

# The Computation of Equivalent Potential Temperature

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## ABSTRACT

A simplified procedure is described for computation of equivalent potential temperature which remains valid in situations such as in the tropics where a term which is omitted in the derivation of the conventional formula can lead to an error of several degrees absolute. The procedure involves new empirical formulas which are introduced for the saturated vapor pressure of water, the lifting condensation level temperature and the equivalent potential temperature. Errors are estimated for each of these, and results are compared with those obtained by the similar, but more complicated procedures of Betts and Dugan (1973) and Simpson (1978).

## 1. Introduction

The equivalent potential temperature (also known as pseudo-equivalent potential temperature) will be taken, as in Holton (1972), to be the final temperature  $\theta_E$  which a parcel of air attains when it is lifted dry adiabatically to its lifting condensation level, then pseudo-wet adiabatically (with respect to water saturation) to a great height (dropping out condensed water as it is formed), then finally brought down dry adiabatically to 1000 mb.

The first two processes are examined by Simpson (1978), the first by Betts and Miller (1975) and the second by Betts and Dugan (1973), hereafter referred to as SS, BM and BD, respectively. SS shows that a term normally neglected in the pseudo-adiabatic ascent can lead to errors of up to 3 K in the computed value of  $\theta_E$  which would, for instance, result in a significant underestimate of the height to which penetrative convection can reach. Both SS and BD give approximate empirical formulas for correcting this error.

In this paper the formulas of SS and BD are examined and compared with simpler empirical formulas which are proposed, and which are found to permit a better fit to values computed by numerical integration. Estimates are made of the errors introduced at each of the stages for attaining the equivalent potential temperature, including the final dry adiabatic descent.

It should be noted that the definition of equivalent potential temperature given above is not the only possible one, since account could have been taken of heat retained by condensed water, and of latent heat of ice formation, which can affect the value by several degrees, as shown in Saunders (1957). Inclusion of these effects, however, would require a

detailed study of cloud microphysics, which is beyond the scope of this paper and, therefore, they have been omitted in keeping with common practice.

## 2. Constants and empirical formulas

Based on List (1951), using the conversion 1 IT cal = 4.18684 J where necessary to convert to SI units, the following constants and empirical formulas will be used as giving best values for the temperature range  $-35^\circ\text{C} \leq T \leq 35^\circ\text{C}$ , and pressure range  $200 \text{ mb} \leq p \leq 1000 \text{ mb}$ . For some constants, the error over this range is negligible; for the specific heats, however, there is an appreciable variation with temperature and pressure, and maximum errors are estimated.

### a. Temperature

$$\text{Absolute temperature: } T_K = T + 273.15, \quad (1)$$

where  $T$  is the temperature in degrees Celsius. For all temperatures apart from the above, a capital subscript will be used to denote absolute, a small subscript Celsius; e.g., for the lifting condensation level temperature,  $T_L = T_l + 273.15$ .

### b. Dry air

Gas constant:

$$R_d = 287.04 \text{ J kg}^{-1} \text{ K}^{-1}.$$

Specific heat at constant pressure:

$$c_{pd} = 1005.7 \pm 2.5 \text{ J kg}^{-1} \text{ K}^{-1}.$$

$$\kappa_d = \frac{R_d}{c_{pd}} = 0.2854 \pm 0.0007.$$

## c. Water

Specific heat:

$$c_w = 4190 \pm 30 \text{ J kg}^{-1} \text{ K}^{-1} \quad (\text{for } T \geq 0^\circ\text{C only}).$$

Latent heat of vaporization:

$$L_w = (2.501 - 0.00237 T) \times 10^6 \text{ J kg}^{-1}. \quad (2)$$

## d. Water vapor

Gas constant:

$$R_v = 461.50 \text{ J kg}^{-1} \text{ K}^{-1}.$$

Specific heat at constant pressure:

$$c_{pv} = 1875 \pm 25 \text{ J kg}^{-1} \text{ K}^{-1}.$$

$$\epsilon = \frac{R_d}{R_v} = 0.6220.$$

## e. Potential temperature for moist air

An unsaturated parcel moving adiabatically satisfies

$$c_{pm} \frac{dT_K}{T_K} + R_m \frac{dp}{p} = 0, \quad (3)$$

where

$$R_m \approx R_d(1 + 0.608 \times 10^{-3} r), \quad (4)$$

$$c_{pm} \approx c_{pd}(p, T_K)[1 + 0.887 \times 10^{-3} r], \quad (5)$$

the latter being obtained as the best fit to the tabulated data of List (1951);  $r$  here is the mixing ratio (expressed in g kg<sup>-1</sup>).Potential temperature  $\theta$  is defined as the temperature an unsaturated parcel would attain if taken adiabatically to 1000 mb. Taking account of the variation of  $c_{pd}$  with pressure and temperature,  $\theta$  may be found approximately by the iterative formula

$$\theta = T_K \left( \frac{1000}{p} \right)^{[R_d/c_{pd}(\bar{p}, \bar{T}_K)(1-0.28 \times 10^{-3} r)]} \quad (6)$$

where  $c_{pd}$  is evaluated at  $\bar{p} = \frac{1}{2}(1000 + p)$ ,  $\bar{T}_K = \frac{1}{2}(\theta + T_K)$  in order to minimize error. Setting  $c_{pd}$  to its value at 1000 mb, 0°C gives

$$\theta = T_K \left( \frac{1000}{p} \right)^{0.2854(1-0.28 \times 10^{-3} r)}, \quad (7)$$

with an error of up to 0.2 K from neglecting variation of  $c_{pd}$ .

## f. Saturated vapor pressure of water

BM and SS both make use of the formula of Tetens (1930) in which the saturated vapor pressure (mb) of water is given by

$$e_s(T) = 6.11 \times 10^{7.5T/(T+237.3)}. \quad (8)$$

As remarked by Murray (1967), this is acceptable for most meteorological purposes, except when extreme accuracy is required at low temperatures where, for example, there is a 2% error at  $T = -30^\circ\text{C}$ , as compared with the value obtained using the formulation of Goff and Gratch (1946). In what follows, unfortunately, an accuracy of better than 0.5% is necessary at low temperatures in order that the lifting condensation level temperature, found by inverting the formula for  $e_s(T)$ , may be sufficiently accurately determined. A modified formula has therefore been derived, based on values obtained using the more recent formula of Wexler (1976).

According to Wexler, the saturated vapor pressure (mb) of water for  $0^\circ\text{C} \leq T \leq 100^\circ\text{C}$  is given to better than 0.005% by

$$\ln e_s = \sum_{i=0}^6 g_i T_K^{-i} + g_7 \ln T_K, \quad (9)$$

where the coefficients  $g_i$  take the following values:

$$\begin{aligned} g_0 &= -2.9912729 \times 10^3, & g_4 &= 1.7838301 \times 10^{-5} \\ g_1 &= -6.0170128 \times 10^3, & g_5 &= -8.4150417 \times 10^{-10} \\ g_2 &= 1.887643854 \times 10^1, & g_6 &= 4.4412543 \times 10^{-13} \\ g_3 &= -2.8354721 \times 10^{-2}, & g_7 &= 2.858487 \times 10^0. \end{aligned}$$

The following formula was fitted to Wexler's results, extrapolated for  $T < 0^\circ\text{C}$ , to an accuracy of 0.1% for  $-30^\circ\text{C} \leq T \leq 35^\circ\text{C}$ :

$$e_s(T) = 6.112 \exp\left(\frac{17.67 T}{T + 243.5}\right). \quad (10)$$

For  $T < 0^\circ\text{C}$ , owing to the lack of accurate experimental data, there is some uncertainty about the validity of extrapolating Wexler's formula. The values of List (1951), obtained by extrapolating the formula of Goff and Gratch (1946) were therefore also taken into account, and the formula fits both sets of data to 0.3% for  $-35^\circ\text{C} \leq T \leq 35^\circ\text{C}$ .

Inverting (10) gives

$$T = \frac{243.5 \ln e_s - 440.8}{19.48 - \ln e_s}, \quad (11)$$

which gives temperatures accurate to 0.03°C for this range.

A similar empirical formula is required for  $T_K^{-1/\kappa_d} e_s(T)$ . Taking  $1/\kappa_d = 3.504$ , the following formula fits Wexler's data to 0.2% for  $-30^\circ\text{C} \leq T \leq 35^\circ\text{C}$ , and fits both Wexler's and List's data to 0.4% for  $-35^\circ\text{C} \leq T \leq 35^\circ\text{C}$ :

$$\frac{e_s(T)}{T_K^{1/\kappa_d}} = 1.7743 \times 10^{-8} \exp\left(\frac{12.992 T}{T + 217.8}\right). \quad (12)$$

Inverting this gives temperatures accurate to 0.05°C.

TABLE 1. Saturated vapor pressure of water calculated from Wexler's formula (9), and percentage errors  $\delta_1$ ,  $\delta_2$  and  $\delta_3$  in Tetens' formula (8), and in (10) and (12), respectively.

$T^\circ\text{C}$	$e_s(T)$ (mb)	$\delta_1$	$\delta_2$	$\delta_3$
-40	0.1905	-3.2	-0.47	-0.91
-30	0.5106	-1.7	-0.05	-0.07
-20	1.2563	-0.77	0.09	0.20
-10	2.8657	-0.26	0.07	0.15
0	6.1121	-0.03	-0.00	-0.00
10	12.279	0.04	-0.06	-0.14
20	23.385	0.02	-0.07	-0.16
30	42.452	-0.02	0.01	-0.01
40	73.813	-0.05	0.18	0.34

Table 1 gives values of saturated vapor pressure obtained using Wexler's formula (9), and the percentage errors obtained when Tetens' (1930) formula (8) and the new formulas (10) and (12) are compared with Wexler's values. The superiority of (10) over Tetens' formula for  $T < 0^\circ\text{C}$  can be clearly seen, while for  $T > 0^\circ\text{C}$ , the loss in accuracy is unimportant.

### 3. The lifting condensation level

The first stage in computing the equivalent potential temperature of a parcel of air is to find the temperature  $T_L$  which it would attain if lifted adiabatically to its condensation level.

By eliminating  $z$  from the equations

$$\frac{dT_D}{dz} = \frac{-gT_D^2}{\epsilon L_w(T_D)T_K} \quad \text{and} \quad \frac{dT_K}{dz} = \frac{-g}{c_{pd}} \quad (13)$$

for the lapse rate of dew point and the dry adiabatic lapse rate, respectively, and integrating making use of (2), it can be shown that  $T_L$  is given by

$$\frac{1}{T_L} - \frac{1}{T_D} + 1.266 \times 10^{-3} \ln\left(\frac{T_L}{T_D}\right) = 0.514 \times 10^{-3} \ln\left(\frac{T_K}{T_D}\right). \quad (14)$$

This, however, can only be solved iteratively for  $T_L$ . By approximating the  $T_L$  and  $T_D$  dependence by rational functions, a direct formula may be obtained

$$T_L = \frac{1}{\frac{1}{T_D - 56} + \frac{\ln(T_K/T_D)}{800}} + 56, \quad (15)$$

where temperatures are all absolute. This gives agreement with (14) to within a tenth of a degree and is therefore adequate.

However, since subsequent work makes use of the mixing ratio  $r$ , it is more convenient (rather than

working in terms of the dew point) to derive a formula directly which uses mixing ratio, or rather, water vapor pressure  $e$ , related to it by

$$e = \frac{pr}{\epsilon + r} = \frac{pr}{622 + r}. \quad (16)$$

The starting point in deriving such a formula is the relationship

$$\frac{T_L}{T_K} = \left[ \frac{e_s(T_L)}{e} \right]^{\kappa_d(1-0.28 \times 10^{-3}r)}, \quad (17)$$

which follows from conservation of potential temperature. SS uses this relationship to derive an implicit formula for  $T_L$  which, making use of (10) instead of Tetens' formula and taking natural logs rather than logs to base 10 as in SS, gives

$$\ln\left(\frac{6.112}{e}\right) + \frac{\ln(T_K/T_L)}{0.2854(1 - 0.28 \times 10^{-3}r)} + \frac{17.67(T_L - 273.15)}{T_L - 29.65} = 0. \quad (18)$$

Such a formula involves the inconvenience of solving iteratively for  $T_L$ , so a direct formula will be derived instead, and will be shown to be equally accurate.

Eq. (17) may be rewritten

$$\frac{e_s(T_L)}{T_L^{1/\kappa_d}} = \frac{e}{T_K^{1/\kappa_d}} \left( \frac{T_K}{T_L} \right)^{0.28 \times 10^{-3}r/\kappa_d} \quad (19)$$

Making use of (12) and taking logarithms gives

$$\ln(1.7743 \times 10^{-8}) + \frac{12.992 T_L}{T_L + 217.8} = \ln e - 3.504 \ln T_K + 0.981 \times 10^{-3} r \ln\left(\frac{T_K}{T_L}\right) \quad (20)$$

(writing  $T_L$  on the left since temperature here is in  $^\circ\text{C}$ ). Now, since  $r$  is never greater than  $40 \text{ g kg}^{-1}$ , the last term on the right is considerably smaller than  $\ln e$ , and will be neglected (whereas SS includes it). Hence, with slight adjustment of the coefficients,

$$T_L = \frac{2840}{3.5 \ln T_K - \ln e - 4.805} + 55. \quad (21)$$

An equally straightforward formula, which makes use of percentage relative humidity  $U$  can be derived in a similar way:

$$T_L = \frac{1}{\frac{1}{T_K - 55} - \frac{\ln(U/100)}{2840}} + 55. \quad (22)$$

The main errors in (21) and (22) are those which arise from the empirical formula (12), and those from neglecting  $r$  and the variation of  $c_{pd}$  (and hence  $\kappa_d$ ) in

TABLE 2. Lifting condensation level temperature  $T_l$  found using the modified Simpson formula (18), and errors  $\epsilon_1$ ,  $\epsilon_2$ ,  $\epsilon_3$  and  $\epsilon_4$  in the original Simpson formula, and in the approximate formulas (15), (21) and (22), respectively.

$T$ (°C)	$U$ (%)	$p$ (mb)	$T_l$ (°C)	$\epsilon_1$	$\epsilon_2$	$\epsilon_3$	$\epsilon_4$
20	75	1000	14.383	-0.003	0.000	-0.006	0.008
20	75	700	14.381	-0.003	0.001	-0.004	0.009
-20	75	1000 & 500	-23.903	0.037	-0.001	-0.027	0.004
20	25	1000	-4.797	0.039	-0.015	-0.016	-0.004
20	25	700	-4.799	0.039	-0.013	-0.013	-0.002
-20	25	1000 & 500	-37.522	0.188	0.008	0.020	0.047

(17); these are less than 0.05 K, 0.03 K and 0.03 K, respectively, so it can be concluded that (21) and (22) determine  $T_L$  correct to 0.1 K.

Table 2 gives values (°C) of  $T_l$  as calculated by the modified version (18) of the formula of SS. The errors for the original formula of SS and for (15), (21) and (22) are also given, the dew-point temperature for (15) being found using (11). In Eqs. (15), (21) and (22),  $T_l$  does not vary with pressure, whereas it does in (18) because of inclusion of the term involving  $r$ ; such variation, however, can be seen to be negligible, and all three formulas agree with (18) to within 0.05°C in all realistic cases. The error in the original formula of SS is significantly larger for low temperatures owing to the error in Tetens' formula.

For the parcel of air at its lifting condensation level, in addition to the temperature, two potential temperatures will be required. The (moist) potential temperature  $\theta$  is conserved in the lifting, so is given simply by (7). The potential temperature  $\theta_D$  of dry air, given by

$$\theta_D = T_K \left( \frac{1000}{p - e} \right)^{\kappa_d}, \quad (23)$$

is not conserved, however. Making use of conservation of  $\theta$ , it can be shown that the value  $\theta_{DL}$  of  $\theta_D$  at the lifting condensation level is given by

$$\theta_{DL} = T_K \left( \frac{1000}{p - e} \right)^{0.2854} \left( \frac{T_K}{T_L} \right)^{0.28 \times 10^{-3} r}, \quad (24)$$

where  $T_K$ ,  $p$  and  $e$  are evaluated at the initial level. The extra term  $(T_K/T_L)^{0.28 \times 10^{-3} r}$  can make a difference of up to 0.2 K in  $\theta_{DL}$ .

#### 4. Equivalent potential temperature

Having brought the air to its lifting condensation level, it is now necessary to raise it to a great height along a water-saturation pseudo-adiabat, i.e., dropping out condensed water as soon as it is formed. Such a process satisfies the equation [derived, for example, in Holton (1972)]

$$c_{pd} \frac{d\theta_D}{\theta_D} + c_w r_s \frac{dT_K}{T_K} + d \left( \frac{r_s L_w}{T_K} \right) = 0, \quad (25)$$

where

$$r_s = \frac{\epsilon e_s(T)}{p - e_s(T)}, \quad (26)$$

$$\theta_D = T_K \left( \frac{1000}{p - e_s(T)} \right)^{\kappa_d}. \quad (27)$$

The equivalent potential temperature  $\theta_E$  is given as the value obtained for  $\theta_D$  when (25) is integrated from the lifting condensation level to a great height. As a simplification, the middle term of (25) has commonly been neglected, giving the following approximation for  $\theta_E$ :

$$\theta'_E = \theta_{DL} \exp \left[ \frac{r_s(T_L) L_w(T_L)}{c_{pd} T_L} \right]. \quad (28)$$

TABLE 3. Values of  $\theta_E$  obtained by numerical integration of (25), together with errors  $\eta_1$ ,  $\eta_2$ ,  $\eta_3$ ,  $\eta_4$ ,  $\eta_5$  and  $\eta_6$  from the conventional formula (28), from Simpson's original formula, and from (33), (35), (38) and (39), respectively, assuming a parcel initially saturated at pressure  $p_i$  (mb), temperature  $T_i$  (°C).

$p_i$ (mb)	$T_i$ (°C)	$\theta_E$ (K)	$\eta_1$	$\eta_2$	$\eta_3$	$\eta_4$	$\eta_5$	$\eta_6$
1000	30	386.28	-3.90	-0.33	0.08	0.39	0.01	-0.018
1000	20	335.61	-1.56	0.04	-0.08	0.18	-0.02	-0.005
1000	0	283.60	-0.27	0.09	0.02	-0.02	0.04	-0.003
1000	-30	244.01	-0.02	0.02	0.02	-0.01	0.01	-0.001
700	20	394.71	-2.83	-0.39	-0.04	-0.10	-0.04	0.015
700	0	319.13	-0.45	0.05	0.02	-0.09	0.03	-0.002
700	-30	270.57	-0.02	0.02	0.03	-0.03	0.02	-0.001
200	-30	391.82	-0.13	-0.06	0.09	-0.19	0.03	-0.006
200	-50	354.11	-0.01	0.01	0.03	-0.03	0.01	-0.002
Maximum error			-4	-0.4	0.1	0.4	0.05	0.02

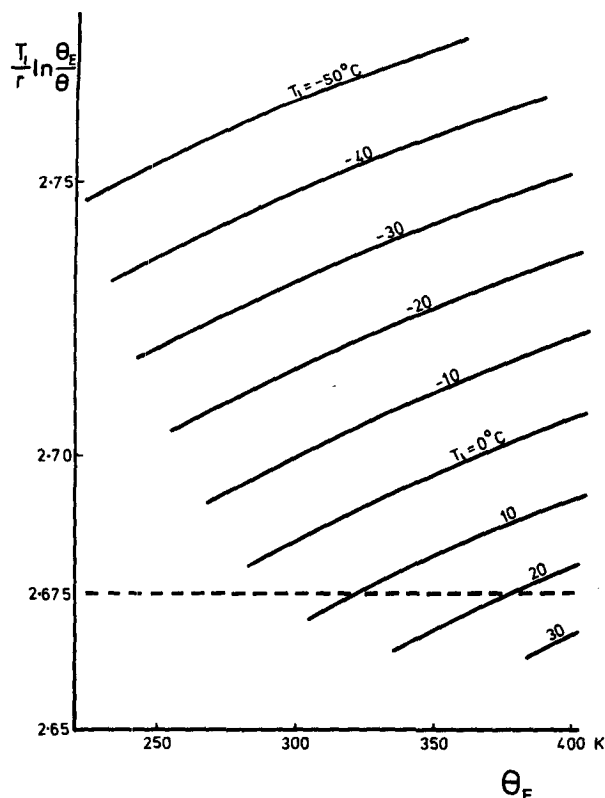


FIG. 1. Plot of  $(T_L/r) \ln(\theta_E/\theta)$  against  $\theta_E$  for various values of  $T_L$ ; the best fit for  $\theta_E$  is shown dashed.

For a more accurate integration of (25) it is necessary to state rather carefully the process by which the water is condensed out; were this carried out maintaining the parcel at a constant temperature, the middle term would vanish and (28) would be exactly the potential temperature attained. However, for a pseudo-adiabatic process as defined earlier, no such simplification exists, and it is necessary to integrate (25) numerically. For this purpose, the following variables will be introduced.

$$\chi = \frac{L_w(T)r_s(T)}{c_{pd}T_K}, \quad (29)$$

$$\theta_x = \theta_D \exp(\chi). \quad (30)$$

Thus (25) becomes

$$\frac{d\theta_x}{\theta_x} + \frac{c_w}{L_w(T)} \chi dT = 0. \quad (31)$$

Eliminating  $p$  and  $r_s$  from (26)–(30) gives

$$\chi \exp(\chi/\kappa_d) = \frac{\epsilon}{1000c_{pd}} \theta_x^{1/\kappa_d} \frac{L_w(T)e_s(T)}{T_K^{1/\kappa_d+1}}. \quad (32)$$

This may be used to solve for  $\chi$ , and hence to

integrate (31) from the lifting condensation level, where  $\theta_x$  takes the value  $\theta'_E$  [given by (28)] to a great height, where  $\theta_E$  is now the value obtained for  $\theta_x$ . Results are given in Table 3, where they are compared with those obtained from the various approximate formulas;  $\eta_1$ , the error in  $\theta'_E$ , can be seen to be as large as 4 K, thus confirming the results of SS.

SS derived a correction to (28) by making a linear approximation in the integration of (25). The errors as compared with the numerical results, given as  $\eta_2$  in Table 3, can be seen to be significant at lower levels for large values of  $\theta_E$ . For consistency in comparison, the errors here and in subsequent formulas were adjusted to give minimax best fits to the numerical results for the range  $200 \text{ mb} \leq p \leq 1000 \text{ mb}$ ,  $\theta_E \leq 400 \text{ K}$ . The formula of SS thus becomes

$$\theta_E = \theta'_E \exp[\exp(3.0 \ln \theta'_E + 13.5 \ln T_L - 99.55)], \quad (33)$$

where  $\theta'_E$  is given by (28). The errors,  $\eta_3$  in Table 3, are now everywhere less than 0.1 K. However, it should be noted that SS overlooks an additional error of up to 0.2 K in assuming that  $\theta_D$  is conserved when bringing air to its lifting condensation level; this extra error can be avoided by making use of (24), but the procedure by this stage is becoming rather cumbersome.

The formula of BD makes use of  $\theta$  not  $\theta_D$  and hence avoids this additional source of error. They fit numerically computed values to a formula of the form

$$\theta_E = \theta \exp\left[\frac{A(\theta_E)r}{T_L}\right], \quad (34)$$

where  $A(\theta_E)$  is a linear function of  $\theta_E$ ; the occurrence of  $\theta_E$  on the right-hand side means that the formula is iterative. However, plotting  $(T_L/r) \ln(\theta_E/\theta)$  against  $\theta_E$  for different  $T_L$  (Fig. 1), using results of the numerical integration, it can be clearly seen that the inclusion of  $\theta_E$  dependence in  $A$  makes negligible improvement to the formula. A best fit to the numerical results is thus given simply by

$$\theta_E = \theta \exp\left[\frac{2.675r}{T_L}\right] \quad (35)$$

(dashed line in Fig. 1), with errors ( $\eta_4$  in Table 3) which can be as large as 0.4 K.

As neither of the formulas of SS and BD was entirely satisfactory, even when modified, new formulas were fitted to the numerical results, which gave negligible error for  $200 \text{ mb} \leq p \leq 1000 \text{ mb}$ ,  $\theta_E \leq 400 \text{ K}$ . The formula of BD was used as a starting point as it has the advantage of using  $\theta$ , not  $\theta_D$ . A graph was plotted of  $(C_{pd}T_L/L_w \times 10^{-3}r) \ln(\theta_E/\theta)$

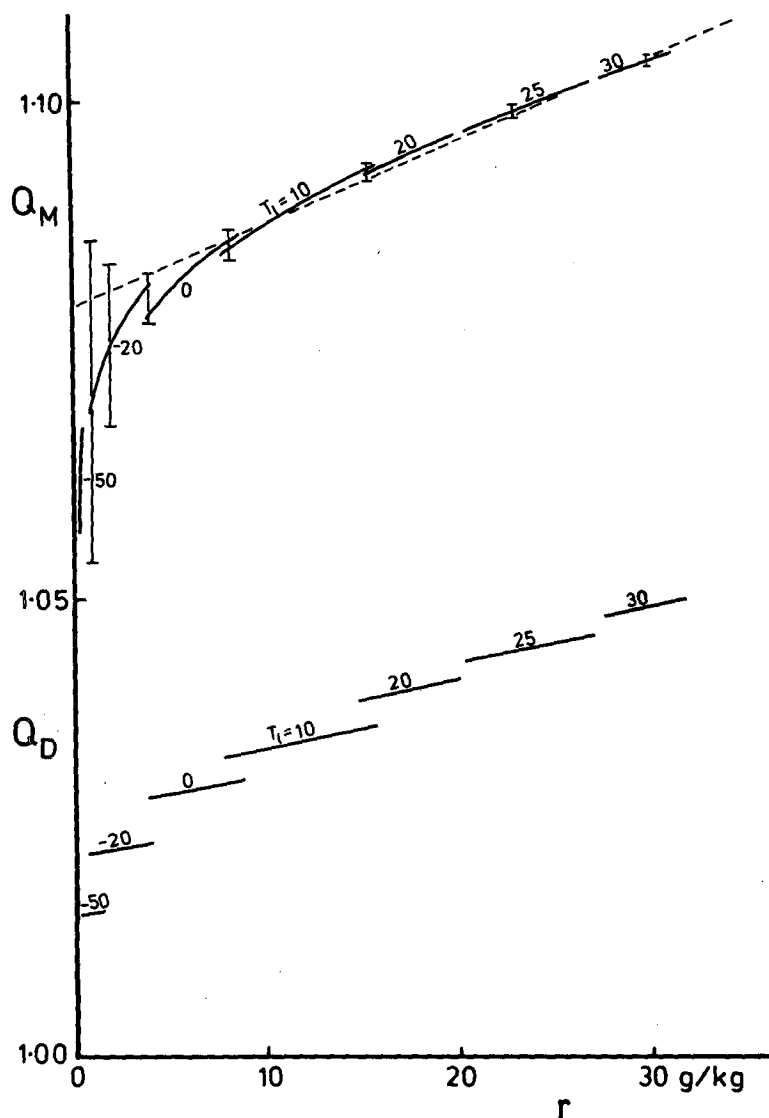


FIG. 2. Plots of  $Q_M = (c_{pd}T_L/L_w \times 10^{-3} r) \ln(\theta_E/\theta)$  (upper graph), and  $Q_D = (c_{pd}T_L/L_w \times 10^{-3} r) \ln(\theta_E/\theta_{DL})$  (lower graph) against  $r$  for various values of  $T_i$ ; the error bars and best fit for  $\theta_E$  (dashed) are shown for the upper graph.

against  $r$  for different  $T_i$  (Fig. 2); it showed very little dependence on  $T_i$ , so a formula of the type

$$\frac{c_{pd}T_L}{L_w \times 10^{-3} r} \ln(\theta_E/\theta) = f(r) \quad (36)$$

could be fitted. Error bars are drawn for an error of  $\pm 0.05$  K in  $\theta_E$ ; they vary in length because  $f(r)$  is multiplied by  $r$  in computing  $\theta_E$ . Hence, although  $f(r)$  shows a rapid fall near  $r = 0$ , a linear fit to  $f(r)$  (dashed) was found to be sufficient to determine  $\theta_E$  to an accuracy of 0.05 K. The best fit thus obtained was

$$f(r) = 1.0784(1 + 0.810 \times 10^{-3} r), \quad (37)$$

giving

$$\begin{aligned} \theta_E &= \theta \exp \left[ 1.0723 \times 10^{-6} \frac{L_w(T_L)}{T_L} \right. \\ &\quad \left. \times r(1 + 0.810 \times 10^{-3} r) \right] \\ &= \theta \exp \left[ \left( \frac{3.376}{T_L} - 0.00254 \right) \right. \\ &\quad \left. \times r(1 + 0.81 \times 10^{-3} r) \right], \quad (38) \end{aligned}$$

where the coefficients in the last line have been adjusted so as to minimize rounding errors. Errors in the computed  $\theta_E$  are given as  $\eta_5$  in Table 3.

Fig. 2 also shows a plot of  $(c_{pd}T_L/L_w \times 10^{-3} r) \times \ln(\theta_E/\theta_{DL})$ , with  $\theta_{DL}$  now replacing  $\theta$ , against  $r$ . There is now a dependence on  $T_L$ , but this and the dependence on  $r$  are both very nearly linear. Hence with a formula similar to (38) but with  $L_w(T_L)$  replaced by another linear function of  $T_L$ , it is possible to evaluate  $\theta_E$  to an accuracy better than 0.02 K, the appropriate formula being

$$\theta_E = \theta_{DL} \exp \left[ \left( \frac{3.036}{T_L} - 0.00178 \right) \times r(1 + 0.448 \times 10^{-3} r) \right]. \quad (39)$$

The errors are given as  $\eta_6$  in Table 3.

For most purposes, Eq. (38) is probably to be preferred, since it makes use of  $\theta$ . As a basis for very accurate work, however, Eq. (39) with  $\theta_{DL}$  given by (24) is suggested. Either is simpler and more accurate than the formulas of SS and BD.

Eq. (38), as it makes use of  $\theta$ , also provides a relationship between  $\theta_E$  and the wet-bulb potential temperature  $\theta_w$  which is attained when a parcel is brought down to 1000 mb from its lifting condensation level along a pseudo-adiabat. This is due to the fact that a saturated parcel at  $T_K = \theta = \theta_w$  at 1000 mb, when taken pseudo-adiabatically to a great height and then returned adiabatically to 1000 mb, will attain a temperature  $\theta_E$ , so that

$$\theta_E = \theta_w \exp \left[ \left( \frac{3.376}{\theta_w} - 0.00254 \right) \times r_s(1 + 0.81 \times 10^{-3} r_s) \right], \quad (40)$$

where

$$r_s = \frac{622 \times e_s(\theta_w)}{1000 - e_s(\theta_w)}, \quad (41)$$

and  $e_s(\theta_w)$  is given by (10).

Apart from the errors arising in the fit to the numerically computed results (i.e., those listed in Table 3), all the approximate formulas are subject to the following errors:

- The most serious computational error is from neglect of the variation of  $c_{pd}$  with temperature and pressure. This affects not only taking a parcel from its lifting condensation level to a great height, but also raising it initially to its lifting condensation level and, most importantly, returning it from a great height to the surface, processes for which the potential temperatures which are conserved should be defined by formulas similar to (6). The cumulative effect of these errors is that the assumed value 1005.7 of  $c_{pd}$  should be replaced wherever it occurs by  $c_{pd}(\bar{p}, \bar{T}_K)$ , where now  $\bar{p} = \frac{1}{2}(1000 + p)$ ,  $\bar{T}_K = \frac{1}{2}(\theta_E + T_K)$  so that, for instance, Eq. (38) would be replaced by

$$\theta_E = T_K \left( \frac{1000}{p} \right)^{[Rd/c_{pd}(\bar{p}, \bar{T}_K)(1 - 0.28 \times 10^{-3} r)]} \times \exp \left[ \frac{1.0784 \times 10^{-3} L_w(T_L)}{c_{pd}(\bar{p}, \bar{T}_K) T_L} \times r(1 + 0.81 \times 10^{-3} r) \right]. \quad (42)$$

The error in assuming a fixed value of  $c_{pd}$  can be as large as 0.2 K.

- The error from assuming a fixed value of  $c_w$  in (25) is by contrast small: less than 0.02 K.

- It was shown in Section 3 that the value of  $T_L$  may be in error by 0.1 K, leading to a possible error of 0.03 K in  $\theta_E$ .

Apart from these errors listed above, it must be remembered that a water saturation pseudo-adiabatic process has been assumed throughout this paper. As mentioned in the Introduction, the answer would be altered by several degrees were heat retained by condensed water or latent heat of ice formation included.

## 5. Conclusion

The following formula is recommended for computation of equivalent potential temperature for a water-saturation pseudo-adiabatic process:

$$\theta_E = T_K \left( \frac{1000}{p} \right)^{0.2854(1 - 0.28 \times 10^{-3} r)} \times \exp \left[ \left( \frac{3.376}{T_L} - 0.00254 \right) \times r(1 + 0.81 \times 10^{-3} r) \right], \quad (43)$$

where  $T_K$ ,  $p$ , and  $r$  are the absolute temperature, pressure and mixing ratio at the initial level, and  $T_L$  is the absolute temperature at the lifting condensation level, given by any of (15), (21) or (22). The maximum error in values thus obtained is 0.3 K, the main contribution to the error arising from neglect of variation of the specific heat of dry air with temperature and pressure, an error which also affects the value of the potential temperature  $\theta$ .

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