

Algorithms

Recursion I

Emanuele Rodolà
rodola@di.uniroma1.it



Recursive algorithms

MERGE-SORT(A, p, r)

1 **if** $p < r$

2 **then** $q \leftarrow \lfloor (p + r)/2 \rfloor$

3 MERGE-SORT(A, p, q)

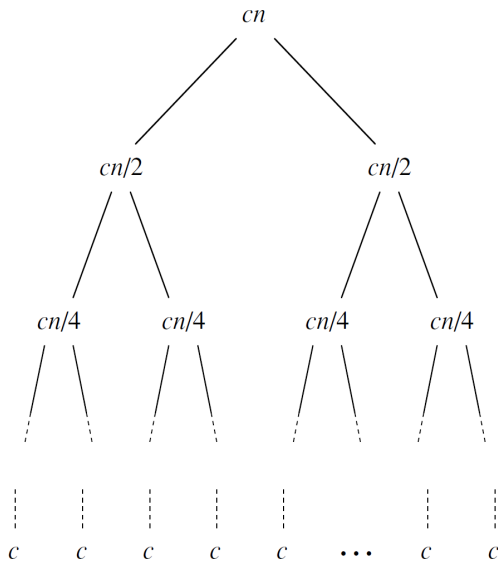
4 MERGE-SORT($A, q + 1, r$)

5 MERGE(A, p, q, r)

Recursive algorithms

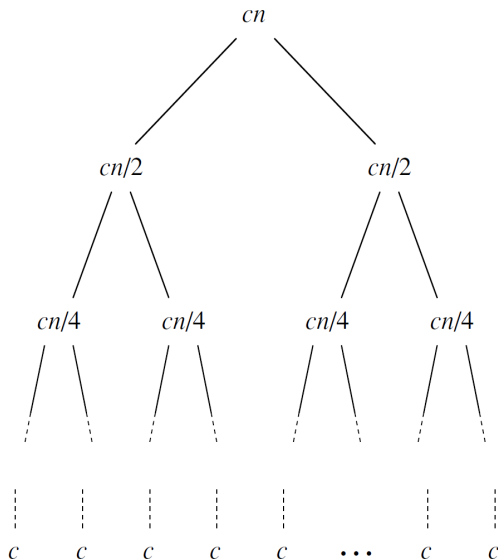
```
9 def binsearch(S, x, i, f):
10     if i < f:
11         q = int((i + f) / 2)
12         if x < S[q]:
13             return binsearch(S, x, i, q)
14         elif x > S[q]:
15             return binsearch(S, x, q + 1, f)
16         elif x == S[q]:
17             return q + 1
18     else:
19         return -1
20
21
22 A = [11, 21, 31, 41, 51, 61, 71, 91]
23 v = 71
24 index = binsearch(A, v, 0, len(A))
25 print(index)
```

Recursion tree



Recursion tree

In general, recursive algorithms trace a path in a recursion tree.



The factorial

$$4! = 4 \times 3 \times 2 \times 1$$

The factorial

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \cdots$$

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$$1! = 1$$

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$$1! = 1 \quad \text{base case}$$

The factorial

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1 factorial(n):  
2     ....if n == 1:  
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Q: What is the complexity $T(n)$ of this algorithm?

Each function call generates a **context**, containing all the data associated with that function call.

The recursion stack

Recursion involves **nested function calls**.

The contexts are memorized one on top of the other, in a **stack**.

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1 factorial(3n):  
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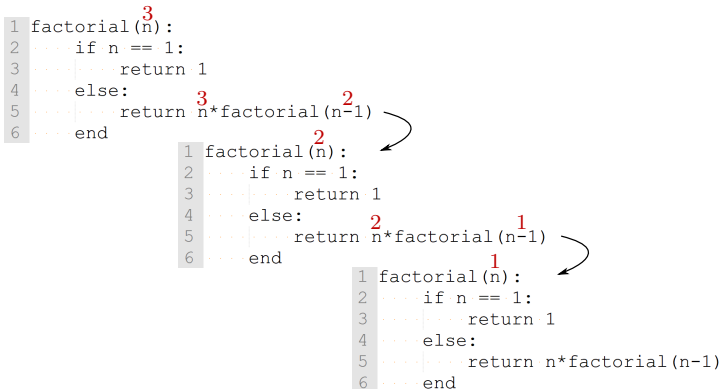


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1 factorial(2n):  
2     ... if n == 1:  
3     ...     return 1  
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```

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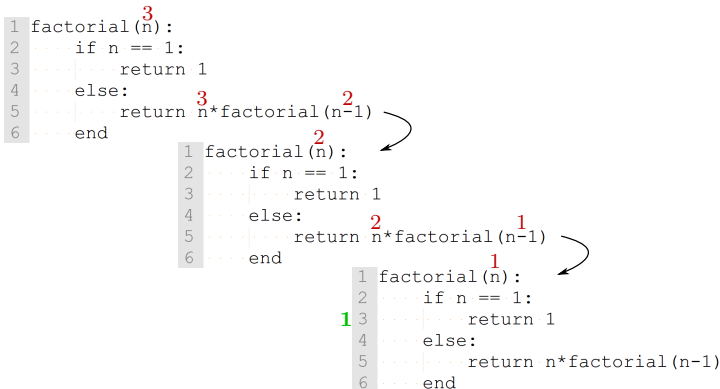
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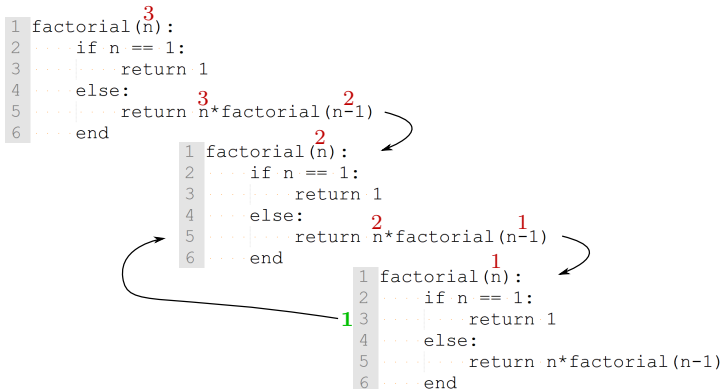
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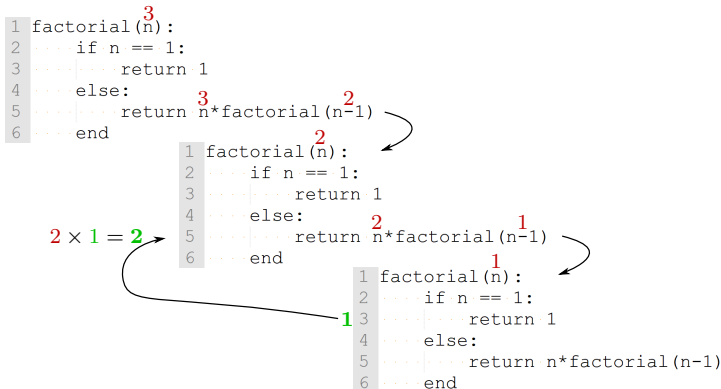
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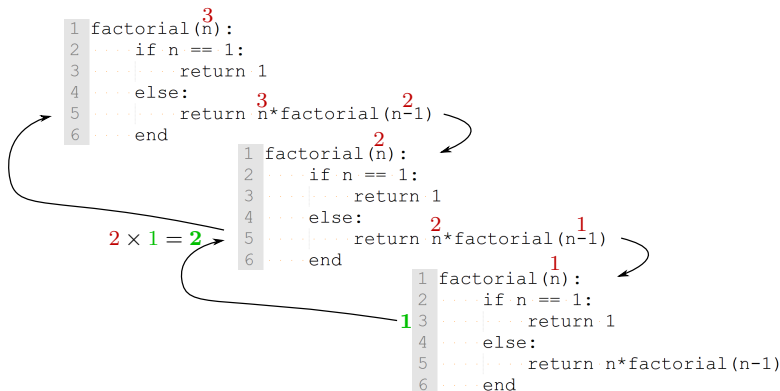
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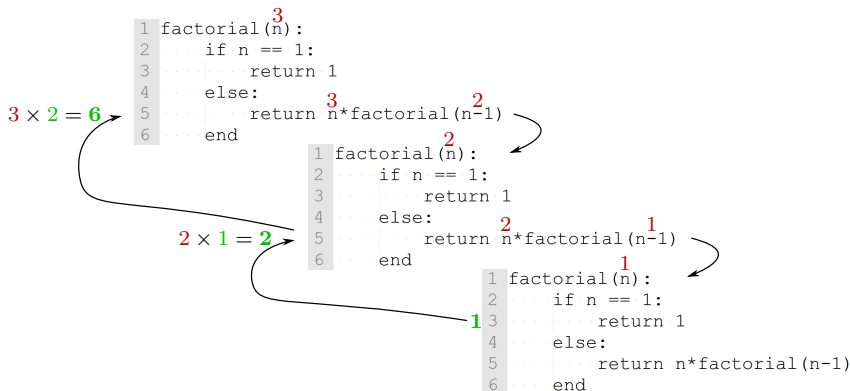
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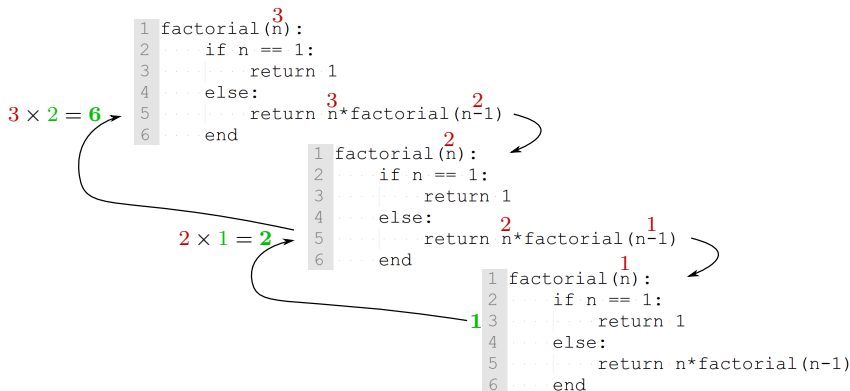
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Tip: Use the **debugger** to trace recursive calls step by step.

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$$3 \times 2 = 6$$

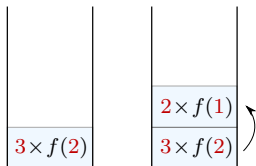
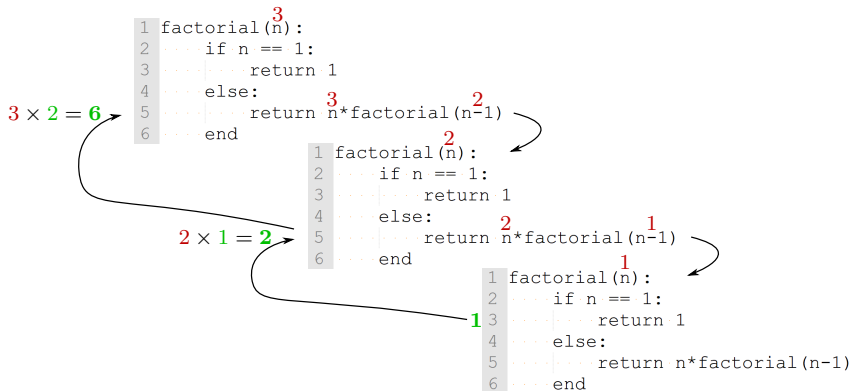
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1 factorial(2n):  
2     ... if n == 1:  
3     ...     return 1  
4     ... else:  
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6     ... end
```

$$2 \times 1 = 2$$

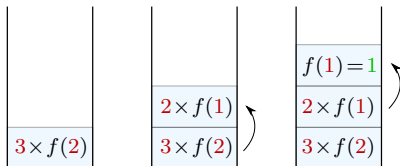
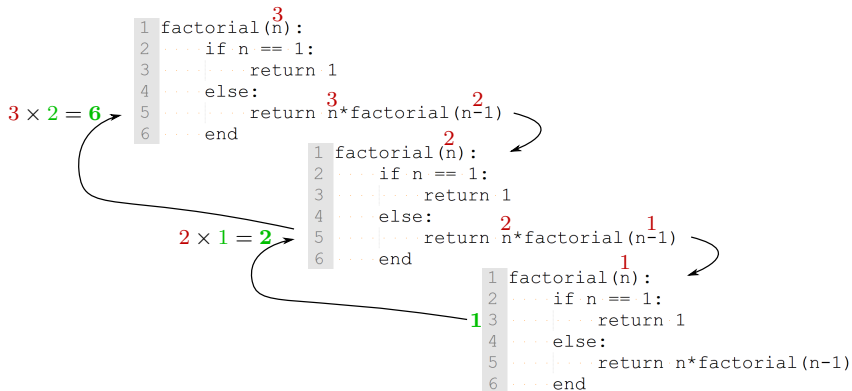
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1 factorial(1n):  
2     ... if n == 1:  
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6     ... end
```

$$3 \times f(2)$$

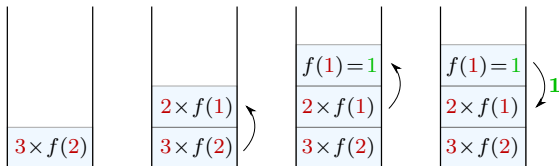
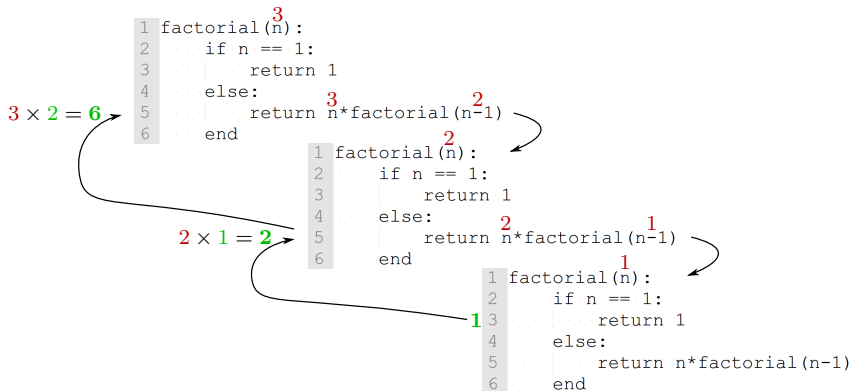
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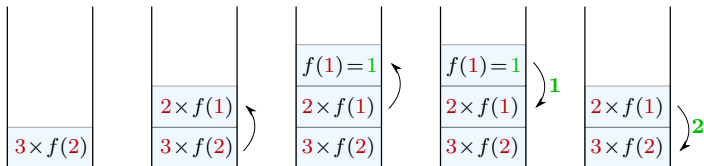
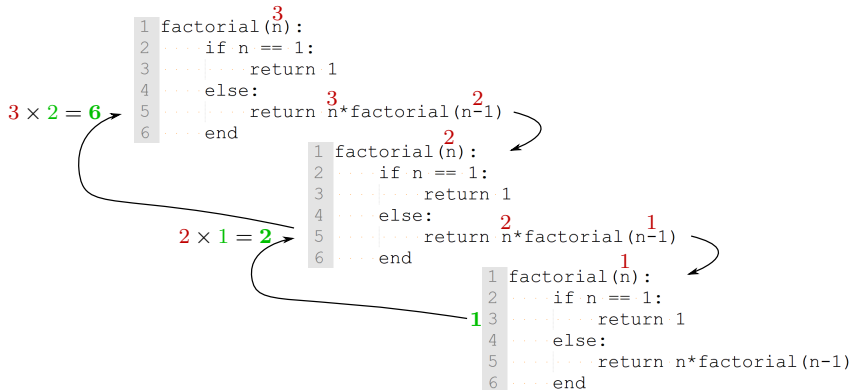
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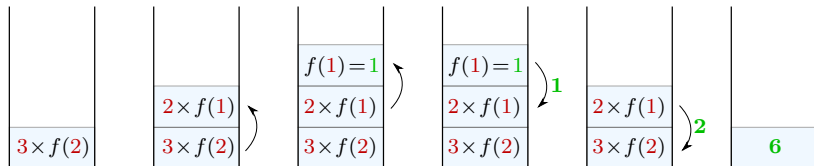
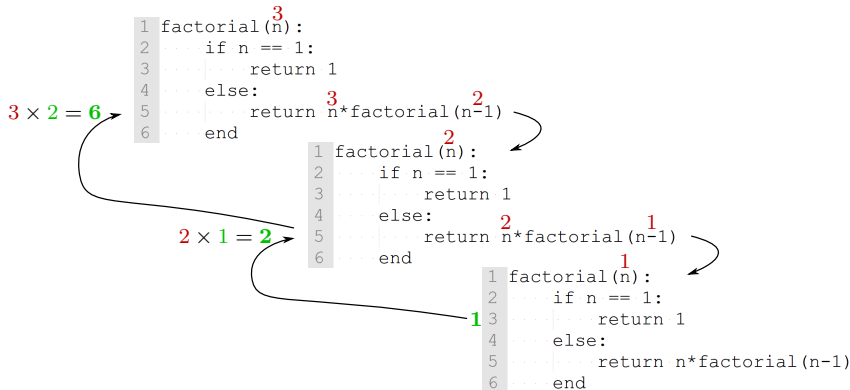
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Backtracking

General algorithm for a wide family of computational problems.

- Incrementally **build** a possible solution.
- Abandon the solution as soon as it does not meet the constraints.
(**backtrack**)

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Each intermediate step is a node of a **search tree**.

Backtracking traverses the tree **recursively**.

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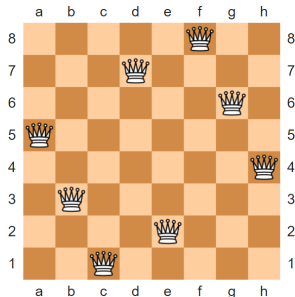
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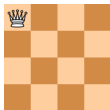
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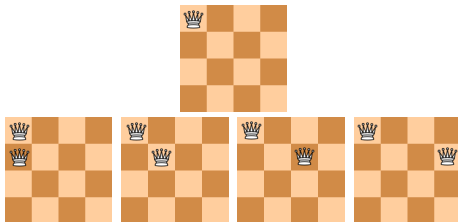
Classical example: **8 queens problem**.



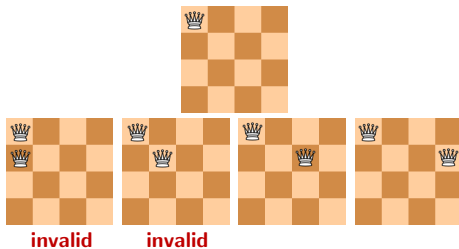
Example: 4 queens



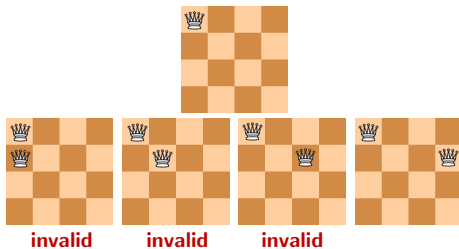
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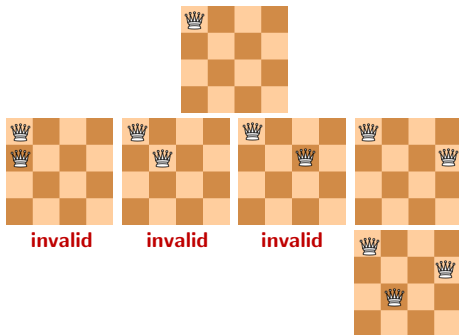
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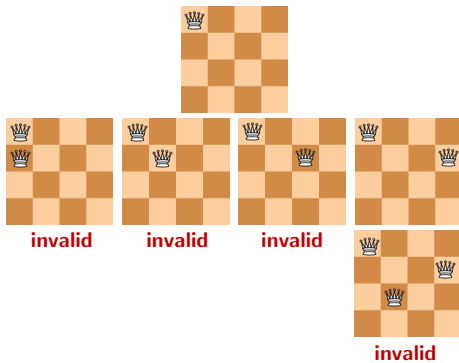
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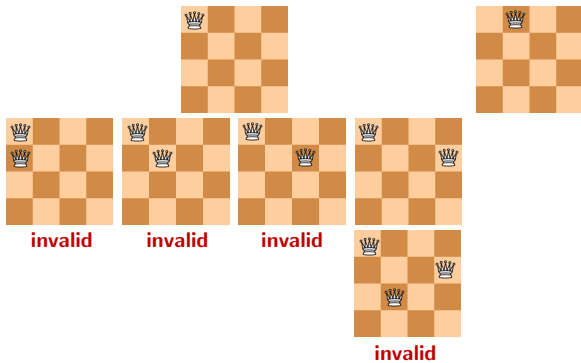
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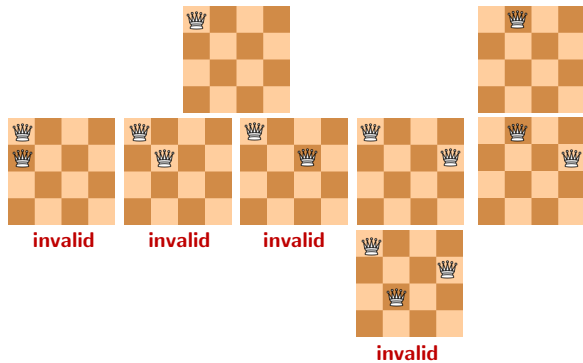
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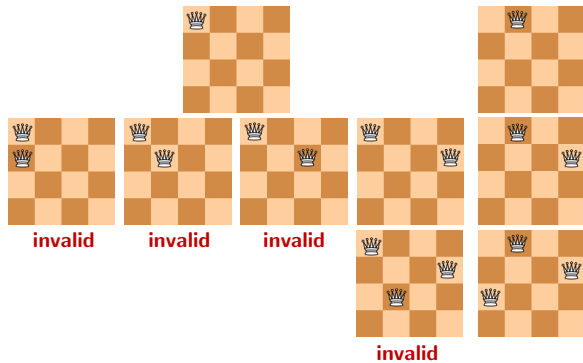
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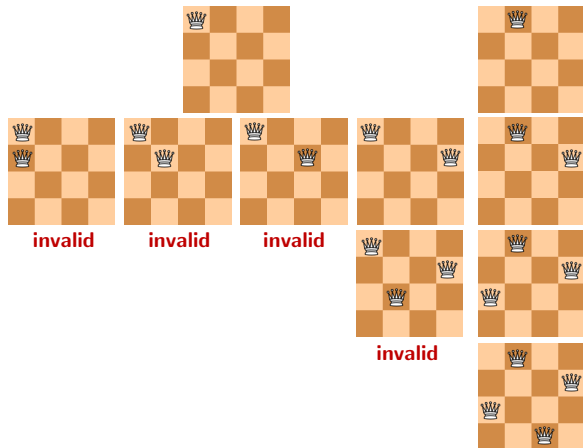
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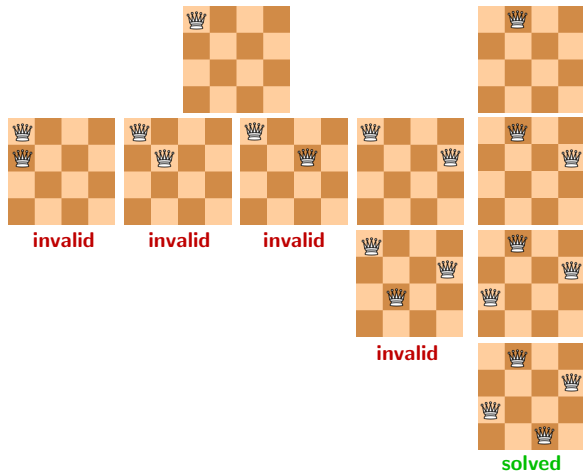
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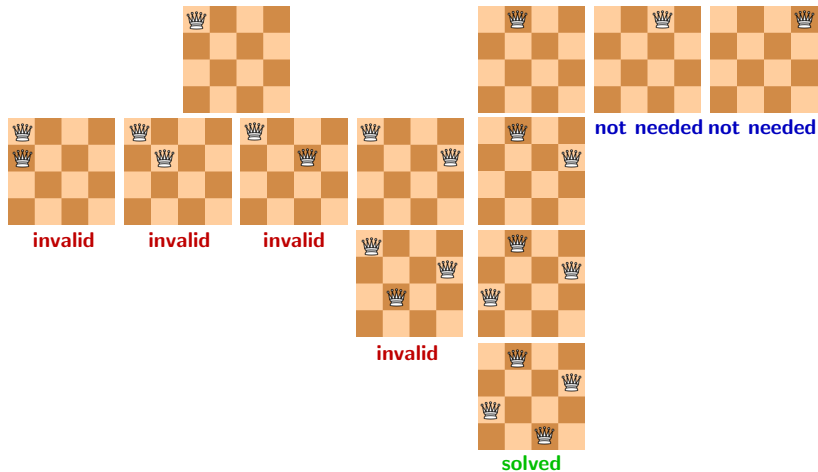
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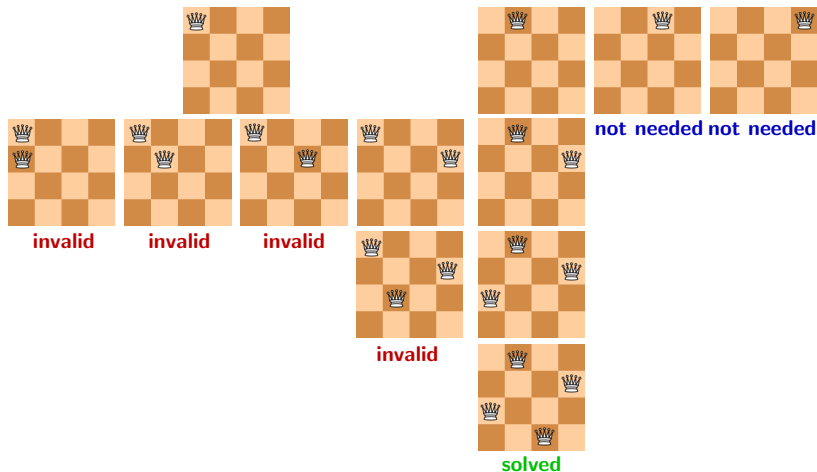
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We only explore a small part of the potential search tree.

This is an example of **depth-first search** of a tree.

Exercises

Write the following **recursive** functions in Python:

- 1 The factorial.
- 2 The Fibonacci function, defined as:

$$F_n = F_{n-1} + F_{n-2}$$

with the base case $F_0 = 0, F_1 = 1$.

- 3 Write all the possible permutations of the nucleobase sequence *GATC*, that is: *GTAC*, *GCAT*, *ATCG*, *ACGT*, ...