Deep Learning & Applied AI

Convolutional neural networks

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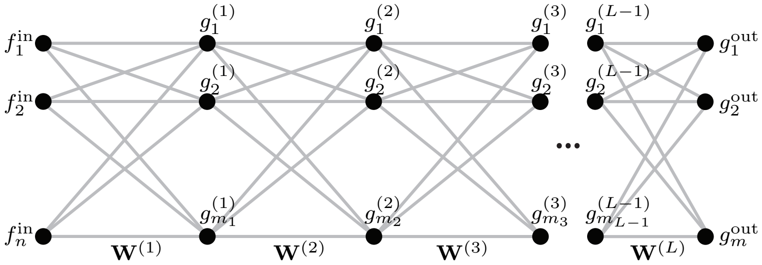
Neural network (NN)



Deep neural network consisting of L layers

Net output $\mathbf{g}^{\text{out}} = \sigma\left(\ldots \mathbf{W}^{(2)}\sigma\left(\mathbf{W}^{(1)}\mathbf{f}^{\text{in}}\right)\right)$

Activation, e.g. $\sigma(x) = \max\{x, 0\}$ rectified linear unit (ReLU)

Parameters weights of all layers $\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(L)}$ (including biases)

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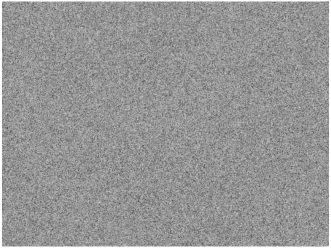
Look for "universal" priors that are task-independent to some extent.

Task-independent priors must come with the data.

Structure as a strong prior

Key insight: Data often carries structural priors in terms of repeating patterns, compositionality, locality, . . .



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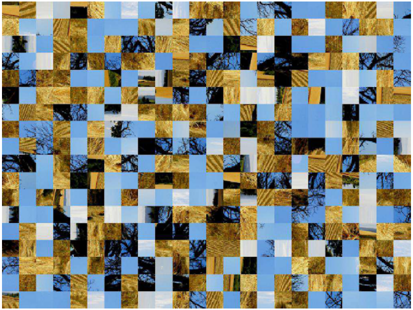

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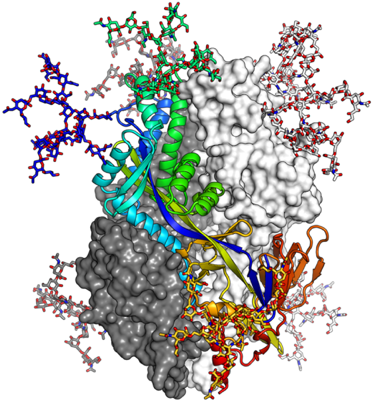

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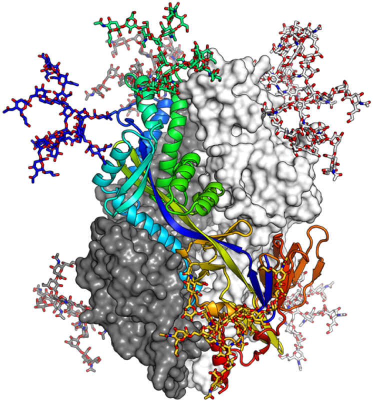
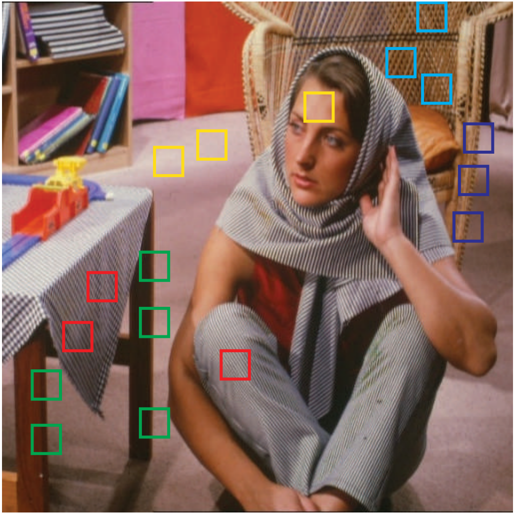


Take advantage of the structure of the data.

Self-similarity

Data tends to be self-similar across the domain:



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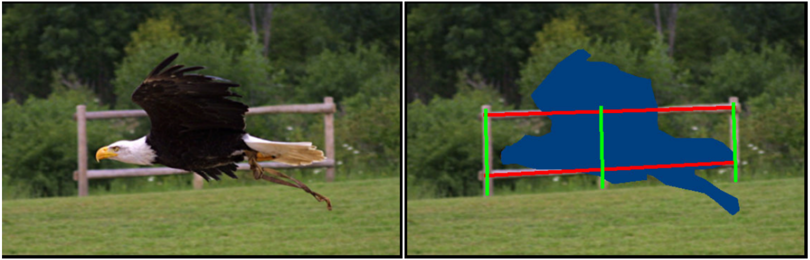

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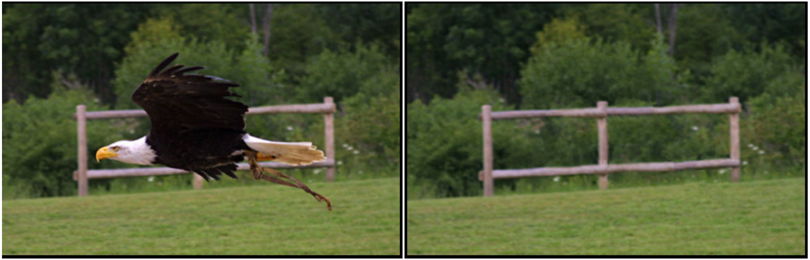

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Define the (linear!) translation operator $\mathcal{T}$ along vector $v \in \mathbb{R}^2$ as:

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Therefore, it is desirable to enforce translation invariance:

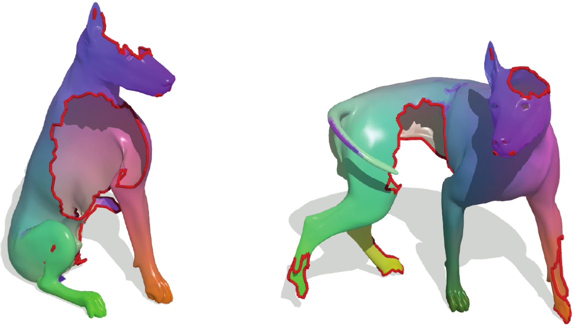
$$\mathcal{C}(\mathcal{T}_v f) = \mathcal{C}(f) \quad \forall f, \mathcal{T}_v$$

where $\mathcal{C}$ is a classification functional.

Deformation invariance

Other types of invariance are possible.

Invariance to partiality and isometric deformations:

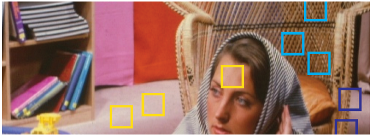


In many cases, invariance can be directly injected into the network.

Today we concentrate on translation invariance.

Hierarchy and compositionality

Translation invariance is desirable across multiple scales:



We expect local features to be invariant to their location in the image:

$$z(\mathcal{T}_v p) = z(p) \quad \forall p, \mathcal{T}_v$$

where p are image patches of variable size.

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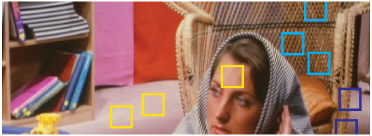

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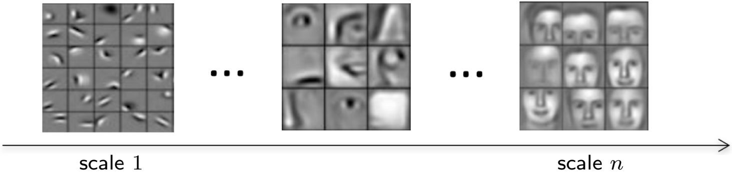

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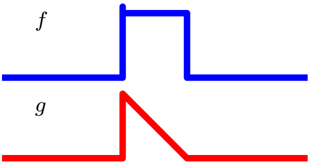
CNNs directly exploit this fact as a prior.

Convolution

Given two functions $f, g : [-\pi, \pi] \to \mathbb{R}$ their convolution is a function:

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(t)g(x-t)dt$$



d'Alembert 1754

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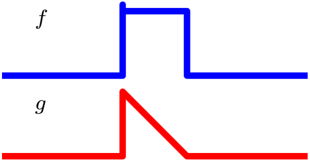

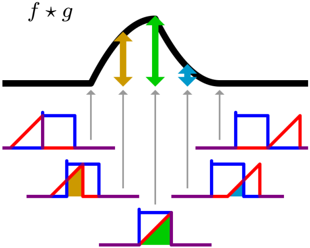

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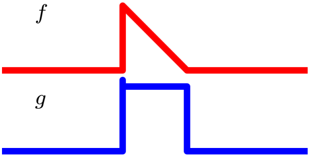

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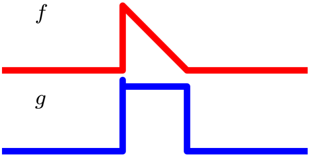

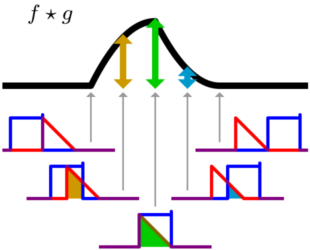

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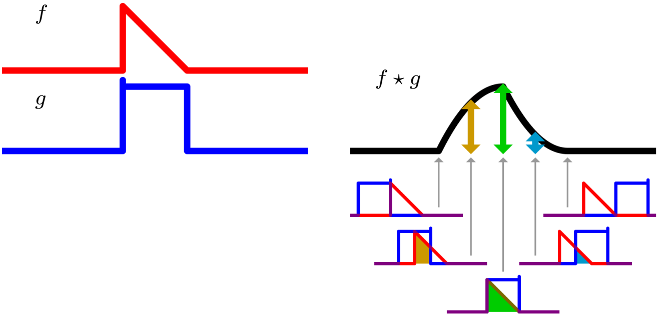

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Convolution: Commutativity

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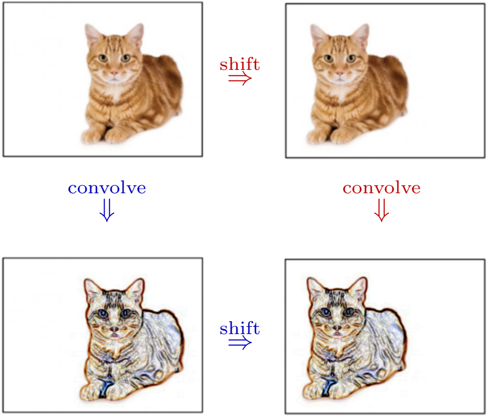
Further, convolution is shift-equivariant (or translation-equivariant):

$$f(x - x_0) \star g(x) = (f \star g)(x - x_0)$$

Convolution: Shift-equivariance



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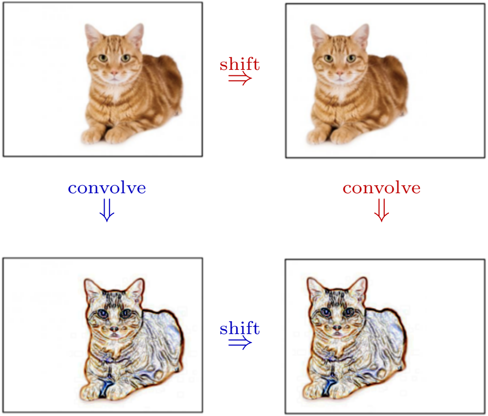

In fact, equivariance is a defining property of convolutions.

(any linear operator that is shift-equivariant, is a convolution)

Convolution: Linearity

We can see convolution as the application of a linear operator $\mathcal{G}$:

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\end{aligned}$$

Translation equivariance can then be phrased as:

$$\mathcal{G}(\mathcal{T}f) = \mathcal{T}(\mathcal{G}f)$$

i.e., the convolution and translation operators commute.

Discrete convolution

In the discrete setting, we deal with vectors $\mathbf{f}, \mathbf{g}$.

We define the convolution sum:

$$(\mathbf{f} \star \mathbf{g})[n] = \sum_{k=-\infty}^{\infty} \mathbf{f}[k]\mathbf{g}[n-k]$$

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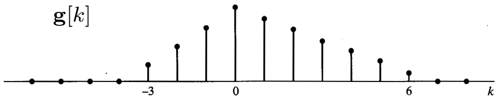

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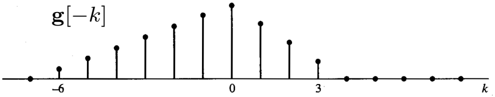

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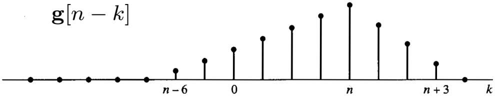

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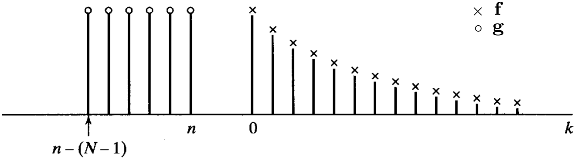

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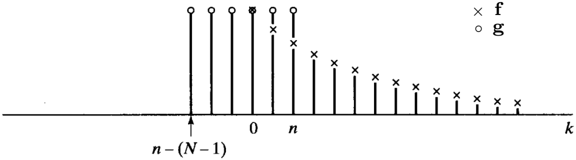

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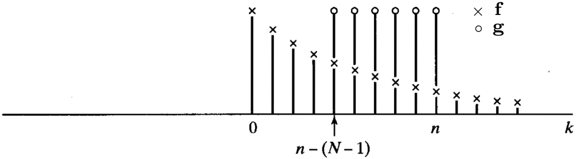

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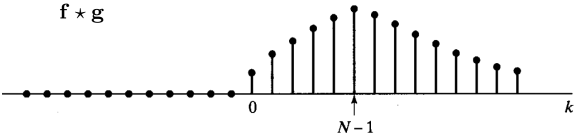

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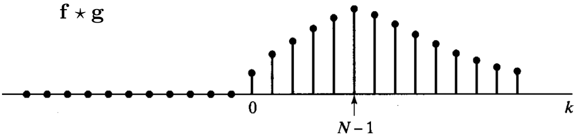

The specific discretization depends on the boundary conditions.

In the example above, $\mathbf{f}$ was zero-padded in order for the products to be well defined for all shifts.

Discrete convolution

On 2D domains (e.g. RGB images $f : \mathbb{R}^2 \to \mathbb{R}^3$), for each channel:

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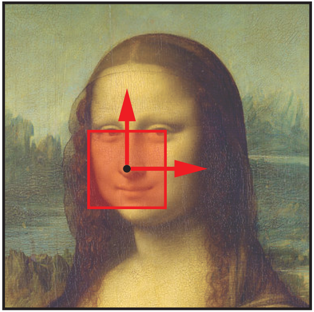

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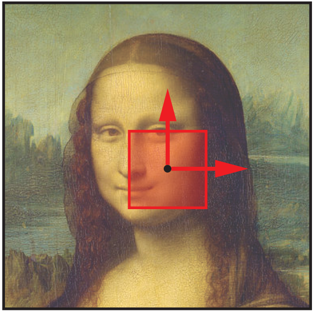
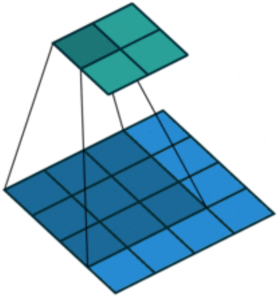
$$(\mathbf{f} \star \mathbf{g})[m, n] = \sum_k \sum_\ell \mathbf{f}[k, \ell]\, \mathbf{g}[m - k, n - \ell]$$

We get the classical interpretation in terms of a moving window:



Boundary conditions and stride

No padding: The convolution kernel is directly applied within the boundaries of the underlying function (an image in this example).

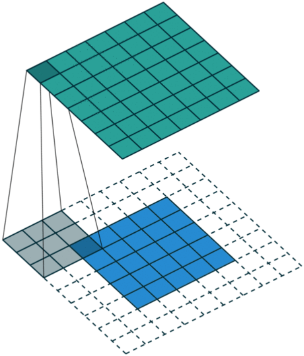


The result of the convolution is a smaller image.

Animations: Vincent Dumoulin

Boundary conditions and stride

Full zero-padding: The domain is enlarged and padded with zeroes. The convolution kernel is applied within the (now larger) boundaries.

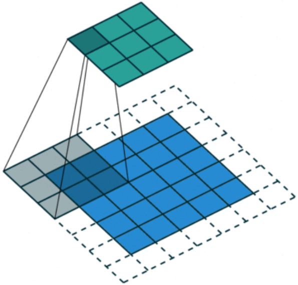


The result of the convolution is a larger image.

Animations: Vincent Dumoulin

Boundary conditions and stride

Arbitrary zero-padding, with stride: The domain is enlarged and padded with zeroes, but not enough to capture the boundary pixels. Further, each discrete step skips one pixel.



The result is the same as no stride followed by downsampling.

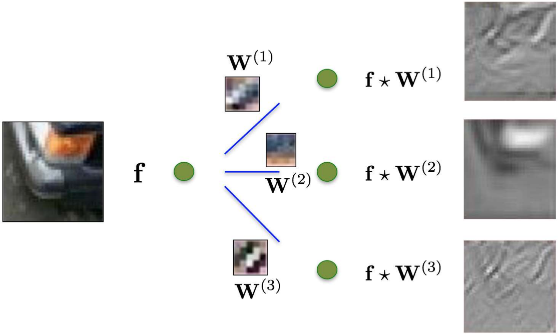
Animations: Vincent Dumoulin

CNN vs. MLP

We are replacing the large matrices of MLPs with small local filters.

CNN vs. MLP

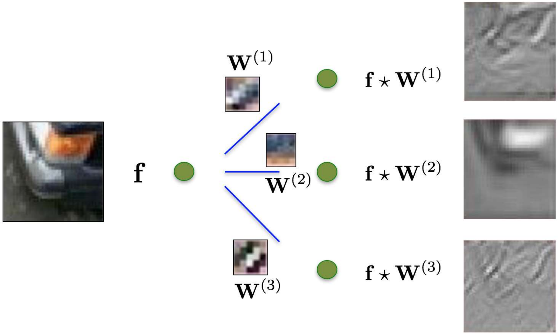
We are replacing the large matrices of MLPs with small local filters.



- $O(1)$ parameters per filter; huge gain compared to the MLP.

CNN vs. MLP

We are replacing the large matrices of MLPs with small local filters.



- $O(1)$ parameters per filter; huge gain compared to the MLP.
- Filter weights are applied across the entire image $\Rightarrow$ weight sharing, which implements the notion of self-similarity and shift-equivariance.

Sparse interactions

Fully-connected layer:

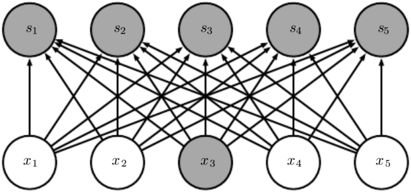


Image: Goodfellow et al, 2016

Sparse interactions

Fully-connected layer:

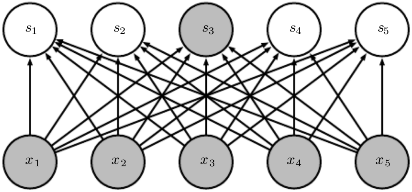
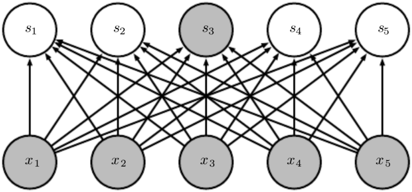


Image: Goodfellow et al, 2016

Sparse interactions

Fully-connected layer:

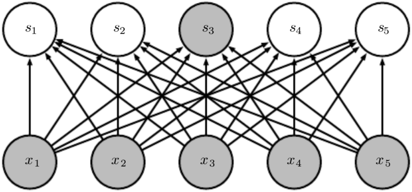


each edge is a different weight

Image: Goodfellow et al, 2016

Sparse interactions

Fully-connected layer:



each edge is a different weight

Convolutional layer:

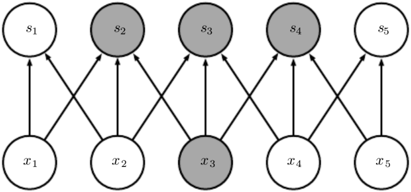
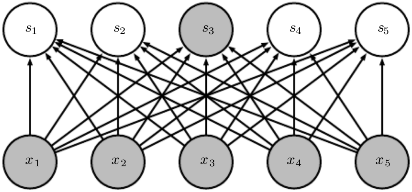


Image: Goodfellow et al, 2016

Sparse interactions

Fully-connected layer:



each edge is a different weight

Convolutional layer:

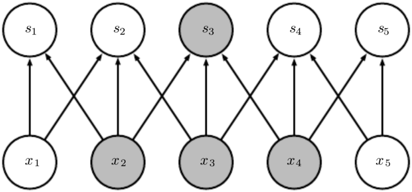
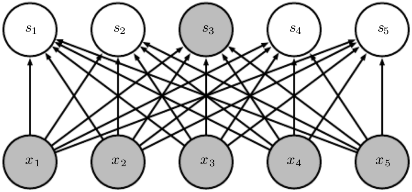


Image: Goodfellow et al, 2016

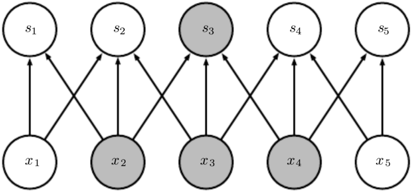
Sparse interactions

Fully-connected layer:



each edge is a different weight

Convolutional layer:



the outgoing edges have the same weights
for each input variable (weight sharing)

Image: Goodfellow et al, 2016

Pooling

At deep layers, filters interact with larger portions of the input.

$$
\begin{bmatrix} 3 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 3 & 2 & 0 & 0 \\ 3 & 3 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{bmatrix}
*
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
=
\begin{bmatrix} 6 & 8 & 6 & 3 & 1 & 0 \\ 9 & 13 & 10 & 5 & 2 & 0 \\ 9 & 14 & 11 & 6 & 3 & 0 \\ 9 & 13 & 11 & 6 & 2 & 0 \\ 8 & 13 & 10 & 5 & 3 & 0 \\ 6 & 7 & 5 & 3 & 1 & 0 \end{bmatrix}
$$

Input data — Filter — Feature map

Pooling

At deep layers, filters interact with larger portions of the input.

3	3	2	1	0	0
3	3	2	1	0	0
3	3	2	1	0	0
3	3	3	2	0	0
3	3	2	1	0	0
3	2	1	1	0	0

Input data

$*$

1	0	1
0	1	0
1	0	1

Filter

$=$

6	8	6	3	1	0
9	13	10	5	2	0
9	14	11	6	3	0
9	13	11	6	2	0
8	13	10	5	3	0
6	7	5	3	1	0

Feature map

13	10	2
14	11	3
13	10	3

Max pooling

Pooling

At deep layers, filters interact with larger portions of the input.

3	3	2	1	0	0
3	3	2	1	0	0
3	3	2	1	0	0
3	3	3	2	0	0
3	3	2	1	0	0
3	2	1	1	0	0

Input data

*

1	0	1
0	1	0
1	0	1

Filter

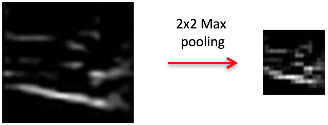
=

6	8	6	3	1	0
9	13	10	5	2	0
9	14	11	6	3	0
9	13	11	6	2	0
8	13	10	5	3	0
6	7	5	3	1	0

Feature map

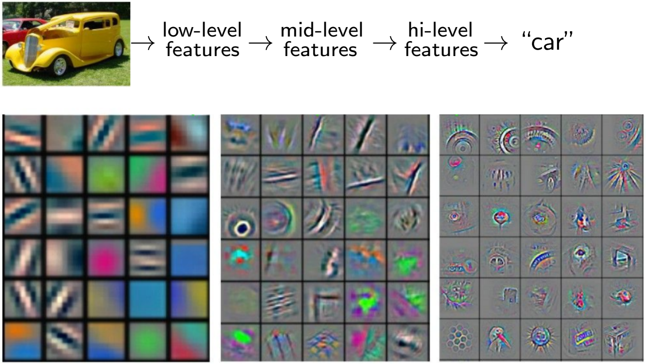
13	10	2
14	11	3
13	10	3

Max pooling



This allows to capture complicated non-local interactions via simple building blocks that only describe sparse interactions.

Learned features



Learned features

https://openai.com/blog/microscope/

Suggested reading

Convolution animations, including variants:
https://github.com/vdumoulin/conv_arithmetic

Seminal paper on CNN, seen as a set of feature detectors:
http://yann.lecun.com/exdb/publis/pdf/lecun-89e.pdf