Deep Learning & Applied AI

Convolutional neural networks

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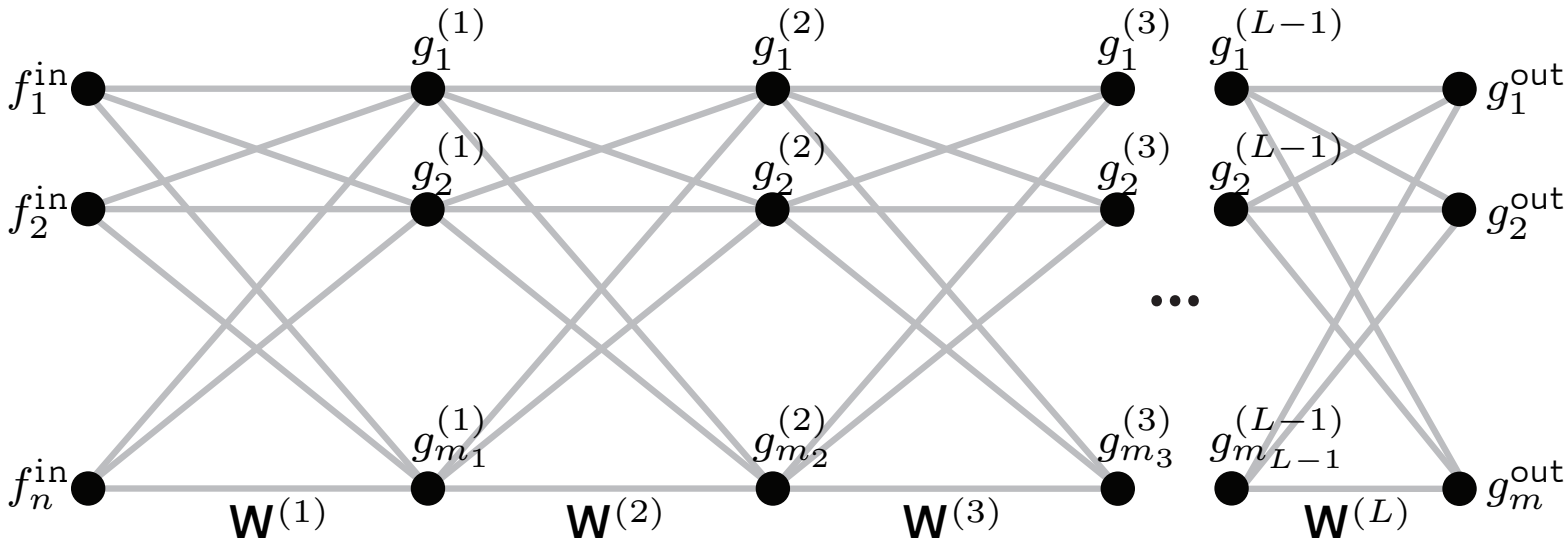
Neural network (NN)



Deep neural network consisting of L layers

Net output $\mathbf{g}^{\text{out}} = \sigma\left(\ldots \mathbf{W}^{(2)} \sigma\left(\mathbf{W}^{(1)} \mathbf{f}^{\text{in}}\right)\right)$

Activation, e.g. $\sigma(x) = \max\{x, 0\}$ ReLU

Parameters weights of all layers $\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(L)}$

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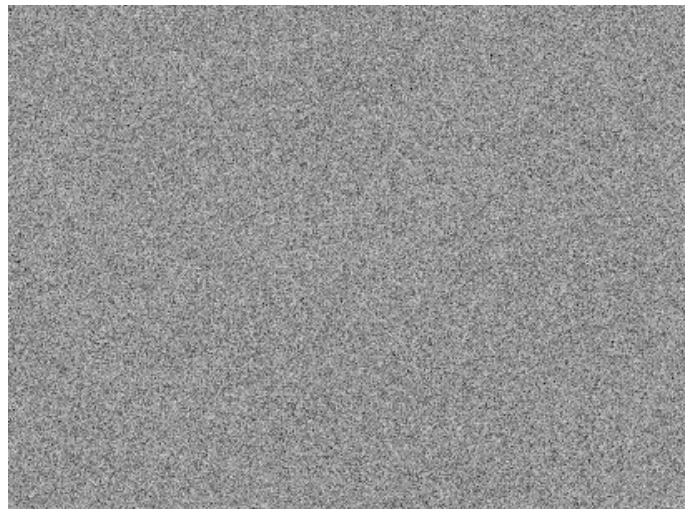
Look for "universal" priors that are task-independent to some extent.

Task-independent priors must come with the data.

Structure as a strong prior

Key insight: Data often carries structural priors in terms of repeating patterns, compositionality, locality, ...



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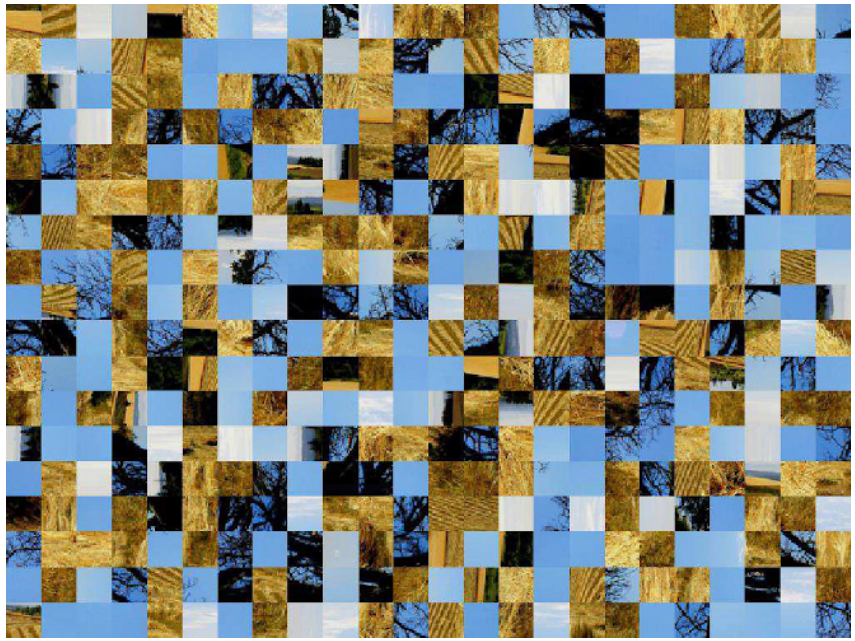

Son et al, "Solving Small-piece Jigsaw Puzzles by Growing Consensus", CVPR 2016

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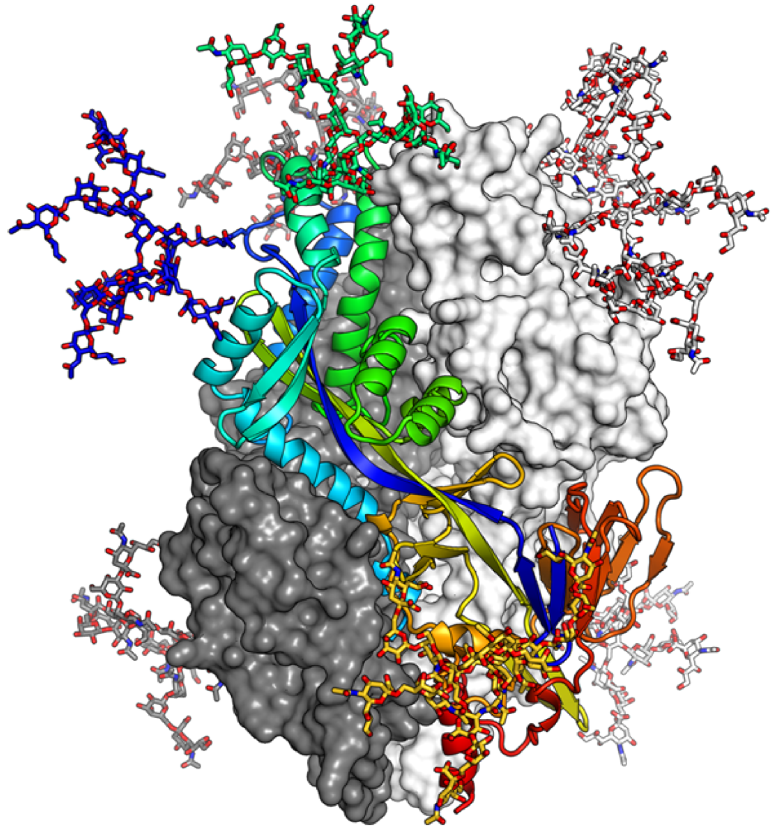

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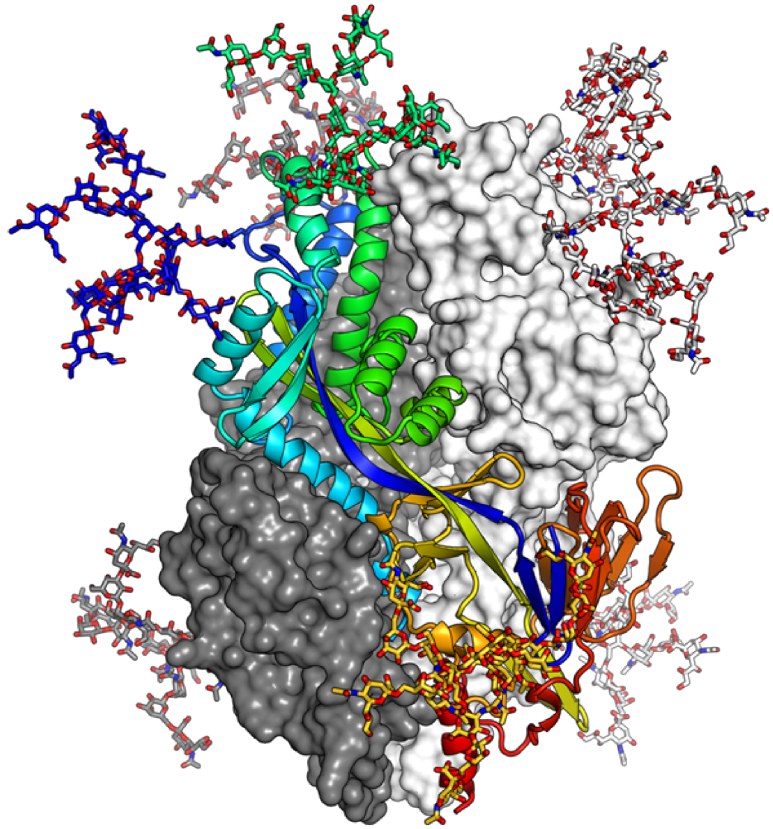
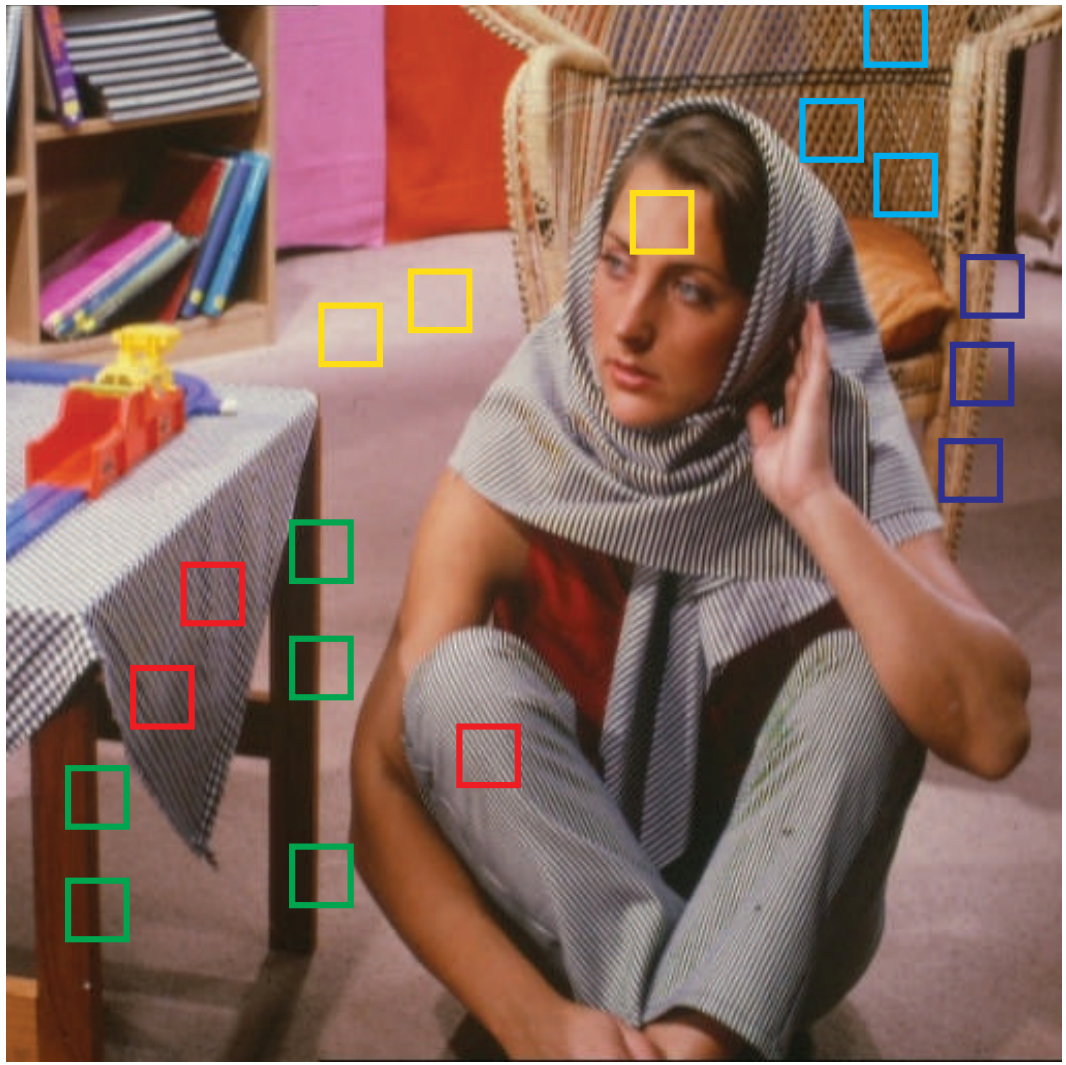


Take advantage of the data structure.

Self-similarity

Data tends to be self-similar across the domain:



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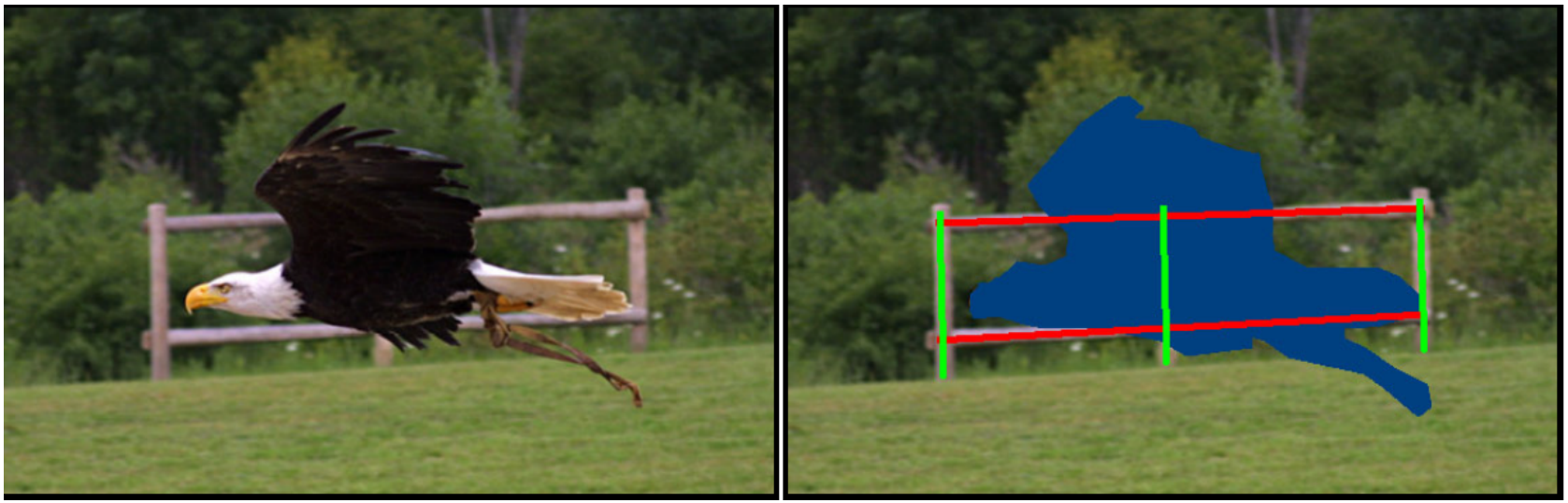

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Translations do not change the image content.

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Define the (linear!) translation operator $\mathcal{T}$ along vector $v \in \mathbb{R}^2$ as:

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Therefore, it is desirable to enforce translation invariance:

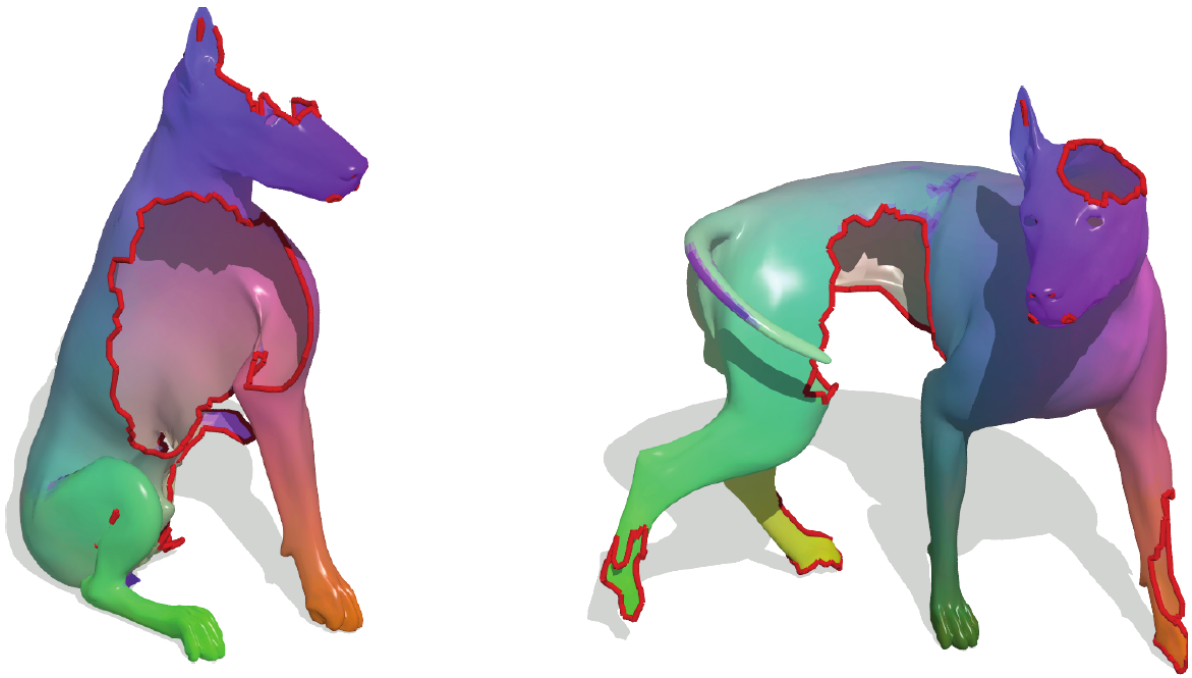
$$\mathcal{C}(\mathcal{T}_v f) = \mathcal{C}(f) \quad \forall f, \mathcal{T}_v$$

where $\mathcal{C}$ is a classification functional.

Deformation invariance

Other types of invariance are possible.

Invariance to partiality and isometric deformations:

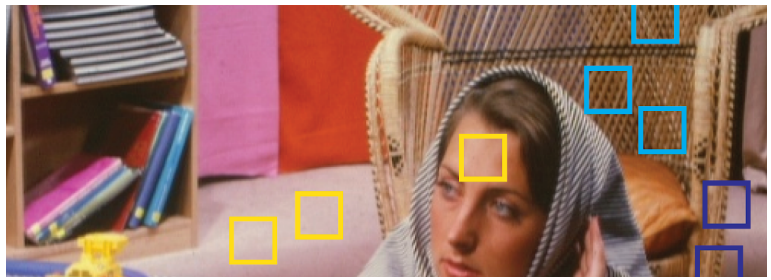


In many cases, invariance can be directly injected into the network.

Today we concentrate on translation invariance.

Hierarchy and compositionality

Translation invariance is desirable across multiple scales:



We expect local features to be invariant to their location in the image:

$$z(\mathcal{T}_v p) = z(p) \quad \forall p, \mathcal{T}_v$$

where p are image patches of variable size.

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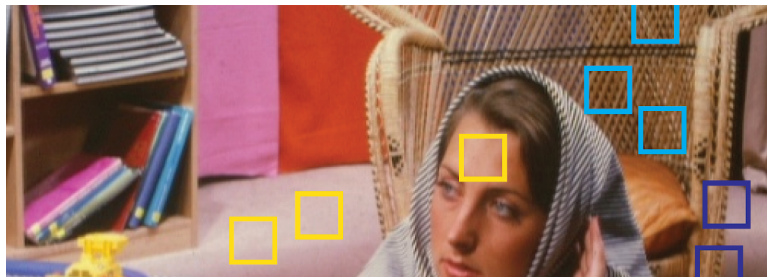

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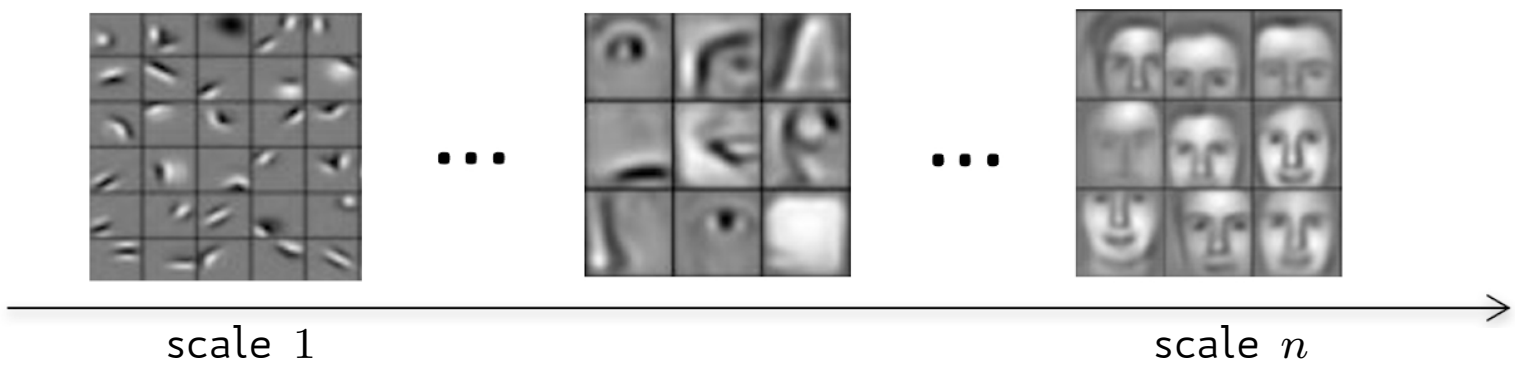

Convolutional neural networks (CNN)

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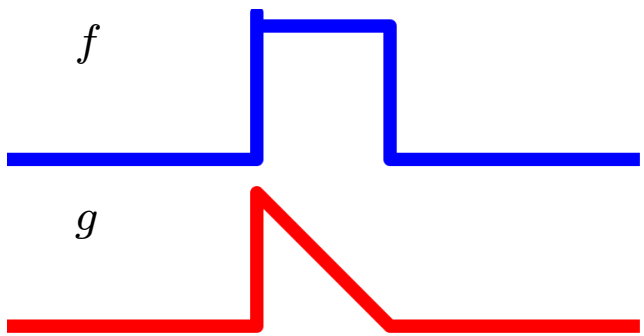
CNNs directly exploit this fact as a prior.

Convolution

Given two functions $f, g : [-\pi, \pi] \to \mathbb{R}$ their convolution is a function:

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(t)g(x-t)dt$$



d'Alembert 1754

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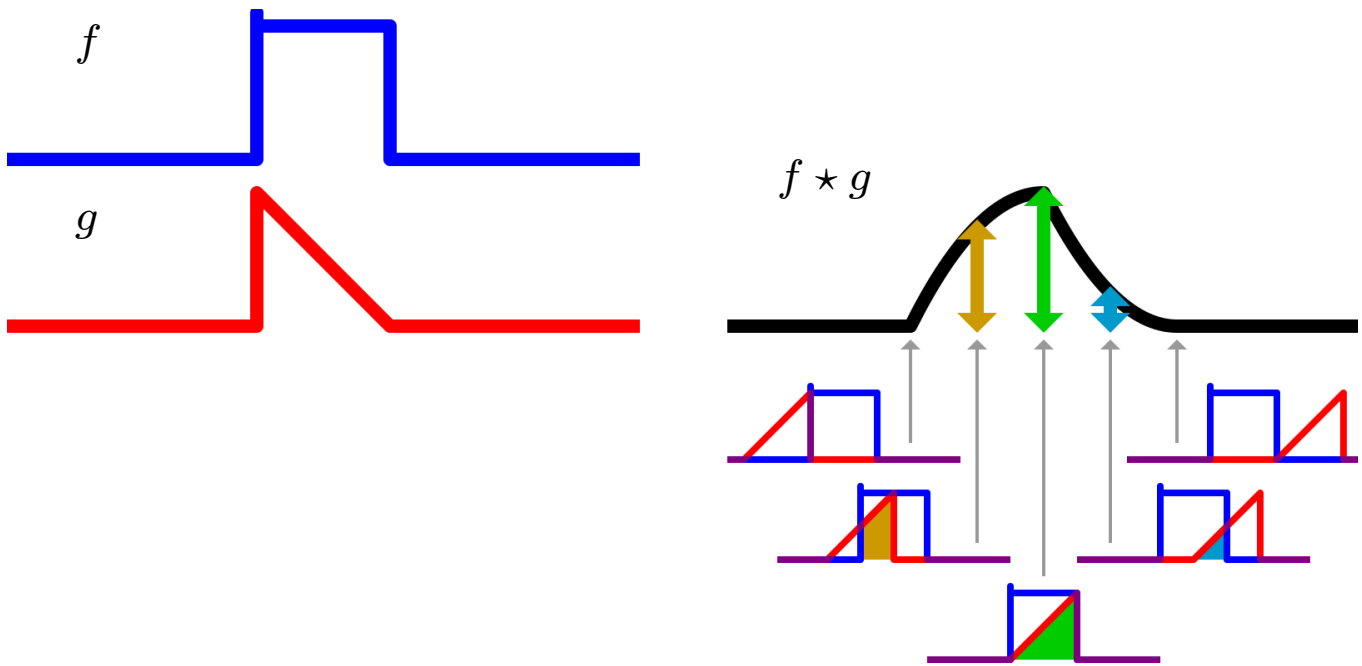

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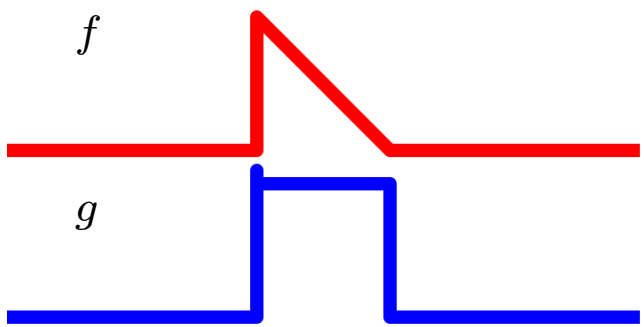

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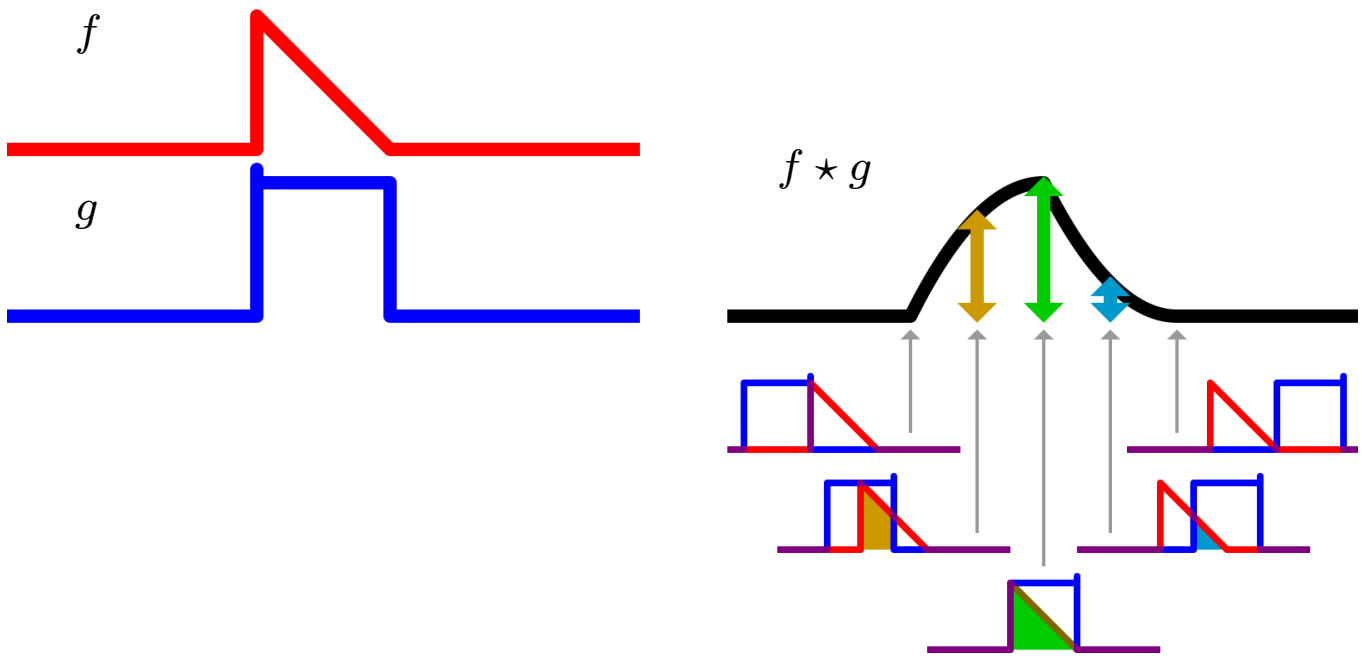

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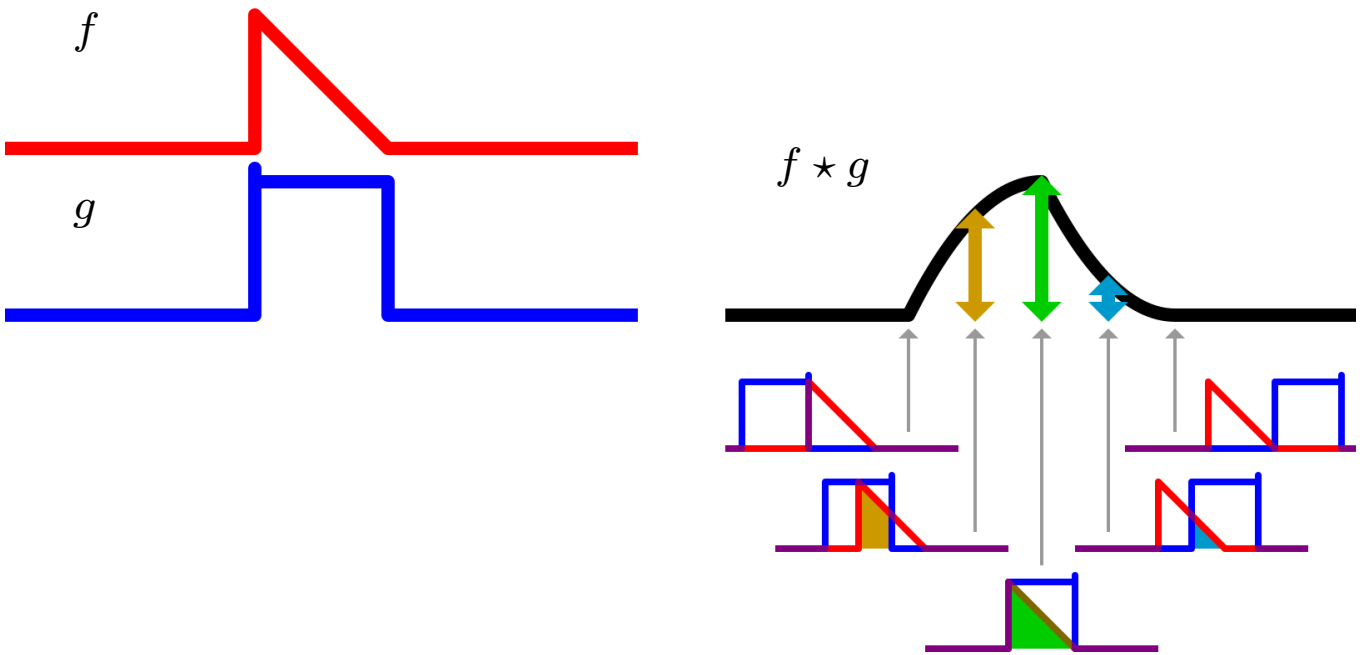
Given two functions $f, g : [-\pi, \pi] \to \mathbb{R}$ their convolution is a function:

$$\underbrace{(f \star g)(x)}_{\text{feature map}} = \int_{-\pi}^{\pi} f(t)\, \underbrace{g(x-t)}_{\text{filter}}\, dt$$



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Convolution: Commutativity

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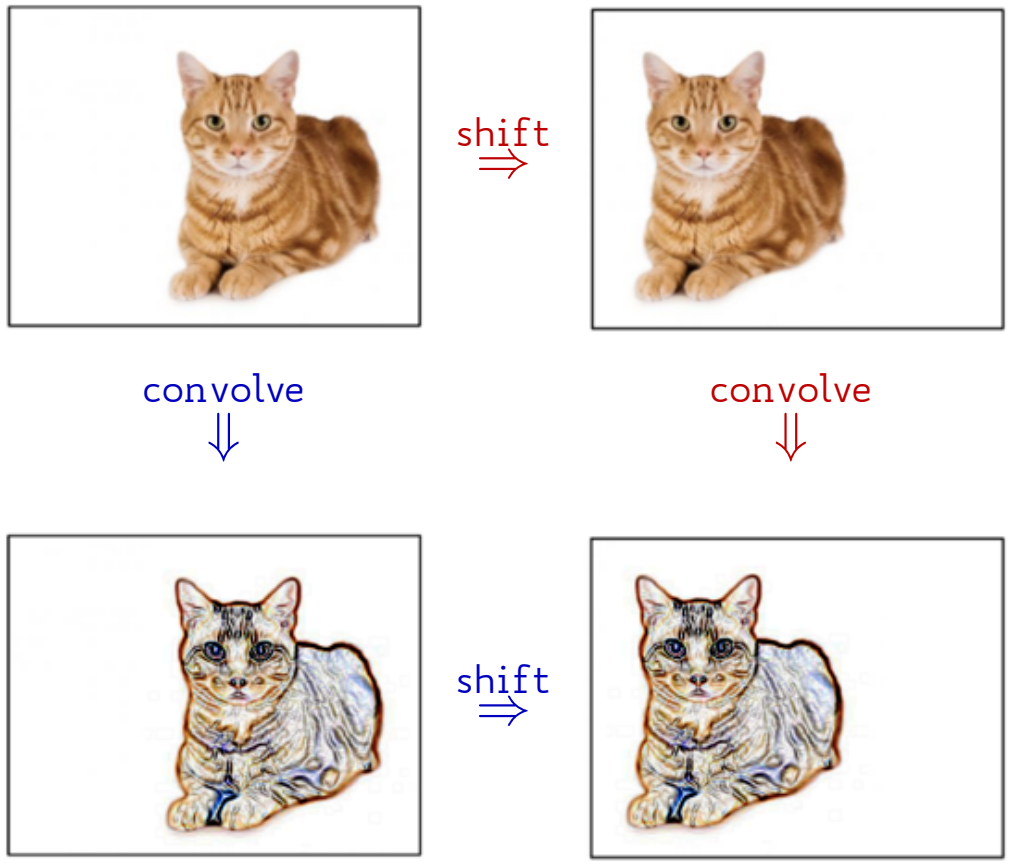
Further, convolution is shift-equivariant (or translation-equivariant):

$$f(x - x_0) \star g(x) = (f \star g)(x - x_0)$$

Convolution: Shift-equivariance



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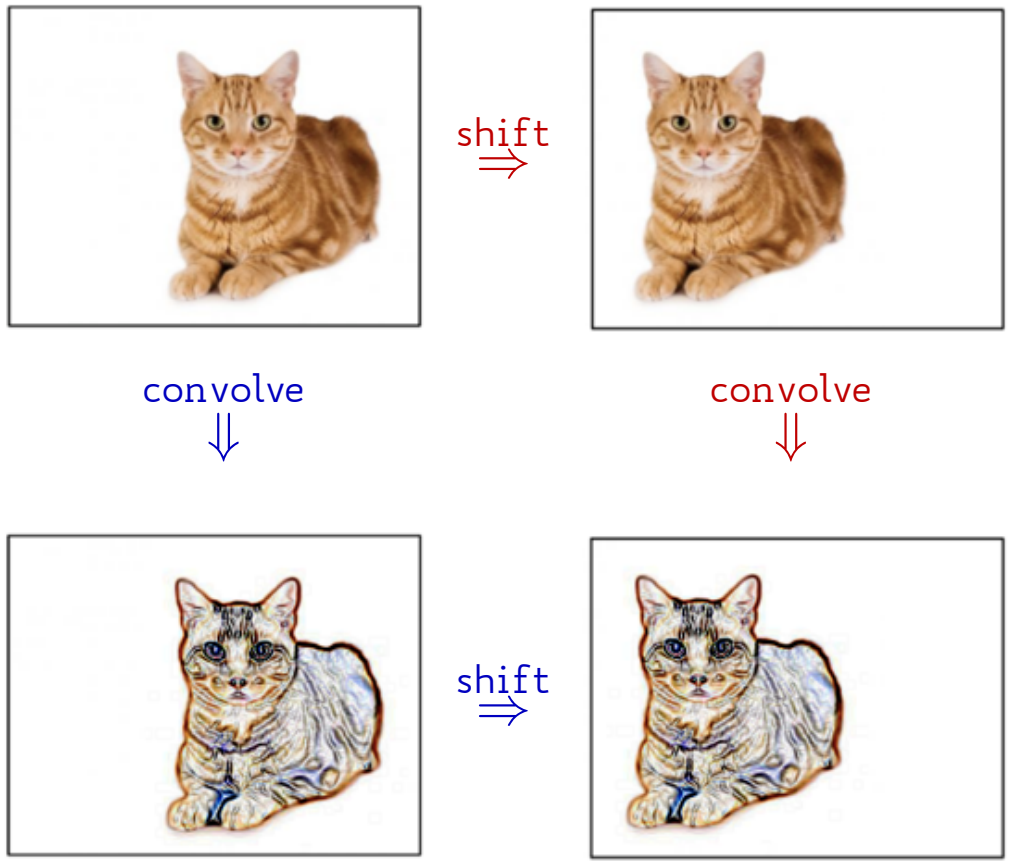

In fact, equivariance is a defining property of convolutions.

(any linear operator that is shift-equivariant, is a convolution)

Convolution: Linearity

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Translation equivariance can then be phrased as:

$$\mathcal{G}(\mathcal{T}f) = \mathcal{T}(\mathcal{G}f)$$

i.e., the convolution and translation operators commute.

Discrete convolution

In the discrete setting, we deal with vectors $\mathbf{f}, \mathbf{g}$.

We define the convolution sum:

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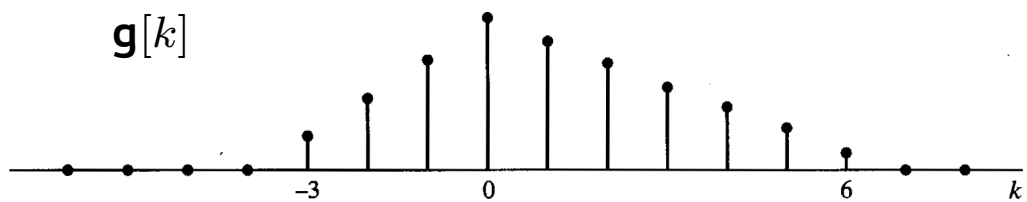

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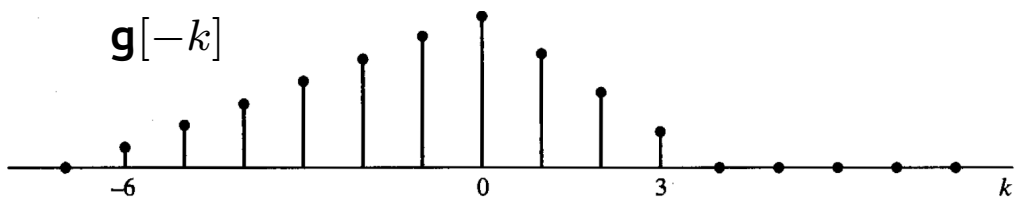

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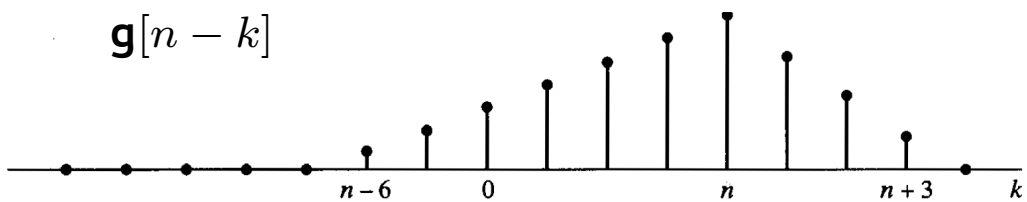

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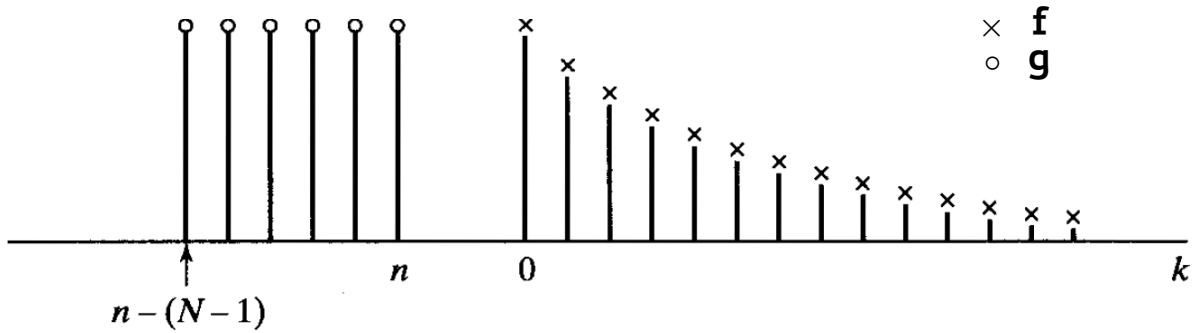

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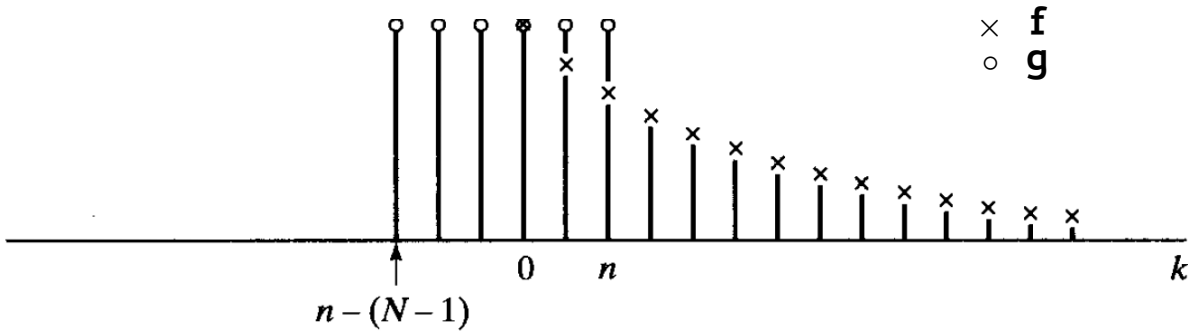

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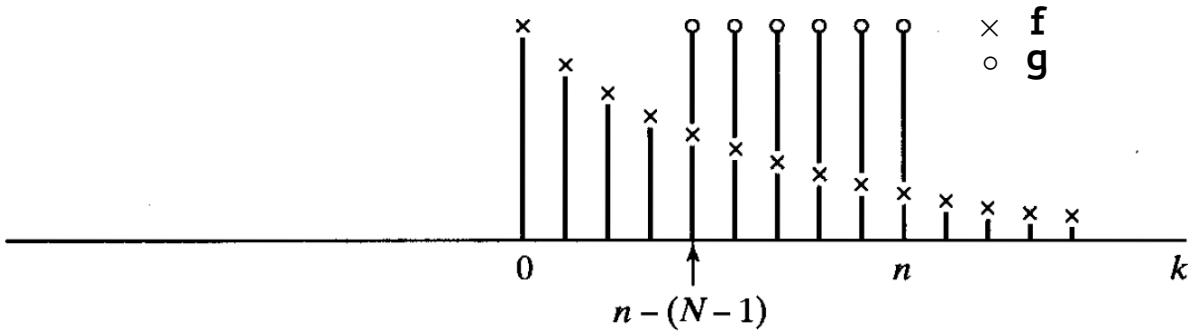

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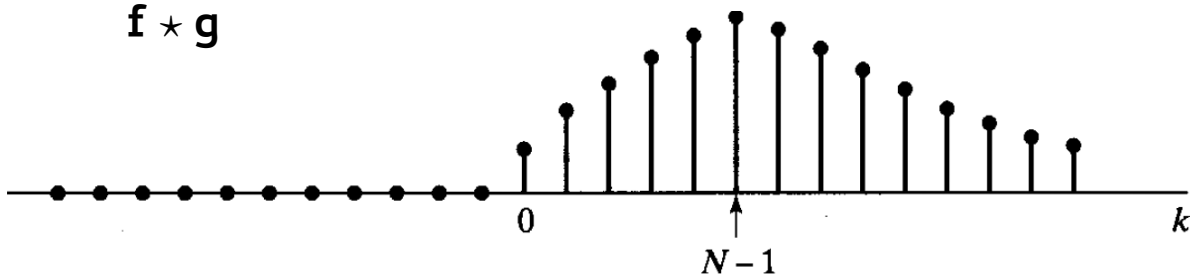
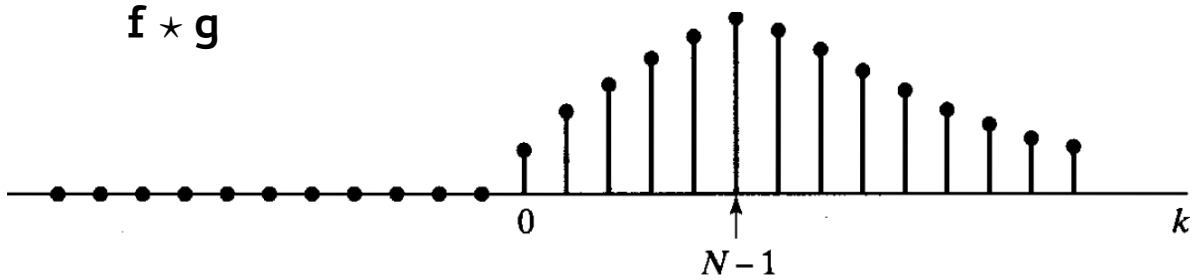


Discrete convolution

In the discrete setting, we deal with vectors $\mathbf{f}, \mathbf{g}$.

We define the convolution sum:

$$(\mathbf{f} \star \mathbf{g})[n] = \sum_{k=-\infty}^{\infty} \mathbf{f}[k]\mathbf{g}[n-k]$$



The specific discretization depends on the boundary conditions.

In the example above, $\mathbf{f}$ was zero-padded in order for the products to be well defined for all shifts.

Discrete convolution

On 2D domains (e.g. RGB images $f : \mathbb{R}^2 \to \mathbb{R}^3$), for each channel:

$$(\mathbf{f} \star \mathbf{g})[m, n] = \sum_k \sum_\ell \mathbf{f}[k, \ell]\, \mathbf{g}[m - k, n - \ell]$$

Discrete convolution

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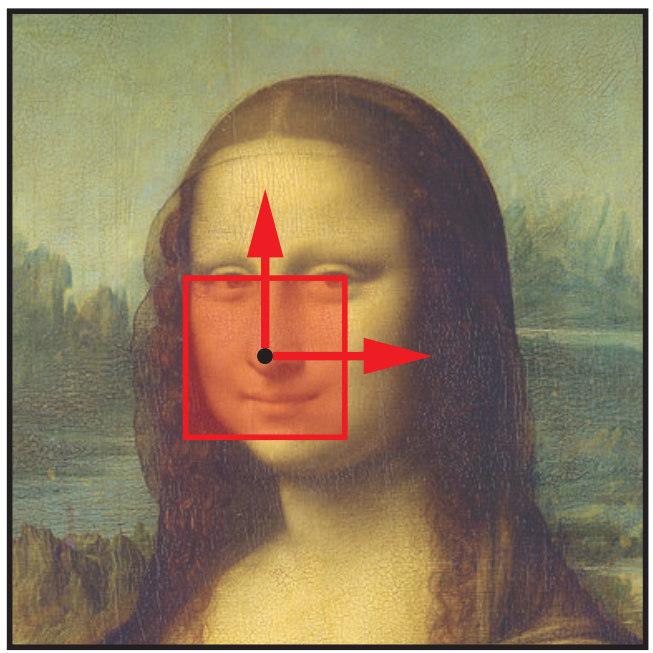
We get the classical interpretation in terms of a moving window:



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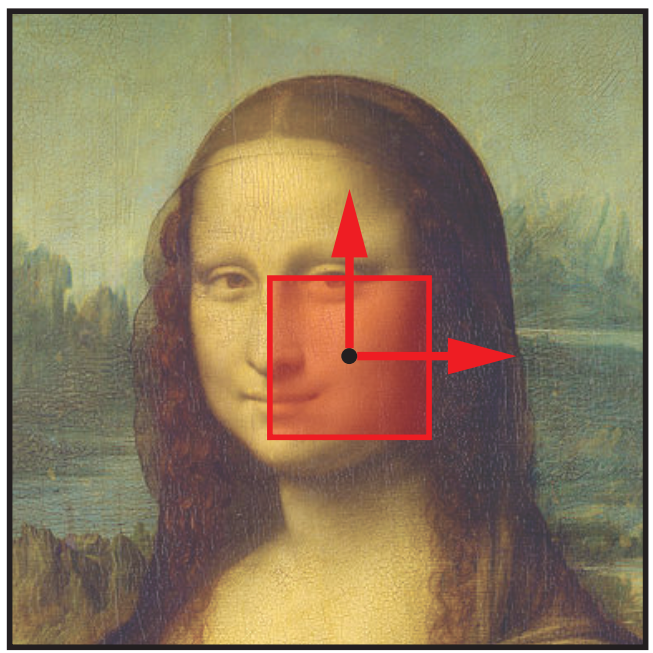
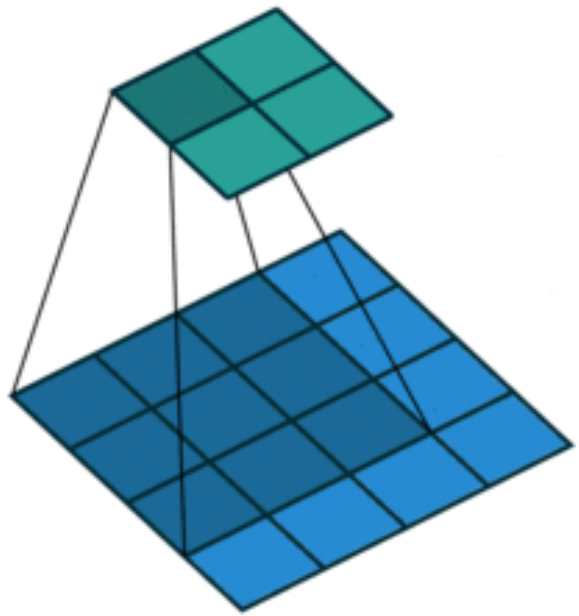


Boundary conditions and stride

No padding: The convolution kernel is directly applied within the boundaries of the underlying function (an image in this example).

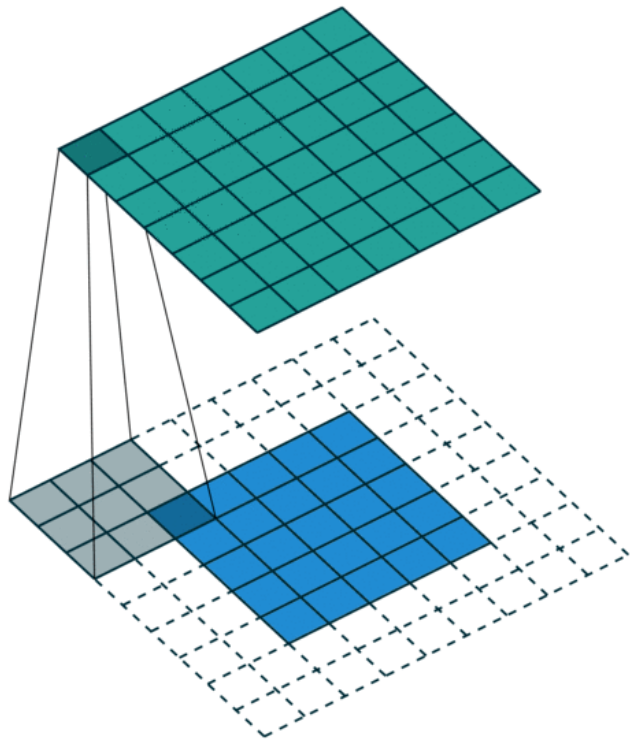


The result of the convolution is a smaller image.

Animations: Vincent Dumoulin

Boundary conditions and stride

Full zero-padding: The domain is enlarged and padded with zeroes. The convolution kernel is applied within the (now larger) boundaries.

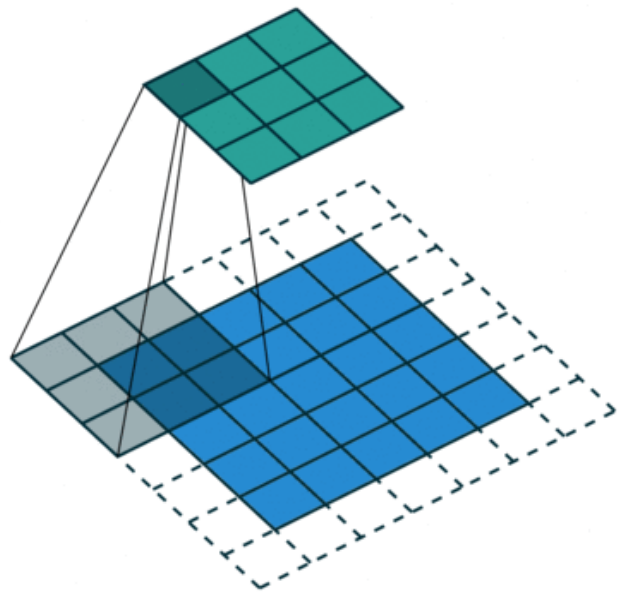


The result of the convolution is a larger image.

Animations: Vincent Dumoulin

Boundary conditions and stride

Arbitrary zero-padding, with stride: The domain is enlarged and padded with zeroes, but not enough to capture the boundary pixels. Further, each discrete step skips one pixel.



The result is the same as no stride followed by downsampling.

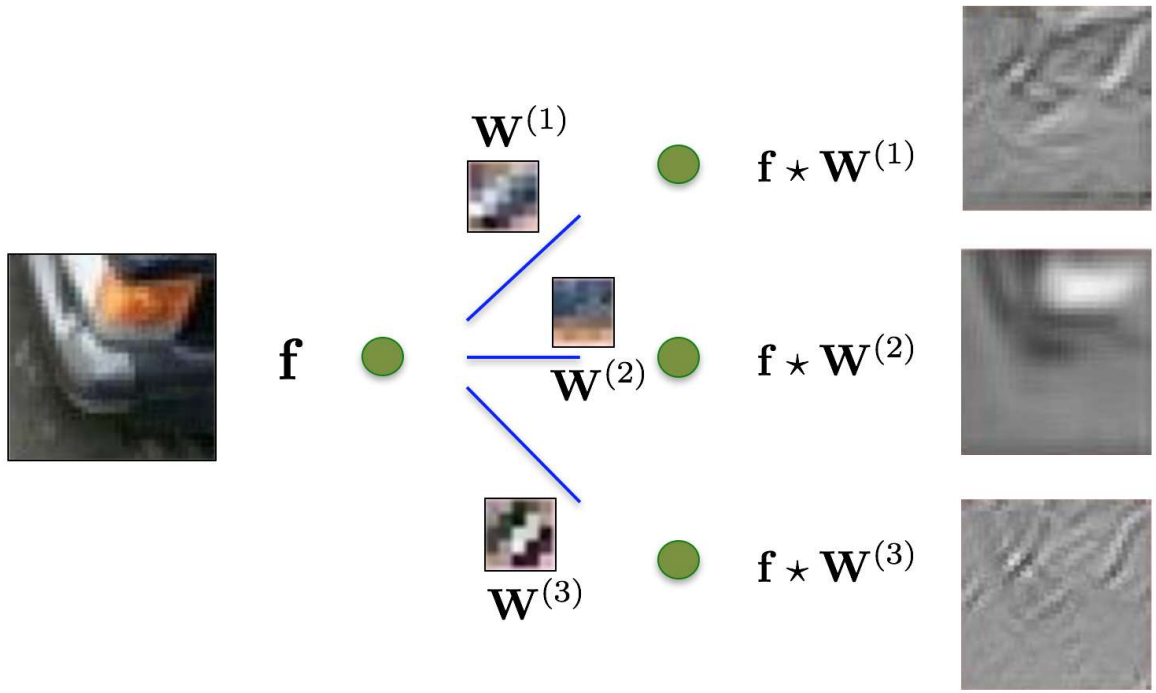
Animations: Vincent Dumoulin

CNN vs. MLP

Replace the large matrices of MLPs with small local filters.

CNN vs. MLP

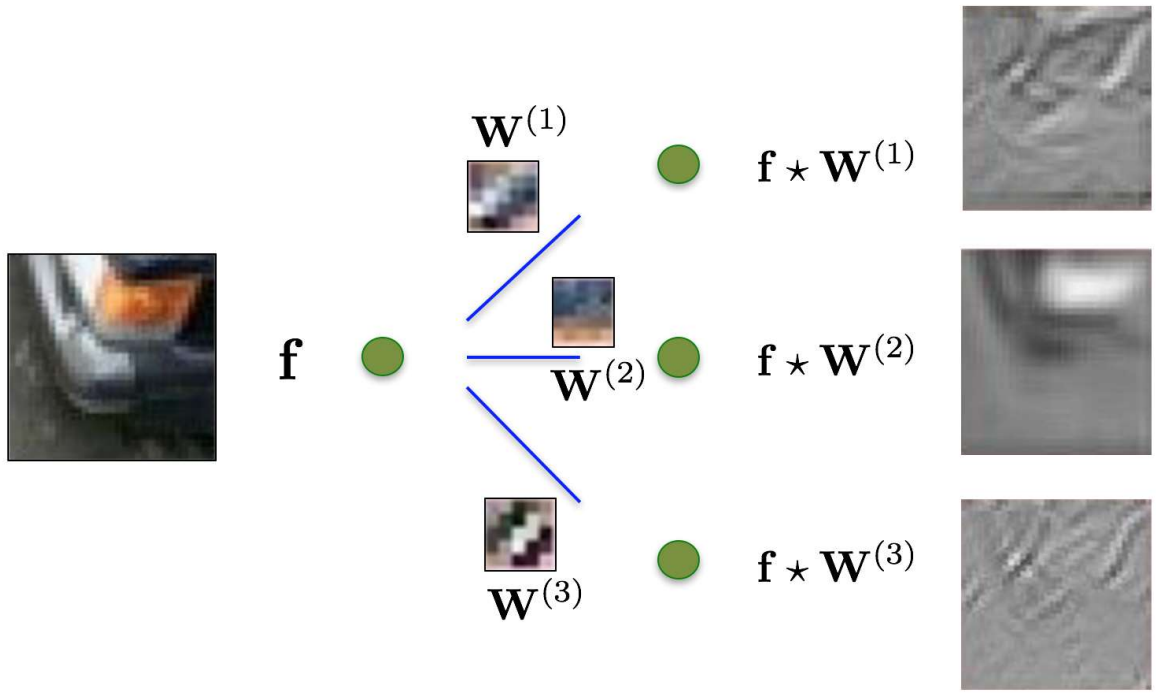
Replace the large matrices of MLPs with small local filters.



- $O(1)$ parameters per filter; huge gain compared to MLP.

CNN vs. MLP

Replace the large matrices of MLPs with small local filters.



- $O(1)$ parameters per filter; huge gain compared to MLP.
- $\mathbf{W}^{(1)}$ is the same for the entire image $\Rightarrow$ weight sharing; exploits self-similarity and shift-equivariance.

Sparse interactions

Fully-connected layer:

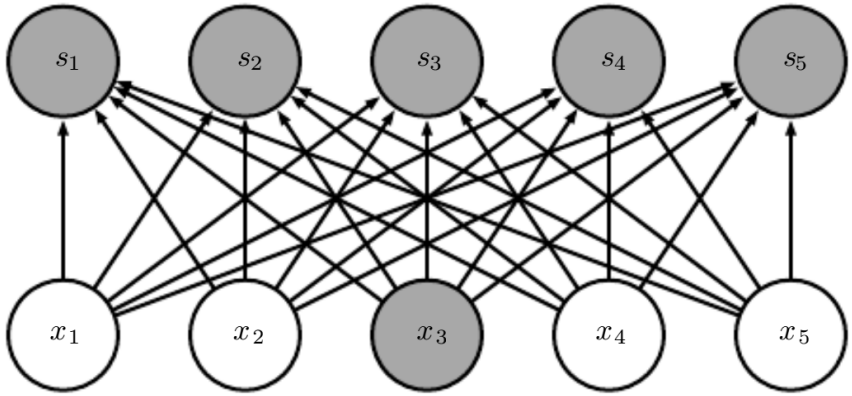


Image: Goodfellow et al, 2016

Sparse interactions

Fully-connected layer:

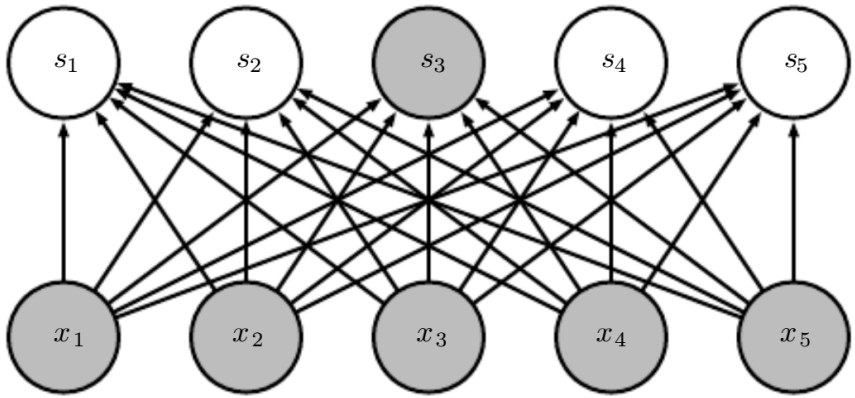


Image: Goodfellow et al, 2016

Sparse interactions

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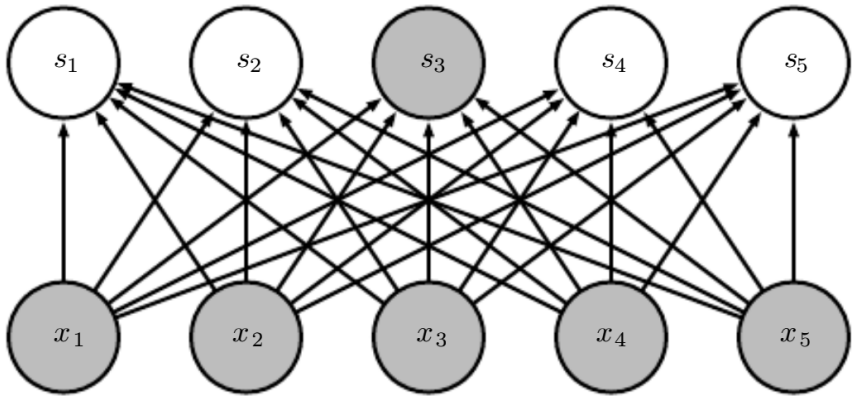

each edge is a different weight

Image: Goodfellow et al, 2016

Sparse interactions

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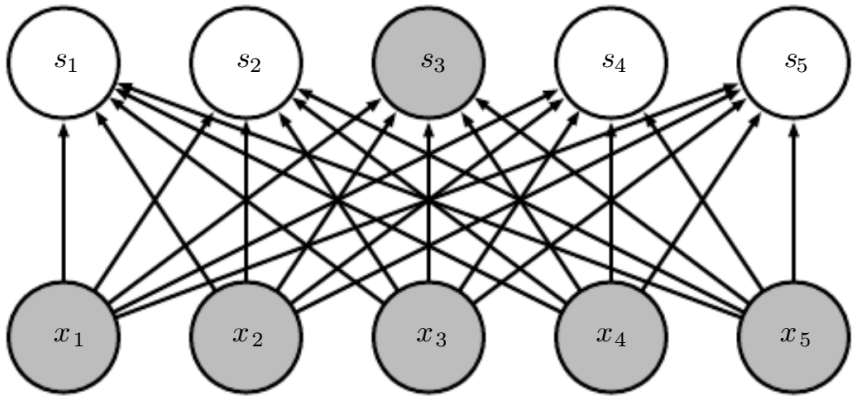

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Convolutional layer:

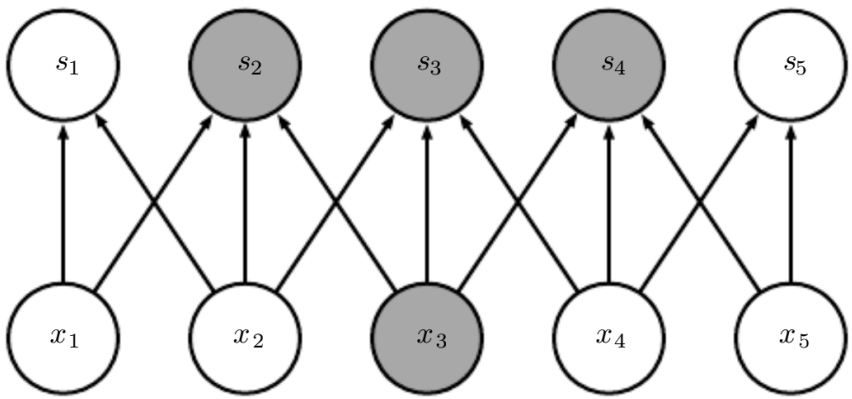
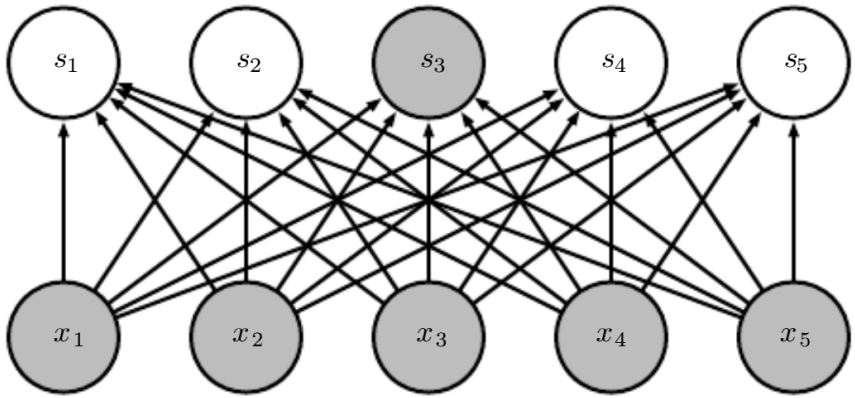


Image: Goodfellow et al, 2016

Sparse interactions

Fully-connected layer:



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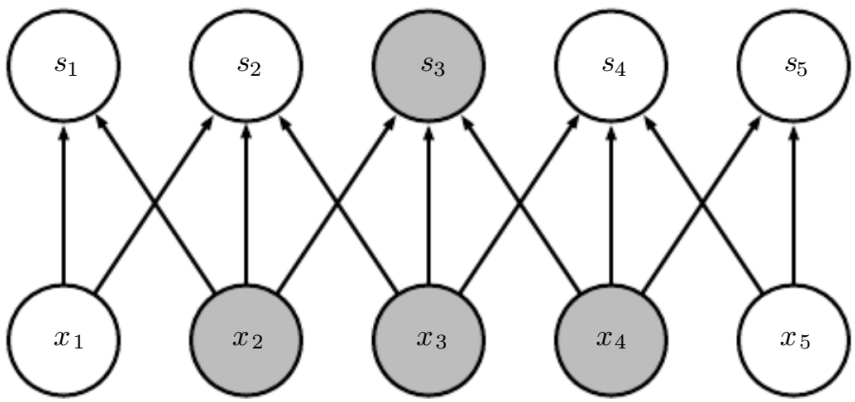
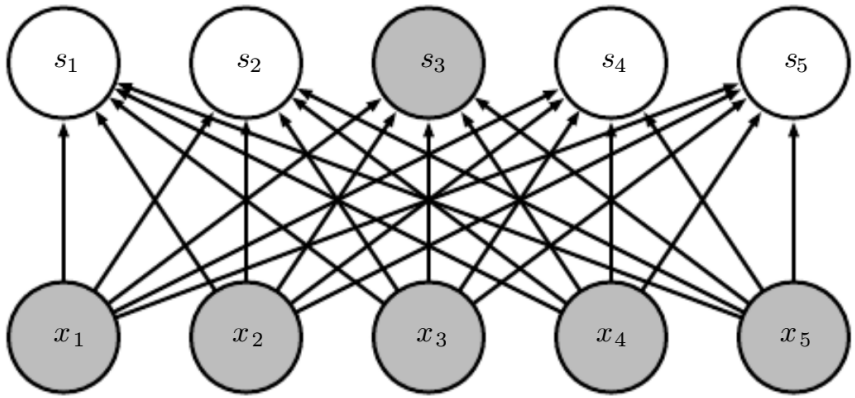


Image: Goodfellow et al, 2016

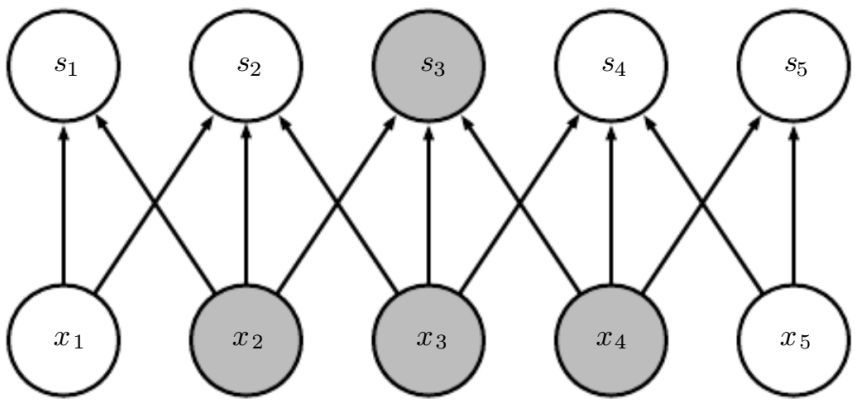
Sparse interactions

Fully-connected layer:



each edge is a different weight

Convolutional layer:

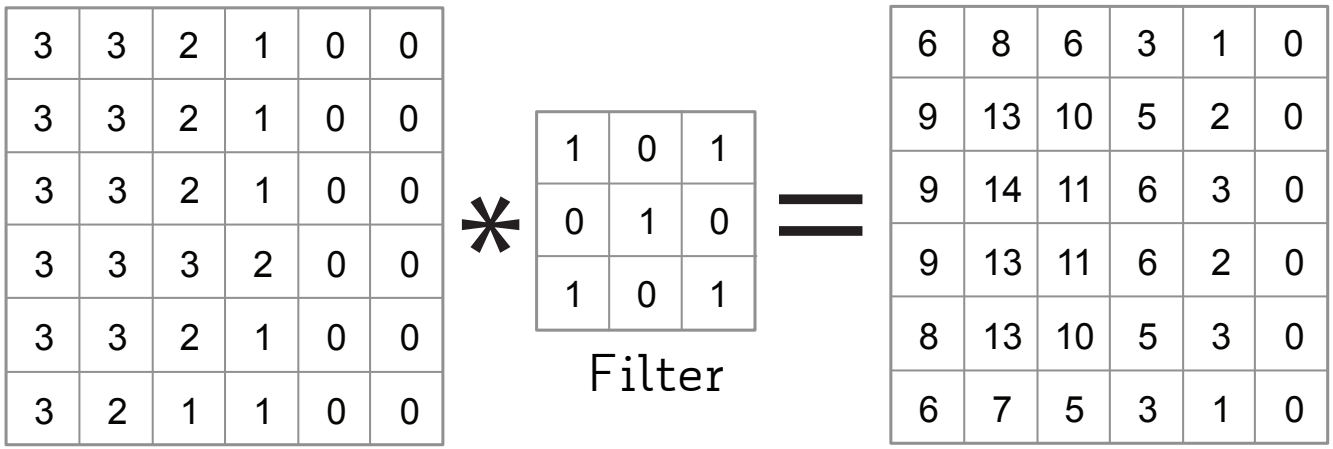


the outgoing edges have the same weights
for each input variable (weight sharing)

Image: Goodfellow et al, 2016

Pooling

Reduces the spatial dimensions of the input for the next convolutional layer.



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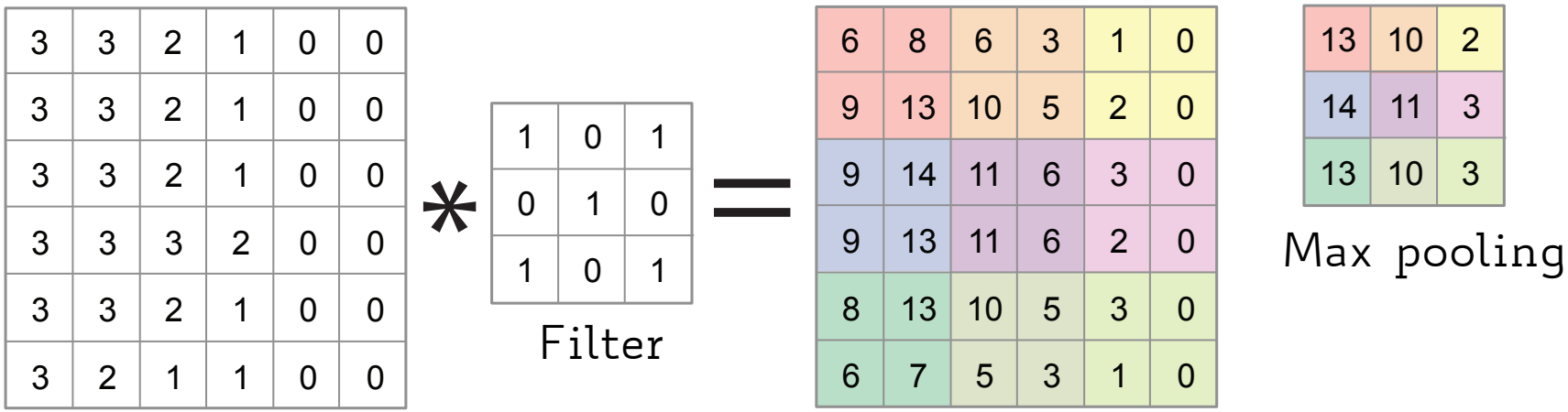


Pooling

Reduces the spatial dimensions of the input for the next convolutional layer.

3	3	2	1	0	0
3	3	2	1	0	0
3	3	2	1	0	0
3	3	3	2	0	0
3	3	2	1	0	0
3	2	1	1	0	0

Input data

$*$

1	0	1
0	1	0
1	0	1

Filter

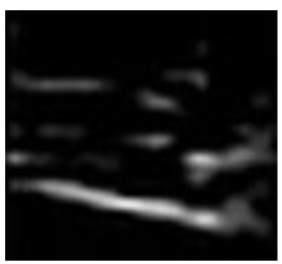
$=$

6	8	6	3	1	0
9	13	10	5	2	0
9	14	11	6	3	0
9	13	11	6	2	0
8	13	10	5	3	0
6	7	5	3	1	0

Feature map

13	10	2
14	11	3
13	10	3

Max pooling

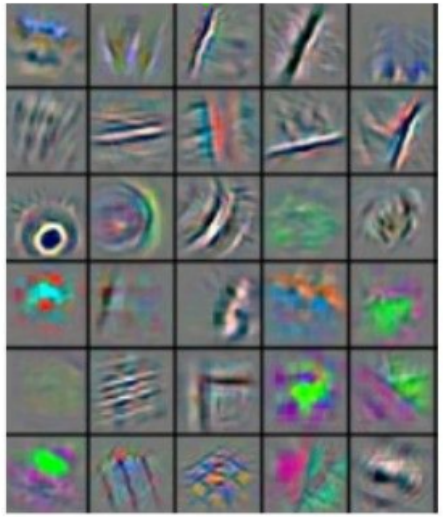


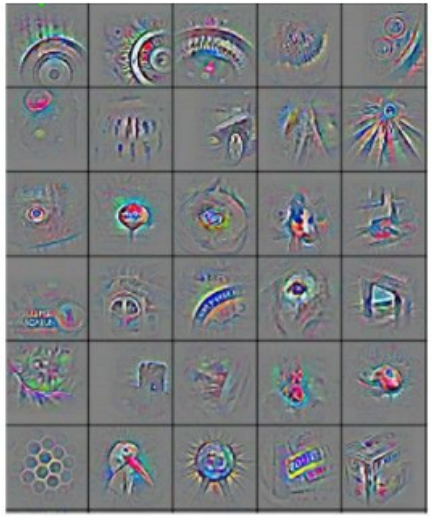
2x2 Max pooling

Captures non-local interactions via simple building blocks.

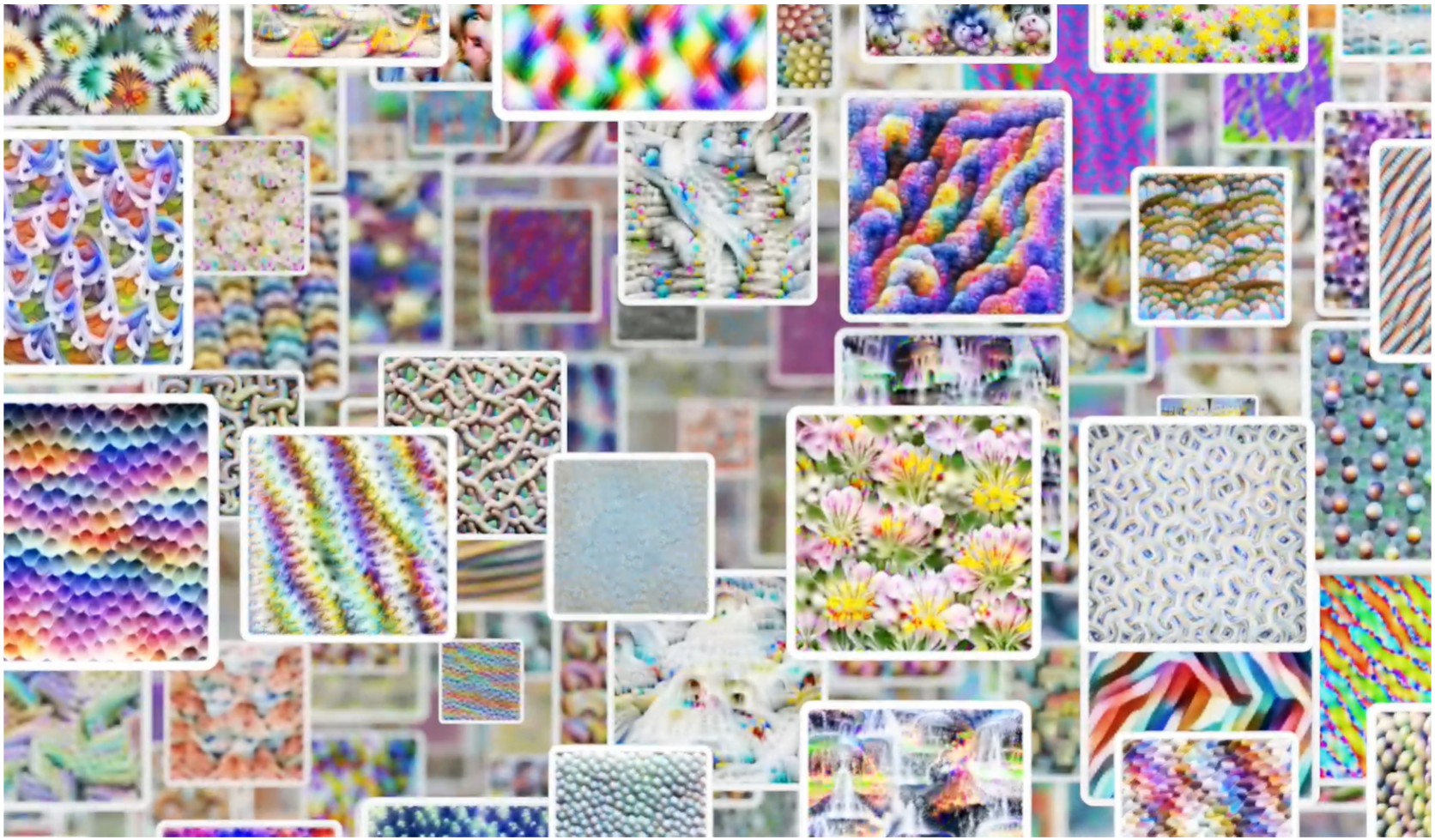
Learned features

$\rightarrow$ low-level features $\rightarrow$ mid-level features $\rightarrow$ hi-level features $\rightarrow$ "car"





Learned features



https://openai.com/blog/microscope/

Suggested reading

Convolution animations, including variants:
https://github.com/vdumoulin/conv_arithmetic

Seminal paper on CNN, seen as a set of feature detectors:
http://yann.lecun.com/exdb/publis/pdf/lecun-89e.pdf