

Isospectralization

or how to hear shape, style and correspondence

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SAPIENZA
UNIVERSITÀ DI ROMA

Sapienza University of Rome, 13 December 2018

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

“La Physique ne nous donne pas seulement
l’occasion de résoudre des problèmes . . . , elle nous
fait présentir la solution.” H. POINCARÉ.

Wave equation

$$\frac{\partial^2 u(x, y; t)}{\partial t^2} = \Delta u(x, y; t)$$

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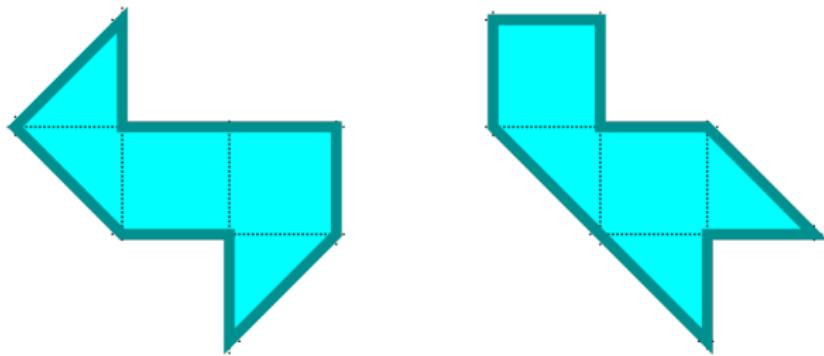
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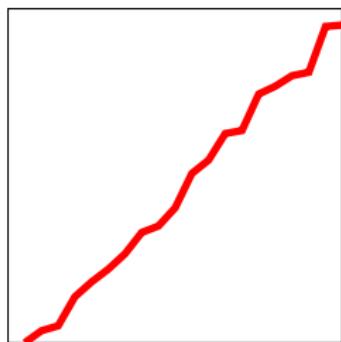
$$u(x, y; t) = \sum_j d_j \phi_j(x, y) \left(\cos \left(\sqrt{\lambda_j} t \right) + i \sin \left(\sqrt{\lambda_j} t \right) \right)$$

Can we reconstruct the shape
from the sequence (λ_i) ?

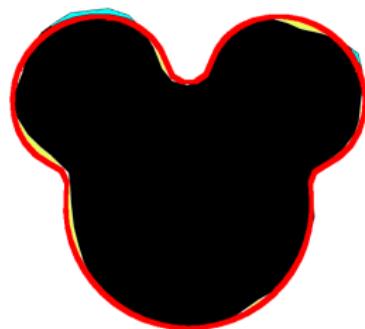
Not technically...



Mickey from spectrum

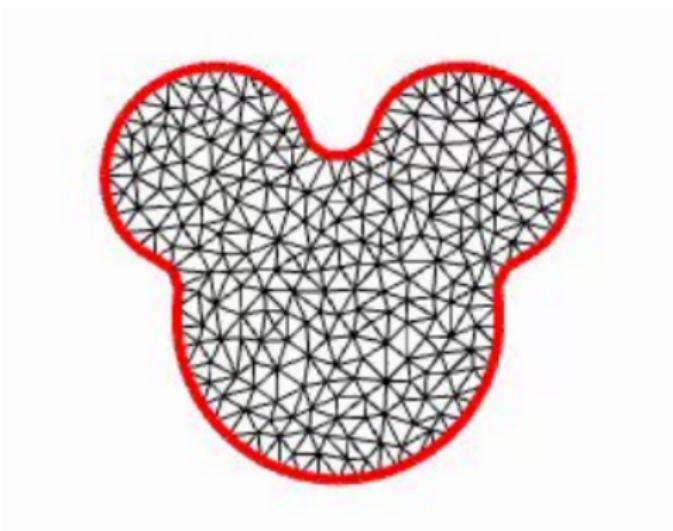


Input



Output

Discretization: shapes



Discretization: Laplace-Beltrami operator

Classical FEM discretization yields:

$$\mathbf{L} = \mathbf{A}^{-1} \mathbf{S}$$

where $\mathbf{S} = (s_{ij})$ is the **stiffness matrix** and $\mathbf{A} = (a_{ij})$ is the **mass matrix**.

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$$s_{ij} = \begin{cases} \frac{-\ell_{ij}^2 + \ell_{jk}^2 + \ell_{ki}^2}{8A_{ijk}} + \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in E_i \\ \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{hi}^2}{8A_{ijh}} & \text{if } e_{ij} \in E_b \\ -\sum_{k \neq i} s_{ik} & \text{if } i = j \end{cases}$$

$$a_{ij} = \begin{cases} \frac{1}{3} \sum_{lk:ilk \in F} A_{ilk} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

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$$\ell_{ij}(\mathbf{V}) = \|\mathbf{v}^i - \mathbf{v}^j\|_2$$

Optimization

$$\min_{\mathbf{V} \in \mathbb{R}^{n \times d}} \|\boldsymbol{\lambda}(\Delta_X(\mathbf{V})) - \boldsymbol{\mu}\|_\omega + \rho_X(\mathbf{V})$$

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A screenshot of a Google search results page. The search bar at the top contains the query "isospectralization". Below the search bar, the "All" tab is selected, followed by other tabs for Maps, Videos, Images, Shopping, and More. To the right of the tabs are links for Settings and Tools. The main search results area is empty, displaying the message "Your search - **isospectralization** - did not match any documents." Below this message, there is a section titled "Suggestions:" with three bullet points: "Make sure that all words are spelled correctly.", "Try different keywords.", and "Try more general keywords."

Your search - **isospectralization** - did not match any documents.

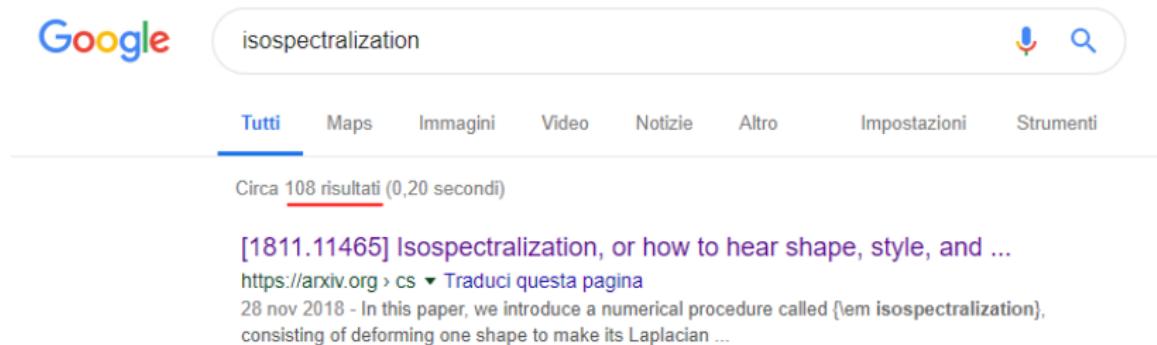
Suggestions:

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Optimization

$$\min_{\mathbf{V} \in \mathbb{R}^{n \times d}} \|\boldsymbol{\lambda}(\Delta_X(\mathbf{V})) - \boldsymbol{\mu}\|_\omega + \rho_X(\mathbf{V})$$

We call this procedure [isospectralization](#)



A screenshot of a Google search results page. The search bar contains the query "isospectralization". Below the search bar, there are several navigation links: "Tutti" (highlighted in blue), "Maps", "Immagini", "Video", "Notizie", "Altro", "Impostazioni", and "Strumenti". A link below the navigation bar indicates "Circa 108 risultati (0,20 secondi)". The first result is a purple link titled "[1811.11465] Isospectralization, or how to hear shape, style, and ...". Below the title, it shows the URL "https://arxiv.org > cs ▾ Traduci questa pagina" and the date "28 nov 2018". The snippet of the page content reads: "In this paper, we introduce a numerical procedure called {em isospectralization}, consisting of deforming one shape to make its Laplacian ...".

Details

- Weighted norm:

$$\|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_{\omega} = \sum_{i=1}^k \frac{1}{i} (\lambda_i - \mu_i)^2$$

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- Regularizers:

$$\rho_X(\mathbf{V}) = \rho_{X,1}(\mathbf{V}) + \rho_{X,2}(\mathbf{V})$$

$$\rho_{X,1}(\mathbf{V}) = \sum_{e_{ij} \in E_b} \ell_{ij}^2(\mathbf{V})$$

$$\rho_{X,2}(\mathbf{V}) = \left(\sum_{ijk \in F} (\mathbf{R}_{\frac{\pi}{2}}(\mathbf{v}^j - \mathbf{v}^i))^{\top} (\mathbf{v}^k - \mathbf{v}^i) \right)_-$$

Implementation

- Only 30 eigenvalues are used

Implementation

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- Alternate optimization of boundary and interior points every 10 iterations

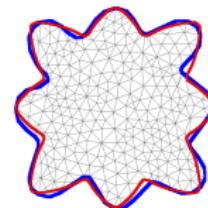
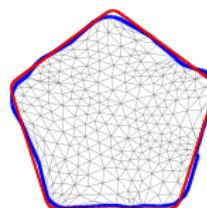
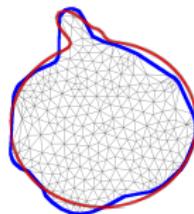
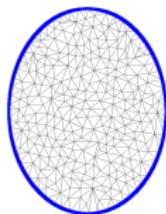
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- Alternate optimization of boundary and interior points every 10 iterations
- Re-sampling step is performed once every 2000 iterations
- Adam optimizer implementation of Tensorflow

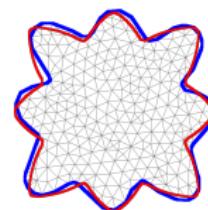
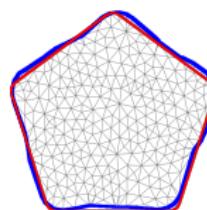
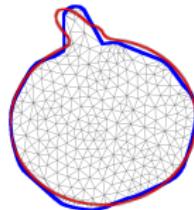
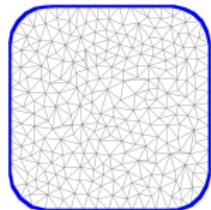
Initialization



0.92

0.95

0.93

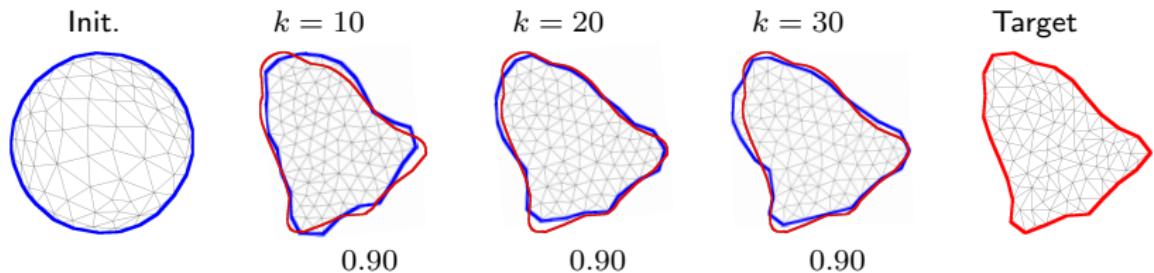


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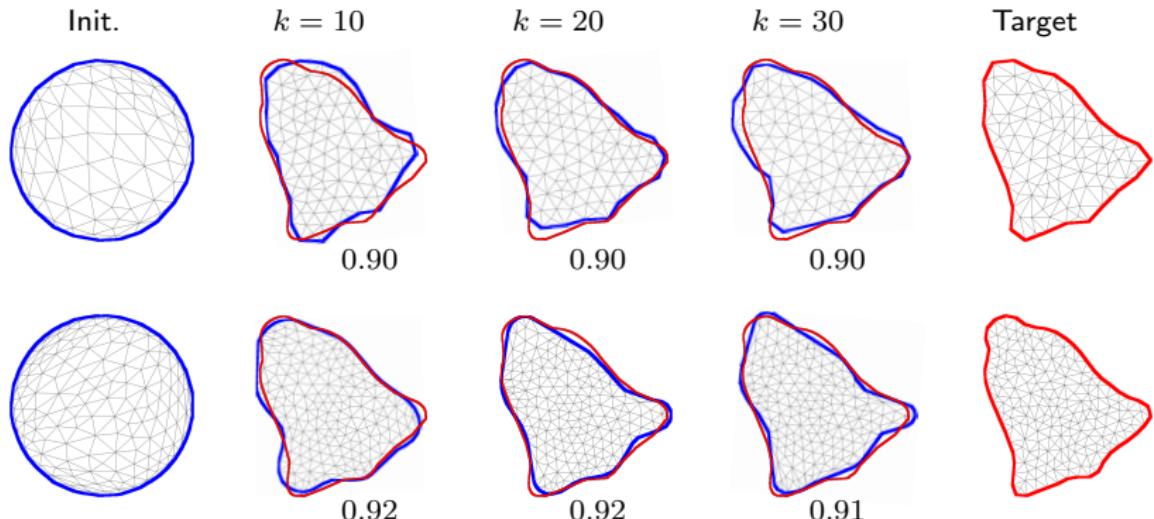
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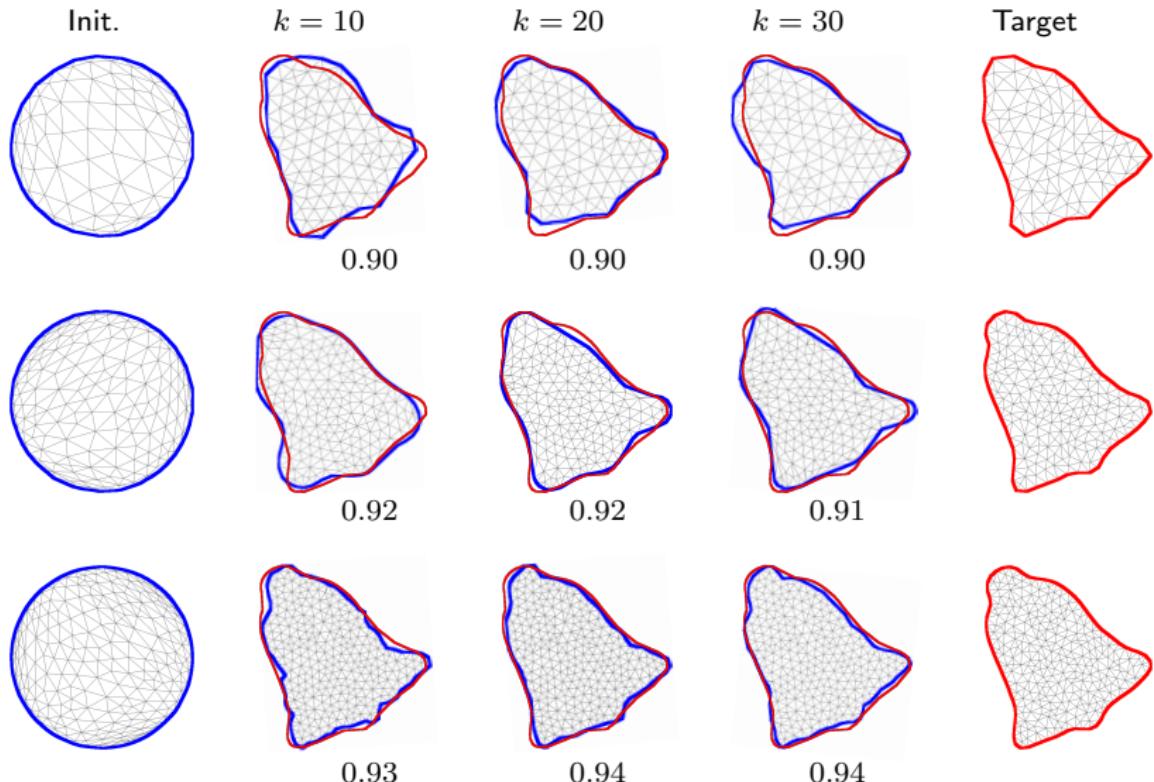
Bandwidth and mesh resolution



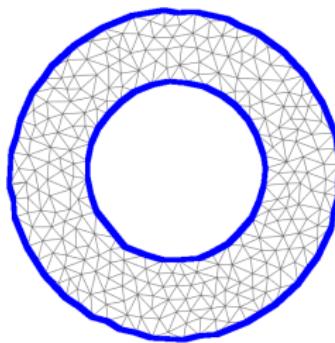
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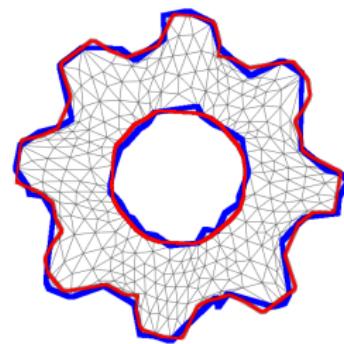
Bandwidth and mesh resolution



Topology

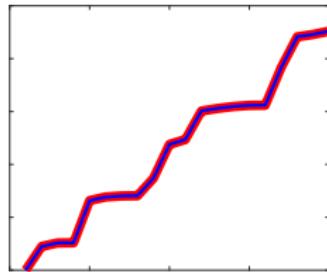
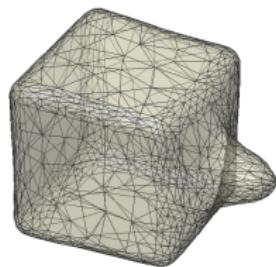
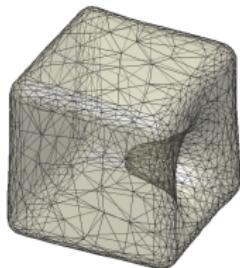


Initialization

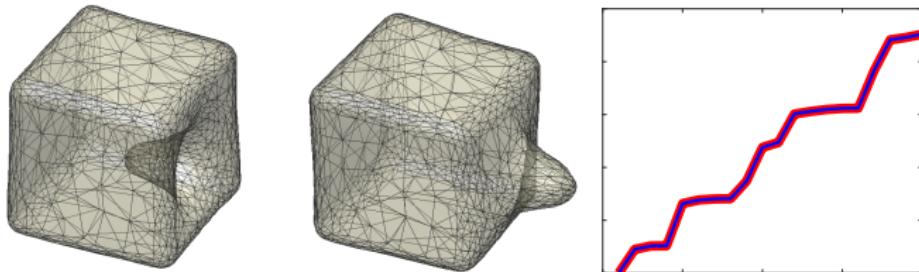


Reconstruction

Reconstruction of surfaces



Reconstruction of surfaces



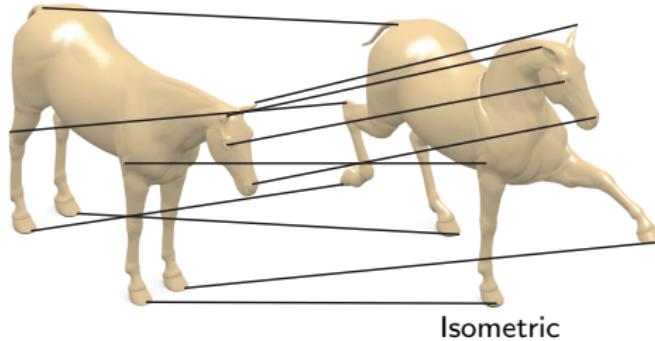
Different regularizers are needed:

$$\rho_X(\mathbf{V}) = \rho_{X,1}(\mathbf{V}) + \rho_{X,2}(\mathbf{V})$$

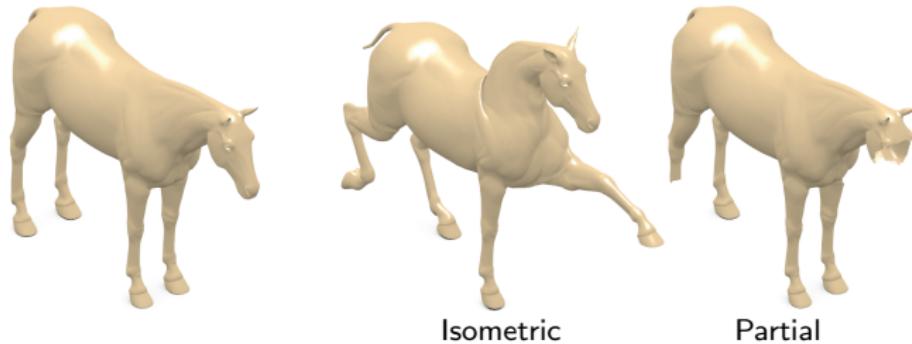
$$\rho_{X,1}(\mathbf{V}) = \|\mathbf{L}\mathbf{V}\|_F^2$$

$$\rho_{X,2}(\mathbf{V}) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)^\top \sum_{ijk \in F} ((\mathbf{v}^j - \mathbf{v}^i) \times (\mathbf{v}^k - \mathbf{v}^j))(\mathbf{v}^i + \mathbf{v}^j + \mathbf{v}^k)$$

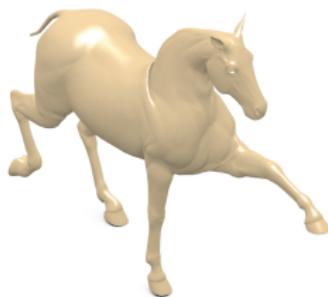
Applications: non-isometric shape matching



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Isometric

Partial



Different representation

Applications: non-isometric shape matching



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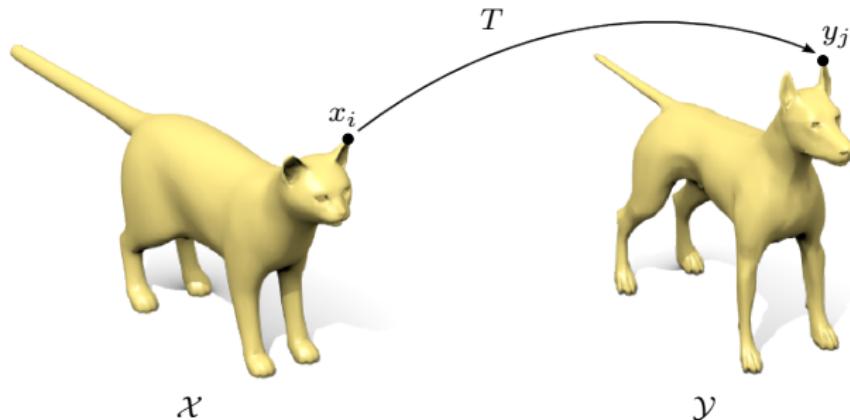
Partial



Different representation

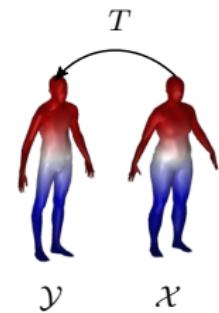
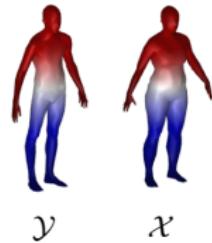
Non-isometric

Applications: non-isometric shape matching

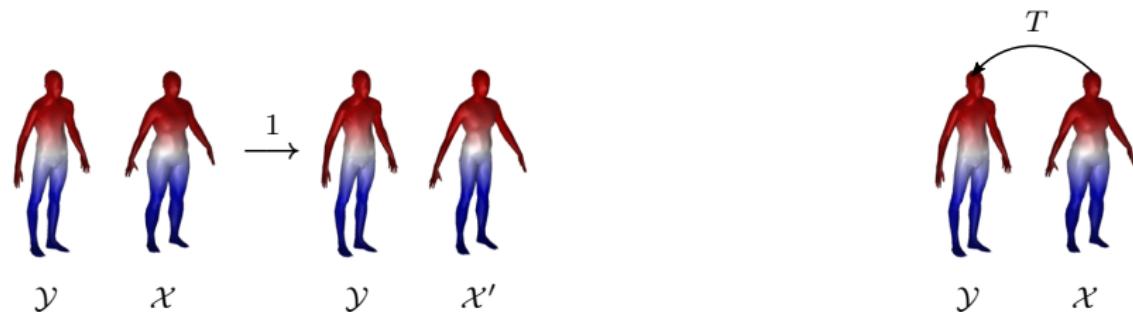


Point-wise map $T: \mathcal{X} \rightarrow \mathcal{Y}$

Our approach

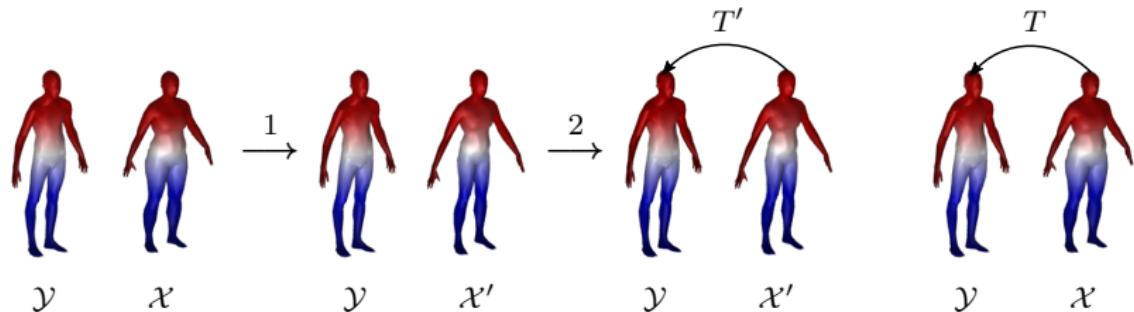


Our approach



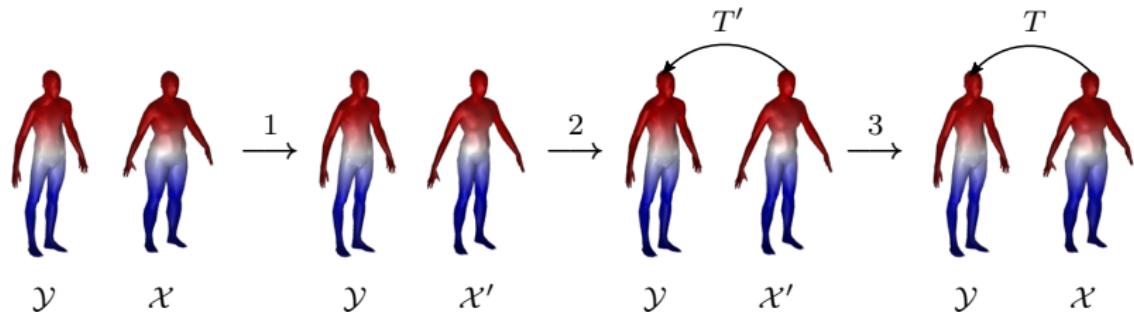
- ➊ Deform \mathcal{X} to obtain \mathcal{X}' whose spectrum $\lambda_{\mathcal{X}'}$ is aligned with $\lambda_{\mathcal{Y}}$

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- ➋ Compute the correspondence $T' : \mathcal{X}' \rightarrow \mathcal{Y}$ (using an existing isometric matching algorithm)

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- ② Compute the correspondence $T' : \mathcal{X}' \rightarrow \mathcal{Y}$ (using an existing isometric matching algorithm)
- ③ Convert T' to $T : \mathcal{X} \rightarrow \mathcal{Y}$ using the identity map between \mathcal{X} and \mathcal{X}' .

Isometric shape matching

Algorithm based on [functional maps](#):

Isometric shape matching

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- ➊ Solve for a functional map between \mathcal{Y} and \mathcal{X}' using several constraints:
 - descriptors-preservation (wave kernel signature)
 - regularization (commutativity with the Laplace operator)

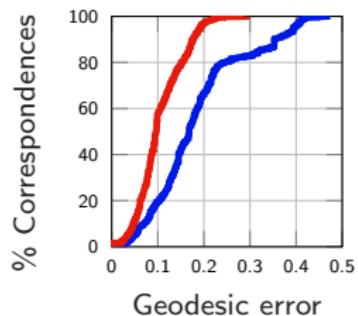
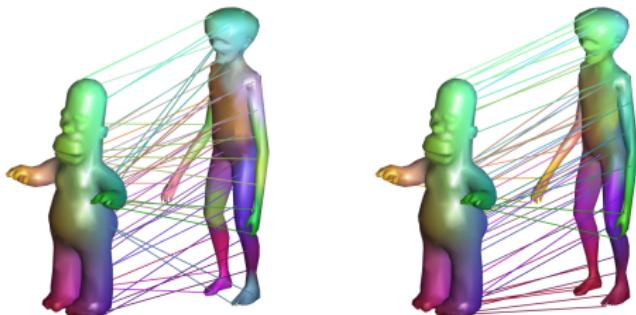
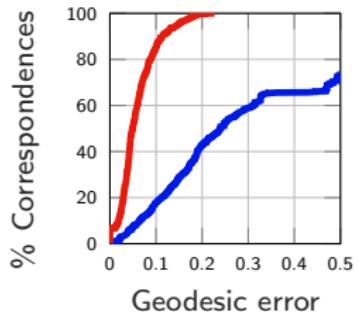
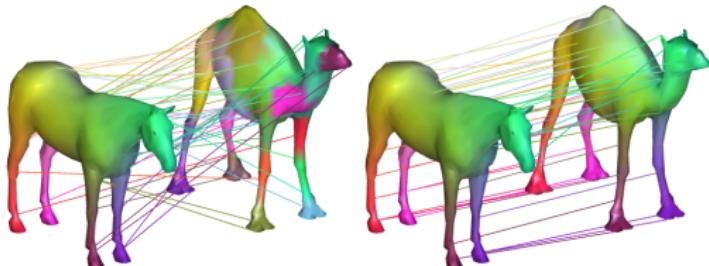
Isometric shape matching

Algorithm based on [functional maps](#):

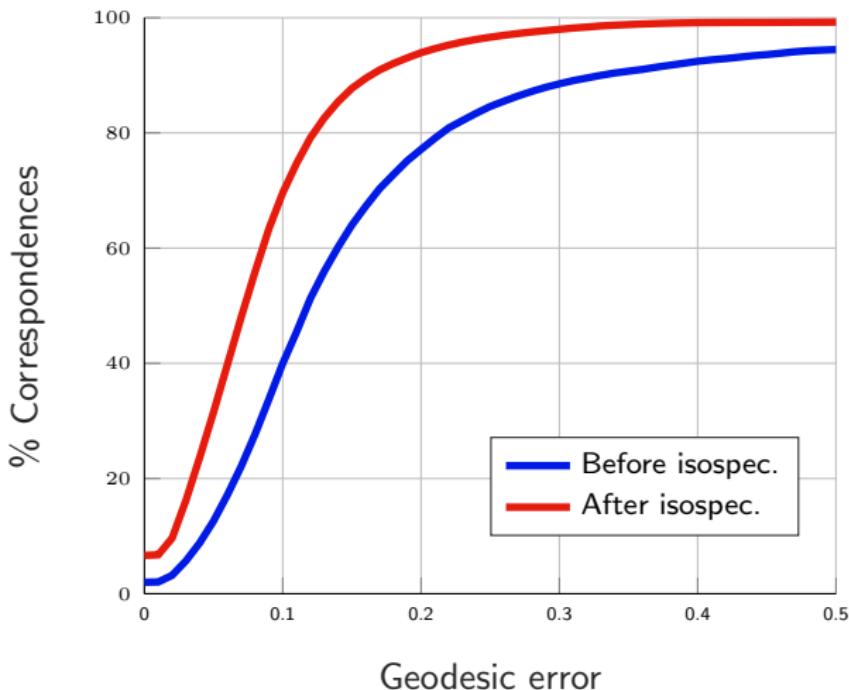
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 - descriptors-preservation (wave kernel signature)
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- ➋ Convert the functional map to a pointwise one:
 - simple nearest neighbor search in the spectral domain

Results: examples

Before isospectralization **After** isospectralization



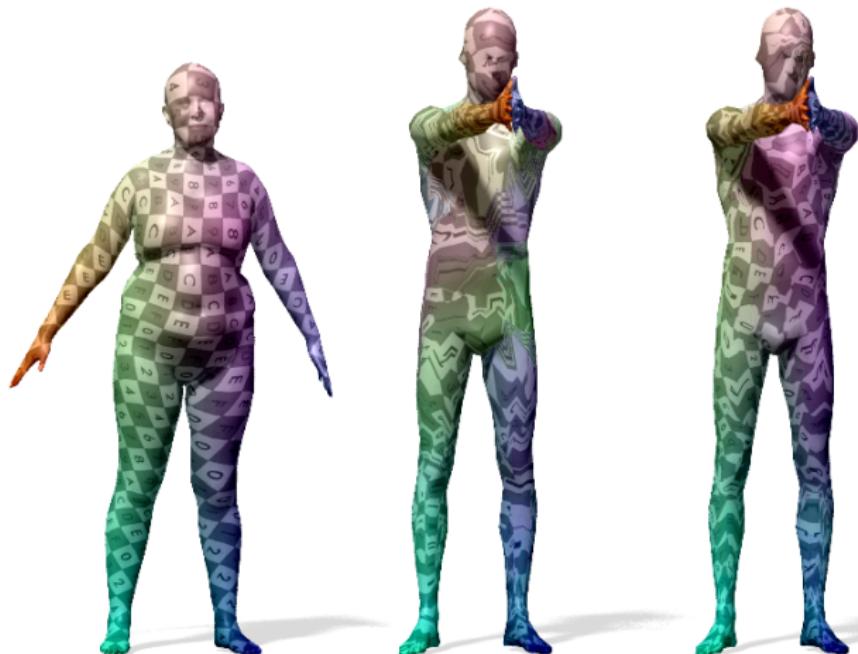
Results: average geodesic error



Results: other examples



Results: other examples



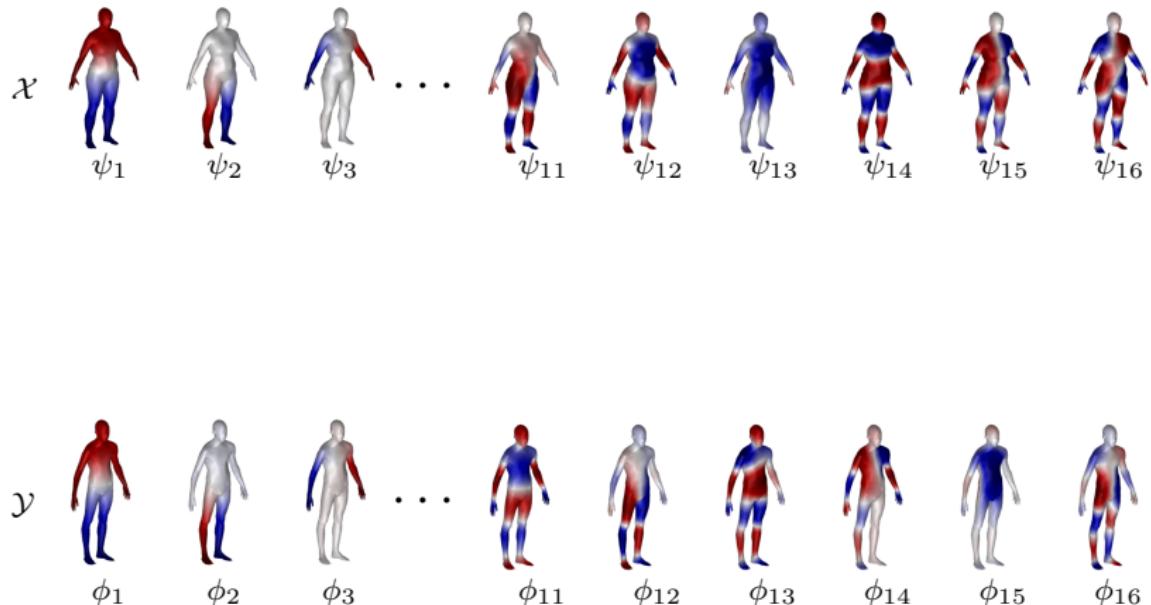
Results: other examples



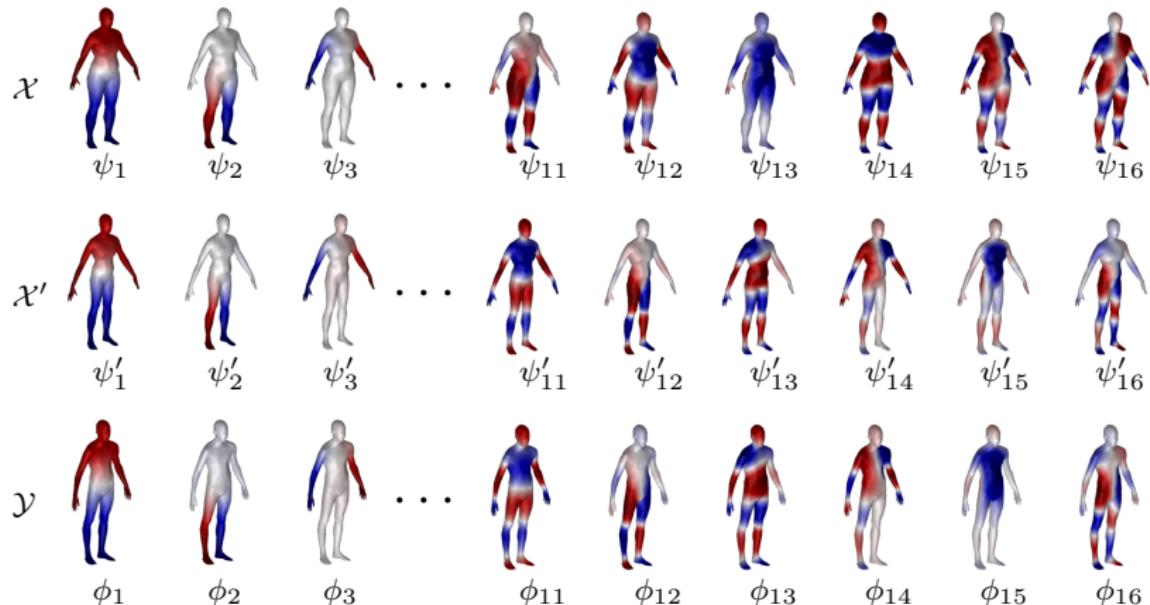
Results: other examples



Why it works

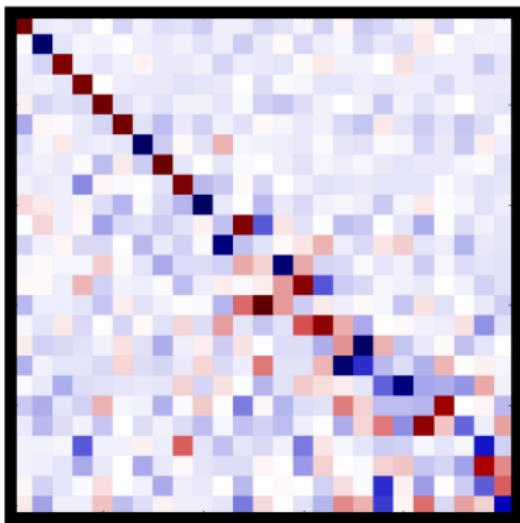


Why it works

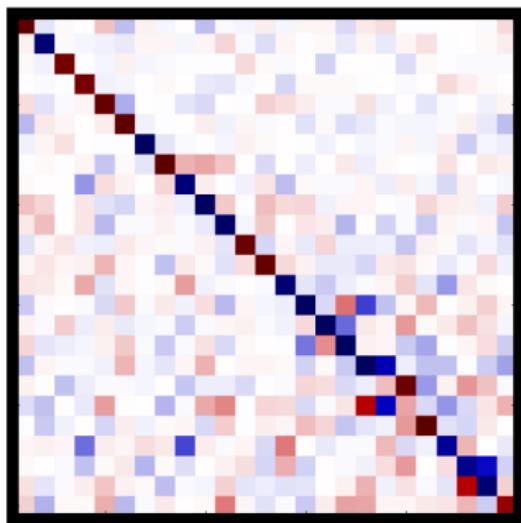


Isospectralization induces an alignment of eigenspaces.

Functional map: before



Functional map: after

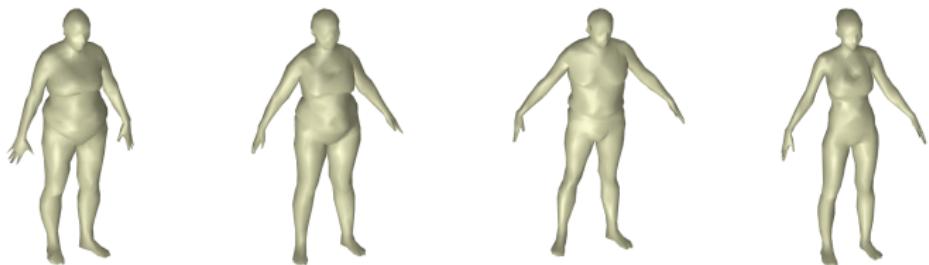


Applications: style transfer

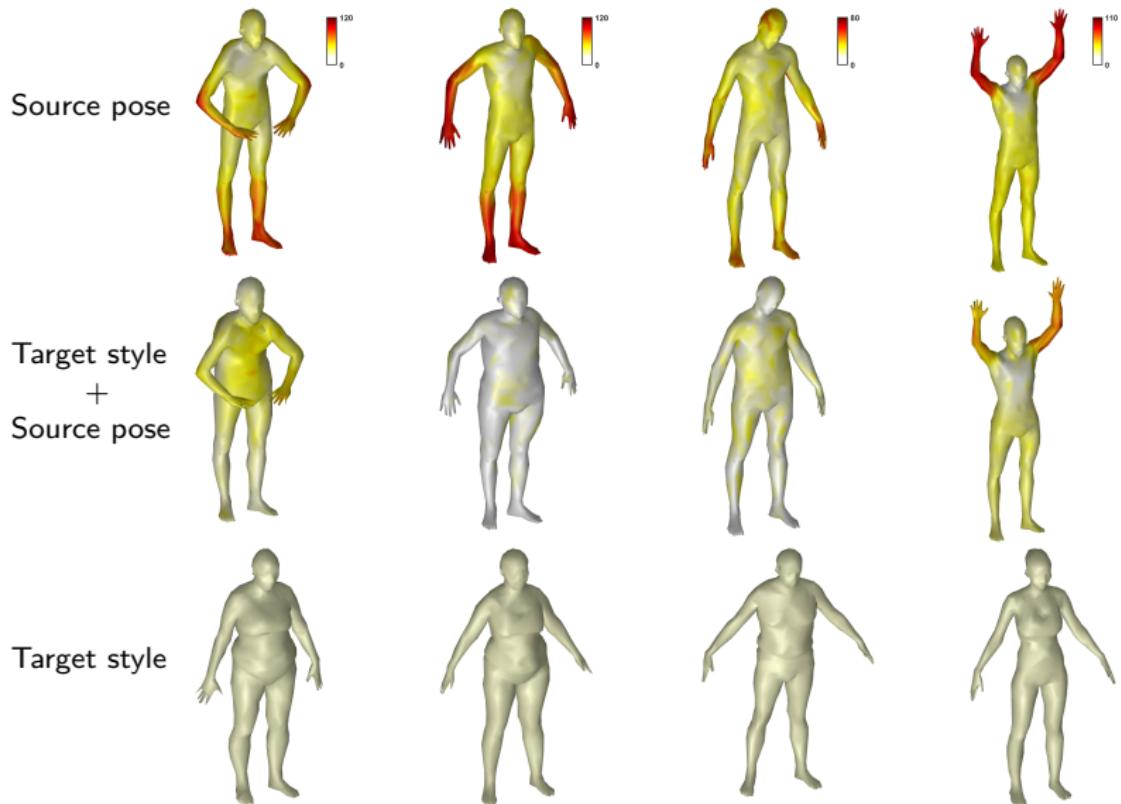
Source pose



Target style



Applications: style transfer



Conclusions and perspectives

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- Countless directions to explore...

Thank you!