Fundamentals of Computer Graphics

Assignment problems

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Exercises

- Voronoi basis

Gromov-Hausdorff distance

$$d_{\mathcal{GH}}(\mathcal{X}, \mathcal{Y}) = \min_{\mathcal{Z}, f, g} d_{\mathcal{H}}^{\mathcal{Z}}(f(\mathcal{X}), g(\mathcal{Y}))$$

where $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are isometric embeddings

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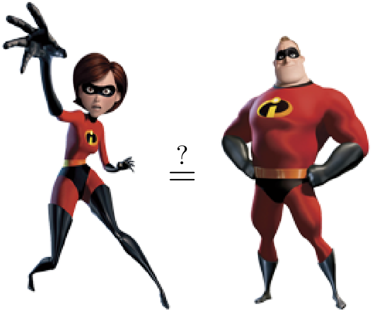

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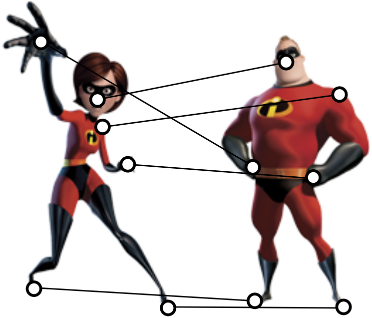
Why should we compute this distance? We can find correspondences



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- It depends on the task (correspondence vs retrieval)

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How to compute it?

- It depends on the task (correspondence vs retrieval)
- It is convenient to compute approximations of $d_{\mathcal{GH}}$
- Still an open problem! AKA shape matching

Example: Texture transfer

With a correspondence we can transfer information (e.g. texture coordinates) across shapes

$\mathcal{X}$

$\mathcal{Y}$

texture of $\mathcal{Y}$
transferred to $\mathcal{X}$

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Correspondence

A correspondence between two sets $\mathcal{X}$ and $\mathcal{Y}$ is a set $\mathcal{R} \subset \mathcal{X} \times \mathcal{Y}$ satisfying:

- For every $x \in \mathcal{X}$, there exists at least one $y \in \mathcal{Y}$ such that $(x, y) \in \mathcal{R}$
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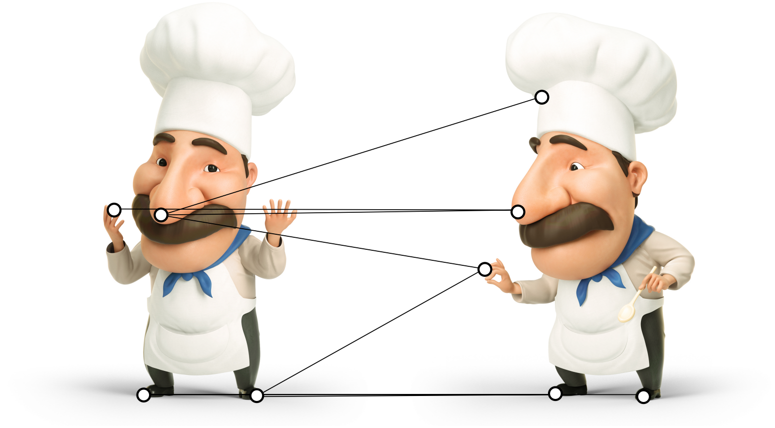
In other words, this is giving at least one match for all points

Later on, we will see that mapping constraints (e.g. one-to-one correspondence) are often necessary

Example: Correspondence



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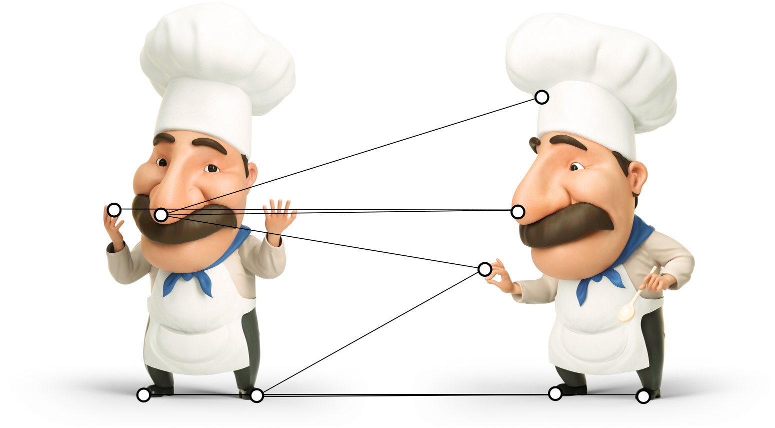

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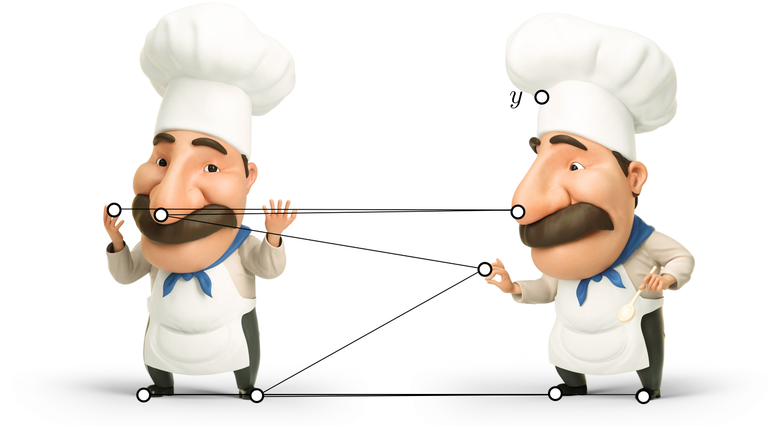

Not a correspondence: point $y \in \mathcal{Y}$ is not mapped

Note: The mathematical definition of correspondence doesn't have anything to do with distortion or semantics

Correspondence vs. maps

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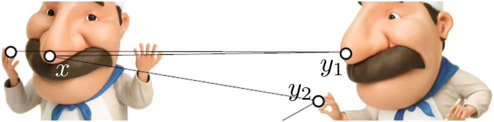

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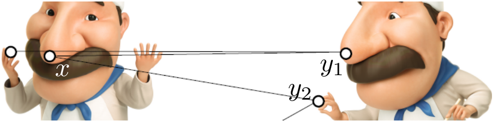

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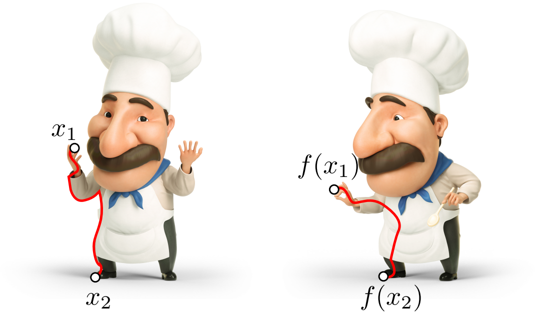
Intuitively, a correspondence is a "multi-valued" map where a single point may have more than one image

Metric distortion

For a map $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_Y)$ we define its (absolute) distortion as:

$$\mathrm{dis}(f) = \max_{x_1, x_2 \in \mathcal{X}} |d_{\mathcal{X}}(x_1, x_2) - d_{\mathcal{Y}}(f(x_1), f(x_2))|$$



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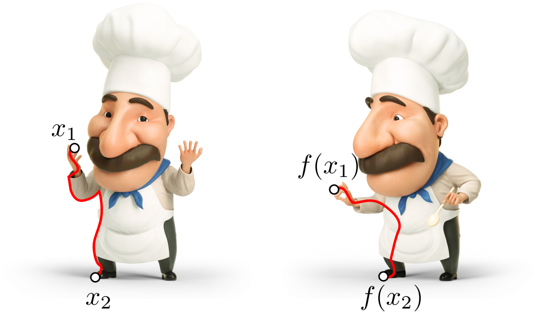

Similarly, for a correspondence $\mathcal{R}$ we define:

$$\mathrm{dis}(\mathcal{R}) = \max\{|d_{\mathcal{X}}(x_1, x_2) - d_{\mathcal{Y}}(y_1, y_2)| \ : \ (x_1, y_1), (x_2, y_2) \in \mathcal{R}\}$$

Distortion of isometries

For a surjective map $f : \mathcal{X} \to \mathcal{Y}$ and the associated correspondence $\mathcal{R} = \{(x, f(x)) \ : \ x \in \mathcal{X}\}$, we have:

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If f is an isometry and $\mathcal{R}$ its induced correspondence, we get:

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Finally, if

$$\mathrm{dis}(f) \leq \epsilon$$

we call f an ϵ-isometry (which is what we see in the real world)

Rigid correspondence

We first consider the simple case where $d_{\mathcal{X}} = d_{\mathcal{Y}} = \| \cdot \|_2$

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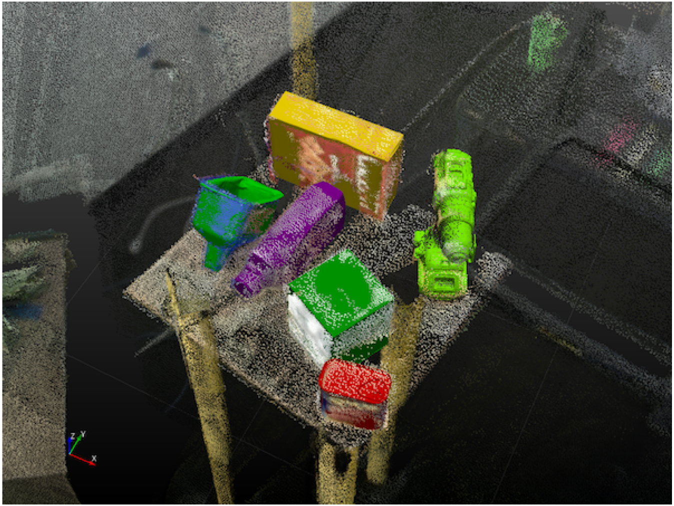
If we want zero distortion, we then expect our shapes to be rigid



Rigid alignment

We consider the Hausdorff distance:

$$d_{\mathcal{H}}(\mathcal{X}, \mathcal{Y}) = \max\{\max_x \mathrm{dist}_{\mathbb{R}^3}(x, \mathcal{Y}), \max_y \mathrm{dist}_{\mathbb{R}^3}(y, \mathcal{X})\}$$

In particular, its non-symmetric version:

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Rigid alignment methods minimize $d_{\mathcal{H}}$ over all rigid isometries:

$$\min_{i \in \mathrm{iso}(\mathbb{R}^3)} d_{\mathcal{H}}(i(\mathcal{X}), \mathcal{Y})$$

Shape $\mathcal{X}$ is moved in $\mathbb{R}^3$ until the Hausdorff distance to $\mathcal{Y}$ is minimum

Iterative closest point (ICP)

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- Repeat

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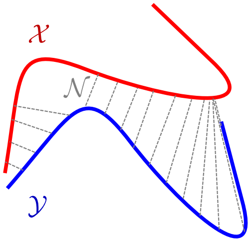
Example: ICP

At each step, we have both a correspondence and a rigid transformation



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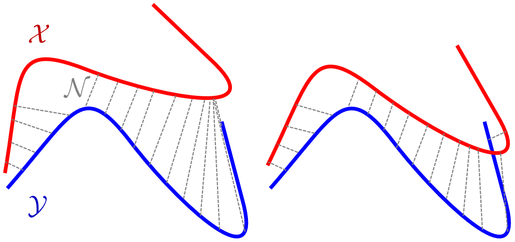

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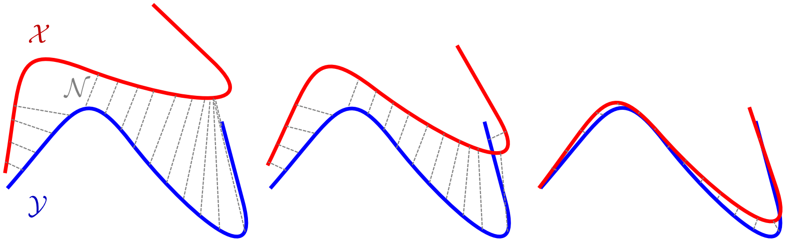

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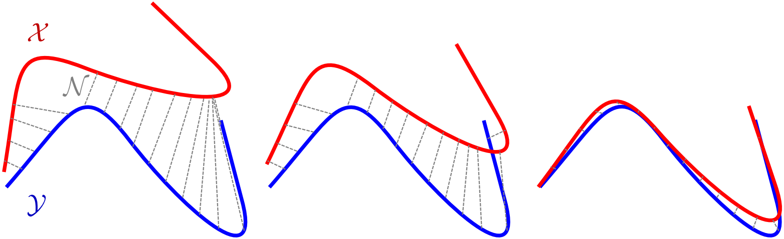

- In practice converges to local optima
- Requires a good initialization
- Only works well in the rigid setting

Correspondence and Gromov-Hausdorff

One can prove the following relationship[1]:

$$d_{\mathcal{GH}}(\mathcal{X},\mathcal{Y}) < r \iff |d_{\mathcal{X}}(x_1,x_2) - d_{\mathcal{Y}}(y_1,y_2)| < 2r$$

for all $(x_1,y_1),(x_2,y_2) \in \mathcal{R}$. More formally, the above holds when such a correspondence $\mathcal{R} \subset \mathcal{X} \times \mathcal{Y}$ exists.

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This allows us to speak about $d_{\mathcal{GH}}$ just by using correspondences $\mathcal{R}$:

$$d_{\mathcal{GH}}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \min_{\mathcal{R}} \operatorname{dis}(\mathcal{R})$$

Still huge!

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Correspondence computation

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Also, restrict the feasible set to the set of one-to-one correspondences; this is done by optimizing over permutations:

$$d_{\mathcal{P}}(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}) = \frac{1}{2} \min_{\pi \in \mathcal{P}_n} \max_{1 \leq i, j \leq n} |d_{\mathcal{X}}(x_1, x_2) - d_{\mathcal{Y}}(y_{\pi i}, y_{\pi j})|$$

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Q: how big is the set of permutations?

Correspondence computation

$$d_{\mathcal{GH}}(\mathcal{X},\mathcal{Y}) = \frac{1}{2}\min_{\mathcal{R}\subset\mathcal{X}\times\mathcal{Y}}\max_{(x_1,y_1),(x_2,y_2)\in\mathcal{R}} |d_{\mathcal{X}}(x_1,x_2) - d_{\mathcal{Y}}(y_1,y_2)|$$

To make it more practical, restrict $\mathcal{X},\mathcal{Y}$ to farthest point samplings $\tilde{\mathcal{X}},\tilde{\mathcal{Y}}$

Also, restrict the feasible set to the set of one-to-one correspondences; this is done by optimizing over permutations:

$$d_{\mathcal{P}}(\tilde{\mathcal{X}},\tilde{\mathcal{Y}}) = \frac{1}{2}\min_{\pi\in\mathcal{P}_n}\max_{1\leq i,j\leq n} |d_{\mathcal{X}}(x_1,x_2) - d_{\mathcal{Y}}(y_{\pi i},y_{\pi j})|$$

Q: how big is the set of permutations? n!

In fact, one can prove that $d_{\mathcal{GH}}(\mathcal{X},\mathcal{Y}) \leq d_{\mathcal{P}}(\tilde{X},\tilde{\mathcal{Y}})$

Example: Sort by distortion

Assume we know the correspondence π, then we can directly compute:

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And we can use it to sort shapes by distortion:

$\mathcal{X}$ $\mathcal{Y}_1$ $\mathcal{Y}_2$ $\mathcal{Y}_3$ $\mathcal{Y}_4$

where

$$d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_1) \leq d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_2) \leq d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_3) \leq d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_4)$$

Correspondence matrix

A common representation is the correspondence matrix $\mathbf{T} \in \{0,1\}^{m \times n}$

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 1 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

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In other words, it maps functions to functions

Permutation matrix

A permutation matrix $\mathbf{P} \in \{0,1\}^{n \times n}$ is a special case where each row and each column have exactly one value equal to 1:

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Permutations encode one-to-one correspondences, meaning that they map indicator functions to indicator functions

In turn, one-to-one correspondences are associated to bijective maps $f : \mathcal{X} \to \mathcal{Y}$

Correspondence problem

$$\pi^* = \arg\min_{\pi \in \mathcal{P}_n} \max_{1 \le i,j \le n} |d_{\mathcal{X}}(x_1, x_2) - d_{\mathcal{Y}}(y_{\pi i}, y_{\pi j})|$$

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The metric distortion terms can be encoded into a matrix $\mathbf{C} \in \mathbb{R}^{n^2 \times n^2}$:

$$C_{(i\ell)(jm)} = |d_{\mathcal{X}}(x_i, x_j) - d_{\mathcal{Y}}(y_\ell, y_m)|$$

where $(i\ell)$ and (jm) identify the matches $(x_i, y_\ell), (x_j, y_m) \in \mathcal{R}$

$$\mathbf{C} = \begin{array}{c} \begin{array}{ccccc} (x_1, y_1) & (x_3, y_5) & \cdots & (x_i, y_\ell) & \cdots \end{array} \\ \left(\begin{array}{ccccc} & & & \vdots & \\ & & & \vdots & \\ & & & \vdots & \\ \cdots & \cdots & \cdots & |d_{\mathcal{X}}(x_i, x_j) - d_{\mathcal{Y}}(y_\ell, y_m)| & \cdots \\ & & & \vdots & \end{array} \right) \end{array} \begin{array}{c} \\ (x_1, y_1) \\ (x_3, y_5) \\ \vdots \\ (x_j, y_m) \\ \vdots \end{array}$$

L_p Gromov-Hausdorff

$$\frac{1}{2} \min_{\mathbf{P} \in \mathcal{P}} \max_{i,j,\ell,m} C_{(i\ell)(jm)} P_{ij} P_{\ell m}$$

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Or, more in general, we can use the L_p distortion

$$C^{(p)}_{(i\ell)(jm)} = |d_{\mathcal{X}}(x_i, x_j) - d_{\mathcal{Y}}(y_\ell, y_m)|^p$$

yielding the minimization problem

$$\frac{1}{2} \min_{\mathbf{P} \in \mathcal{P}} \sum_{i,j,\ell,m} C^{(p)}_{(i\ell)(jm)} P_{ij} P_{\ell m}$$

Quadratic assignment problem

Rewriting in matrix notation, we get:

$$\min_{\mathbf{P} \in \{0,1\}^{n \times n}} \operatorname{vec}(\mathbf{P})^\top \mathbf{C} \operatorname{vec}(\mathbf{P})$$
$$\text{s.t.} \quad \mathbf{P1} = \mathbf{1}\,, \mathbf{P}^\top \mathbf{1} = \mathbf{1}$$

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also known as quadratic assignment problem (QAP)

- The problem is quadratic in $\mathbf{P}$
- It is NP-hard
- Still an open problem (a survey might be a final project)

Continuous relaxation

Replace $\{0, 1\}$ with continuous values in $[0, 1]$

$$\begin{aligned} \min_{\mathbf{S} \in [0,1]^{n \times n}} \quad & \mathrm{vec}(\mathbf{S})^\top \mathbf{C} \mathrm{vec}(\mathbf{S}) \\ \text{s.t.} \quad & \mathbf{S}\mathbf{1} = \mathbf{1}\,, \mathbf{S}^\top \mathbf{1} = \mathbf{1} \end{aligned}$$

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We are replacing permutations by doubly-stochastic matrices:

$$\mathbf{S} = \begin{pmatrix} 0.2 & 0.3 & \cdots & 0.5 \\ 0.1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0.7 & 0 & \cdots & 0.1 \end{pmatrix}$$

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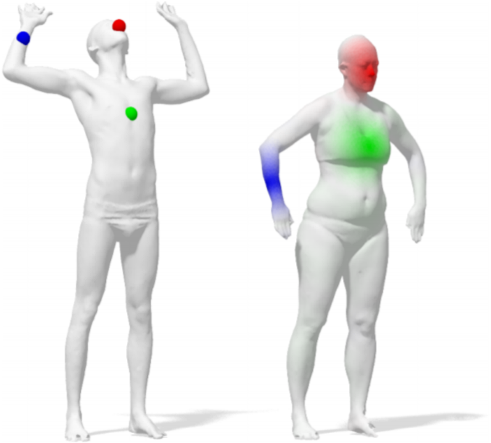
Doubly-stochastic matrices are convex combinations of permutations:

$$\mathbf{S} = t\mathbf{P}_1 + (1-t)\mathbf{P}_2\,, \quad t\in[0,1]$$

Example: Soft maps

Each row / column of a soft map can be interpreted as a probability distribution on the shape:



Exercise: Sort by distortion

Load the shapes tr_reg_000, tr_reg_001, tr_reg_002, and tr_reg_003 (download from course website).

- The ground-truth correspondence $T^{\mathcal{X},\mathcal{Y}}$ between each pair of shapes $(\mathcal{X},\mathcal{Y})$ is the identity map

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- Sort the shapes by increasing distortion

Exercise: ICP

Load the shapes bun000 and bun045 (download from course website).

- Compute a FPS of $n =$ a few hundred points for each scan

Visualize the alignment with the entire shapes (not just the farthest point samplings) without running ICP on the entire shapes

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- Print the alignment error across the iterations

Visualize the alignment with the entire shapes (not just the farthest point samplings) without running ICP on the entire shapes

Suggested reading

- An extensive treatment of Gromov-Hausdorff distances and correspondences is in Chapter 7 of:

  Burago, Burago, Ivanov, "A course in metric geometry". AMS, 2001

- A very accessible survey of ICP and its variants:

  Rusinkiewicz and Levoy, "Efficient variants of the ICP algorithm". Proc. 3DIM, 2011