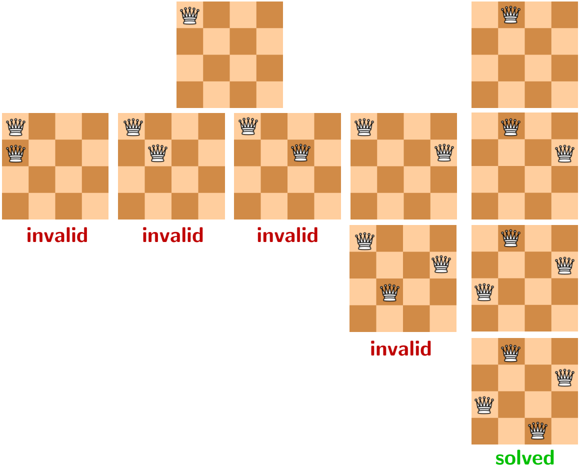

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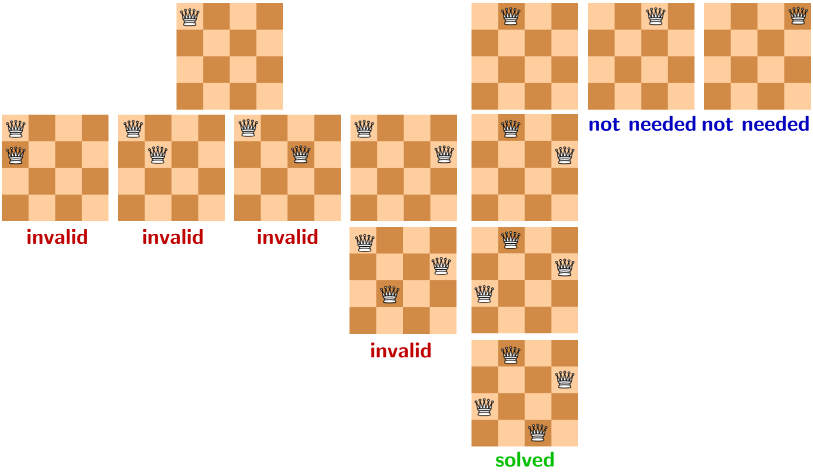

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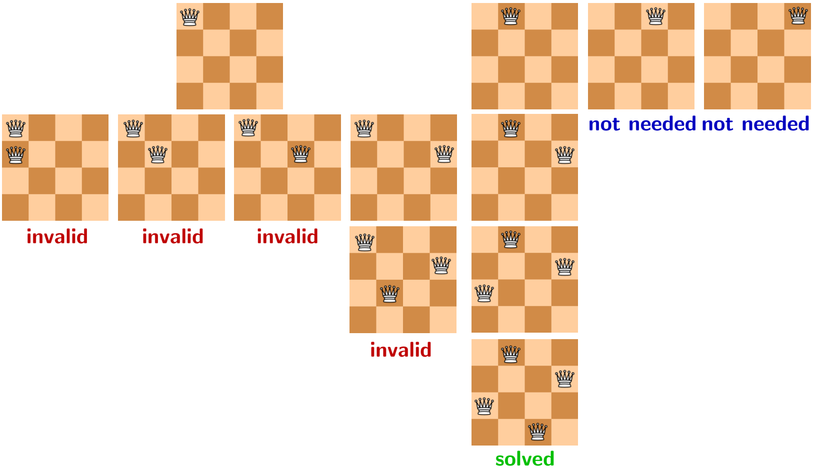

We only explore a small part of the potential search tree.

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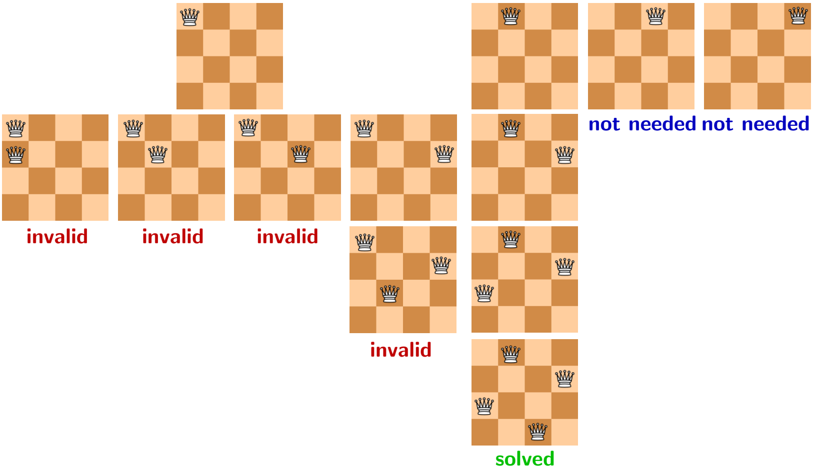

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This is an example of depth-first search of a tree.

Exercises

Write the following recursive functions in Python:

1. The factorial.
2. The Fibonacci function, defined as:

   $$F_n = F_{n-1} + F_{n-2}$$

   with the base case $F_0 = 0, F_1 = 1$.
3. Write all the possible permutations of the nucleobase sequence $GATC$, that is: $GTAC$, $GCAT$, $ATCG$, $ACGT$, ...