Computational Spectral Geometry:
The Inverse Problem in Applications

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Outline

- The inverse eigenvalue problem
- Methods: optimization
- Methods: data-driven approaches
- Applications

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CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

«Can one hear the shape of the drum?»

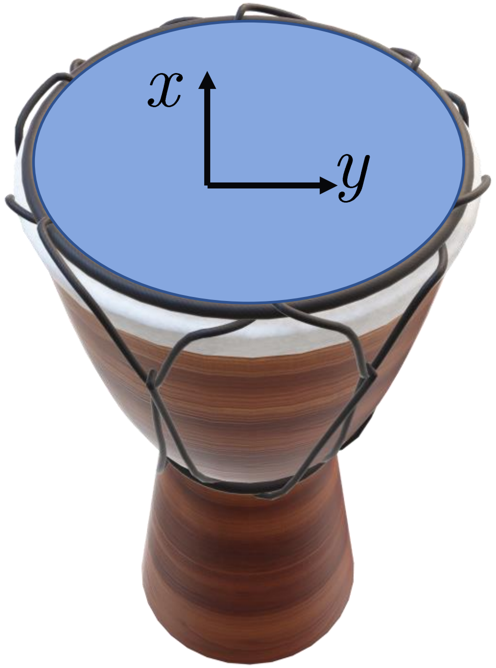
«Can one hear the shape of the drum?»

Wave equation:

$$\Delta u(x, y; t) = -\frac{\partial^2 u(x,y;t)}{\partial t^2}$$

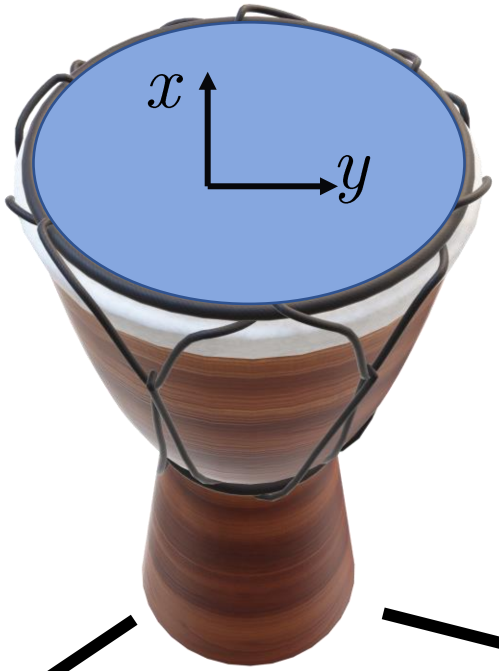
$u(x,y;t)$ is the displacement field at point (x,y) and time t

Δ is the Laplacian

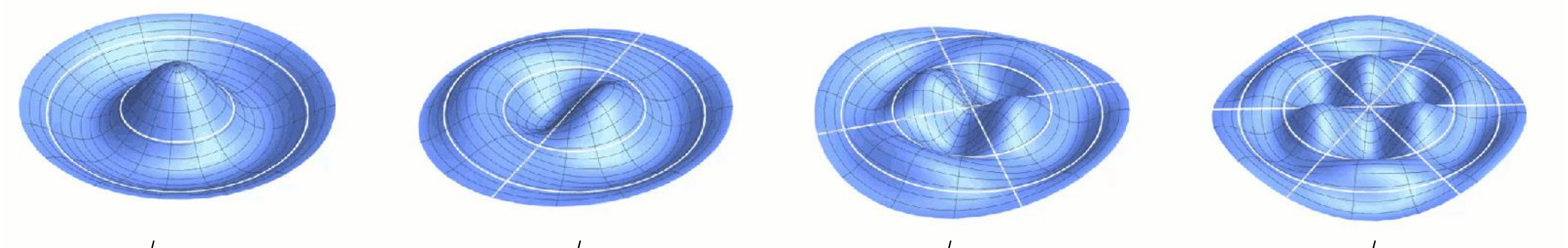


Eigenvalue problem for the Laplacian

$$\Delta\phi = \lambda\phi$$

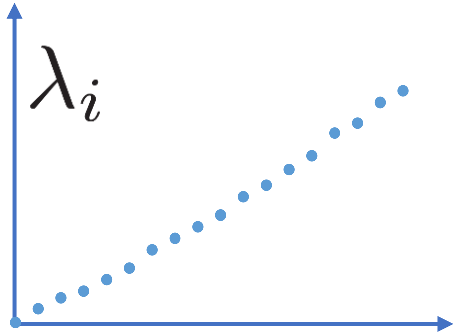


Eigenfunctions



ϕ_1 ϕ_2 ϕ_3 ϕ_4

Spectrum / eigenvalues



Why the eigenvalue problem?

$$\Delta u = -\frac{\partial^2 u}{\partial t^2}$$

$$\Delta \phi f = -\frac{\partial^2 \phi f}{\partial t^2}$$

$$\frac{\Delta \phi}{\phi} = -\frac{f''}{f}$$

$$\boxed{\Delta \phi = \lambda \phi}$$

$$-\frac{f''}{f} = \lambda$$

$$f(t) = e^{it\sqrt{\lambda}}$$

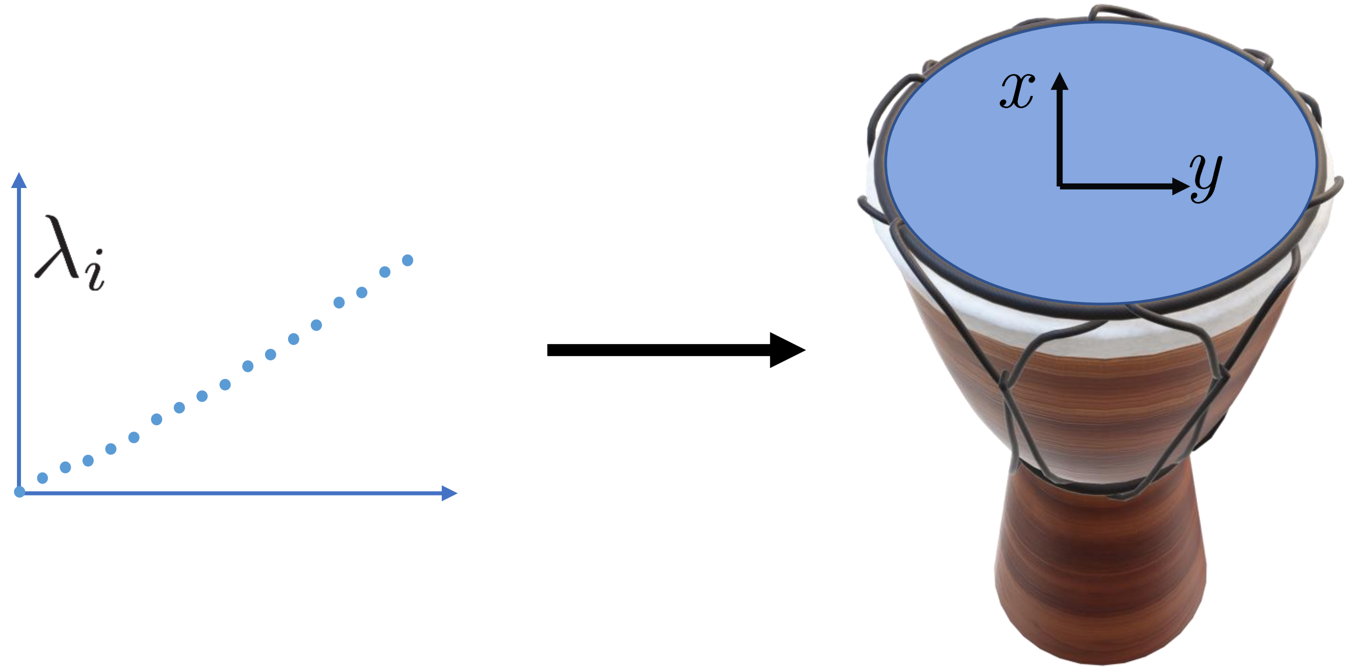
$$u(x, y; t) = \phi(x, y) f(t)$$

Laplacian eigenfunction

Oscillating function with frequency λ

Hearing the shape of the drum

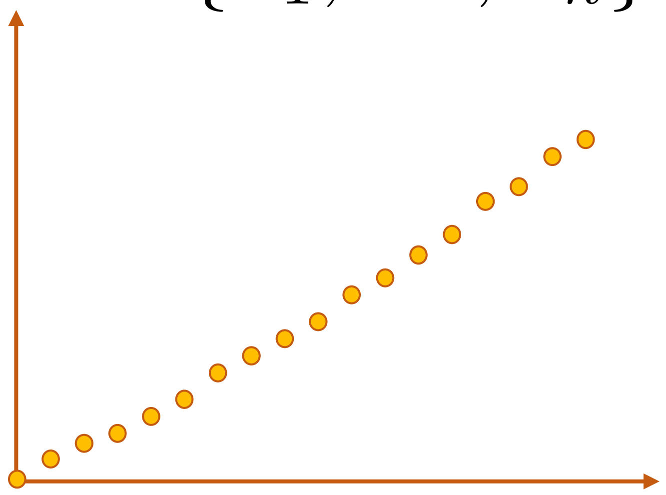
Reconstruction of a domain from its Laplacian eigenvalues



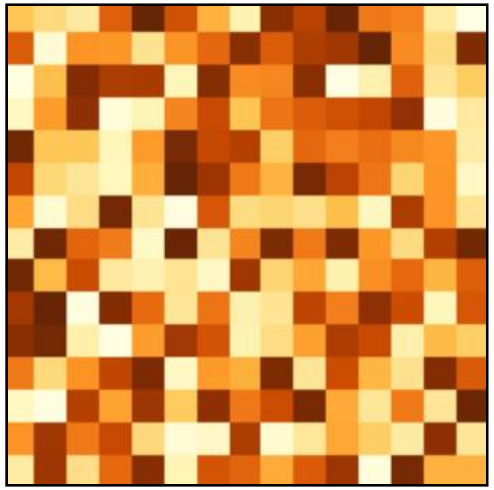
The inverse eigenvalue problem

Reconstruction of a matrix from prescribed spectral data:

$$\lambda = \{\lambda_1, \ldots, \lambda_n\}$$



$$A$$



$$\sigma(A) = \lambda$$

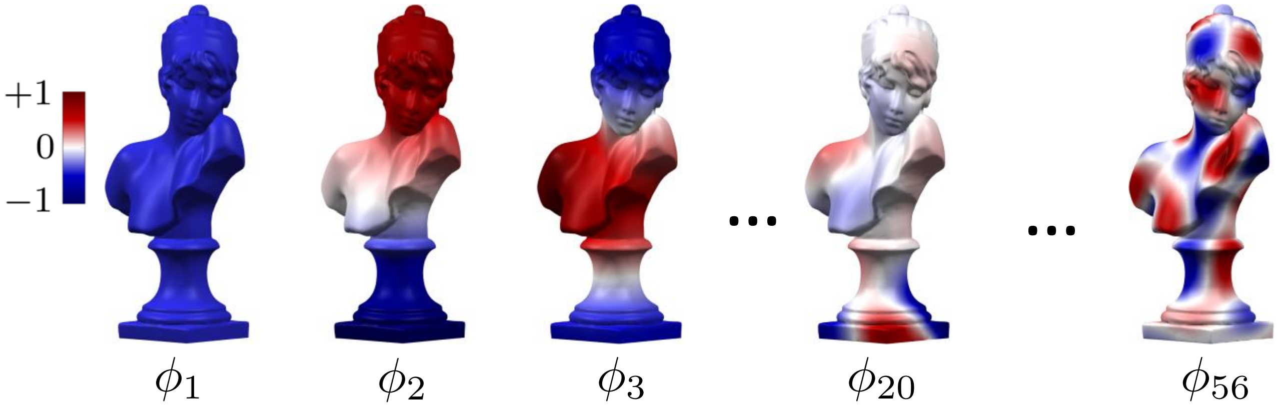
Chu M.T. *Inverse eigenvalue problems*. SIAM review. 1998.

Our drum

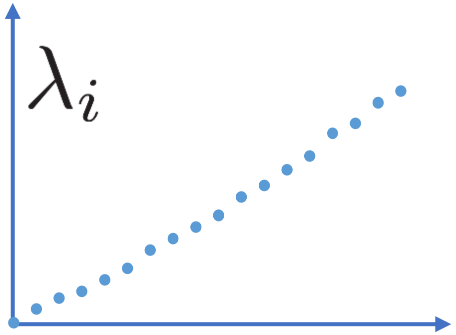
The eigenvalue problem on shapes

$$\Delta\phi_i(x) = \lambda_i \phi_i(x)$$

Eigenfunctions

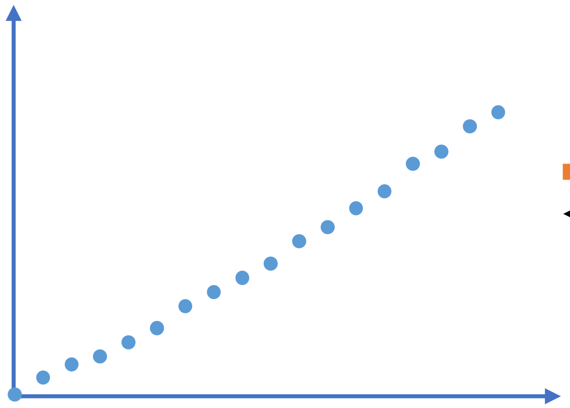


Spectrum / eigenvalues



Solvability

Can we recover the shape from the eigenvalues?

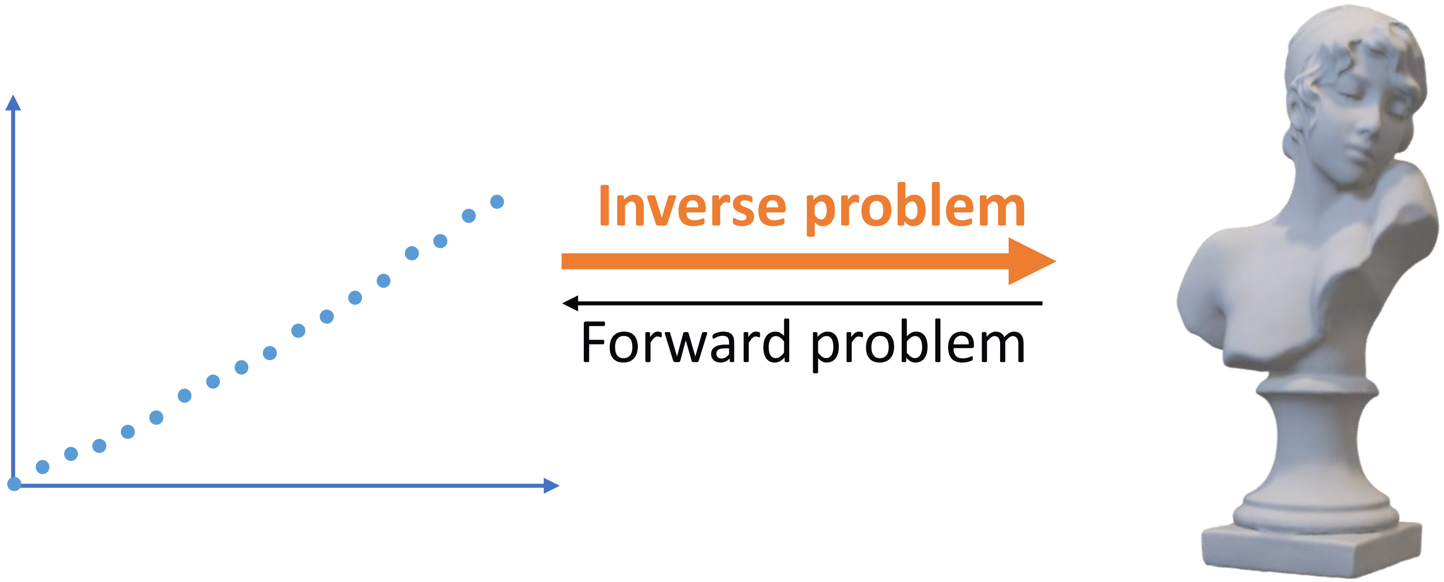


Inverse problem

Forward problem

Computability

…how?



Outline

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- Methods: optimization
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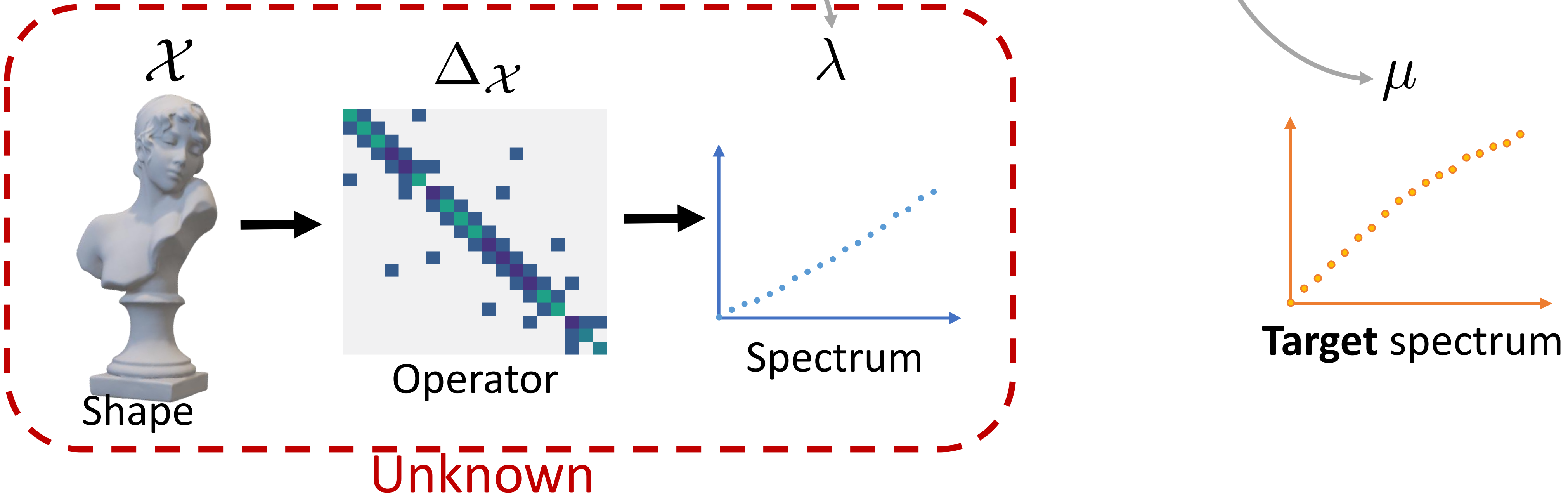
Isospectralization

Optimization directly on the 3D coordinates:

$$\arg\min_{\mathcal{X}} \left\| \boldsymbol{\lambda}\left(\boldsymbol{\Delta}_{\mathcal{X}}\right) - \boldsymbol{\mu} \right\|_{\omega}$$



Cosmo et al., *Isospectralization, or how to hear shape, style and correspondence* (CVPR 2019)

Optimization

- The spectrum is differentiable

If $\mathbf{A}$ is a real and symmetric matrix, λ_i and $\mathbf{v}_i$ are eigenvalues and eigenvectors of $\mathbf{A}$, then:

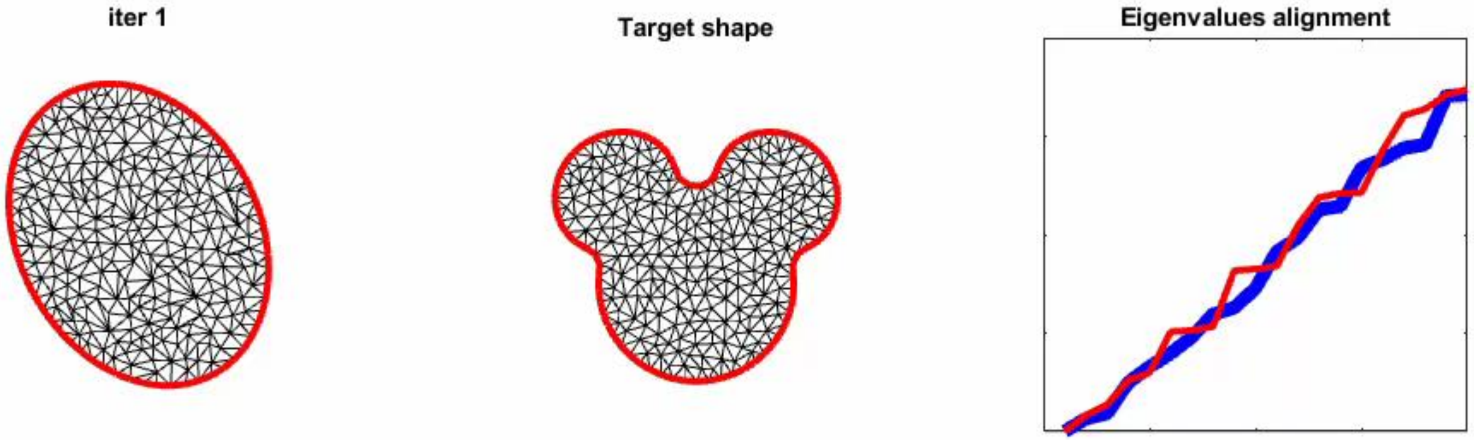
$$\partial\lambda_i = \mathbf{v}_i^T \partial(\mathbf{A})\mathbf{v}_i$$

- Gradient-based minimization algorithm (Adam optimizer)

Petersen KB, Pedersen MS. *The matrix cookbook.* 2008.

Mickey from spectrum

$$\arg\min_{\mathcal{X}} \|\boldsymbol{\lambda}(\boldsymbol{\Delta}_{\mathcal{X}}) - \boldsymbol{\mu}\|_{\omega}$$



3D: several local minima

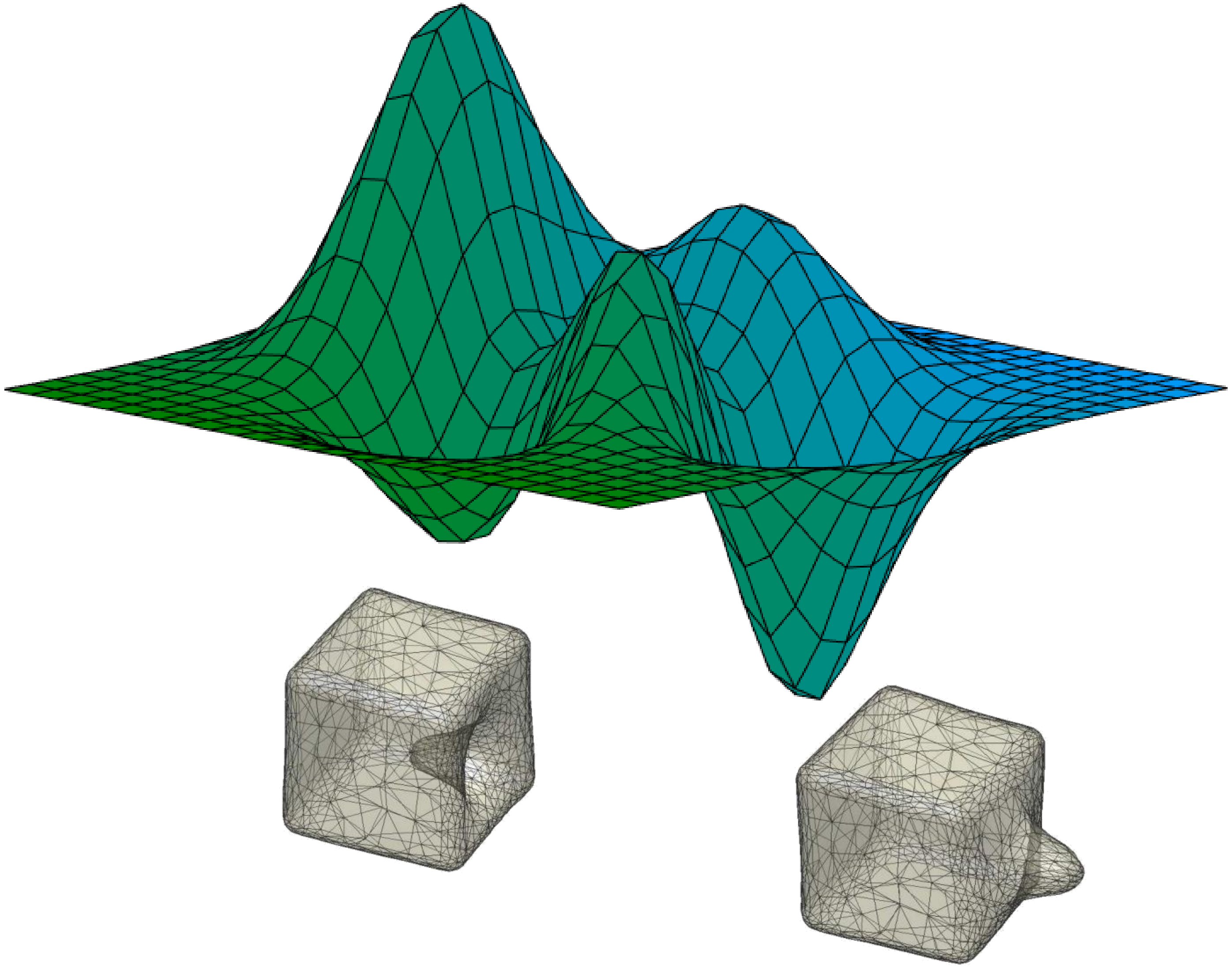
symmetries and **isometries**

↓

existence of several

local minima in the

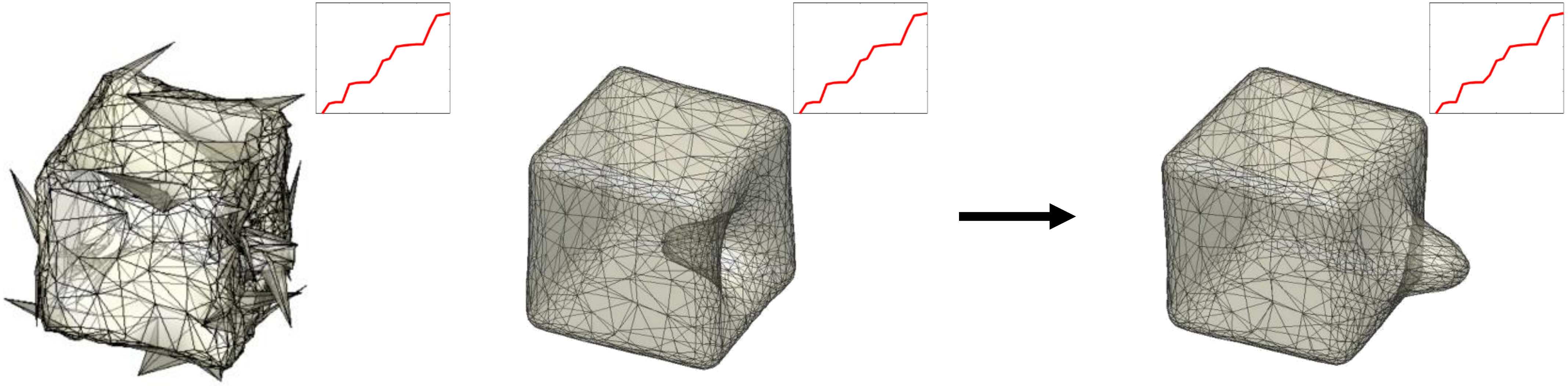
isospectralization problem



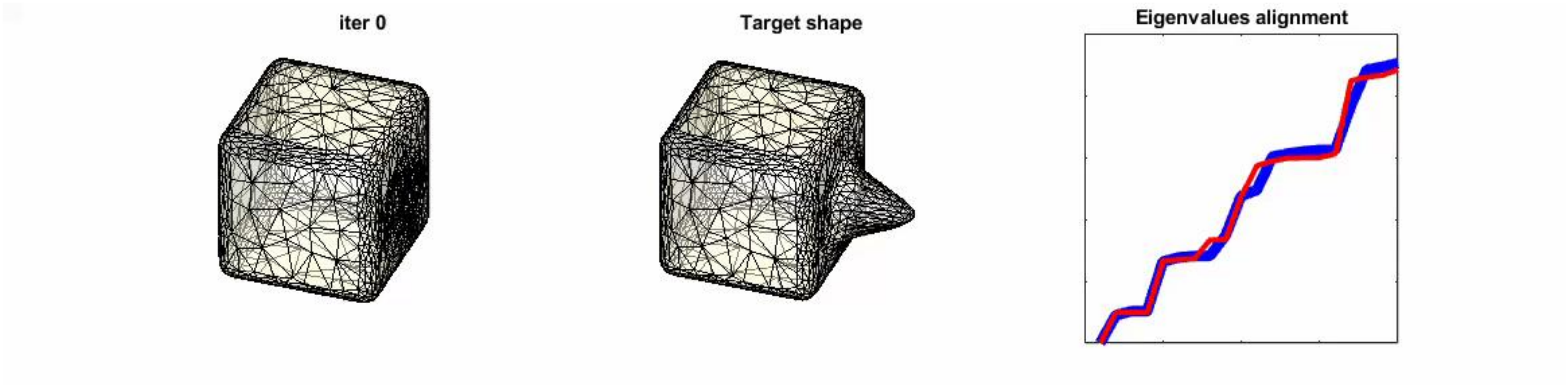
Regularizers

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times d}} \|\boldsymbol{\lambda}(\boldsymbol{\Delta}(\mathbf{X})) - \boldsymbol{\mu}\|_{\omega} + \rho_X(\mathbf{X})$$

To promote smoothness and maximize volume:



Isospectralization on surfaces



Cosmo et al., *Isospectralization, or how to hear shape, style and correspondence* (CVPR 2019)

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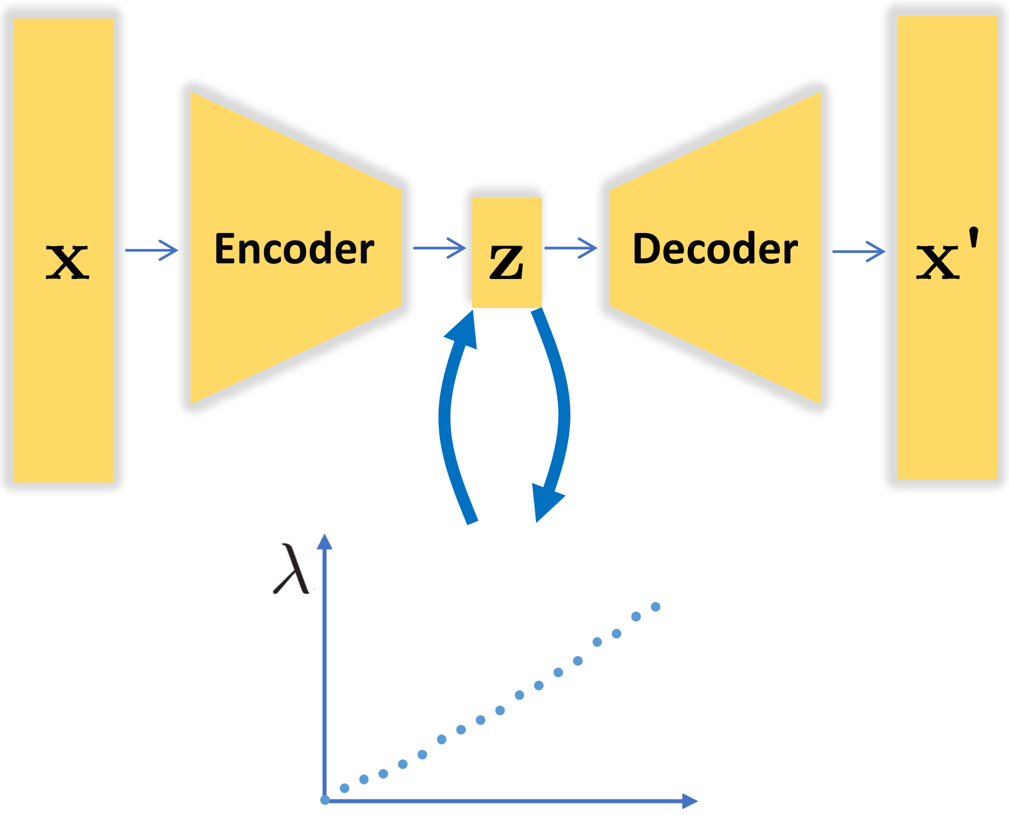
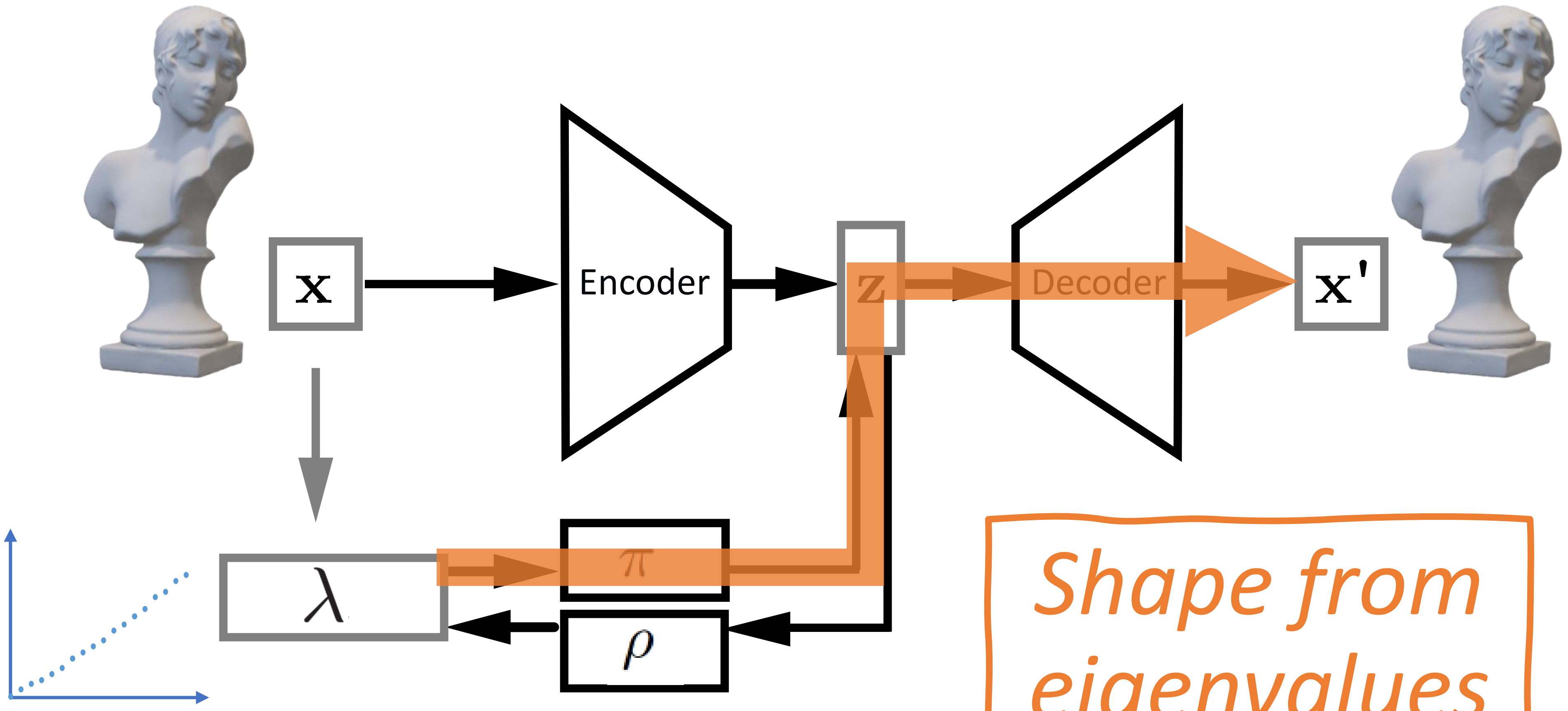
Learning to hear shapes

We can use a **cycle-consistent module** to map latent vectors to spectra:



Marin et al., *Instant recovery of shape from spectrum via latent space connections* (3DV 2020)

Learning to hear shapes

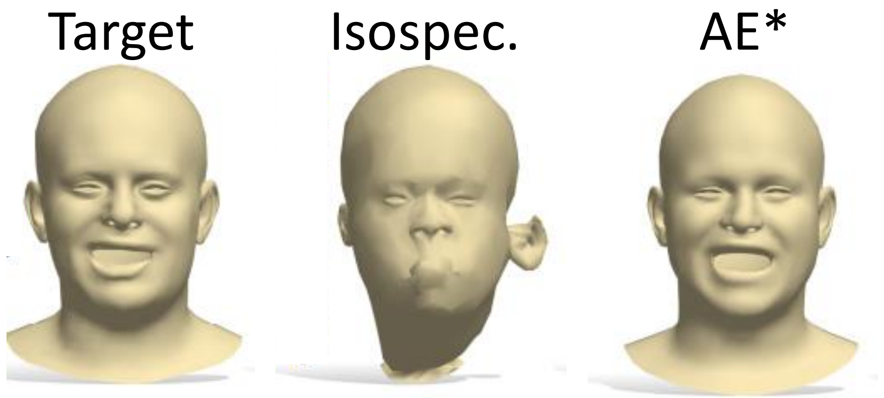


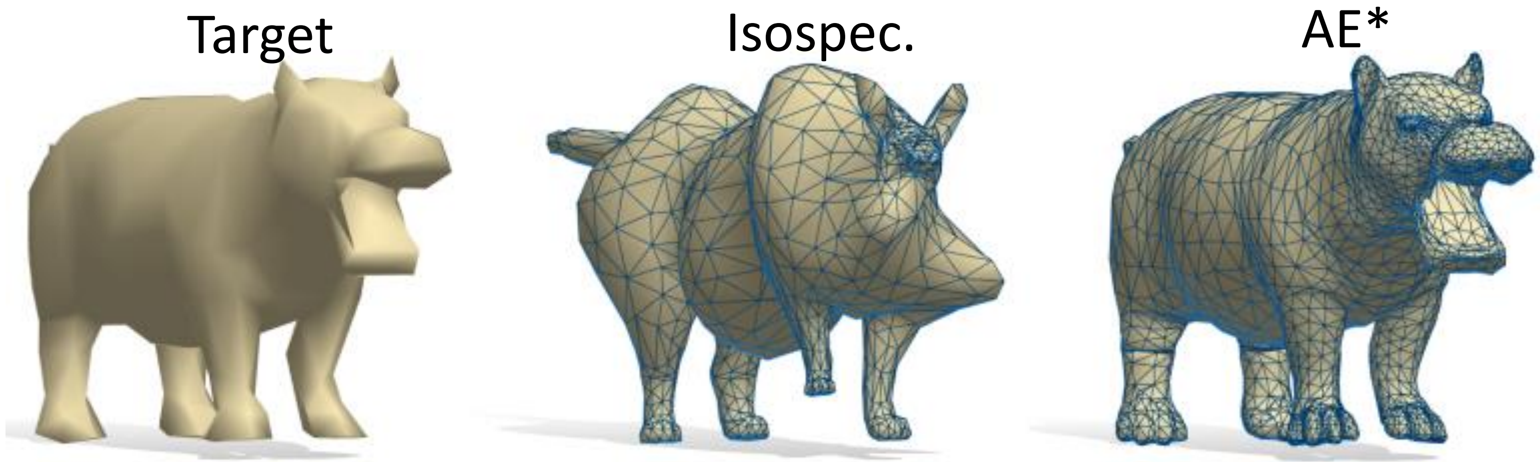
Shape from eigenvalues

Marin et al., *Instant recovery of shape from spectrum via latent space connections* (3DV 2020)

Pros of data-driven approach

- Fast: *instant* recovery
- Accuracy
- No dependence from initialization
- Larger meshes and point clouds
- No need of regularizers

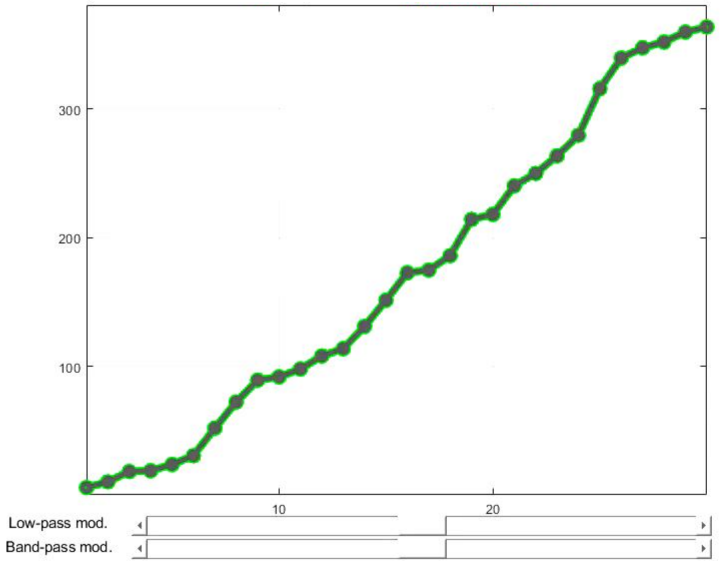




* Marin et al., Instant recovery of shape from spectrum via latent space connections (3DV 2020)

Pros of data-driven approach

Input: **spectrum**



Output: **shape**

Marin et al., *Instant recovery of shape from spectrum via latent space connections* (3DV 2020)

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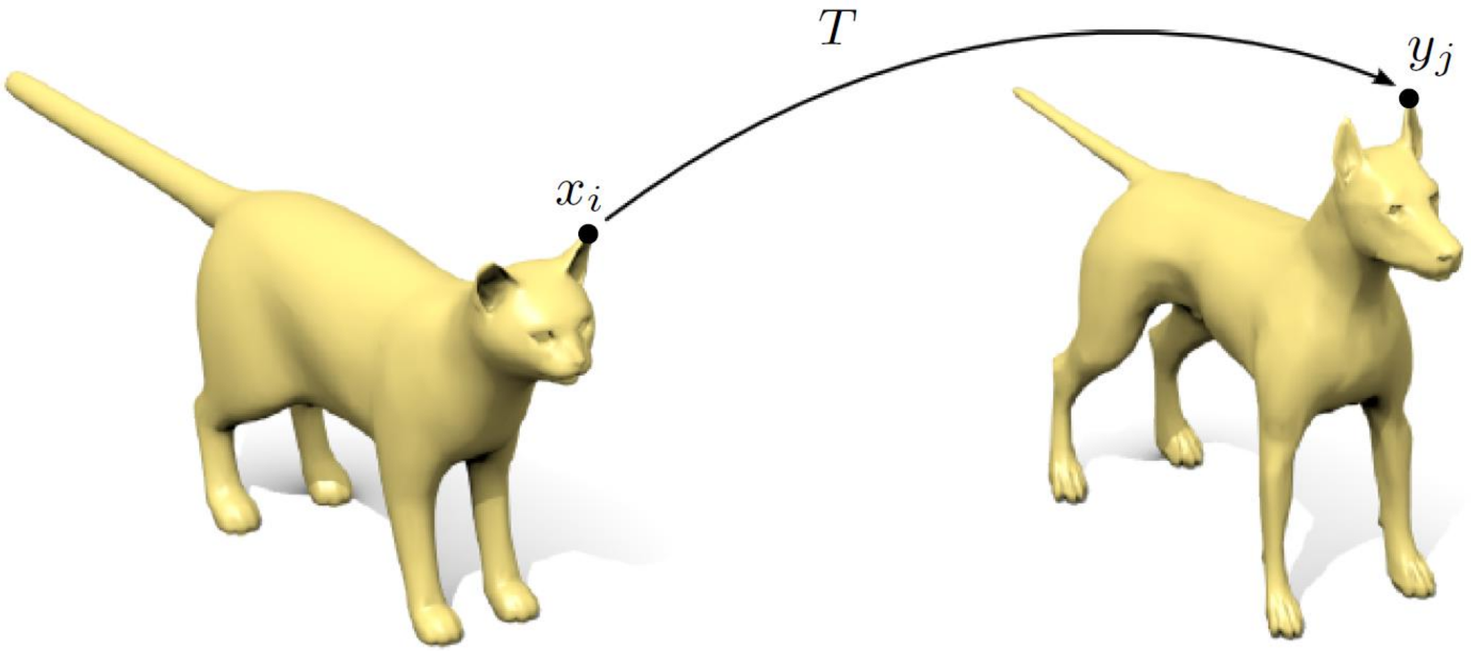
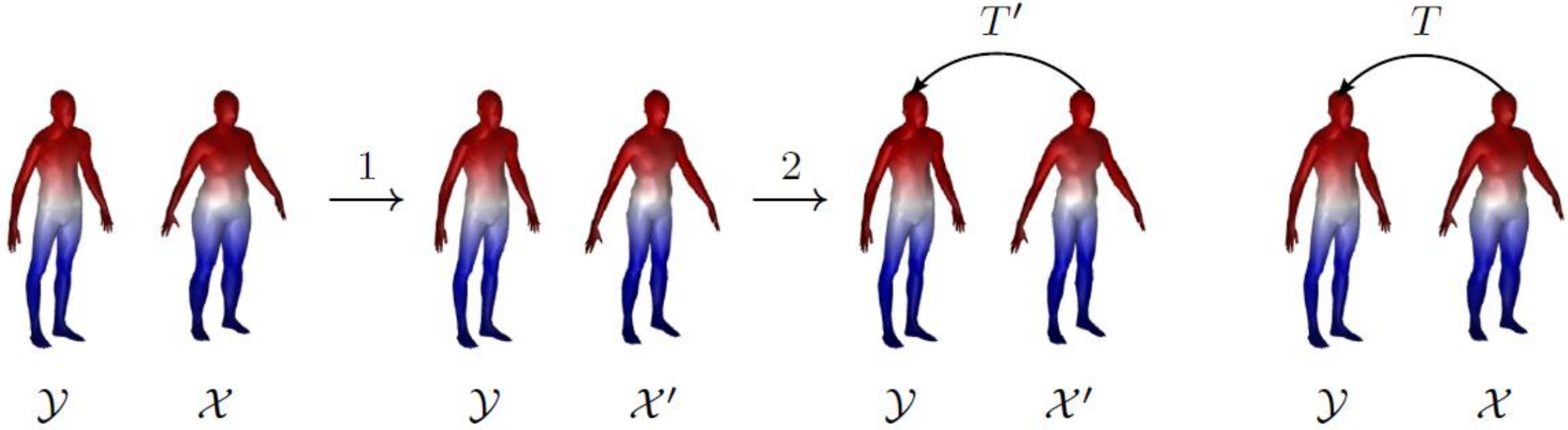
Non-isometric shape matching

Goal



Isospectralization

- Preprocessing step in Functional Map based matching algorithms
- Isospectralization induces isometry



Isospectralization

- Preprocessing step in Functional Map based matching algorithms
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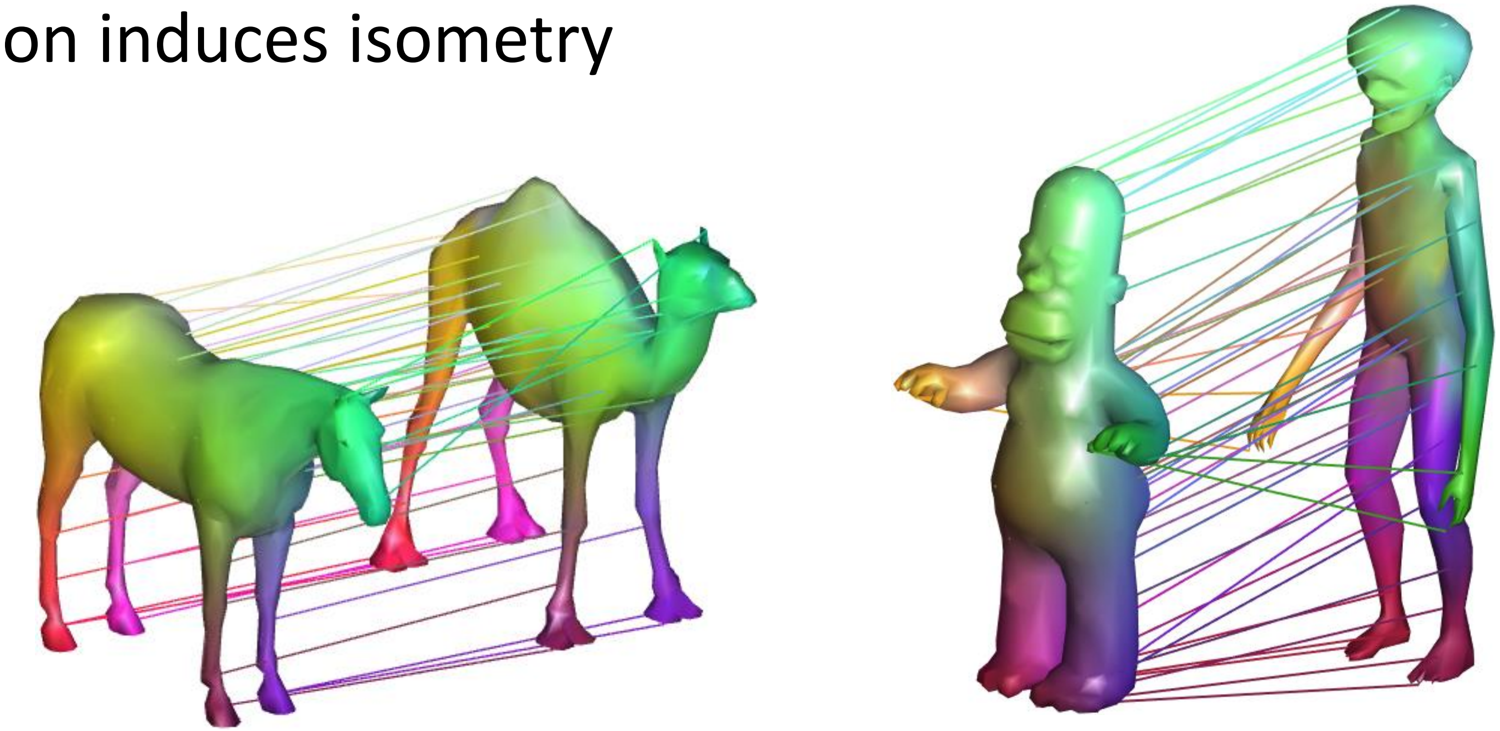
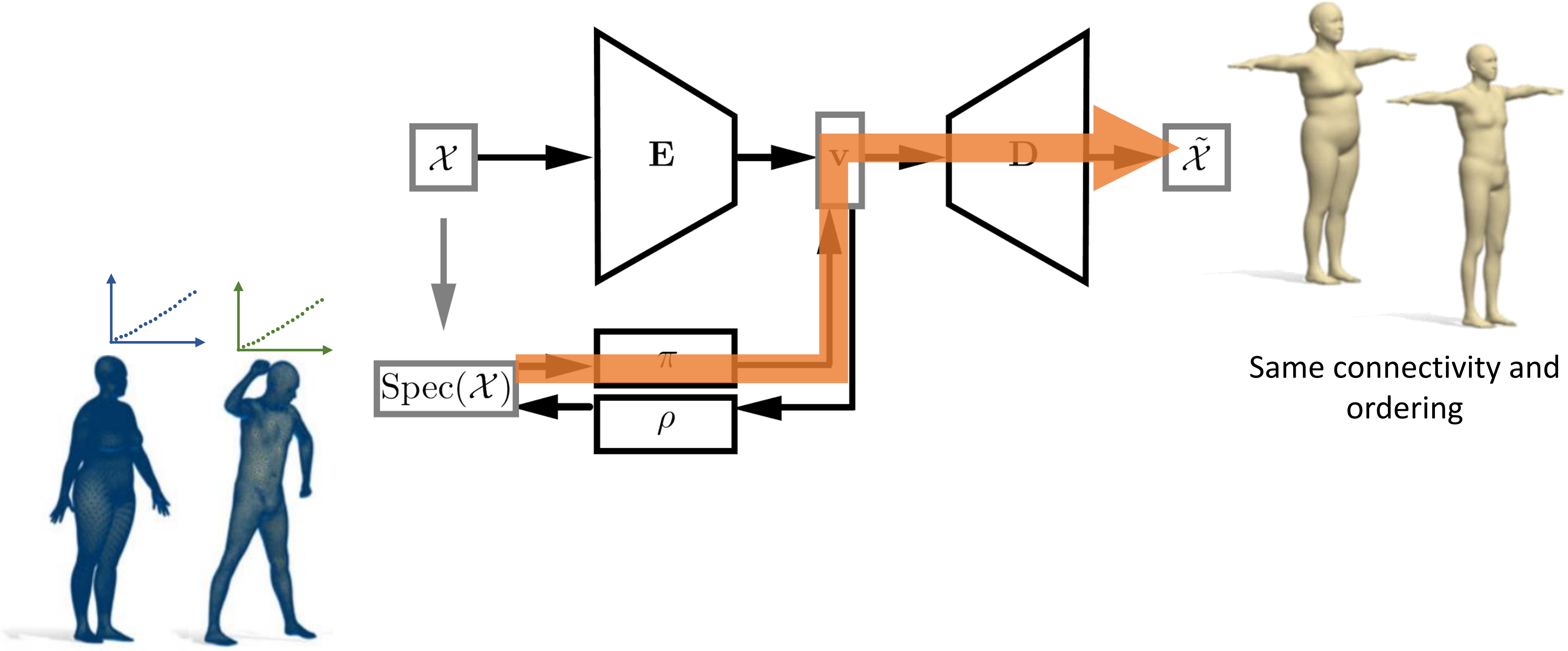


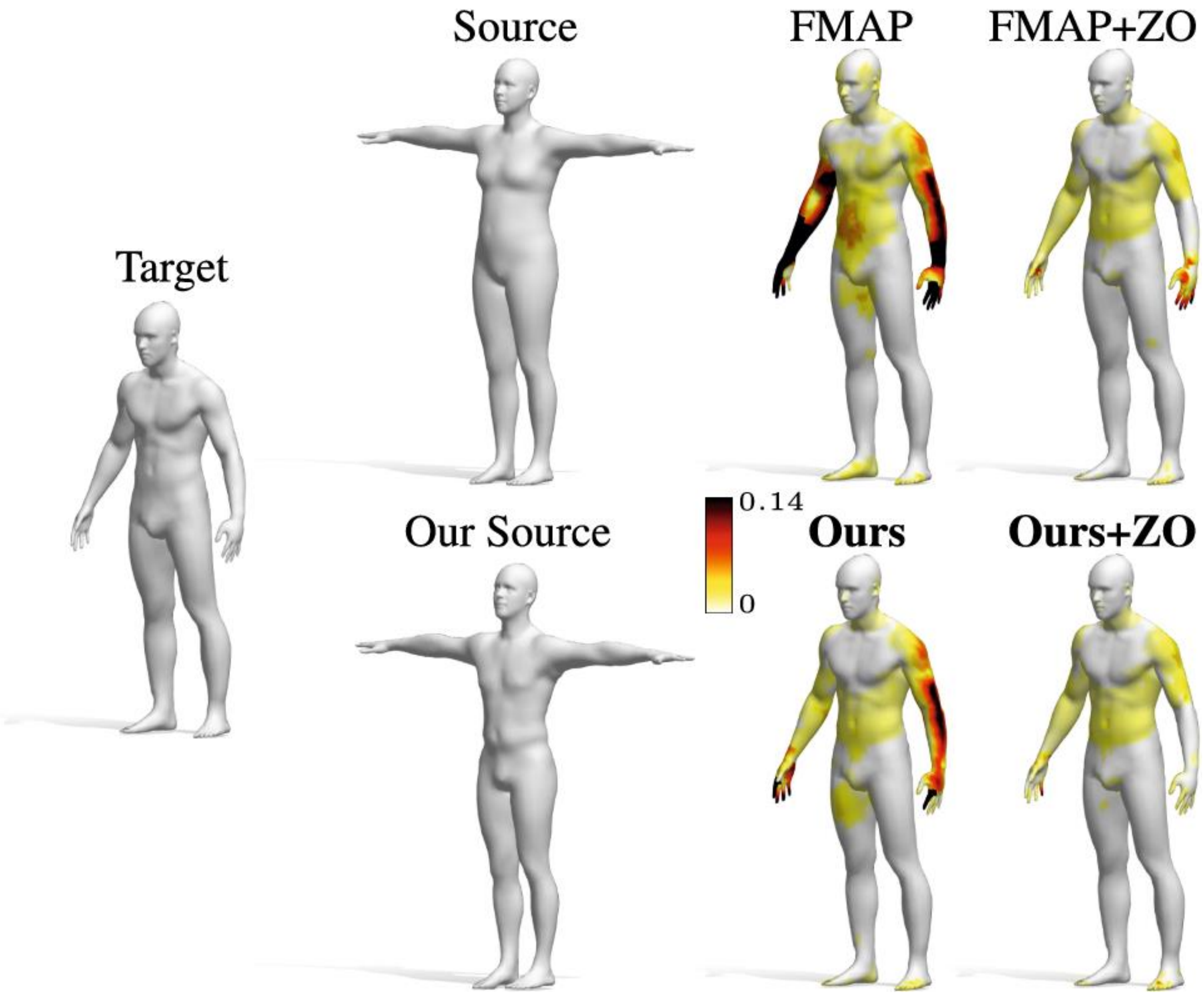
After isospectralization

Data-driven approach



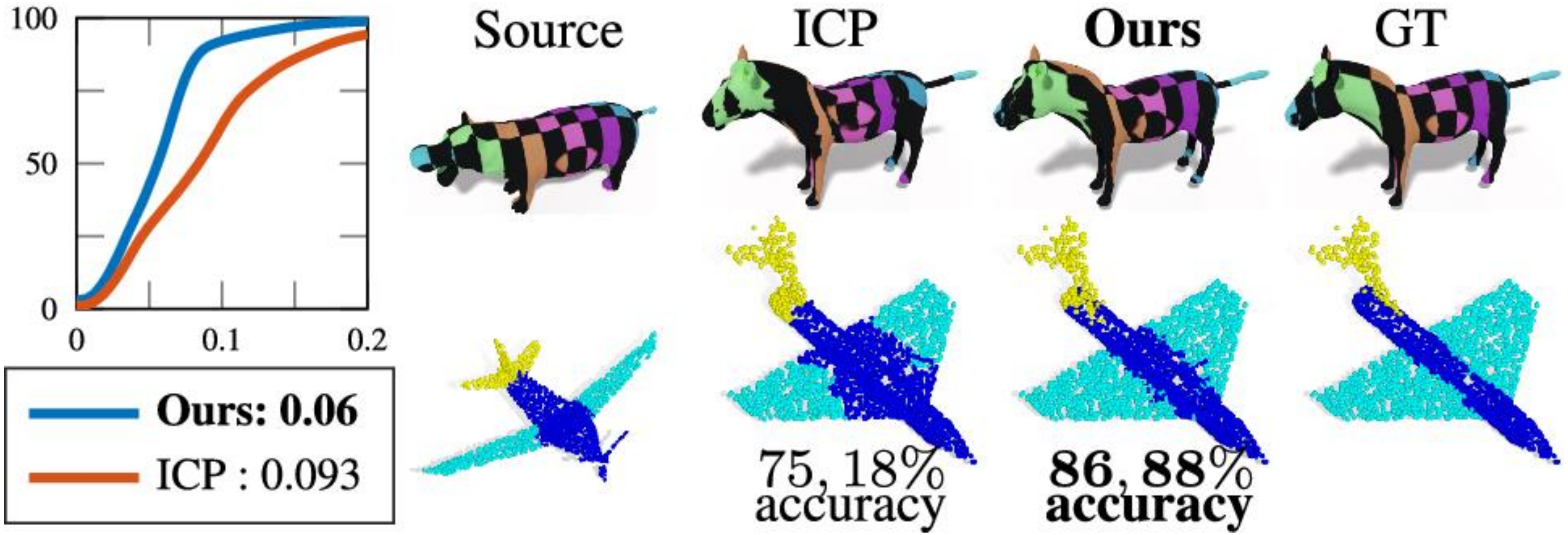
Same connectivity and ordering

Results



Marin et al., *Spectral Shape Recovery and Analysis Via Data-driven Connections* (IJCV 2021)

Results: segmentation and texture transfer

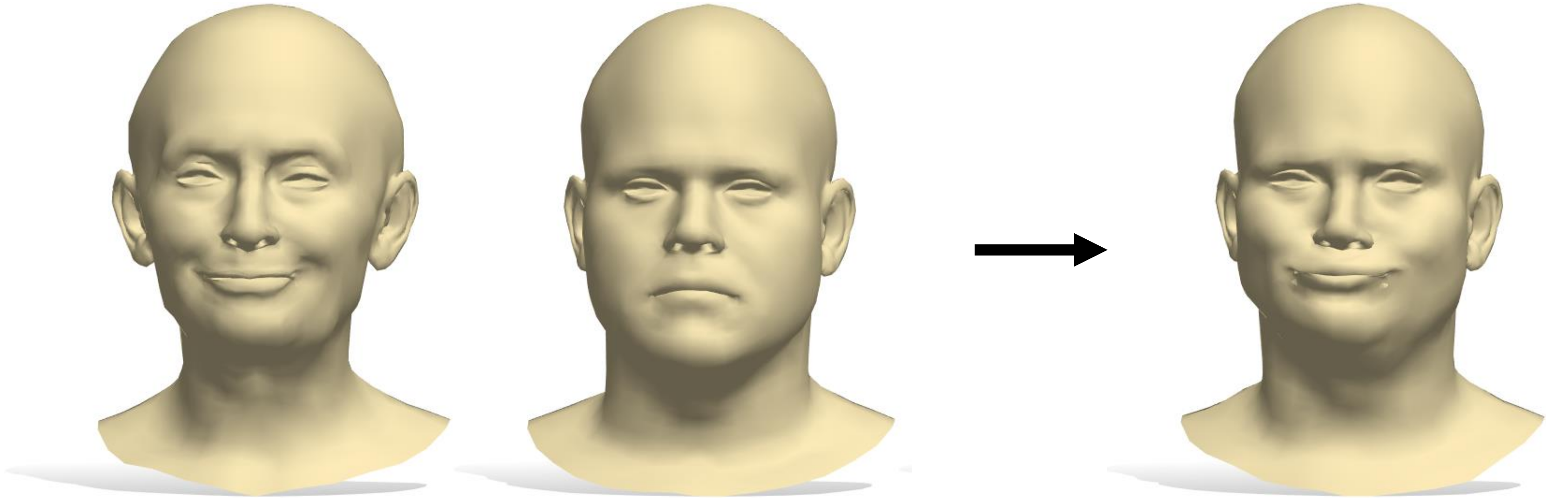


Marin et al., *Spectral Shape Recovery and Analysis Via Data-driven Connections* (IJCV 2021)

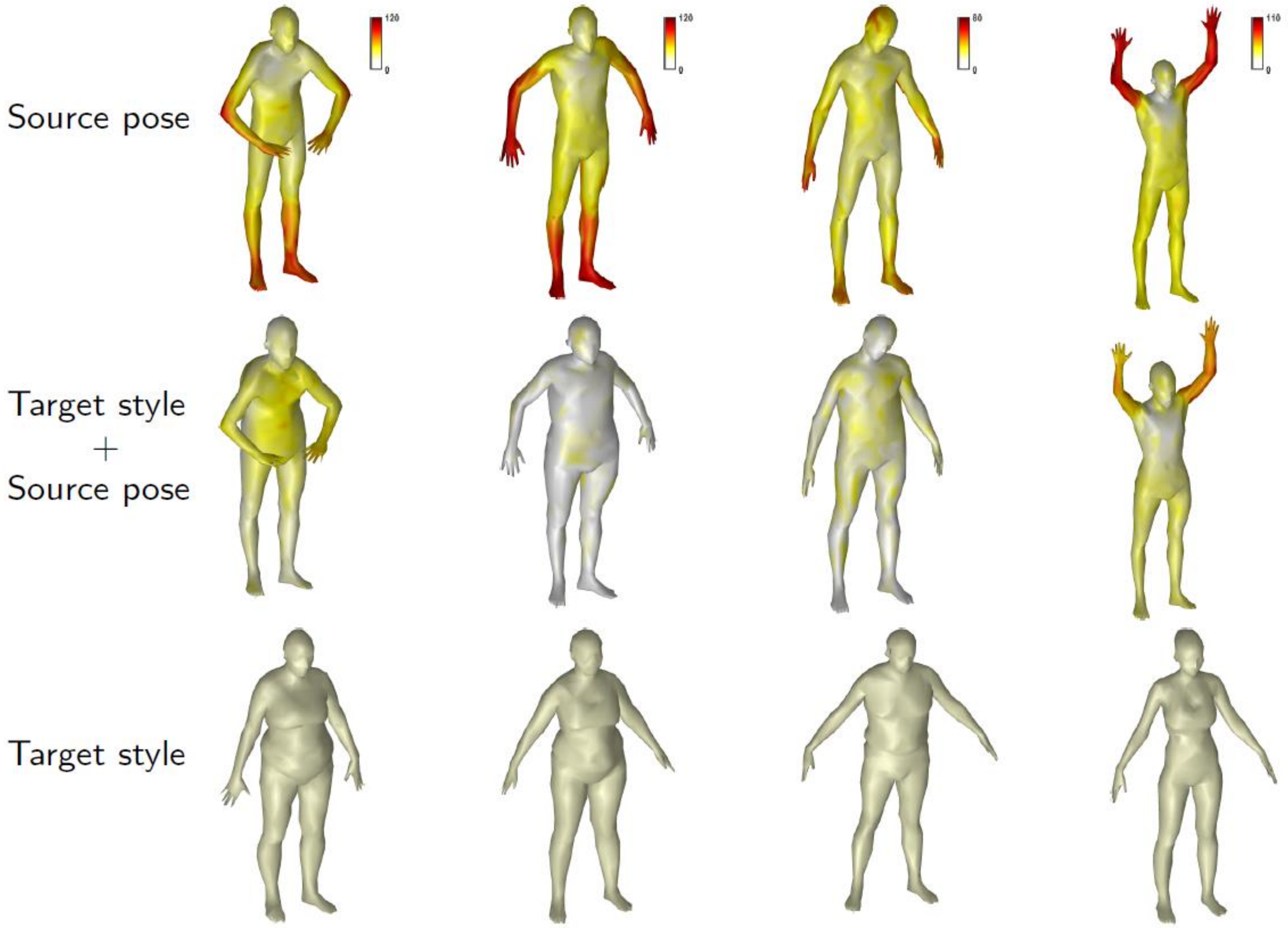
Style transfer

Goal

Input: **pose and style donors** Output: **new shape**



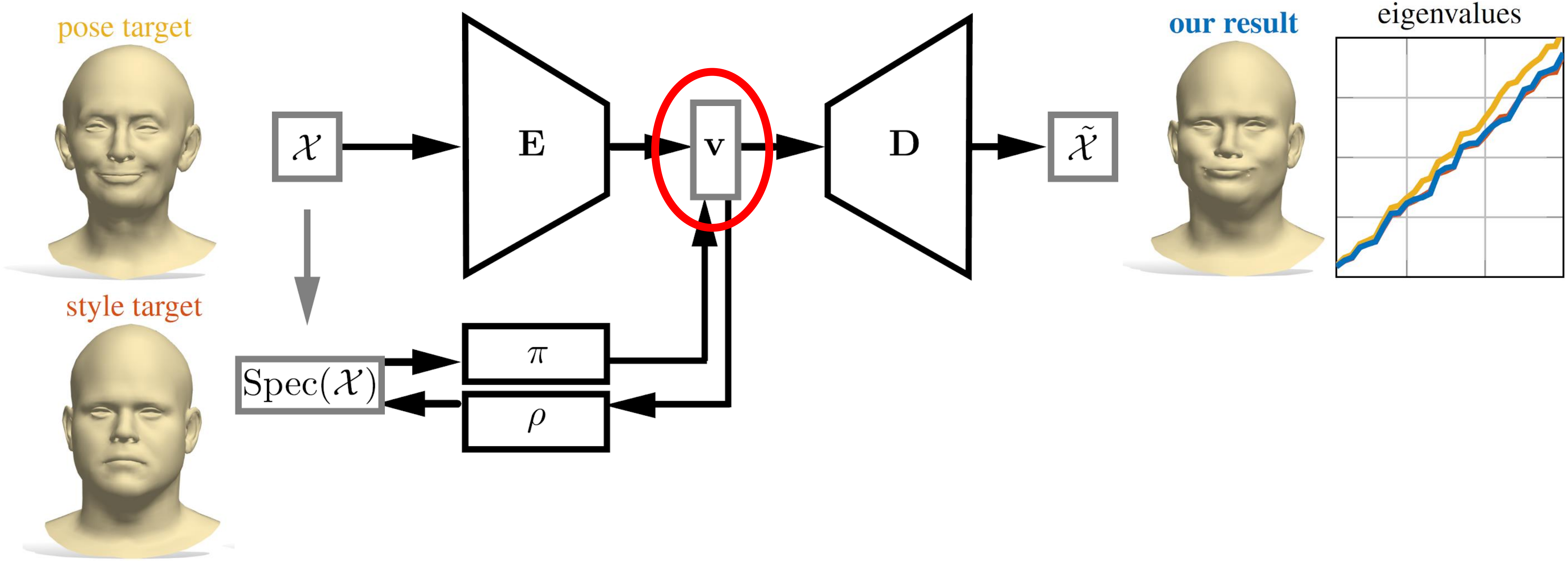
Isospectralization



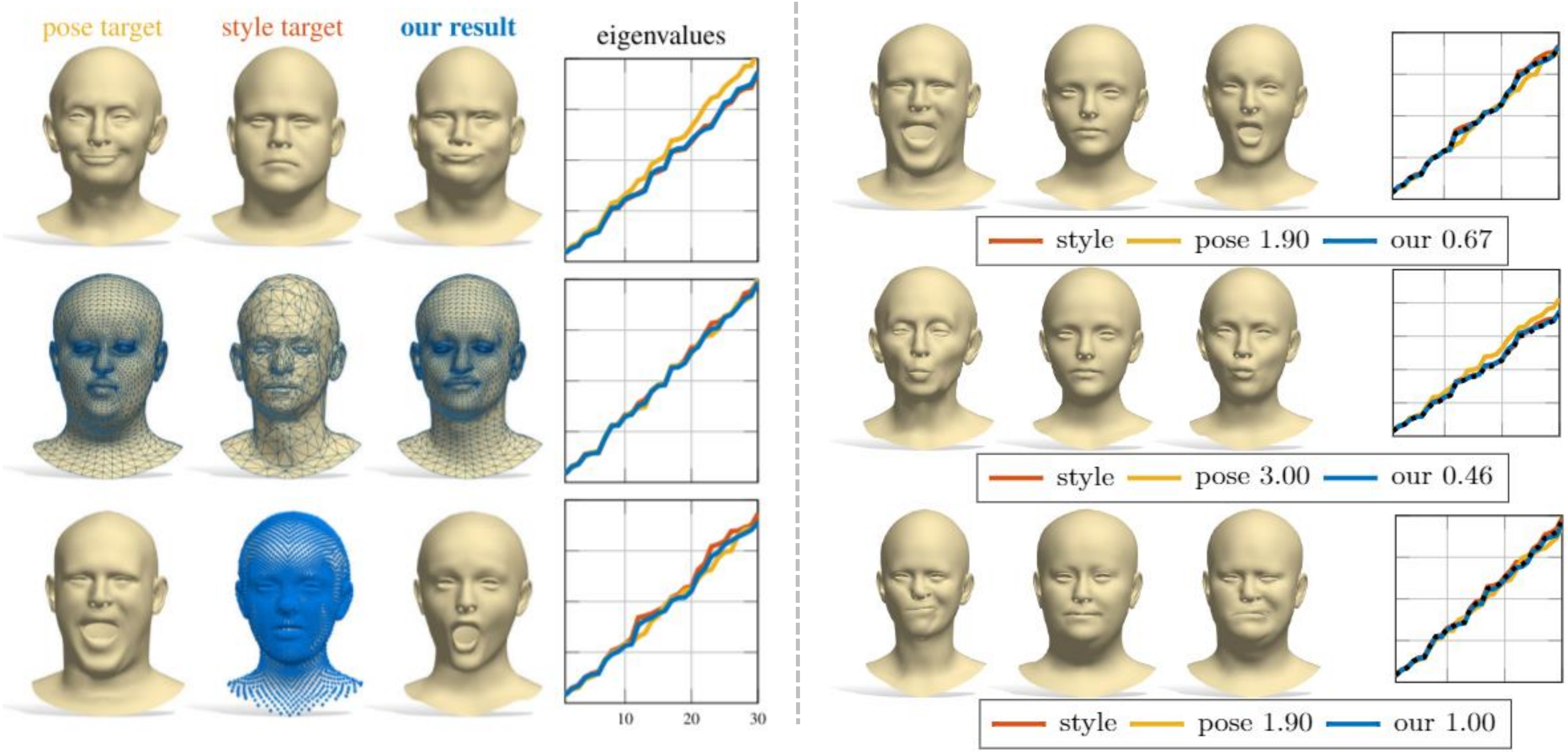
Cosmo et al., *Isospectralization, or how to hear shape, style and correspondence* (CVPR 2019)

Data-driven approach

$$\min_{\mathbf{v}} \|\mathrm{Spec}(\mathcal{X}_{\mathrm{style}}) - \rho(\mathbf{v})\|_2^2 + w\|\mathbf{v} - E(\mathcal{X}_{\mathrm{pose}})\|_2^2$$



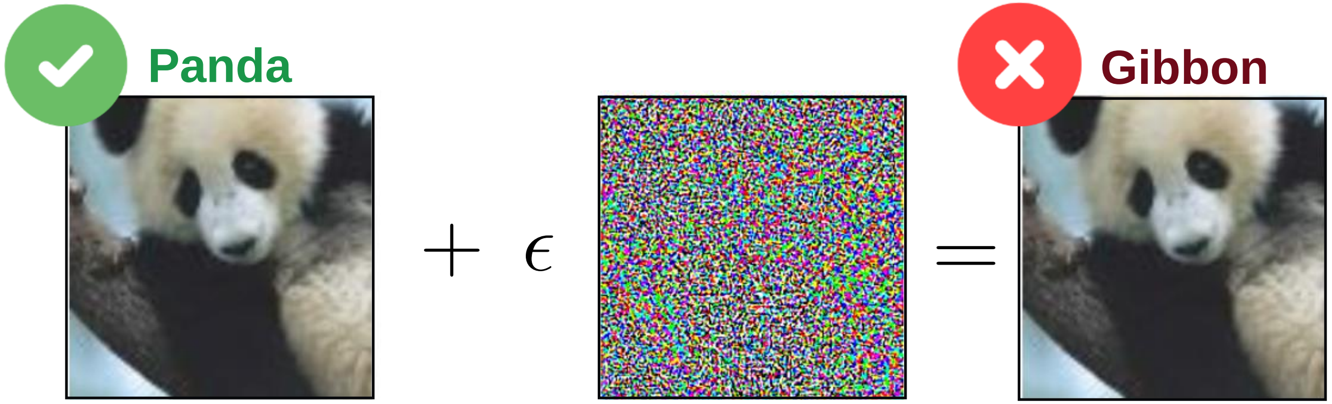
Results



Marin et al., *Instant recovery of shape from spectrum via latent space connections* (3DV 2020)

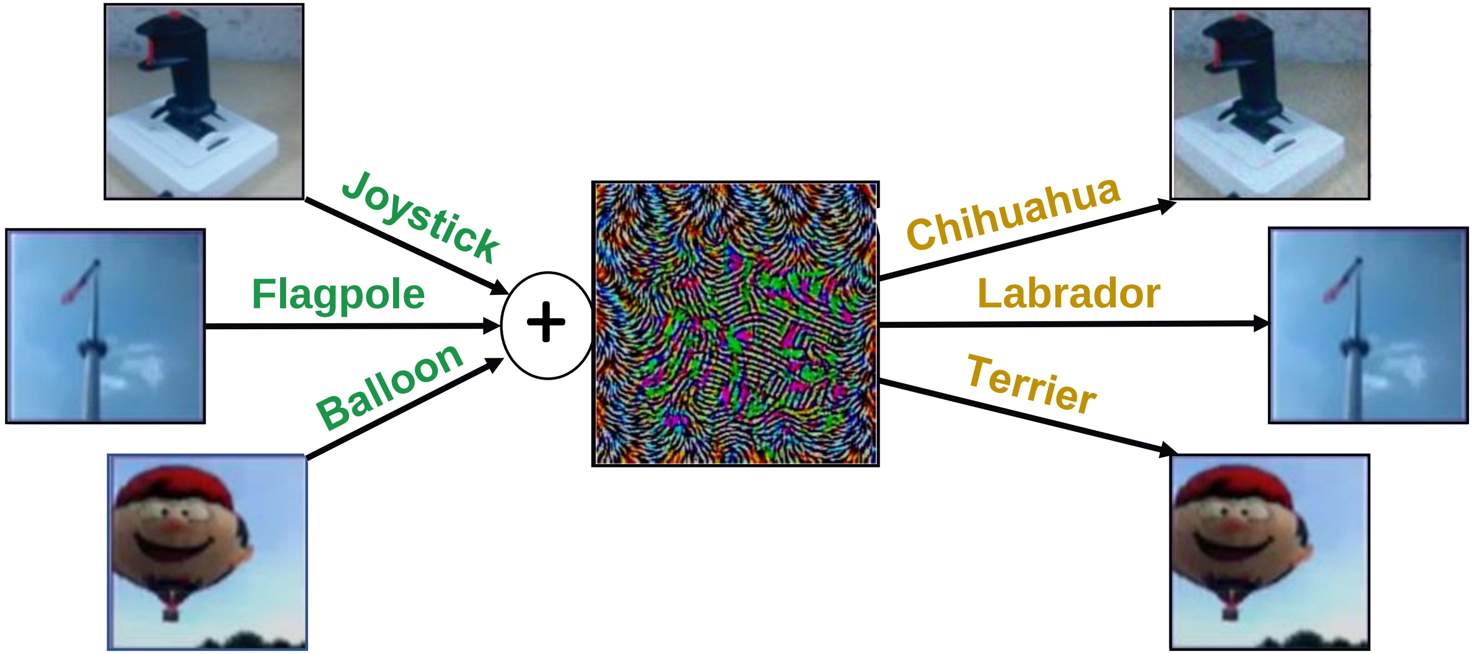
Adversarial attacks

Adversarial attacks



Goodfellow et al., *Explaining and harnessing adversarial examples*, ICLR (2014).

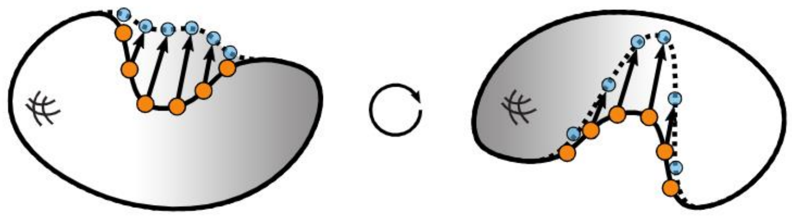
Universal adversarial attacks



Moosavi-Dezfooli et al. "Universal adversarial perturbations". CVPR (2017)

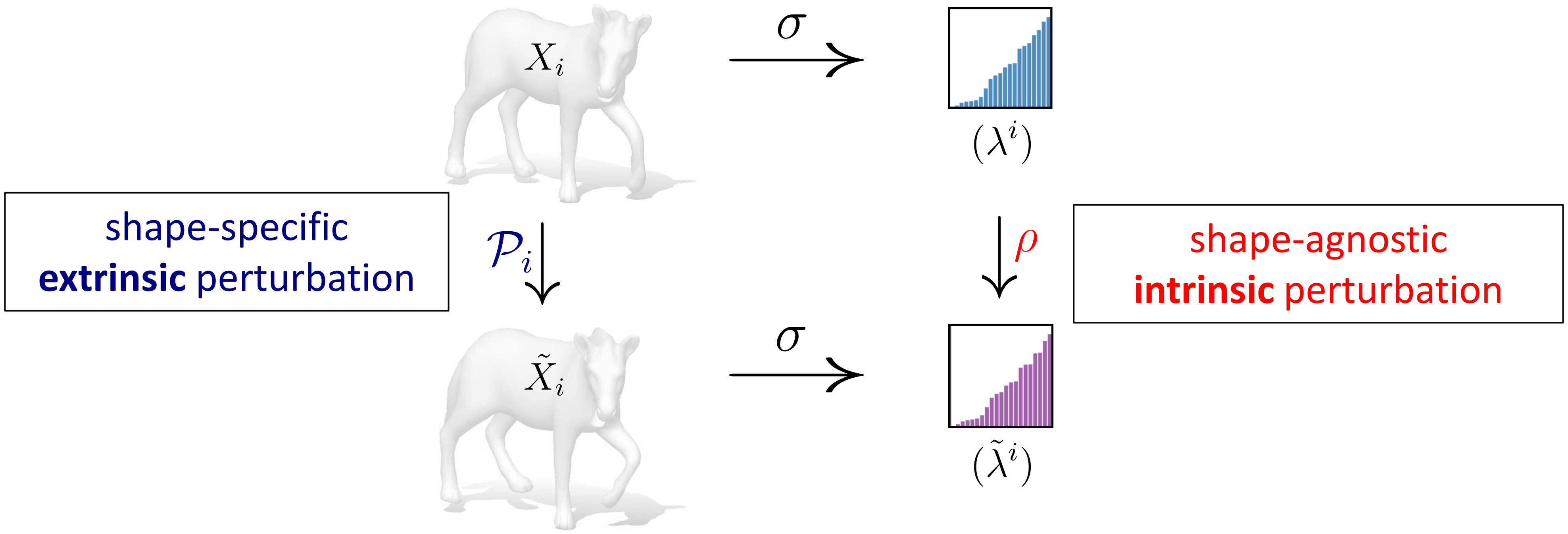
Universal attacks for deformable shapes

An extrinsic perturbation needs correspondence and can not be *deformation-invariant*.

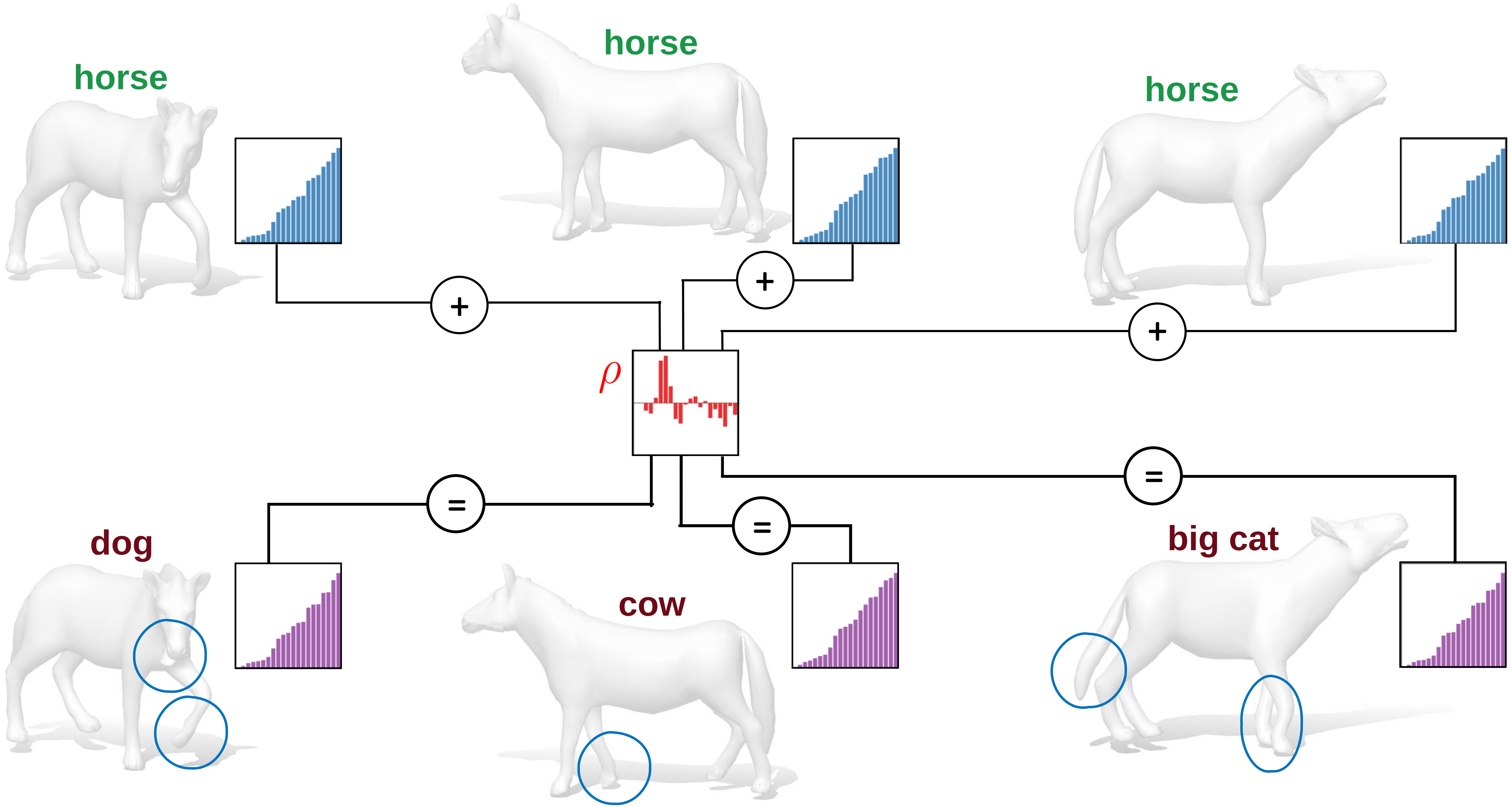


spectral domain

Perturbation in the spectral domain

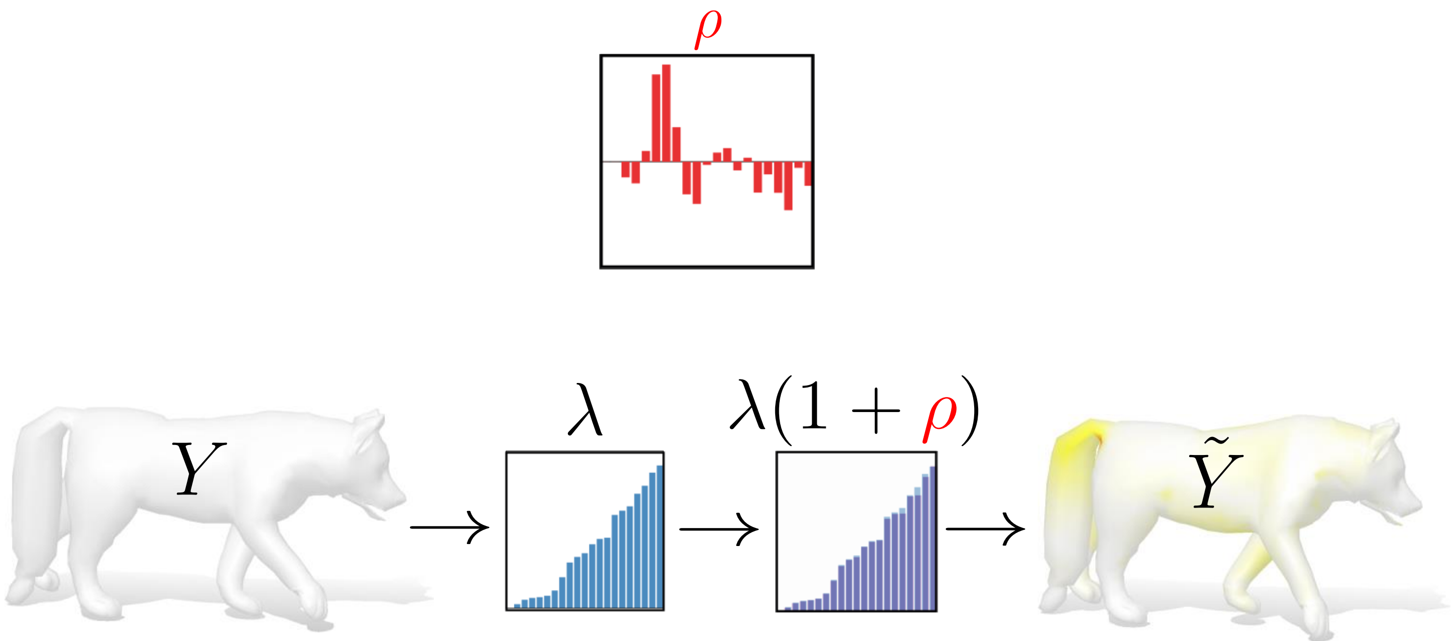


Rampini et al. "Universal Spectral Adversarial Attacks for Deformable Shapes" (CVPR 2021)



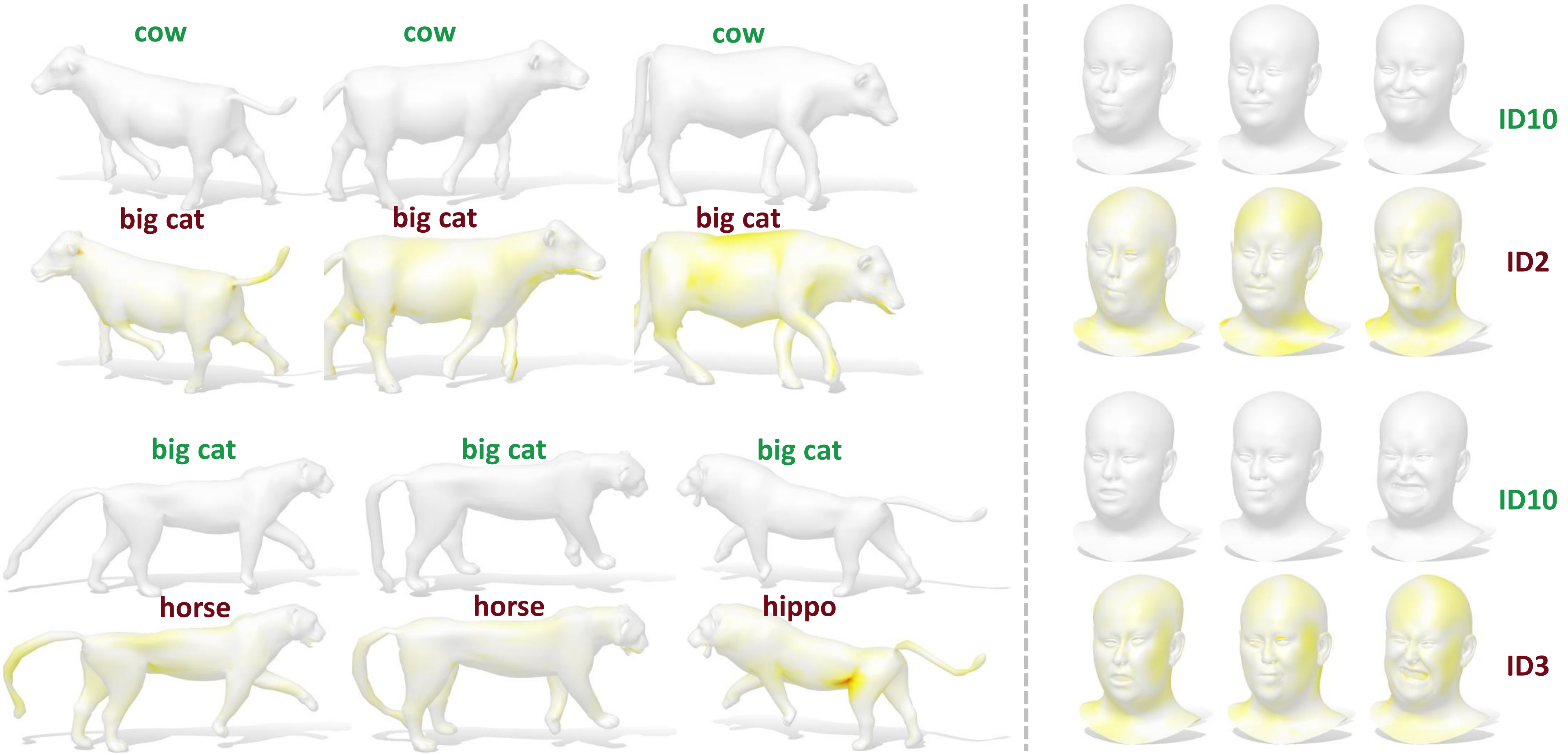
Rampini et al., *Universal Spectral Adversarial Attacks for Deformable Shapes* (CVPR 2021)

Generalization to unseen shapes



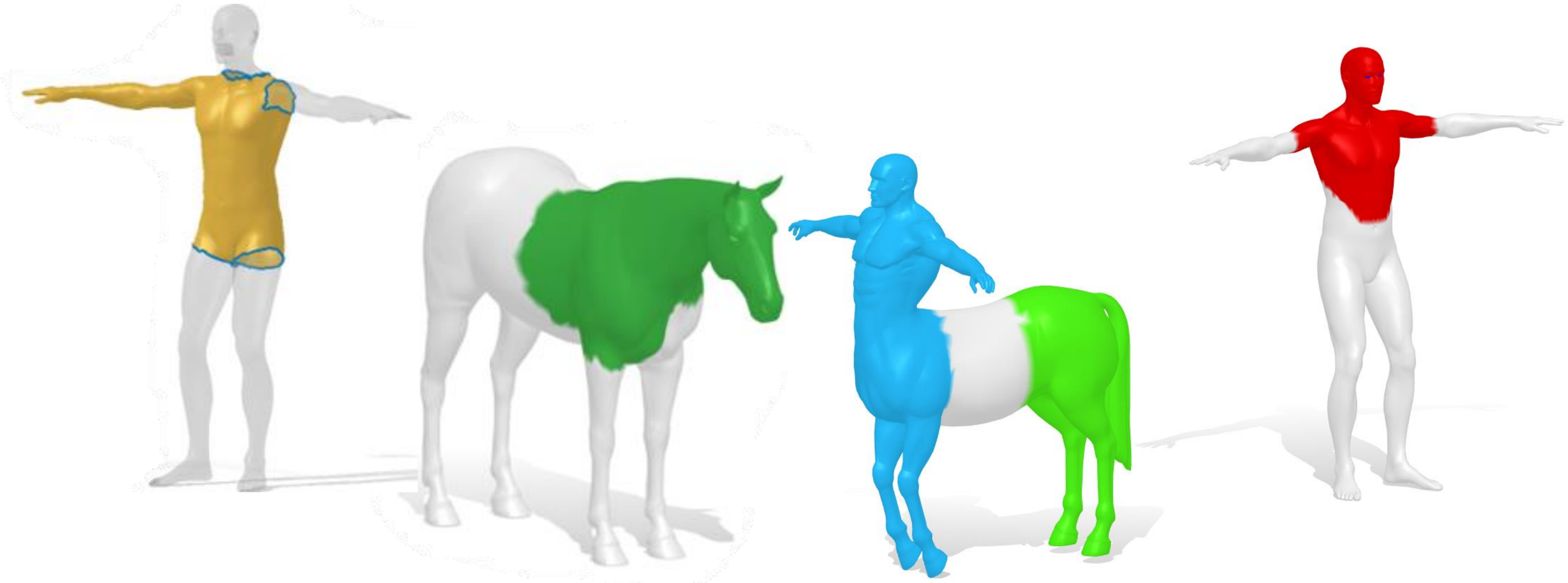
Rampini et al., *Universal Spectral Adversarial Attacks for Deformable Shapes* (CVPR 2021)

Examples

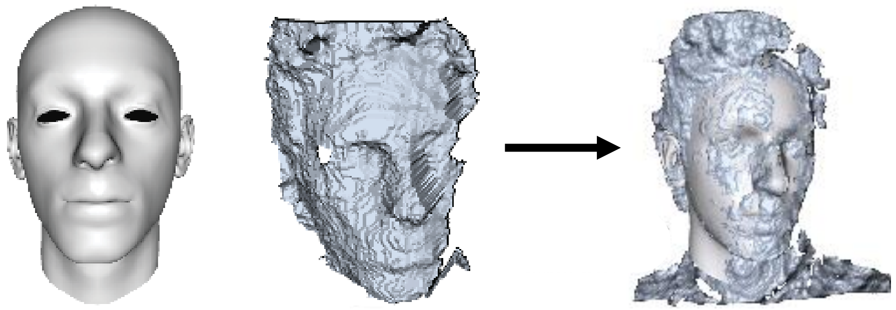

cow
cow
cow
big cat
big cat
big cat
big cat
big cat
big cat
horse
horse
hippo
ID10
ID2
ID10
ID3

Partial shape localization

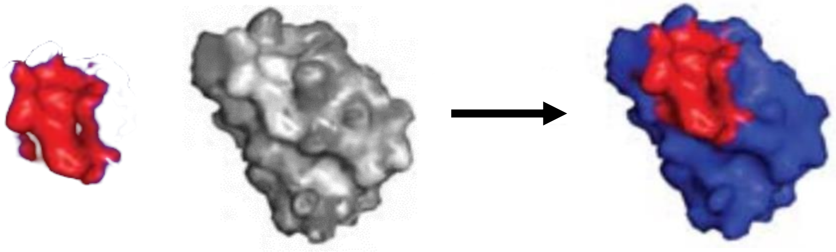
Subregion of a given shape



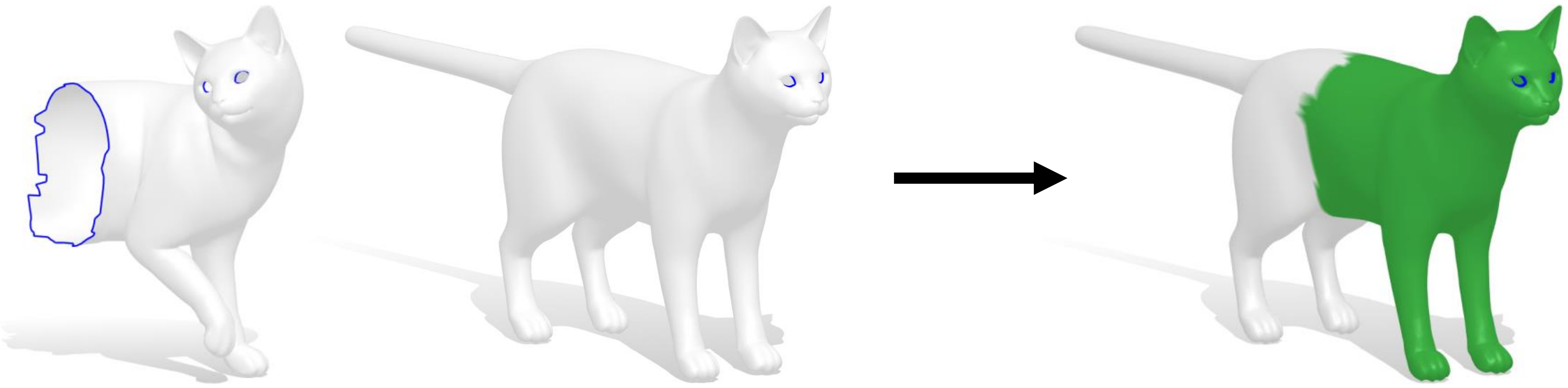
Motivation



Template Scan Registration



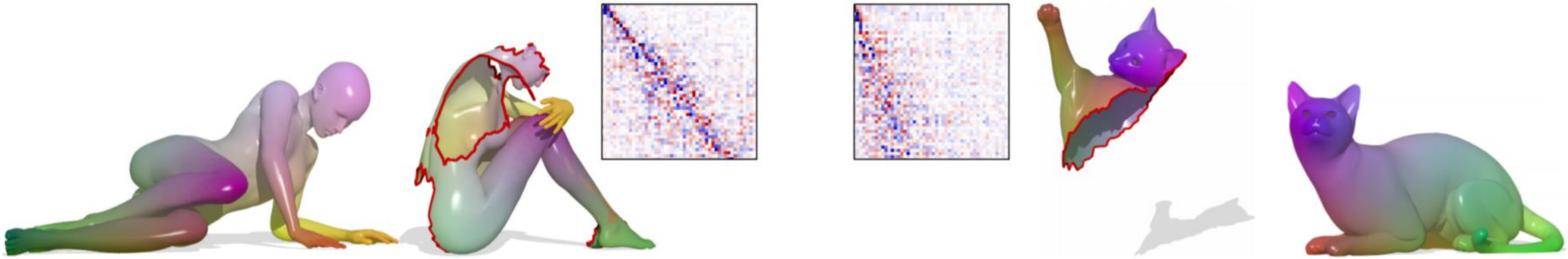
Protein A Protein B Interface / Non-interface



Partial shape Full shape Mask

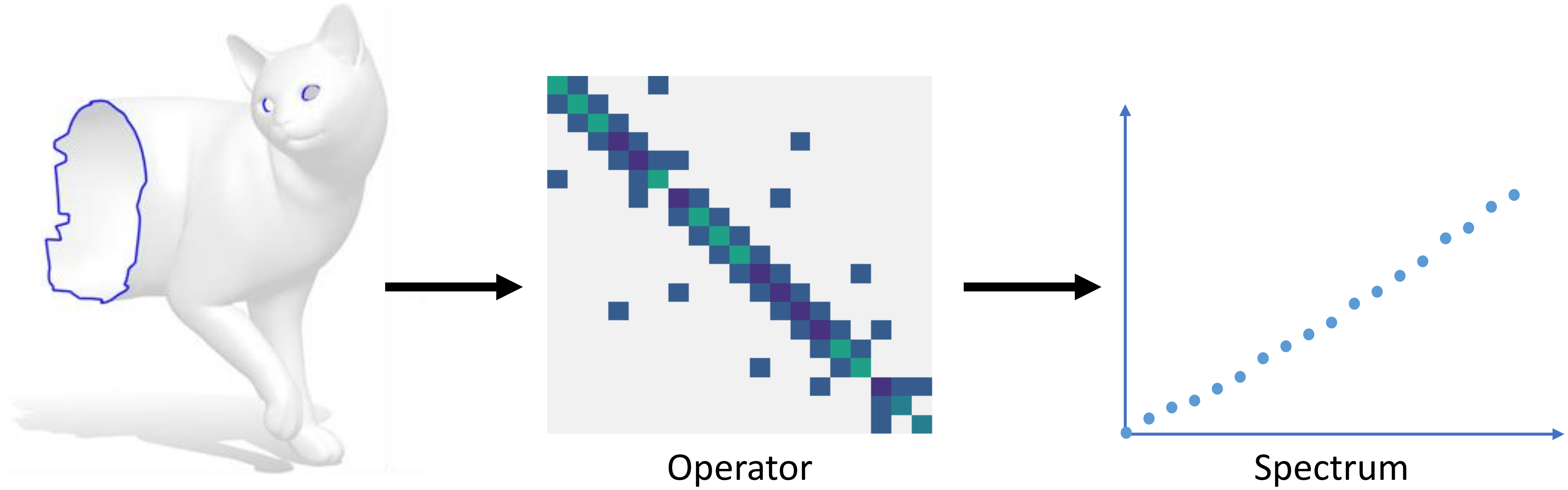
Remark

- Spectral quantities can be used to analyze partialities of 3D objects

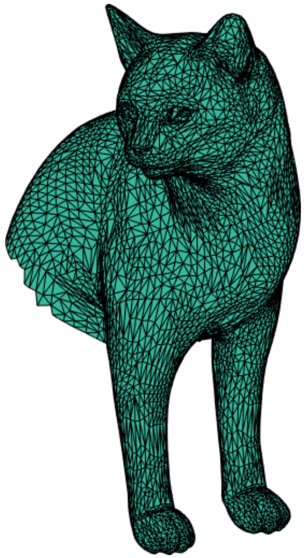


Rodolà et al., *Partial functional correspondence* (2017)

Which operator?

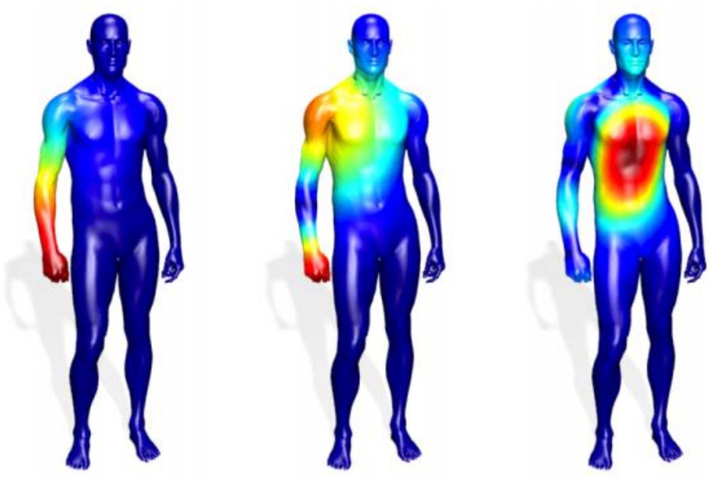


Which operator?



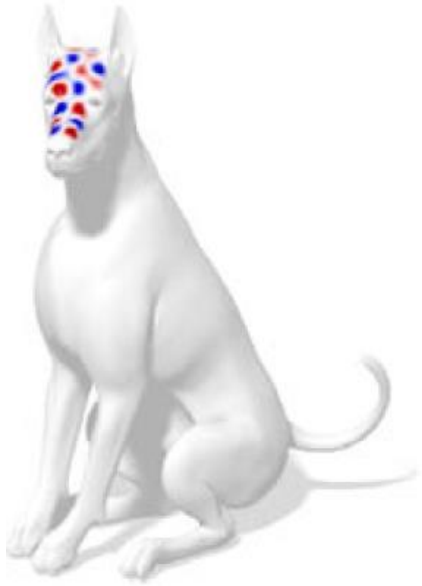
Laplacian of the patch

"Computing Discrete Minimal Surfaces and Their Conjugates", U. Pinkall et al. 1993.



Hamiltonian

"Hamiltonian operator for spectral shape analysis", Y. Choukroun et al. 2018.



LMH

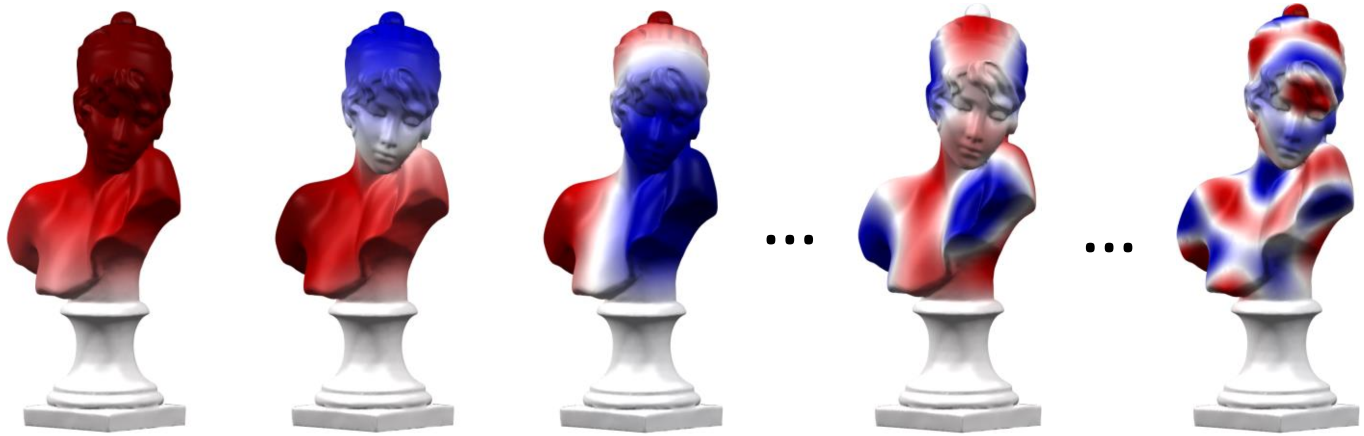
"Localized Manifold Harmonics for Spectral Shape Analysis", S. Melzi et al. 2018.

The Hamiltonian operator

$$H\psi_i(x) = \lambda_i \psi_i(x)$$

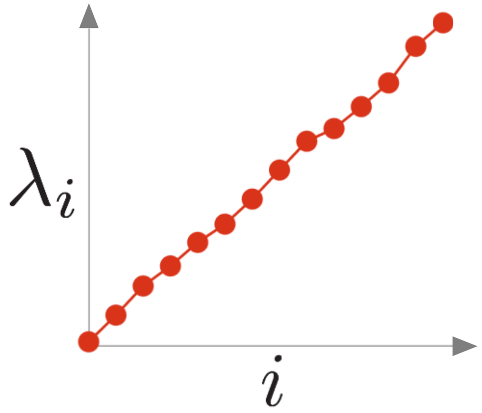
Step potential

$v_\tau : \mathcal{X} \to \{0, \tau\}$



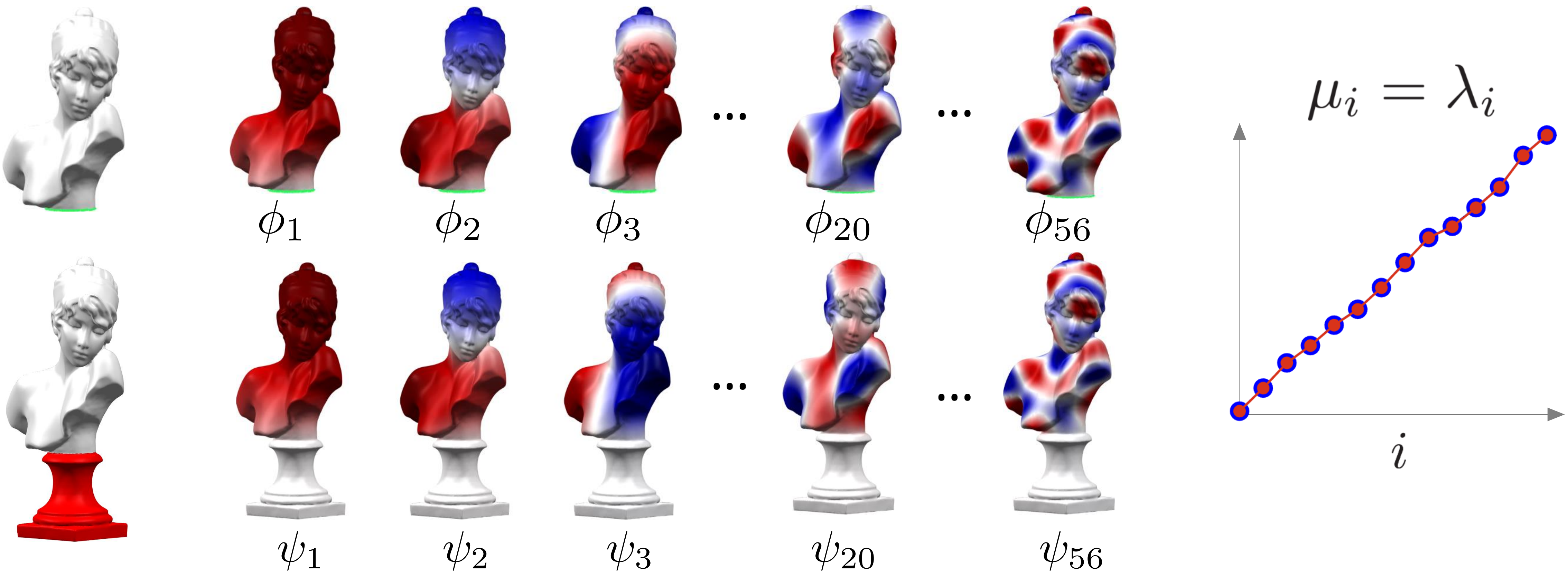
ψ_1 ψ_2 ψ_3 ... ψ_{20} ... ψ_{56}

Hamiltonian spectrum



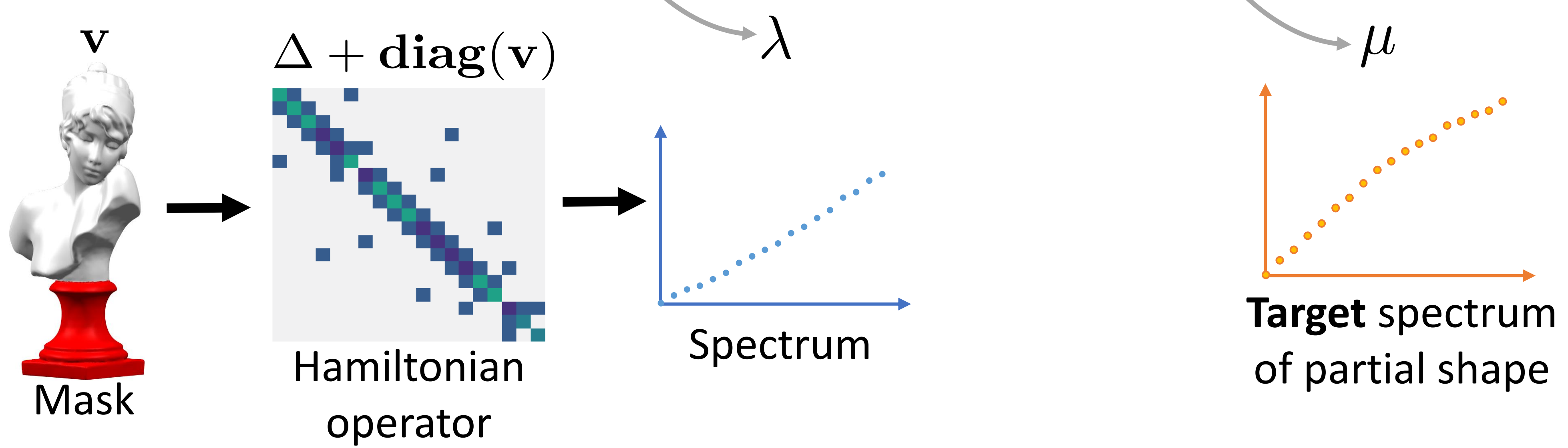
The Hamiltonian operator

Theorem: There exists a step potential for which the Hamiltonian on the full shape and the LBO on the partial shape share the same spectrum:

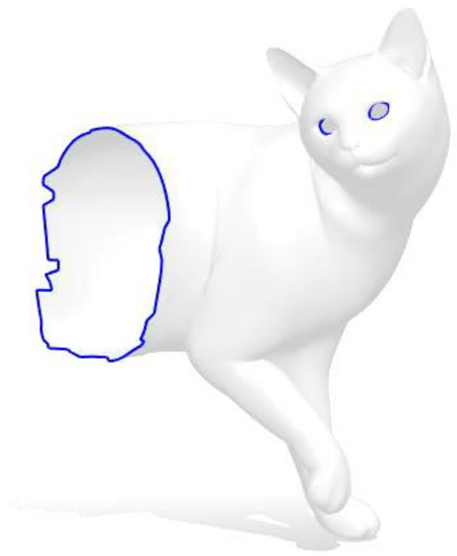
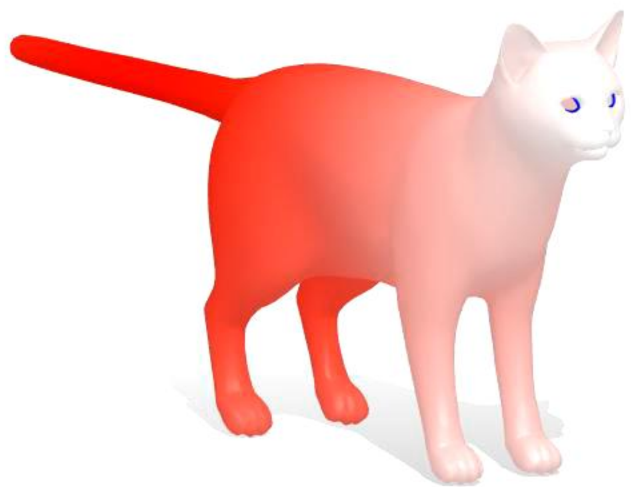


Optimization problem

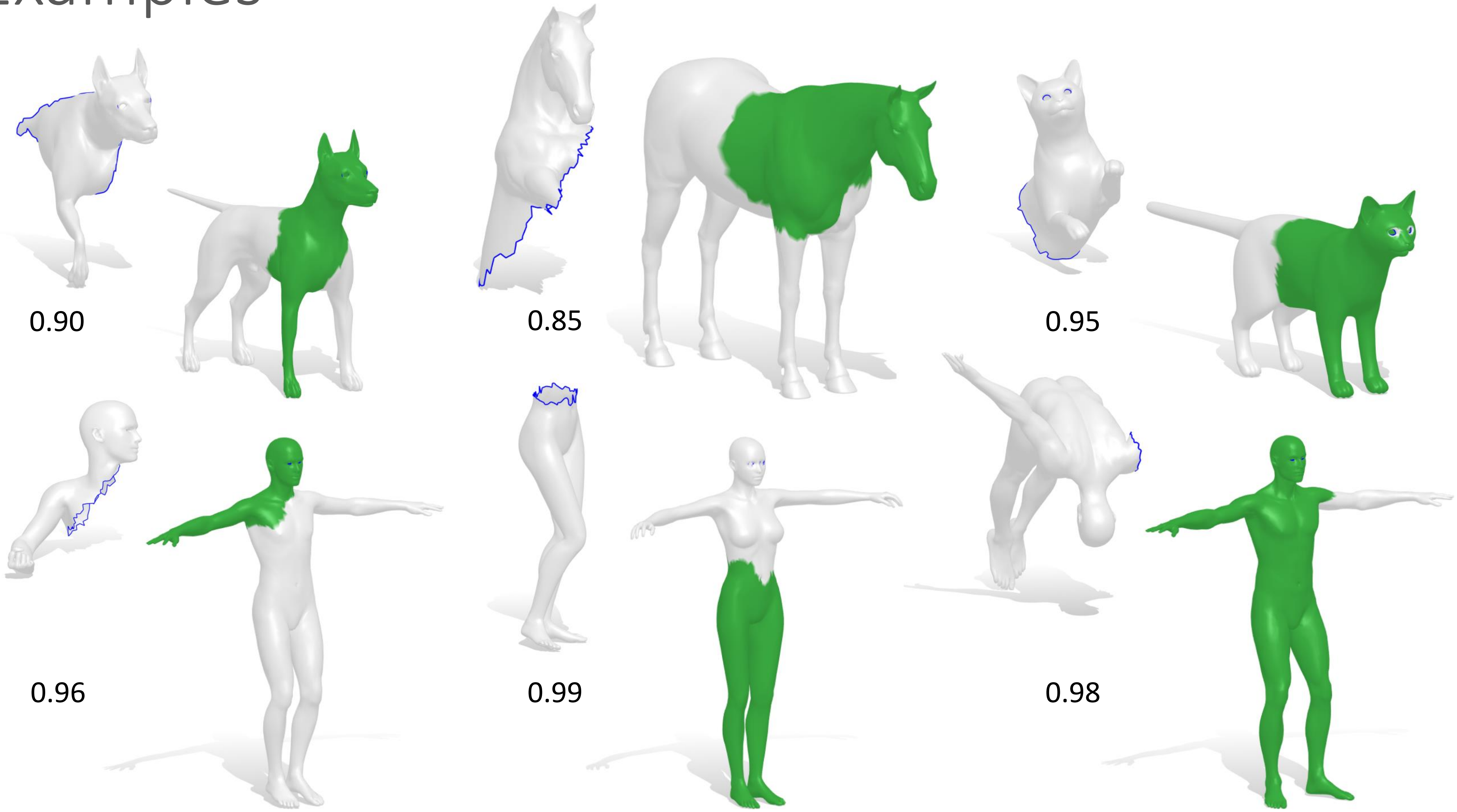
$$\min_{\mathbf{v}\geq 0} \;\|\lambda(\boldsymbol{\Delta} + \mathbf{diag}(\mathbf{v})) - \mu\|_w^2$$



Rampini et al., *Correspondence-Free Region Localization for Partial Shape Similarity via Hamiltonian Spectrum Alignment* (3DV 2019)

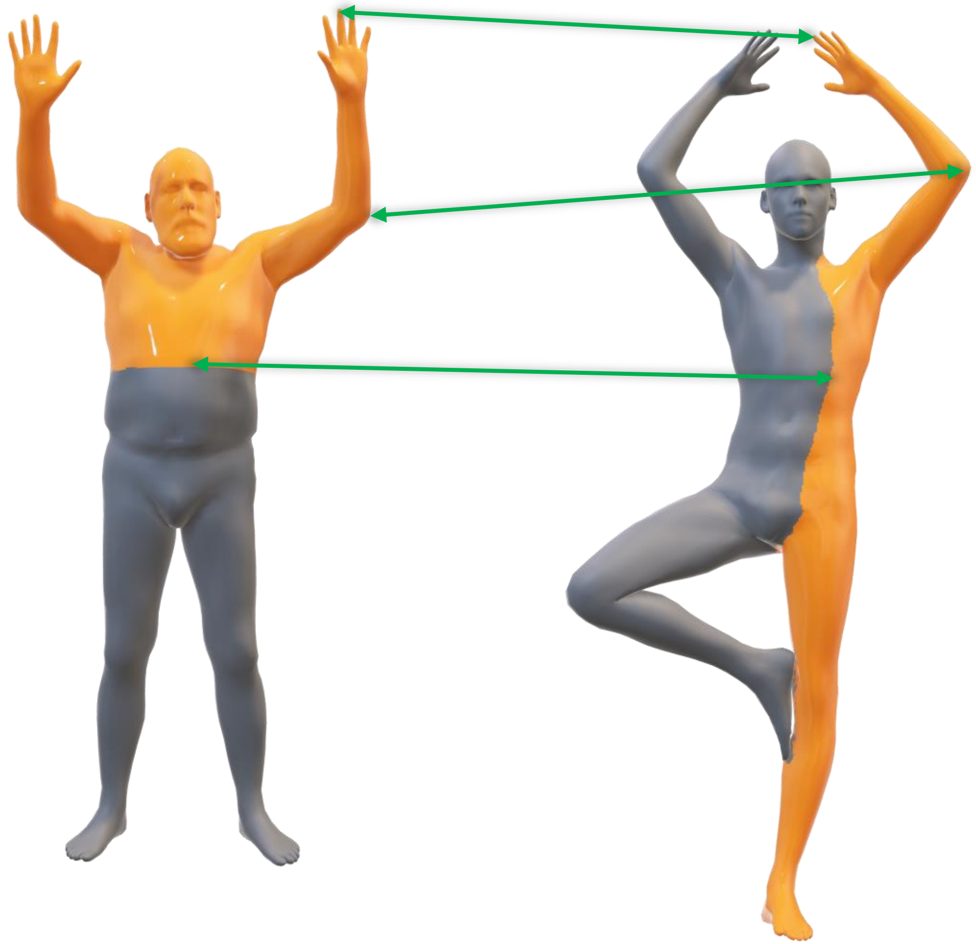



Examples

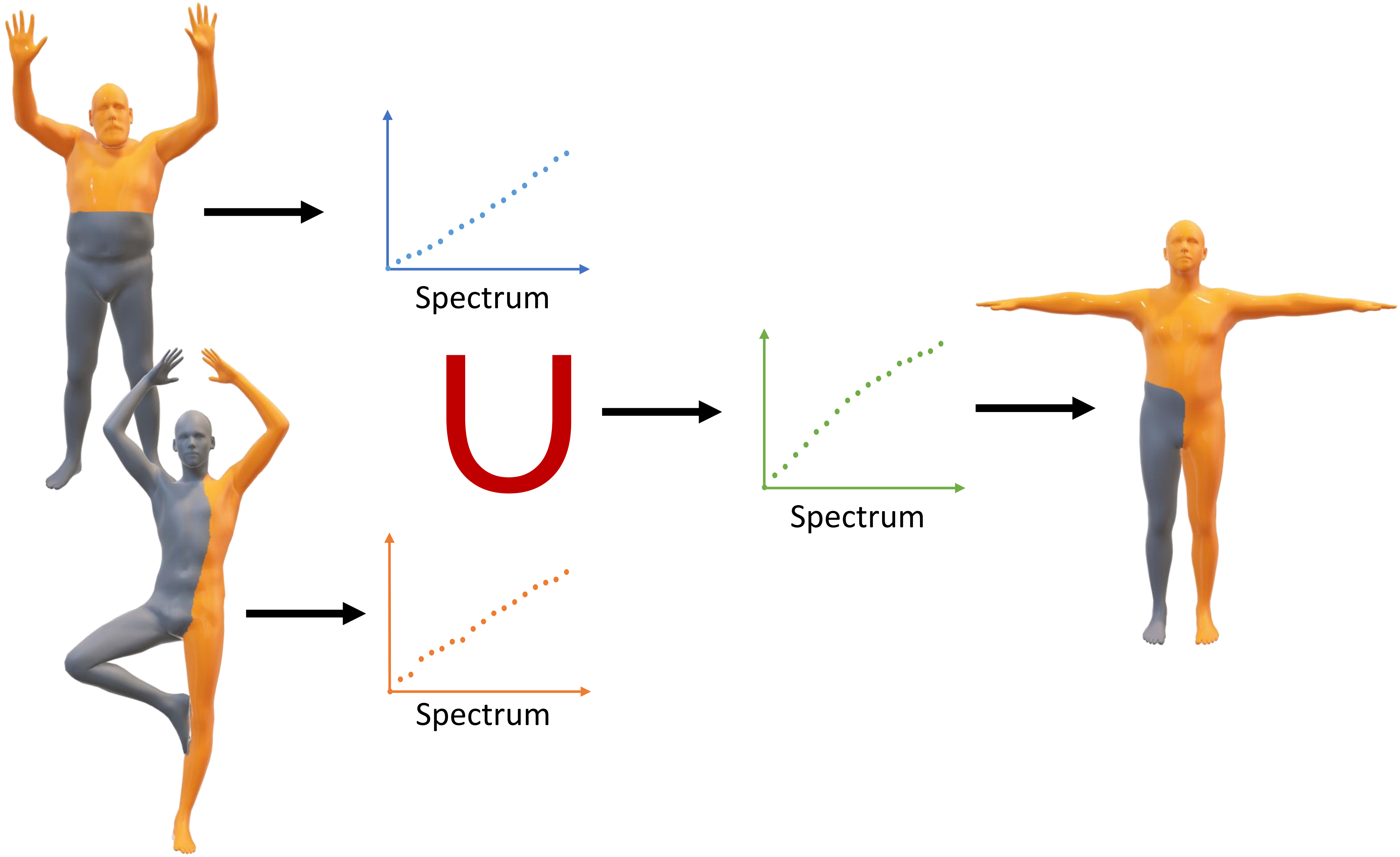


Set operations

Typical pipeline



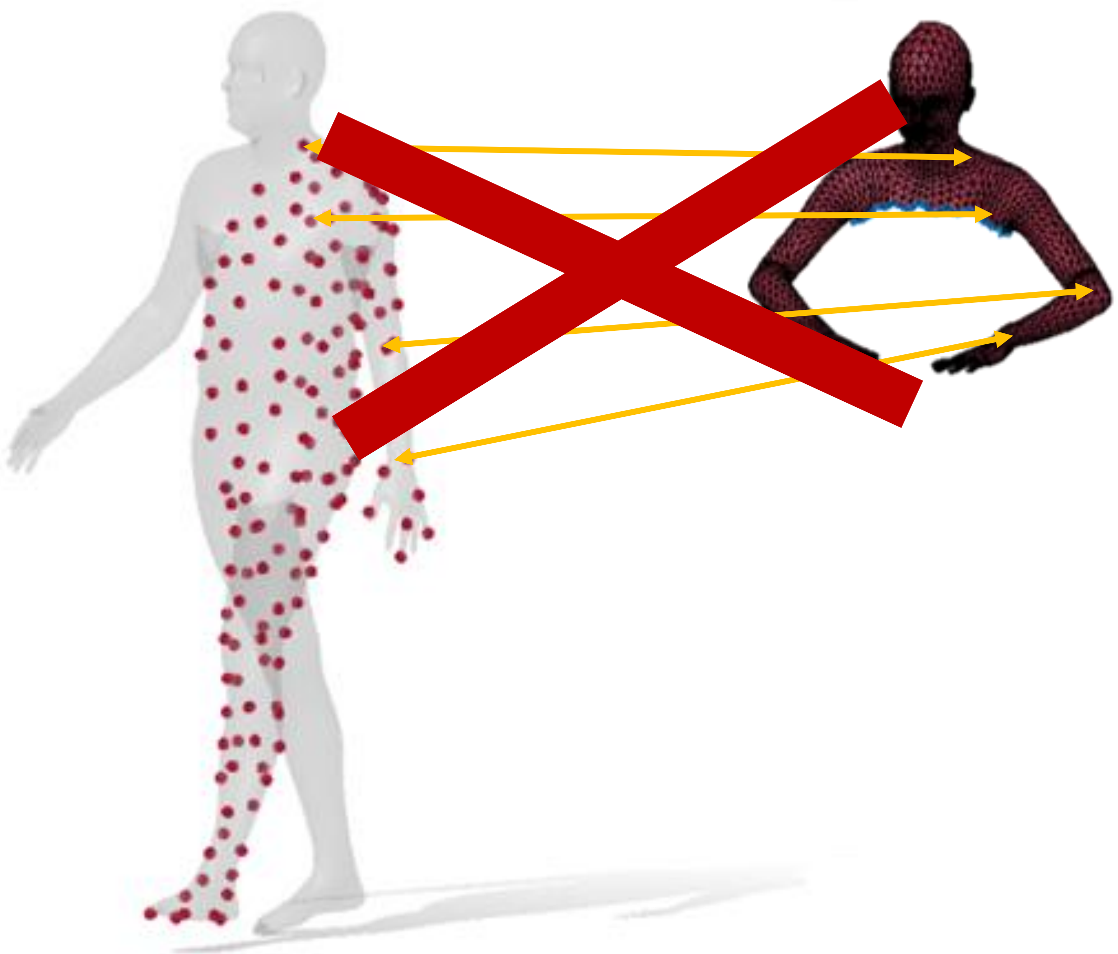
1. Find partial correspondence
2. Extract non-rigid transformation
3. Merge partial views into a consistent discretization



Moschella et al., *Spectral Unions of Partial Deformable 3D Shapes* (EUROGRAPHICS 2022)

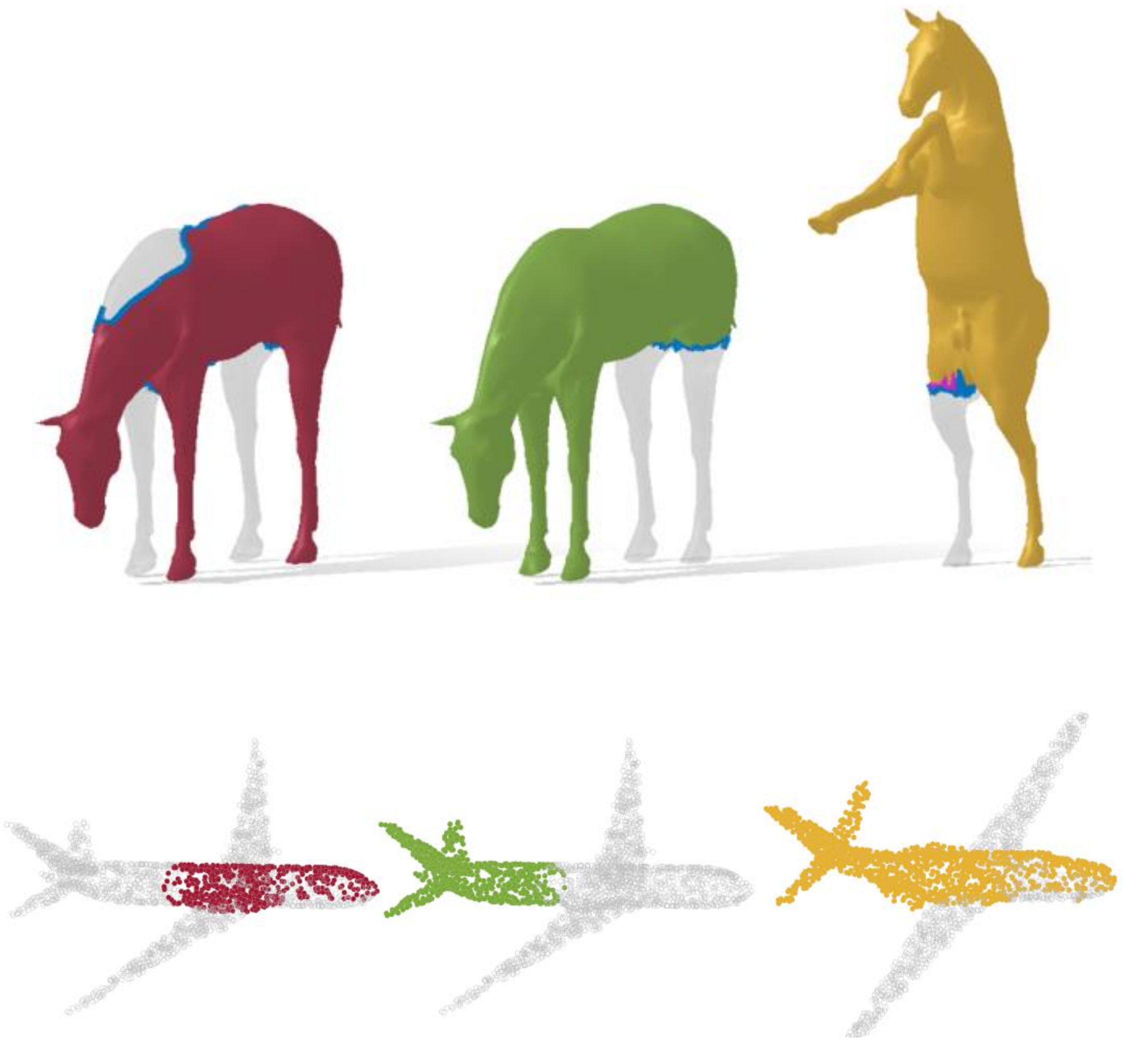
The spectrum is the right tool

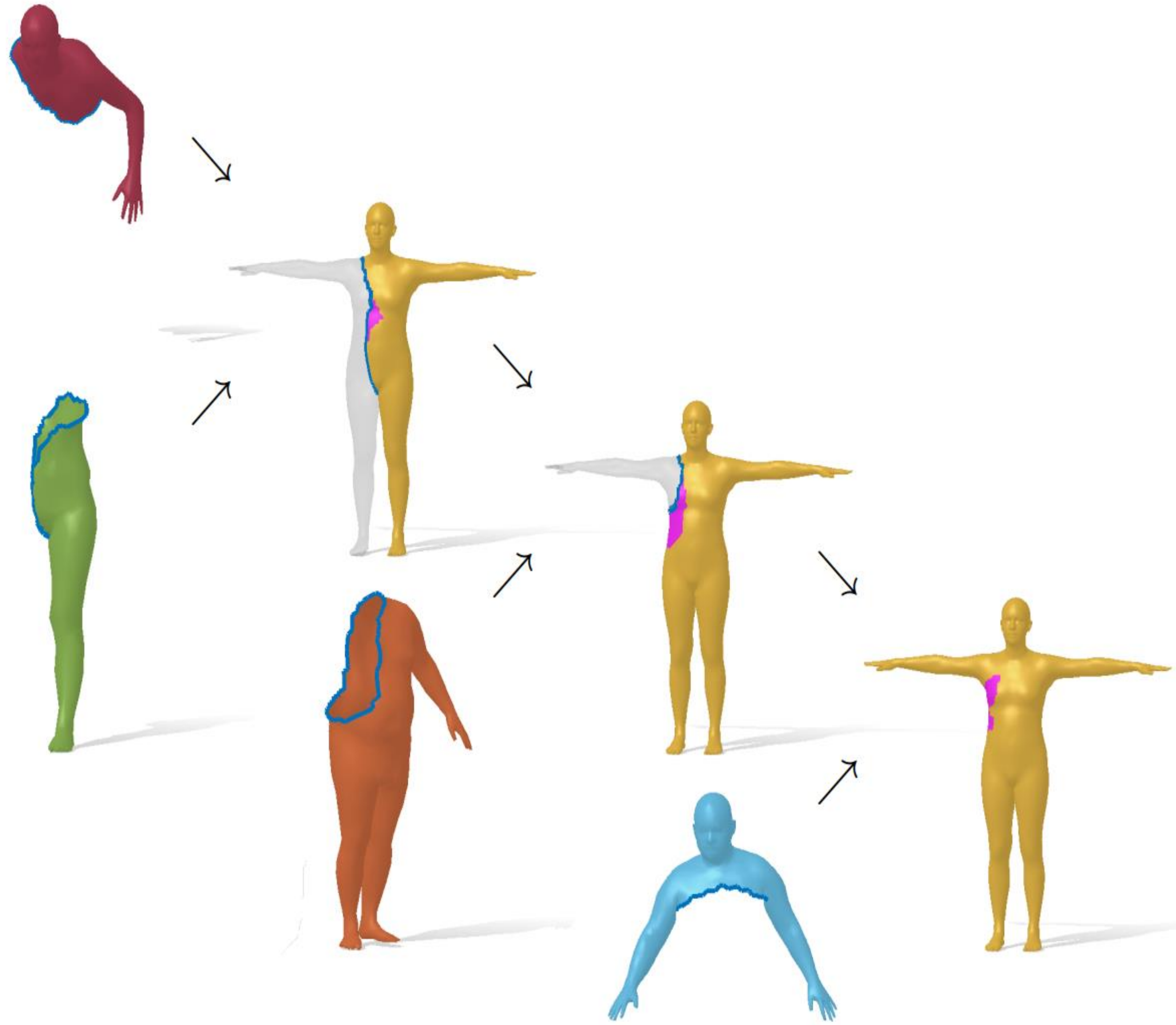
- Invariant to isometries
- Invariant to different representations
- Does not require a correspondence



Moschella et al., *Spectral Unions of Partial Deformable 3D Shapes* (EUROGRAPHICS 2022)

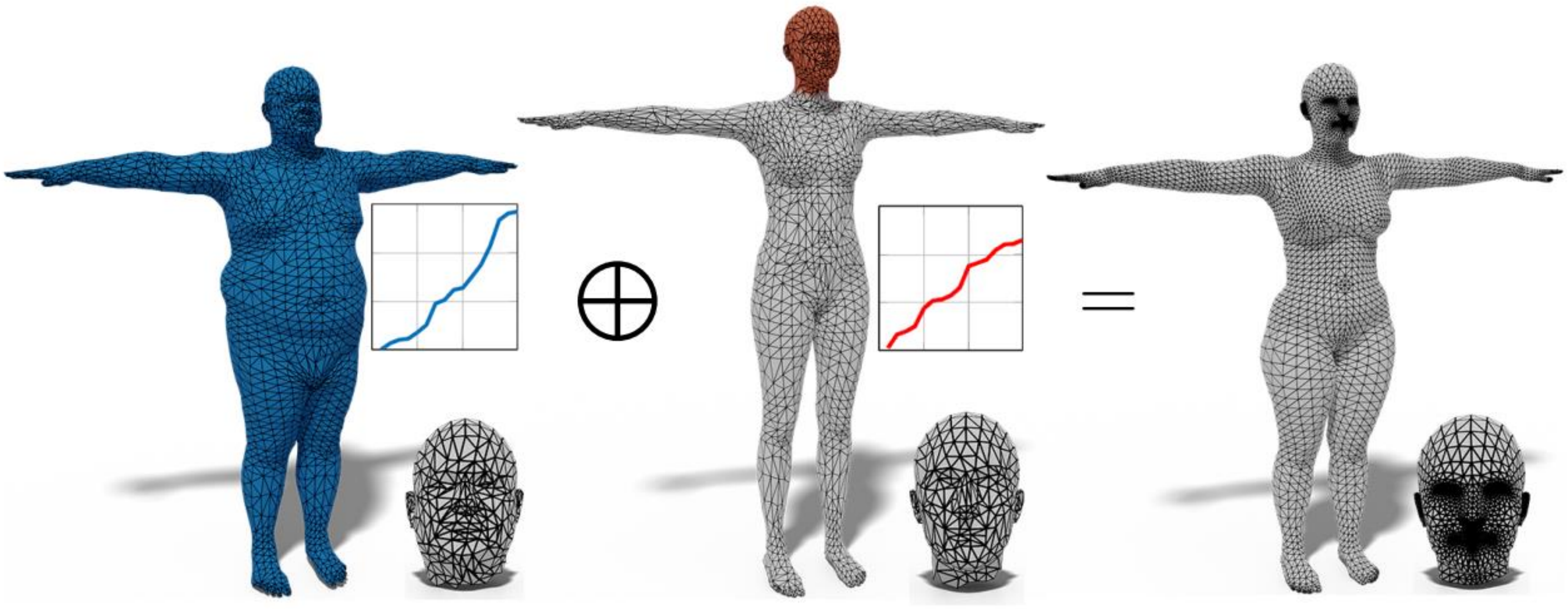
Results





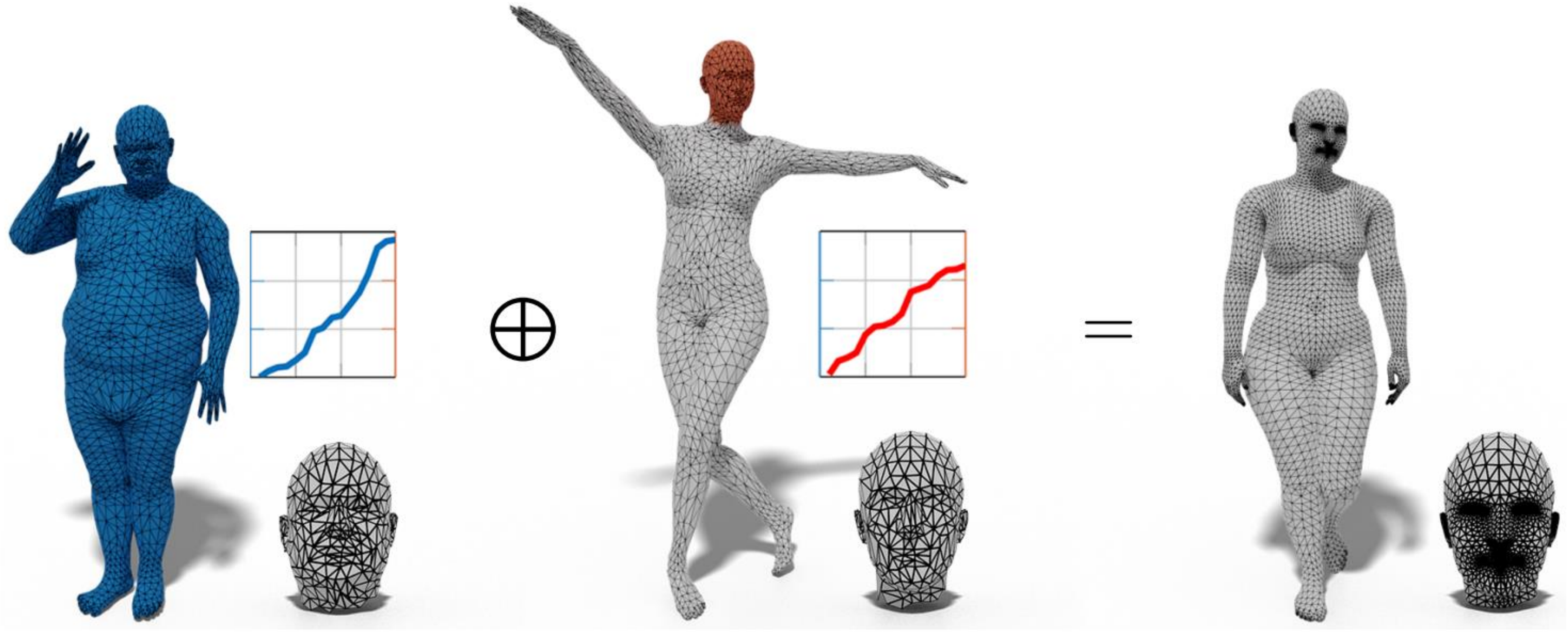
Moschella et al., *Spectral Unions of Partial Deformable 3D Shapes* (EUROGRAPHICS 2022)

Shape generation



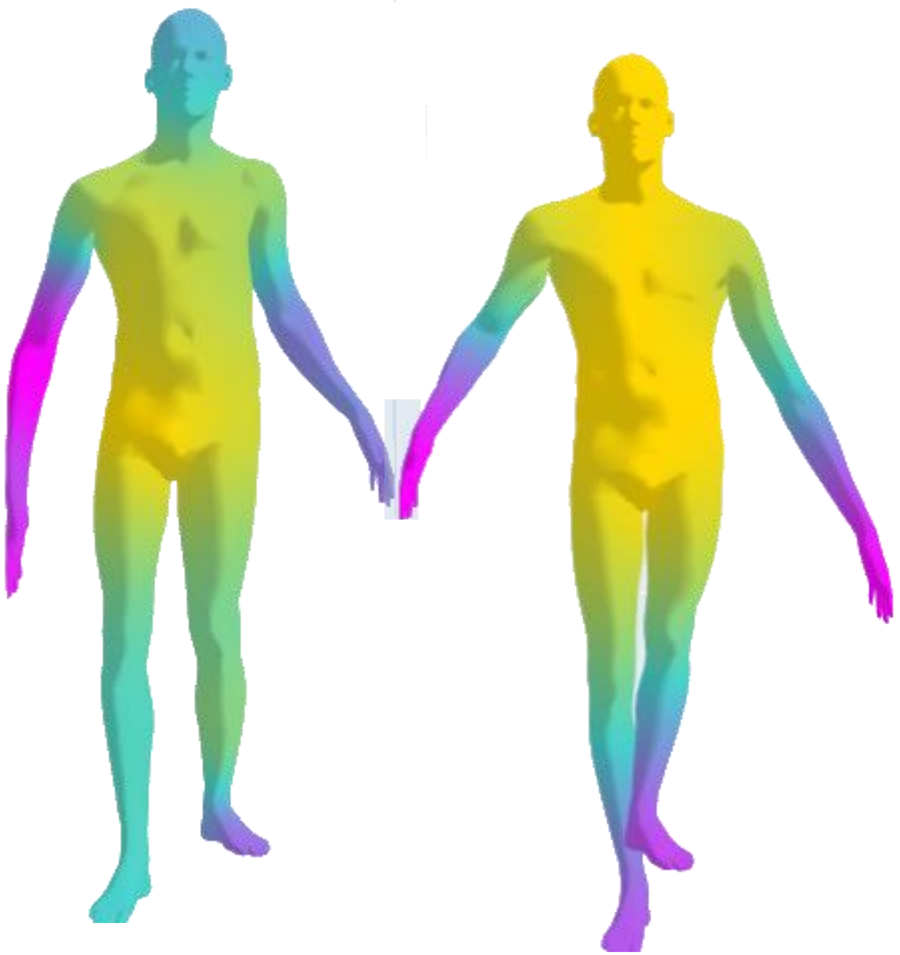
Pegoraro et al., *Learning to generate shape from global-local spectra* (2021)

Shape generation



Pegoraro et al., *Learning to generate shape from global-local spectra* (2021)

Hearing shapes with PyTorch



https://github.com/AriannaRampini/InverseSpectralGeometry_3DVTutorial

GLADIA

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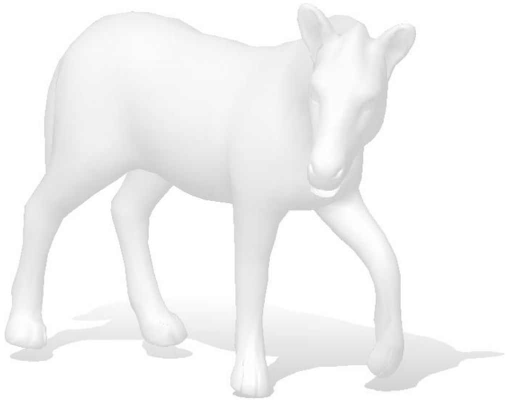
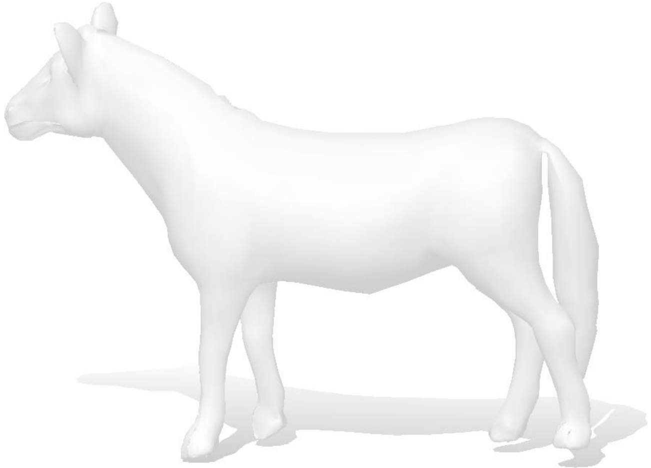


Thank you!



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