Algorithms

Divide et impera

Emanuele Rodolà
rodola@di.uniroma1.it

SAPIENZA
UNIVERSITÀ DI ROMA

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Breaking the problem

Overall idea:

1. Break the main problem into many subproblems

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   - Each subproblem is smaller in size
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This general approach is called divide et impera or also divide and conquer.

Merge sort

Again, we consider the sorting problem as a toy example.

Input: A sequence of n numbers $(a_1, a_2, \dots, a_n)$

Output: A reordered sequence $(a'_1, a'_2, \dots, a'_n)$ such that:

$$a'_1 \le a'_2 \le \cdots \le a'_n$$

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At some point, the generated subsequences will have length **1**
$\Rightarrow$ no more splitting needed!

We call this the base case.

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Divide-et-impera

We translate the original problem into a different and easier subproblem of smaller size.

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Sorting problem $\Rightarrow$ Merging problem

Complete algorithm

```
MERGE-SORT(A, p, r)
if p < r
   then q ← ⌊(p + r)/2⌋
        MERGE-SORT(A, p, q)
        MERGE-SORT(A, q + 1, r)
        MERGE(A, p, q, r)
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Complete algorithm

MERGE-SORT(A, p, r)

if $p < r$
  then $q \leftarrow \lfloor (p + r)/2 \rfloor$
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- The procedure calls itself recursively.

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- The procedure calls itself recursively.
- Start: call merge sort on $(A, 1, length[A])$.

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```

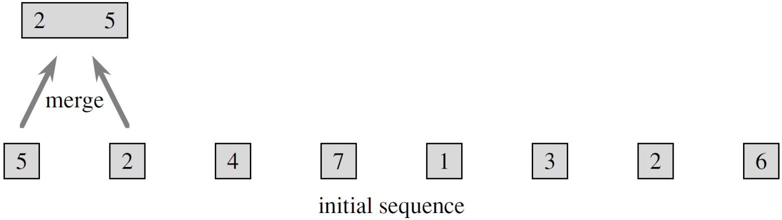
- The procedure calls itself recursively.
- Start: call merge sort on $(A, 1, length[A])$.
- Keep splitting until $p < r$ is false...
- ...which is the base case: the two halves have length 1.
- If $length[A]$ is a power of 2, we always split in equal halves.
  Q: Why doesn't this happen for even $length[A]$?

Example

5	2	4	7	1	3	2	6

initial sequence

Example



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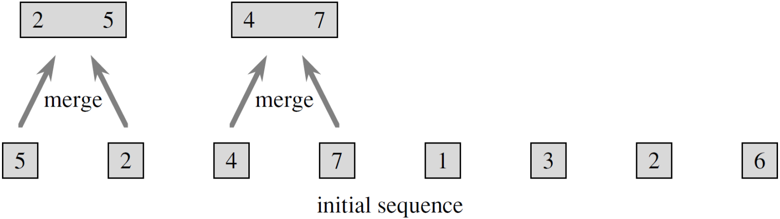

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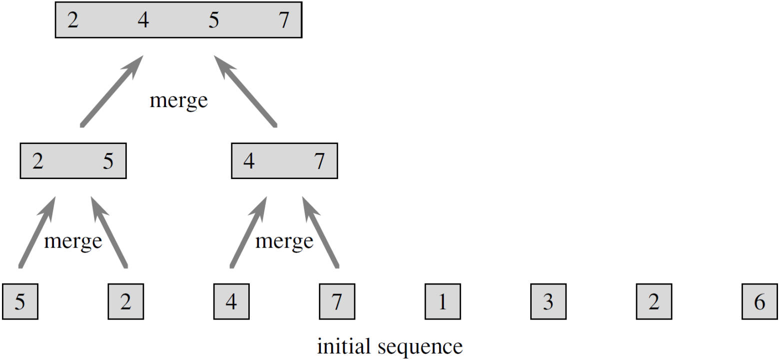

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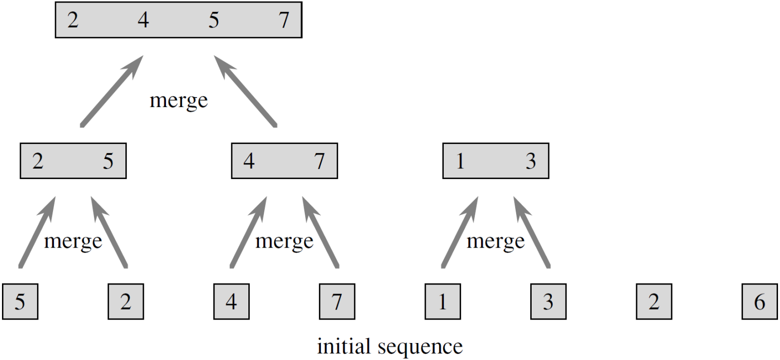
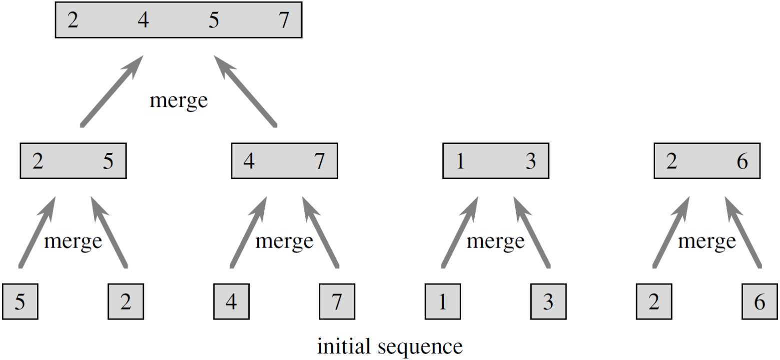

2 4 5 7
merge
2 5
4 7
1 3
merge
merge
merge
5
2
4
7
1
3
2
6
initial sequence

Example


2 4 5 7
merge
2 5
4 7
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merge
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merge
merge
5
2
4
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1
3
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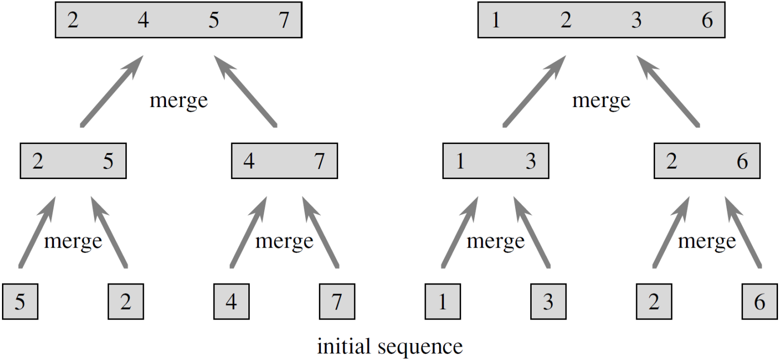
Example



2 4 5 7
1 2 3 6
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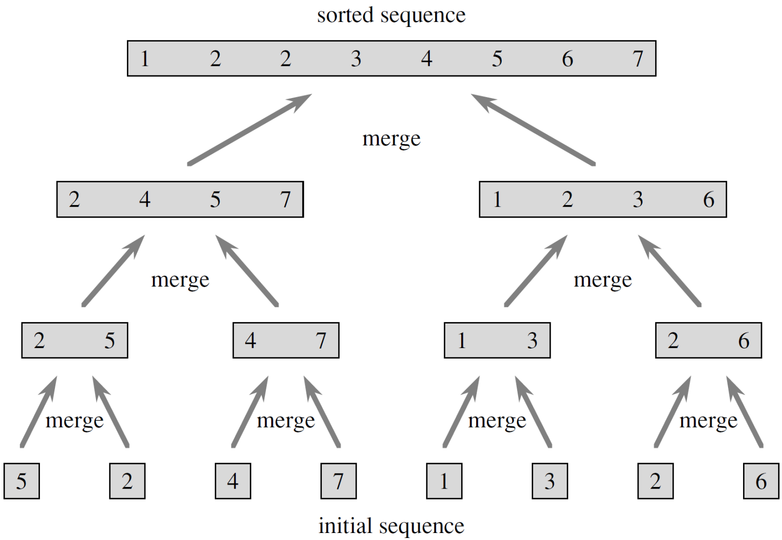

sorted sequence
1 2 2 3 4 5 6 7
merge
2 4 5 7
1 2 3 6
merge
merge
2 5
4 7
1 3
2 6
merge
merge
merge
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Merge operation

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Main idea:

1. Compare only the first element of the two sequences
2. Take out the smaller one, and put it in the output sequence
3. Go back to step (1) until one of the two sequences is empty

MERGE(A, p, q, r)

```
n_1 ← q − p + 1
n_2 ← r − q
```

$n_1 \leftarrow q - p + 1$

$n_2 \leftarrow r - q$

MERGE(A, p, q, r)

$n_1 \leftarrow q - p + 1$
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create arrays $L[1 \,..\, n_1 + 1]$ and $R[1 \,..\, n_2 + 1]$

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for $i \leftarrow 1$ **to** n_1
    do $L[i] \leftarrow A[p + i - 1]$
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L[n1 + 1] ← ∞
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i ← 1
j ← 1
for k ← p to r
     do if L[i] ≤ R[j]
           then A[k] ← L[i]
                i ← i + 1
           else A[k] ← R[j]
                j ← j + 1
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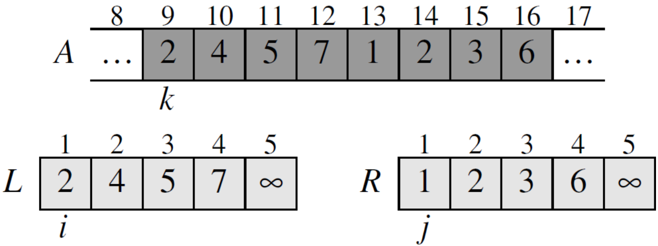
Merge example

	8	9	10	11	12	13	14	15	16	17
A	…	2	4	5	7	1	2	3	6	…
		k								

Merge example



The light gray numbers make up the original sequence

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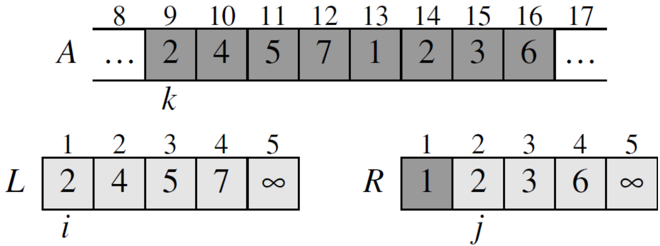

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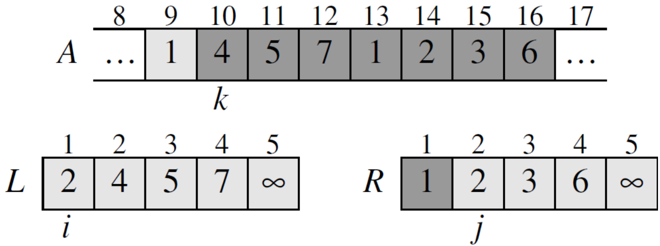

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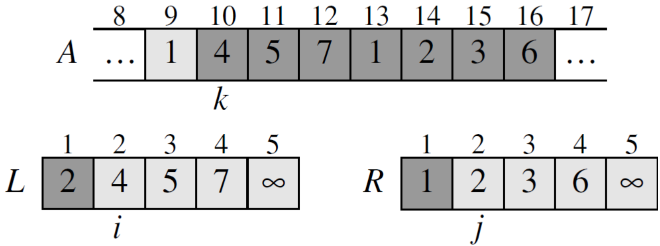

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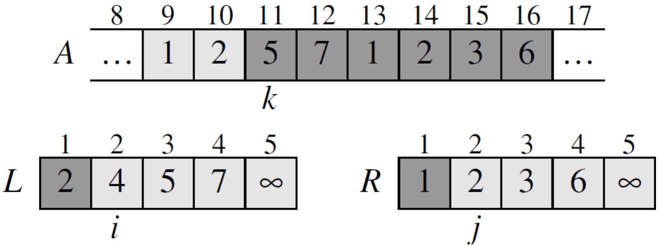

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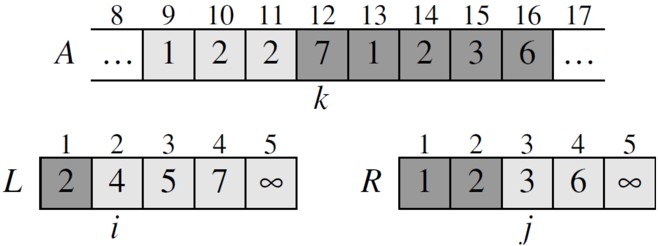

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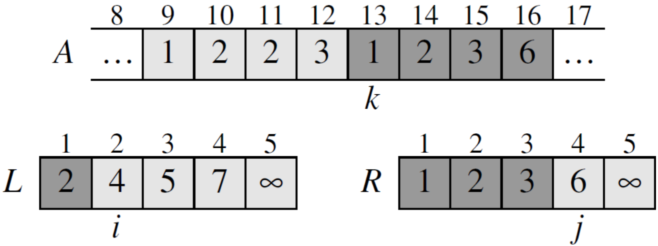

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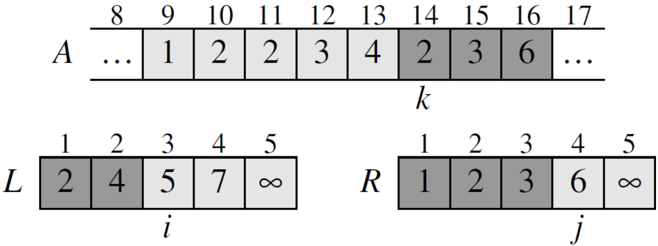

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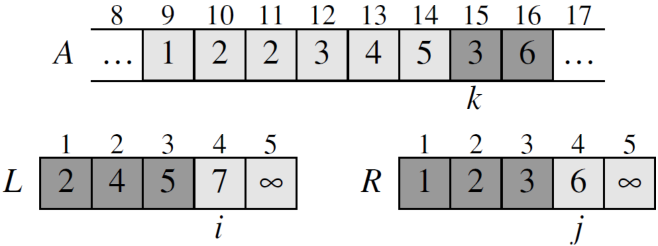

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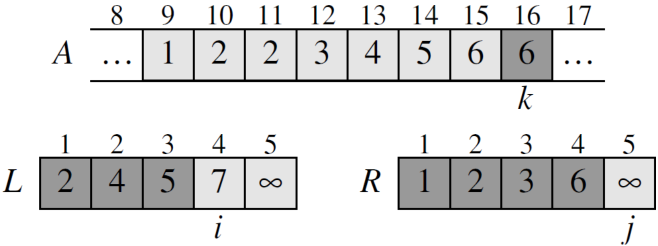

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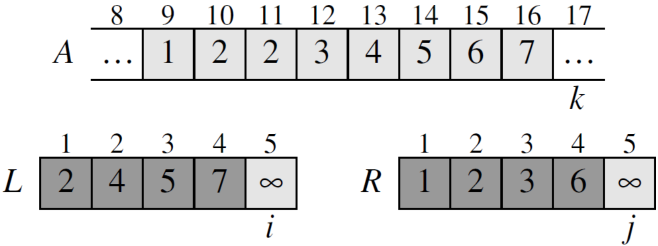

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Order of growth (merge operation only)

```
          MERGE(A, p, q, r)
            n1 ← q − p + 1
constant    n2 ← r − q
            create arrays L[1 .. n1 + 1] and R[1 .. n2 + 1]
            for i ← 1 to n1
                do L[i] ← A[p + i − 1]
            for j ← 1 to n2
                do R[j] ← A[q + j]
            L[n1 + 1] ← ∞
constant    R[n2 + 1] ← ∞
            i ← 1
            j ← 1
            for k ← p to r
                do if L[i] ≤ R[j]
                      then A[k] ← L[i]
                           i ← i + 1
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  -- constant
  n1 ← q − p + 1
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  create arrays L[1 .. n1 + 1] and R[1 .. n2 + 1]
  -- linear in n1
  for i ← 1 to n1
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  for j ← 1 to n2
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  -- constant
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MERGE(A, p, q, r)
Θ(1)   n1 ← q − p + 1
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       create arrays L[1 .. n1 + 1] and R[1 .. n2 + 1]
Θ(n)   for i ← 1 to n1
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Order of growth (merge operation only)

$\Theta(n)$

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$$T(n) =$$

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$$T(n) = \begin{cases} \Theta(1) & n \leq a \\ \underbrace{D(n)}_{\text{split}} + \qquad\qquad\qquad + \underbrace{C(n)}_{\text{merge}} & n > a \end{cases}$$

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$$T(n) = \begin{cases} \Theta(1) & n \leq a \\ \underbrace{D(n)}_{\text{split}} + \underbrace{2T(\frac{n}{2})}_{\substack{\text{cost on the} \\ \text{two halves}}} + \underbrace{C(n)}_{\text{merge}} & n > a \end{cases}$$

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Divide: $\Theta(1)$

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Divide: $\Theta(1)$

Conquer: $2T(\frac{n}{2})$

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Divide: $\Theta(1)$

Conquer: $2T(\frac{n}{2})$

Combine: $\Theta(n)$

Order of growth (complete merge sort algorithm)

$$T(n) = \begin{cases} \Theta(1) & n \le a \\ \underbrace{D(n)}_{\textbf{divide}} + \underbrace{2T(\frac{n}{2})}_{\textbf{conquer}} + \underbrace{C(n)}_{\textbf{combine}} & n > a \end{cases}$$

Divide: $\Theta(1)$

Conquer: $2T(\frac{n}{2})$

Combine: $\Theta(n)$

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ 2T(\frac{n}{2}) + \Theta(n) & n > 1 \end{cases}$$

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Conquer: $2T(\frac{n}{2})$

Combine: $\Theta(n)$

$$T(n) = \begin{cases} c & n = 1 \\ 2T(\frac{n}{2}) + cn & n > 1 \end{cases}$$

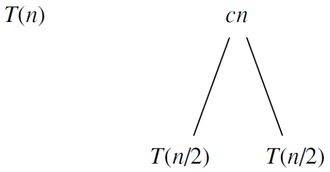
We are not done yet!

Recursion tree

$T(n)$

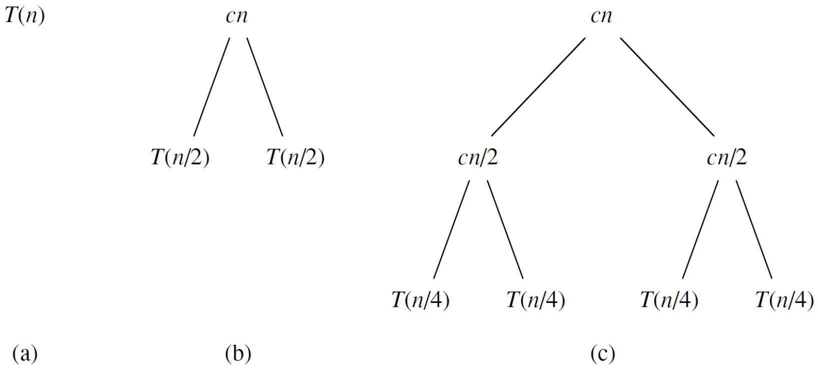
(a)

Recursion tree

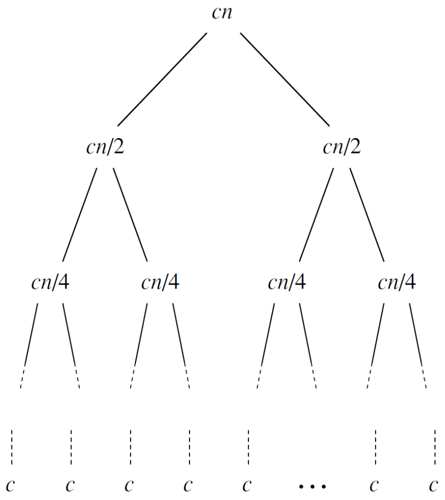


(a) (b)

Recursion tree

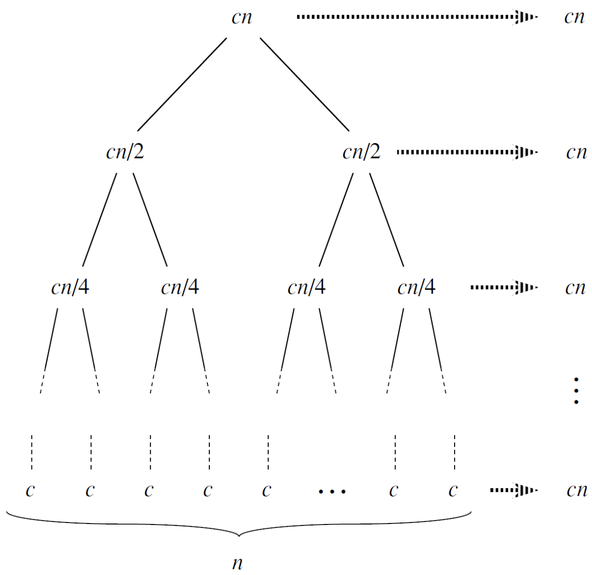


Recursion tree



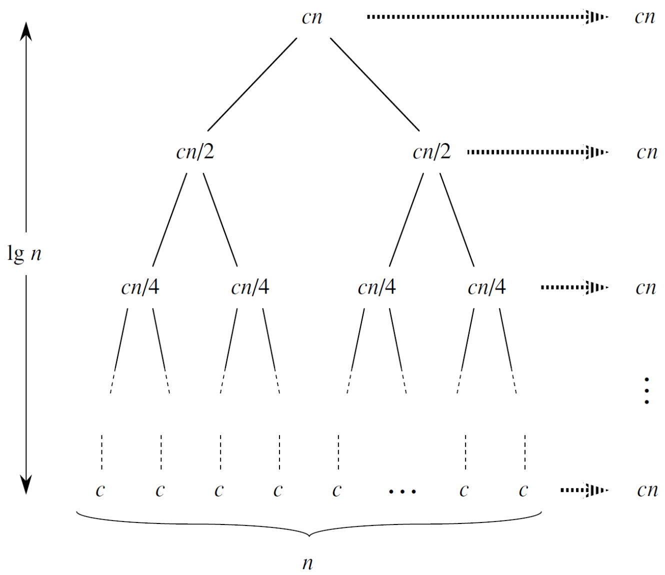
(d)

Recursion tree



(d)

Recursion tree



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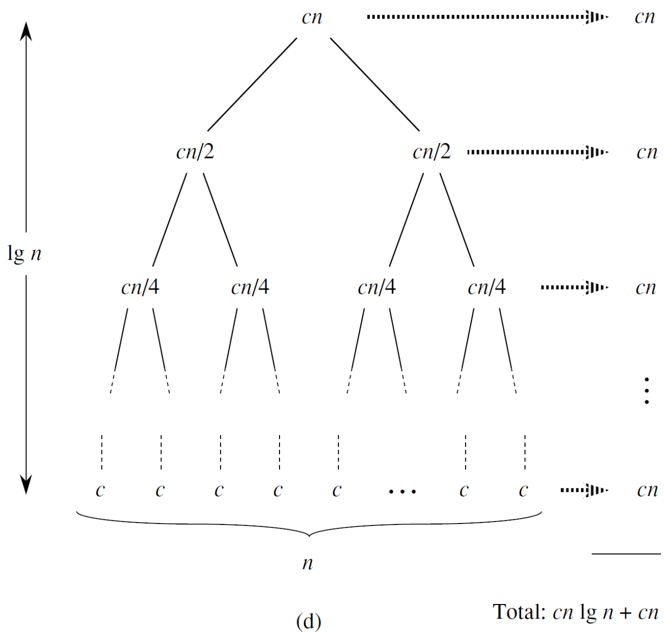
Recursion tree



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Therefore, merge sort grows like:

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Therefore, merge sort grows like:

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We will see that this is true also if n is not a power of 2.

For now, we observe that dividing seems to be beneficial!

Exercises

Write pseudocode for binary search. The task is the same as the one for linear search:

Given a sequence of numbers $A = (a_1, \ldots, a_n)$ and a number v, find an index i such that $v = A[i]$. Return a special number if v can not be found in the sequence.

However, the algorithm is different:

1. Assume that the sequence is sorted
2. Check at the midpoint, and eliminate half of the sequence
3. Keep halving, either iteratively or recursively
4. Check that the worst-case running time is $\Theta(\lg n)$. Is this more or less efficient than linear search?

Suggested reading

Chapters 2.3.1 and 2.3.2 of:

"Introduction to Algorithms – 2nd Ed.", Cormen et al.