TTT4115 Kommunikasjonsteori

Exercise 2

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Problem 1

a)

$$
\begin{aligned}
R_y(\tau) &= y(t) * y(\tau - t) && (1)\\
&= h(t) * x(t) * h(\tau - t) * x(\tau - t) && (2)\\
&= x(t) * x(\tau - t) * h(t) * h(\tau - t) && (3)\\
&= R_x(\tau) * \left(\int_{-\infty}^{\infty} h(t)h(t+\tau)\,dt \right) && (4)
\end{aligned}
$$

b) The rightmost part of the righthand side of the result in a) is the autocorrelation of the filter transfer function and gives the leftmost part of the righthand side of the equation in b) as a power spectrum when fourier transformed.

The leftmost part of the righthand side of the result in a) obviously gives the input power spectral density when fourier transformed.

Lastly, a convolution is replaced with a multiplication when fourier transform is performed.

c)

$$
\begin{aligned}
\sigma_y^2 &= E\{y^2\} - E\{y\}^2 = E\{y^2\} && (5)\\
&= R_y(0) && (6)\\
&= R_x(0) * \left(\int_{-\infty}^{\infty} h(t)h(t)\,dt \right) && (7)\\
&= R_x(0) \cdot \left(\int_{-\infty}^{\infty} h(t)^2\,dt \right) && (8)
\end{aligned}
$$

$$\sigma_y^2 = \int_0^\infty S_y(f)\,df \tag{9}$$
$$= \int_0^\infty |H(f)|^2\,S_x(f)\,df \tag{10}$$

d) X having unit variance gives:

$$\sigma_x^2 = 1 \tag{11}$$
$$R_x(0) = 1 \tag{12}$$

Remembering c) we have:

$$\sigma_y^2 = R_x(0)\cdot\left(\int_{-\infty}^{\infty} h(t)^2\,dt\right) \tag{13}$$
$$\sigma_y^2 = \int_{-\infty}^{\infty} h(t)^2\,dt \tag{14}$$

$h(t)$ is still real, so the absolute value operator makes no significant difference to the expression.

Problem 2

a)

$$D(f) = X(f) - G(f)\big[N(f) + H(f)X(f)\big] \tag{15}$$

b) We remember that X(f) is independent of N(f), thus the cross-correlation of the two is zero.

$$S_D(f) = D(f)\cdot D^*(f) \tag{16}$$
$$\begin{aligned} &= S_X(f) + |G(f)|^2\left[\,|H(f)|^2\,S_X(f) + S_N(f) + 2\mathrm{Re}\left\{H(f)X(f)N^*(f)\right\}\right] \\ &\qquad -2\mathrm{Re}\left\{X^*(f)G(f)\big[H(f)X(f)+N(f)\big]\right\} \end{aligned} \tag{17}$$
$$\begin{aligned} &= S_X(f) + |G(f)|^2\left[\,|H(f)|^2\,S_X(f) + S_N(f)\right] \\ &\qquad -2\mathrm{Re}\left\{G(f)\big[H(f)S_X(f) + X^*(f)N(f)\big]\right\} \end{aligned} \tag{18}$$
$$= S_X(f) + |G(f)|^2\left[\,|H(f)|^2\,S_X(f) + S_N(f)\right] - 2\mathrm{Re}\left\{G(f)H(f)\right\}S_X(f) \tag{19}$$

This can of course be simplified further by assuming one or more of $G(f)$ and $H(f)$ are real.

c)

$$|G(f)|^2 = \frac{|H(f)|^2 S_X^2(f)}{|H(f)|^4 S_X^2(f) + S_N^2(f) + |H(f)|^2 S_X(f) S_N(f)} \tag{20}$$

$$= \frac{|H(f)|^2 S_X^2(f)}{|H(f)|^4 S_X^2(f) + S_N^2(f)} \tag{21}$$

$$\begin{aligned} S_D(f) = S_X(f) + \frac{|H(f)|^2 S_X^2(f)}{|H(f)|^4 S_X^2(f) + S_N^2(f)} (|H(f)|^2 S_X(f) + S_N(f)) \\ -2\mathrm{Re}\left\{ \frac{H^*(f) S_X(f)}{|H(f)|^2 S_X(f) + S_N(f)} H(f) S_X(f) \right\} \end{aligned} \tag{22}$$

$$\begin{aligned} = S_X(f) + \frac{|H(f)|^2 S_X^2(f)}{|H(f)|^4 S_X^2(f) + S_N^2(f)} (|H(f)|^2 S_X(f) + S_N(f)) \\ -2 \frac{|H(f)|^2 S_X^2(f)}{|H(f)|^2 S_X(f) + S_N(f)} \end{aligned} \tag{23}$$

$$\begin{aligned} = S_X(f) + \frac{|H(f)|^2 S_X^2(f) \, |H(f)|^2 S_X(f) + |H(f)|^2 S_X^2(f) S_N(f)}{|H(f)|^4 S_X^2(f) + S_N^2(f)} \\ -2 \frac{|H(f)|^2 S_X^2(f)}{|H(f)|^2 S_X(f) + S_N(f)} \end{aligned} \tag{24}$$

$$= S_X(f) + \frac{|H(f)|^4 S_X^3(f)}{|H(f)|^4 S_X^2(f) + S_N^2(f)} - 2 \frac{|H(f)|^2 S_X^2(f)}{|H(f)|^2 S_X(f) + S_N(f)} \tag{25}$$

d) •

$$S_N(f) = \left| \frac{N_0}{2} \right|^2 = \frac{|N_0|^2}{4} \tag{26}$$

$$S_D = S_X(f) + \frac{|A|^4 S_X^3(f)}{|A|^4 S_X^2(f) + \frac{|N_0|^4}{16}} - 2 \frac{|A|^2 S_X^2(f)}{|A|^2 S_X(f) + \frac{|N_0|^2}{4}} \tag{27}$$

- $$S_D = S_X(f) + \frac{\left|\frac{B}{\sqrt{S_X(f)}}\right|^4 S_X^3(f)}{\left|\frac{B}{\sqrt{S_X(f)}}\right|^4 S_X^2(f) + \frac{|N_0|^4}{16}} - 2\frac{\left|\frac{B}{\sqrt{S_X(f)}}\right|^2 S_X^2(f)}{\left|\frac{B}{\sqrt{S_X(f)}}\right|^2 S_X(f) + \frac{|N_0|^2}{4}} \tag{28}$$

$$S_D = S_X(f) + \frac{\frac{|B|^4}{S_X^2(f)} S_X^3(f)}{\frac{|B|^4}{S_X^2(f)} S_X^2(f) + \frac{|N_0|^4}{16}} - 2\frac{\frac{|B|^2}{S_X(f)} S_X^2(f)}{\frac{|B|^2}{S_X(f)} S_X(f) + \frac{|N_0|^2}{4}} \tag{29}$$

$$S_D = S_X(f) + \frac{|B|^4 S_X(f)}{|B|^4 + \frac{|N_0|^4}{16}} - 2\frac{|B|^2 S_X(f)}{|B|^2 + \frac{|N_0|^2}{4}} \tag{30}$$

$$S_D = S_X(f)\left(1 + \frac{|B|^4}{|B|^4 + \frac{|N_0|^4}{16}} - 2\frac{|B|^2}{|B|^2 + \frac{|N_0|^2}{4}}\right) \tag{31}$$

- $$A = \frac{C}{\sqrt[4]{S_X(f)}} \tag{32}$$

$$S_D = S_X(f) + \frac{\left|\frac{C}{\sqrt[4]{S_X(f)}}\right|^4 S_X^3(f)}{\left|\frac{C}{\sqrt[4]{S_X(f)}}\right|^4 S_X^2(f) + \frac{|N_0|^4}{16}} - 2\frac{\left|\frac{C}{\sqrt[4]{S_X(f)}}\right|^2 S_X^2(f)}{\left|\frac{C}{\sqrt[4]{S_X(f)}}\right|^2 S_X(f) + \frac{|N_0|^2}{4}} \tag{33}$$

$$S_D = S_X(f) + \frac{\frac{|C|^4}{S_X(f)} S_X^3(f)}{\frac{|C|^4}{S_X(f)} S_X^2(f) + \frac{|N_0|^4}{16}} - 2\frac{\frac{|C|^2}{\sqrt{S_X(f)}} S_X^2(f)}{\frac{|C|^2}{\sqrt{S_X(f)}} S_X(f) + \frac{|N_0|^2}{4}} \tag{34}$$

$$S_D = S_X(f) + \frac{|C|^4 S_X^2(f)}{|C|^4 S_X(f) + \frac{|N_0|^4}{16}} - 2\frac{|C|^2 S_X^{\frac{3}{2}}(f)}{|C|^2 \sqrt{S_X(f)} + \frac{|N_0|^2}{4}} \tag{35}$$

e)

f)

g) Sub-problems e) – g) are not answered, due to time constraints.