Machine Learning

Regression problems

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UNIVERSITÀ DI ROMA

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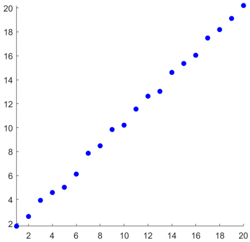
Linear regression

We start from the simplest non-trivial case for a learning model:



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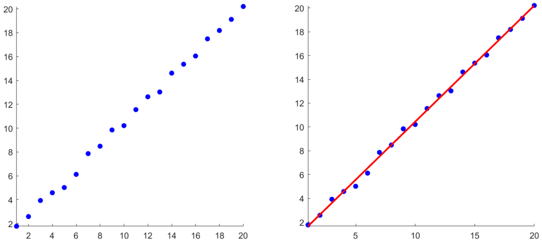

$$y_i = ax_i + b$$

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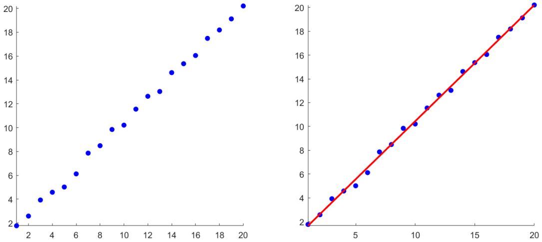

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Data: n pairs (x_i, y_i); the x_i are called the regressors

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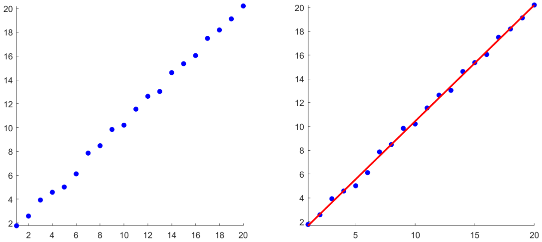

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Given a and b, we have a mapping that gives new output from new input.

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When f_Θ is linear, this is called a least-squares approximation problem.

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The error criterion w.r.t. the parameters is also called a loss or energy function, usually denoted by ℓ:

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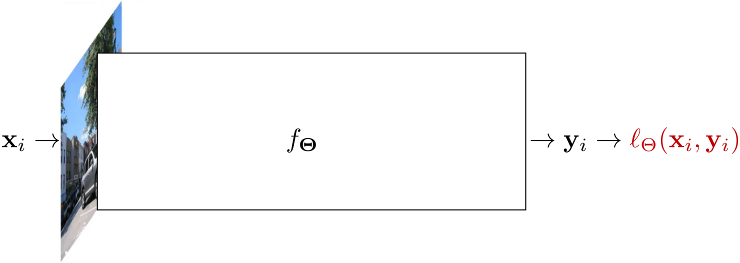
Remark: We minimize the energy w.r.t. the parameters Θ, and **not** w.r.t. the data (x_i, y_i).

Linear regression

We are considering the following case:



where $f_{\boldsymbol{\Theta}}$ is linear, and ℓ_{Θ} is quadratic.

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Let's see what optimization problems we can solve easily!

Convexity and gradients

Convex functions

Jensen's inequality:

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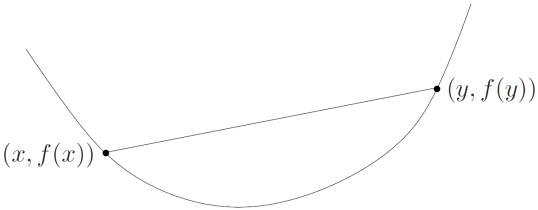

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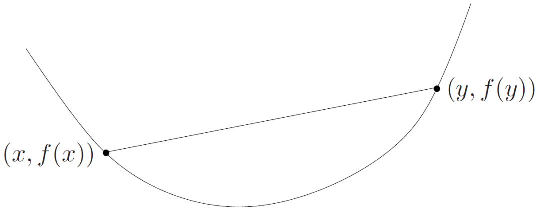

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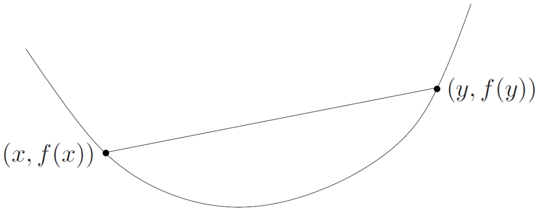

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Theorem: the global minimizer x is where $\frac{df(x)}{dx} = 0$.

Convex functions on $\mathbb{R}^n$

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Convexity is defined as before:

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$$

and we also have the global optimality condition:

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \mathbf{0} \quad \Longrightarrow \quad f(\mathbf{x}) \leq f(\mathbf{y}) \text{ for all } \mathbf{y} \in \mathbb{R}^n$$

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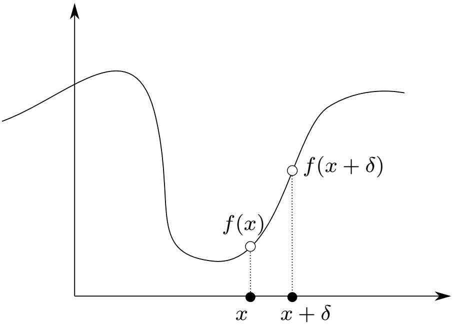

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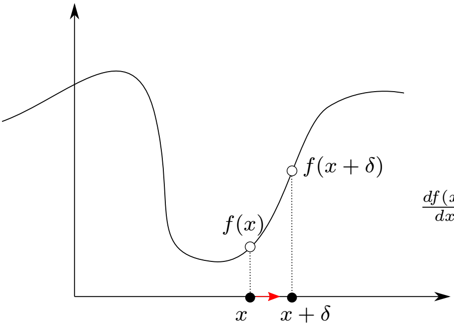

$$\frac{df(x)}{dx} = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

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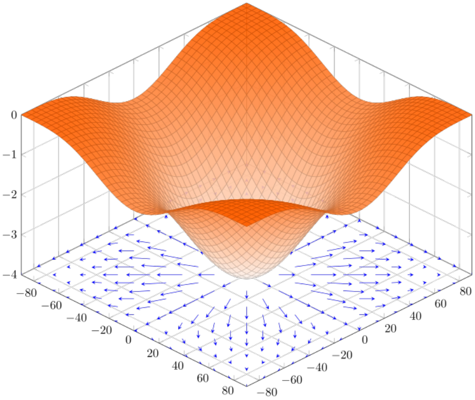

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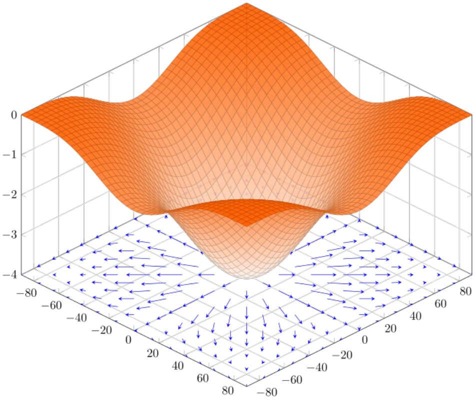


The length of the gradient vector encodes its steepness.

Convex functions: Global minima

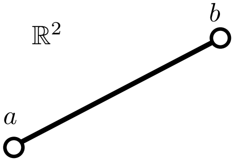
To summarize:

If $f(x)$ is convex, then a global minimizer is found by setting $\frac{df(x)}{dx} = 0$ and solving for x.

If $f(\mathbf{x})$ is convex, then a global minimizer is found by setting $\nabla_{\mathbf{x}} f(\mathbf{x}) = \mathbf{0}$ and solving for $\mathbf{x}$.

Vector lengths

How to measure the length of the gradient? Let's first start from the definition of Euclidean distance, which measures the length of any straight line connecting two points:



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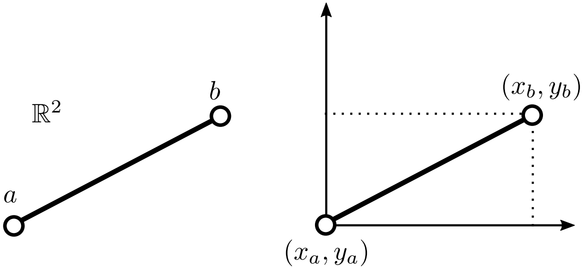

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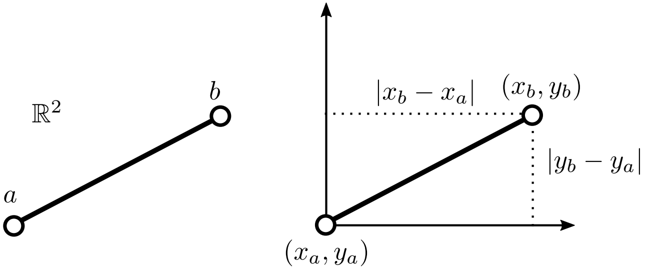

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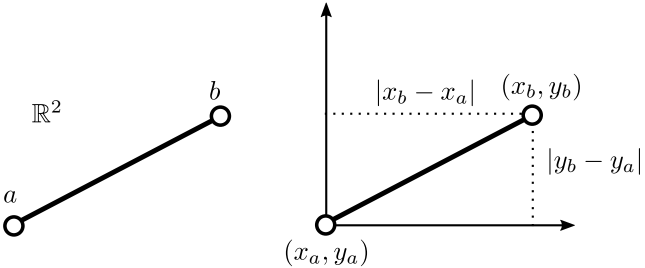

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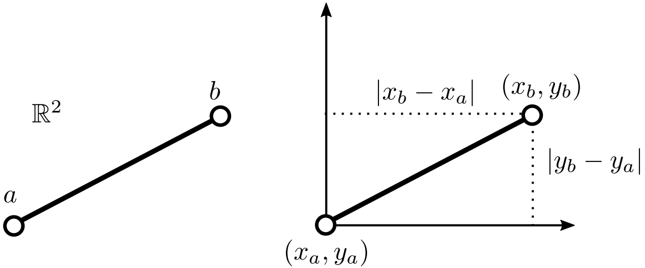

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In matrix notation:

$$d(\mathbf{a},\mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_2$$

where $\mathbf{a} = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_b \\ y_b \end{pmatrix}$

L_p distance in $\mathbb{R}^k$

One can generalize to different power coefficients $p \geq 1$:

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The length (or norm) of a vector is simply its distance from the origin:

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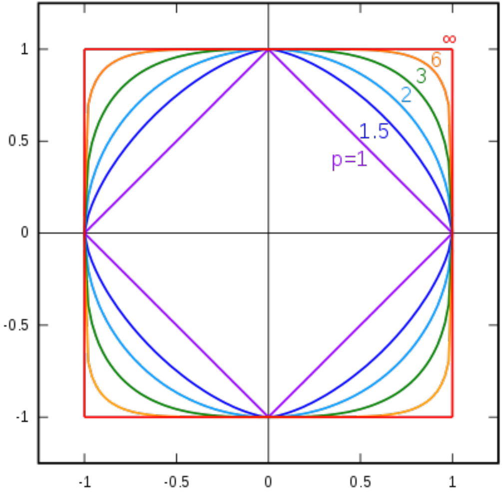
L_p unit balls in $\mathbb{R}^2$



Unit balls on manifolds

The notion of unit ball makes sense in any metric space, as it only depends on the presence of a distance function.

anisotropic

geodesic

L_7 in $\mathbb{R}^3$

Each isoline identifies points at the same distance from the source

Normal equation

Linear regression: Finding a solution

$$\min_{a,b\in\mathbb{R}} \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

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$$\mathbf{\Theta}^* = \arg\min_{\mathbf{\Theta} \in \mathbb{R}^2} \ell(\mathbf{\Theta})$$

where $\ell : \mathbb{R}^2 \to \mathbb{R}$ is defined as:

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A solution is found by setting $\nabla_{\boldsymbol{\Theta}}\ell(\boldsymbol{\Theta}) = \mathbf{0}$:

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&= \sum_{i=1}^{n} \begin{pmatrix} 2ax_i^2 - 2x_iy_i + 2bx_i \\ 2b - 2y_i + 2ax_i \end{pmatrix}
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We get 2 linear equations in the 2 unknowns a, b:

$$\begin{pmatrix} \sum_{i=1}^{n} ax_i^2 + bx_i - x_iy_i \\ \sum_{i=1}^{n} ax_i + b - y_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Linear regression: Matrix notation

The learning model of linear regression is linear in the parameters (while it is **not** linear in x, due to the bias).

Therefore, in matrix notation the equations $y_i = ax_i + b$ read:

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}}_{\mathbf{X}} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\boldsymbol{\theta}}$$

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Remark: Deep learning frameworks frequently use the alternative expression with the bias encoded separately:

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\mathbf{y}} = a \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{X}} + b$$

Familiarize with matrix calculus.

When implementing a ML system, we manipulate matrices, vectors, and tensors.

Linear regression: Matrix notation

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}}_{\mathbf{X}} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\boldsymbol{\theta}}$$

This expresses all the equations $y_i = ax_i + b$ at once and makes the linearity w.r.t. a, b evident.

The MSE is simply:

$$\ell(\boldsymbol{\theta}) = \|\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\|_2^2$$

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$$-2\mathbf{X}^\top\mathbf{y} + 2\mathbf{X}^\top\mathbf{X}\boldsymbol{\theta} = \mathbf{0}$$

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Setting $\nabla_{\boldsymbol{\theta}} \ell = \mathbf{0}$ we get:

$$\boldsymbol{\theta} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$$

We get a closed form solution to our problem (aka normal equation).

Linear regression: Higher dimensions

Until now we have seen the case where:

$$y_i = ax_i + b \quad \text{for } i = 1, \ldots, n$$

that is, each data point is one-dimensional (just one number).

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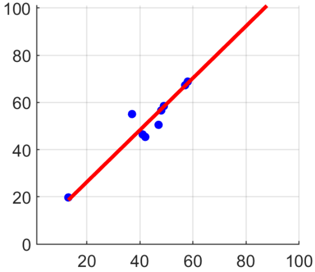
Defining the matrices $\mathbf{X} = \begin{pmatrix} | & | & \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots \\ | & | & \\ 1 & 1 & \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} | & | & \\ \mathbf{y}_1 & \mathbf{y}_2 & \cdots \\ | & | & \end{pmatrix}, \boldsymbol{\Theta} = \begin{pmatrix} \mathbf{A} \\ \mathbf{b}^\top \end{pmatrix}$,

we get a closed-form solution to $\nabla_{\boldsymbol{\Theta}} \ell(\boldsymbol{\Theta}) = \mathbf{0}$:

$$\boldsymbol{\Theta} = (\mathbf{X}\mathbf{X}^\top)^{-1}\mathbf{X}\mathbf{Y}^\top$$

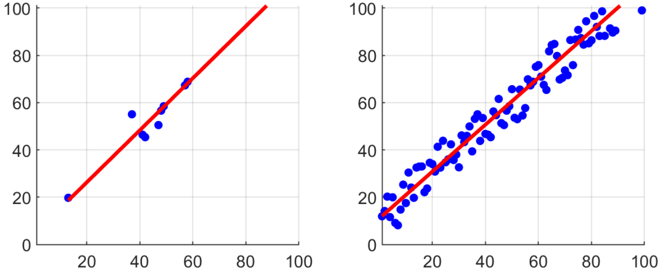
Polynomial regression

Data distribution



Assumption: linear model

Data distribution



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More data allows us to improve our prediction

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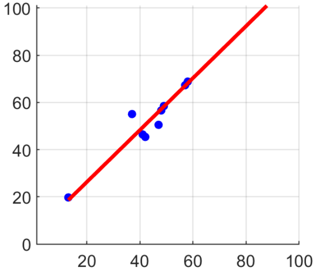

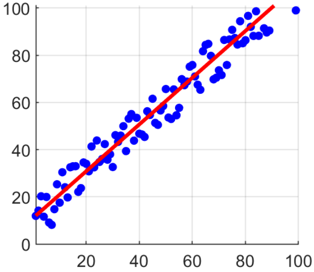

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What if the assumption (i.e. linear prior here) is **wrong**?

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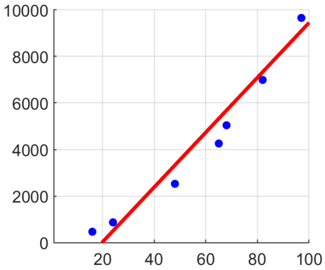

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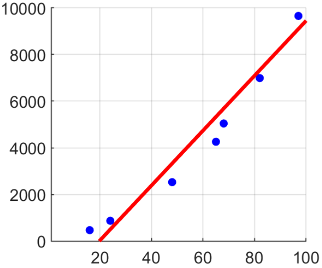

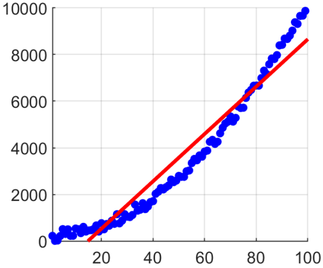

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More data confutes our assumptions

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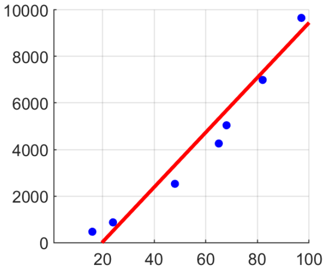

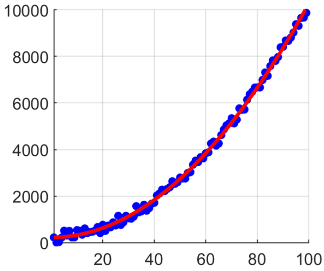

Assumption: quadratic model

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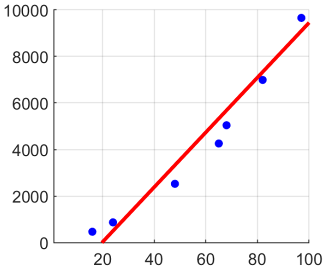

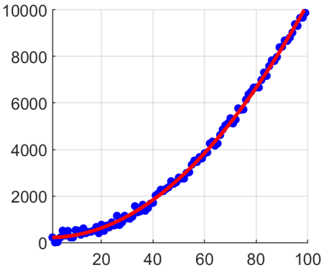

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Key questions:

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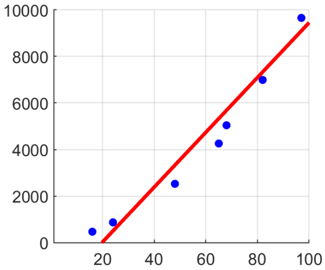

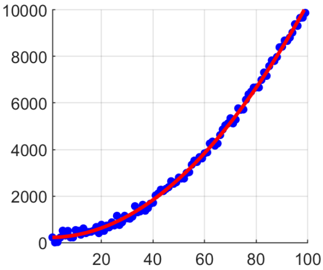

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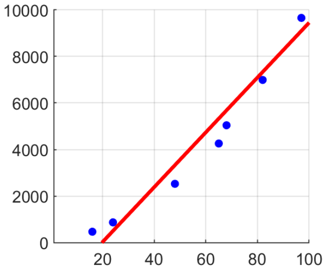

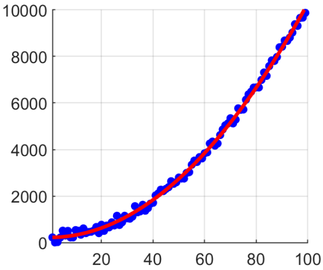

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Key questions:

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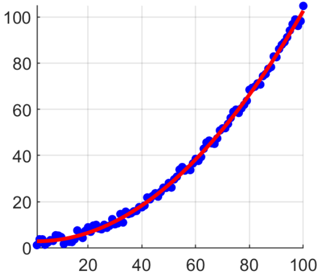
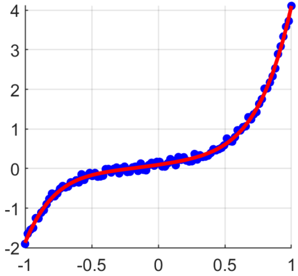
- How much data do we need?
- What if the correct distribution does not admit a simple expression?

Polynomial regression

After the linear model, the simplest thing is a polynomial model.



2nd-order



5th-order

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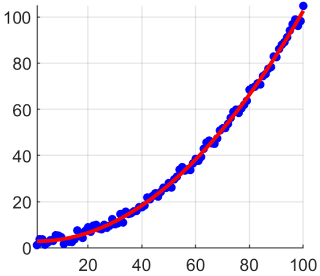

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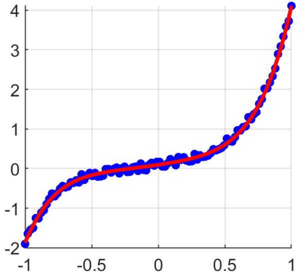

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The number of parameters grows with the order.

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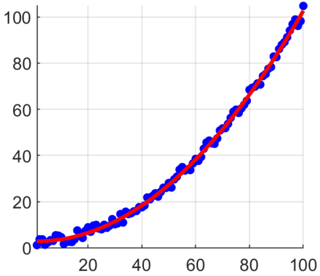

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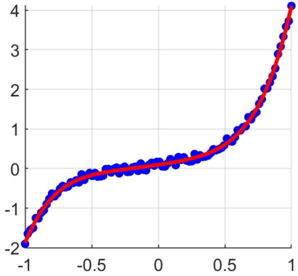

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The number of parameters grows with the order.

More data are needed to make an informed decision on the order.

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Linear regression with polynomial features

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$$y_i = b + \sum_{j=1}^{k} a_j x_i^j \quad \text{for all data points } i = 1, \ldots, n$$

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The same exact least-squares solution as with linear regression applies, with the requirement that $k < n$.

Underfitting and overfitting

Polynomial fitting

An application of the Stone-Weierstrass theorem tells us:

If f is continuous on the interval $[a, b]$, then for every $\epsilon > 0$ there exists a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all x.

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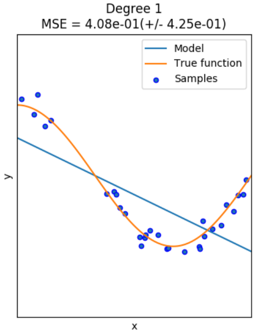

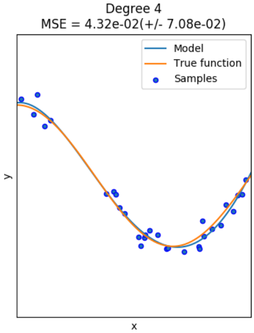

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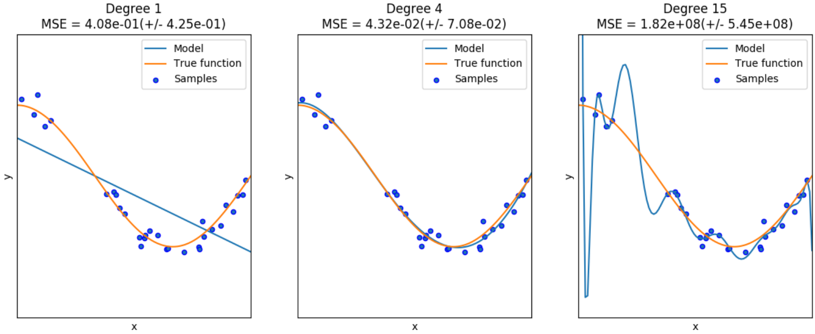
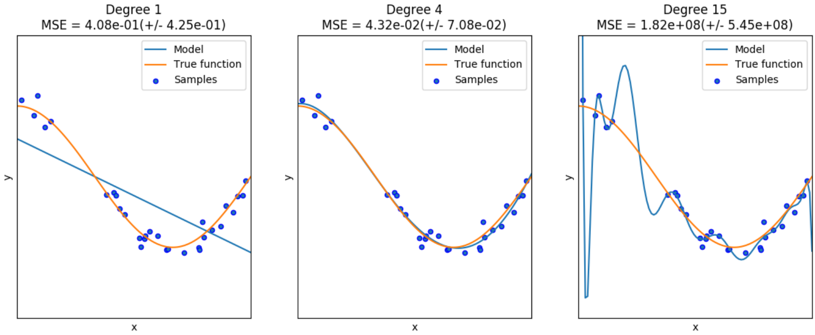


Underfitting vs. Overfitting



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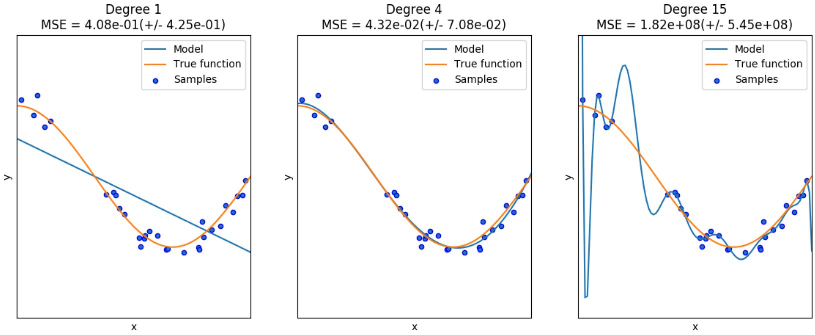

- Underfitting: Not sufficiently fitting the data (large MSE).

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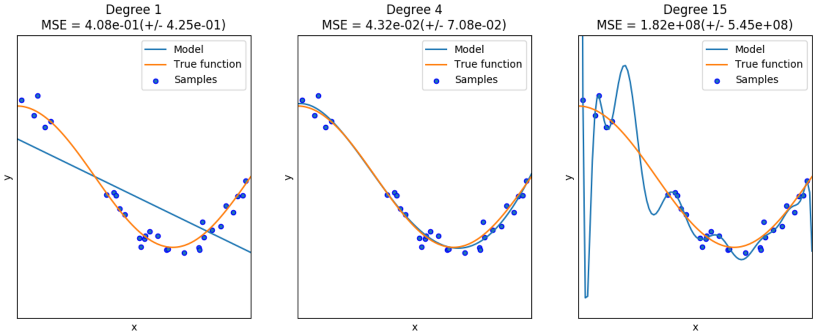

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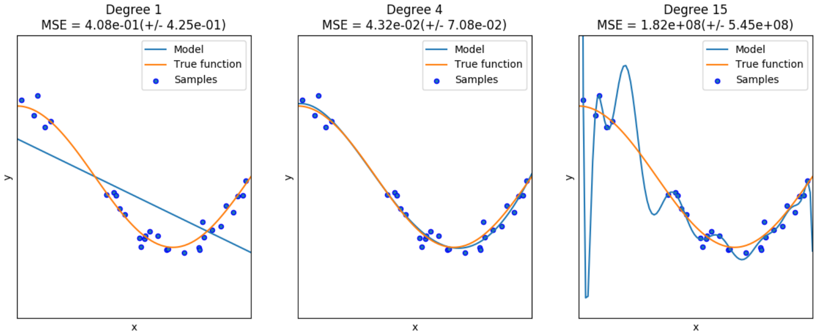

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Adding complexity can lead to overfitting and thus worse generalization.

Underfitting vs. Overfitting

Mitigation measures:

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3. Small MSE $\Rightarrow$ Test on a validation set
4. Large MSE on the validation $\Rightarrow$ overfitting $\Rightarrow$ bad generalization

To recap, overfitting happens with small training error and large validation error

n-fold cross-validation

We do not want to overfit on the validation set either!

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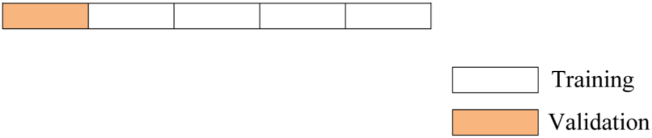

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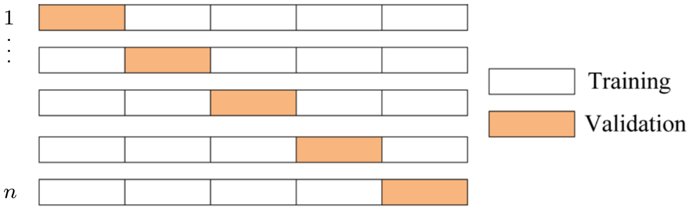

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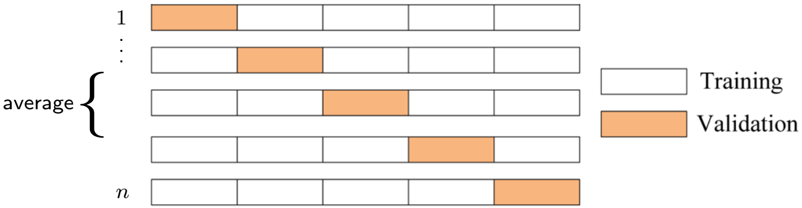

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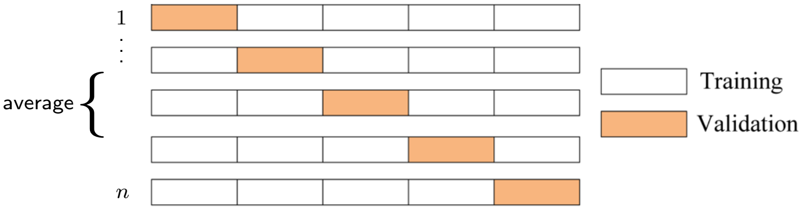

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Example: For polynomial regression, do the above many times with different degrees, choose the run with the smallest average MSE.

Not done yet

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So is polynomial regression all we need?

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So is polynomial regression all we need?

Not really!

- Different loss than MSE
- Regularization
- Additional priors
- Intermediate features
- Flexibility
- Regression (predict a value) vs. classification (predict a category)

Regularization penalties

Sometimes our prior knowledge can be expressed in terms of an energy.

For example, avoid large parameters to counteract overfitting:

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More in general:

$$\min_{\boldsymbol{\Theta}} \ell_{\boldsymbol{\Theta}} + \lambda\|\Theta\|_p$$

Controlling parameter growth is generally known as shrinkage.

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More in general:

$$\min_{\boldsymbol{\Theta}} \underbrace{\ell_{\boldsymbol{\Theta}}}_{\text{convex}} + \lambda \underbrace{\|\Theta\|_p}_{\text{convex}}$$

Controlling parameter growth is generally known as shrinkage.

Regularization penalties

Sometimes our prior knowledge can be expressed in terms of an energy.

For example, avoid large parameters to counteract overfitting:

$$\min_{\boldsymbol{\Theta}} \underbrace{\ell_{\boldsymbol{\Theta}}}_{\text{data term}} + \underbrace{\lambda}_{\text{trade-off}} \cdot \underbrace{\|\Theta\|_F^2}_{\text{regularizer}}$$

Adding a quadratic penalty to the loss is also known as weight decay, ridge, or Tikhonov regularization.

More in general:

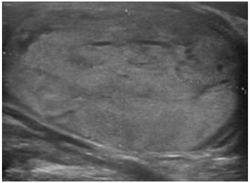
$$\min_{\boldsymbol{\Theta}} \underbrace{\ell_{\boldsymbol{\Theta}} + \lambda \|\Theta\|_p}_{\text{convex}}$$

Controlling parameter growth is generally known as shrinkage.

Logistic regression

Classification

What if we want to predict a category instead of a value?

$f_\Theta($  $) = \{0, 1\}$

Classification

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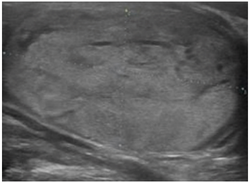
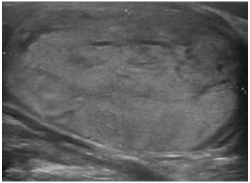
$$f_\Theta($$  $$) = \{0, 1\}$$

Possible solution: Do post-processing (e.g. thresholding) to convert linear regression to a binary output.

Classification

What if we want to predict a category instead of a value?

$f_\Theta($  $) = \{0, 1\}$

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$\Rightarrow$ The solution is not necessarily an optimum anymore.

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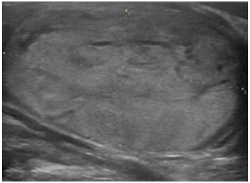
$f_\Theta($  $) = \{0, 1\}$

Possible solution: Do post-processing (e.g. thresholding) to convert linear regression to a binary output.

$\Rightarrow$ The solution is not necessarily an optimum anymore.

Instead: Modify the loss to minimize over categorical values directly.

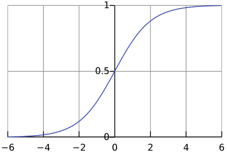
Logistic regression

New loss:

$$\ell_\Theta(\{x_i, y_i\}) = \sum_{i=1}^{n} (y_i - \sigma(\underbrace{ax_i + b}_{\text{linear}}))^2$$

Here, σ is the nonlinear logistic sigmoid:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



Logistic regression

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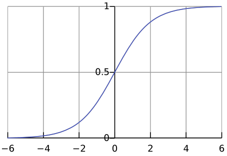


σ has a saturation effect as it maps $\mathbb{R} \mapsto (0, 1)$.

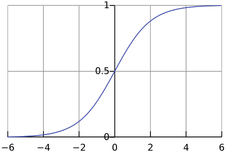
Logistic regression

New loss:

$$\ell_\Theta(\{x_i, y_i\}) = \sum_{i=1}^{n} (y_i - \sigma(\underbrace{ax_i + b}_{\text{linear}}))^2 \qquad \text{non-convex in } a, b$$

Here, σ is the nonlinear logistic sigmoid:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



σ has a saturation effect as it maps $\mathbb{R} \mapsto (0, 1)$.

Logistic regression

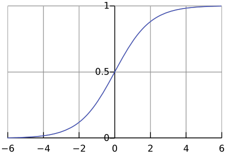
New loss:

$$\ell_\Theta(\{x_i, y_i\}) = \sum_{i=1}^{n} c(x_i, y_i)\,, \quad \text{with}$$

$$c(x_i, y_i) = \begin{cases} -\ln(\sigma(ax_i + b)) & y_i = 1 \\ -\ln(1 - \sigma(ax_i + b)) & y_i = 0 \end{cases} \qquad \text{convex}$$

Here, σ is the nonlinear logistic sigmoid:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



σ has a saturation effect as it maps $\mathbb{R} \mapsto (0, 1)$.

Logistic regression

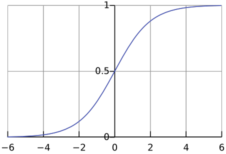
New loss:

$$\ell_\Theta(\{x_i, y_i\}) = \sum_{i=1}^{n} c(x_i, y_i)\,, \quad \text{with}$$

$$c(x_i, y_i) = -y_i \ln(\sigma(ax_i + b)) - (1 - y_i) \ln(1 - \sigma(ax_i + b)) \quad \text{convex}$$

Here, σ is the nonlinear logistic sigmoid:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



σ has a saturation effect as it maps $\mathbb{R} \mapsto (0, 1)$.

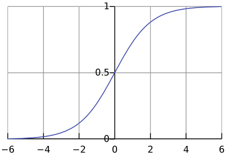
Logistic regression

New convex loss:

$$\ell_\Theta(\{x_i, y_i\}) = -\sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i)\ln(1 - \sigma(ax_i + b))$$

Here, σ is the nonlinear logistic sigmoid:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



σ has a saturation effect as it maps $\mathbb{R} \mapsto (0, 1)$.

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \ell_\Theta = 0$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) = 0$$

where $\Theta = \{a, b\}$.

Logistic regression: Finding a solution

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$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) = 0$$

where $\Theta = \{a, b\}$.

Consider the gradient of each term in the summation:

$$\nabla_\Theta \left(y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) \right)$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

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Consider the gradient of each term in the summation:

$$\nabla_\Theta y_i \ln(\sigma(ax_i + b)) + \nabla_\Theta (1 - y_i) \ln(1 - \sigma(ax_i + b))$$

Logistic regression: Finding a solution

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Consider the gradient of each term in the summation:

$$y_i \nabla_\Theta \underbrace{\ln(\sigma(ax_i + b))}_{f(g(h(\Theta)))} + (1 - y_i) \nabla_\Theta \ln(1 - \sigma(ax_i + b))$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) = 0$$

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial h} \cdot \frac{\partial h}{\partial a}$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial h} \cdot \frac{\partial}{\partial a} a x_i + b$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

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$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial h} \cdot x_i$$

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{\partial \sigma(ax_i + b)}{\partial(ax_i + b)} \cdot x_i$$

Logistic regression: Finding a solution

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{\partial}{\partial (ax_i + b)} \frac{1}{1 + e^{-(ax_i + b)}} \cdot x_i$$

Logistic regression: Finding a solution

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{e^{-(ax_i + b)}}{(1 + e^{-(ax_i + b)})^2} \cdot x_i$$

Logistic regression: Finding a solution

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{1}{1 + e^{-(ax_i + b)}} \frac{e^{-(ax_i + b)}}{1 + e^{-(ax_i + b)}} \cdot x_i$$

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{1}{1 + e^{-(ax_i + b)}} \frac{(1 + e^{-(ax_i + b)}) - 1}{1 + e^{-(ax_i + b)}} \cdot x_i$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i)\ln(1 - \sigma(ax_i + b)) = 0$$

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \frac{1}{1 + e^{-(ax_i + b)}}(1 - \frac{1}{1 + e^{-(ax_i + b)}}) \cdot x_i$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) = 0$$

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{\partial f}{\partial g} \cdot \sigma(ax_i + b)(1 - \sigma(ax_i + b)) \cdot x_i$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

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Logistic regression: Finding a solution

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Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i)\ln(1 - \sigma(ax_i + b)) = 0$$

where $\Theta = \{a, b\}$.

Consider the gradient of each term in the summation:

$$y_i \nabla_\Theta \underbrace{\ln(\sigma(ax_i + b))}_{f(g(h(\Theta)))} + (1 - y_i)\nabla_\Theta \ln(1 - \sigma(ax_i + b))$$

Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} f(g(h(a, b))) = \frac{1}{\sigma(ax_i + b)} \cdot \sigma(ax_i + b)(1 - \sigma(ax_i + b)) \cdot x_i$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) = 0$$

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Logistic regression: Finding a solution

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Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(ax_i + b)) = (1 - \sigma(ax_i + b))x_i$$

Logistic regression: Finding a solution

Since the loss is convex, the first-order conditions apply:

$$\nabla_\Theta \sum_{i=1}^{n} y_i \ln(\sigma(ax_i + b)) + (1 - y_i) \ln(1 - \sigma(ax_i + b)) = 0$$

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$$y_i \nabla_\Theta \underbrace{\ln(\sigma(ax_i + b))}_{f(g(h(\Theta)))} + (1 - y_i) \nabla_\Theta \ln(1 - \sigma(ax_i + b))$$

Apply the chain rule to each partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(ax_i + b)) = (1 - \sigma(ax_i + b)) x_i$$

...and so on for the second term and for parameter b.

Logistic regression: Finding a solution

By looking at the partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(a x_i + b)) = (1 - \sigma(a x_i + b)) x_i$$

we see that the parameters enter the gradient in a nonlinear way.

Logistic regression: Finding a solution

By looking at the partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(a x_i + b)) = (1 - \sigma(a x_i + b)) x_i$$

we see that the parameters enter the gradient in a nonlinear way.

Thus:

- $\nabla \ell_\Theta = 0$ is not a linear system that we can solve easily.

Logistic regression: Finding a solution

By looking at the partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(ax_i + b)) = (1 - \sigma(ax_i + b))x_i$$

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Thus:

- $\nabla \ell_\Theta = 0$ is not a linear system that we can solve easily.
- $\nabla \ell_\Theta = 0$ is a transcendental equation $\Rightarrow$ no analytical solution.

Logistic regression: Finding a solution

By looking at the partial derivative:

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model	**loss**	**solution**
linear regression		
linear regression + Tikhonov		
logistic regression		

Logistic regression: Finding a solution

By looking at the partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(ax_i + b)) = (1 - \sigma(ax_i + b))x_i$$

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Thus:

- $\nabla \ell_\Theta = 0$ is not a linear system that we can solve easily.
- $\nabla \ell_\Theta = 0$ is a transcendental equation $\Rightarrow$ no analytical solution.

model	**loss**	**solution**
linear regression	convex	
linear regression + Tikhonov	convex	
logistic regression	convex	

Logistic regression: Finding a solution

By looking at the partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(ax_i + b)) = (1 - \sigma(ax_i + b))x_i$$

we see that the parameters enter the gradient in a nonlinear way.

Thus:

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model	**loss**	**solution**
linear regression	convex	least squares
linear regression + Tikhonov	convex	
logistic regression	convex	

Logistic regression: Finding a solution

By looking at the partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(ax_i + b)) = (1 - \sigma(ax_i + b))x_i$$

we see that the parameters enter the gradient in a nonlinear way.

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model	**loss**	**solution**
linear regression	convex	least squares
linear regression + Tikhonov	convex	least squares
logistic regression	convex	

Logistic regression: Finding a solution

By looking at the partial derivative:

$$\frac{\partial}{\partial a} \ln(\sigma(ax_i + b)) = (1 - \sigma(ax_i + b))x_i$$

we see that the parameters enter the gradient in a nonlinear way.

Thus:

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- $\nabla \ell_\Theta = 0$ is a transcendental equation $\Rightarrow$ no analytical solution.

model	**loss**	**solution**
linear regression	convex	least squares
linear regression + Tikhonov	convex	least squares
logistic regression	convex	nonlinear optimization

Suggested reading

For convexity and optimality, read Sections 3.1.1 and 3.1.3 of the book:

S. Boyd & L. Vandenberghe, "Convex optimization". Cambridge University Press, 2009

Public download link: `https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf`

On polynomial regression vs. neural nets:
`https://ar5iv.org/abs/1806.06850`

Proof that the logistic loss is convex:
`https://math.stackexchange.com/questions/1582452/logistic-regression-prove-that-the-cost-function-is-convex`