Fundamentals of Computer Graphics

Recap of linear algebra II

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Recap: Bases

A basis of V is a collection of vectors in V that is linearly independent and spans V

- $\text{span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots a_n v_n : a_1, \ldots, a_n \in \mathbb{R}\}$
- $v_1, \ldots, v_n \in V$ are linearly independent if and only if each $v \in \text{span}(v_1, \ldots, v_n)$ has only one representation as a linear combination of $v_1, \ldots, v_n$

So every vector $v \in V$ can be expressed uniquely as a linear combination

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

You can think of a basis as the minimal set of vectors that generates the entire space

Recap: Matrices

Consider a linear map $T : V \to W$, a basis $v_1, \dots, v_n \in V$ and a basis $w_1, \dots, w_m \in W$.

The matrix of T in these bases is the $m \times n$ array of values in $\mathbb{R}$

$$\mathbf{T} = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}$$

whose entries $T_{i,j}$ are defined by

$$Tv_j = T_{1,j} w_1 + \cdots + T_{m,j} w_m$$

In other words, the matrix encodes how basis vectors are mapped, and this is enough to map all other vectors in their span, since:

$$Tv = T(\sum_j \alpha_j v_j) = \sum_j T(\alpha_j v_j) = \sum_j \alpha_j Tv_j$$

Recap: Matrix of a vector

Suppose $v \in V$ is an arbitrary vector, while $v_1, \ldots, v_n$ is a basis of V. The matrix of v wrt this basis is the $n \times 1$ matrix:

$$\mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

so that

$$v = c_1 v_1 + \cdots c_n v_n$$

Once again, we see that the matrix depends on the choice of basis for V

Recap: Product of "map matrix" and "vector matrix"

$$\underbrace{\begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}}_{\mathbf{T}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} = \sum_{j=1}^{n} c_j \underbrace{\begin{pmatrix} \color{red}{T_{1,j}} \\ \vdots \\ \color{red}{T_{m,j}} \end{pmatrix}}_{\mathrm{Tv_j \ wrt \ (w_1, \ldots, w_m)}}$$

Because recall that, for bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$:

$$Tv_j = \color{red}{T_{1,j}}w_1 + \cdots + \color{red}{T_{m,j}}w_m$$

We see then that vector $c = \sum_j c_j v_j$ is mapped to $Tc = \sum_j c_j T v_j$

In other words, matrix product is behaving as expected

Rank of a matrix

The rank of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the dimension of the span of its columns

Example:

$$\mathbf{A} = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}$$

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$$\operatorname{span}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix} \right) \in \mathbb{R}^2$$

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Note that this result does not depend on a choice of basis, i.e., change of basis preserves the rank

Example: Reduced bases

Consider the $\mathbb{R}^{n\times k}$ matrix

$$\mathbf{V} = \begin{pmatrix} | & \cdots & \cdots & | \\ \mathbf{v}_1 & \cdots & \cdots & \mathbf{v}_k \\ | & \cdots & \cdots & | \end{pmatrix}$$

containing Voronoi basis vectors as its columns, and the $\mathbb{R}^{n\times k'}$ matrix

$$\mathbf{V}' = \begin{pmatrix} | & \cdots & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_{k'} \\ | & \cdots & | \end{pmatrix}$$

obtained by truncating $\mathbf{V}$ to the first $k' < k$ columns

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Then, $k = \mathrm{rank}(\mathbf{V}) > \mathrm{rank}(\mathbf{V}') = k'$

The rank reflects the expressive power of the full $(\mathbf{V})$ and reduced $(\mathbf{V}')$ bases

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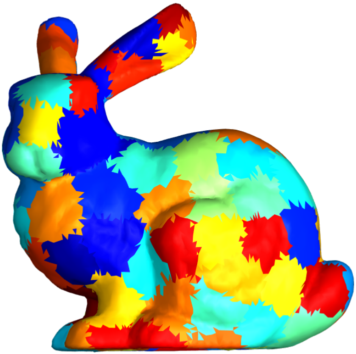
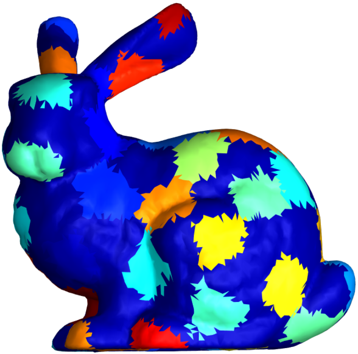


full basis
$\mathrm{rank}(\mathbf{V}) = k$



reduced basis
$\mathrm{rank}(\mathbf{V}') = k' < k$

Example: Rank of a map

In the standard basis, a one-to-one correspondence is written as a permutation matrix in $\mathbb{R}^{n\times n}$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

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Each column is a basis vector, so $\operatorname{rank}(\mathbf{P}) = n$, and this is independent of the choice of a basis

Example: Rank of a map

In the k-dimensional Voronoi basis, a one-to-one correspondence is written as a generic matrix in $\mathbb{R}^{k\times k}$

$$\tilde{\mathbf{P}} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \vdots & & & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{pmatrix}$$

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Functions mapped via $\tilde{\mathbf{P}}$ span a subspace of those mapped via $\mathbf{P}$; so the rank of the matrix encodes how precisely we can map functions to functions

Example: Rank of a map

Consider a correspondence matrix from $\mathcal{F}(\mathcal{X})$ to $\mathcal{F}(\mathcal{Y})$, where:

- The standard basis is chosen for $\mathcal{F}(\mathcal{X})$
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$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kn} \end{pmatrix} \in \mathbb{R}^{k \times n}$$

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$\mathbf{C}$ maps functions expressed in the n-dim. standard basis to functions expressed in the k-dim. Voronoi basis; thus, $\mathrm{rank}(\mathbf{C}) = k$

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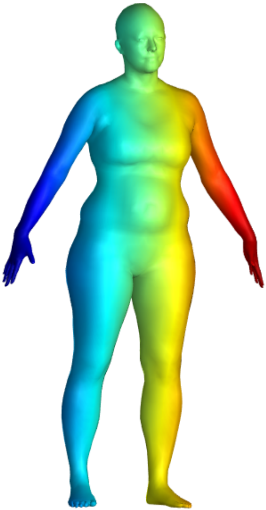

standard basis

Voronoi basis

Invariant subspaces

A subspace U of V is called invariant under $T : V \to V$ if:

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Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbb{R}$, then $\operatorname{span}(v)$ is a 1-dimensional subspace of V invariant under T

Eigenvalues and eigenvectors

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If the equation holds for m distinct eigenvalues and eigenvectors:

$$\begin{gathered} Tv_1 = \lambda_1 v_1 \\ \vdots \\ Tv_m = \lambda_m v_m \end{gathered}$$

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$$m \leq \dim(V)$$

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Additional notes:

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Additional notes:

- This decomposition only makes sense for $T : V \to V$
- If V is a function space, eigenvectors are called eigenfunctions

Eigenspaces

If distinct eigenvectors $E = (v_1, \ldots, v_m)$ correspond to the same eigenvalue λ, then E spans an eigenspace of T

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- Eigenspaces provide a form of decomposition of V

Inner product

We want to be able to measure lengths and angles among vectors

Note: We assume our vectors are real-valued

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To do so, we define the inner product as a function $\langle u, v \rangle : V \times V \to \mathbb{R}$ with the properties:

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- **symmetry:** $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$

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Examples: Inner products

- **Lists:**

  The Euclidean inner product (or dot product) is defined by

  $$\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = u_1 v_1 + \cdots u_n v_n$$

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- **Functions:**

  On the vector space of continuous functions $f : [-1, 1] \to \mathbb{R}$

  $$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

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For $u, v \in \mathbb{R}^2$, it can be shown that

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From this, we can think of the inner product as encoding a general notion of angle between two vectors:

$$\theta = \arccos \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

For example, we can now think of "angle between two functions"

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Given an orthonormal basis, $v \in V$ can be written as a linear combination:

$$v = \langle v, v_1 \rangle v_1 + \cdots \langle v, v_n \rangle v_n$$

So the combination coefficients are simply given by inner products

Inner product in matrix notation

For vectors $u, v \in V$ in the standard basis $\{e_i\}$, we can write:

$$\langle u, v \rangle = \langle \sum_i u_i e_i, \sum_j v_j e_j \rangle$$

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In matrix notation, we can thus write

$$\langle u, v \rangle = \mathbf{u}^\top \mathbf{v}$$

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Example: Voronoi basis

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$$v_i \mapsto \frac{v_i}{\|v_i\|}$$

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This is true for any orthonormal basis

Exercise: Rank of a map

Implement the example of slide number 22 (download shapes tr_reg_010 and tr_reg_031 from the course website)

For these shapes, the ground-truth correspondence is the identity.

- Use the standard basis U on the source
- Use the Voronoi basis V on the target, based on 50 FPS
- Encode the ground-truth map as a matrix $\mathbf{C}$ wrt bases U and V
- Map the x coordinate function from source to target via $\mathbf{C}$

Visualize the function on source and target using the jet colormap; you should get a similar rendering as the one shown in slide 22.

Suggested reading

See sections 3.F, 5.A – 6.B of:

S. Axler, "Linear algebra done right – 3rd edition". Springer, 2015