

Deep Learning & Applied AI

Convolutional neural networks

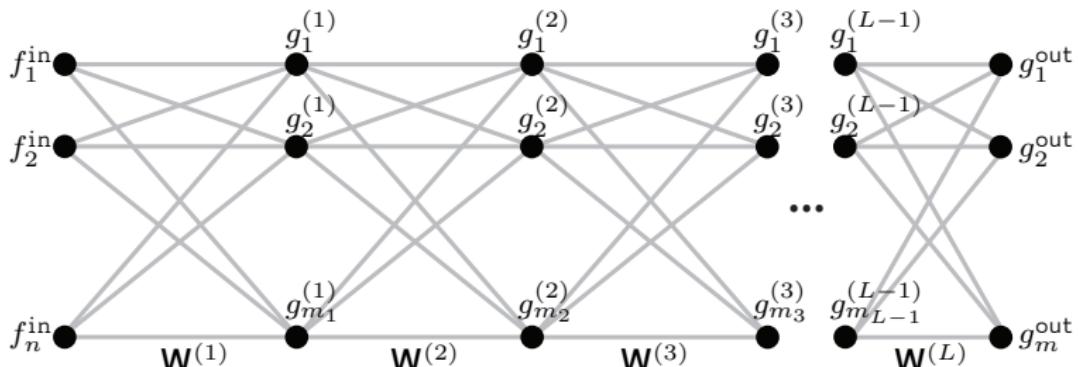
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SAPIENZA
UNIVERSITÀ DI ROMA

2nd semester a.y. 2024/2025 · April 7, 2025

Neural network (NN)



Deep neural network consisting of L layers

Net output

$$\mathbf{g}^{\text{out}} = \sigma(\dots \mathbf{W}^{(2)} \sigma(\mathbf{W}^{(1)} \mathbf{f}^{\text{in}}))$$

Activation, e.g. $\sigma(x) = \max\{x, 0\}$ ReLU

Parameters

weights of all layers $\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(L)}$

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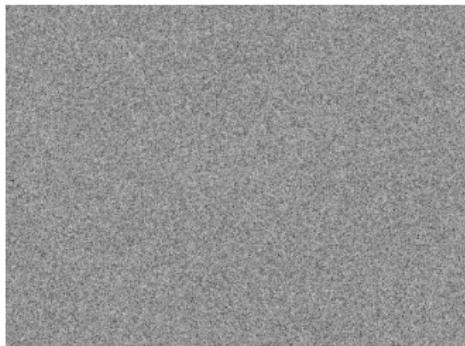
We need additional priors as a (partial) remedy to the above.

Look for "universal" priors that are task-independent to some extent.

Task-independent priors must come with the data.

Structure as a strong prior

Key insight: Data often carries **structural priors** in terms of repeating patterns, compositionality, locality, ...



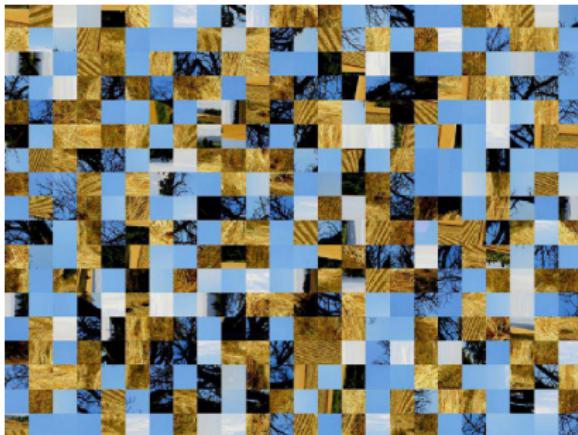
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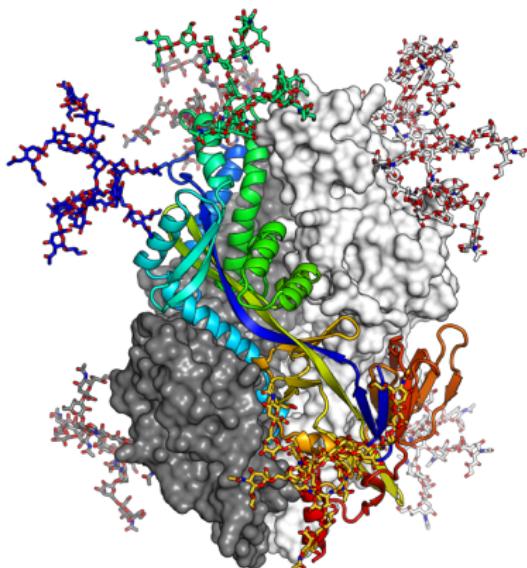
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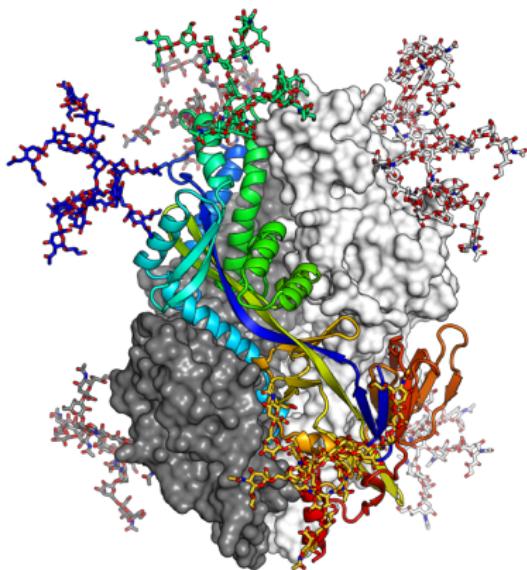
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Take advantage of the **data structure**.

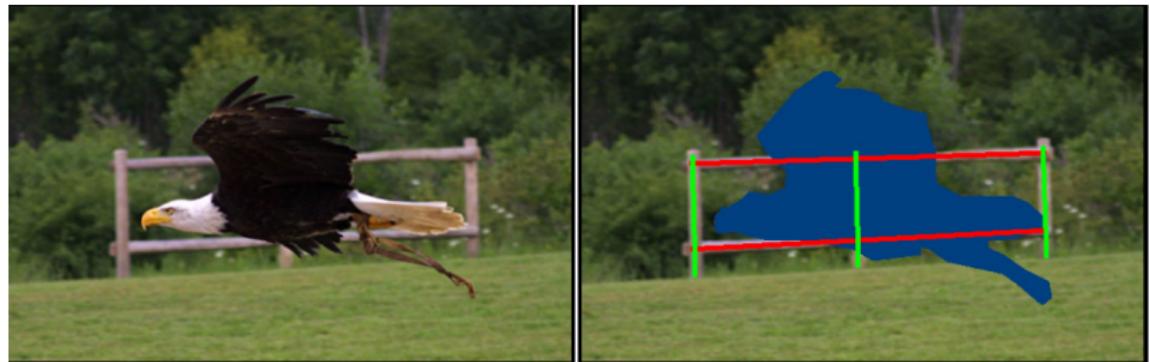
Self-similarity

Data tends to be **self-similar** across the domain:



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Translations do not change the image content.



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Define the (linear!) **translation operator** \mathcal{T} along vector $v \in \mathbb{R}^2$ as:

$$\mathcal{T}_v f(x) = f(x - v)$$



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Therefore, it is desirable to enforce **translation invariance**:

$$\mathcal{C}(\mathcal{T}_v f) = \mathcal{C}(f) \quad \forall f, \mathcal{T}_v$$

where \mathcal{C} is a classification functional.

Deformation invariance

Other types of invariance are possible.

Invariance to [partiality](#) and [isometric deformations](#):



In many cases, invariance can be directly injected into the network.

Today we concentrate on [translation](#) invariance.

Hierarchy and compositionality

Translation invariance is desirable across multiple scales:



We expect **local features** to be invariant to their location in the image:

$$z(\mathcal{T}_v p) = z(p) \quad \forall p, \mathcal{T}_v$$

where p are image patches of variable size.

Hierarchy and compositionality

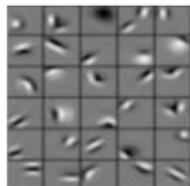
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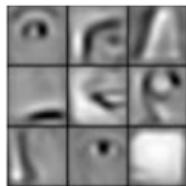
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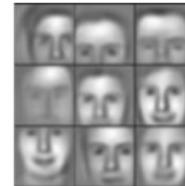
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scale 1

scale n

Convolutional neural networks (CNN)

Data is often composed of hierarchical, local, shift-invariant patterns.

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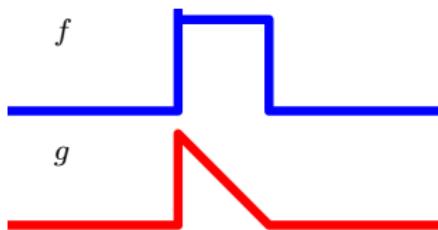
Data is often composed of hierarchical, local, shift-invariant patterns.

CNNs directly exploit this fact as a prior.

Convolution

Given two functions $f, g : [-\pi, \pi] \rightarrow \mathbb{R}$ their **convolution** is a function:

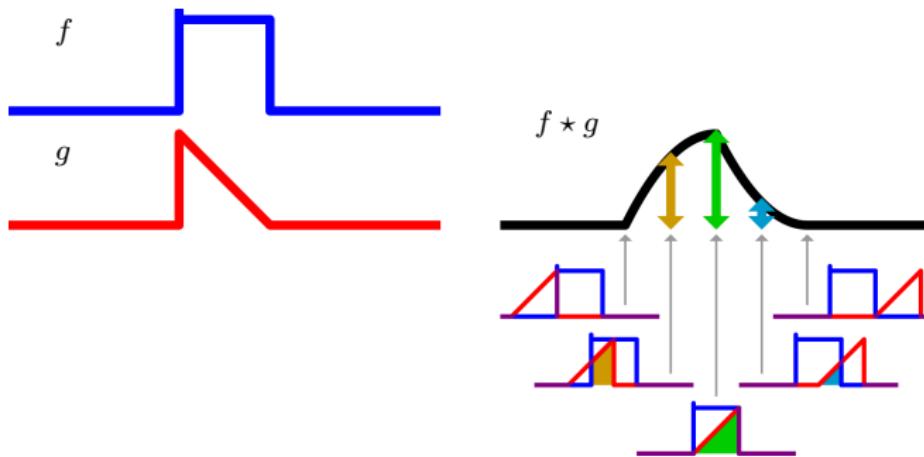
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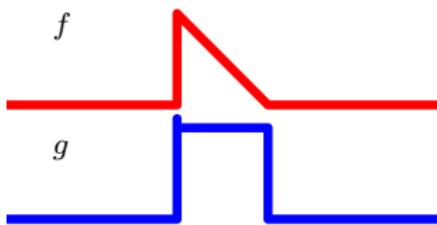
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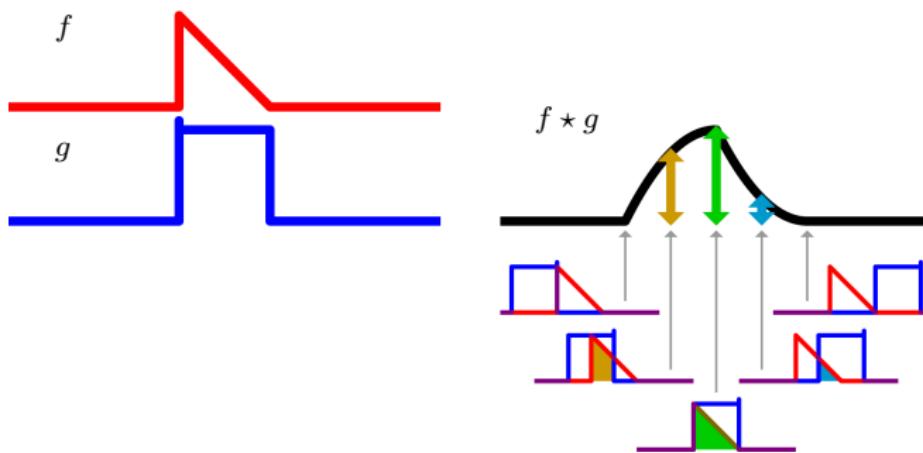
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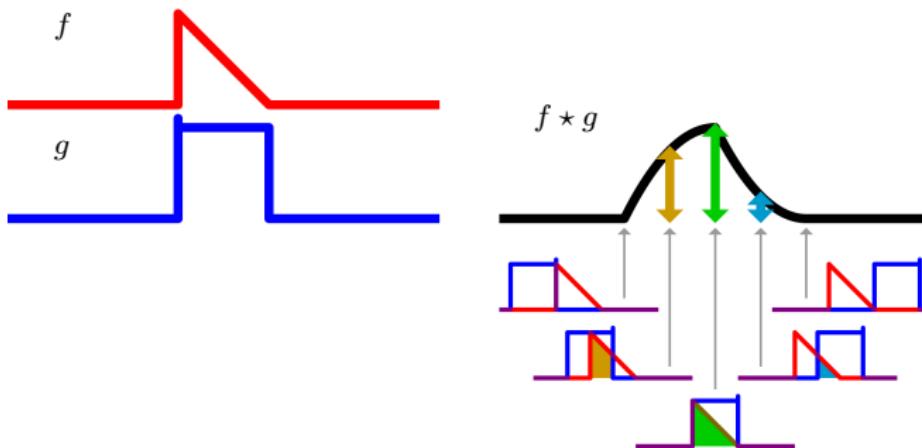
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Further, convolution is **shift-equivariant** (or **translation-equivariant**):

$$f(x - x_0) \star g(x) = (f \star g)(x - x_0)$$

Convolution: Shift-equivariance



shift
⇒



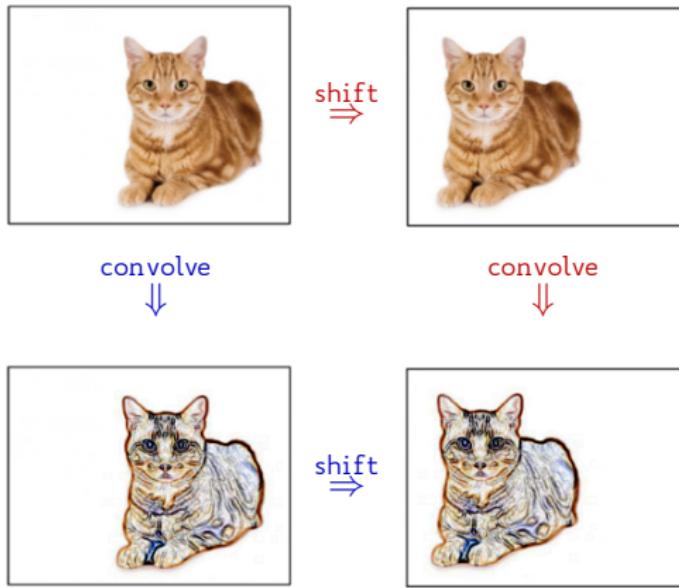
convolve
↓



shift
⇒



Convolution: Shift-equivariance



In fact, equivariance is a **defining property** of convolutions.
(any linear operator that is shift-equivariant, is a convolution)

Convolution: Linearity

We can see convolution as the application of a [linear operator](#) \mathcal{G} :

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Translation **equivariance** can then be phrased as:

$$\mathcal{G}(\mathcal{T}f) = \mathcal{T}(\mathcal{G}f)$$

i.e., the convolution and translation operators **commute**.

Discrete convolution

In the **discrete** setting, we deal with vectors \mathbf{f}, \mathbf{g} .

We define the **convolution sum**:

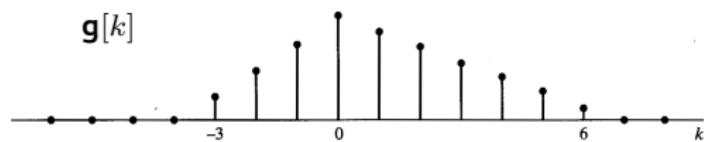
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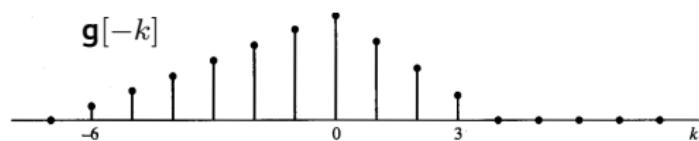


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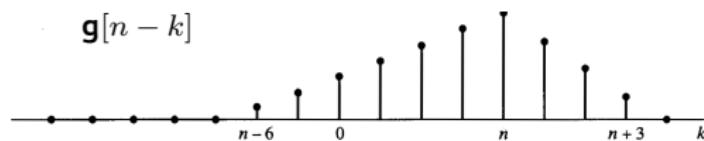


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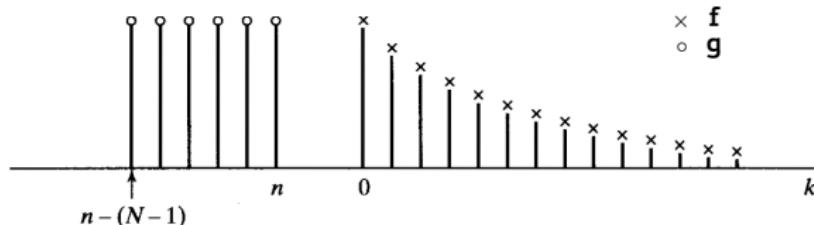


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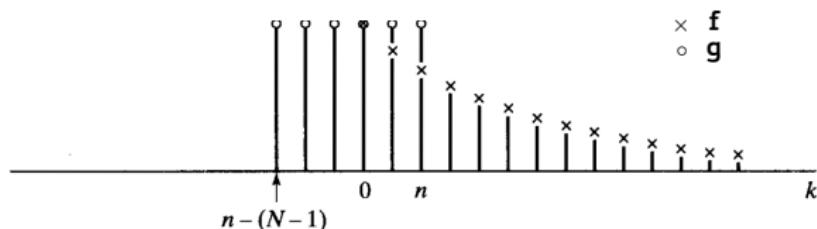


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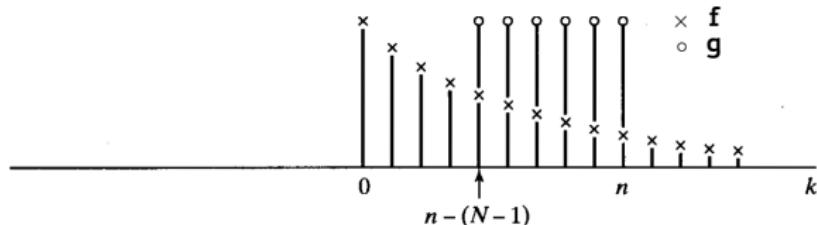


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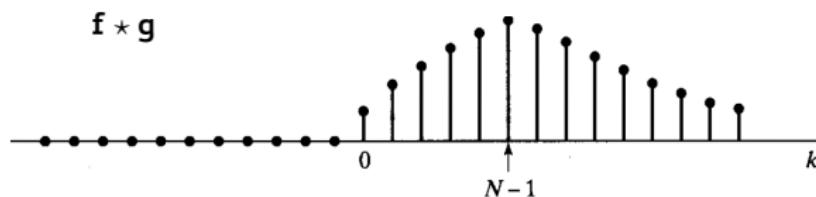


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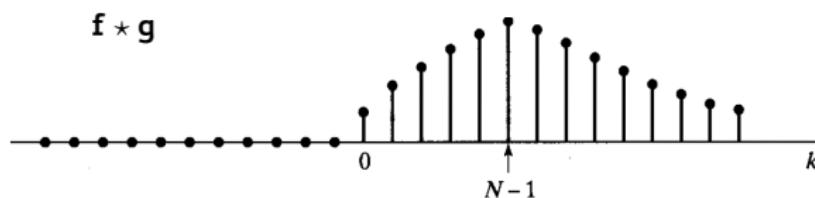


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The specific discretization depends on the **boundary conditions**.

In the example above, \mathbf{f} was **zero-padded** in order for the products to be well defined for all shifts.

Discrete convolution

On 2D domains (e.g. RGB images $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$), for each channel:

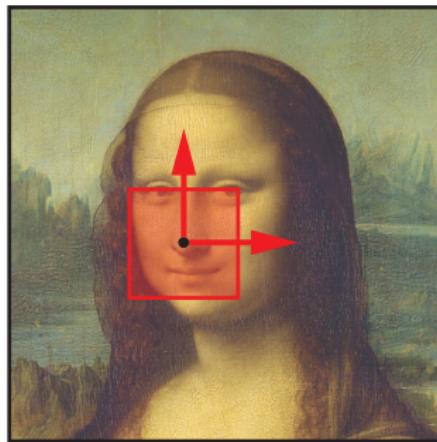
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We get the classical interpretation in terms of a moving window:

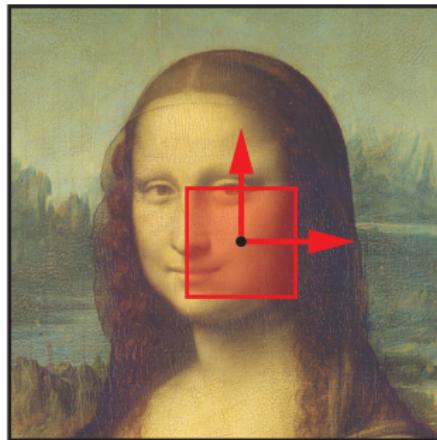


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We get the classical interpretation in terms of a moving window:



Boundary conditions and stride

No padding: The convolution kernel is directly applied within the boundaries of the underlying function (an image in this example).

The result of the convolution is a smaller image.

Boundary conditions and stride

Full zero-padding: The domain is enlarged and padded with zeroes. The convolution kernel is applied within the (now larger) boundaries.

The result of the convolution is a larger image.

Animations: Vincent Dumoulin

Boundary conditions and stride

Arbitrary zero-padding, with stride: The domain is enlarged and padded with zeroes, but not enough to capture the boundary pixels. Further, each discrete step skips one pixel.

The result is the same as no stride followed by downsampling.

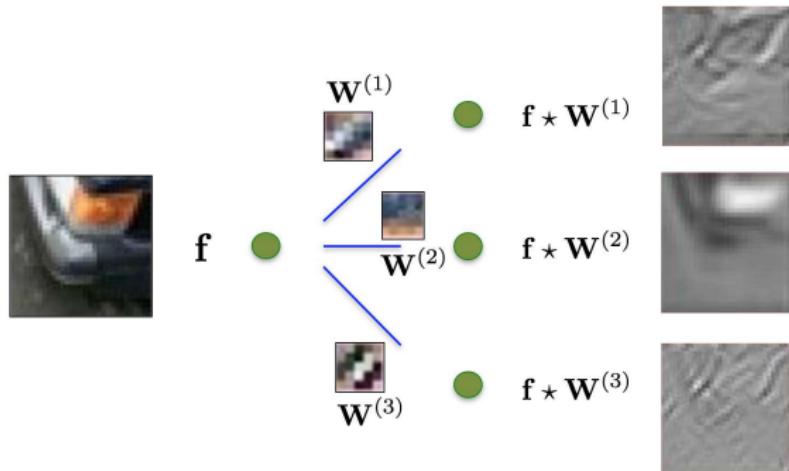
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CNN vs. MLP

Replace the large matrices of MLPs with small local filters.

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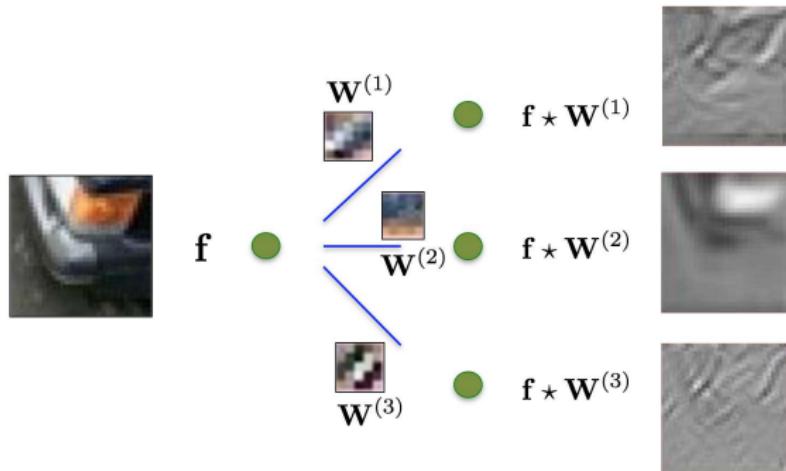
Replace the large matrices of MLPs with small **local filters**.



- $O(1)$ parameters per filter; huge gain compared to MLP.

CNN vs. MLP

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- $O(1)$ parameters per filter; huge gain compared to MLP.
- $\mathbf{W}^{(1)}$ is the same for the entire image \Rightarrow weight sharing; exploits self-similarity and shift-equivariance.

Sparse interactions

Fully-connected layer:

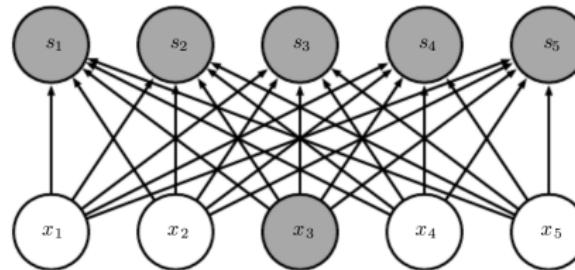


Image: Goodfellow et al, 2016

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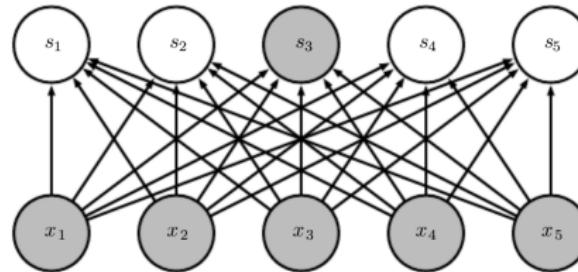


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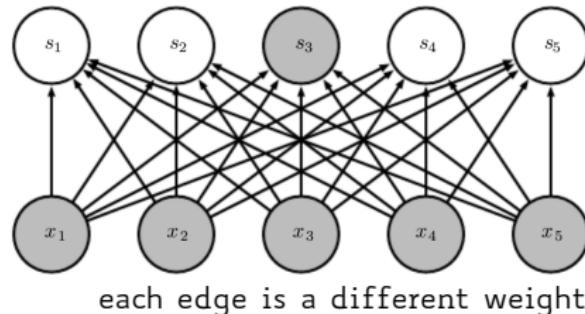
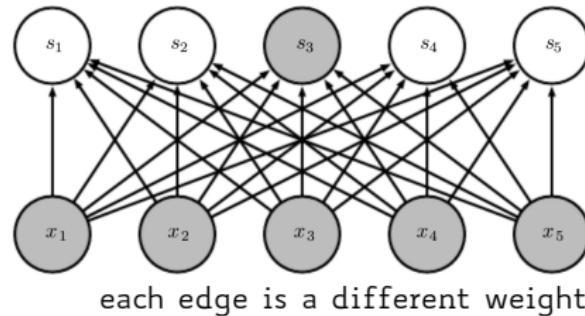


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Sparse interactions

Fully-connected layer:



Convolutional layer:

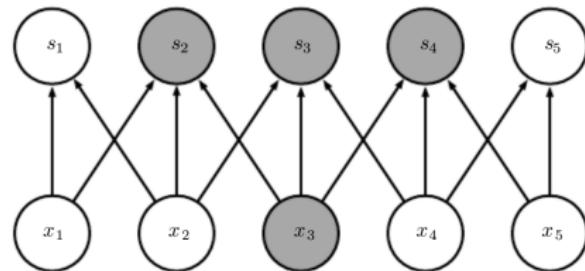
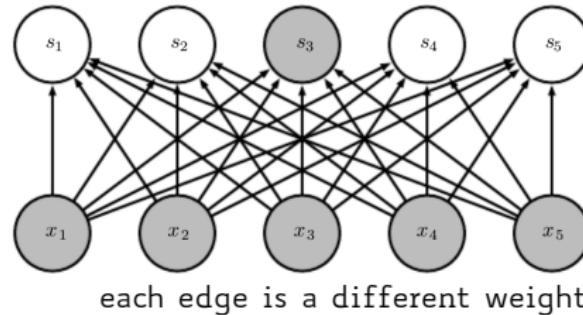


Image: Goodfellow et al, 2016

Sparse interactions

Fully-connected layer:



Convolutional layer:

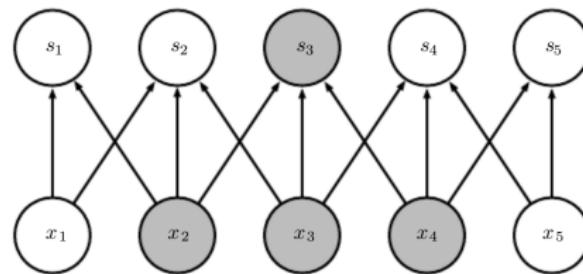
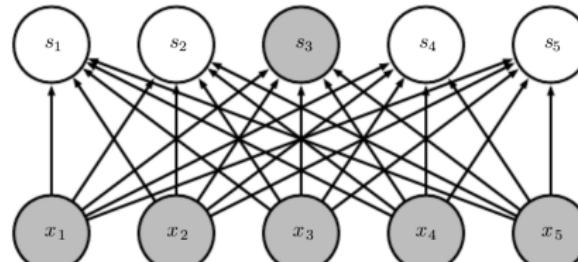


Image: Goodfellow et al, 2016

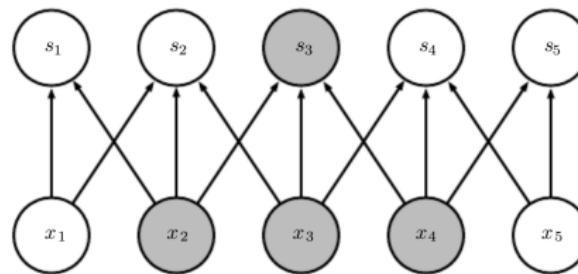
Sparse interactions

Fully-connected layer:



each edge is a different weight

Convolutional layer:



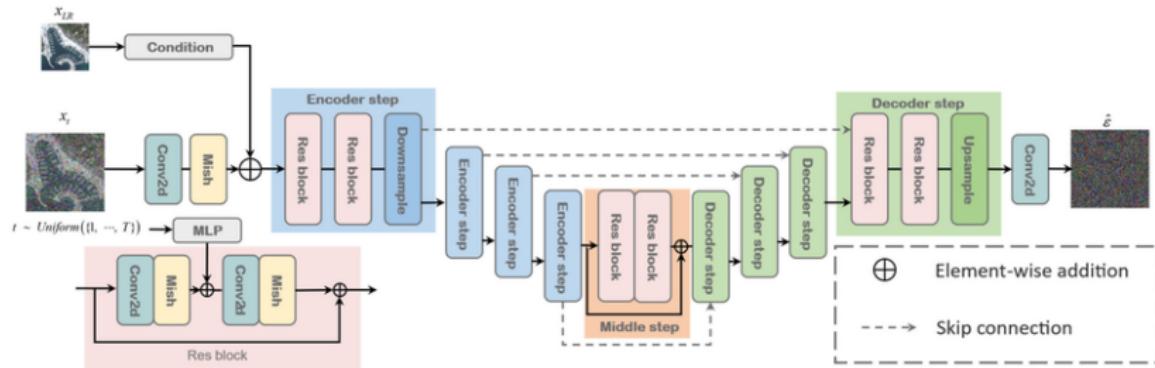
the outgoing edges have the same weights
for each input variable (weight sharing)

Image: Goodfellow et al, 2016

CNNs today

As of 2025, convolutional blocks are **still relevant!**

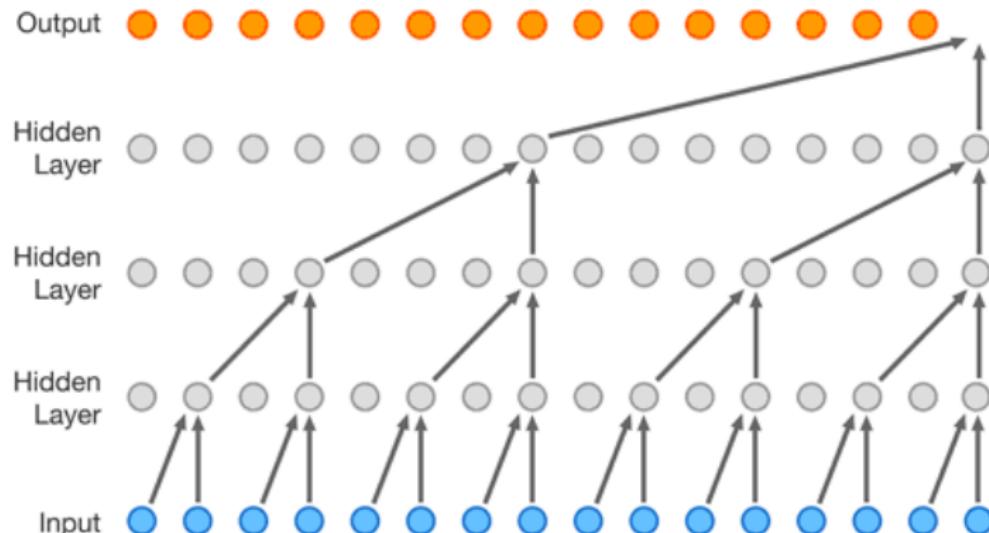
- U-Nets in diffusion models (Stable Diffusion, Imagen, etc.)



CNNs today

As of 2025, convolutional blocks are **still relevant!**

- WaveNet



Pooling

Reduces the spatial dimensions of the input for the next convolutional layer.

3	3	2	1	0	0
3	3	2	1	0	0
3	3	2	1	0	0
3	3	3	2	0	0
3	3	2	1	0	0
3	2	1	1	0	0

Input data



1	0	1
0	1	0
1	0	1

Filter



6	8	6	3	1	0
9	13	10	5	2	0
9	14	11	6	3	0
9	13	11	6	2	0
8	13	10	5	3	0
6	7	5	3	1	0

Feature map

Pooling

Reduces the spatial dimensions of the input for the next convolutional layer.

3	3	2	1	0	0
3	3	2	1	0	0
3	3	2	1	0	0
3	3	3	2	0	0
3	3	2	1	0	0
3	2	1	1	0	0

Input data



1	0	1
0	1	0
1	0	1

Filter



6	8	6	3	1	0
9	13	10	5	2	0
9	14	11	6	3	0
9	13	11	6	2	0
8	13	10	5	3	0
6	7	5	3	1	0

Feature map

13	10	2
14	11	3
13	10	3

Max pooling

Pooling

Reduces the spatial dimensions of the input for the next convolutional layer.

3	3	2	1	0	0
3	3	2	1	0	0
3	3	2	1	0	0
3	3	3	2	0	0
3	3	2	1	0	0
3	2	1	1	0	0

Input data



1	0	1
0	1	0
1	0	1

Filter

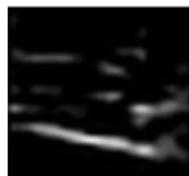


6	8	6	3	1	0
9	13	10	5	2	0
9	14	11	6	3	0
9	13	11	6	2	0
8	13	10	5	3	0
6	7	5	3	1	0

Feature map

13	10	2
14	11	3
13	10	3

Max pooling



2x2 Max
pooling

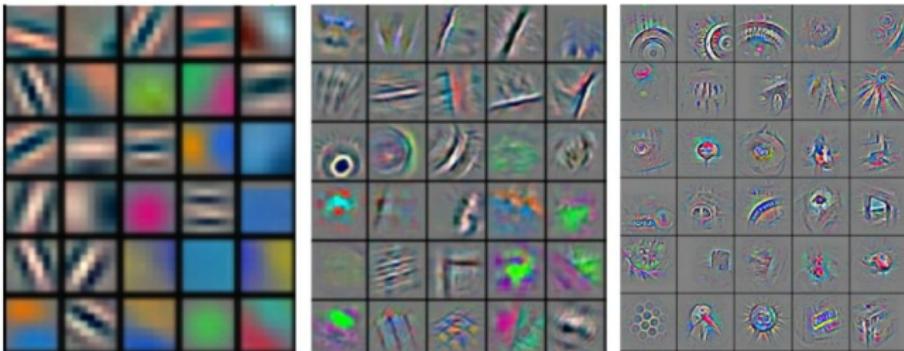


Captures **non-local interactions** via simple building blocks.

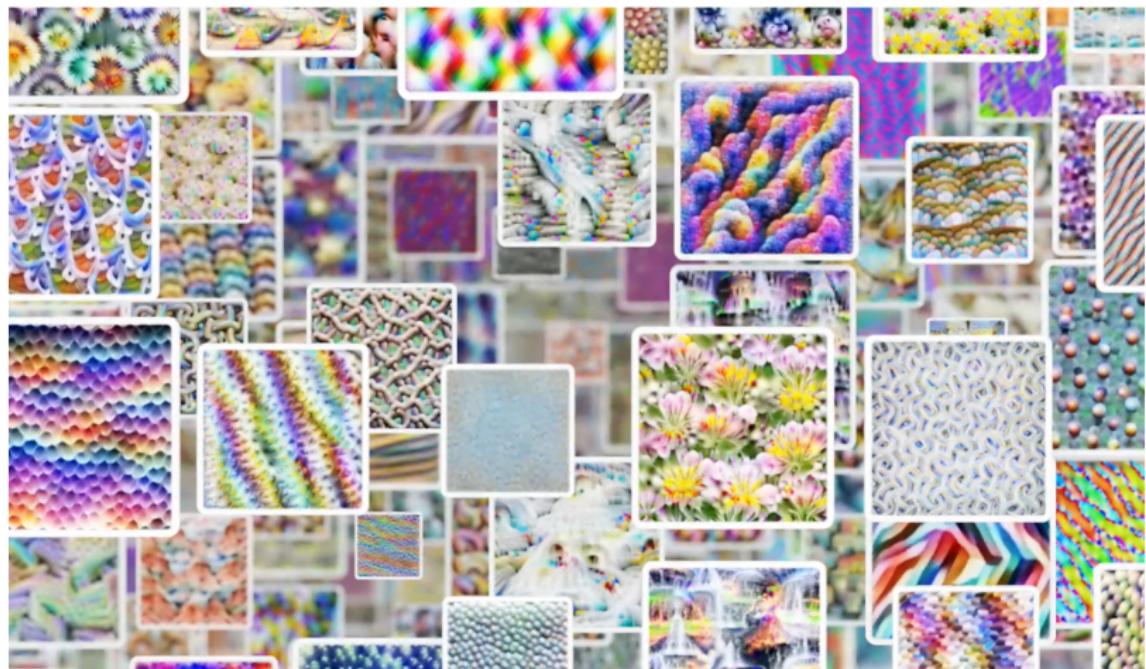
Learned features



→ low-level features → mid-level features → hi-level features → "car"



Learned features



<https://openai.com/blog/microscope/>

Suggested reading

Convolution animations, including variants:

https://github.com/vdumoulin/conv_arithmetic

Seminal paper on CNN, seen as a set of feature detectors:

<http://yann.lecun.com/exdb/publis/pdf/lecun-89e.pdf>