Fundamentals of Computer Graphics

Heat diffusion

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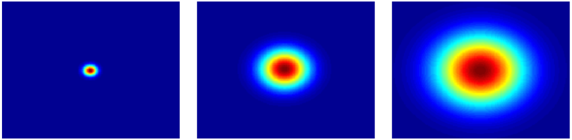
Heat equation

Heat diffusion is governed by the heat equation

$$\frac{\partial}{\partial t}u(x,t) = -\Delta u(x,t)$$
$$u(x,0) = u_0(x)$$

where function u describes the heat distribution at point x after time t



$t = 0$ $t = 1.3$ $t = 3.7$

Heat kernel

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- $k_t(x,y)$ describes the amount of heat transferred from point x to point y in time t
- It is a property of the manifold $\mathcal{X}$ and does not depend on the initial distribution $u_0(x)$

Dirac initialization

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$$\int_{\mathcal{X}} f(x) \delta_z(x) dx = f(z)$$

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$$u(x,t) = \int_{\mathcal{X}} k_t(x,y)\delta_z(y)dy = k_t(x,z)$$

Note that z is fixed, so here $k_t(x,z)$ is a function of x

Heat kernel: Properties

The heat kernel has some useful properties:

- In $\mathbb{R}^n$ it is given by

$$k_t(x,y) = \frac{1}{(\sqrt{4\pi t})^n} \exp(-\frac{\|x-y\|^2}{4t})$$

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- If $T : \mathcal{X} \to \mathcal{Y}$ is an isometry, then

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$$\Delta u(x,t) = \sum_{i=0}^{k} c_i(t)\Delta\phi_i(x) = \sum_{i=0}^{k} c_i(t)\lambda_i\phi_i(x)$$

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Almost there: we need to find an expression for the coefficients d_i

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We use the given initial condition to solve for d_i

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$$\delta_y(x) \approx \sum_{i=0}^{k} d_i \phi_i(x) \Longrightarrow d_i = \langle \phi_i, \delta_y \rangle_{\mathcal{X}}$$

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$$k_t(x, y) \approx \sum_{i=0}^{k} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

Recall (see "Dirac initialization" slide) that this also defines the heat kernel between x and y

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In matrix notation, the heat kernel can be written as a $k \times k$ matrix

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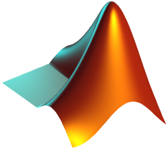

Heat kernel signature

We stated that:

If $T : \mathcal{X} \to \mathcal{Y}$ is an isometry, then

$$k_t^{\mathcal{X}}(x, y) = k_t^{\mathcal{Y}}(T(x), T(y))$$

and the vice-versa also holds.

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We use this property to define a local descriptor based on the heat kernel;

Consider the diagonal of the heat kernel:

$$k_t(x,x) = \sum_{i=0}^{k} e^{-\lambda_i t} \phi_i(x)^2$$

Sun, Ovsjanikov and Guibas 2009

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The heat kernel signature is then defined as:

$$\text{hks}(x) = (k_{t_1}(x, x), \dots, k_{t_T}(x, x)) \in \mathbb{R}^T$$

Sun, Ovsjanikov and Guibas 2009

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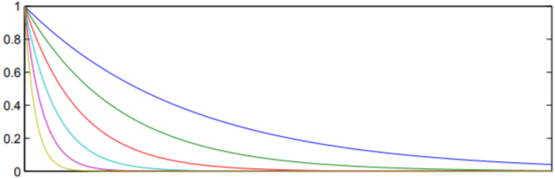

If $\mathcal{X}$ and $\mathcal{Y}$ are isometric, corresponding points $(x, y) \in \mathcal{X} \times \mathcal{Y}$ are expected to have similar signatures

Sun, Ovsjanikov and Guibas 2009

Diffusion distance

Does the heat kernel define a distance function?

$$k_t(x, y) \approx \sum_{i=0}^{k} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

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A family of diffusion distances can be defined by:

$$d_t^2(x, y) = \|k_t(x, \cdot) - k_t(\cdot, y)\|^2$$

Coifman et al. 2005

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- Diffusion time $t \geq 0$ plays the role of a scale parameter

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- The definition above satisfies all properties of a metric
- Diffusion time $t \geq 0$ plays the role of a scale parameter
- Interpretation: If two points x and y are close, there is a large probability of transition from x to y and vice-versa

Coifman et al. 2005

Diffusion distance

$$d_t^2(x,y) = \|k_t(x,\cdot) - k_t(\cdot,y)\|^2$$

Coifman et al. 2005

Diffusion distance

$$
\begin{aligned}
d_t^2(x,y) &= \|k_t(x,\cdot) - k_t(\cdot,y)\|^2 \\
&= \int_{\mathcal{X}} (k_t(x,z) - k_t(z,y))^2 dz
\end{aligned}
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Coifman et al. 2005

Diffusion distance

$$\begin{aligned} d_t^2(x,y) &= \|k_t(x,\cdot) - k_t(\cdot,y)\|^2 \\ &= \int_{\mathcal{X}} (k_t(x,z) - k_t(z,y))^2 dz \\ &= \int_{\mathcal{X}} k_t(x,z)^2 + k_t(z,y)^2 - 2k_t(x,z)k_t(z,y) dz \end{aligned}$$

Coifman et al. 2005

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&= \int_{\mathcal{X}} k_t(x,z)k_t(z,x) + k_t(y,z)k_t(z,y) - 2k_t(x,z)k_t(z,y) dz
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Coifman et al. 2005

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$$

$$
\begin{aligned}
d_t^2(x,y) &= \|k_t(x,\cdot) - k_t(\cdot,y)\|^2 \\
&= \cdots \\
&= \sum_{i=0}^{k} e^{-2\lambda_i t} (\phi_i(x) - \phi_i(y))^2
\end{aligned}
$$

Coifman et al. 2005

Exercise: Functional maps and HKS

Solve for a functional map between two deformable shapes by using HKS as corresponding functions.

Specifically:

- Use shapes $\mathcal{X}$= tr_reg_001 and $\mathcal{Y}$= tr_reg_002
- Compute 100-dimensional heat kernel signatures for all vertices of $\mathcal{X}$ and $\mathcal{Y}$ (choose your own diffusion times)
- Express the two descriptor fields as $k \times 100$ spectral coefficient matrices $\mathbf{A}$ and $\mathbf{B}$, where k is the eigenbasis dimension for both $\mathcal{X}$ and $\mathcal{Y}$
- Solve for the $k \times k$ functional map matrix $\mathbf{C}$ in a least squares sense
- Use $\mathbf{C}$ to transfer delta functions from $\mathcal{X}$ to $\mathcal{Y}$, and visually evaluate the quality of the estimated functional map