TTT4115 Communication Theory
Solution 01 – Spring 2011

Problem 1

a) The received signal can be represented by

$$y = \begin{cases} +1 + w, & x_0 \text{ is sent} \\ -1 + w, & x_1 \text{ is sent} \end{cases}$$

where $w \sim N(0; \sigma^2)$. Then, the PDF of observed y as x_0 was sent is given by

$$p_y(y|x_0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-1)^2}{2\sigma^2}}$$

and the PDF of observed y as x_1 was sent is given by

$$p_y(y|x_1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y+1)^2}{2\sigma^2}}$$

then the PDF required can be represented by

$$p_y(y) = p_y(y|x_0)p(x_0) + p_y(y|x_1)p(x_1)$$

Assume that $p(x_0) = p(x_1)$, then

$$\begin{aligned} p_y(y) &= p_y(y|x_0)p(x_0) + p_y(y|x_1)p(x_1) \\ &= \frac{1}{2}\big(p_y(y|x_0) + p_y(y|x_1)\big) \\ &= \frac{1}{2\sqrt{2\pi}\sigma}\Big(e^{-\frac{(y-1)^2}{2\sigma^2}} + e^{-\frac{(y+1)^2}{2\sigma^2}}\Big) \end{aligned}$$

b) Since $p(y \geq 0) = \int_0^\infty p_y(y)dy$, $\int_{-\infty}^\infty p_y(y)dy = 1$, then $p(y \geq 0) = 1/2$. But

$$p(y \geq 0) = p(y \geq \alpha) + p(0 \leq y < \alpha)$$

then

$$p(y \geq \alpha) = \frac{1}{2} - p(0 \leq y < \alpha)$$

Since

$$\begin{aligned} p(0 \leq y < \alpha) &= \int_0^\alpha p_y(y)dy \\ &= \frac{1}{2}\int_0^\alpha \big(p_y(y|x_0) + p_y(y|x_1)\big)dy \\ &= \frac{1}{2\sqrt{2\pi}\sigma}\int_0^\alpha e^{-\frac{(y-1)^2}{2\sigma^2}}dy + \frac{1}{2\sqrt{2\pi}\sigma}\int_0^\alpha e^{-\frac{(y+1)^2}{2\sigma^2}}dy \end{aligned}$$

Let

$$u = \frac{y-1}{\sqrt{2}\sigma}, \qquad v = \frac{y+1}{\sqrt{2}\sigma}$$

then

$$\begin{aligned}
p(0 \le y < \alpha) &= \frac{1}{2\sqrt{2\pi}\sigma}\int_0^\alpha e^{-\frac{(y-1)^2}{2\sigma^2}}dy + \frac{1}{2\sqrt{2\pi}\sigma}\int_0^\alpha e^{-\frac{(y+1)^2}{2\sigma^2}}dy \\
&= \frac{2}{4\sqrt{\pi}}\int_0^{\frac{\alpha-1}{\sqrt{2}\sigma}} e^{-u^2}du + \frac{2}{4\sqrt{\pi}}\int_0^{\frac{\alpha+1}{\sqrt{2}\sigma}} e^{-v^2}dv \\
&= \frac{1}{4}\text{erf}\Big(\frac{\alpha-1}{\sqrt{2}\sigma}\Big) + \frac{1}{4}\text{erf}\Big(\frac{\alpha+1}{\sqrt{2}\sigma}\Big)
\end{aligned}$$

then

$$\begin{aligned}
p(y \ge \alpha) &= \frac{1}{2} - p(0 \le y < \alpha) \\
&= \frac{1}{2} - \frac{1}{4}\text{erf}\Big(\frac{\alpha-1}{\sqrt{2}\sigma}\Big) - \frac{1}{4}\text{erf}\Big(\frac{\alpha+1}{\sqrt{2}\sigma}\Big) \\
&= \frac{1}{4}\Big(2 - \text{erf}(\frac{\alpha-1}{\sqrt{2}\sigma}) - \text{erf}(\frac{\alpha+1}{\sqrt{2}\sigma})\Big)
\end{aligned}$$

Problem 2

a) Since

$$\begin{aligned}
Y(t) &= \int_0^t X(\tau)d\tau \\
&= \int_0^t A\cos(2\pi f_c\tau)d\tau \\
&= \frac{A\sin(2\pi f_c t)}{2\pi f_c}
\end{aligned}$$

and $A \sim N(0;\sigma_A^2)$, then

$$\begin{aligned}
E\Big[Y(t)\Big] &= \frac{\sin(2\pi f_c t)}{2\pi f_c}E\Big[A\Big] \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\text{var}(Y(t)) &= E\Big[Y^2(t)\Big] \\
&= \frac{\sin^2(2\pi f_c t)}{4\pi^2 f_c^2}E\Big[A^2\Big] \\
&= \frac{\sin^2(2\pi f_c t)}{4\pi^2 f_c^2}\sigma_A^2
\end{aligned}$$

then the PDF is

$$f_{Y(t)}(y(t_k)) = \frac{2\pi f_c}{\sqrt{2\pi}\sin(2\pi f_c t)\sigma_A}e^{-\frac{2\pi^2 f_c^2 y^2(t_k)}{\sin^2(2\pi f_c t)\sigma_A^2}}$$

b) By definition, the autocorrelation of $Y(t)$ is

$$
\begin{aligned}
R_Y(t_1, t_2) &= E\Big[Y(t_1)Y(t_2)\Big] \\
&= \frac{\sin(2\pi f_c t_1)\sin(2\pi f_c t_2)}{4\pi^2 f_c^2} E\Big[A^2\Big] \\
&= \frac{\sigma_A^2}{8\pi^2 f_c^2}\Big(\cos\big(2\pi f_c(t_2 - t_1)\big) - \cos\big(2\pi f_c(t_1 + t_2)\big)\Big)
\end{aligned}
$$

Obviously, $R_Y(t_1, t_2)$ is a function of t_1 and t_2, but not only $t_2 - t_1$. It is nonstationary.

c) It is not ergodic, since each sample function of the stochastic process will depend on A, which is different for each sample function.

Problem 3

a) By definition, the autocorrelation of $X(t)$ is

$$
R_X(\tau) = E\Big[x(t)x(t-\tau)\Big]
$$

Let $\tau = 0$, then we get

$$
R_X(0) = E\Big[x^2(t)\Big]
$$

then

$$
\begin{aligned}
E\Big[Y(t)\Big] &= E\Big[x^2(t)\Big] \\
&= R_X(0)
\end{aligned}
$$

b) By definition, the autocovariance of $Y(t)$ is

$$
\begin{aligned}
C_Y(\tau) &= E\Big[\big(Y(t) - R_X(0)\big)\big(Y(t-\tau) - R_X(0)\big)\Big] \\
&= E\Big[\big(X^2(t) - R_X(0)\big)\big(X^2(t-\tau) - R_X(0)\big)\Big] \\
&= E\Big[X^2(t)X^2(t-\tau)\Big] - R_X^2(0)
\end{aligned}
$$

Since

$$
\begin{aligned}
E\Big[X^2(t)X^2(t-\tau)\Big] &= E\Big[X^2(t)\Big]E\Big[X^2(t-\tau)\Big] + 2E^2\Big[X(t)X(t-\tau)\Big] \\
&= R_X(0)R_X(0) + 2\Big(R_x(\tau)\Big)^2 \\
&= R_X^2(0) + 2R_X^2(\tau)
\end{aligned}
$$

then

$$
\begin{aligned}
C_Y(\tau) &= E\Big[X^2(t)X^2(t-\tau)\Big] - R_X^2(0) \\
&= R_X^2(0) + 2R_X^2(\tau) - R_X^2(0) \\
&= 2R_X^2(\tau)
\end{aligned}
$$