

Deep Learning & Applied AI

Linear algebra revisited

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Linear algebra is the study of linear maps on finite dimensional vector spaces

Linear algebra is about matrices as much as
astronomy is about telescopes

Vector space

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- **multiplicative identity:** $1v = v$ for all $v \in V$
- **distributive properties:** $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$

Example: Lists of numbers

\mathbb{R}^n is defined to be the set of all n -long sequences of numbers in \mathbb{R} :

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, 2, \dots, n\}$$

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Addition and multiplication are defined as expected:

$$\begin{aligned}(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \lambda(x_1, x_2, \dots, x_n) &= (\lambda x_1, \lambda x_2, \dots, \lambda x_n)\end{aligned}$$

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With these definitions, \mathbb{R}^n is a vector space

Example: Functions

Consider the set of all functions $f : [0, 1] \rightarrow \mathbb{R}$ with the standard definitions for sum and scalar product:

$$(f + g)(x) = f(x) + g(x)$$

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The above forms a vector space. In fact, **any** set of functions $f : S \rightarrow \mathbb{R}$ with $S \neq \emptyset$ (Q: why?) and the definitions above forms a vector space.

Vector spaces

Elements of a vector space (called **vectors**)
are not necessarily lists

A vector space is an **abstract** entity whose elements
might be lists, functions, or weird objects

Example: Curved surfaces

Do surfaces form a vector space?



Example: Curved surfaces

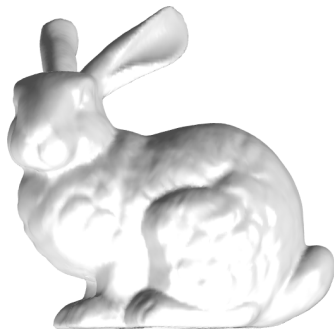
Do surfaces form a vector space? Not really – if you sum the coordinates of two points, you may get a third point that is not on the surface.



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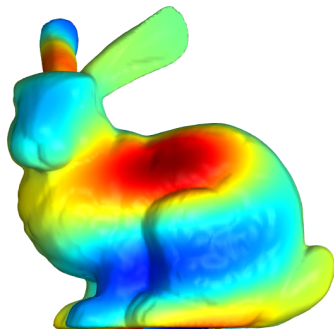
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We can still use linear algebra to study [functions on surfaces](#)

Bases

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- $v_1, \dots, v_n \in V$ are **linearly independent** if and only if each $v \in \text{span}(v_1, \dots, v_n)$ has only one representation as a linear combination of v_1, \dots, v_n

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So every vector $v \in V$ can be expressed **uniquely** as a linear combination

$$v = \sum_{i=1}^n \alpha_i v_i$$

You can think of a basis as the minimal set of vectors that generates the entire space

Example: Bases

- $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ is a basis of \mathbb{R}^n called the **standard basis**; its vectors are called the **indicator vectors**.

In deep learning, also called **one-hot** representation.

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$$f_1(x) = \begin{cases} 1 & \text{if } x = x_1 \\ 0 & \text{else} \end{cases}$$


$$f_2(x) = \begin{cases} 1 & \text{if } x = x_2 \\ 0 & \text{else} \end{cases}$$

$$\vdots$$

is the standard basis for the set of functions $f : \mathbb{R} \rightarrow \mathbb{R}$; the basis vectors are also called **indicator functions**

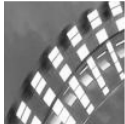
Examples

An image expressed in the **standard basis**:

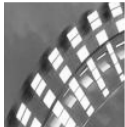

$$= \alpha_1 \begin{array}{|c|} \hline \cdot \\ \hline \end{array} + \alpha_2 \begin{array}{|c|} \hline \\ \hline \cdot \\ \hline \end{array} + \alpha_3 \begin{array}{|c|} \hline \\ \hline \\ \hline \cdot \\ \hline \end{array} + \dots$$

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The same image, expressed in terms of a **nonlinear** map σ :


$$= \sigma \left(\begin{array}{|c|} \hline \text{gray square} \\ \hline \end{array}, \square, \text{—} \right)$$

The image is **not** in the span of the three features.

Dimension

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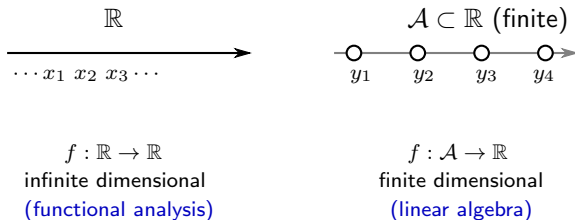
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Linear maps

A **linear map** from V to W is a function $T : V \rightarrow W$ with the properties:

- **additivity:** $T(u + v) = Tu + Tv$ for all $u, v \in V$
- **homogeneity:** $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{R}$ and all $v \in V$

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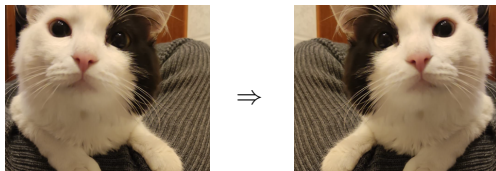
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Reflection operation on an image:

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T(x, y) = (-x, y)$$



Linear maps as a vector space

Linear maps $T : V \rightarrow W$ form a **vector space**, with addition and multiplication (Q: what is the additive identity?) defined as:

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We also have a useful definition of **product** between linear maps.

If $T : U \rightarrow V$ and $S : V \rightarrow W$, their product $ST : U \rightarrow W$ is defined by

$$(ST)(u) = S(Tu)$$

In other words, ST is just the usual composition $S \circ T$ of two functions

Algebraic properties of products of linear maps

- **associativity:** $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
- **identity:** $TI = IT = T$
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Keep in mind that composition of linear maps **is not commutative**, i.e.

$$ST \neq TS$$

in general (although there are special cases)

Example: Take $Sf = f'$ and $(Tf)(x) = x^2 f(x)$

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The **matrix** of T in these bases is the $m \times n$ array of values in \mathbb{R}

$$\mathbf{T} = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}$$

whose entries $T_{i,j}$ are defined by

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Hence each column of \mathbf{T} contains the **linear combination coefficients** for the **image via T of a basis vector from V**

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In other words, the matrix encodes **how basis vectors are mapped**, and this is enough to map all other vectors in their span, since:

$$Tv = T\left(\sum_j \alpha_j v_j\right) = \sum_j T(\alpha_j v_j) = \sum_j \alpha_j Tv_j$$

Matrices

The matrix is a **representation** for a linear map, and
it **depends on the choice of bases**

Matrix of a vector

Suppose $v \in V$ is an arbitrary vector, while v_1, \dots, v_n is a basis of V .
The matrix of v wrt this basis is the $n \times 1$ matrix:

$$\mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

so that

$$v = c_1 v_1 + \cdots c_n v_n$$

Once again, we see that the matrix **depends on the choice of basis** for V

Product of “map matrix” and “vector matrix”

$$\underbrace{\begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}}_{\mathbf{T}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} = \sum_{j=1}^n c_j \underbrace{\begin{pmatrix} T_{1,j} \\ \vdots \\ T_{m,j} \end{pmatrix}}_{Tv_j \text{ wrt } (w_1, \dots, w_m)}$$

Because recall that, for bases $v_1, \dots, v_n \in V$ and $w_1, \dots, w_m \in W$:

$$Tv_j = T_{1,j}w_1 + \cdots + T_{m,j}w_m$$

Product of “map matrix” and “vector matrix”

$$\underbrace{\begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}}_{\mathbf{T}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} = \sum_{j=1}^n c_j \underbrace{\begin{pmatrix} T_{1,j} \\ \vdots \\ T_{m,j} \end{pmatrix}}_{Tv_j \text{ wrt } (w_1, \dots, w_m)}$$

Because recall that, for bases $v_1, \dots, v_n \in V$ and $w_1, \dots, w_m \in W$:

$$Tv_j = T_{1,j}w_1 + \cdots + T_{m,j}w_m$$

We see then that vector $c = \sum_j c_j v_j$ is mapped to $Tc = \sum_j c_j Tv_j$.

In other words, matrix product is behaving as expected.

Suggested reading

Sections 1.A – 3.D of the textbook:

S. Axler, “Linear algebra done right – 3rd edition”. Springer, 2015