Isospectralization
or how to hear shape, style and correspondence

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Sapienza University of Rome, 13 December 2018

Marvel's Daredevil, 2003

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Wave equation

$$\frac{\partial^2 u(x,y;t)}{\partial t^2} = \Delta u(x,y;t)$$

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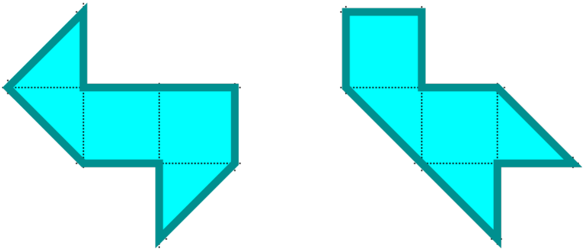
$$u(x,y;t) = \sum_j d_j \phi_j(x,y) \left(\cos\left(\sqrt{\lambda_j} t\right) + i \sin\left(\sqrt{\lambda_j} t\right)\right)$$

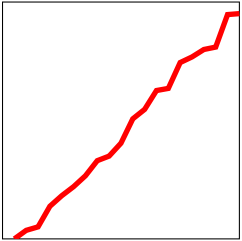
Can we reconstruct the shape from the sequence (λ_i)?

Not technically...

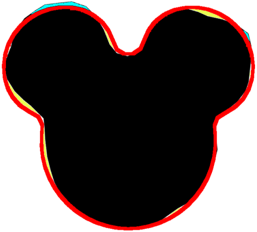


Gordon et al., 1991

Mickey from spectrum

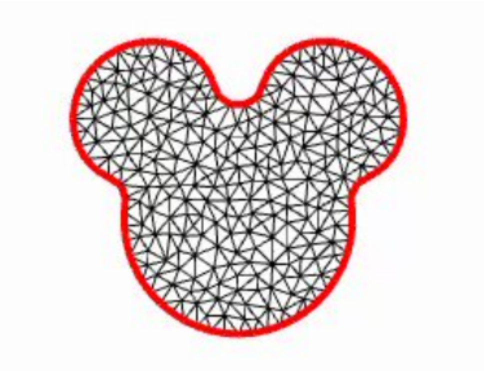


Input



Output

Discretization: shapes



Discretization: Laplace-Beltrami operator

Classical FEM discretization yields:

$$\mathbf{L} = \mathbf{A}^{-1}\mathbf{S}$$

where $\mathbf{S} = (s_{ij})$ is the stiffness matrix and $\mathbf{A} = (a_{ij})$ is the mass matrix.

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$$s_{ij} = \begin{cases} \frac{-\ell_{ij}^2 + \ell_{jk}^2 + \ell_{ki}^2}{8A_{ijk}} + \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in E_i \\ \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in E_b \\ -\sum_{k \neq i} s_{ik} & \text{if } i = j \end{cases}$$

$$a_{ij} = \begin{cases} \frac{1}{3} \sum_{lk:ilk \in F} A_{ilk} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

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$$\ell_{ij}(\mathbf{V}) = \|\mathbf{v}^i - \mathbf{v}^j\|_2$$

Optimization

$$\min_{\mathbf{V}\in\mathbb{R}^{n\times d}} \|\boldsymbol{\lambda}(\boldsymbol{\Delta}_X(\mathbf{V})) - \boldsymbol{\mu}\|_\omega + \rho_X(\mathbf{V})$$

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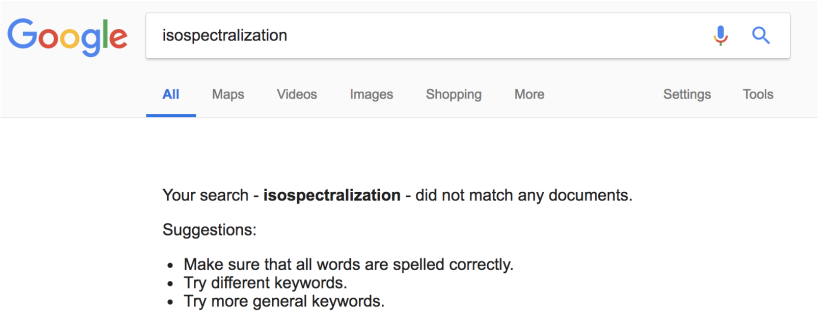

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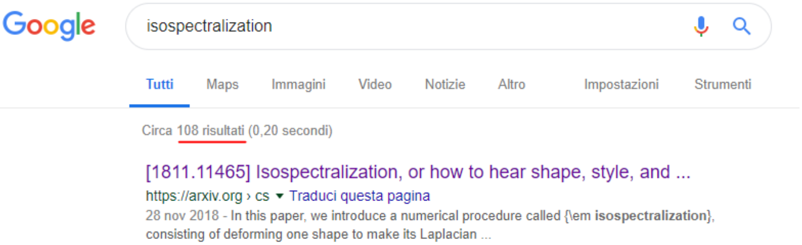

Details

- Weighted norm:

$$\|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_\omega = \sum_{i=1}^{k} \frac{1}{i}(\lambda_i - \mu_i)^2$$

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- Regularizers:

$$\rho_X(\mathbf{V}) = \rho_{X,1}(\mathbf{V}) + \rho_{X,2}(\mathbf{V})$$

$$\rho_{X,1}(\mathbf{V}) = \sum_{e_{ij} \in E_b} \ell_{ij}^2(\mathbf{V})$$

$$\rho_{X,2}(\mathbf{V}) = (\sum_{ijk \in F} (\mathbf{R}_{\frac{\pi}{2}}(\mathbf{v}^j - \mathbf{v}^i))^\top (\mathbf{v}^k - \mathbf{v}^i))_-$$

Implementation

- Only 30 eigenvalues are used

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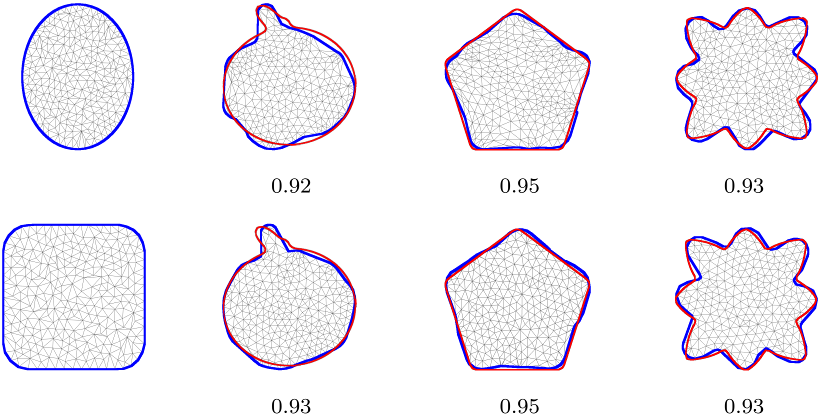
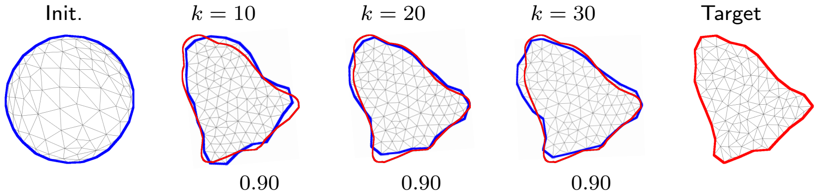
- Adam optimizer implementation of Tensorflow

Kingma and Ba, 2014

Initialization



Bandwidth and mesh resolution



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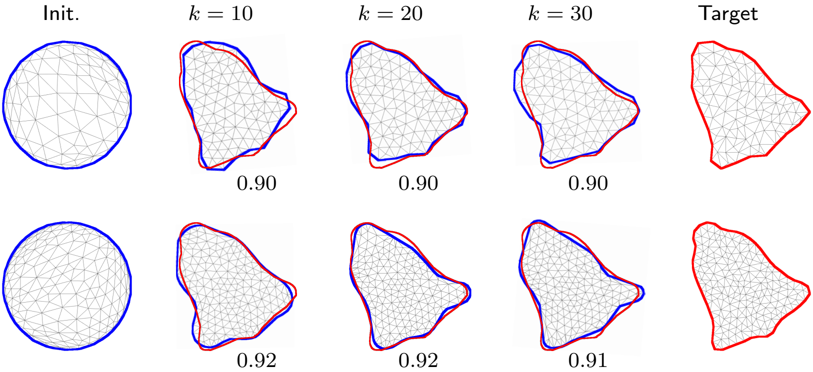

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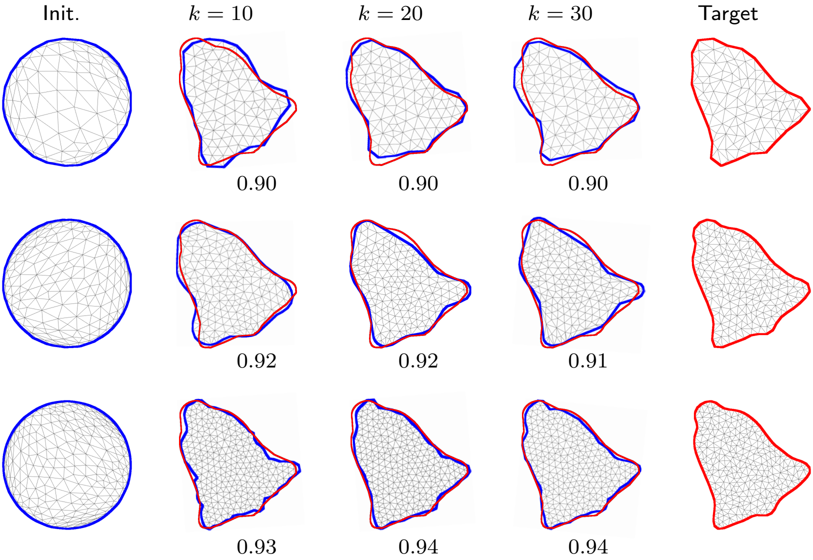
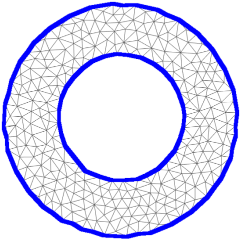
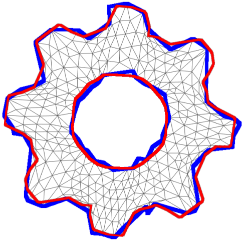


Topology

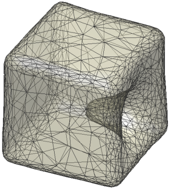
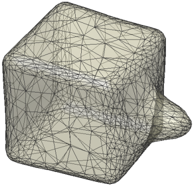
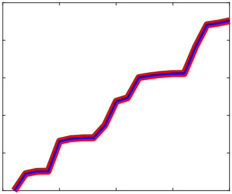


Initialization



Reconstruction

Reconstruction of surfaces

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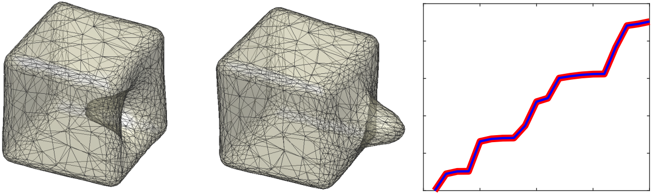


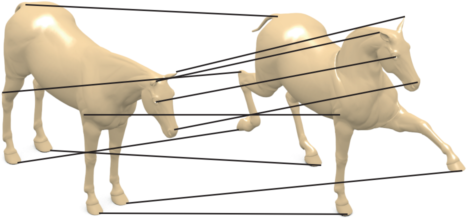
Different regularizers are needed:

$$\rho_X(\mathbf{V}) = \rho_{X,1}(\mathbf{V}) + \rho_{X,2}(\mathbf{V})$$

$$\rho_{X,1}(\mathbf{V}) = \|\mathbf{L}\mathbf{V}\|_F^2$$

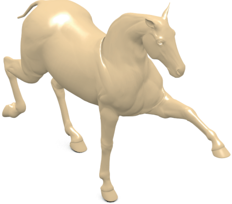
$$\rho_{X,2}(\mathbf{V}) = \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)^\top \sum_{ijk\in F} ((\mathbf{v}^j - \mathbf{v}^i) \times (\mathbf{v}^k - \mathbf{v}^j))(\mathbf{v}^i + \mathbf{v}^j + \mathbf{v}^k)$$

Applications: non-isometric shape matching



Isometric

Applications: non-isometric shape matching



Isometric

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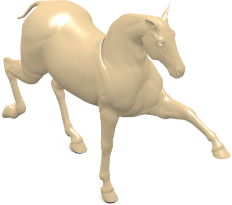

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Different representation

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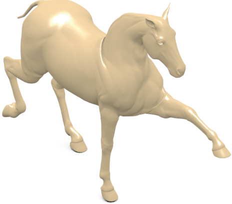

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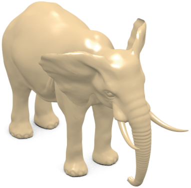
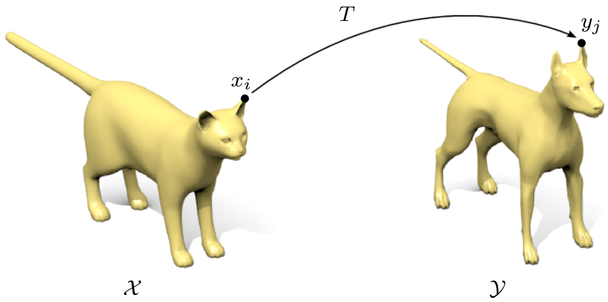


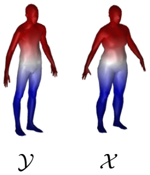
Non-isometric

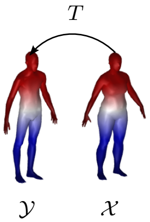
Applications: non-isometric shape matching



Point-wise map $T\colon \mathcal{X} \to \mathcal{Y}$

Our approach





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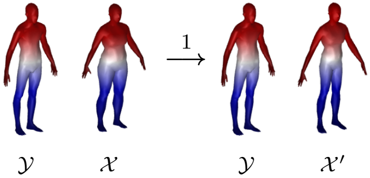
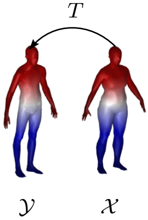
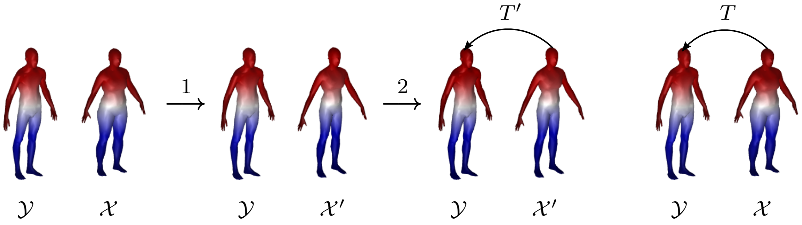




1. Deform $\mathcal{X}$ to obtain $\mathcal{X}'$ whose spectrum $\boldsymbol{\lambda}_{\mathcal{X}'}$ is aligned with $\boldsymbol{\lambda}_{\mathcal{Y}}$

Our approach



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2. Compute the correspondence $T' : \mathcal{X}' \to \mathcal{Y}$ (using an existing isometric matching algorithm)

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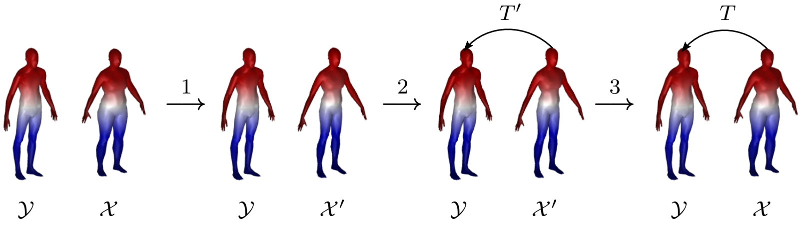

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3. Convert T' to $T : \mathcal{X} \to \mathcal{Y}$ using the identity map between $\mathcal{X}$ and $\mathcal{X}'$.

Isometric shape matching

Algorithm based on functional maps:

Nogneng and Ovsjanikov, 2017

Isometric shape matching

Algorithm based on functional maps:

1. Solve for a functional map between $\mathcal{Y}$ and $\mathcal{X}'$ using several constraints:
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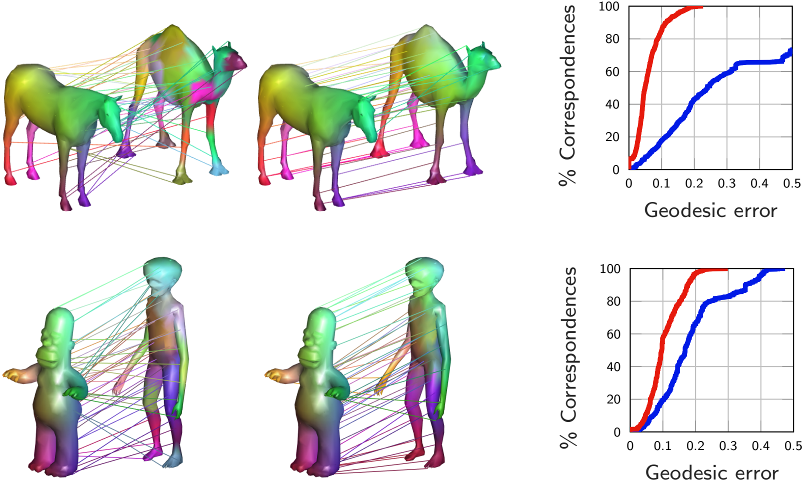
    - regularization (commutativity with the Laplace operator)
2. Convert the functional map to a pointwise one:
    - simple nearest neighbor search in the spectral domain

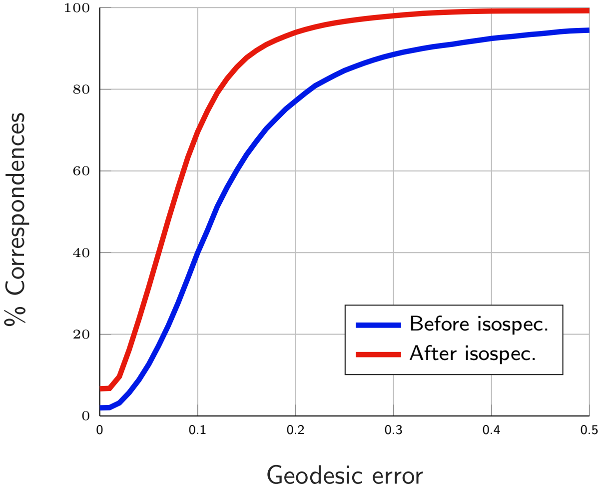
Nogneng and Ovsjanikov, 2017

Results: examples

Before isospectralization **After** isospectralization



Results: average geodesic error



Results: other examples

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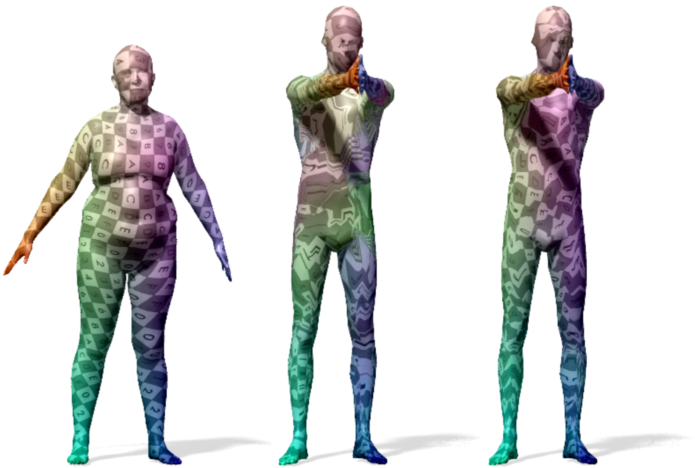

Results: other examples

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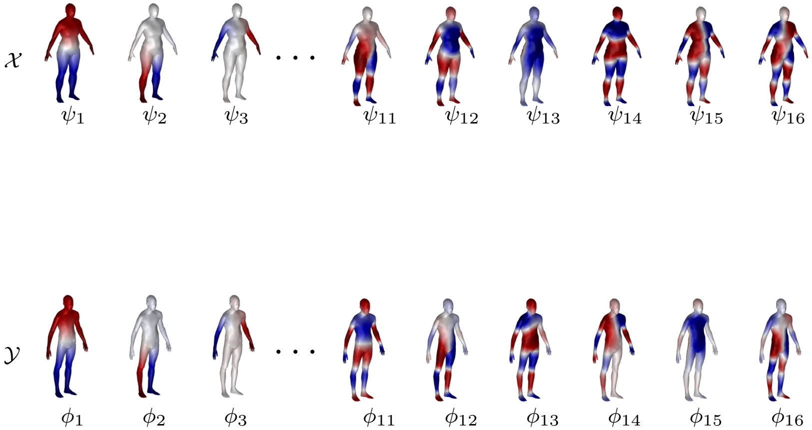
Why it works



$\mathcal{X}$
ψ_1
ψ_2
ψ_3
$\cdots$
ψ_{11}
ψ_{12}
ψ_{13}
ψ_{14}
ψ_{15}
ψ_{16}
$\mathcal{Y}$
ϕ_1
ϕ_2
ϕ_3
$\cdots$
ϕ_{11}
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ϕ_{14}
ϕ_{15}
ϕ_{16}

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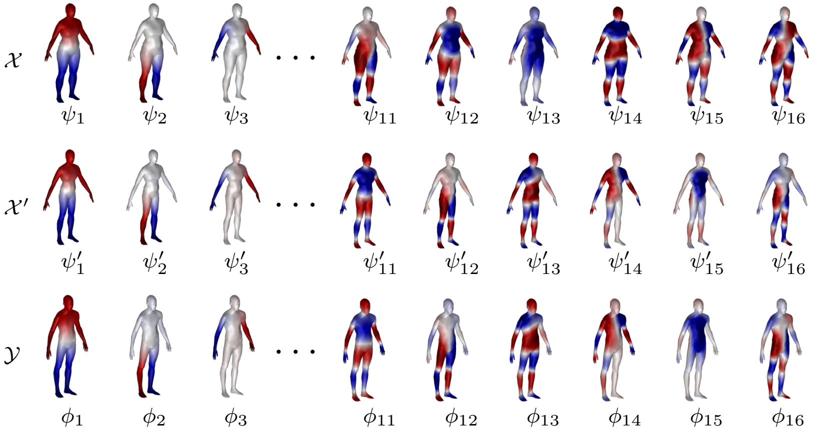
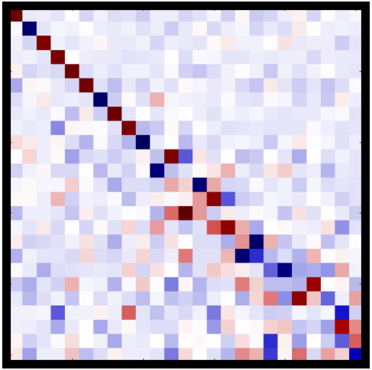
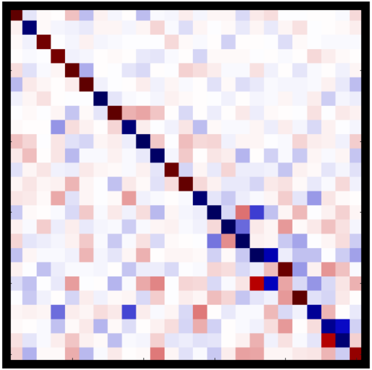


Isospectralization induces an alignment of eigenspaces.

Functional map: before



Functional map: after

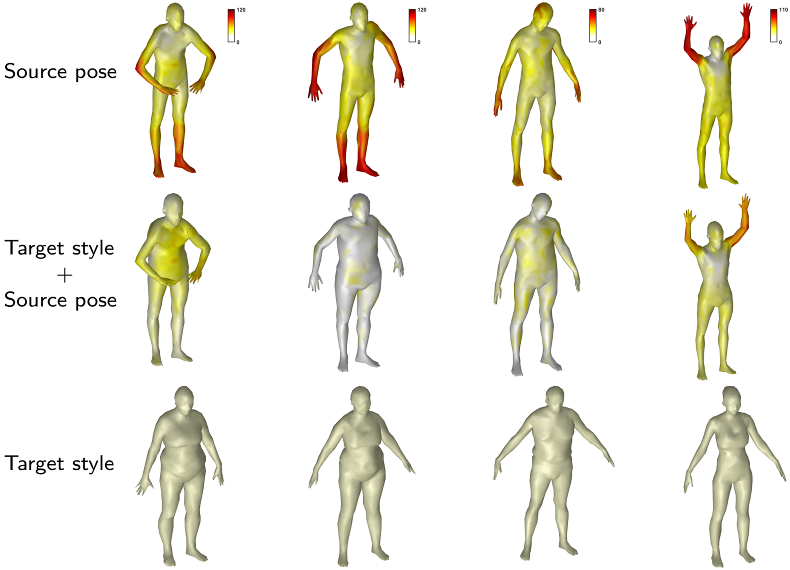


Applications: style transfer

Source pose

Target style

Applications: style transfer



Conclusions and perspectives

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- Remarkable applications in computer vision, graphics, and more
- Countless directions to explore...

Thank you!