

# Deep Learning & Applied AI

Stochastic gradient descent

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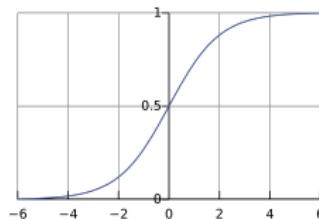
# Logistic regression

We got to the following **convex** loss:

$$\ell_{\Theta}(\{x_i, y_i\}) = - \sum_{i=1}^n y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b))$$

Here,  $\sigma$  is the nonlinear **logistic sigmoid**:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



$\sigma$  has a **saturation** effect as it maps  $\mathbb{R} \mapsto (0, 1)$ .

## Logistic regression: Finding a solution

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$$\nabla_{\Theta} \sum_{i=1}^n y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) = 0$$

where  $\Theta = \{a, b\}$ .

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Consider the gradient of each term in the summation:

$$\nabla_{\Theta} (y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)))$$

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Apply the **chain rule** to each partial derivative:

$$\frac{\partial}{\partial \textcolor{red}{a}} f(g(h(\textcolor{red}{a}, b))) = \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial h} \cdot \frac{\partial h}{\partial \textcolor{red}{a}}$$

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Apply the **chain rule** to each partial derivative:

$$\frac{\partial}{\partial \textcolor{red}{a}} f(g(h(\textcolor{red}{a}, b))) = \frac{\partial f}{\partial g} \cdot \frac{\partial}{\partial (\textcolor{blue}{ax}_i + \textcolor{blue}{b})} \frac{1}{1 + e^{-(\textcolor{blue}{ax}_i + \textcolor{blue}{b})}} \cdot x_i$$

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And similarly for the **second term** and for all parameters.

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By looking at the partial derivative:

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we see that the parameters enter the gradient in a **nonlinear** way.

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logistic regression	convex	<b>nonlinear optimization</b>

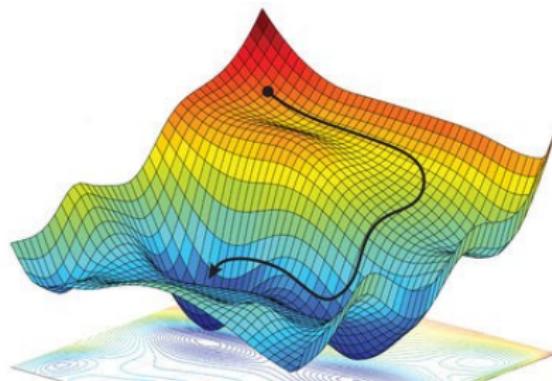
## Gradient descent: Intuition

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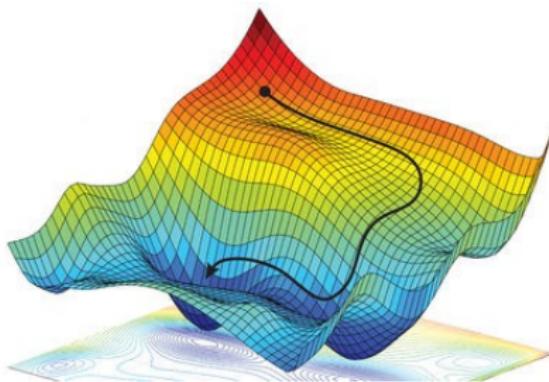
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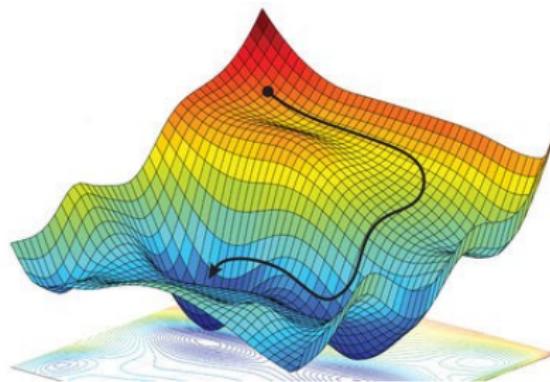
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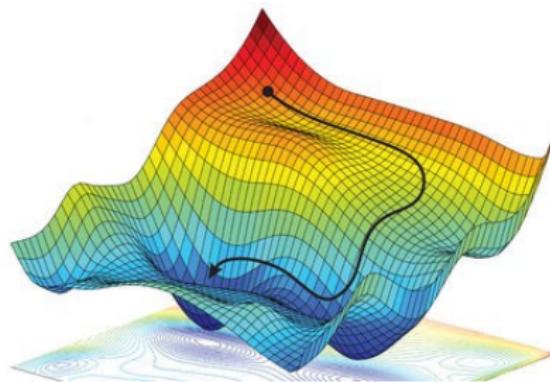
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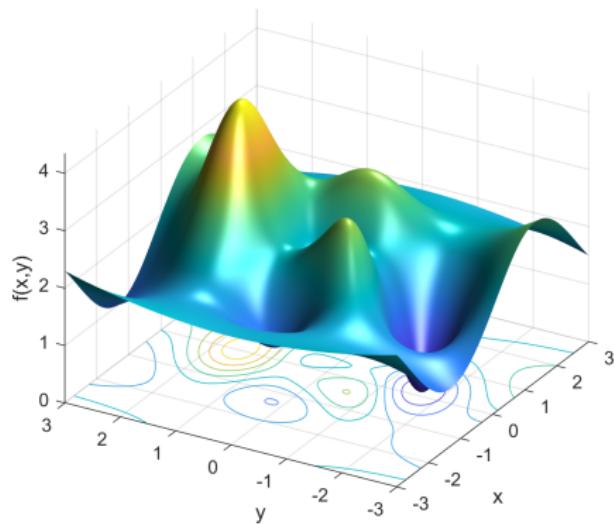
- ③ Stop when a minimum is reached.

# Gradient descent

$$\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} - \alpha \nabla f(\mathbf{x}^{(t)})$$

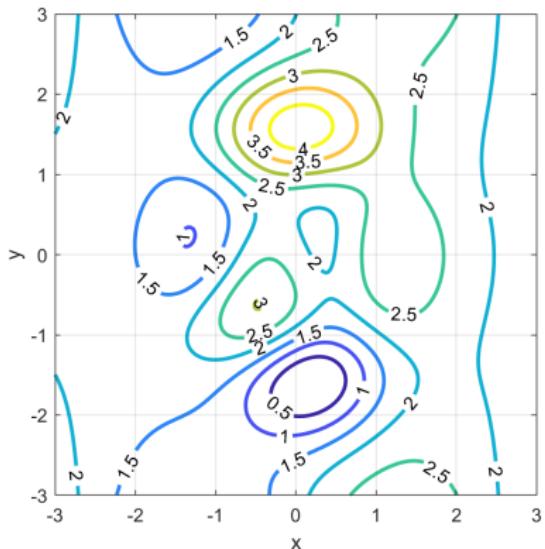
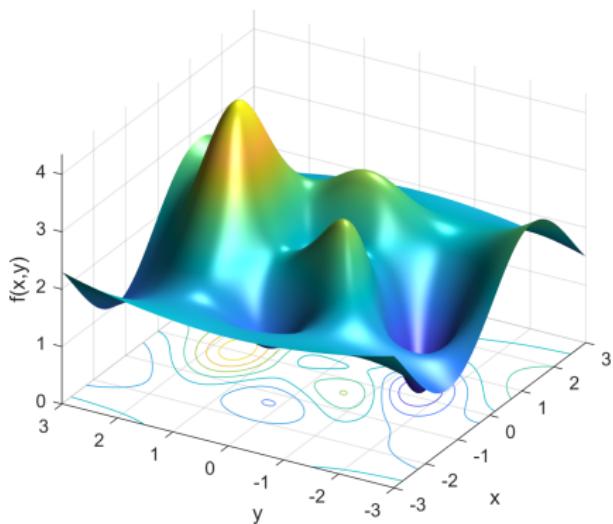
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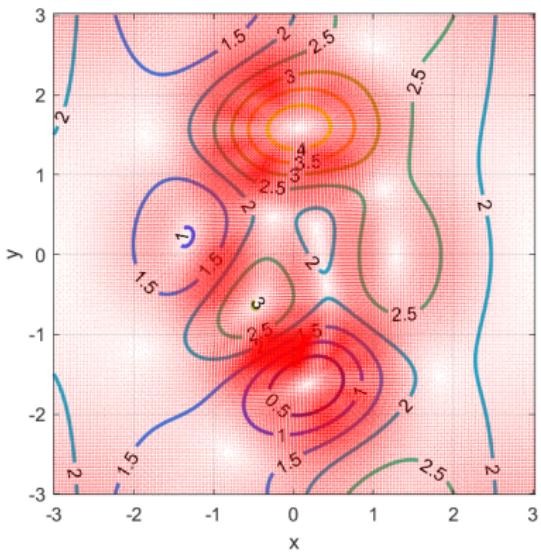
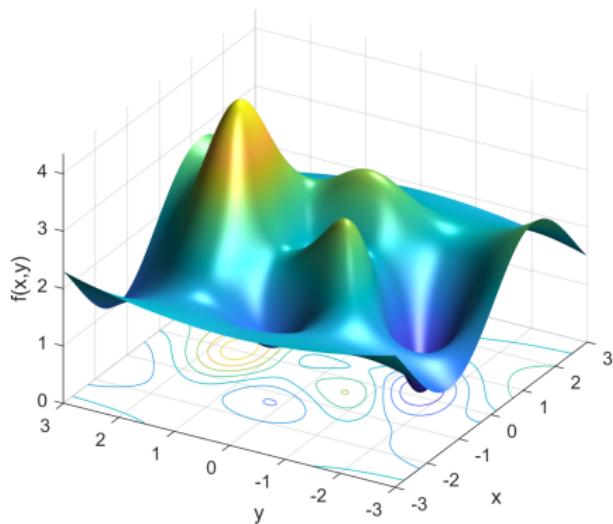
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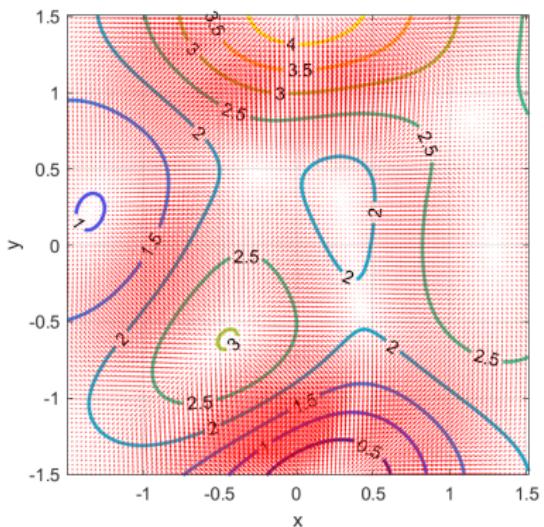
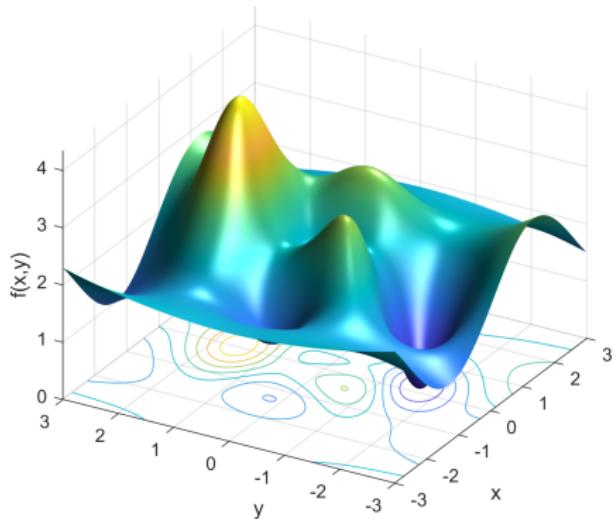
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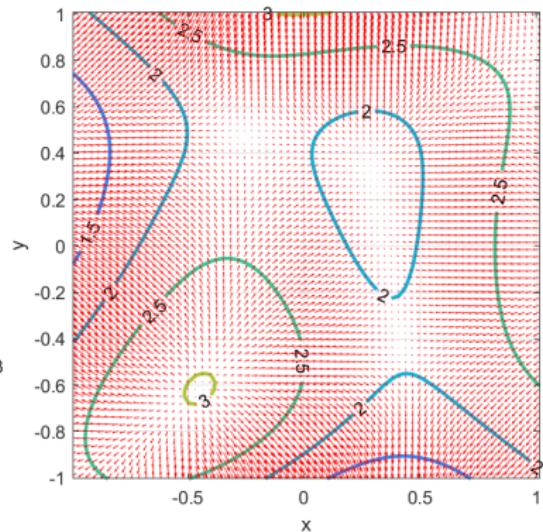
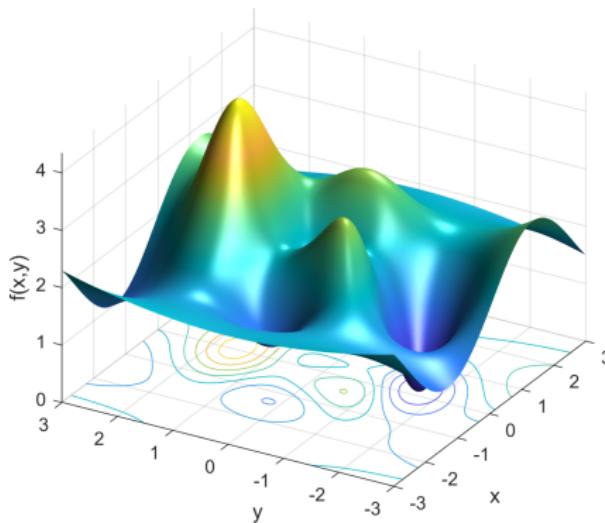
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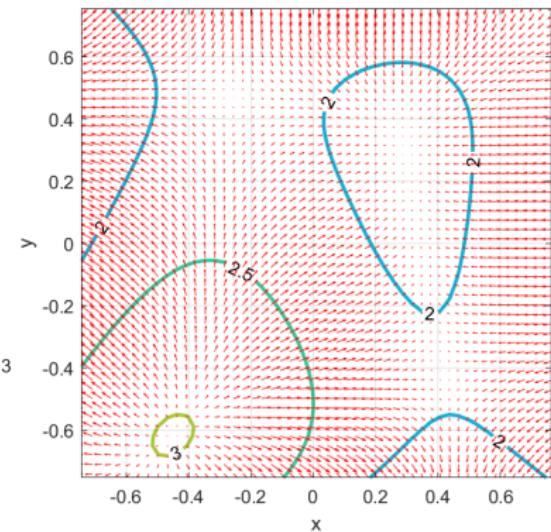
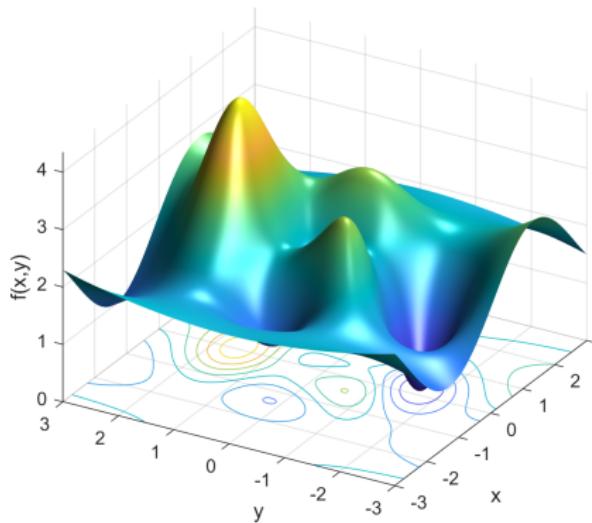
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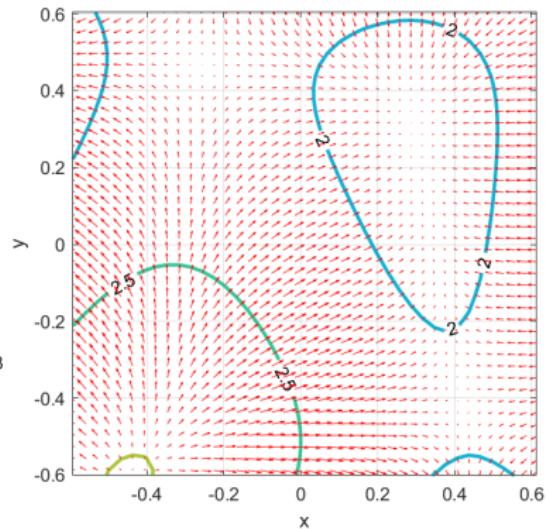
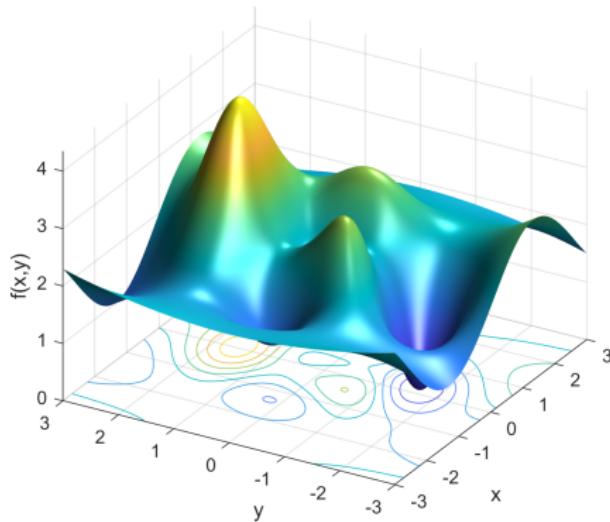
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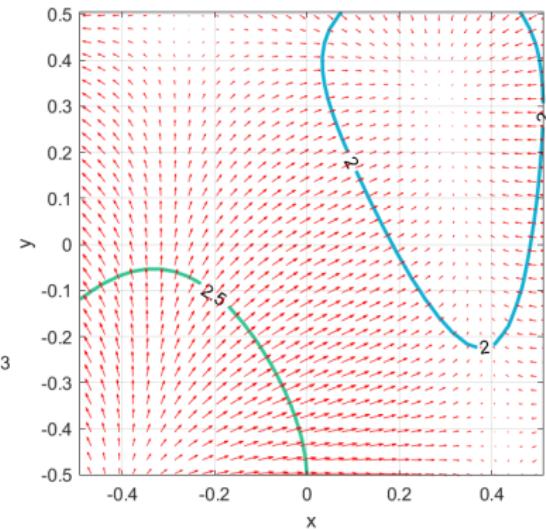
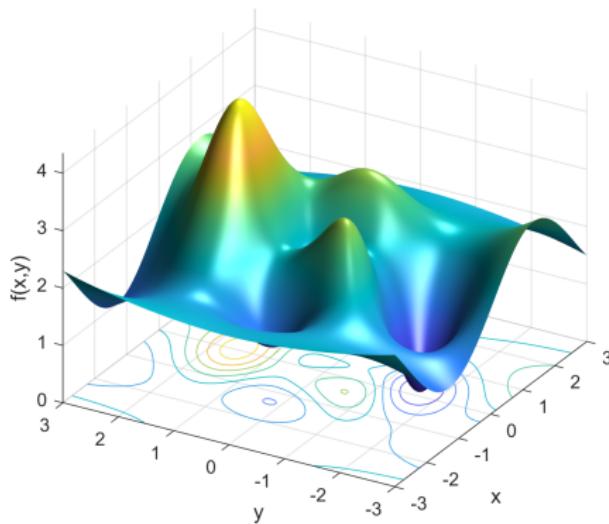
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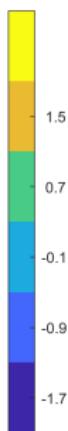
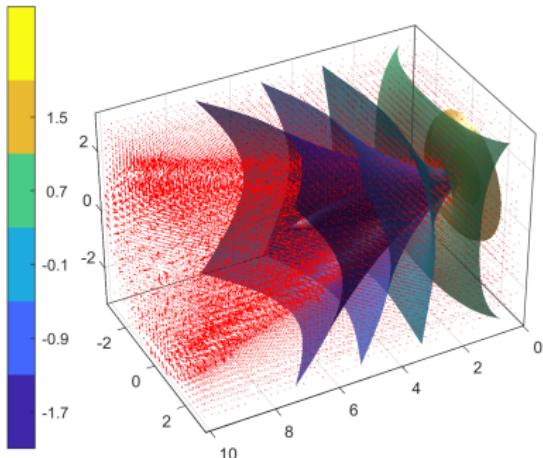
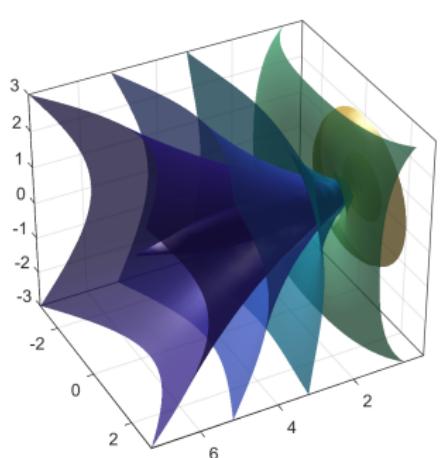
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# Gradient descent: High dimensions

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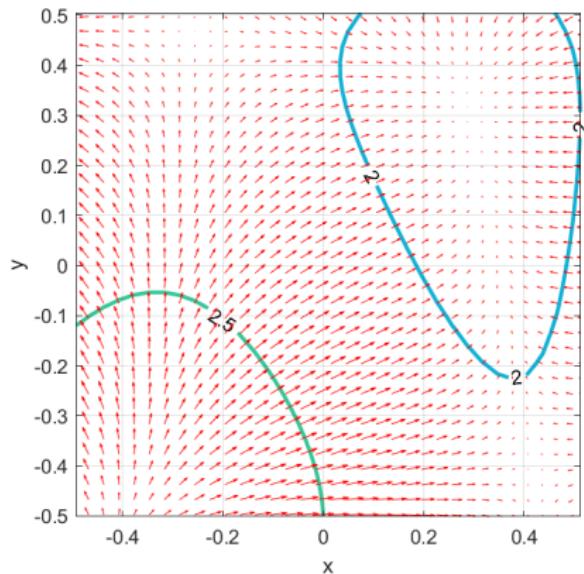


All we say is also valid in  $\gg 2$  dimensions.

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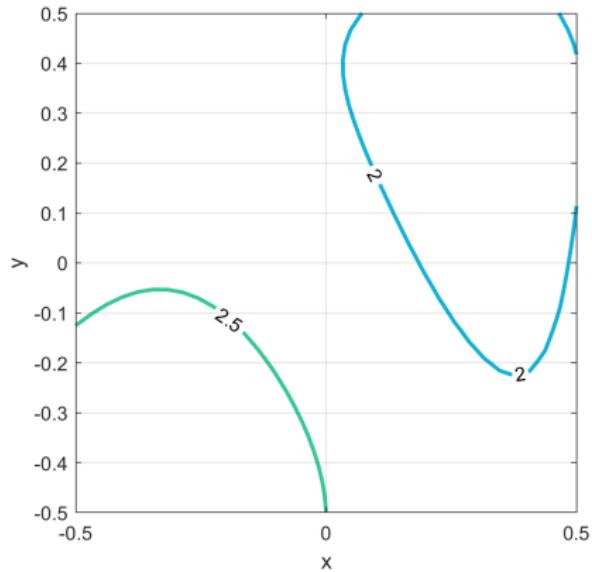
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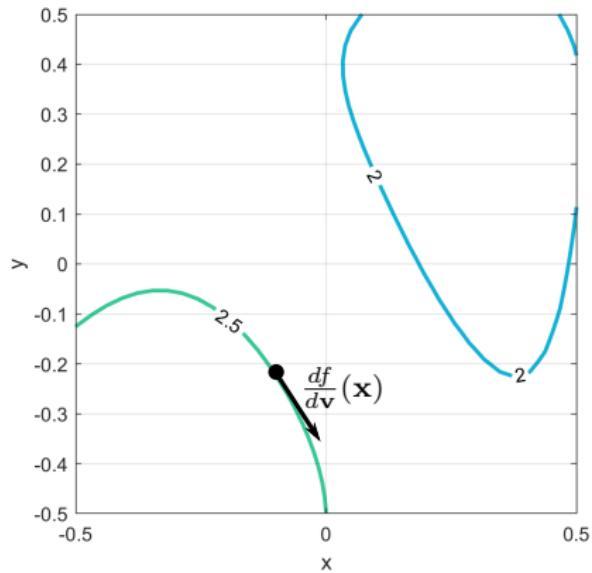
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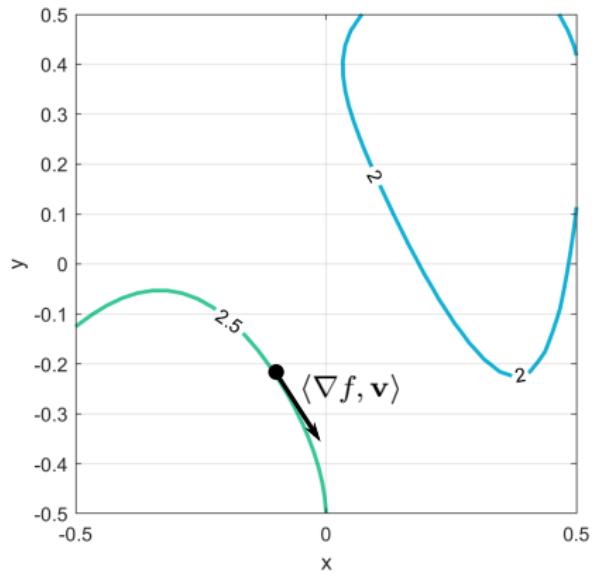
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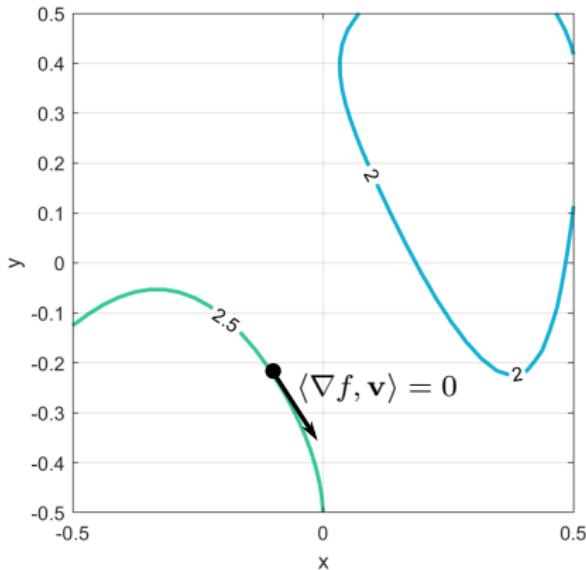
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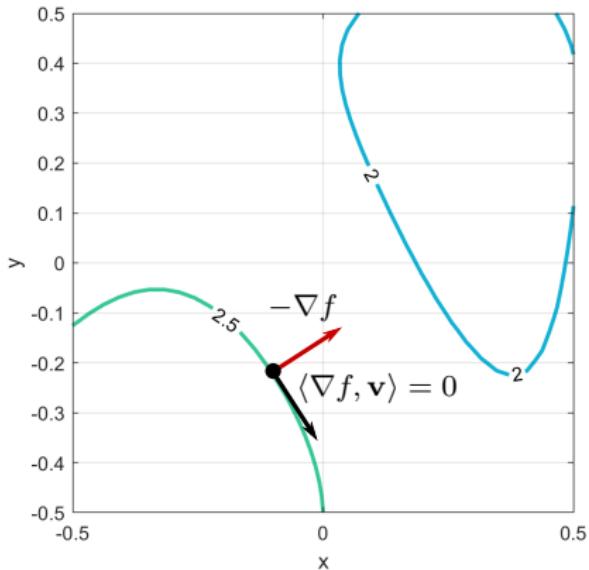


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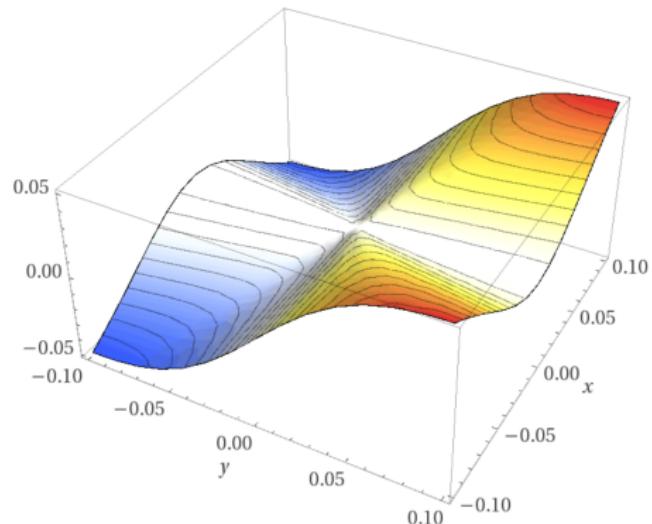
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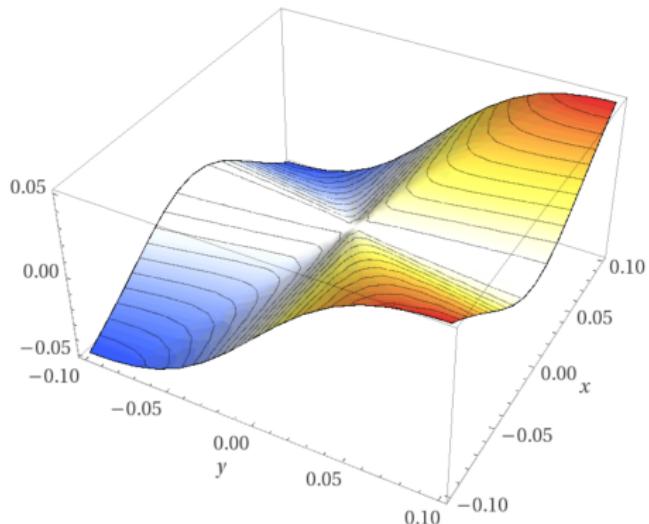
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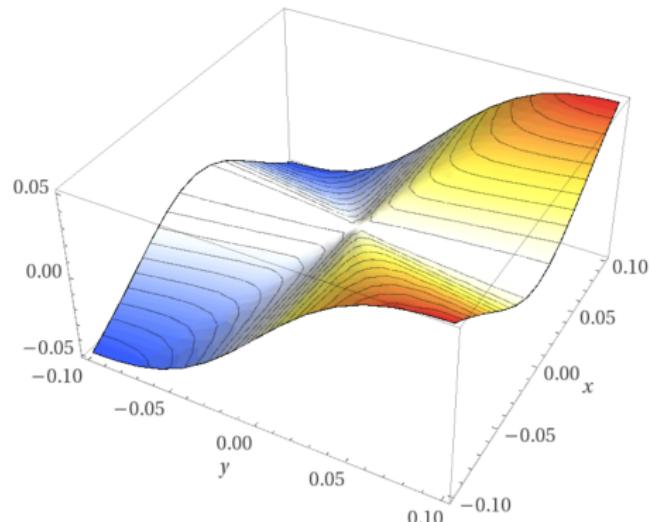
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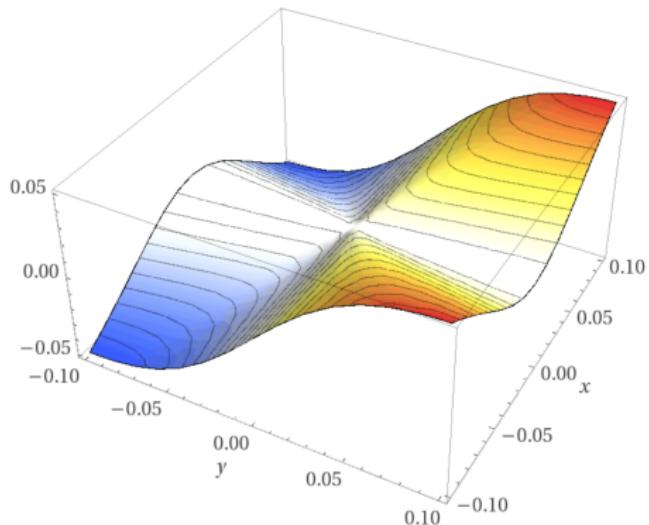
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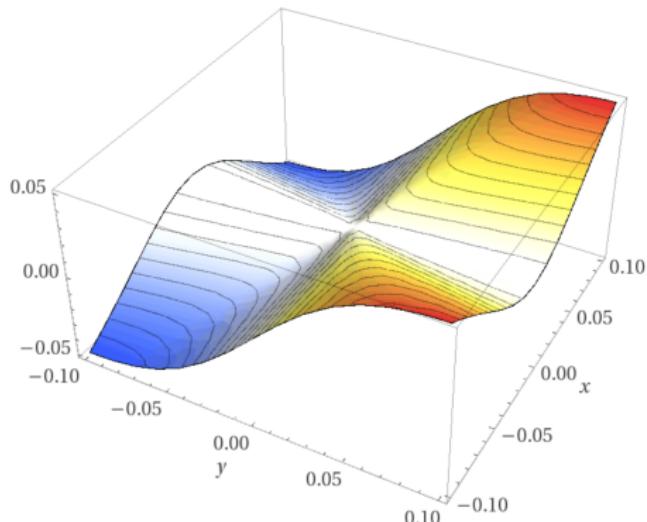
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the partial derivatives of  $f$  are **defined everywhere**, but are **discontinuous** at the origin  
 $\Rightarrow f$  is **not differentiable**



## Gradient descent: Stationary points

A **stationary point** is such that:

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Gradient descent “gets stuck” at stationary points.

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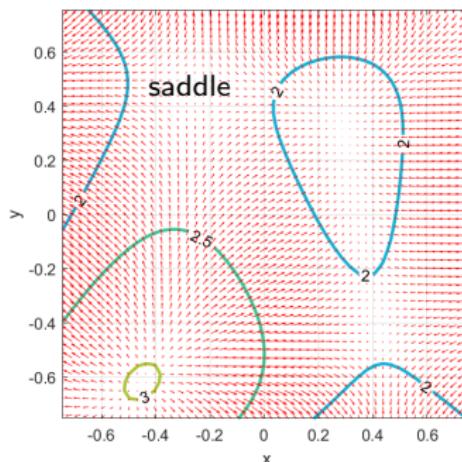
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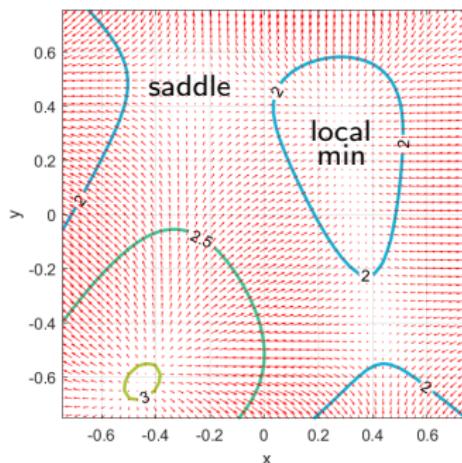
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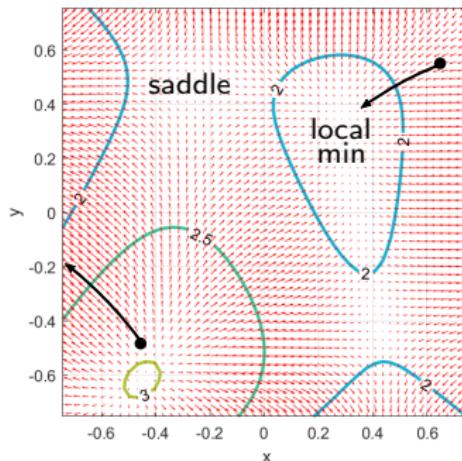
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- Which stationary point depends on the **initialization**.



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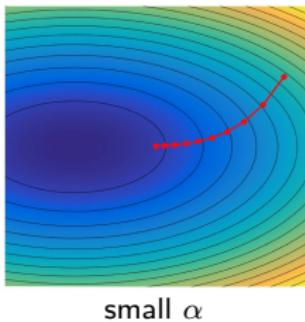
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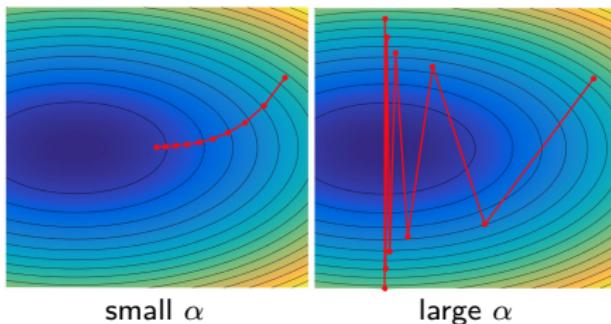
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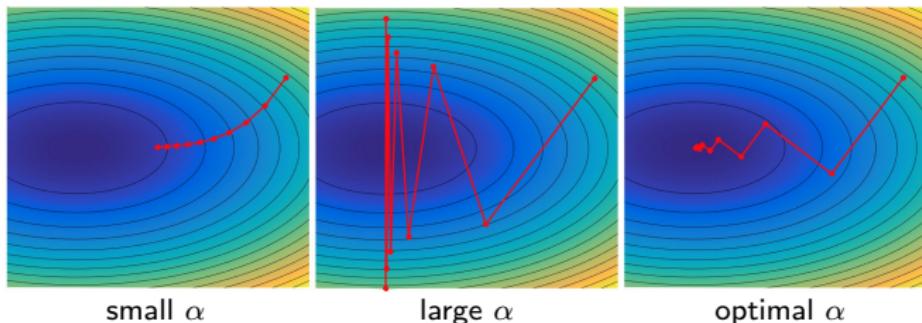
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- Optimal values can be found via **line search** algorithms



$$\arg \min_{\alpha} f(\mathbf{x}^{(t)} - \alpha \nabla f(\mathbf{x}^{(t)}))$$

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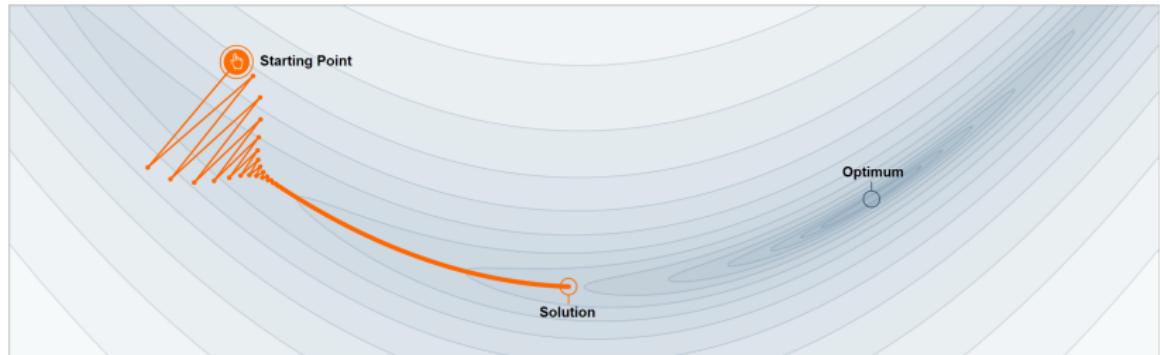
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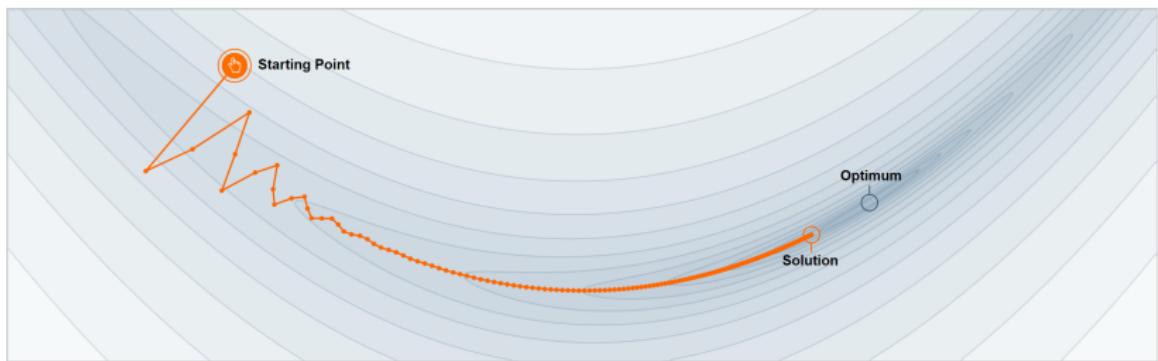
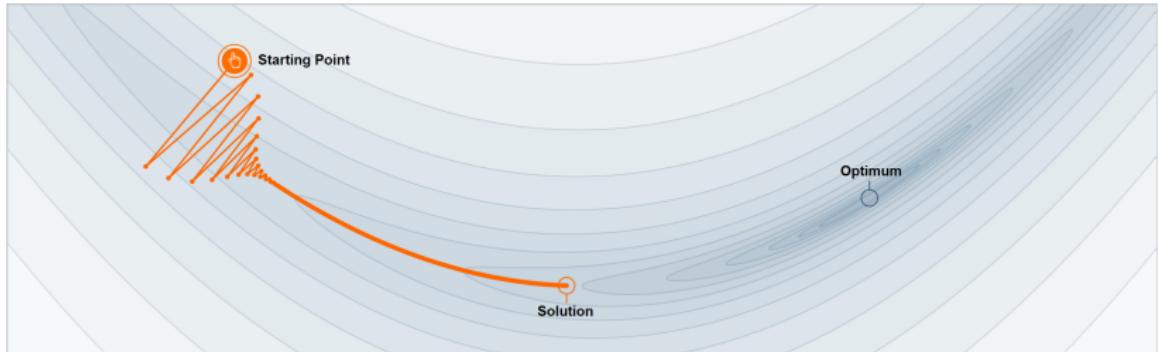
**Acceleration** effect for big  $\lambda$  + escape from local minima.

# Momentum



Goh, "Why momentum really works", Distill 2017

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# First-order acceleration methods

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generalizes optimization algorithms like ADAM, AdaGrad, etc.

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Very often in DL we confront problems that have well defined solutions in mathematical terms, but require extra care and tweaking in practice.

## Stochastic gradient descent

Recall that the loss is usually defined over  $n$  training examples:

$$\ell_{\Theta}(\{x_i, y_i\}) = \frac{1}{n} \sum_{i=1}^n (y_i - f_{\Theta}(x_i))^2$$

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Two **bottlenecks** make gradient descent impractical:

- Number of examples
- Number of parameters

Wilson and Martinez, "The general inefficiency of batch training for gradient descent learning", Neural Networks 2003

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Compute  $\nabla \ell_{\Theta}$  for a **small** representative subset of  $m \ll n$  examples:

$$\frac{1}{m} \sum_{i=1}^m \nabla \hat{\ell}_{\Theta}(\mathcal{B}) \approx \frac{1}{n} \sum_{i=1}^n \nabla \hat{\ell}_{\Theta}(\mathcal{T})$$

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The true gradient  $\nabla \ell_{\Theta}$  is approximated, but with a significant **speed-up**.

**Example:** MNIST dataset

$$n = 60,000, \ m = 10 \quad \Rightarrow \quad 6,000 \times \text{speedup}$$

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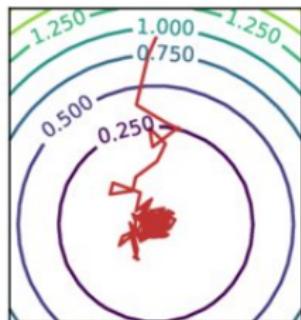
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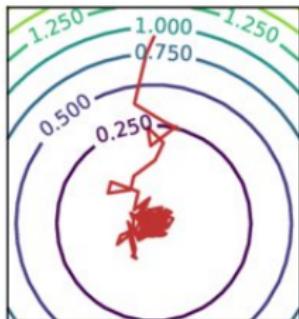
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**Remark:** The update cost is [constant](#) regardless of the size of  $\mathcal{T}$ .

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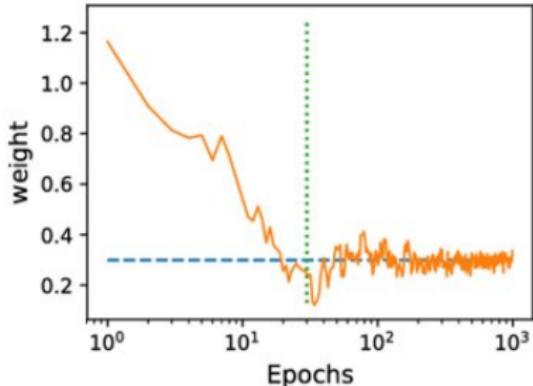
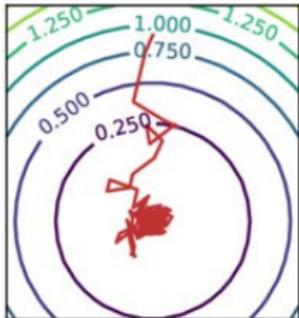
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$$f^* = \arg \min_f \ell(f)$$

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$d$  parameters

$\kappa, \nu$  are constants related to the conditioning of the problem

	<b>cost per iteration</b>	<b>iterations to reach <math>\rho</math></b>
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implying better generalization

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SGD can find a **low value** of the loss quickly enough to be useful, even if it's not a minimum.

## Suggested reading

Approachable explanation of differentiability in higher dimensions:

[https://mathinsight.org/differentiability\\_multivariable\\_subtleties](https://mathinsight.org/differentiability_multivariable_subtleties)

Distill article on why momentum really works:

<https://distill.pub/2017/momentum/>

Seminal paper on using mini-batches for training:

<http://axon.cs.byu.edu/papers/Wilson.nn03.batch.pdf>

Seminal paper on GD vs. SGD performance:

<https://papers.nips.cc/paper/3323-the-tradeoffs-of-large-scale-learning.pdf>