Algorithms

Shortest paths

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SAPIENZA
UNIVERSITÀ DI ROMA

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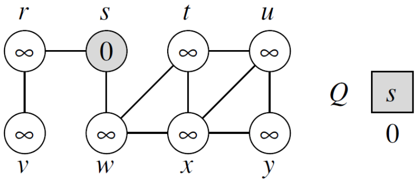
Breadth-first search (BFS)



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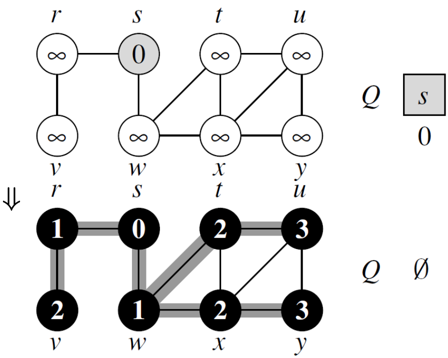

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Each vertex has its distance to s (# edges) computed incrementally.

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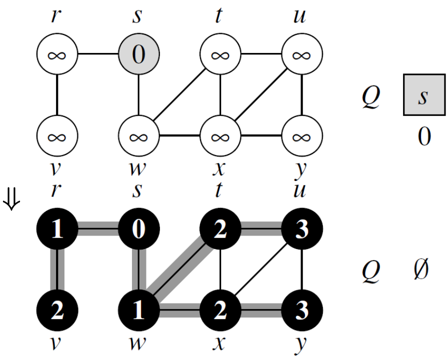

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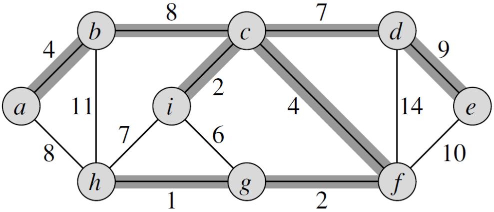
Each vertex has its distance to s (# edges) computed incrementally.

A breadth-first tree is obtained as a side-product.

Minimum spanning tree (MST)



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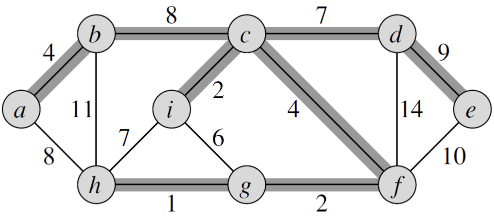

For a weighted graph $G = (V, E)$, find a subset of edges $T \subseteq E$ s.t.:

- The set is acyclic.
- The set covers all the vertices.
- The sum of edge weights is minimized.

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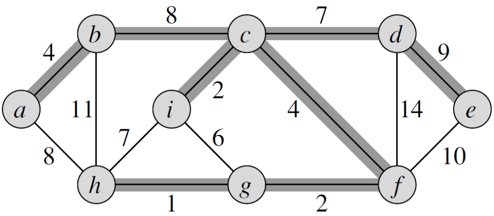

For a weighted graph $G = (V, E)$, find a subset of edges $T \subseteq E$ s.t.:

- The set is acyclic.
- The set covers all the vertices.
- The sum of edge weights is minimized.

The MST is not unique: you can replace (b, c) with (a, h).

Overall idea

```
GENERIC-MST(G, w)
A ← ∅
while A does not form a spanning tree
      do find an edge (u, v) that is safe for A
         A ← A ∪ {(u, v)}
return A
```

A safe edge (u, v) ensures that $A \cup \{(u, v)\}$ is a subset of some MST.

At the end, the set A must then be a MST.

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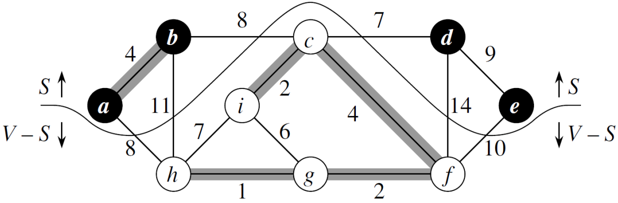
How to find safe edges?

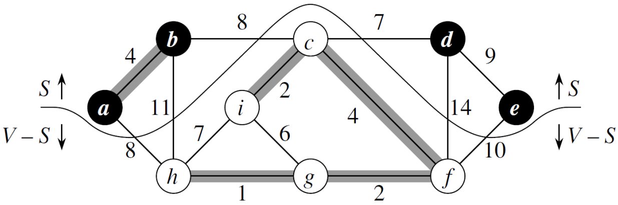
Definitions and theorem



A cut $(S, V - S)$ partitions the vertices V.

The edge (a, h) crosses the cut.

Definitions and theorem



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The cut respects A since none of the edges in A crosses the cut.

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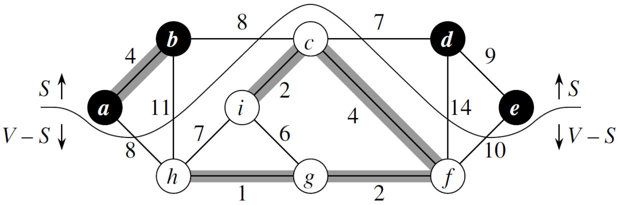

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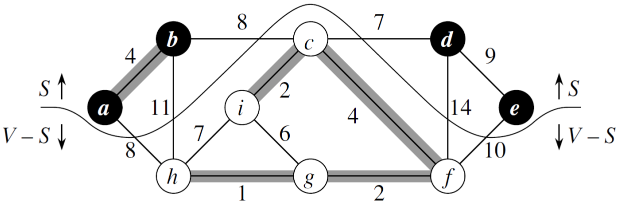

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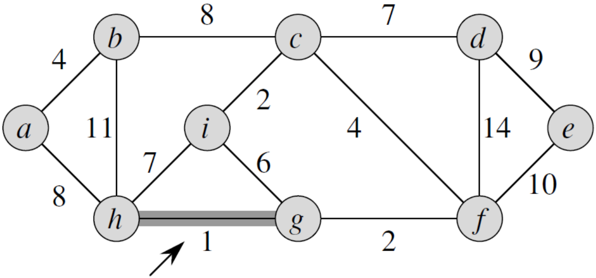
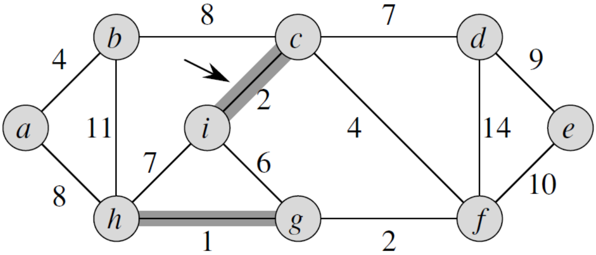
> If A is included in a MST, then each light edge is safe for A.

Kruskal's algorithm



Start exploring each edge by increasing weight.

Kruskal's algorithm



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Each edge generates its own tree

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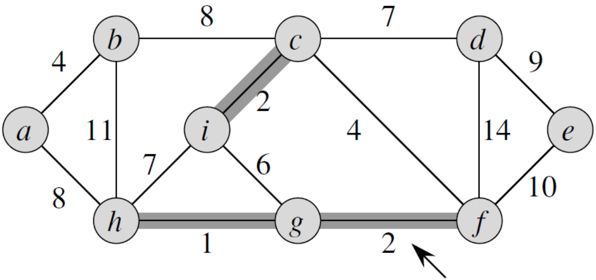

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Each edge generates its own tree, or enters an existing tree.

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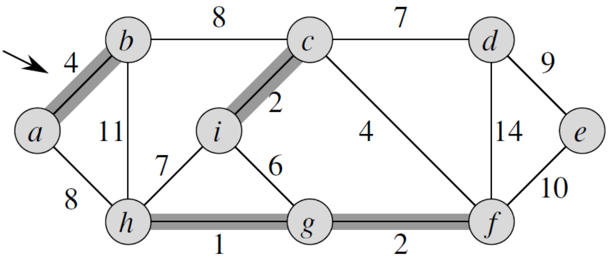

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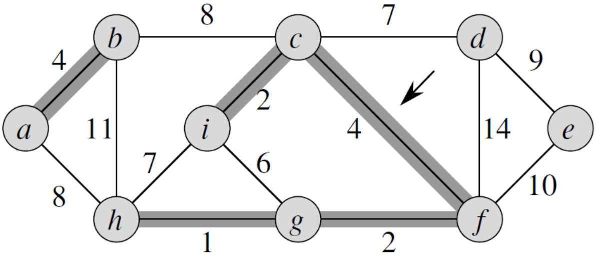

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A new edge may merge two existing trees.

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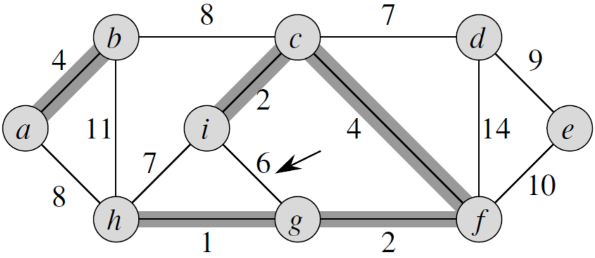

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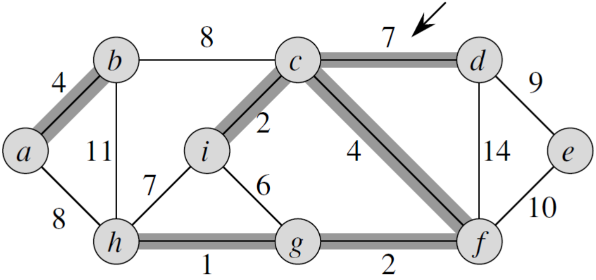

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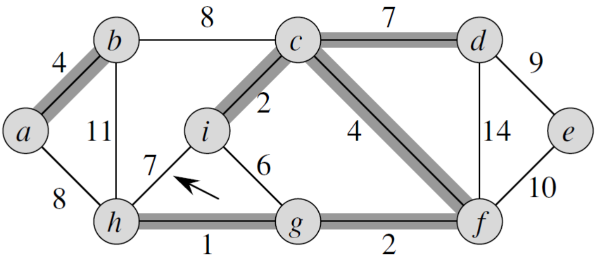

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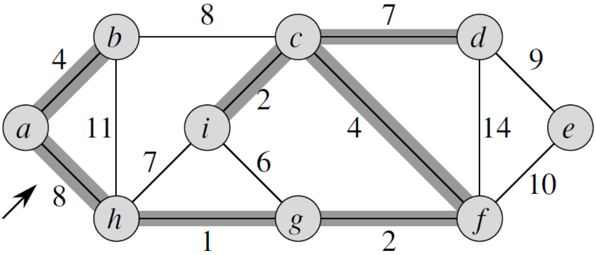

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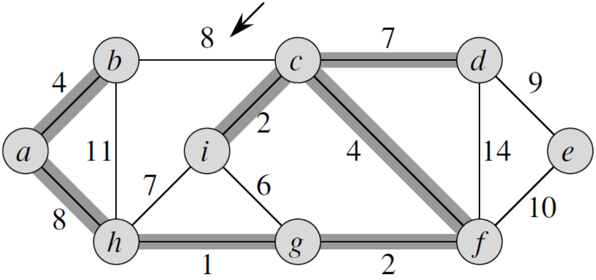

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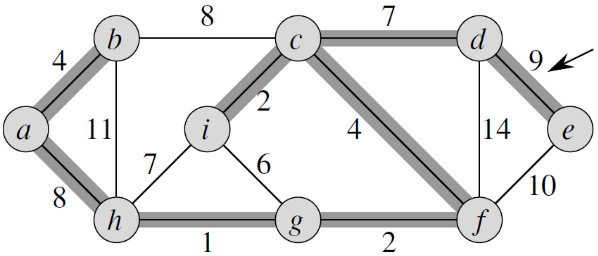

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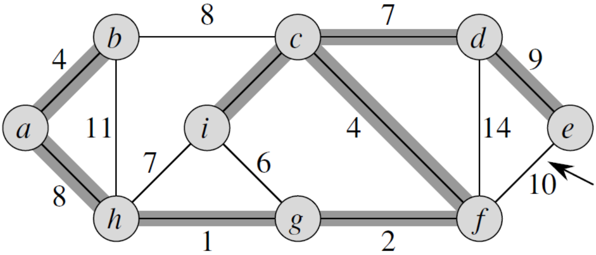

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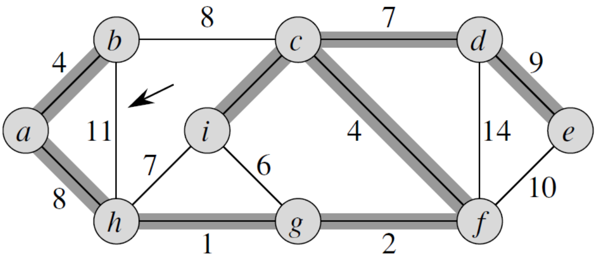

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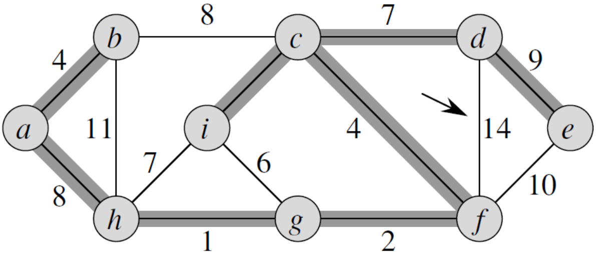

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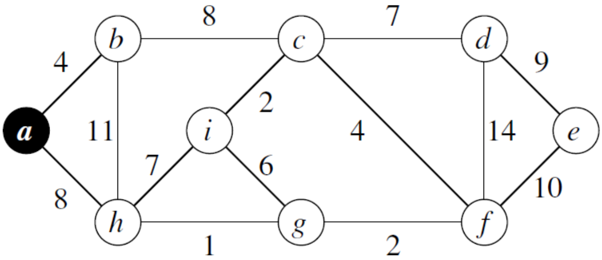
Kruskal's algorithm

```
MST-KRUSKAL(G, w)
1  A ← ∅
2  for each vertex v ∈ V[G]
3        do MAKE-SET(v)
4  sort the edges of E into nondecreasing order by weight w
5  for each edge (u, v) ∈ E, taken in nondecreasing order by weight
6        do if FIND-SET(u) ≠ FIND-SET(v)
7              then A ← A ∪ {(u, v)}
8                    UNION(u, v)
9  return A
```

FIND-SET(u) returns the tree to which u belongs.

The check at line 6 avoids the creation of cycles.

Prim's algorithm



Start from an arbitrary root.

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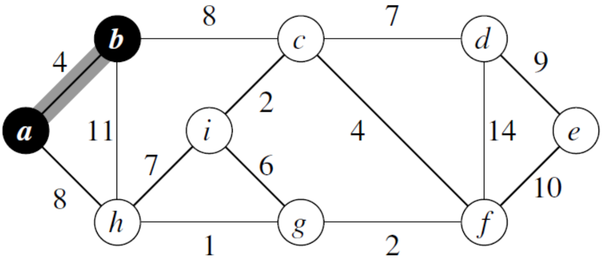

Start from an arbitrary root.

Grow the tree by adding a safe edge at each step greedily.

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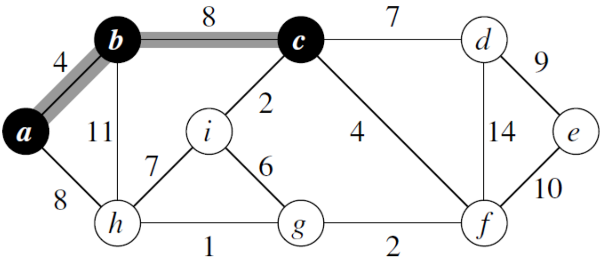

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We can choose between edge (b, c) or edge (a, h), since they both are at the frontier of the current tree and have equal weight.

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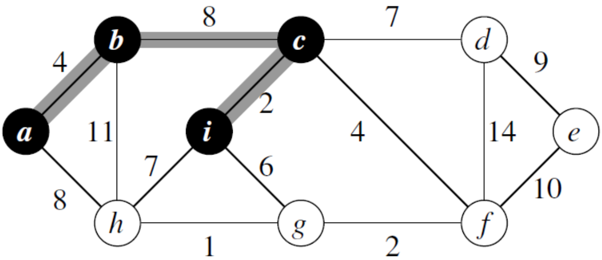

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At each step the selected vertices determine a cut, and a light edge crossing the cut is added to the tree.

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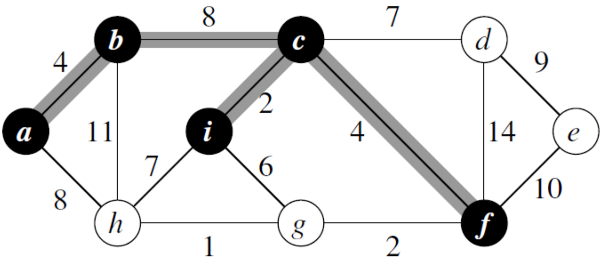

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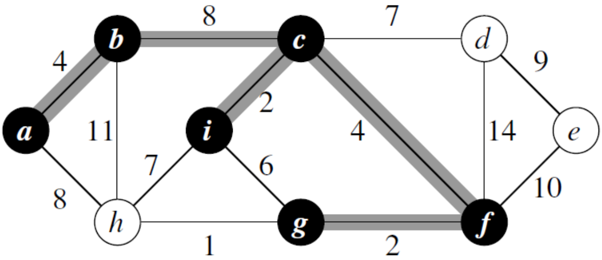

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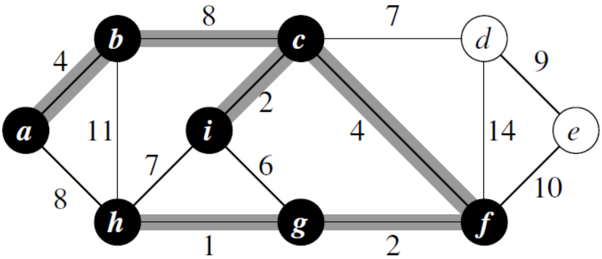

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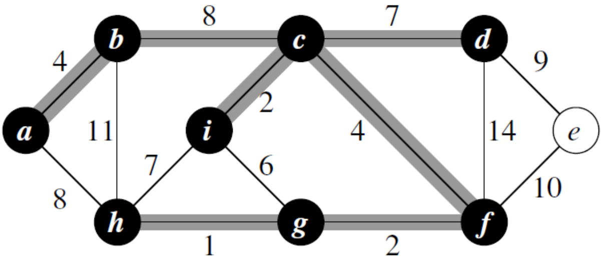

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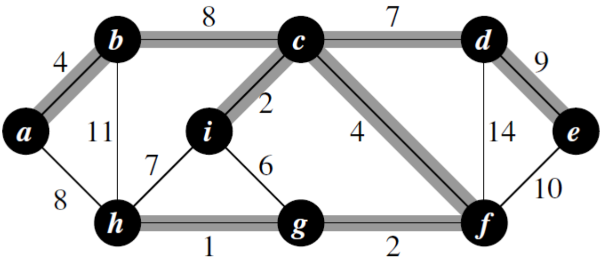

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MST-PRIM(G, w, r)

for each $u \in V[G]$
  do $key[u] \leftarrow \infty$
    $\pi[u] \leftarrow$ NIL

Each vertex will have a parent (in the tree) and a key.

Prim's algorithm

MST-PRIM(G, w, r)

for each $u \in V[G]$
    do $key[u] \leftarrow \infty$
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$key[r] \leftarrow 0$ — r is the chosen root
$Q \leftarrow V[G]$ — init. a priority queue with all vertices

Each vertex will have a parent (in the tree) and a key.

The priority queue is based on the value of the key.

Prim's algorithm

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for each u ∈ V[G]
    do key[u] ← ∞
       π[u] ← NIL
key[r] ← 0
Q ← V[G]
while Q ≠ ∅
    do u ← EXTRACT-MIN(Q)
       for each v ∈ Adj[u]
           do if v ∈ Q and w(u, v) < key[v]
                 then π[v] ← u
                      key[v] ← w(u, v)
```

Each vertex will have a parent (in the tree) and a key.

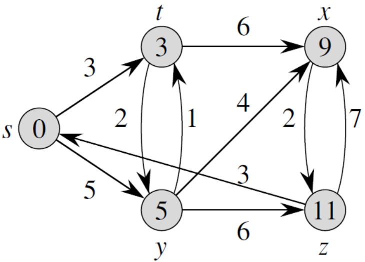
The priority queue is based on the value of the key.

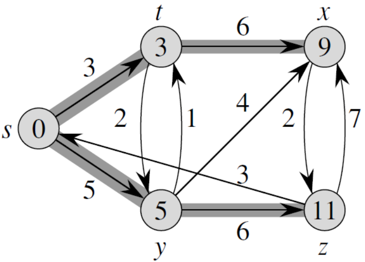
The key of a vertex v will hold the minimum weight connecting v to a vertex that is already in the tree.

Shortest paths

We now want to compute the shortest path (i.e. the sequence of vertices) from a single source to all the other vertices.

To keep track of the path, we store a predecessor for each vertex (similarly to what we did with the BFS tree).





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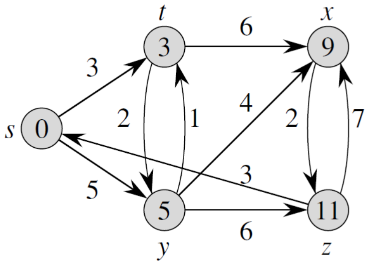

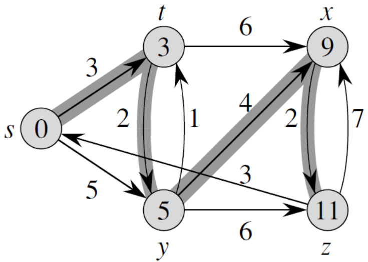

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Relaxation

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An edge (u, v) with weight w is relaxed as:

```
RELAX(u, v, w)
if d[v] > d[u] + w(u, v)          "can we improve the path to v?"
   then d[v] ← d[u] + w(u, v)
        π[v] ← u
```

It tests if we can improve the shortest path to v by going through u and, if so, decreases $d[v]$ and updates $\pi[v]$.

Dijkstra's algorithm

```
DIJKSTRA(G, w, s)
    INITIALIZE-SINGLE-SOURCE(G, s)
    S ← ∅
    Q ← V[G]
    while Q ≠ ∅
shaded vertex →  do u ← EXTRACT-MIN(Q)
          S ← S ∪ {u}
          for each vertex v ∈ Adj[u]
              do RELAX(u, v, w)
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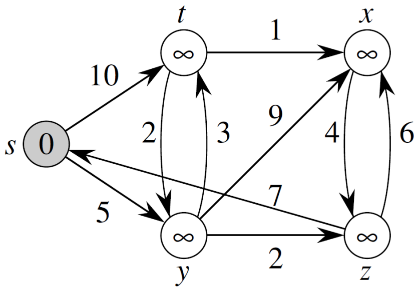

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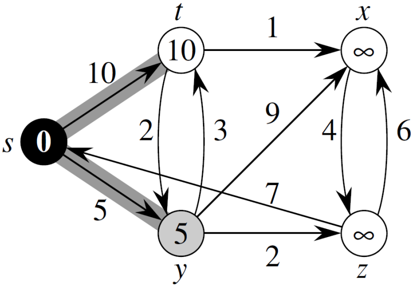

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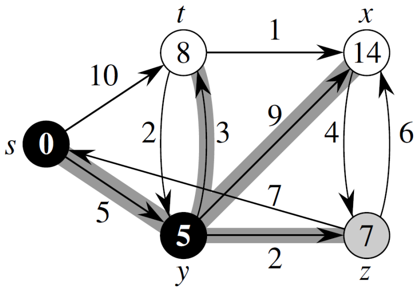
shaded vertex → 5



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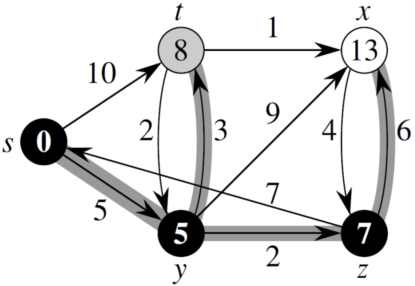
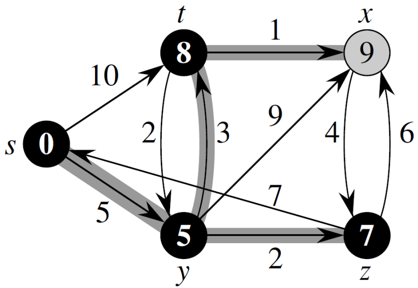
shaded vertex → 5      do u ← EXTRACT-MIN(Q)
   6         S ← S ∪ {u}
   7         for each vertex v ∈ Adj[u]
   8             do RELAX(u, v, w)
```



Dijkstra's algorithm

```
DIJKSTRA(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S ← ∅
3  Q ← V[G]
4  while Q ≠ ∅
5      do u ← EXTRACT-MIN(Q)
6         S ← S ∪ {u}
7         for each vertex v ∈ Adj[u]
8             do RELAX(u, v, w)
```

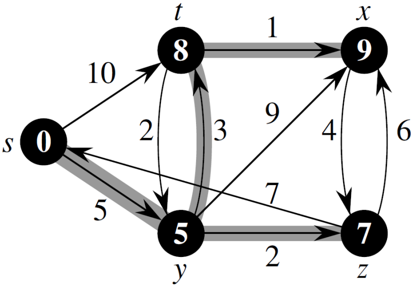
shaded vertex → 5



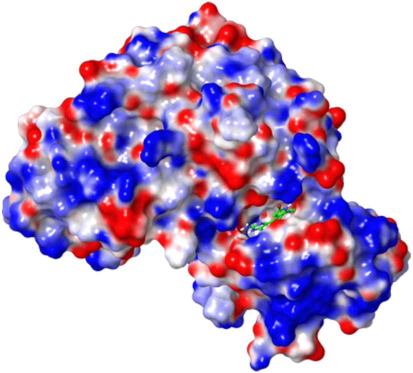
Dijkstra's algorithm

DIJKSTRA(G, w, s)

```
              1  INITIALIZE-SINGLE-SOURCE(G, s)
              2  S ← ∅
              3  Q ← V[G]
              4  while Q ≠ ∅
shaded vertex → 5      do u ← EXTRACT-MIN(Q)
              6         S ← S ∪ {u}
              7         for each vertex v ∈ Adj[u]
              8             do RELAX(u, v, w)
```



Demo



Exercises

Implement Dijkstra's algorithm.

Test it on a random, undirected, weighted graphs generated by yourself.

Suggested reading

Chapters 23.1, 23.2, 24 ("Representing shortest paths" and "Relaxation" paragraphs) and 24.3 of:

"Introduction to Algorithms – 2nd Ed.", Cormen et al.