Algorithms

Insertion sort

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Exercises

In pseudocode or in your favorite language, write an algorithm to solve each of the following problems.

1. Reverse any given sequence of length n:

$$(3, 7, 9, 14) \to (14, 9, 7, 3)$$

2. Given a number x and a sequence $(a_i)_{i=1}^{n}$, find the closest number to x in the sequence.

$$12.1\,,\;\; (3, 31, 7, 11, 52) \to 11$$

3. Given a sequence $(a_i)_{i=1}^{n}$ and a smaller sequence $(b_i)_{i=1}^{m}, m < n$, find the latter inside the former, and return the index of the first occurrence as output.

$$(C, G, A, T, T, G, C, \dots)\,,\;\; (T, T, G \dots) \to 4$$

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	1	2	3	4	5	6
(a)	5	**2**	4	6	1	3

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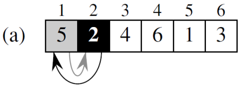

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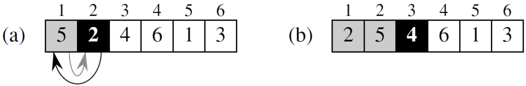

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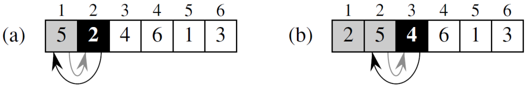

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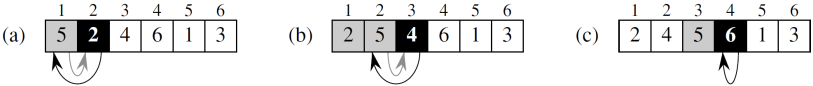

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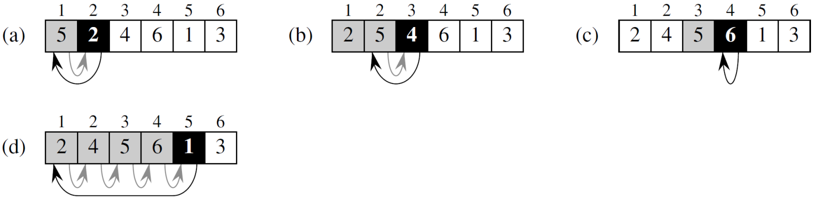

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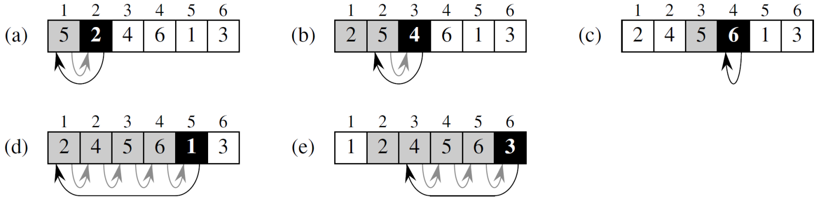

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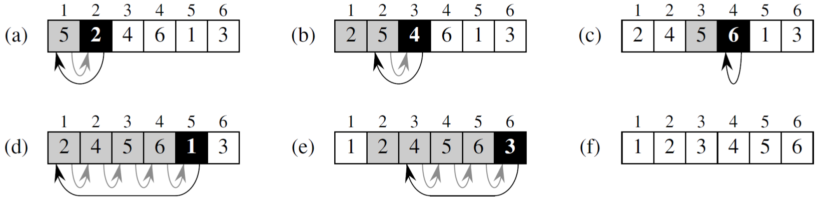

Pseudocode

INSERTION-SORT(A)

for $j \leftarrow 2$ **to** $length[A]$
    do $key \leftarrow A[j]$
        $\triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.

Remark: The sequence is sorted in-place.

Pseudocode

```
INSERTION-SORT(A)
for j ← 2 to length[A]
    do key ← A[j]
       ▷ Insert A[j] into the sorted sequence A[1 .. j − 1].
       i ← j − 1
       while i > 0 and A[i] > key
           do A[i + 1] ← A[i]
              i ← i − 1
       A[i + 1] ← key
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Analysis

We want to predict the resources that the algorithm requires.

- Memory
- Bandwidth
- Time
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Most of these factors ultimately depend on the size of the input.

As we have seen, we will measure the running time (i.e., the number of steps) as a function of size.

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INSERTION-SORT(A)	*cost*	*times*
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$i \leftarrow i-1$	c_7	

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while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
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Total running time $T(n)$:

$$c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n}(t_j-1) + c_7 \sum_{j=2}^{n}(t_j-1) + c_8(n-1)$$

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where a and b are constants.

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When the input sequence is already sorted.

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$$\sum_{i=1}^{n} i = (1+n) + \underbrace{(2+(n-1))}_{1+n} + \underbrace{(3+(n-2))}_{1+n} + \cdots = (1+n)\frac{n}{2}$$

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What if the input sequence is sorted in decreasing order?

In this case, $t_j = j$ since we will compare each number with the entire sorted subsequence.

$$T(n) = an^2 + bn + c$$

where a, b, c are constants.

The worst-case cost of insertion sort is quadratic in n.

Average-case analysis

For a given random sequence of numbers, we observe that, on average:

- $A[j] >$ half the elements in $A[1 \ldots j-1]$, and
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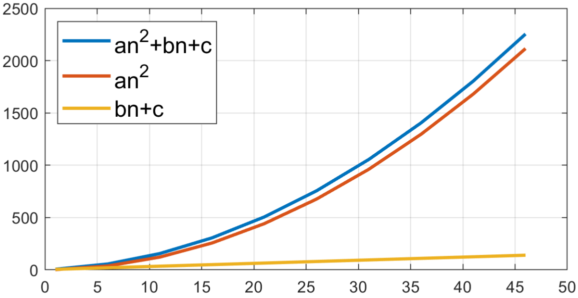
In our analysis, we will often concentrate on studying the worst case, since it gives us a guaranteed upper bound on the total cost.

Order of growth



an^2 dominates the lower-order terms.

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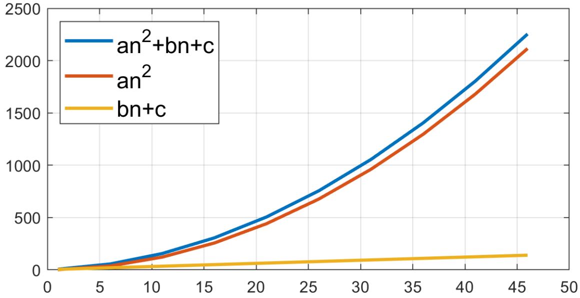

an^2 dominates the lower-order terms.

We say that insertion sort has worst-case running time of $\Theta(n^2)$.

Exercises

Solve the following exercises:

1. Write an algorithm in pseudocode to perform linear search.

   Given a sequence of numbers $A = (a_1, \ldots, a_n)$ and a number v, find an index i such that $v = A[i]$. Return a special number if v can not be found in the sequence.

   The search must be done by simple linear scanning through the sequence.

2. How many elements must be checked on average, and in the best and worst cases?

3. What are the average-case, best-case, and worst-case running times of linear search in Θ-notation?

Suggested reading

"Introduction to Algorithms – 2nd Ed.", Cormen et al.

- Chapter 2.1, skipping the "Loop invariants" and "Pseudocode conventions" paragraphs
- Chapter 2.2